Exercises

(0.1) **Fractal Dimensions.** (Math, Complexity) (With Myers. [72])

There are many strange sets that emerge in science. In statistical mechanics, such sets often arise at continuous phase transitions, where self–similar spatial structures arise (chapter 13. In chaotic dynamical systems, the attractor (the set of points occupied at long times after the transients have disappeared) is often a fractal (called a *strange attractor*. These sets often are tenuous and jagged, with holes on all length scales: see figures 13.2, 13.3, and 13.14.

We often try to characterize these strange sets by a dimension. The dimensions of two extremely different sets can be the same: the path exhibited by a random walk (embedded in three or more dimensions) is arguably a two–dimensional set (note 6 on page 15), but does not locally look like a surface! However, if two sets have different spatial dimensions (measured in the same way) they surely are qualitatively different.

There is more than one way to define a dimension. Roughly speaking, strange sets are often spatially inhomogeneous, and what dimension you measure depends upon how you weight different regions of the set. In this exercise, we will calculate the *information dimension* (closely connected to the non-equilibrium entropy!), and the *capacity dimension* (originally called the *Hausdorff dimension*, also sometimes called the *fractal dimension*).

To generate our strange set – along with some more ordinary sets – we will use the logistic map[1]

$$f(x) = 4\mu x(1-x) \tag{1}$$

that we also study in exercises 5.11, 4.3, and 13.8. The attractor for the logistic map is a periodic orbit (dimension zero) at $\mu = 0.8$, and a chaotic, cusped density filling two intervals (dimension one)[2] at $\mu = 0.9$. At the onset of chaos at $\mu = \mu_\infty \approx 0.892486418$ (exercise 13.8) the dimension becomes intermediate between zero and one: the attractor is strange, self–similar set.

Both the information dimension and the capacity dimension are defined in terms of the occupation P_n of cells of size ϵ in the limit as $\epsilon \to 0$.

(a) Write a routine which, given μ and a set of bin sizes ϵ,

- *Iterates f hundreds or thousands of times (to get on the attractor)*
- *Iterates f many more times, collecting points on the attractor. (For $\mu \le \mu_\infty$, you could just integrate 2^n times for n fairly large.)*
- *For each ϵ, use a histogram to calculate the probability P_n that the points fall in the n^{th} bin*
- *Return the set of vectors $P_n[\epsilon]$.*

You may wish to test your routine by using it for $\mu = 1$ (where the distribution should look like $\rho(x) = \frac{1}{\pi\sqrt{x(1-x)}}$, exercise 4.3(b)) and $\mu = 0.8$ (where the distribution should look like two δ-functions, each with half of the points).

The Capacity Dimension. The definition of the capacity dimension is motivated by the idea that it takes at least

$$N_{\text{cover}} = V/\epsilon^D \tag{2}$$

bins of size ϵ^D to cover a D-dimensional set of volume V.[3] By taking logs of both sides we find $\log N_{\text{cover}} \approx \log V + D \log \epsilon$. The capacity dimension is defined as the limit

$$D_{\text{capacity}} = \lim_{\epsilon\to 0} \frac{\log N_{\text{cover}}}{\log \epsilon} \tag{3}$$

but the convergence is slow (the error goes roughly as $\log V/\log\epsilon$). Faster convergence is given by calculating the slope of $\log N$ versus $\log \epsilon$:

$$\begin{aligned} D_{\text{capacity}} &= \lim_{\epsilon\to 0} \frac{d\log N_{\text{cover}}}{d\log \epsilon} \\ &= \lim_{\epsilon\to 0} \frac{\log N_{i+1} - \log N_i}{\log \epsilon_{i+1} - \log \epsilon_i}. \end{aligned} \tag{4}$$

[1] We also study this map in exercises 4.3, 5.11, and 13.8.

[2] See exercise 4.3. The chaotic region for the logistic map isn't a strange attractor because it's confined to one dimension: period doubling cascades for dynamical systems in higher spatial dimensions likely will have fractal, strange attractors in the chaotic region.

[3] Imagine covering the surface of a sphere in 3D with tiny cubes: the number of cubes will go as the surface area [2D-volume] divided by ϵ^2.

(b) Use your routine from part (a), write a routine to calculate $N[\epsilon]$ by counting non-empty bins. Plot D_{capacity} from the fast convergence equation 5.55 versus the midpoint $\frac{1}{2}(\log \epsilon_{i+1} + \log \epsilon_i)$. Does it appear to extrapolate to $D = 1$ for $\mu = 0.9$?[4] *Does it appear to extrapolate to $D = 0$ for $\mu = 0.8$? Plot these two curves together with the curve for μ_∞. Does the last one appear to converge to $D_1 \approx 0.538$, the capacity dimension for the Feigenbaum attractor gleaned from the literature? How small a deviation from μ_∞ does it take to see the numerical crossover to integer dimensions?*

Entropy and the Information Dimension. The entropy of a statistical mechanical system is given by equation 5.23, $S = -k_B \mathrm{Tr}(\rho \log \rho)$. In the chaotic regime this works fine. Our probabilities $P_n \approx \rho(x_n)\epsilon$, so converting the entropy integral into a sum $\int f(x)\,dx \approx \sum_n f(x_n)\epsilon$ gives

$$S = -k_B \int \rho(x) \log(\rho(x))\,dx \qquad (5)$$
$$\approx -\sum_n P_n \log(P_n/\epsilon) = -\sum_n P_n \log P_n + \log \epsilon$$

(setting the conversion factor $k_B = 1$ for convenience).

You might imagine that the entropy for a fixed point would be zero, and the entropy for a period-n cycle would be $k_B \log n$. But this is incorrect: when there is a fixed point or a periodic limit cycle, the attractor is on a set of dimension zero (a bunch of points) rather than dimension one. The entropy must go to minus infinity – since we have precise information about where the trajectory sits at long times. To estimate the "zero–dimensional" entropy $k_B \log n$ on the computer, we would take the same bins as above but sum over bins P_n instead of integrating over x:

$$S_{d=0} = -\sum_n P_n \log(P_n) = S_{d=1} - \log(\epsilon). \qquad (6)$$

More generally, the 'natural' measure of the entropy for a set with D dimensions might be defined as

$$S_D = -\sum_n P_n \log(P_n) + D \log(\epsilon). \qquad (7)$$

Instead of using this formula to define the entropy, mathematicians use it to define the information dimension

$$D_{\mathrm{inf}} = \lim_{\epsilon \to 0} \left(\sum P_n \log P_n \right) / \log(\epsilon). \qquad (8)$$

The information dimension agrees with the ordinary dimension for sets that locally look like $\mathbb{R}^D$. It's different from the capacity dimension because the information dimension weights each part (bin) of the attractor by the time spent in it. Again, we can speed up the convergence by noting that equation 5.58 says that $\sum_n P_n \log P_n$ is a linear function of $\log \epsilon$ with slope D and intercept S_D. Measuring the slope directly, we find

$$D_{\mathrm{inf}} = \lim_{\epsilon \to 0} \frac{d \sum_n P_n(\epsilon) \log P_n(\epsilon)}{d \log \epsilon}. \qquad (9)$$

(c) As in part (b), write a routine that plots D_{inf} from equation 5.60 as a function of the midpoint $\log \epsilon$, as we increase the number of bins. Plot the curves for $\mu = 0.9$, $\mu = 0.8$, and μ_∞. Does the information dimension agree with the ordinary one for the first two? Does the last one appear to converge to $D_1 \approx 0.517098$, the information dimension for the Feigenbaum attractor from the literature?

Most 'real world' fractals have a whole spectrum of different characteristic spatial dimensions: they are *multifractal*.)

[4] In the chaotic regions, keep the number of bins small compared to the number of iterates in your sample, or you start finding empty bins between points and eventually get a dimension of zero.

