Exercises

(0.1) **Chaos, Lyapunov, and Entropy Increase.** (Math, Complexity) (With Myers. [72])

Let's consider a simple dynamical system, given by a mappingfrom the unit interval $(0, 1)$ into itself:[1]

$$f(x) = 4\mu x(1 - x). \tag{1}$$

where the time evolution is given by iterating the map:

$$x_0, x_1, x_2, \dots = x_0, f(x_0), f(f(x_0)), \dots \tag{2}$$

In particular, for $\mu = 1$ it precisely folds the unit interval in half, and stretches it (non-uniformly) to cover the original domain.

The mathematics community lumps together continuous dynamical evolution laws and discrete mappings as both being dynamical systems. You can motivate the relationship using the Poincaré sections (figure 4.3), which connect a continuous recirculating dynamical system to the once–return map. The mapping 4.11 is not invertible, so it isn't directly given by a Poincaré section of a smooth differential equation[2] but the general stretching and folding exhibited by our map is often seen in driven physical systems without conservation laws.

In this problem, we will focus on values of μ near one, where the motion is mostly *chaotic*. Chaos is sometimes defined as motion where the final position depends sensitively on the initial conditions. Two trajectories, starting a distance ϵ apart, will typically drift apart in time as $\epsilon e^{\lambda t}$, where λ is the Lyapunov exponent for the chaotic dynamics.

Start with $\mu = 0.9$ and two nearby points x_0 and $y_0 = x_0 + \epsilon$ somewhere between zero and one. Investigate the two trajectories $x_0, f(x_0), f(f(x_0)), \dots f^{[n]}(x_0)$ and $y_0, f(y_0), \dots$. How fast do they separate? Estimate the Lyapunov exponent.

Many Hamiltonian systems are also chaotic. Two configurations of classical atoms or billiard balls, with initial positions and velocities that are almost identical, will rapidly diverge as the collisions magnify small initial deviations in angle and velocity into large ones. It is this chaos that stretches, folds, and kneads phase space (as in the Poincaré cat map of exercise 5.4) that is at root our explanation that entropy increases.[3]

[1] We also study this map in exercises 4.3, 5.13, and 13.8.

[2] Remember the existence and uniqueness theorems from math class? The invertibility follows from uniqueness.

[3] There have been speculations by some physicists that entropy increases through information dropping into black holes – either real ones or tiny virtual black–hole fluctuations (see exercise 5.12. Recent work has cast doubt that the information is really lost even then: we're told it's just scrambled, presumably much as in chaotic systems.