Exercises

(0.1) **Period Doubling.** (Math, Complexity) (With Myers. [72])

Chaos is often associated with dynamics which stretch and fold: when a batch of taffy is being pulled, the motion of a speck in the taffy depends sensitively on the initial conditions. A simple representation of this physics is provided by the map[1]

$$f(x) = 4\mu x(1-x) \tag{1}$$

restricted to the domain $(0, 1)$. It takes $f(0) = f(1) = 0$, and $f(1/2) = \mu$. Thus, for $\mu = 1$ it precisely folds the unit interval in half, and stretches it to cover the original domain.

The study of dynamical systems (*e.g.*, differential equations and maps like equation 13.53) often focuses on the behavior after long times, where the trajectory moves along the *attractor*.[2] We can study the onset and behavior of chaos in our system by observing the evolution of the attractor as we change μ. For small enough μ, all points shrink to the origin: the origin is a stable fixed point which attracts the entire interval $x \in (0, 1)$. For larger μ, we first get a stable fixed point inside the interval, and then period doubling.

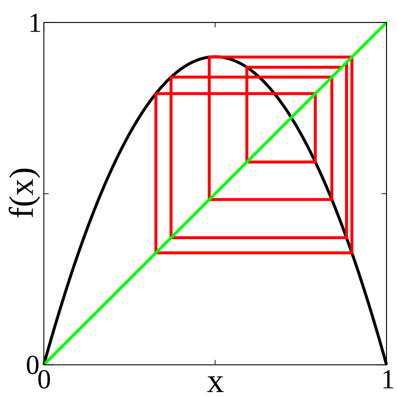

Fig. 1 Period–eight cycle. Iterating around the attractor of the Feigenbaum map at $\mu = 0.9$.

(a) **Iteration:** *Set $\mu = 0.2$; iterate f for some initial points x_0 of your choosing, and convince yourself that they all are attracted to zero. Plot f and the diagonal $y = x$ on the same plot. Are there any fixed points other than $x = 0$? Repeat for $\mu = 0.3$, $\mu = 0.7$, and 0.8. What happens?*

On the same graph, plot f, the diagonal $y = x$, and the segments $\{x_0, x_0\}$, $\{x_0, f(x_0)\}$, $\{f(x_0), f(x_0)\}$, $\{f(x_0), f(f(x_0))\}$, ... (representing the convergence of the trajectory to the attractor: see figure 13.26). See how $\mu = 0.7$ and 0.8 differ. Try other values of μ.

By iterating the map many times, find a point a_0 on the attractor. As above, then plot the successive iterates of a_0 for $\mu = 0.7$, 0.8, 0.88, 0.89, 0.9, and 1.0.

You can see at higher μ that the system no longer settles into a stationary state at long times. The fixed point where $f(x) = x$ exists for all $\mu > 1/4$ – but for larger μ it is no longer *stable*. If x^* is a fixed point (so $f(x^*) = x^*$) we can add a small perturbation $f(x^* + \epsilon) \approx f(x^*) + f'(x^*)\epsilon = x^* + f'(x^*)\epsilon$; the fixed point is stable (perturbations die away) if $|f'(x^*)| < 1$.[3]

In this particular case, once the fixed point goes unstable the motion after many iterations becomes periodic, repeating itself after *two* iterations of the map – so $f(f(x))$ has two fixed points. Notice that by the chain rule $\frac{d\,f(f(x))}{dx} = f'(x)f'(f(x))$, and indeed

$$\frac{d\,f^{[n]}}{dx} = \frac{d\,f(f(\ldots f(x)\ldots))}{dx} \tag{2}$$
$$= f'(x)f'(f(x))\ldots f'(f(\ldots f(x)\ldots)) \tag{3}$$

[1] We also study this map in exercises 4.3, 5.11, and 5.13; parts (a) and (b) below overlap somewhat with exercise 4.3.

[2] In statistical mechanics, we also focus on the behavior at long times, which we call the equilibrium state. Microscopically our systems do not settle down onto attractors: Liouville's theorem 4.1 guarantees that no points of phase space attract others. Of course we have attractors for the macroscopic variables in statistical mechanics.

[3] In a continuous evolution, perturbations die away if the Jacobian of the derivative at the fixed point has all negative eigenvalues. For mappings, perturbations die away if all eigenvalues of the Jacobian have magnitude less than one.

so the stability of a period N orbit is determined by the product of the derivatives of f at each point along the orbit.

(b) **Analytics:** *Find the fixed point $x^*(\mu)$ of the map 13.53, and show that it exists and is stable for $1/4 < \mu < 3/4$. If you're ambitious or have a computer algebra program, show that there is a stable period–two cycle for $3/4 < \mu < (1+\sqrt{6})/4$.*

(c) **Bifurcation Diagram:** *Plot the attractor as a function of μ, for $0 < \mu < 1$: compare with figure 13.22. (Pick regularly spaced $\delta\mu$, run $n_{\mathrm{transient}}$ steps, record n_{cycles} steps, and plot. After the routine is working, you should be able to push $n_{\mathrm{transient}}$ and n_{cycles} both larger than 100, and $\delta\mu < 0.01$.) Also plot the attractor for another one–humped map*

$$f_{\sin}(x) = B \sin(\pi x), \tag{4}$$

for $0 < B < 1$. Do the bifurcation diagrams appear similar to one another?

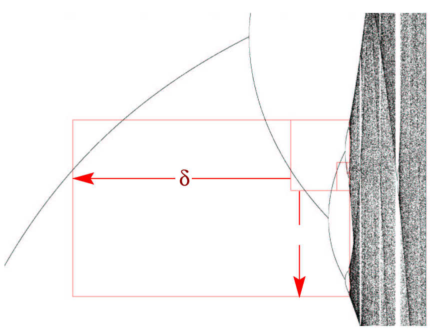

Fig. 2 Self–similarity in period doubling bifurcations. The period doublings occur at geometrically spaced values of the control parameter $\mu_\infty - \mu_n \propto \delta^n$, and the attractor at the n^{th} period doubling is similar to one half of the attractor at the $(n+1)^{\mathrm{th}}$ period doubling, except inverted and larger, rescaled by a factor of α. The boxes shown in the diagram illustrate this self-similarity: each box looks like the next, except expanded by δ along the horizontal μ axis and flipped and expanded by α along the vertical axis.

Notice the complex, structured, chaotic region for large μ (which we study in exercise 4.3). How do we get from a stable fixed point $\mu < 3/4$ to chaos? The onset of chaos in this system occurs through a cascade of *period doublings*. There is the sequence of bifurcations as μ increases – the period two cycle starting at $\mu_1 = 3/4$, followed by a period four cycle starting at μ_2, period eight at μ_3 – a whole period doubling cascade. The convergence appears geometrical, to a fixed point μ_∞:

$$\mu_n \approx \mu_\infty - A\delta^n \tag{5}$$

so

$$\delta = \lim(\mu_{n-1} - \mu_{n-2})/(\mu_n - \mu_{n-1}) \tag{6}$$

and there is a similar geometrical self-similarity along the x axis, with a (negative) scale factor α relating each generation of the tree (figure 13.27).

In exercise 4.3, we explained the boundaries in the chaotic region as images of $x = 1/2$. These special points are also convenient for studying period doubling. Since $x = 1/2$ is the maximum in the curve, $f'(1/2) = 0$. If it were a fixed point (as it is for $\mu = 1/2$), it would not only be stable, but unusually so: a shift by ϵ away from the fixed point converges after one step of the map to a distance $\epsilon f'(1/2) + \epsilon^2/2 f''(1/2) = O(\epsilon^2)$. We say that such a fixed point is *superstable*. If we have a period N orbit that passes through $x = 1/2$, so that the N^{th} iterate $f^N(1/2) \equiv f(\ldots f(1/2)\ldots) = 1/2$, then the orbit is also superstable, since (by equation 13.54) the derivative of the iterated map is the product of the derivatives along the orbit, and hence is also zero.[4]

These superstable points happen roughly halfway between the period–doubling bifurcations, and are easy to locate – since we know that $x = 1/2$ is on the orbit. Let's use them to investigate the geometrical convergence and self-similarity of the period doubling bifurcation diagram from part (d). For this part and part (h), you'll need a routine that finds the roots $G(y) = 0$ for functions G of one variable y.

(d) **The Feigenbaum Numbers and Universality:** *Numerically, find the values of μ_n^s at which the 2^n–cycle is superstable, for the first few values of n.* (Hint: define a function $G(\mu) = f_\mu^{[2^n]}(1/2) - 1/2$, and find the root as a function of μ. In searching for μ_n^s, you'll want to search in a range $(\mu_{n-1}^s + \epsilon, \mu_n^s + (\mu_n^s - \mu_{n-1}^s)/A)$ where $A \sim 3$ works pretty well. Calculate μ_0 and μ_1 by hand.) *Calculate the ratios $\frac{\mu_{n-1}^s - \mu_{n-2}^s}{\mu_n^s - \mu_{n-1}^s}$: do they appear to converge to the*

[4] This is also why there is a cusp in the density at the boundaries in the chaotic region: the derivative of the function is zero, so points near $x = 1/2$ become compressed into a small region to one side of $f(1/2)$.

Feigenbaum number $\delta = 4.6692016091029909\ldots$*? Extrapolate the series to* μ_∞ *by using your last two reliable values of* μ_n^s *and equation 13.58. In the superstable orbit with* 2^n *points, the nearest point to* $x = 1/2$ *is* $f^{[2^{n-1}]}(1/2)$.[5] *Calculate the ratios of the amplitudes* $f^{[2^{n-1}]}(1/2) - 1/2$ *at successive values of* n*; do they appear to converge to the universal value* $\alpha = -2.50290787509589284\ldots$*? Calculate the same ratios for the map* $f_2(x) = B\sin(\pi x)$*: do* α *and* δ *appear to be universal (independent of the mapping)?*

The limits α and δ are independent of the map, so long as it folds (one hump) with a quadratic maximum. They are the same, also, for experimental systems with many degrees of freedom which undergo the period-doubling cascade. This self–similarity and universality suggests that we should look for a renormalization–group explanation.

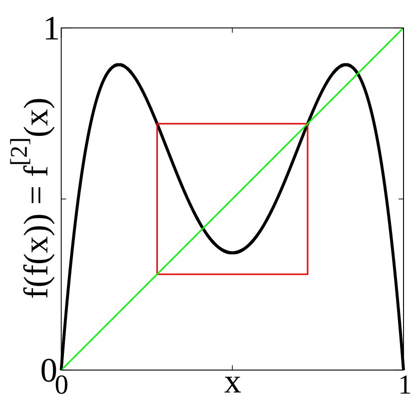

Fig. 3 Renormalization–group transformation. The renormalization–group transformation takes $g(g(x))$ in the small window with upper corner x^* and inverts and stretches it to fill the whole initial domain and range $(0,1) \times (0,1)$.

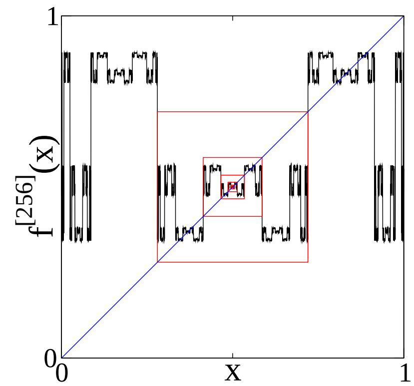

Fig. 4 Self-similar iterated function in period doubling. The function $\lim_{n\to\infty} f^{[2^n]}(x)$ is self-similar at the period–doubling fixed point μ_∞.

(e) **Coarse–graining in time.** *Plot* $f(f(x))$ *vs.* x *for* $\mu = 0.8$*, together with the line* $y = x$ *(or see figure 13.28). Notice that the period–two cycle of* f*, naturally, become a pair of stable fixed points for* $f^{[2]}$*. (We're coarse–graining in time – removing every other point in the time series, by studying* $f(f(x))$ *rather than* f*.) Compare the plot with that for* $f(x)$ *vs.* x *for* $\mu = 0.5$*. Notice that the region around the stable fixed points for* $f^{[2]} = f(f(x))$ *looks quite a bit like that around the fixed point for* f *at the smaller value of* μ*. Plot* $f^{[4]}(x)$ *at* $\mu = 0.875$*; notice again the small one–humped map near* $x = 1/2$*.*

The fact that the one–humped map reappears in smaller form just after the period–doubling bifurcation is the basic reason that succeeding bifurcations so often follow one another. The fact that many things are universal is due to the fact that the little one–humped maps have a shape which becomes *independent of the original map* after several period doublings.

Let's define this renormalization–group transformation T, taking function space into itself. Roughly speaking, T will take the small upside–down hump in $f(f(x))$ (figure 13.28), invert it, and stretch it to cover the interval from $(0,1)$. Notice in your graphs for part (g) that the line $y = x$ crosses the plot $f(f(x))$ not only at the two points on the period–two attractor, but also (of course) at

[5] That's easy to see, since at the previous superstable orbit, 2^{n-1} iterates was the period, and this iterate equaled $x = 1/2$.

[6] For asymmetric maps, we would need to locate this other corner $f(f(x_c)) = x^*$ numerically. As it happens, this asymmetry is irrelevant at the fixed point.

the old fixed point $x^*[f]$ for $f(x)$. This unstable fixed point plays the role for $f^{[2]}$ that the origin played for f: our renormalization–group rescaling must map $(x^*[f], f(x^*)) = (x^*, x^*)$ to the origin. The corner of the window that maps to $(1,0)$ is conveniently located at $1 - x^*$, since our map happens to be symmetric[6] about $x = 1/2$. For a general one–humped map $g(x)$ with fixed point $x^*[g]$ the side of the window is thus of length $2(x^*[g] - 1/2)$. To invert and stretch, we must thus rescale by a factor $\alpha[g] = -1/(2(x^*[g] - 1/2))$. Our renormalization–group transformation is thus a mapping $T[g]$ taking function space into itself, where

$$T[g](x) = \alpha[g]\left(g\left(g(x/\alpha[g] + x^*[g])\right) - x^*[g]\right). \quad (7)$$

(This is just rescaling x to squeeze into the window, applying g twice, shifting the corner of the window to the origin, and then rescaling by α to fill the original range $(0,1) \times (0,1)$.)

(f) **Scaling and the Renormalization Group:** *Write routines that calculate $x^*[g]$ and $\alpha[g]$, and define the renormalization–group transformation $T[g]$. Plot $T[f]$, $T[T[f]]$, ... and compare them. Are we approaching a fixed point f^* in function space?*

This explains the self–similarity: in particular, the value of $\alpha[g]$ as g iterates to f^* becomes the Feigenbaum number $\alpha = -2.5029\ldots$

(g) **Universality and the Renormalization Group:** *Using the sine function of equation 13.56, plot $T[T[f_{\sin}]]$ and compare with $T[T[f]]$. Are they approaching the same fixed point?*

By using this rapid convergence in function space, one can prove both that there will (often) be an infinite geometrical series of period doubling bifurcations leading to chaos, and that this series will share universal features (exponents α and δ and features) that are independent of the original dynamics.