Exercises

(0.1) **Invariant Measures.** (Math, Complexity) (With Myers. [72])

Reading: Reference [47], Roderick V. Jensen and Christopher R. Myers, "Images of the critical points of nonlinear maps" *Physical Review A* **32**, 1222-1224 (1985).

Liouville's theorem tells us that all available points in phase space are equally weighted when a Hamiltonian system is averaged over all times. What happens for systems that evolve according to laws that are not Hamiltonian? Usually, the system does not continue to explore all points in its state space: at long times it is confined a subset of the original space known as the *attractor*.

We consider the behavior of the 'logistic' mapping from the unit interval $(0, 1)$ into itself.[1]

$$f(x) = 4\mu x(1 - x). \tag{1}$$

We talk of the trajectory of an initial point x_0 as the sequence of points x_0, $f(x_0)$, $f(f(x_0))$, ..., $f^{[n]}(x_0)$, ... Iteration can be thought of as a time step (one iteration of a Poincaré return map of exercise 4.2 or one step Δt in a time-step algorithm as in exercise 8.9).

Attracting Fixed Point: For small μ, our mapping has an attracting fixed point. A fixed point of a mapping is a value $x^* = f(x^*)$; a fixed point is stable if under small perturbations shrink:

$$|f(x^* + \epsilon) - x^*| \approx |f'(x^*)|\epsilon < \epsilon, \tag{2}$$

which happens if the derivative $|f'(x^*)| < 1$.[2]

(a) **Iteration:** *Set $\mu = 0.2$; iterate f for some initial points $0 < x_0 < 1$ of your choosing, and convince yourself that they all are attracted to zero. Plot f and the diagonal $y = x$ on the same plot. Are there any fixed points other than $x = 0$? Repeat for $\mu = 0.4$, and 0.6. What happens?*

Analytics: *Find the non-zero fixed point $x^*(\mu)$ of the map 4.11, and show that it exists and is stable for $1/4 < \mu < 3/4$. If you're ambitious or have a computer algebra program, show that there is a stable period–two cycle for $3/4 < \mu < (1 + \sqrt{6})/4$.*

An attracting fixed point is the antithesis of Liouville's theorem: all initial conditions are transient except one, and all systems lead eventually to the same, time–independent state. (On the other hand, this is precisely the behavior we expect in statistical mechanics *on the macroscopic scale*: the system settles down into a time–independent equilibrium state! All microstates are equivalent, but the vast majority of accessible microstates have the same macroscopic behavior in most large systems). We could define a rather trivial "equilibrium ensemble" for this system, which consists of the single point x^*: any property $A(x)$ will have the long–time average $\langle A\rangle = A(x^*)$.

For larger values of μ, more complicated things happen. At $\mu = 1$, the dynamics can be shown to fill the entire interval: the dynamics is ergodic, and the attractor fills the entire set of available states. However, unlike the case of Hamiltonian systems, not all states are weighted equally.

We can find time averages for functions of x in two ways: by averaging over time (many iterates of the map) or by weighting an integral over x by the *invariant density* $\rho(x)$. The invariant density $\rho(x)\,dx$ is the probability that a point on a long trajectory will lie between x and $x + dx$. To find it numerically, we iterate a typical point[3] x_0 a thousand or so times ($N_{\text{transient}}$) to find a point x_a on the attractor, and then collect a long trajectory of perhaps a million points (N_{cycles}). A histogram of this trajectory gives $\rho(x)$. Clearly averaging over this density is the same as averaging over the trajectory of a million points. We call $\rho(x)$ an invariant measure

[1] We also study this map in exercises 5.11, 5.13, and 13.8.

[2] For many dimensional mappings, a sufficient criterion for stability is that all the eigenvalues of the Jacobian have magnitude smaller than one. A continuous time evolution $dy/dt = F(y)$, will be stable if dF/dy is smaller than zero, or (for multidimensional systems) if the Jacobian DF has eigenvalues whose real parts are all less than zero. This is all precisely analogous to discrete and continuous–time Markov chains, see section 8.2

[3] For example, we must not choose an unstable fixed point or unstable periodic orbit!

because it's left invariant under the mapping f: iterating our million–point approximation for ρ once under f only removes the first point x_a and adds one extra point to the end.

(b) **Invariant Density:** *Set $\mu = 1$; iterate f many times, and form a histogram of values giving the density $\rho(x)$ of points along the trajectory. You should find that points x near the boundaries are approached more often than points near the center.*

Analytics: *Use the fact that the long time average $\rho(x)$ must be independent of time, verify for $\mu = 1$ that the density of points is*[4]

$$\rho(x) = \frac{1}{\pi\sqrt{x(1-x)}}. \tag{3}$$

Plot this theoretical curve with your numerical histogram. (Hint: The points in a range dx of a point x map under f to a range $dy = f'(x)\,dx$ around the image $y = f(x)$. Each iteration maps two points x_a and $x_b = 1-x_a$ to y, and thus maps all the density $\rho(x_a)|dx_a|$ and $\rho(x_b)|dx_b|$ into dy. Hence the probability $\rho(y)dy$ must equal $\rho(x_a)|dx_a| + \rho(x_b)|dx_b|$, so

$$\rho(f(x_a)) = \rho(x_a)/|f'(x_a)| + \rho(x_b)/|f'(x_b)| \tag{4}$$

Plug equation 4.13 into equation 4.14. You'll need to factor a polynomial.)

Mathematicians call this probability density $\rho(x)dx$ the *invariant measure* on the attractor.[5] To get the long term average of any function $A(x)$, one can use

$$\langle A \rangle = \int A(x)\rho(x)dx \tag{5}$$

To a mathematician, a measure is a way of weighting different regions in doing integrals – precisely our $\rho(x)dx$. Notice that, for the case of an attracting fixed point, we would have $\rho(x) = \delta(x^*)$.[6]

Cusps in the invariant density: At values of μ slightly smaller than one, our mapping has a rather complex invariant density.

(c) Find the invariant density (as described above) for $\mu = 0.9$. Make your trajectory length N_{cycles} big enough and the bin size small enough to see the interesting structures. Notice that the attractor no longer fills the whole range $(0,1)$: locate roughly where the edges are. Notice also the cusps in $\rho(x)$ at the edges of the attractor, and also at places inside the attractor (called boundaries*, see reprint above). Locate some of the more prominent cusps.*

Analytics of cusps: Notice that $f'(\frac{1}{2}) = 0$. so by equation 4.14 we know that $\rho(f(x)) \geq \rho(x)/|f'(x)|$ must have a singularity near $x = \frac{1}{2}$: all the points near $x = \frac{1}{2}$ are squeezed together and folded to one side by f. Further iterates of this singularity produce more cusps: the crease after one fold stays a crease after being further stretched and kneaded.

(d) Set $\mu = 0.9$. Calculate $f(\frac{1}{2})$, $f(f(\frac{1}{2}))$, ... and compare these iterates to the locations of the edges and cusps from part (c). (You may wish to include them both on the same plot.)

Bifurcation Diagram: The evolution of the attractor and its invariant density as μ varies is plotted in the *bifurcation diagram*, which is shown for large μ in figure 4.4. One of the striking features in this plot are the sharp boundaries formed by the cusps.

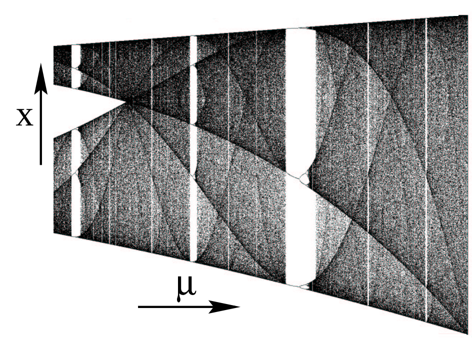

Fig. 1 Bifurcation diagram in the chaotic region. Notice the boundary lines threading through the diagram, images of the crease formed by the folding at $x = \frac{1}{2}$ in our map (see reprint above).

(e) **Bifurcation Diagram:** *Plot the attractor (duplicating figure 4.4) as a function of μ, for $0.8 < \mu < 1$. (Pick regularly spaced $\delta\mu$, run $n_{\text{transient}}$*

[4]You need not derive the factor $1/\pi$, which normalizes the probability density to one.

[5]There are actually many possible invariant measures on some attractors: this one is the SRB measure.[39]

[6]The case of a fixed point then becomes mathematically a measure with a point mass at x^*.

steps, record n_{cycles} *steps, and plot. After the routine is working, you should be able to push* $n_{\text{transient}}$ *and* n_{cycles} *both larger than 100, and* $\delta\mu < 0.01$.*)*

On the same plot, for the same μ*s, plot the first eight images of* $x = {}^1\!/\!_2$, *that is,* $f({}^1\!/\!_2), f(f({}^1\!/\!_2)), \dots$. *Are the boundaries you see just the cusps? What happens in the bifurcation diagram when two boundaries touch? (See the reprint above.)*