axiom™

The 30 Year Horizon

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Volume 10: Axiom Algebra: Packages

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Contents

New Foreword

On October 1, 2001 Axiom was withdrawn from the market and ended life as a commercial product. On September 3, 2002 Axiom was released under the Modified BSD license, including this document. On August 27, 2003 Axiom was released as free and open source software available for download from the Free Software Foundation's website, Savannah.

Work on Axiom has had the generous support of the Center for Algorithms and Interactive Scientific Computation (CAISS) at City College of New York. Special thanks go to Dr. Gilbert Baumslag for his support of the long term goal.

The online version of this documentation is roughly 1000 pages. In order to make printed versions we've broken it up into three volumes. The first volume is tutorial in nature. The second volume is for programmers. The third volume is reference material. We've also added a fourth volume for developers. All of these changes represent an experiment in print-on-demand delivery of documentation. Time will tell whether the experiment succeeded.

Axiom has been in existence for over thirty years. It is estimated to contain about three hundred man-years of research and has, as of September 3, 2003, 143 people listed in the credits. All of these people have contributed directly or indirectly to making Axiom available. Axiom is being passed to the next generation. I'm looking forward to future milestones.

With that in mind I've introduced the theme of the "30 year horizon". We must invent the tools that support the Computational Mathematician working 30 years from now. How will research be done when every bit of mathematical knowledge is online and instantly available? What happens when we scale Axiom by a factor of 100, giving us 1.1 million domains? How can we integrate theory with code? How will we integrate theorems and proofs of the mathematics with space-time complexity proofs and running code? What visualization tools are needed? How do we support the conceptual structures and semantics of mathematics in effective ways? How do we support results from the sciences? How do we teach the next generation to be effective Computational Mathematicians?

The "30 year horizon" is much nearer than it appears.

Tim Daly
CAISS, City College of New York
November 10, 2003 ((iHy))

Chapter 1

Chapter Overview

This book contains the domains in Axiom, in alphabetical order.

Each domain has an associated 'dotpic' chunk which only lists the domains, categories, and packages that are in the layer immediately below in the build order. For the full list see the algebra Makefile where this information is maintained.

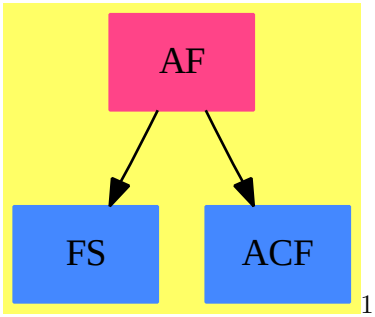
Each domain is preceded by a picture. The picture indicates several things. The colors indicate whether the name refers to a category, domain, or package. An ellipse means that the name refers to something in the bootstrap set. Thus,

Chapter 2

Chapter A

2.1 package AF AlgebraicFunction

2.2 AlgebraicFunction



1.00

Exports:

belong?	droot	definingPolynomial	inrootof	iroot
minPoly	operator	rootOf	?**?	

⟨*package AF AlgebraicFunction*⟩≡

```
)abbrev package AF AlgebraicFunction
++ Author: Manuel Bronstein
++ Date Created: 21 March 1988
++ Date Last Updated: 11 November 1993
++ Description:
++   This package provides algebraic functions over an integral domain.
++ Keywords: algebraic, function.

AlgebraicFunction(R, F): Exports == Implementation where
  R: Join(OrderedSet, IntegralDomain)
```

```
F: FunctionSpace R

SE  ==> Symbol
Z   ==> Integer
Q   ==> Fraction Z
OP  ==> BasicOperator
K   ==> Kernel F
P   ==> SparseMultivariatePolynomial(R, K)
UP  ==> SparseUnivariatePolynomial F
UPR ==> SparseUnivariatePolynomial R
ALGOP       ==> "%alg"
SPECIALDISP ==> "%specialDisp"
SPECIALDIFF ==> "%specialDiff"

Exports ==> with
  rootOf  : (UP, SE) -> F
    ++ rootOf(p, y) returns y such that \spad{p(y) = 0}.
    ++ The object returned displays as \spad{'y}.
  operator: OP -> OP
    ++ operator(op) returns a copy of \spad{op} with the domain-dependent
    ++ properties appropriate for \spad{F}.
    ++ Error: if op is not an algebraic operator, that is,
    ++ an nth root or implicit algebraic operator.
  belong? : OP -> Boolean
    ++ belong?(op) is true if \spad{op} is an algebraic operator, that is,
    ++ an nth root or implicit algebraic operator.
  inrootof: (UP, F) -> F
    ++ inrootof(p, x) should be a non-exported function.
    -- un-export when the compiler accepts conditional local functions!
  droot : List F -> OutputForm
    ++ droot(l) should be a non-exported function.
    -- un-export when the compiler accepts conditional local functions!
  if R has RetractableTo Integer then
    "**"   : (F, Q) -> F
      ++ x ** q is \spad{x} raised to the rational power \spad{q}.
    minPoly: K  -> UP
      ++ minPoly(k) returns the defining polynomial of \spad{k}.
    definingPolynomial: F -> F
      ++ definingPolynomial(f) returns the defining polynomial of \spad{f}
      ++ as an element of \spad{F}.
      ++ Error: if f is not a kernel.
    iroot : (R, Z) -> F
      ++ iroot(p, n) should be a non-exported function.
      -- un-export when the compiler accepts conditional local functions!

Implementation ==> add
```

```
    ialg : List F -> F
    dvalg: (List F, SE) -> F
    dalg : List F -> OutputForm

    opalg  := operator("rootOf"::Symbol)$CommonOperators
    oproot := operator("nthRoot"::Symbol)$CommonOperators

    belong? op == has?(op, ALGOP)
    dalg l     == second(l)::OutputForm

    rootOf(p, x) ==
      k := kernel(x)$K
      (r := retractIfCan(p)@Union(F, "failed")) case "failed" =>
        inrootof(p, k::F)
      n := numer(f := univariate(r::F, k))
      degree denom f > 0 => error "roofOf: variable appears in denom"
      inrootof(n, k::F)

    dvalg(l, x) ==
      p := numer univariate(first l, retract(second l)@K)
      alpha := kernel(opalg, l)
      - (map((s:F):F +-> differentiate(s, x), p) alpha)_
          / ((differentiate p) alpha)

    ialg l ==
      f := univariate(p := first l, retract(x := second l)@K)
      degree denom f > 0 => error "roofOf: variable appears in denom"
      inrootof(numer f, x)

    operator op ==
      is?(op,  "rootOf"::Symbol) => opalg
      is?(op, "nthRoot"::Symbol) => oproot
      error "Unknown operator"

    if R has AlgebraicallyClosedField then
      UP2R: UP -> Union(UPR, "failed")

      inrootof(q, x) ==
        monomial? q => 0

        (d := degree q) <= 0 => error "rootOf: constant polynomial"
--        one? d=> - leadingCoefficient(reductum q) / leadingCoefficient q
        (d = 1) => - leadingCoefficient(reductum q) / leadingCoefficient q
        ((rx := retractIfCan(x)@Union(SE, "failed")) case SE) and
          ((r := UP2R q) case UPR) => rootOf(r::UPR, rx::SE)::F
        kernel(opalg, [q x, x])
```

```
      UP2R p ==
        ans:UPR := 0
        while p ^= 0 repeat
          (r := retractIfCan(leadingCoefficient p)@Union(R, "failed"))
            case "failed" => return "failed"
          ans := ans + monomial(r::R, degree p)
          p   := reductum p
        ans

    else
      inrootof(q, x) ==
        monomial? q => 0
        (d := degree q) <= 0 => error "rootOf: constant polynomial"
--        one? d => - leadingCoefficient(reductum q) /leadingCoefficient q
        (d = 1) => - leadingCoefficient(reductum q) /leadingCoefficient q
        kernel(opalg, [q x, x])

    evaluate(opalg, ialg)$BasicOperatorFunctions1(F)
    setProperty(opalg, SPECIALDIFF,
                              dvalg@((List F, SE) -> F) pretend None)
    setProperty(opalg, SPECIALDISP,
                              dalg@(List F -> OutputForm) pretend None)

    if R has RetractableTo Integer then
      import PolynomialRoots(IndexedExponents K, K, R, P, F)

      dumvar := "%%var"::Symbol::F

      lzero   : List F -> F
      dvroot  : List F -> F
      inroot  : List F -> F
      hackroot: (F, Z) -> F
      inroot0 : (F, Z, Boolean, Boolean) -> F

      lzero l == 0

      droot l ==
        x := first(l)::OutputForm
        (n := retract(second l)@Z) = 2 => root x
        root(x, n::OutputForm)

      dvroot l ==
        n := retract(second l)@Z
        (first(l) ** ((1 - n) / n)) / (n::F)
```

```
      x ** q ==
        qr := divide(numer q, denom q)
        x ** qr.quotient * inroot([x, (denom q)::F]) ** qr.remainder

      hackroot(x, n) ==
        (n = 1) or (x = 1) => x
        (((dx := denom x) ^= 1) and
           ((rx := retractIfCan(dx)@Union(Integer,"failed")) case Integer) and
           positive?(rx))
           => hackroot((numer x)::F, n)/hackroot(rx::Integer::F, n)
        (x = -1) and n = 4 =>
          ((-1::F) ** (1::Q / 2::Q) + 1) / ((2::F) ** (1::Q / 2::Q))
        kernel(oproot, [x, n::F])

      inroot l ==
        zero?(n := retract(second l)@Z) => error "root: exponent = 0"
--        one?(x := first l) or one? n => x
        ((x := first l) = 1) or (n = 1) => x
        (r := retractIfCan(x)@Union(R,"failed")) case R => iroot(r::R,n)
        (u := isExpt(x, oproot)) case Record(var:K, exponent:Z) =>
          pr := u::Record(var:K, exponent:Z)
          (first argument(pr.var)) **
              (pr.exponent /$Fraction(Z)
                   (n * retract(second argument(pr.var))@Z))
        inroot0(x, n, false, false)

-- removes powers of positive integers from numer and denom
-- num? or den? is true if numer or denom already processed
      inroot0(x, n, num?, den?) ==
        rn:Union(Z, "failed") := (num? => "failed"; retractIfCan numer x)
        rd:Union(Z, "failed") := (den? => "failed"; retractIfCan denom x)
        (rn case Z) and (rd case Z) =>
          rec := qroot(rn::Z / rd::Z, n::NonNegativeInteger)
          rec.coef * hackroot(rec.radicand, rec.exponent)
        rn case Z =>
          rec := qroot(rn::Z::Fraction(Z), n::NonNegativeInteger)
          rec.coef * inroot0((rec.radicand**(n exquo rec.exponent)::Z)
                                / (denom(x)::F), n, true, den?)
        rd case Z =>
          rec := qroot(rd::Z::Fraction(Z), n::NonNegativeInteger)
          inroot0((numer(x)::F) /
                  (rec.radicand ** (n exquo rec.exponent)::Z),
                   n, num?, true) / rec.coef
        hackroot(x, n)

      if R has AlgebraicallyClosedField then iroot(r, n) == nthRoot(r, n)::F
```

```
    else
      iroot0: (R, Z) -> F

      if R has RadicalCategory then
        if R has imaginary:() -> R then iroot(r, n) == nthRoot(r, n)::F
        else
          iroot(r, n) ==
            odd? n or r >= 0 => nthRoot(r, n)::F
            iroot0(r, n)

      else iroot(r, n) == iroot0(r, n)

      iroot0(r, n) ==
        rec := rroot(r, n::NonNegativeInteger)
        rec.coef * hackroot(rec.radicand, rec.exponent)

      definingPolynomial x ==
        (r := retractIfCan(x)@Union(K, "failed")) case K =>
          is?(k := r::K, opalg) => first argument k
          is?(k, oproot) =>
            dumvar ** retract(second argument k)@Z - first argument k
          dumvar - x
        dumvar - x

      minPoly k ==
        is?(k, opalg)  =>
           numer univariate(first argument k,
                                          retract(second argument k)@K)
        is?(k, oproot) =>
           monomial(1,retract(second argument k)@Z :: NonNegativeInteger)
             - first(argument k)::UP
        monomial(1, 1) - k::F::UP

      evaluate(oproot, inroot)$BasicOperatorFunctions1(F)
      derivative(oproot, [dvroot, lzero])

    else   -- R is not retractable to Integer
      droot l ==
        x := first(l)::OutputForm
        (n := second l) = 2::F => root x
        root(x, n::OutputForm)

      minPoly k ==
        is?(k, opalg)  =>
           numer univariate(first argument k,
                                          retract(second argument k)@K)
```

```
        monomial(1, 1) - k::F::UP

    setProperty(oproot, SPECIALDISP,
                            droot@(List F -> OutputForm) pretend None)
```

⟨*AF.dotabb*⟩≡

```
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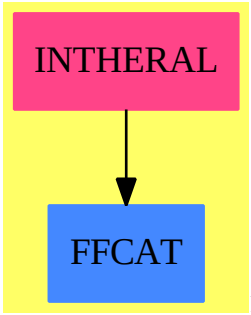
"ACF" [color="#4488FF",href="bookvol10.2.pdf#nameddest=ACF"]
"AF" -> "FS"
"AF" -> "ACF"
```

2.3 package INTHERAL AlgebraicHermiteIntegration

2.4 AlgebraicHermiteIntegration



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Exports:
HermiteIntegrate

⟨*package INTHERAL AlgebraicHermiteIntegration*⟩≡

```
)abbrev package INTHERAL AlgebraicHermiteIntegration
++ Hermite integration, algebraic case
++ Author: Manuel Bronstein
++ Date Created: 1987
++ Date Last Updated: 25 July 1990
++ Description: algebraic Hermite redution.
AlgebraicHermiteIntegration(F,UP,UPUP,R):Exports == Implementation where
  F   : Field
  UP  : UnivariatePolynomialCategory F
  UPUP: UnivariatePolynomialCategory Fraction UP
  R   : FunctionFieldCategory(F, UP, UPUP)

  N   ==> NonNegativeInteger
  RF  ==> Fraction UP

  Exports ==> with
    HermiteIntegrate: (R, UP -> UP) -> Record(answer:R, logpart:R)
      ++ HermiteIntegrate(f, ') returns \spad{[g,h]} such that
      ++ \spad{f = g' + h} and h has a only simple finite normal poles.

  Implementation ==> add
    localsolve: (Matrix UP, Vector UP, UP) -> Vector UP

-- the denominator of f should have no prime factor P s.t. P | P'
-- (which happens only for P = t in the exponential case)
    HermiteIntegrate(f, derivation) ==
```

```
    ratform:R := 0
    n    := rank()
    m    := transpose((mat:= integralDerivationMatrix derivation).num)
    inum := (cform := integralCoordinates f).num
    if ((iden := cform.den) exquo (e := mat.den)) case "failed" then
      iden := (coef := (e exquo gcd(e, iden))::UP) * iden
      inum := coef * inum
    for trm in factors squareFree iden | (j:= trm.exponent) > 1 repeat
      u':=(u:=(iden exquo (v:=trm.factor)**(j::N))::UP) * derivation v
      sys := ((u * v) exquo e)::UP * m
      nn := minRowIndex sys - minIndex inum
      while j > 1 repeat
        j := j - 1
        p := - j * u'
        sol := localsolve(sys + scalarMatrix(n, p), inum, v)
        ratform := ratform + integralRepresents(sol, v ** (j::N))
        inum    := [((qelt(inum, i) - p * qelt(sol, i) -
                      dot(row(sys, i - nn), sol))
                        exquo v)::UP - u * derivation qelt(sol, i)
                           for i in minIndex inum .. maxIndex inum]
      iden := u * v
    [ratform, integralRepresents(inum, iden)]

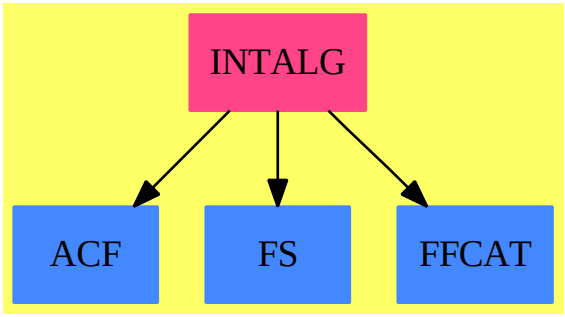
  localsolve(mat, vec, modulus) ==
    ans:Vector(UP) := new(nrows mat, 0)
    diagonal? mat =>
      for i in minIndex ans .. maxIndex ans
        for j in minRowIndex mat .. maxRowIndex mat
          for k in minColIndex mat .. maxColIndex mat repeat
            (bc := extendedEuclidean(qelt(mat, j, k), modulus,
              qelt(vec, i))) case "failed" => return new(0, 0)
            qsetelt_!(ans, i, bc.coef1)
      ans
    sol := particularSolution(
                    map(x+->x::RF, mat)$MatrixCategoryFunctions2(UP,
                      Vector UP, Vector UP, Matrix UP, RF,
                        Vector RF, Vector RF, Matrix RF),
                          map(x+->x::RF, vec)$VectorFunctions2(UP,
                            RF))$LinearSystemMatrixPackage(RF,
                                     Vector RF, Vector RF, Matrix RF)
    sol case "failed" => new(0, 0)
    for i in minIndex ans .. maxIndex ans repeat
      (bc := extendedEuclidean(denom qelt(sol, i), modulus, 1))
        case "failed" => return new(0, 0)
      qsetelt_!(ans, i, (numer qelt(sol, i) * bc.coef1) rem modulus)
    ans
```

⟨*INTHERAL.dotabb*⟩≡

```
"INTHERAL" [color="#FF4488",href="bookvol10.4.pdf#nameddest=INTHERAL"]
"FFCAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=FFCAT"]
"INTHERAL" -> "FFCAT"
```

2.5 package INTALG AlgebraicIntegrate

2.6 AlgebraicIntegrate



1.00

Exports:
algintegrate palginfieldint palgintegrate

⟨*package INTALG AlgebraicIntegrate*⟩≡

```
)abbrev package INTALG AlgebraicIntegrate
++ Integration of an algebraic function
++ Author: Manuel Bronstein
++ Date Created: 1987
++ Date Last Updated: 19 May 1993
++ Description:
++ This package provides functions for integrating a function
++ on an algebraic curve.
AlgebraicIntegrate(R0, F, UP, UPUP, R): Exports == Implementation where
  R0   : Join(OrderedSet, IntegralDomain, RetractableTo Integer)
  F    : Join(AlgebraicallyClosedField, FunctionSpace R0)
  UP   : UnivariatePolynomialCategory F
  UPUP : UnivariatePolynomialCategory Fraction UP
  R    : FunctionFieldCategory(F, UP, UPUP)

  SE  ==> Symbol
  Z   ==> Integer
  Q   ==> Fraction Z
  SUP ==> SparseUnivariatePolynomial F
  QF  ==> Fraction UP
  GP  ==> LaurentPolynomial(F, UP)
  K   ==> Kernel F
  IR  ==> IntegrationResult R
  UPQ ==> SparseUnivariatePolynomial Q
  UPR ==> SparseUnivariatePolynomial R
  FRQ ==> Factored UPQ
  FD  ==> FiniteDivisor(F, UP, UPUP, R)
```

```
FAC ==> Record(factor:UPQ, exponent:Z)
LOG ==> Record(scalar:Q, coeff:UPR, logand:UPR)
DIV ==> Record(num:R, den:UP, derivden:UP, gd:UP)
FAIL0 ==> error "integrate: implementation incomplete (constant residues)"
FAIL1==> error "integrate: implementation incomplete (non-algebraic residues)"
FAIL2 ==> error "integrate: implementation incomplete (residue poly has multipl
FAIL3 ==> error "integrate: implementation incomplete (has polynomial part)"
NOTI  ==> error "Not integrable (provided residues have no relations)"

Exports ==> with
  algintegrate  : (R, UP -> UP) -> IR
    ++ algintegrate(f, d) integrates f with respect to the derivation d.
  palgintegrate : (R, UP -> UP) -> IR
    ++ palgintegrate(f, d) integrates f with respect to the derivation d.
    ++ Argument f must be a pure algebraic function.
  palginfieldint: (R, UP -> UP) -> Union(R, "failed")
    ++ palginfieldint(f, d) returns an algebraic function g
    ++ such that \spad{dg = f} if such a g exists, "failed" otherwise.
    ++ Argument f must be a pure algebraic function.

Implementation ==> add
  import FD
  import DoubleResultantPackage(F, UP, UPUP, R)
  import PointsOfFiniteOrder(R0, F, UP, UPUP, R)
  import AlgebraicHermiteIntegration(F, UP, UPUP, R)
  import InnerCommonDenominator(Z, Q, List Z, List Q)
  import FunctionSpaceUnivariatePolynomialFactor(R0, F, UP)
  import PolynomialCategoryQuotientFunctions(IndexedExponents K,
                       K, R0, SparseMultivariatePolynomial(R0, K), F)

  F2R        : F  -> R
  F2UPR      : F  -> UPR
  UP2SUP     : UP -> SUP
  SUP2UP     : SUP -> UP
  UPQ2F      : UPQ -> UP
  univ       : (F, K) -> QF
  pLogDeriv  : (LOG, R -> R) -> R
  nonLinear  : List FAC -> Union(FAC, "failed")
  mkLog      : (UP, Q, R, F) -> List LOG
  R2UP       : (R, K) -> UPR
  alglogint  : (R, UP -> UP) -> Union(List LOG, "failed")
  palglogint : (R, UP -> UP) -> Union(List LOG, "failed")
  trace00    : (DIV, UP, List LOG) -> Union(List LOG,"failed")
  trace0     : (DIV, UP, Q, FD)    -> Union(List LOG, "failed")
  trace1     : (DIV, UP, List Q, List FD, Q) -> Union(List LOG, "failed")
  nonQ       : (DIV, UP)           -> Union(List LOG, "failed")
```

```
    rlift      : (F, K, K) -> R
    varRoot?   : (UP, F -> F) -> Boolean
    algintexp  : (R, UP -> UP) -> IR
    algintprim : (R, UP -> UP) -> IR

    dummy:R := 0

    dumx  := kernel(new()$SE)$K
    dumy  := kernel(new()$SE)$K

    F2UPR f == F2R(f)::UPR
    F2R f   == f::UP::QF::R

    algintexp(f, derivation) ==
      d := (c := integralCoordinates f).den
      v := c.num
      vp:Vector(GP) := new(n := #v, 0)
      vf:Vector(QF) := new(n, 0)
      for i in minIndex v .. maxIndex v repeat
        r := separate(qelt(v, i) / d)$GP
        qsetelt_!(vf, i, r.fracPart)
        qsetelt_!(vp, i, r.polyPart)
      ff := represents(vf, w := integralBasis())
      h := HermiteIntegrate(ff, derivation)
      p := represents(
             map((x1:GP):QF+->convert(x1)@QF, vp)$VectorFunctions2(GP, QF), w)
      zero?(h.logpart) and zero? p => h.answer::IR
      (u := alglogint(h.logpart, derivation)) case "failed" =>
                      mkAnswer(h.answer, empty(), [[p + h.logpart, dummy]])
      zero? p => mkAnswer(h.answer, u::List(LOG), empty())
      FAIL3

    algintprim(f, derivation) ==
      h := HermiteIntegrate(f, derivation)
      zero?(h.logpart) => h.answer::IR
      (u := alglogint(h.logpart, derivation)) case "failed" =>
                      mkAnswer(h.answer, empty(), [[h.logpart, dummy]])
      mkAnswer(h.answer, u::List(LOG), empty())

    -- checks whether f = +/[ci (ui)'/(ui)]
    -- f dx must have no pole at infinity
    palglogint(f, derivation) ==
      rec := algSplitSimple(f, derivation)
      ground?(r := doubleResultant(f, derivation)) => "failed"
-- r(z) has roots which are the residues of f at all its poles
      (u  := qfactor r) case "failed" => nonQ(rec, r)
```

```
      (fc := nonLinear(lf := factors(u::FRQ))) case "failed" => FAIL2
-- at this point r(z) = fc(z) (z - b1)^e1 .. (z - bk)^ek
-- where the ri's are rational numbers, and fc(z) is arbitrary
-- (fc can be linear too)
-- la = [b1....,bk]  (all rational residues)
      la := [- coefficient(q.factor, 0) for q in remove_!(fc::FAC, lf)]
-- ld = [D1,...,Dk] where Di is the sum of places where f has residue bi
      ld  := [divisor(rec.num, rec.den, rec.derivden, rec.gd, b::F) for b in la]
      pp  := UPQ2F(fc.factor)
-- bb = - sum of all the roots of fc (i.e. the other residues)
      zero?(bb := coefficient(fc.factor,
           (degree(fc.factor) - 1)::NonNegativeInteger)) =>
              -- cd = [[a1,...,ak], d]  such that bi = ai/d
              cd  := splitDenominator la
              -- g = gcd(a1,...,ak), so bi = (g/d) ci  with ci = bi / g
              -- so [g/d] is a basis for [a1,...,ak] over the integers
              g   := gcd(cd.num)
              -- dv0 is the divisor +/[ci Di] corresponding to all the residues
              -- of f except the ones which are root of fc(z)
              dv0 := +/[(a quo g) * dv for a in cd.num for dv in ld]
              trace0(rec, pp, g / cd.den, dv0)
      trace1(rec, pp, la, ld, bb)


    UPQ2F p ==
      map((x:Q):F+->x::F,p)$UnivariatePolynomialCategoryFunctions2(Q,UPQ,F,UP)

    UP2SUP p ==
      map((x:F):F+->x,p)$UnivariatePolynomialCategoryFunctions2(F, UP, F, SUP)

    SUP2UP p ==
      map((x:F):F+->x,p)$UnivariatePolynomialCategoryFunctions2(F, SUP, F, UP)

    varRoot?(p, derivation) ==
      for c in coefficients primitivePart p repeat
        derivation(c) ^= 0 => return true
      false

    pLogDeriv(log, derivation) ==
      map(derivation, log.coeff) ^= 0 =>
                 error "can only handle logs with constant coefficients"
--      one?(n := degree(log.coeff)) =>
      ((n := degree(log.coeff)) = 1) =>
        c := - (leadingCoefficient reductum log.coeff)
             / (leadingCoefficient log.coeff)
        ans := (log.logand) c
```

```
        (log.scalar)::R * c * derivation(ans) / ans
      numlog := map(derivation, log.logand)
      (diflog := extendedEuclidean(log.logand, log.coeff, numlog)) case
          "failed" => error "this shouldn't happen"
      algans := diflog.coef1
      ans:R := 0
      for i in 0..n-1 repeat
        algans := (algans * monomial(1, 1)) rem log.coeff
        ans    := ans + coefficient(algans, i)
      (log.scalar)::R * ans

    R2UP(f, k) ==
      x := dumx :: F
      g :=
       (map((f1:QF):F+->f1(x), lift f)_
        $UnivariatePolynomialCategoryFunctions2(QF,UPUP,F,UP))
          (y := dumy::F)
      map((x1:F):R+->rlift(x1, dumx, dumy), univariate(g, k, minPoly k))_
        $UnivariatePolynomialCategoryFunctions2(F,SUP,R,UPR)

    univ(f, k) ==
      g := univariate(f, k)
      (SUP2UP numer g) / (SUP2UP denom g)

    rlift(f, kx, ky) ==
      reduce map(x1+->univ(x1, kx), retract(univariate(f, ky))@SUP)_
        $UnivariatePolynomialCategoryFunctions2(F,SUP,QF,UPUP)

    nonQ(rec, p) ==
      empty? rest(lf := factors ffactor primitivePart p) =>
                       trace00(rec, first(lf).factor, empty()$List(LOG))
      FAIL1

-- case when the irreducible factor p has roots which sum to 0
-- p is assumed doubly transitive for now
    trace0(rec, q, r, dv0) ==
      lg:List(LOG) :=
        zero? dv0 => empty()
        (rc0 := torsionIfCan dv0) case "failed" => NOTI
        mkLog(1, r / (rc0.order::Q), rc0.function, 1)
      trace00(rec, q, lg)

    trace00(rec, pp, lg) ==
      p0 := divisor(rec.num, rec.den, rec.derivden, rec.gd,
                    alpha0 := zeroOf UP2SUP pp)
      q  := (pp exquo (monomial(1, 1)$UP - alpha0::UP))::UP
```

```
      alpha := rootOf UP2SUP q
      dvr := divisor(rec.num, rec.den, rec.derivden, rec.gd, alpha) - p0
      (rc := torsionIfCan dvr) case "failed" =>
        degree(pp) <= 2 => "failed"
        NOTI
      concat(lg, mkLog(q, inv(rc.order::Q), rc.function, alpha))

-- case when the irreducible factor p has roots which sum <> 0
-- the residues of f are of the form [a1,...,ak] rational numbers
-- plus all the roots of q(z), which is squarefree
-- la is the list of residues la := [a1,...,ak]
-- ld is the list of divisors [D1,...Dk] where Di is the sum of all the
-- places where f has residue ai
-- q(z) is assumed doubly transitive for now.
-- let [alpha_1,...,alpha_m] be the roots of q(z)
-- in this function, b = - alpha_1 - ... - alpha_m is <> 0
-- which implies only one generic log term
    trace1(rec, q, la, ld, b) ==
-- cd = [[b1,...,bk], d]  such that ai / b = bi / d
      cd  := splitDenominator [a / b for a in la]
-- then, a basis for all the residues of f over the integers is
-- [beta_1 = - alpha_1 / d,..., beta_m = - alpha_m / d], since:
--      alpha_i = - d beta_i
--      ai = (ai / b) * b = (bi / d) * b = b1 * beta_1 + ... + bm * beta_m
-- linear independence is a consequence of the doubly transitive assumption
-- v0 is the divisor +/[bi Di] corresponding to the residues [a1,...,ak]
      v0 := +/[a * dv for a in cd.num for dv in ld]
-- alpha is a generic root of q(z)
      alpha := rootOf UP2SUP q
-- v is the divisor corresponding to all the residues
      v := v0 - cd.den * divisor(rec.num, rec.den, rec.derivden, rec.gd, alpha)
      (rc := torsionIfCan v) case "failed" =>   -- non-torsion case
        degree(q) <= 2 => "failed"       -- guaranteed doubly-transitive
        NOTI                             -- maybe doubly-transitive
      mkLog(q, inv((- rc.order * cd.den)::Q), rc.function, alpha)

    mkLog(q, scalr, lgd, alpha) ==
      degree(q) <= 1 =>
        [[scalr, monomial(1, 1)$UPR - F2UPR alpha, lgd::UPR]]
      [[scalr,
         map(F2R, q)$UnivariatePolynomialCategoryFunctions2(F,UP,R,UPR),
                                          R2UP(lgd, retract(alpha)@K)]]

-- return the non-linear factor, if unique
-- or any linear factor if they are all linear
    nonLinear l ==
```

```
        found:Boolean := false
        ans := first l
        for q in l repeat
          if degree(q.factor) > 1 then
            found => return "failed"
            found := true
            ans   := q
        ans

  -- f dx must be locally integral at infinity
      palginfieldint(f, derivation) ==
        h := HermiteIntegrate(f, derivation)
        zero?(h.logpart) => h.answer
        "failed"

  -- f dx must be locally integral at infinity
      palgintegrate(f, derivation) ==
        h := HermiteIntegrate(f, derivation)
        zero?(h.logpart) => h.answer::IR
        (not integralAtInfinity?(h.logpart)) or
          ((u := palglogint(h.logpart, derivation)) case "failed") =>
                        mkAnswer(h.answer, empty(), [[h.logpart, dummy]])
        zero?(difFirstKind := h.logpart - +/[pLogDeriv(lg,
              x1+->differentiate(x1, derivation)) for lg in u::List(LOG)]) =>
                  mkAnswer(h.answer, u::List(LOG), empty())
        mkAnswer(h.answer, u::List(LOG), [[difFirstKind, dummy]])

  -- for mixed functions. f dx not assumed locally integral at infinity
      algintegrate(f, derivation) ==
        zero? degree(x' := derivation(x := monomial(1, 1)$UP)) =>
           algintprim(f, derivation)
        ((xx := x' exquo x) case UP) and
          (retractIfCan(xx::UP)@Union(F, "failed") case F) =>
            algintexp(f, derivation)
        error "should not happen"

      alglogint(f, derivation) ==
        varRoot?(doubleResultant(f, derivation),
                           x1+->retract(derivation(x1::UP))@F) => "failed"
        FAIL0
```

⟨*INTALG.dotabb*⟩≡

```
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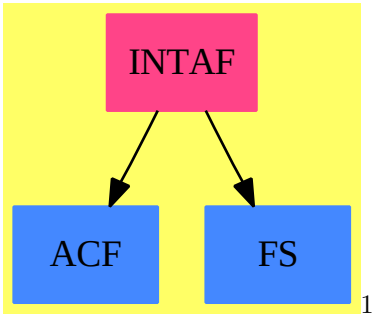
"ACF" [color="#4488FF",href="bookvol10.2.pdf#nameddest=ACF"]
"FS" [color="#4488FF",href="bookvol10.2.pdf#nameddest=FS"]
"FFCAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=FFCAT"]
"INTALG" -> "ACF"
"INTALG" -> "FS"
"INTALG" -> "FFCAT"
```

2.7 package INTAF AlgebraicIntegration

2.8 AlgebraicIntegration



1.00

Exports:
algint

⟨*package INTAF AlgebraicIntegration*⟩≡
```
)abbrev package INTAF AlgebraicIntegration
++ Mixed algebraic integration;
++ Author: Manuel Bronstein
++ Date Created: 12 October 1988
++ Date Last Updated: 4 June 1988
++ Description:
++ This package provides functions for the integration of
++ algebraic integrands over transcendental functions;
AlgebraicIntegration(R, F): Exports == Implementation where
  R : Join(OrderedSet, IntegralDomain)
  F : Join(AlgebraicallyClosedField, FunctionSpace R)

  SY  ==> Symbol
  N   ==> NonNegativeInteger
  K   ==> Kernel F
  P   ==> SparseMultivariatePolynomial(R, K)
  UP  ==> SparseUnivariatePolynomial F
  RF  ==> Fraction UP
  UPUP==> SparseUnivariatePolynomial RF
  IR  ==> IntegrationResult F
  IR2 ==> IntegrationResultFunctions2(curve, F)
  ALG ==> AlgebraicIntegrate(R, F, UP, UPUP, curve)
  FAIL==> error "failed - cannot handle that integrand"

  Exports ==> with
    algint: (F, K, K, UP -> UP) -> IR
      ++ algint(f, x, y, d) returns the integral of \spad{f(x,y)dx}
```

```
    ++ where y is an algebraic function of x;
    ++ d is the derivation to use on \spad{k[x]}.

  Implementation ==> add
    import ChangeOfVariable(F, UP, UPUP)
    import PolynomialCategoryQuotientFunctions(IndexedExponents K,
                                                    K, R, P, F)

    rootintegrate: (F, K, K, UP -> UP) -> IR
    algintegrate : (F, K, K, UP -> UP) -> IR
    UPUP2F       : (UPUP, RF, K, K) -> F
    F2UPUP       : (F, K, K, UP) -> UPUP
    UP2UPUP      : (UP, K) -> UPUP

    F2UPUP(f, kx, k, p) == UP2UPUP(univariate(f, k, p), kx)

    rootintegrate(f, t, k, derivation) ==
      r1     := mkIntegral(modulus := UP2UPUP(p := minPoly k, t))
      f1     := F2UPUP(f, t, k, p) monomial(inv(r1.coef), 1)
      r      := radPoly(r1.poly)::Record(radicand:RF, deg:N)
      q      := retract(r.radicand)
      curve  := RadicalFunctionField(F, UP, UPUP, q::RF, r.deg)
      map(x1+->UPUP2F(lift x1, r1.coef, t, k),
                           algintegrate(reduce f1, derivation)$ALG)$IR2

    algintegrate(f, t, k, derivation) ==
      r1     := mkIntegral(modulus := UP2UPUP(p := minPoly k, t))
      f1     := F2UPUP(f, t, k, p) monomial(inv(r1.coef), 1)
      modulus:= UP2UPUP(p := minPoly k, t)
      curve  := AlgebraicFunctionField(F, UP, UPUP, r1.poly)
      map(x1+->UPUP2F(lift x1, r1.coef, t, k),
                           algintegrate(reduce f1, derivation)$ALG)$IR2

    UP2UPUP(p, k) ==
      map(x1+->univariate(x1,k),p)$SparseUnivariatePolynomialFunctions2(F,RF)

    UPUP2F(p, cf, t, k) ==
      map((x1:RF):F+->multivariate(x1, t),
         p)$SparseUnivariatePolynomialFunctions2(RF, F)
                                          (multivariate(cf, t) * k::F)

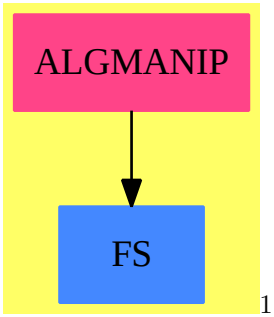
    algint(f, t, y, derivation) ==
      is?(y, "nthRoot"::SY) => rootintegrate(f, t, y, derivation)
      is?(y, "rootOf"::SY)  => algintegrate(f, t, y, derivation)
      FAIL
```

⟨*INTAF.dotabb*⟩≡

```
"INTAF" [color="#FF4488",href="bookvol10.4.pdf#nameddest=INTAF"]
"ACF" [color="#4488FF",href="bookvol10.2.pdf#nameddest=ACF"]
"FS" [color="#4488FF",href="bookvol10.2.pdf#nameddest=FS"]
"INTAF" -> "ACF"
"INTAF" -> "FS"
```

2.9 package ALGMANIP AlgebraicManipulations

2.10 AlgebraicManipulations



1.00

Exports:

ratDenom	ratPoly	rootKerSimp	rootPower
rootProduct	rootSimp	rootSplit	

⟨*package ALGMANIP AlgebraicManipulations*⟩≡

```
)abbrev package ALGMANIP AlgebraicManipulations
++ Author: Manuel Bronstein
++ Date Created: 28 Mar 1988
++ Date Last Updated: 5 August 1993
++ Description:
++ AlgebraicManipulations provides functions to simplify and expand
++ expressions involving algebraic operators.
++ Keywords: algebraic, manipulation.
AlgebraicManipulations(R, F): Exports == Implementation where
  R : IntegralDomain
  F : Join(Field, ExpressionSpace) with
    numer  : $ -> SparseMultivariatePolynomial(R, Kernel $)
        ++ numer(x) \undocumented
    denom  : $ -> SparseMultivariatePolynomial(R, Kernel $)
        ++ denom(x) \undocumented
    coerce : SparseMultivariatePolynomial(R, Kernel $) -> $
        ++ coerce(x) \undocumented

  N  ==> NonNegativeInteger
  Z  ==> Integer
  OP ==> BasicOperator
  SY ==> Symbol
  K  ==> Kernel F
  P  ==> SparseMultivariatePolynomial(R, K)
  RF ==> Fraction P
  REC ==> Record(ker:List K, exponent: List Z)
```

```
ALGOP ==> "%alg"
NTHR  ==> "nthRoot"

Exports ==> with
  rootSplit: F -> F
    ++ rootSplit(f) transforms every radical of the form
    ++ \spad{(a/b)**(1/n)} appearing in f into \spad{a**(1/n) / b**(1/n)}.
    ++ This transformation is not in general valid for all
    ++ complex numbers \spad{a} and b.
  ratDenom  : F -> F
    ++ ratDenom(f) rationalizes the denominators appearing in f
    ++ by moving all the algebraic quantities into the numerators.
  ratDenom  : (F, F) -> F
    ++ ratDenom(f, a) removes \spad{a} from the denominators in f
    ++ if \spad{a} is an algebraic kernel.
  ratDenom  : (F, List F) -> F
    ++ ratDenom(f, [a1,...,an]) removes the ai's which are
    ++ algebraic kernels from the denominators in f.
  ratDenom  : (F, List K) -> F
    ++ ratDenom(f, [a1,...,an]) removes the ai's which are
    ++ algebraic from the denominators in f.
  ratPoly  : F -> SparseUnivariatePolynomial F
    ++ ratPoly(f) returns a polynomial p such that p has no
    ++ algebraic coefficients, and \spad{p(f) = 0}.
  if R has Join(OrderedSet, GcdDomain, RetractableTo Integer)
    and F has FunctionSpace(R) then
      rootPower  : F -> F
        ++ rootPower(f) transforms every radical power of the form
        ++ \spad{(a**(1/n))**m} into a simpler form if \spad{m} and
        ++ \spad{n} have a common factor.
      rootProduct: F -> F
        ++ rootProduct(f) combines every product of the form
        ++ \spad{(a**(1/n))**m * (a**(1/s))**t} into a single power
        ++ of a root of \spad{a}, and transforms every radical power
        ++ of the form \spad{(a**(1/n))**m} into a simpler form.
      rootSimp   : F -> F
        ++ rootSimp(f) transforms every radical of the form
        ++ \spad{(a * b**(q*n+r))**(1/n)} appearing in f into
        ++ \spad{b**q * (a * b**r)**(1/n)}.
        ++ This transformation is not in general valid for all
        ++ complex numbers b.
      rootKerSimp: (OP, F, N) -> F
        ++ rootKerSimp(op,f,n) should be local but conditional.

Implementation ==> add
  import PolynomialCategoryQuotientFunctions(IndexedExponents K,K,R,P,F)
```

```
    innerRF    : (F, List K) -> F
    rootExpand : K -> F
    algkernels : List K -> List K
    rootkernels: List K -> List K

    dummy := kernel(new()$SY)$K

    ratDenom x                == innerRF(x, algkernels tower x)
    ratDenom(x:F, l:List K):F == innerRF(x, algkernels l)
    ratDenom(x:F, y:F)        == ratDenom(x, [y])
    ratDenom(x:F, l:List F)   == ratDenom(x, [retract(y)@K for y in l]$List(K))
    algkernels l  == select_!((z1:K):Boolean +-> has?(operator z1, ALGOP), l)
    rootkernels l == select_!((z1:K):Boolean +-> is?(operator z1, NTHR::SY), l)

    ratPoly x ==
      numer univariate(denom(ratDenom inv(dummy::P::F - x))::F, dummy)

    rootSplit x ==
      lk := rootkernels tower x
      eval(x, lk, [rootExpand k for k in lk])

    rootExpand k ==
      x  := first argument k
      n  := second argument k
      op := operator k
      op(numer(x)::F, n) / op(denom(x)::F, n)

  -- all the kernels in ll must be algebraic
    innerRF(x, ll) ==
      empty?(l := sort_!((z1:K,z2:K):Boolean +-> z1 > z2,kernels x)$List(K)) or
        empty? setIntersection(ll, tower x) => x
      lk := empty()$List(K)
      while not member?(k := first l, ll) repeat
        lk := concat(k, lk)
        empty?(l := rest l) =>
          return eval(x, lk, [map((z3:F):F+->innerRF(z3,ll), kk) for kk in lk])
      q := univariate(eval(x, lk,
             [map((z4:F):F+->innerRF(z4,ll),kk) for kk in lk]),k,minPoly k)
      map((z5:F):F+->innerRF(z5, ll), q) (map((z6:F):F+->innerRF(z6, ll), k))

    if R has Join(OrderedSet, GcdDomain, RetractableTo Integer)
     and F has FunctionSpace(R) then
      import PolynomialRoots(IndexedExponents K, K, R, P, F)

      sroot  : K -> F
```

```
        inroot : (OP, F, N) -> F
        radeval: (P, K) -> F
        breakup: List K -> List REC

        if R has RadicalCategory then
          rootKerSimp(op, x, n) ==
            (r := retractIfCan(x)@Union(R, "failed")) case R =>
               nthRoot(r::R, n)::F
            inroot(op, x, n)
        else
          rootKerSimp(op, x, n) == inroot(op, x, n)

  -- l is a list of nth-roots, returns a list of records of the form
  -- [a**(1/n1),a**(1/n2),...], [n1,n2,...]]
  -- such that the whole list covers l exactly
        breakup l ==
          empty? l => empty()
          k := first l
          a := first(arg := argument(k := first l))
          n := retract(second arg)@Z
          expo := empty()$List(Z)
          others := same := empty()$List(K)
          for kk in rest l repeat
            if (a = first(arg := argument kk)) then
              same := concat(kk, same)
              expo := concat(retract(second arg)@Z, expo)
            else others := concat(kk, others)
          ll := breakup others
          concat([concat(k, same), concat(n, expo)], ll)

        rootProduct x ==
          for rec in breakup rootkernels tower x repeat
            k0 := first(l := rec.ker)
            nx := numer x; dx := denom x
            if empty? rest l then x := radeval(nx, k0) / radeval(dx, k0)
            else
              n  := lcm(rec.exponent)
              k  := kernel(operator k0, [first argument k0, n::F], height k0)$K
              lv := [monomial(1, k, (n quo m)::N) for m in rec.exponent]$List(P)
              x  := radeval(eval(nx, l, lv), k) / radeval(eval(dx, l, lv), k)
          x

        rootPower x ==
          for k in rootkernels tower x repeat
            x := radeval(numer x, k) / radeval(denom x, k)
          x
```

```
  -- replaces (a**(1/n))**m in p by a power of a simpler radical of a if
  -- n and m have a common factor
        radeval(p, k) ==
          a := first(arg := argument k)
          n := (retract(second arg)@Integer)::NonNegativeInteger
          ans:F := 0
          q := univariate(p, k)
          while (d := degree q) > 0 repeat
            term :=
  --            one?(g := gcd(d, n)) => monomial(1, k, d)
              ((g := gcd(d, n)) = 1) => monomial(1, k, d)
              monomial(1, kernel(operator k, [a,(n quo g)::F], height k), d quo g)
            ans := ans + leadingCoefficient(q)::F * term::F
            q := reductum q
          leadingCoefficient(q)::F + ans

        inroot(op, x, n) ==
  --        one? x => x
          (x = 1) => x
  --        (x ^= -1) and (one?(num := numer x) or (num = -1)) =>
          (x ^= -1) and (((num := numer x) = 1) or (num = -1)) =>
            inv inroot(op, (num * denom x)::F, n)
          (u := isExpt(x, op)) case "failed" => kernel(op, [x, n::F])
          pr := u::Record(var:K, exponent:Integer)
          q := pr.exponent /$Fraction(Z)
                                  (n * retract(second argument(pr.var))@Z)
          qr := divide(numer q, denom q)
          x  := first argument(pr.var)
          x ** qr.quotient * rootKerSimp(op,x,denom(q)::N) ** qr.remainder

        sroot k ==
          pr := froot(first(arg := argument k),(retract(second arg)@Z)::N)
          pr.coef * rootKerSimp(operator k, pr.radicand, pr.exponent)

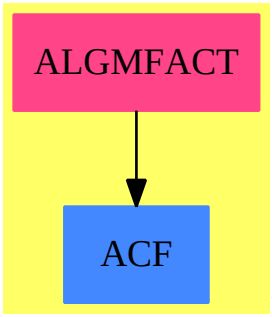
        rootSimp x ==
          lk := rootkernels tower x
          eval(x, lk, [sroot k for k in lk])
```

⟨*ALGMANIP.dotabb*⟩≡

```
  "ALGMANIP" [color="#FF4488",href="bookvol10.4.pdf#nameddest=ALGMANIP"]
  "FS" [color="#4488FF",href="bookvol10.2.pdf#nameddest=FS"]
  "ALGMANIP" -> "FS"
```

2.11 package ALGMFACT AlgebraicMultFact

2.12 AlgebraicMultFact



1.00

Exports:
factor

⟨*package ALGMFACT AlgebraicMultFact*⟩≡

```
)abbrev package ALGMFACT AlgebraicMultFact
++ Author: P. Gianni
++ Date Created: 1990
++ Date Last Updated:
++ Basic Functions:
++ Related Constructors:
++ Also See:
++ AMS Classifications:
++ Keywords:
++ References:
++ Description:
++ This package factors multivariate polynomials over the
++ domain of \spadtype{AlgebraicNumber} by allowing the user
++ to specify a list of algebraic numbers generating the particular
++ extension to factor over.

AlgebraicMultFact(OV,E,P) : C == T
 where
  AN          ==> AlgebraicNumber
  OV          :   OrderedSet
  E           :   OrderedAbelianMonoidSup
  P           :   PolynomialCategory(AN,E,OV)
  BP          ==> SparseUnivariatePolynomial AN
  Z             ==> Integer
  MParFact      ==> Record(irr:P,pow:Z)
  USP           ==> SparseUnivariatePolynomial P
  SUParFact     ==> Record(irr:USP,pow:Z)
```

```
SUPFinalFact ==> Record(contp:R,factors:List SUParFact)
MFinalFact   ==> Record(contp:R,factors:List MParFact)

               --  contp   =  content,
               --  factors =  List of irreducible factors with exponent
L          ==> List

C == with
  factor      :   (P,L AN)  ->  Factored P
    ++ factor(p,lan) factors the polynomial p over the extension
    ++ generated by the algebraic numbers given by the list lan.
  factor      :   (USP,L AN)  ->  Factored USP
    ++ factor(p,lan) factors the polynomial p over the extension
    ++ generated by the algebraic numbers given by the list lan.
    ++ p is presented as a univariate polynomial with multivariate
    ++ coefficients.
T == add

  AF := AlgFactor(BP)

  INNER ==> InnerMultFact(OV,E,AN,P)

  factor(p:P,lalg:L AN) : Factored P ==
    factor(p,(z1:BP):Factored(BP) +-> factor(z1,lalg)$AF)$INNER

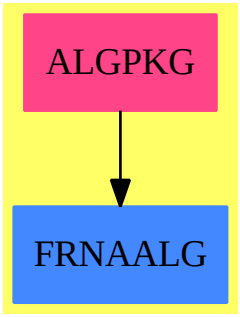
  factor(up:USP,lalg:L AN) : Factored USP ==
    factor(up,(z1:BP):Factored(BP) +-> factor(z1,lalg)$AF)$INNER
```

⟨*ALGMFACT.dotabb*⟩≡

```
"ALGMFACT" [color="#FF4488",href="bookvol10.4.pdf#nameddest=ALGMFACT"]
"ACF" [color="#4488FF",href="bookvol10.2.pdf#nameddest=ACF"]
"ALGMFACT" -> "ACF"
```

2.13 package ALGPKG AlgebraPackage

2.14 AlgebraPackage



1.00

Exports:

basis	basisOfCenter	basisOfCentroid	basisOfCommutingElements	basis
basisOfLeftNucleus	basisOfLeftNucloid	basisOfMiddleNucleus	basisOfNucleus	basis
basisOfRightNucleus	basisOfRightNucloid	biRank	doubleRank	leftR
radicalOfLeftTraceForm	rightRank	weakBiRank		

⟨*package ALGPKG AlgebraPackage*⟩≡

```
)abbrev package ALGPKG AlgebraPackage
++ Authors: J. Grabmeier, R. Wisbauer
++ Date Created: 04 March 1991
++ Date Last Updated: 04 April 1992
++ Basic Operations:
++ Related Constructors:
++ Also See:
++ AMS Classifications:
++ Keywords: rank, nucleus, nucloid, structural constants
++ Reference:
++  R.S. Pierce: Associative Algebras
++  Graduate Texts in Mathematics 88
++  Springer-Verlag,  Heidelberg, 1982, ISBN 0-387-90693-2
++
++  R.D. Schafer: An Introduction to Nonassociative Algebras
++  Academic Press, New York, 1966
++
++  A. Woerz-Busekros: Algebra in Genetics
++  Lectures Notes in Biomathematics 36,
++  Springer-Verlag,  Heidelberg, 1980
++ Description:
++  AlgebraPackage assembles a variety of useful functions for
++  general algebras.
AlgebraPackage(R:IntegralDomain, A: FramedNonAssociativeAlgebra(R)): _
```

```
 public == private where

V  ==> Vector
M  ==> Matrix
I  ==> Integer
NNI  ==> NonNegativeInteger
REC  ==> Record(particular: Union(V R,"failed"),basis: List V R)
LSMP ==> LinearSystemMatrixPackage(R,V R,V R, M R)

public ==>  with

    leftRank: A -> NonNegativeInteger
      ++ leftRank(x) determines the number of linearly independent elements
      ++ in \spad{x*b1},...,\spad{x*bn},
      ++ where \spad{b=[b1,...,bn]} is a basis.
    rightRank: A -> NonNegativeInteger
      ++ rightRank(x) determines the number of linearly independent elements
      ++ in \spad{b1*x},...,\spad{bn*x},
      ++ where \spad{b=[b1,...,bn]} is a basis.
    doubleRank: A -> NonNegativeInteger
      ++ doubleRank(x) determines the number of linearly
      ++ independent elements
      ++ in \spad{b1*x},...,\spad{x*bn},
      ++ where \spad{b=[b1,...,bn]} is a basis.
    weakBiRank: A -> NonNegativeInteger
      ++ weakBiRank(x) determines the number of
      ++ linearly independent elements
      ++ in the \spad{bi*x*bj}, \spad{i,j=1,...,n},
      ++ where \spad{b=[b1,...,bn]} is a basis.
    biRank: A -> NonNegativeInteger
      ++ biRank(x) determines the number of linearly independent elements
      ++ in \spad{x}, \spad{x*bi}, \spad{bi*x}, \spad{bi*x*bj},
      ++ \spad{i,j=1,...,n},
      ++ where \spad{b=[b1,...,bn]} is a basis.
      ++ Note: if \spad{A} has a unit,
      ++ then \spadfunFrom{doubleRank}{AlgebraPackage},
      ++ \spadfunFrom{weakBiRank}{AlgebraPackage}
      ++ and \spadfunFrom{biRank}{AlgebraPackage} coincide.
    basisOfCommutingElements: () -> List A
      ++ basisOfCommutingElements() returns a basis of the space of
      ++ all x of \spad{A} satisfying \spad{0 = commutator(x,a)} for all
      ++ \spad{a} in \spad{A}.
    basisOfLeftAnnihilator: A -> List A
      ++ basisOfLeftAnnihilator(a) returns a basis of the space of
      ++ all x of \spad{A} satisfying \spad{0 = x*a}.
    basisOfRightAnnihilator: A -> List A
```

```
    ++ basisOfRightAnnihilator(a) returns a basis of the space of
    ++ all x of \spad{A} satisfying \spad{0 = a*x}.
  basisOfLeftNucleus: () -> List A
    ++ basisOfLeftNucleus() returns a basis of the space of
    ++ all x of \spad{A} satisfying \spad{0 = associator(x,a,b)}
    ++ for all \spad{a},b in \spad{A}.
  basisOfRightNucleus: () -> List A
    ++ basisOfRightNucleus() returns a basis of the space of
    ++ all x of \spad{A} satisfying \spad{0 = associator(a,b,x)}
    ++ for all \spad{a},b in \spad{A}.
  basisOfMiddleNucleus: () -> List A
    ++ basisOfMiddleNucleus() returns a basis of the space of
    ++ all x of \spad{A} satisfying \spad{0 = associator(a,x,b)}
    ++ for all \spad{a},b in \spad{A}.
  basisOfNucleus: () -> List A
    ++ basisOfNucleus() returns a basis of the space of
    ++ all x of \spad{A} satisfying
    ++ \spad{associator(x,a,b) = associator(a,x,b) = associator(a,b,x) = 0}
    ++ for all \spad{a},b in \spad{A}.
  basisOfCenter: () -> List A
    ++ basisOfCenter() returns a basis of the space of
    ++ all x of \spad{A} satisfying \spad{commutator(x,a) = 0} and
    ++ \spad{associator(x,a,b) = associator(a,x,b) = associator(a,b,x) = 0}
    ++ for all \spad{a},b in \spad{A}.
  basisOfLeftNucloid:()-> List Matrix R
    ++ basisOfLeftNucloid() returns a basis of the space of
    ++ endomorphisms of \spad{A} as right module.
    ++ Note: left nucloid coincides with left nucleus
    ++ if \spad{A} has a unit.
  basisOfRightNucloid:()-> List Matrix R
    ++ basisOfRightNucloid() returns a basis of the space of
    ++ endomorphisms of \spad{A} as left module.
    ++ Note: right nucloid coincides with right nucleus
    ++ if \spad{A} has a unit.
  basisOfCentroid:()-> List Matrix R
    ++ basisOfCentroid() returns a basis of the centroid, i.e. the
    ++ endomorphism ring of \spad{A} considered as \spad{(A,A)}-bimodule.
  radicalOfLeftTraceForm: () -> List A
    ++ radicalOfLeftTraceForm() returns basis for null space of
    ++ \spad{leftTraceMatrix()}, if the algebra is
    ++ associative, alternative or a Jordan algebra, then this
    ++ space equals the radical (maximal nil ideal) of the algebra.
  if R has EuclideanDomain then
    basis : V A ->  V A
      ++ basis(va) selects a basis from the elements of va.
```

```
private ==>  add

    -- constants

    n  : PositiveInteger := rank()$A
    n2 : PositiveInteger := n*n
    n3 : PositiveInteger := n*n2
    gamma : Vector Matrix R  := structuralConstants()$A


    -- local functions

    convVM : Vector R -> Matrix R
      -- converts n2-vector to (n,n)-matrix row by row
    convMV : Matrix R -> Vector R
      -- converts n-square matrix to  n2-vector row by row
    convVM v  ==
      cond : Matrix(R) := new(n,n,0$R)$M(R)
      z : Integer := 0
      for i in 1..n repeat
        for j in 1..n  repeat
          z := z+1
          setelt(cond,i,j,v.z)
      cond


    -- convMV m ==
    --     vec : Vector(R) := new(n*n,0$R)
    --     z : Integer := 0
    --     for i in 1..n repeat
    --       for j in 1..n  repeat
    --         z := z+1
    --         setelt(vec,z,elt(m,i,j))
    --     vec


    radicalOfLeftTraceForm() ==
      ma : M R := leftTraceMatrix()$A
      map(represents, nullSpace ma)$ListFunctions2(Vector R, A)


    basisOfLeftAnnihilator a ==
      ca : M R := transpose (coordinates(a) :: M R)
      cond : M R := reduce(vertConcat$(M R),
        [ca*transpose(gamma.i) for i in 1..#gamma])
```

```
    map(represents, nullSpace cond)$ListFunctions2(Vector R, A)

  basisOfRightAnnihilator a ==
    ca : M R := transpose (coordinates(a) :: M R)
    cond : M R := reduce(vertConcat$(M R),
      [ca*(gamma.i) for i in 1..#gamma])
    map(represents, nullSpace cond)$ListFunctions2(Vector R, A)

  basisOfLeftNucloid() ==
    cond : Matrix(R) := new(n3,n2,0$R)$M(R)
    condo: Matrix(R) := new(n3,n2,0$R)$M(R)
    z : Integer := 0
    for i in 1..n repeat
      for j in 1..n repeat
        r1  : Integer := 0
        for k in 1..n repeat
          z := z + 1
          -- z equals (i-1)*n*n+(j-1)*n+k (loop-invariant)
          r2 : Integer := i
          for r in 1..n repeat
            r1 := r1 + 1
            -- here r1 equals (k-1)*n+r (loop-invariant)
            setelt(cond,z,r1,elt(gamma.r,i,j))
            -- here r2 equals (r-1)*n+i (loop-invariant)
            setelt(condo,z,r2,-elt(gamma.k,r,j))
            r2 := r2 + n
    [convVM(sol) for sol in nullSpace(cond+condo)]

  basisOfCommutingElements() ==
    --gamma1 := first gamma
    --gamma1 := gamma1 - transpose gamma1
    --cond : Matrix(R) := gamma1 :: Matrix(R)
    --for  i in  2..n repeat
    --  gammak := gamma.i
    --  gammak := gammak - transpose gammak
    --  cond :=  vertConcat(cond, gammak :: Matrix(R))$Matrix(R)
    --map(represents, nullSpace cond)$ListFunctions2(Vector R, A)

    cond : M R := reduce(vertConcat$(M R),
      [(gam := gamma.i) - transpose gam for i in 1..#gamma])
    map(represents, nullSpace cond)$ListFunctions2(Vector R, A)

  basisOfLeftNucleus() ==
    condi: Matrix(R) := new(n3,n,0$R)$Matrix(R)
    z : Integer := 0
    for k in 1..n repeat
```

```
   for j in 1..n repeat
    for s in 1..n repeat
      z := z+1
      for i in 1..n repeat
        entry : R := 0
        for l in 1..n repeat
          entry :=  entry+elt(gamma.l,j,k)*elt(gamma.s,i,l)_
                         -elt(gamma.l,i,j)*elt(gamma.s,l,k)
        setelt(condi,z,i,entry)$Matrix(R)
  map(represents, nullSpace condi)$ListFunctions2(Vector R,A)

basisOfRightNucleus() ==
  condo : Matrix(R) := new(n3,n,0$R)$Matrix(R)
  z : Integer := 0
  for k in 1..n repeat
   for j in 1..n repeat
    for s in 1..n repeat
      z := z+1
      for i in 1..n repeat
        entry : R := 0
        for l in 1..n repeat
          entry :=  entry+elt(gamma.l,k,i)*elt(gamma.s,j,l) _
                         -elt(gamma.l,j,k)*elt(gamma.s,l,i)
        setelt(condo,z,i,entry)$Matrix(R)
  map(represents, nullSpace condo)$ListFunctions2(Vector R,A)

basisOfMiddleNucleus() ==
  conda : Matrix(R) := new(n3,n,0$R)$Matrix(R)
  z : Integer := 0
  for k in 1..n repeat
   for j in 1..n repeat
    for s in 1..n repeat
      z := z+1
      for i in 1..n repeat
        entry : R := 0
        for l in 1..n repeat
          entry :=  entry+elt(gamma.l,j,i)*elt(gamma.s,l,k)
                         -elt(gamma.l,i,k)*elt(gamma.s,j,l)
        setelt(conda,z,i,entry)$Matrix(R)
  map(represents, nullSpace conda)$ListFunctions2(Vector R,A)


basisOfNucleus() ==
  condi: Matrix(R) := new(3*n3,n,0$R)$Matrix(R)
  z : Integer := 0
  u : Integer := n3
```

```
    w : Integer := 2*n3
    for k in 1..n repeat
     for j in 1..n repeat
      for s in 1..n repeat
        z := z+1
        u := u+1
        w := w+1
        for i in 1..n repeat
          entry : R := 0
          enter : R := 0
          ent   : R := 0
          for l in 1..n repeat
            entry :=  entry + elt(gamma.l,j,k)*elt(gamma.s,i,l) _
                            - elt(gamma.l,i,j)*elt(gamma.s,l,k)
            enter :=  enter + elt(gamma.l,k,i)*elt(gamma.s,j,l) _
                            - elt(gamma.l,j,k)*elt(gamma.s,l,i)
            ent :=  ent  +  elt(gamma.l,j,k)*elt(gamma.s,i,l) _
                         -  elt(gamma.l,j,i)*elt(gamma.s,l,k)
          setelt(condi,z,i,entry)$Matrix(R)
          setelt(condi,u,i,enter)$Matrix(R)
          setelt(condi,w,i,ent)$Matrix(R)
    map(represents, nullSpace condi)$ListFunctions2(Vector R,A)

  basisOfCenter() ==
    gamma1 := first gamma
    gamma1 := gamma1 - transpose gamma1
    cond : Matrix(R) := gamma1 :: Matrix(R)
    for  i in  2..n repeat
      gammak := gamma.i
      gammak := gammak - transpose gammak
      cond :=  vertConcat(cond, gammak :: Matrix(R))$Matrix(R)
    B := cond :: Matrix(R)
    condi: Matrix(R) := new(2*n3,n,0$R)$Matrix(R)
    z : Integer := 0
    u : Integer := n3
    for k in 1..n repeat
     for j in 1..n repeat
      for s in 1..n repeat
        z := z+1
        u := u+1
        for i in 1..n repeat
          entry : R := 0
          enter : R := 0
          for l in 1..n repeat
            entry :=  entry + elt(gamma.l,j,k)*elt(gamma.s,i,l) _
                            - elt(gamma.l,i,j)*elt(gamma.s,l,k)
```

```
            enter :=  enter + elt(gamma.l,k,i)*elt(gamma.s,j,l) _
                            - elt(gamma.l,j,k)*elt(gamma.s,l,i)
          setelt(condi,z,i,entry)$Matrix(R)
          setelt(condi,u,i,enter)$Matrix(R)
    D := vertConcat(condi,B)$Matrix(R)
    map(represents, nullSpace D)$ListFunctions2(Vector R, A)

  basisOfRightNucloid() ==
    cond : Matrix(R) := new(n3,n2,0$R)$M(R)
    condo: Matrix(R) := new(n3,n2,0$R)$M(R)
    z : Integer := 0
    for i in 1..n repeat
      for j in 1..n repeat
        r1  : Integer := 0
        for k in 1..n repeat
          z := z + 1
          -- z equals (i-1)*n*n+(j-1)*n+k (loop-invariant)
          r2 : Integer := i
          for r in 1..n repeat
            r1 := r1 + 1
            -- here r1 equals (k-1)*n+r (loop-invariant)
            setelt(cond,z,r1,elt(gamma.r,j,i))
            -- here r2 equals (r-1)*n+i (loop-invariant)
            setelt(condo,z,r2,-elt(gamma.k,j,r))
            r2 := r2 + n
    [convVM(sol) for sol in nullSpace(cond+condo)]

  basisOfCentroid() ==
    cond : Matrix(R) := new(2*n3,n2,0$R)$M(R)
    condo: Matrix(R) := new(2*n3,n2,0$R)$M(R)
    z : Integer := 0
    u : Integer := n3
    for i in 1..n repeat
      for j in 1..n repeat
        r1  : Integer := 0
        for k in 1..n repeat
          z := z + 1
          u := u + 1
          -- z equals (i-1)*n*n+(j-1)*n+k (loop-invariant)
          -- u equals n**3 + (i-1)*n*n+(j-1)*n+k (loop-invariant)
          r2 : Integer := i
          for r in 1..n repeat
            r1 := r1 + 1
            -- here r1 equals (k-1)*n+r (loop-invariant)
            setelt(cond,z,r1,elt(gamma.r,i,j))
            setelt(cond,u,r1,elt(gamma.r,j,i))
```

```
            -- here r2 equals (r-1)*n+i (loop-invariant)
            setelt(condo,z,r2,-elt(gamma.k,r,j))
            setelt(condo,u,r2,-elt(gamma.k,j,r))
            r2 := r2 + n
    [convVM(sol) for sol in nullSpace(cond+condo)]


doubleRank x ==
    cond : Matrix(R) := new(2*n,n,0$R)
    for k in 1..n repeat
     z : Integer := 0
     u : Integer := n
     for j in 1..n repeat
       z := z+1
       u := u+1
       entry : R := 0
       enter : R := 0
       for i in 1..n repeat
         entry := entry + elt(x,i)*elt(gamma.k,j,i)
         enter := enter + elt(x,i)*elt(gamma.k,i,j)
       setelt(cond,z,k,entry)$Matrix(R)
       setelt(cond,u,k,enter)$Matrix(R)
    rank(cond)$(M R)

weakBiRank(x) ==
    cond : Matrix(R) := new(n2,n,0$R)$Matrix(R)
    z : Integer := 0
    for i in 1..n repeat
      for j in 1..n repeat
        z := z+1
        for k in 1..n repeat
          entry : R := 0
          for l in 1..n repeat
           for s in 1..n repeat
            entry:=entry+elt(x,l)*elt(gamma.s,i,l)*elt(gamma.k,s,j)
          setelt(cond,z,k,entry)$Matrix(R)
    rank(cond)$(M R)

biRank(x) ==
    cond : Matrix(R) := new(n2+2*n+1,n,0$R)$Matrix(R)
    z : Integer := 0
    for j in 1..n repeat
      for i in 1..n repeat
        z := z+1
        for k in 1..n repeat
          entry : R := 0
```

```
          for l in 1..n repeat
           for s in 1..n repeat
            entry:=entry+elt(x,l)*elt(gamma.s,i,l)*elt(gamma.k,s,j)
          setelt(cond,z,k,entry)$Matrix(R)
    u : Integer := n*n
    w : Integer := n*(n+1)
    c := n2 + 2*n + 1
    for j in 1..n repeat
       u := u+1
       w := w+1
       for k in 1..n repeat
         entry : R := 0
         enter : R := 0
         for i in 1..n repeat
           entry := entry + elt(x,i)*elt(gamma.k,j,i)
           enter := enter + elt(x,i)*elt(gamma.k,i,j)
         setelt(cond,u,k,entry)$Matrix(R)
         setelt(cond,w,k,enter)$Matrix(R)
       setelt(cond,c,j, elt(x,j))
    rank(cond)$(M R)

  leftRank x ==
    cond : Matrix(R) := new(n,n,0$R)
    for k in 1..n repeat
     for j in 1..n repeat
       entry : R := 0
       for i in 1..n repeat
         entry := entry + elt(x,i)*elt(gamma.k,i,j)
       setelt(cond,j,k,entry)$Matrix(R)
    rank(cond)$(M R)

  rightRank x ==
    cond : Matrix(R) := new(n,n,0$R)
    for k in 1..n repeat
     for j in 1..n repeat
       entry : R := 0
       for i in 1..n repeat
         entry := entry + elt(x,i)*elt(gamma.k,j,i)
       setelt(cond,j,k,entry)$Matrix(R)
    rank(cond)$(M R)


  if R has EuclideanDomain then
    basis va ==
      v : V A := remove(zero?, va)$(V A)
      v : V A := removeDuplicates v
```

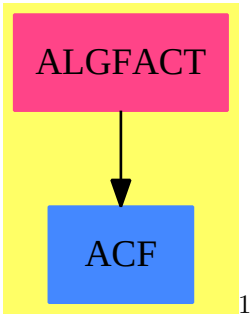
```
empty? v =>  [0$A]
m : Matrix R := coerce(coordinates(v.1))$(Matrix R)
for i in 2..maxIndex v repeat
  m := horizConcat(m,coerce(coordinates(v.i))$(Matrix R) )
m := rowEchelon m
lj : List Integer := []
h : Integer := 1
mRI : Integer := maxRowIndex m
mCI : Integer := maxColIndex m
finished? : Boolean := false
j : Integer := 1
while not finished? repeat
  not zero? m(h,j) =>  -- corner found
    lj := cons(j,lj)
    h := mRI
    while zero? m(h,j) repeat h := h-1
    finished? := (h = mRI)
    if not finished? then h := h+1
  if j < mCI then
    j := j + 1
  else
    finished? := true
[v.j for j in reverse lj]
```

⟨*ALGPKG.dotabb*⟩≡

```
"ALGPKG" [color="#FF4488",href="bookvol10.4.pdf#nameddest=ALGPKG"]
"FRNAALG" [color="#4488FF",href="bookvol10.2.pdf#nameddest=FRNAALG"]
"ALGPKG" -> "FRNAALG"
```

2.15 package ALGFACT AlgFactor

2.16 AlgFactor



1.00

Exports:
doublyTransitive? factor split

⟨*package ALGFACT AlgFactor*⟩≡

```
)abbrev package ALGFACT AlgFactor
++ Factorization of UP AN;
++ Author: Manuel Bronstein
++ Date Created: ???
++ Date Last Updated: ???
++ Description:
++ Factorization of univariate polynomials with coefficients in
++ \spadtype{AlgebraicNumber}.

AlgFactor(UP): Exports == Implementation where
  UP: UnivariatePolynomialCategory AlgebraicNumber

  N   ==> NonNegativeInteger
  Z   ==> Integer
  Q   ==> Fraction Integer
  AN  ==> AlgebraicNumber
  K   ==> Kernel AN
  UPQ ==> SparseUnivariatePolynomial Q
  SUP ==> SparseUnivariatePolynomial AN
  FR  ==> Factored UP

  Exports ==> with
    factor: (UP, List AN) -> FR
      ++ factor(p, [a1,...,an]) returns a prime factorisation of p
      ++ over the field generated by its coefficients and a1,...,an.
    factor: UP            -> FR
      ++ factor(p) returns a prime factorisation of p
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      ++ over the field generated by its coefficients.
    split : UP            -> FR
      ++ split(p) returns a prime factorisation of p
      ++ over its splitting field.
    doublyTransitive?: UP -> Boolean
      ++ doublyTransitive?(p) is true if p is irreducible over
      ++ over the field K generated by its coefficients, and
      ++ if \spad{p(X) / (X - a)} is irreducible over
      ++ \spad{K(a)} where \spad{p(a) = 0}.

  Implementation ==> add
    import PolynomialCategoryQuotientFunctions(IndexedExponents K,
                          K, Z, SparseMultivariatePolynomial(Z, K), AN)

    UPCF2 ==> UnivariatePolynomialCategoryFunctions2

    fact    : (UP,  List K) -> FR
    ifactor : (SUP, List K) -> Factored SUP
    extend  : (UP, Z) -> FR
    allk    : List AN -> List K
    downpoly: UP  -> UPQ
    liftpoly: UPQ -> UP
    irred?  : UP  -> Boolean

    allk l       == removeDuplicates concat [kernels x for x in l]
    liftpoly p   == map(x +-> x::AN,  p)$UPCF2(Q, UPQ, AN, UP)
    downpoly p   == map(x +-> retract(x)@Q, p)$UPCF2(AN, UP ,Q, UPQ)
    ifactor(p,l) == (fact(p pretend UP, l)) pretend Factored(SUP)
    factor p     == fact(p, allk coefficients p)

    factor(p, l) ==
      fact(p, allk removeDuplicates concat(l, coefficients p))

    split p ==
      fp := factor p
      unit(fp) *
            _*/[extend(fc.factor, fc.exponent) for fc in factors fp]

    extend(p, n) ==
--      one? degree p => primeFactor(p, n)
      (degree p = 1) => primeFactor(p, n)
      q := monomial(1, 1)$UP - zeroOf(p pretend SUP)::UP
      primeFactor(q, n) * split((p exquo q)::UP) ** (n::N)

    doublyTransitive? p ==
      irred? p and irred?((p exquo
```

```
        (monomial(1, 1)$UP - zeroOf(p pretend SUP)::UP))::UP)

    irred? p ==
      fp := factor p
--      one? numberOfFactors fp and one? nthExponent(fp, 1)
      (numberOfFactors fp = 1) and  (nthExponent(fp, 1) = 1)

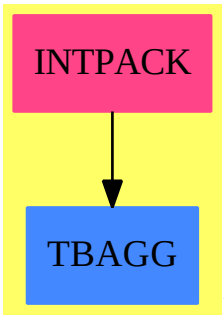
    fact(p, l) ==
--      one? degree p => primeFactor(p, 1)
      (degree p = 1) => primeFactor(p, 1)
      empty? l =>
        dr := factor(downpoly p)$RationalFactorize(UPQ)
        (liftpoly unit dr) *
          _*/[primeFactor(liftpoly dc.factor,dc.exponent)
            for dc in factors dr]
      q   := minPoly(alpha := "max"/l)$AN
      newl  := remove((x:K):Boolean +-> alpha = x, l)
      sae := SimpleAlgebraicExtension(AN, SUP, q)
      ups := SparseUnivariatePolynomial sae
      fr  := factor(map(x +-> reduce univariate(x, alpha, q),p)_
               $UPCF2(AN, UP, sae, ups),_
                 x +-> ifactor(x, newl))$InnerAlgFactor(AN, SUP, sae, ups)
      newalpha := alpha::AN
      map((x:sae):AN +-> (lift(x)$sae) newalpha, unit fr)_
        $UPCF2(sae, ups, AN, UP) *
         _*/[primeFactor(map((y:sae):AN +-> (lift(y)$sae) newalpha,fc.factor)_
            $UPCF2(sae, ups, AN, UP),
              fc.exponent) for fc in factors fr]
```

⟨*ALGFACT.dotabb*⟩≡

```
"ALGFACT" [color="#FF4488",href="bookvol10.4.pdf#nameddest=ALGFACT"]
"ACF" [color="#4488FF",href="bookvol10.2.pdf#nameddest=ACF"]
"ALGFACT" -> "ACF"
```

2.17 package INTPACK AnnaNumericalIntegrationPackage

2.18 AnnaNumericalIntegrationPackage



1.00

Exports:
integrate measure

⟨*package INTPACK AnnaNumericalIntegrationPackage*⟩≡

```
)abbrev package INTPACK AnnaNumericalIntegrationPackage
++ Author: Brian Dupee
++ Date Created: August 1994
++ Date Last Updated: December 1997
++ Basic Operations: integrate, measure
++ Related Constructors: Result, RoutinesTable
++ Also See:
++ AMS Classifications:
++ Keywords:
++ References:
++ Description:
++ \axiomType{AnnaNumericalIntegrationPackage} is a \axiom{package}
++ of functions for the \axiom{category}
++ \axiomType{NumericalIntegrationCategory}
++ with \axiom{measure}, and \axiom{integrate}.
EDF     ==> Expression DoubleFloat
DF      ==> DoubleFloat
EF      ==> Expression Float
F       ==> Float
INT     ==> Integer
SOCDF   ==> Segment OrderedCompletion DoubleFloat
OCDF    ==> OrderedCompletion DoubleFloat
SBOCF   ==> SegmentBinding OrderedCompletion Float
LSOCF   ==> List Segment OrderedCompletion Float
SOCF    ==> Segment OrderedCompletion Float
OCF     ==> OrderedCompletion Float
```

```
LS      ==> List Symbol
S       ==> Symbol
LST     ==> List String
ST      ==> String
RT      ==> RoutinesTable
NIA     ==> Record(var:S, fn:EDF, range:SOCDF, abserr:DF, relerr:DF)
MDNIA   ==> Record(fn:EDF,range:List SOCDF,abserr:DF,relerr:DF)
IFL     ==> List(Record(ifail:Integer,instruction:String))
Entry   ==> Record(chapter:String, type:String, domainName: String,
                   defaultMin:F, measure:F, failList:IFL, explList:List String)
Measure ==> Record(measure:F, name:ST, explanations:LST, extra:Result)


AnnaNumericalIntegrationPackage(): with

  integrate: (EF,SOCF,F,F,RT) -> Result
    ++ integrate(exp, a..b, epsrel, routines) is a top level ANNA function
    ++ to integrate an expression, {\tt exp}, over a given range {\tt a}
    ++ to {\tt b} to the required absolute and relative accuracy using
    ++ the routines available in the RoutinesTable provided.
    ++
    ++ It iterates over the \axiom{domains} of
    ++ \axiomType{NumericalIntegrationCategory}
    ++ to get the name and other
    ++ relevant information of the the (domain of the) numerical
    ++ routine likely to be the most appropriate,
    ++ i.e. have the best \axiom{measure}.
    ++
    ++ It then performs the integration of the given expression
    ++ on that \axiom{domain}.
  integrate: NumericalIntegrationProblem -> Result
    ++ integrate(IntegrationProblem) is a top level ANNA function
    ++ to integrate an expression over a given range or ranges
    ++ to the required absolute and relative accuracy.
    ++
    ++ It iterates over the \axiom{domains} of
    ++ \axiomType{NumericalIntegrationCategory} to get the name and other
    ++ relevant information of the the (domain of the) numerical
    ++ routine likely to be the most appropriate,
    ++ i.e. have the best \axiom{measure}.
    ++
    ++ It then performs the integration of the given expression
    ++ on that \axiom{domain}.

  integrate: (EF,SOCF,F,F) -> Result
    ++ integrate(exp, a..b, epsabs, epsrel) is a top level ANNA function
```

```
    ++ to integrate an expression, {\tt exp}, over a given range {\tt a}
    ++ to {\tt b} to the required absolute and relative accuracy.
    ++
    ++ It iterates over the \axiom{domains} of
    ++ \axiomType{NumericalIntegrationCategory} to get the name and other
    ++ relevant information of the the (domain of the) numerical
    ++ routine likely to be the most appropriate,
    ++ i.e. have the best \axiom{measure}.
    ++
    ++ It then performs the integration of the given expression
    ++ on that \axiom{domain}.

  integrate: (EF,SOCF,F) -> Result
    ++ integrate(exp, a..b, epsrel) is a top level ANNA
    ++ function to integrate an expression, {\tt exp}, over a given
    ++ range {\tt a} to {\tt b} to the required relative accuracy.
    ++
    ++ It iterates over the \axiom{domains} of
    ++ \axiomType{NumericalIntegrationCategory} to get the name and other
    ++ relevant information of the the (domain of the) numerical
    ++ routine likely to be the most appropriate,
    ++ i.e. have the best \axiom{measure}.
    ++
    ++ It then performs the integration of the given expression
    ++ on that \axiom{domain}.
    ++
    ++ If epsrel = 0, a default absolute accuracy is used.

  integrate: (EF,SOCF) -> Result
    ++ integrate(exp, a..b) is a top
    ++ level ANNA function to integrate an expression, {\tt exp},
    ++ over a given range {\tt a} to {\tt b}.
    ++
    ++ It iterates over the \axiom{domains} of
    ++ \axiomType{NumericalIntegrationCategory} to get the name and other
    ++ relevant information of the the (domain of the) numerical
    ++ routine likely to be the most appropriate,
    ++ i.e. have the best \axiom{measure}.
    ++
    ++ It then performs the integration of the given expression
    ++ on that \axiom{domain}.
    ++
    ++ Default values for the absolute and relative error are used.

  integrate:(EF,LSOCF) -> Result
    ++ integrate(exp, [a..b,c..d,...]) is a top
```

```
  ++ level ANNA function to integrate a multivariate expression, {\tt exp},
  ++ over a given set of ranges.
  ++
  ++ It iterates over the \axiom{domains} of
  ++ \axiomType{NumericalIntegrationCategory} to get the name and other
  ++ relevant information of the the (domain of the) numerical
  ++ routine likely to be the most appropriate,
  ++ i.e. have the best \axiom{measure}.
  ++
  ++ It then performs the integration of the given expression
  ++ on that \axiom{domain}.
  ++
  ++ Default values for the absolute and relative error are used.

integrate:(EF,LSOCF,F) -> Result
  ++ integrate(exp, [a..b,c..d,...], epsrel) is a top
  ++ level ANNA function to integrate a multivariate expression, {\tt exp},
  ++ over a given set of ranges to the required relative
  ++ accuracy.
  ++
  ++ It iterates over the \axiom{domains} of
  ++ \axiomType{NumericalIntegrationCategory} to get the name and other
  ++ relevant information of the the (domain of the) numerical
  ++ routine likely to be the most appropriate,
  ++ i.e. have the best \axiom{measure}.
  ++
  ++ It then performs the integration of the given expression
  ++ on that \axiom{domain}.
  ++
  ++ If epsrel = 0, a default absolute accuracy is used.

integrate:(EF,LSOCF,F,F) -> Result
  ++ integrate(exp, [a..b,c..d,...], epsabs, epsrel) is a top
  ++ level ANNA function to integrate a multivariate expression, {\tt exp},
  ++ over a given set of ranges to the required absolute and relative
  ++ accuracy.
  ++
  ++ It iterates over the \axiom{domains} of
  ++ \axiomType{NumericalIntegrationCategory} to get the name and other
  ++ relevant information of the the (domain of the) numerical
  ++ routine likely to be the most appropriate,
  ++ i.e. have the best \axiom{measure}.
  ++
  ++ It then performs the integration of the given expression
  ++ on that \axiom{domain}.
```

```
integrate:(EF,LSOCF,F,F,RT) -> Result
  ++ integrate(exp, [a..b,c..d,...], epsabs, epsrel, routines) is a top
  ++ level ANNA function to integrate a multivariate expression, {\tt exp},
  ++ over a given set of ranges to the required absolute and relative
  ++ accuracy, using the routines available in the RoutinesTable provided.
  ++
  ++ It iterates over the \axiom{domains} of
  ++ \axiomType{NumericalIntegrationCategory} to get the name and other
  ++ relevant information of the the (domain of the) numerical
  ++ routine likely to be the most appropriate,
  ++ i.e. have the best \axiom{measure}.
  ++
  ++ It then performs the integration of the given expression
  ++ on that \axiom{domain}.

measure:NumericalIntegrationProblem -> Measure
  ++ measure(prob) is a top level ANNA function for identifying the most
  ++ appropriate numerical routine for solving the numerical integration
  ++ problem defined by \axiom{prob}.
  ++
  ++ It calls each \axiom{domain} of \axiom{category}
  ++ \axiomType{NumericalIntegrationCategory} in turn to calculate all measures
  ++ and returns the best
  ++ i.e. the name of the most appropriate domain and any other relevant
  ++ information.
measure:(NumericalIntegrationProblem,RT) -> Measure
  ++ measure(prob,R) is a top level ANNA function for identifying the most
  ++ appropriate numerical routine from those in the routines table
  ++ provided for solving the numerical integration
  ++ problem defined by \axiom{prob}.
  ++
  ++ It calls each \axiom{domain} listed in \axiom{R} of \axiom{category}
  ++ \axiomType{NumericalIntegrationCategory} in turn to calculate all measures
  ++ and returns the best
  ++ i.e. the name of the most appropriate domain and any other relevant
  ++ information.
integrate:(EF,SBOCF,ST) -> Union(Result,"failed")
  ++ integrate(exp, x = a..b, "numerical") is a top level ANNA function to
  ++ integrate an expression, {\tt exp}, over a given range, {\tt a}
  ++ to {\tt b}.
  ++
  ++ It iterates over the \axiom{domains} of
  ++ \axiomType{NumericalIntegrationCategory} to get the name and other
  ++ relevant information of the the (domain of the) numerical
  ++ routine likely to be the most appropriate,
  ++ i.e. have the best \axiom{measure}.
```

```
    ++
    ++ It then performs the integration of the given expression
    ++ on that \axiom{domain}.\newline
    ++
    ++ Default values for the absolute and relative error are used.
    ++
    ++ It is an error of the last argument is not {\tt "numerical"}.
   integrate:(EF,SBOCF,S) -> Union(Result,"failed")
    ++ integrate(exp, x = a..b, numerical) is a top level ANNA function to
    ++ integrate an expression, {\tt exp}, over a given range, {\tt a}
    ++ to {\tt b}.
    ++
    ++ It iterates over the \axiom{domains} of
    ++ \axiomType{NumericalIntegrationCategory} to get the name and other
    ++ relevant information of the the (domain of the) numerical
    ++ routine likely to be the most appropriate,
    ++ i.e. have the best \axiom{measure}.
    ++
    ++ It then performs the integration of the given expression
    ++ on that \axiom{domain}.\newline
    ++
    ++ Default values for the absolute and relative error are used.
    ++
    ++ It is an error if the last argument is not {\tt numerical}.

  ==  add

   zeroMeasure: Measure -> Result
   scriptedVariables?: MDNIA -> Boolean
   preAnalysis:(Union(nia:NIA,mdnia:MDNIA),RT) -> RT
   measureSpecific:(ST,RT,Union(nia:NIA,mdnia:MDNIA)) -> Record(measure:F,explanat
   changeName:(Result,ST) -> Result
   recoverAfterFail:(Union(nia:NIA,mdnia:MDNIA),RT,Measure,INT,Result) -> Record(a
   better?:(Result,Result) -> Boolean
   integrateConstant:(EF,SOCF) -> Result
   integrateConstantList: (EF,LSOCF) -> Result
   integrateArgs:(NumericalIntegrationProblem,RT) -> Result
   integrateSpecific:(Union(nia:NIA,mdnia:MDNIA),ST,Result) -> Result

   import ExpertSystemToolsPackage

   integrateConstantList(exp:EF,ras:LSOCF):Result ==
     c:OCF := ((retract(exp)@F)$EF)::OCF
     b := [hi(j)-lo(j) for j in ras]
     c := c*reduce((x,y) +-> x*y,b)
     a := coerce(c)$AnyFunctions1(OCF)
```

```
    text := coerce("Constant Function")$AnyFunctions1(ST)
    construct([[result@S,a],[method@S,text]])$Result

  integrateConstant(exp:EF,ra:SOCF):Result ==
    c := (retract(exp)@F)$EF
    r:OCF := (c::OCF)*(hi(ra)-lo(ra))
    a := coerce(r)$AnyFunctions1(OCF)
    text := coerce("Constant Function")$AnyFunctions1(ST)
    construct([[result@S,a],[method@S,text]])$Result

  zeroMeasure(m:Measure):Result ==
    a := coerce(0$DF)$AnyFunctions1(DF)
    text := coerce("Constant Function")$AnyFunctions1(String)
    r := construct([[result@Symbol,a],[method@Symbol,text]])$Result
    concat(measure2Result m,r)$ExpertSystemToolsPackage

  scriptedVariables?(mdnia:MDNIA):Boolean ==
    vars:List Symbol := variables(mdnia.fn)$EDF
    var1 := first(vars)$(List Symbol)
    not scripted?(var1) => false
    name1 := name(var1)$Symbol
    for i in 2..# vars repeat
      not ((scripted?(vars.i)$Symbol) and (name1 = name(vars.i)$Symbol)) =>
         return false
    true

  preAnalysis(args:Union(nia:NIA,mdnia:MDNIA),t:RT):RT ==
    import RT
    r:RT := selectIntegrationRoutines t
    args case nia =>
      arg:NIA := args.nia
      rangeIsFinite(arg)$d01AgentsPackage case finite =>
        selectFiniteRoutines r
      selectNonFiniteRoutines r
    selectMultiDimensionalRoutines r

  changeName(ans:Result,name:ST):Result ==
    sy:S := coerce(name "Answer")$S
    anyAns:Any := coerce(ans)$AnyFunctions1(Result)
    construct([[sy,anyAns]])$Result

  measureSpecific(name:ST,R:RT,args:Union(nia:NIA,mdnia:MDNIA)):
      Record(measure:F,explanations:ST,extra:Result) ==
    args case nia =>
      arg:NIA := args.nia
      name = "d01ajfAnnaType" => measure(R,arg)$d01ajfAnnaType
```

```
      name = "d01akfAnnaType" => measure(R,arg)$d01akfAnnaType
      name = "d01alfAnnaType" => measure(R,arg)$d01alfAnnaType
      name = "d01amfAnnaType" => measure(R,arg)$d01amfAnnaType
      name = "d01anfAnnaType" => measure(R,arg)$d01anfAnnaType
      name = "d01apfAnnaType" => measure(R,arg)$d01apfAnnaType
      name = "d01aqfAnnaType" => measure(R,arg)$d01aqfAnnaType
      name = "d01asfAnnaType" => measure(R,arg)$d01asfAnnaType
      name = "d01TransformFunctionType" =>
                    measure(R,arg)$d01TransformFunctionType
      error("measureSpecific","invalid type name: " name)$ErrorFunctions
    args case mdnia =>
      arg2:MDNIA := args.mdnia
      name = "d01gbfAnnaType" => measure(R,arg2)$d01gbfAnnaType
      name = "d01fcfAnnaType" => measure(R,arg2)$d01fcfAnnaType
      error("measureSpecific","invalid type name: " name)$ErrorFunctions
    error("measureSpecific","invalid type name")$ErrorFunctions

  measure(a:NumericalIntegrationProblem,R:RT):Measure ==
    args:Union(nia:NIA,mdnia:MDNIA) := retract(a)$NumericalIntegrationProblem
    sofar := 0$F
    best := "none" :: ST
    routs := copy R
    routs := preAnalysis(args,routs)
    empty?(routs)$RT =>
      error("measure", "no routines found")$ErrorFunctions
    rout := inspect(routs)$RT
    e := retract(rout.entry)$AnyFunctions1(Entry)
    meth:LST := ["Trying " e.type " integration routines"]
    ext := empty()$Result
    for i in 1..# routs repeat
      rout := extract!(routs)$RT
      e := retract(rout.entry)$AnyFunctions1(Entry)
      n := e.domainName
      if e.defaultMin > sofar then
        m := measureSpecific(n,R,args)
        if m.measure > sofar then
          sofar := m.measure
          best := n
        ext := concat(m.extra,ext)$ExpertSystemToolsPackage
        str:LST := [string(rout.key)$S "measure: " outputMeasure(m.measure)
                    " - " m.explanations]
      else
        str:LST :=  [string(rout.key)$S " is no better than other routines"]
      meth := append(meth,str)$LST
    [sofar,best,meth,ext]
```

```
measure(a:NumericalIntegrationProblem):Measure ==
  measure(a,routines()$RT)

integrateSpecific(args:Union(nia:NIA,mdnia:MDNIA),n:ST,ex:Result):Result ==
  args case nia =>
    arg:NIA := args.nia
    n = "d01ajfAnnaType" => numericalIntegration(arg,ex)$d01ajfAnnaType
    n = "d01TransformFunctionType" =>
      numericalIntegration(arg,ex)$d01TransformFunctionType
    n = "d01amfAnnaType" => numericalIntegration(arg,ex)$d01amfAnnaType
    n = "d01apfAnnaType" => numericalIntegration(arg,ex)$d01apfAnnaType
    n = "d01aqfAnnaType" => numericalIntegration(arg,ex)$d01aqfAnnaType
    n = "d01alfAnnaType" => numericalIntegration(arg,ex)$d01alfAnnaType
    n = "d01akfAnnaType" => numericalIntegration(arg,ex)$d01akfAnnaType
    n = "d01anfAnnaType" => numericalIntegration(arg,ex)$d01anfAnnaType
    n = "d01asfAnnaType" => numericalIntegration(arg,ex)$d01asfAnnaType
    error("integrateSpecific","invalid type name: " n)$ErrorFunctions
  args case mdnia =>
    arg2:MDNIA := args.mdnia
    n = "d01gbfAnnaType" => numericalIntegration(arg2,ex)$d01gbfAnnaType
    n = "d01fcfAnnaType" => numericalIntegration(arg2,ex)$d01fcfAnnaType
    error("integrateSpecific","invalid type name: " n)$ErrorFunctions
  error("integrateSpecific","invalid type name: " n)$ErrorFunctions

better?(r:Result,s:Result):Boolean ==
  a1 := search("abserr"::S,r)$Result
  a1 case "failed" => false
  abserr1 := retract(a1)$AnyFunctions1(DF)
  negative?(abserr1) => false
  a2 := search("abserr"::S,s)$Result
  a2 case "failed" => true
  abserr2 := retract(a2)$AnyFunctions1(DF)
  negative?(abserr2) => true
  (abserr1 < abserr2) -- true if r.abserr better than s.abserr

recoverAfterFail(n:Union(nia:NIA,mdnia:MDNIA),routs:RT,m:Measure,iint:INT,
                      r:Result):Record(a:Result,b:Measure) ==
  bestName := m.name
  while positive?(iint) repeat
    routineName := m.name
    s := recoverAfterFail(routs,routineName(1..6),iint)$RoutinesTable
    s case "failed" => iint := 0
    if s = "changeEps" then
      nn := n.nia
      zero?(nn.abserr) =>
        nn.abserr := 1.0e-8 :: DF
```

```
          m := measure(n::NumericalIntegrationProblem,routs)
          zero?(m.measure) => iint := 0
          r := integrateSpecific(n,m.name,m.extra)
          iint := 0
      rn := routineName(1..6)
      buttVal := getButtonValue(rn,"functionEvaluations")$AttributeButtons
      if (s = "incrFunEvals") and (buttVal < 0.8) then
        increase(rn,"functionEvaluations")$AttributeButtons
      if s = "increase tolerance" then
        (n.nia).relerr := (n.nia).relerr*(10.0::DF)
      if s = "decrease tolerance" then
        (n.nia).relerr := (n.nia).relerr/(10.0::DF)
      fl := coerce(s)$AnyFunctions1(ST)
      flrec:Record(key:S,entry:Any):=[failure@S,fl]
      m2 := measure(n::NumericalIntegrationProblem,routs)
      zero?(m2.measure) => iint := 0
      r2:Result := integrateSpecific(n,m2.name,m2.extra)
      better?(r,r2) =>
        m.name := m2.name
        insert!(flrec,r)$Result
      bestName := m2.name
      m := m2
      insert!(flrec,r2)$Result
      r := concat(r2,changeName(r,routineName))$ExpertSystemToolsPackage
      iany := search(ifail@S,r2)$Result
      iany case "failed" => iint := 0
      iint := retract(iany)$AnyFunctions1(INT)
    m.name := bestName
    [r,m]

  integrateArgs(prob:NumericalIntegrationProblem,t:RT):Result ==
    args:Union(nia:NIA,mdnia:MDNIA) := retract(prob)$NumericalIntegrationProblem
    routs := copy(t)$RT
    if args case mdnia then
      arg := args.mdnia
      v := (# variables(arg.fn))
      not scriptedVariables?(arg) =>
        error("MultiDimensionalNumericalIntegrationPackage",
                "invalid variable names")$ErrorFunctions
      (v ~= # arg.range)@Boolean =>
        error("MultiDimensionalNumericalIntegrationPackage",
          "number of variables do not match number of ranges")$ErrorFunctions
    m := measure(prob,routs)
    zero?(m.measure) => zeroMeasure m
    r := integrateSpecific(args,m.name,m.extra)
    iany := search(ifail@S,r)$Result
```

```
    iint := 0$INT
    if (iany case Any) then
      iint := retract(iany)$AnyFunctions1(INT)
    if positive?(iint) then
      tu:Record(a:Result,b:Measure) := recoverAfterFail(args,routs,m,iint,r)
      r := tu.a
      m := tu.b
    r := concat(measure2Result m,r)$ExpertSystemToolsPackage
    n := m.name
    nn:ST :=
      (# n > 14) => "d01transform"
      n(1..6)
    expl := getExplanations(routs,nn)$RoutinesTable
    expla := coerce(expl)$AnyFunctions1(LST)
    explaa:Record(key:Symbol,entry:Any) := ["explanations"::Symbol,expla]
    r := concat(construct([explaa]),r)
    args case nia =>
      att := showAttributes(args.nia)$IntegrationFunctionsTable
      att case "failed" => r
      concat(att2Result att,r)$ExpertSystemToolsPackage
    r

  integrate(args:NumericalIntegrationProblem):Result ==
    integrateArgs(args,routines()$RT)

  integrate(exp:EF,ra:SOCF,epsabs:F,epsrel:F,r:RT):Result ==
    Var:LS := variables(exp)$EF
    empty?(Var)$LS => integrateConstant(exp,ra)
    args:NIA := [first(Var)$LS,ef2edf exp,socf2socdf ra,f2df epsabs,f2df epsrel]
    integrateArgs(args::NumericalIntegrationProblem,r)

  integrate(exp:EF,ra:SOCF,epsabs:F,epsrel:F):Result ==
    integrate(exp,ra,epsabs,epsrel,routines()$RT)

  integrate(exp:EF,ra:SOCF,err:F):Result ==
    positive?(err)$F => integrate(exp,ra,0$F,err)
    integrate(exp,ra,1.0E-5,err)

  integrate(exp:EF,ra:SOCF):Result == integrate(exp,ra,0$F,1.0E-5)

  integrate(exp:EF,sb:SBOCF, st:ST) ==
    st = "numerical" => integrate(exp,segment sb)
    "failed"

  integrate(exp:EF,sb:SBOCF, s:S) ==
    s = (numerical::Symbol) => integrate(exp,segment sb)
```

```
      "failed"

    integrate(exp:EF,ra:LSOCF,epsabs:F,epsrel:F,r:RT):Result ==
      vars := variables(exp)$EF
      empty?(vars)$LS => integrateConstantList(exp,ra)
      args:MDNIA := [ef2edf exp,convert ra,f2df epsabs,f2df epsrel]
      integrateArgs(args::NumericalIntegrationProblem,r)

    integrate(exp:EF,ra:LSOCF,epsabs:F,epsrel:F):Result ==
      integrate(exp,ra,epsabs,epsrel,routines()$RT)

    integrate(exp:EF,ra:LSOCF,epsrel:F):Result ==
      zero? epsrel => integrate(exp,ra,1.0e-6,epsrel)
      integrate(exp,ra,0$F,epsrel)

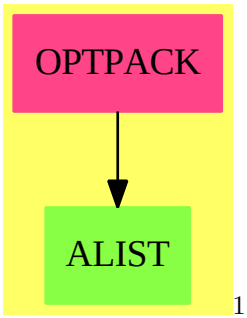
    integrate(exp:EF,ra:LSOCF):Result == integrate(exp,ra,1.0e-4)
```

⟨*INTPACK.dotabb*⟩≡

```
"INTPACK" [color="#FF4488",href="bookvol10.4.pdf#nameddest=INTPACK"]
"TBAGG" [color="#4488FF",href="bookvol10.2.pdf#nameddest=TBAGG"]
"INTPACK" -> "TBAGG"
```

2.19 package OPTPACK AnnaNumericalOptimizationPackage

2.20 AnnaNumericalOptimizationPackage



1.00

Exports:
goodnessOfFit measure optimize

⟨*package OPTPACK AnnaNumericalOptimizationPackage*⟩≡

```
)abbrev package OPTPACK AnnaNumericalOptimizationPackage
++ Author: Brian Dupee
++ Date Created: February 1995
++ Date Last Updated: December 1997
++ Basic Operations: measure, optimize, goodnessOfFit.
++ Description:
++ \axiomType{AnnaNumericalOptimizationPackage} is a \axiom{package} of
++ functions for the \axiomType{NumericalOptimizationCategory}
++ with \axiom{measure} and \axiom{optimize}.
EDF     ==> Expression DoubleFloat
LEDF    ==> List Expression DoubleFloat
LDF     ==> List DoubleFloat
MDF     ==> Matrix DoubleFloat
DF      ==> DoubleFloat
LOCDF   ==> List OrderedCompletion DoubleFloat
OCDF    ==> OrderedCompletion DoubleFloat
LOCF    ==> List OrderedCompletion Float
OCF     ==> OrderedCompletion Float
LEF     ==> List Expression Float
EF      ==> Expression Float
LF      ==> List Float
F       ==> Float
LS      ==> List Symbol
LST     ==> List String
INT     ==> Integer
NOA     ==> Record(fn:EDF, init:LDF, lb:LOCDF, cf:LEDF, ub:LOCDF)
```

```
LSA     ==> Record(lfn:LEDF, init:LDF)
IFL     ==> List(Record(ifail:Integer,instruction:String))
Entry   ==> Record(chapter:String, type:String, domainName: String,
                   defaultMin:F, measure:F, failList:IFL, explList:LST)
Measure ==> Record(measure:F,name:String, explanations:List String)
Measure2        ==> Record(measure:F,explanations:String)
RT      ==> RoutinesTable
UNOALSA ==> Union(noa:NOA,lsa:LSA)

AnnaNumericalOptimizationPackage(): with
  measure:NumericalOptimizationProblem -> Measure
    ++ measure(prob) is a top level ANNA function for identifying the most
    ++ appropriate numerical routine from those in the routines table
    ++ provided for solving the numerical optimization problem defined by
    ++ \axiom{prob} by checking various attributes of the functions and
    ++ calculating a measure of compatibility of each routine to these
    ++ attributes.
    ++
    ++ It calls each \axiom{domain} of \axiom{category}
    ++ \axiomType{NumericalOptimizationCategory} in turn to calculate all
    ++ measures and returns the best i.e. the name of the most
    ++ appropriate domain and any other relevant information.

  measure:(NumericalOptimizationProblem,RT) -> Measure
    ++ measure(prob,R) is a top level ANNA function for identifying the most
    ++ appropriate numerical routine from those in the routines table
    ++ provided for solving the numerical optimization problem defined by
    ++ \axiom{prob} by checking various attributes of the functions and
    ++ calculating a measure of compatibility of each routine to these
    ++ attributes.
    ++
    ++ It calls each \axiom{domain} listed in \axiom{R} of \axiom{category}
    ++ \axiomType{NumericalOptimizationCategory} in turn to calculate all
    ++ measures and returns the best i.e. the name of the most
    ++ appropriate domain and any other relevant information.

  optimize:(NumericalOptimizationProblem,RT) -> Result
    ++ optimize(prob,routines) is a top level ANNA function to
    ++ minimize a function or a set of functions with any constraints
    ++ as defined within \axiom{prob}.
    ++
    ++ It iterates over the \axiom{domains} listed in \axiom{routines} of
    ++ \axiomType{NumericalOptimizationCategory}
    ++ to get the name and other relevant information of the best
    ++ \axiom{measure} and then optimize the function on that \axiom{domain}.
```

```
optimize:NumericalOptimizationProblem -> Result
  ++ optimize(prob) is a top level ANNA function to
  ++ minimize a function or a set of functions with any constraints
  ++ as defined within \axiom{prob}.
  ++
  ++ It iterates over the \axiom{domains} of
  ++ \axiomType{NumericalOptimizationCategory}
  ++ to get the name and other relevant information of the best
  ++ \axiom{measure} and then optimize the function on that \axiom{domain}.

goodnessOfFit:NumericalOptimizationProblem -> Result
  ++ goodnessOfFit(prob) is a top level ANNA function to
  ++ check to goodness of fit of a least squares model
  ++ as defined within \axiom{prob}.
  ++
  ++ It iterates over the \axiom{domains} of
  ++ \axiomType{NumericalOptimizationCategory}
  ++ to get the name and other relevant information of the best
  ++ \axiom{measure} and then optimize the function on that \axiom{domain}.
  ++ It then calls the numerical routine \axiomType{E04YCF} to get estimates
  ++ of the variance-covariance matrix of the regression coefficients of
  ++ the least-squares problem.
  ++
  ++ It thus returns both the results of the optimization and the
  ++ variance-covariance calculation.

optimize:(EF,LF,LOCF,LEF,LOCF) -> Result
  ++ optimize(f,start,lower,cons,upper) is a top level ANNA function to
  ++ minimize a function, \axiom{f}, of one or more variables with the
  ++ given constraints.
  ++
  ++ These constraints may be simple constraints on the variables
  ++ in which case \axiom{cons} would be an empty list and the bounds on
  ++ those variables defined in \axiom{lower} and \axiom{upper}, or a
  ++ mixture of simple, linear and non-linear constraints, where
  ++ \axiom{cons} contains the linear and non-linear constraints and
  ++ the bounds on these are added to \axiom{upper} and \axiom{lower}.
  ++
  ++ The parameter \axiom{start} is a list of the initial guesses of the
  ++ values of the variables.
  ++
  ++ It iterates over the \axiom{domains} of
  ++ \axiomType{NumericalOptimizationCategory}
  ++ to get the name and other relevant information of the best
  ++ \axiom{measure} and then optimize the function on that \axiom{domain}.
```

```
optimize:(EF,LF,LOCF,LOCF) -> Result
  ++ optimize(f,start,lower,upper) is a top level ANNA function to
  ++ minimize a function, \axiom{f}, of one or more variables with
  ++ simple constraints.  The bounds on
  ++ the variables are defined in \axiom{lower} and \axiom{upper}.
  ++
  ++ The parameter \axiom{start} is a list of the initial guesses of the
  ++ values of the variables.
  ++
  ++ It iterates over the \axiom{domains} of
  ++ \axiomType{NumericalOptimizationCategory}
  ++ to get the name and other relevant information of the best
  ++ \axiom{measure} and then optimize the function on that \axiom{domain}.

optimize:(EF,LF) -> Result
  ++ optimize(f,start) is a top level ANNA function to
  ++ minimize a function, \axiom{f}, of one or more variables without
  ++ constraints.
  ++
  ++ The parameter \axiom{start} is a list of the initial guesses of the
  ++ values of the variables.
  ++
  ++ It iterates over the \axiom{domains} of
  ++ \axiomType{NumericalOptimizationCategory}
  ++ to get the name and other relevant information of the best
  ++ \axiom{measure} and then optimize the function on that \axiom{domain}.

optimize:(LEF,LF) -> Result
  ++ optimize(lf,start) is a top level ANNA function to
  ++ minimize a set of functions, \axiom{lf}, of one or more variables
  ++ without constraints i.e. a least-squares problem.
  ++
  ++ The parameter \axiom{start} is a list of the initial guesses of the
  ++ values of the variables.
  ++
  ++ It iterates over the \axiom{domains} of
  ++ \axiomType{NumericalOptimizationCategory}
  ++ to get the name and other relevant information of the best
  ++ \axiom{measure} and then optimize the function on that \axiom{domain}.

goodnessOfFit:(LEF,LF) -> Result
  ++ goodnessOfFit(lf,start) is a top level ANNA function to
  ++ check to goodness of fit of a least squares model i.e. the minimization
  ++ of a set of functions, \axiom{lf}, of one or more variables without
  ++ constraints.
  ++
```

```
    ++ The parameter \axiom{start} is a list of the initial guesses of the
    ++ values of the variables.
    ++
    ++ It iterates over the \axiom{domains} of
    ++ \axiomType{NumericalOptimizationCategory}
    ++ to get the name and other relevant information of the best
    ++ \axiom{measure} and then optimize the function on that \axiom{domain}.
    ++ It then calls the numerical routine \axiomType{E04YCF} to get estimates
    ++ of the variance-covariance matrix of the regression coefficients of
    ++ the least-squares problem.
    ++
    ++ It thus returns both the results of the optimization and the
    ++ variance-covariance calculation.

    ++ goodnessOfFit(lf,start) is a top level function to iterate over
    ++ the \axiom{domains} of \axiomType{NumericalOptimizationCategory}
    ++ to get the name and other relevant information of the best
    ++ \axiom{measure} and then optimize the function on that \axiom{domain}.
    ++ It then checks the goodness of fit of the least squares model.

  == add

   preAnalysis:RT -> RT
   zeroMeasure:Measure -> Result
   optimizeSpecific:(UNOALSA,String) -> Result
   measureSpecific:(String,RT,UNOALSA) -> Measure2
   changeName:(Result,String) -> Result
   recoverAfterFail:(UNOALSA,RT,Measure,INT,Result) -> Record(a:Result,b:Measure)
   constant:UNOALSA -> Union(DF, "failed")
   optimizeConstant:DF -> Result

   import ExpertSystemToolsPackage,e04AgentsPackage,NumericalOptimizationProblem

   constant(args:UNOALSA):Union(DF,"failed") ==
     args case noa =>
       Args := args.noa
       f := Args.fn
       retractIfCan(f)@Union(DoubleFloat,"failed")
     "failed"

   optimizeConstant(c:DF): Result ==
     a := coerce(c)$AnyFunctions1(DF)
     text := coerce("Constant Function")$AnyFunctions1(String)
     construct([[objf@Symbol,a],[method@Symbol,text]])$Result

   preAnalysis(args:UNOALSA,t:RT):RT ==
```

```
    r := selectOptimizationRoutines(t)$RT
    args case lsa =>
      selectSumOfSquaresRoutines(r)$RT
    r

  zeroMeasure(m:Measure):Result ==
    a := coerce(0$F)$AnyFunctions1(F)
    text := coerce("Zero Measure")$AnyFunctions1(String)
    r := construct([[objf@Symbol,a],[method@Symbol,text]])$Result
    concat(measure2Result m,r)

  measureSpecific(name:String,R:RT,args:UNOALSA): Measure2 ==
    args case noa =>
      arg:NOA := args.noa
      name = "e04dgfAnnaType" => measure(R,arg)$e04dgfAnnaType
      name = "e04fdfAnnaType" => measure(R,arg)$e04fdfAnnaType
      name = "e04gcfAnnaType" => measure(R,arg)$e04gcfAnnaType
      name = "e04jafAnnaType" => measure(R,arg)$e04jafAnnaType
      name = "e04mbfAnnaType" => measure(R,arg)$e04mbfAnnaType
      name = "e04nafAnnaType" => measure(R,arg)$e04nafAnnaType
      name = "e04ucfAnnaType" => measure(R,arg)$e04ucfAnnaType
      error("measureSpecific","invalid type name: " name)$ErrorFunctions
    args case lsa =>
      arg2:LSA := args.lsa
      name = "e04fdfAnnaType" => measure(R,arg2)$e04fdfAnnaType
      name = "e04gcfAnnaType" => measure(R,arg2)$e04gcfAnnaType
      error("measureSpecific","invalid type name: " name)$ErrorFunctions
    error("measureSpecific","invalid argument type")$ErrorFunctions

  measure(Args:NumericalOptimizationProblem,R:RT):Measure ==
    args:UNOALSA := retract(Args)$NumericalOptimizationProblem
    sofar := 0$F
    best := "none" :: String
    routs := copy R
    routs := preAnalysis(args,routs)
    empty?(routs)$RT =>
      error("measure", "no routines found")$ErrorFunctions
    rout := inspect(routs)$RT
    e := retract(rout.entry)$AnyFunctions1(Entry)
    meth := empty()$(List String)
    for i in 1..# routs repeat
      rout := extract!(routs)$RT
      e := retract(rout.entry)$AnyFunctions1(Entry)
      n := e.domainName
      if e.defaultMin > sofar then
        m := measureSpecific(n,R,args)
```

```
      if m.measure > sofar then
        sofar := m.measure
        best := n
      str := [concat(concat([string(rout.key)$Symbol,"measure: ",
               outputMeasure(m.measure)," - "],
                 m.explanations)$(List String))$String]
    else
      str := [concat([string(rout.key)$Symbol
                      ," is no better than other routines"])$String]
    meth := append(meth,str)$(List String)
  [sofar,best,meth]

measure(args:NumericalOptimizationProblem):Measure == measure(args,routines()$RT)

optimizeSpecific(args:UNOALSA,name:String):Result ==
  args case noa =>
    arg:NOA := args.noa
    name = "e04dgfAnnaType" => numericalOptimization(arg)$e04dgfAnnaType
    name = "e04fdfAnnaType" => numericalOptimization(arg)$e04fdfAnnaType
    name = "e04gcfAnnaType" => numericalOptimization(arg)$e04gcfAnnaType
    name = "e04jafAnnaType" => numericalOptimization(arg)$e04jafAnnaType
    name = "e04mbfAnnaType" => numericalOptimization(arg)$e04mbfAnnaType
    name = "e04nafAnnaType" => numericalOptimization(arg)$e04nafAnnaType
    name = "e04ucfAnnaType" => numericalOptimization(arg)$e04ucfAnnaType
    error("optimizeSpecific","invalid type name: " name)$ErrorFunctions
  args case lsa =>
    arg2:LSA := args.lsa
    name = "e04fdfAnnaType" => numericalOptimization(arg2)$e04fdfAnnaType
    name = "e04gcfAnnaType" => numericalOptimization(arg2)$e04gcfAnnaType
    error("optimizeSpecific","invalid type name: " name)$ErrorFunctions
  error("optimizeSpecific","invalid type name: " name)$ErrorFunctions

changeName(ans:Result,name:String):Result ==
  st:String := concat([name,"Answer"])$String
  sy:Symbol := coerce(st)$Symbol
  anyAns:Any := coerce(ans)$AnyFunctions1(Result)
  construct([[sy,anyAns]])$Result

recoverAfterFail(args:UNOALSA,routs:RT,m:Measure,
                  iint:INT,r:Result):Record(a:Result,b:Measure) ==
  while positive?(iint) repeat
    routineName := m.name
    s := recoverAfterFail(routs,routineName(1..6),iint)$RT
    s case "failed" => iint := 0
    (s = "no action")@Boolean => iint := 0
    fl := coerce(s)$AnyFunctions1(String)
```

```
      flrec:Record(key:Symbol,entry:Any):=[failure@Symbol,fl]
      m2 := measure(args::NumericalOptimizationProblem,routs)
      zero?(m2.measure) => iint := 0
      r2:Result := optimizeSpecific(args,m2.name)
      m := m2
      insert!(flrec,r2)$Result
      r := concat(r2,changeName(r,routineName))
      iany := search(ifail@Symbol,r2)$Result
      iany case "failed" => iint := 0
      iint := retract(iany)$AnyFunctions1(INT)
    [r,m]

  optimize(Args:NumericalOptimizationProblem,t:RT):Result ==
    args:UNOALSA := retract(Args)$NumericalOptimizationProblem
    routs := copy(t)$RT
    c:Union(DF,"failed") := constant(args)
    c case DF => optimizeConstant(c)
    m := measure(Args,routs)
    zero?(m.measure) => zeroMeasure m
    r := optimizeSpecific(args,n := m.name)
    iany := search(ifail@Symbol,r)$Result
    iint := 0$INT
    if (iany case Any) then
      iint := retract(iany)$AnyFunctions1(INT)
    if positive?(iint) then
      tu:Record(a:Result,b:Measure) := recoverAfterFail(args,routs,m,iint,r)
      r := tu.a
      m := tu.b
    r := concat(measure2Result m,r)
    expl := getExplanations(routs,n(1..6))$RoutinesTable
    expla := coerce(expl)$AnyFunctions1(LST)
    explaa:Record(key:Symbol,entry:Any) := ["explanations"::Symbol,expla]
    r := concat(construct([explaa]),r)
    att:List String := optAttributes(args)
    atta := coerce(att)$AnyFunctions1(List String)
    attr:Record(key:Symbol,entry:Any) := [attributes@Symbol,atta]
    insert!(attr,r)$Result

  optimize(args:NumericalOptimizationProblem):Result == optimize(args,routines()$

  goodnessOfFit(Args:NumericalOptimizationProblem):Result ==
    r := optimize(Args)
    args1:UNOALSA := retract(Args)$NumericalOptimizationProblem
    args1 case noa => error("goodnessOfFit","Not an appropriate problem")
    args:LSA := args1.lsa
    lf := args.lfn
```

```
    n:INT := #(variables(args))
    m:INT := # lf
    me := search(method,r)$Result
    me case "failed" => r
    meth := retract(me)$AnyFunctions1(Result)
    na := search(nameOfRoutine,meth)$Result
    na case "failed" => r
    name := retract(na)$AnyFunctions1(String)
    temp:INT := (n*(n-1)) quo 2
    ns:INT :=
      name = "e04fdfAnnaType" => 6*n+(2+n)*m+1+max(1,temp)
      8*n+(n+2)*m+temp+1+max(1,temp)
    nv:INT := ns+n
    ww := search(w,r)$Result
    ww case "failed" => r
    ws:MDF := retract(ww)$AnyFunctions1(MDF)
    fr := search(objf,r)$Result
    fr case "failed" => r
    f := retract(fr)$AnyFunctions1(DF)
    s := subMatrix(ws,1,1,ns,nv-1)$MDF
    v := subMatrix(ws,1,1,nv,nv+n*n-1)$MDF
    r2 := e04ycf(0,m,n,f,s,n,v,-1)$NagOptimisationPackage
    concat(r,r2)

  optimize(f:EF,start:LF,lower:LOCF,cons:LEF,upper:LOCF):Result ==
    args:NOA := [ef2edf(f),[f2df i for i in start],[ocf2ocdf j for j in lower],
                 [ef2edf k for k in cons], [ocf2ocdf l for l in upper]]
    optimize(args::NumericalOptimizationProblem)

  optimize(f:EF,start:LF,lower:LOCF,upper:LOCF):Result ==
    optimize(f,start,lower,empty()$LEF,upper)

  optimize(f:EF,start:LF):Result ==
    optimize(f,start,empty()$LOCF,empty()$LOCF)

  optimize(lf:LEF,start:LF):Result ==
    args:LSA := [[ef2edf i for i in lf],[f2df j for j in start]]
    optimize(args::NumericalOptimizationProblem)

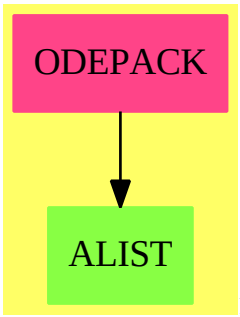
  goodnessOfFit(lf:LEF,start:LF):Result ==
    args:LSA := [[ef2edf i for i in lf],[f2df j for j in start]]
    goodnessOfFit(args::NumericalOptimizationProblem)
```

⟨*OPTPACK.dotabb*⟩≡

```
"OPTPACK" [color="#FF4488",href="bookvol10.4.pdf#nameddest=OPTPACK"]
"ALIST" [color="#88FF44",href="bookvol10.3.pdf#nameddest=ALIST"]
"OPTPACK" -> "ALIST"
```

2.21 package ODEPACK AnnaOrdinaryDifferentialEquationPackage

2.22 AnnaOrdinaryDifferentialEquationPackage



1.00

Exports:
measure solve

⟨*package ODEPACK AnnaOrdinaryDifferentialEquationPackage*⟩≡

```
)abbrev package ODEPACK AnnaOrdinaryDifferentialEquationPackage
++ Author: Brian Dupee
++ Date Created: February 1995
++ Date Last Updated: December 1997
++ Basic Operations: solve, measure
++ Description:
++ \axiomType{AnnaOrdinaryDifferentialEquationPackage} is a \axiom{package}
++ of functions for the \axiom{category} \axiomType{OrdinaryDifferentialEquationsSolverCateg
++ with \axiom{measure}, and \axiom{solve}.
++
EDF     ==> Expression DoubleFloat
LDF     ==> List DoubleFloat
MDF     ==> Matrix DoubleFloat
DF      ==> DoubleFloat
FI      ==> Fraction Integer
EFI     ==> Expression Fraction Integer
SOCDF   ==> Segment OrderedCompletion DoubleFloat
VEDF    ==> Vector Expression DoubleFloat
VEF     ==> Vector Expression Float
EF      ==> Expression Float
LF      ==> List Float
F       ==> Float
VDF     ==> Vector DoubleFloat
VMF     ==> Vector MachineFloat
MF      ==> MachineFloat
LS      ==> List Symbol
```

```
ST      ==> String
LST     ==> List String
INT     ==> Integer
RT      ==> RoutinesTable
ODEA    ==> Record(xinit:DF,xend:DF,fn:VEDF,yinit:LDF,intvals:LDF,_
                   g:EDF,abserr:DF,relerr:DF)
IFL     ==> List(Record(ifail:Integer,instruction:String))
Entry   ==> Record(chapter:String, type:String, domainName: String,
                    defaultMin:F, measure:F, failList:IFL, explList:LST)
Measure ==> Record(measure:F,name:String, explanations:List String)

AnnaOrdinaryDifferentialEquationPackage(): with
  solve:(NumericalODEProblem) -> Result
    ++ solve(odeProblem) is a top level ANNA function to solve numerically a
    ++ system of ordinary differential equations i.e. equations for the
    ++ derivatives y[1]'..y[n]' defined in terms of x,y[1]..y[n], together
    ++ with starting values for x and y[1]..y[n] (called the initial
    ++ conditions), a final value of x, an accuracy requirement and any
    ++ intermediate points at which the result is required.
    ++
    ++ It iterates over the \axiom{domains} of
    ++ \axiomType{OrdinaryDifferentialEquationsSolverCategory}
    ++ to get the name and other
    ++ relevant information of the the (domain of the) numerical
    ++ routine likely to be the most appropriate,
    ++ i.e. have the best \axiom{measure}.
    ++
    ++ The method used to perform the numerical
    ++ process will be one of the routines contained in the NAG numerical
    ++ Library.  The function predicts the likely most effective routine
    ++ by checking various attributes of the system of ODE's and calculating
    ++ a measure of compatibility of each routine to these attributes.
    ++
    ++ It then calls the resulting 'best' routine.
  solve:(NumericalODEProblem,RT) -> Result
    ++ solve(odeProblem,R) is a top level ANNA function to solve numerically a
    ++ system of ordinary differential equations i.e. equations for the
    ++ derivatives y[1]'..y[n]' defined in terms of x,y[1]..y[n], together
    ++ with starting values for x and y[1]..y[n] (called the initial
    ++ conditions), a final value of x, an accuracy requirement and any
    ++ intermediate points at which the result is required.
    ++
    ++ It iterates over the \axiom{domains} of
    ++ \axiomType{OrdinaryDifferentialEquationsSolverCategory} contained in
    ++ the table of routines \axiom{R} to get the name and other
    ++ relevant information of the the (domain of the) numerical
```

```
    ++ routine likely to be the most appropriate,
    ++ i.e. have the best \axiom{measure}.
    ++
    ++ The method used to perform the numerical
    ++ process will be one of the routines contained in the NAG numerical
    ++ Library.  The function predicts the likely most effective routine
    ++ by checking various attributes of the system of ODE's and calculating
    ++ a measure of compatibility of each routine to these attributes.
    ++
    ++ It then calls the resulting 'best' routine.
  solve:(VEF,F,F,LF) -> Result
    ++ solve(f,xStart,xEnd,yInitial) is a top level ANNA function to solve
    ++ numerically a system of ordinary differential equations i.e. equations
    ++ for the derivatives y[1]'..y[n]' defined in terms of x,y[1]..y[n],
    ++ together with a starting value for x and y[1]..y[n] (called the initial
    ++ conditions) and a final value of x.  A default value
    ++ is used for the accuracy requirement.
    ++
    ++ It iterates over the \axiom{domains} of
    ++ \axiomType{OrdinaryDifferentialEquationsSolverCategory} contained in
    ++ the table of routines \axiom{R} to get the name and other
    ++ relevant information of the the (domain of the) numerical
    ++ routine likely to be the most appropriate,
    ++ i.e. have the best \axiom{measure}.
    ++
    ++ The method used to perform the numerical
    ++ process will be one of the routines contained in the NAG numerical
    ++ Library.  The function predicts the likely most effective routine
    ++ by checking various attributes of the system of ODE's and calculating
    ++ a measure of compatibility of each routine to these attributes.
    ++
    ++ It then calls the resulting 'best' routine.
  solve:(VEF,F,F,LF,F) -> Result
    ++ solve(f,xStart,xEnd,yInitial,tol) is a top level ANNA function to solve
    ++ numerically a system of ordinary differential equations, \axiom{f},
    ++ i.e. equations for the derivatives y[1]'..y[n]' defined in terms
    ++ of x,y[1]..y[n] from \axiom{xStart} to \axiom{xEnd} with the initial
    ++ values for y[1]..y[n] (\axiom{yInitial}) to a tolerance \axiom{tol}.
    ++
    ++ It iterates over the \axiom{domains} of
    ++ \axiomType{OrdinaryDifferentialEquationsSolverCategory} contained in
    ++ the table of routines \axiom{R} to get the name and other
    ++ relevant information of the the (domain of the) numerical
    ++ routine likely to be the most appropriate,
    ++ i.e. have the best \axiom{measure}.
    ++
```

```
  ++ The method used to perform the numerical
  ++ process will be one of the routines contained in the NAG numerical
  ++ Library.  The function predicts the likely most effective routine
  ++ by checking various attributes of the system of ODE's and calculating
  ++ a measure of compatibility of each routine to these attributes.
  ++
  ++ It then calls the resulting 'best' routine.
solve:(VEF,F,F,LF,EF,F) -> Result
  ++ solve(f,xStart,xEnd,yInitial,G,tol) is a top level ANNA function to
  ++ solve numerically a system of ordinary differential equations,
  ++ \axiom{f}, i.e. equations for the derivatives y[1]'..y[n]' defined in
  ++ terms of x,y[1]..y[n] from \axiom{xStart} to \axiom{xEnd} with the
  ++ initial values for y[1]..y[n] (\axiom{yInitial}) to a tolerance
  ++ \axiom{tol}. The calculation will stop if the function
  ++ G(x,y[1],..,y[n]) evaluates to zero before x = xEnd.
  ++
  ++ It iterates over the \axiom{domains} of
  ++ \axiomType{OrdinaryDifferentialEquationsSolverCategory} contained in
  ++ the table of routines \axiom{R} to get the name and other
  ++ relevant information of the the (domain of the) numerical
  ++ routine likely to be the most appropriate,
  ++ i.e. have the best \axiom{measure}.
  ++
  ++ The method used to perform the numerical process will be one of the
  ++ routines contained in the NAG numerical Library.  The function
  ++ predicts the likely most effective routine by checking various
  ++ attributes of the system of ODE's and calculating a measure of
  ++ compatibility of each routine to these attributes.
  ++
  ++ It then calls the resulting 'best' routine.
solve:(VEF,F,F,LF,LF,F) -> Result
  ++ solve(f,xStart,xEnd,yInitial,intVals,tol) is a top level ANNA function
  ++ to solve numerically a system of ordinary differential equations,
  ++ \axiom{f}, i.e. equations for the derivatives y[1]'..y[n]' defined in
  ++ terms of x,y[1]..y[n] from \axiom{xStart} to \axiom{xEnd} with the
  ++ initial values for y[1]..y[n] (\axiom{yInitial}) to a tolerance
  ++ \axiom{tol}. The values of y[1]..y[n] will be output for the values
  ++ of x in \axiom{intVals}.
  ++
  ++ It iterates over the \axiom{domains} of
  ++ \axiomType{OrdinaryDifferentialEquationsSolverCategory} contained in
  ++ the table of routines \axiom{R} to get the name and other
  ++ relevant information of the the (domain of the) numerical
  ++ routine likely to be the most appropriate,
  ++ i.e. have the best \axiom{measure}.
  ++
```

```
    ++ The method used to perform the numerical
    ++ process will be one of the routines contained in the NAG numerical
    ++ Library.  The function predicts the likely most effective routine
    ++ by checking various attributes of the system of ODE's and calculating
    ++ a measure of compatibility of each routine to these attributes.
    ++
    ++ It then calls the resulting 'best' routine.
  solve:(VEF,F,F,LF,EF,LF,F) -> Result
    ++ solve(f,xStart,xEnd,yInitial,G,intVals,tol) is a top level ANNA
    ++ function to solve numerically a system of ordinary differential
    ++ equations, \axiom{f}, i.e. equations for the derivatives y[1]'..y[n]'
    ++ defined in terms of x,y[1]..y[n] from \axiom{xStart} to \axiom{xEnd}
    ++ with the initial values for y[1]..y[n] (\axiom{yInitial}) to a
    ++ tolerance \axiom{tol}. The values of y[1]..y[n] will be output for
    ++ the values of x in \axiom{intVals}.  The calculation will stop if the
    ++ function G(x,y[1],..,y[n]) evaluates to zero before x = xEnd.
    ++
    ++ It iterates over the \axiom{domains} of
    ++ \axiomType{OrdinaryDifferentialEquationsSolverCategory} contained in
    ++ the table of routines \axiom{R} to get the name and other
    ++ relevant information of the the (domain of the) numerical
    ++ routine likely to be the most appropriate,
    ++ i.e. have the best \axiom{measure}.
    ++
    ++ The method used to perform the numerical
    ++ process will be one of the routines contained in the NAG numerical
    ++ Library.  The function predicts the likely most effective routine
    ++ by checking various attributes of the system of ODE's and calculating
    ++ a measure of compatibility of each routine to these attributes.
    ++
    ++ It then calls the resulting 'best' routine.
  solve:(VEF,F,F,LF,EF,LF,F,F) -> Result
    ++ solve(f,xStart,xEnd,yInitial,G,intVals,epsabs,epsrel) is a top level
    ++ ANNA function to solve numerically a system of ordinary differential
    ++ equations, \axiom{f}, i.e.
    ++ equations for the derivatives y[1]'..y[n]' defined in terms
    ++ of x,y[1]..y[n] from \axiom{xStart} to \axiom{xEnd} with the initial
    ++ values for y[1]..y[n] (\axiom{yInitial}) to an absolute error
    ++ requirement \axiom{epsabs} and relative error \axiom{epsrel}.
    ++ The values of y[1]..y[n] will be output for the values of x in
    ++ \axiom{intVals}.  The calculation will stop if the function
    ++ G(x,y[1],..,y[n]) evaluates to zero before x = xEnd.
    ++
    ++ It iterates over the \axiom{domains} of
    ++ \axiomType{OrdinaryDifferentialEquationsSolverCategory} contained in
    ++ the table of routines \axiom{R} to get the name and other
```

```
    ++ relevant information of the the (domain of the) numerical
    ++ routine likely to be the most appropriate,
    ++ i.e. have the best \axiom{measure}.
    ++
    ++ The method used to perform the numerical
    ++ process will be one of the routines contained in the NAG numerical
    ++ Library.  The function predicts the likely most effective routine
    ++ by checking various attributes of the system of ODE's and calculating
    ++ a measure of compatibility of each routine to these attributes.
    ++
    ++ It then calls the resulting 'best' routine.
  measure:(NumericalODEProblem) -> Measure
    ++ measure(prob) is a top level ANNA function for identifying the most
    ++ appropriate numerical routine from those in the routines table
    ++ provided for solving the numerical ODE
    ++ problem defined by \axiom{prob}.
    ++
    ++ It calls each \axiom{domain} of \axiom{category}
    ++ \axiomType{OrdinaryDifferentialEquationsSolverCategory} in turn to
    ++ calculate all measures and returns the best i.e. the name of
    ++ the most appropriate domain and any other relevant information.
    ++ It predicts the likely most effective NAG numerical
    ++ Library routine to solve the input set of ODEs
    ++ by checking various attributes of the system of ODEs and calculating
    ++ a measure of compatibility of each routine to these attributes.
  measure:(NumericalODEProblem,RT) -> Measure
    ++ measure(prob,R) is a top level ANNA function for identifying the most
    ++ appropriate numerical routine from those in the routines table
    ++ provided for solving the numerical ODE
    ++ problem defined by \axiom{prob}.
    ++
    ++ It calls each \axiom{domain} listed in \axiom{R} of \axiom{category}
    ++ \axiomType{OrdinaryDifferentialEquationsSolverCategory} in turn to
    ++ calculate all measures and returns the best i.e. the name of
    ++ the most appropriate domain and any other relevant information.
    ++ It predicts the likely most effective NAG numerical
    ++ Library routine to solve the input set of ODEs
    ++ by checking various attributes of the system of ODEs and calculating
    ++ a measure of compatibility of each routine to these attributes.

  == add

   import ODEA,NumericalODEProblem

   f2df:F -> DF
   ef2edf:EF -> EDF
```

```
preAnalysis:(ODEA,RT) -> RT
zeroMeasure:Measure -> Result
measureSpecific:(ST,RT,ODEA) -> Record(measure:F,explanations:ST)
solveSpecific:(ODEA,ST) -> Result
changeName:(Result,ST) -> Result
recoverAfterFail:(ODEA,RT,Measure,Integer,Result) -> Record(a:Result,b:Measure)

f2df(f:F):DF == (convert(f)@DF)$F

ef2edf(f:EF):EDF == map(f2df,f)$ExpressionFunctions2(F,DF)

preAnalysis(args:ODEA,t:RT):RT ==
  rt := selectODEIVPRoutines(t)$RT
  if positive?(# variables(args.g)) then
    changeMeasure(rt,d02bbf@Symbol,getMeasure(rt,d02bbf@Symbol)*0.8)
  if positive?(# args.intvals) then
    changeMeasure(rt,d02bhf@Symbol,getMeasure(rt,d02bhf@Symbol)*0.8)
  rt

zeroMeasure(m:Measure):Result ==
  a := coerce(0$F)$AnyFunctions1(F)
  text := coerce("Zero Measure")$AnyFunctions1(ST)
  r := construct([[result@Symbol,a],[method@Symbol,text]])$Result
  concat(measure2Result m,r)$ExpertSystemToolsPackage

measureSpecific(name:ST,R:RT,ode:ODEA):Record(measure:F,explanations:ST) ==
  name = "d02bbfAnnaType" => measure(R,ode)$d02bbfAnnaType
  name = "d02bhfAnnaType" => measure(R,ode)$d02bhfAnnaType
  name = "d02cjfAnnaType" => measure(R,ode)$d02cjfAnnaType
  name = "d02ejfAnnaType" => measure(R,ode)$d02ejfAnnaType
  error("measureSpecific","invalid type name: " name)$ErrorFunctions

measure(Ode:NumericalODEProblem,R:RT):Measure ==
  ode:ODEA := retract(Ode)$NumericalODEProblem
  sofar := 0$F
  best := "none" :: ST
  routs := copy R
  routs := preAnalysis(ode,routs)
  empty?(routs)$RT =>
    error("measure", "no routines found")$ErrorFunctions
  rout := inspect(routs)$RT
  e := retract(rout.entry)$AnyFunctions1(Entry)
  meth := empty()$LST
  for i in 1..# routs repeat
    rout := extract!(routs)$RT
    e := retract(rout.entry)$AnyFunctions1(Entry)
```

```
      n := e.domainName
      if e.defaultMin > sofar then
        m := measureSpecific(n,R,ode)
        if m.measure > sofar then
          sofar := m.measure
          best := n
        str:LST := [string(rout.key)$Symbol "measure: "
                    outputMeasure(m.measure)$ExpertSystemToolsPackage " - "
                     m.explanations]
      else
        str := [string(rout.key)$Symbol " is no better than other routines"]
      meth := append(meth,str)$LST
    [sofar,best,meth]

  measure(ode:NumericalODEProblem):Measure == measure(ode,routines()$RT)

  solveSpecific(ode:ODEA,n:ST):Result ==
    n = "d02bbfAnnaType" => ODESolve(ode)$d02bbfAnnaType
    n = "d02bhfAnnaType" => ODESolve(ode)$d02bhfAnnaType
    n = "d02cjfAnnaType" => ODESolve(ode)$d02cjfAnnaType
    n = "d02ejfAnnaType" => ODESolve(ode)$d02ejfAnnaType
    error("solveSpecific","invalid type name: " n)$ErrorFunctions

  changeName(ans:Result,name:ST):Result ==
    sy:Symbol := coerce(name "Answer")$Symbol
    anyAns:Any := coerce(ans)$AnyFunctions1(Result)
    construct([[sy,anyAns]])$Result

  recoverAfterFail(ode:ODEA,routs:RT,m:Measure,iint:Integer,r:Result):
                                          Record(a:Result,b:Measure) ==
    while positive?(iint) repeat
      routineName := m.name
      s := recoverAfterFail(routs,routineName(1..6),iint)$RT
      s case "failed" => iint := 0
      if s = "increase tolerance" then
        ode.relerr := ode.relerr*(10.0::DF)
        ode.abserr := ode.abserr*(10.0::DF)
      if s = "decrease tolerance" then
        ode.relerr := ode.relerr/(10.0::DF)
        ode.abserr := ode.abserr/(10.0::DF)
      (s = "no action")@Boolean => iint := 0
      fl := coerce(s)$AnyFunctions1(ST)
      flrec:Record(key:Symbol,entry:Any):=[failure@Symbol,fl]
      m2 := measure(ode::NumericalODEProblem,routs)
      zero?(m2.measure) => iint := 0
      r2:Result := solveSpecific(ode,m2.name)
```

```
      m := m2
      insert!(flrec,r2)$Result
      r := concat(r2,changeName(r,routineName))$ExpertSystemToolsPackage
      iany := search(ifail@Symbol,r2)$Result
      iany case "failed" => iint := 0
      iint := retract(iany)$AnyFunctions1(Integer)
    [r,m]

  solve(Ode:NumericalODEProblem,t:RT):Result ==
    ode:ODEA := retract(Ode)$NumericalODEProblem
    routs := copy(t)$RT
    m := measure(Ode,routs)
    zero?(m.measure) => zeroMeasure m
    r := solveSpecific(ode,n := m.name)
    iany := search(ifail@Symbol,r)$Result
    iint := 0$Integer
    if (iany case Any) then
      iint := retract(iany)$AnyFunctions1(Integer)
    if positive?(iint) then
      tu:Record(a:Result,b:Measure) := recoverAfterFail(ode,routs,m,iint,r)
      r := tu.a
      m := tu.b
    r := concat(measure2Result m,r)$ExpertSystemToolsPackage
    expl := getExplanations(routs,n(1..6))$RoutinesTable
    expla := coerce(expl)$AnyFunctions1(LST)
    explaa:Record(key:Symbol,entry:Any) := ["explanations"::Symbol,expla]
    r := concat(construct([explaa]),r)
    iflist := showIntensityFunctions(ode)$ODEIntensityFunctionsTable
    iflist case "failed" => r
    concat(iflist2Result iflist, r)$ExpertSystemToolsPackage

  solve(ode:NumericalODEProblem):Result == solve(ode,routines()$RT)

  solve(f:VEF,xStart:F,xEnd:F,yInitial:LF,G:EF,intVals:LF,epsabs:F,epsrel:F):Result ==
    d:ODEA := [f2df xStart,f2df xEnd,vector([ef2edf e for e in members f])$VEDF,
               [f2df i for i in yInitial], [f2df j for j in intVals],
                ef2edf G,f2df epsabs,f2df epsrel]
    solve(d::NumericalODEProblem,routines()$RT)

  solve(f:VEF,xStart:F,xEnd:F,yInitial:LF,G:EF,intVals:LF,tol:F):Result ==
    solve(f,xStart,xEnd,yInitial,G,intVals,tol,tol)

  solve(f:VEF,xStart:F,xEnd:F,yInitial:LF,intVals:LF,tol:F):Result ==
    solve(f,xStart,xEnd,yInitial,1$EF,intVals,tol)

  solve(f:VEF,xStart:F,xEnd:F,y:LF,G:EF,tol:F):Result ==
```

```
      solve(f,xStart,xEnd,y,G,empty()$LF,tol)

    solve(f:VEF,xStart:F,xEnd:F,yInitial:LF,tol:F):Result ==
      solve(f,xStart,xEnd,yInitial,1$EF,empty()$LF,tol)

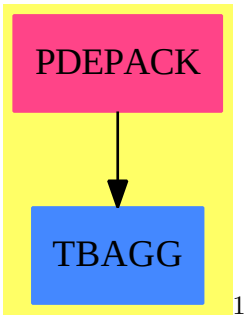
    solve(f:VEF,xStart:F,xEnd:F,yInitial:LF):Result == solve(f,xStart,xEnd,yInitial
```

⟨*ODEPACK.dotabb*⟩≡

```
  "ODEPACK" [color="#FF4488",href="bookvol10.4.pdf#nameddest=ODEPACK"]
  "ALIST" [color="#88FF44",href="bookvol10.3.pdf#nameddest=ALIST"]
  "ODEPACK" -> "ALIST"
```

2.23 package PDEPACK AnnaPartialDifferentialEquationPackage

2.24 AnnaPartialDifferentialEquationPackage



1.00

Exports:
measure solve

⟨*package PDEPACK AnnaPartialDifferentialEquationPackage*⟩≡

```
)abbrev package PDEPACK AnnaPartialDifferentialEquationPackage
++ Author: Brian Dupee
++ Date Created: June 1996
++ Date Last Updated: December 1997
++ Basic Operations:
++ Description: AnnaPartialDifferentialEquationPackage is an uncompleted
++ package for the interface to NAG PDE routines.  It has been realised that
++ a new approach to solving PDEs will need to be created.
++
LEDF    ==> List Expression DoubleFloat
EDF     ==> Expression DoubleFloat
LDF     ==> List DoubleFloat
MDF     ==> Matrix DoubleFloat
DF      ==> DoubleFloat
LEF     ==> List Expression Float
EF      ==> Expression Float
MEF     ==> Matrix Expression Float
LF      ==> List Float
F       ==> Float
LS      ==> List Symbol
ST      ==> String
LST     ==> List String
INT     ==> Integer
NNI     ==> NonNegativeInteger
RT      ==> RoutinesTable
PDEC    ==> Record(start:DF, finish:DF, grid:NNI, boundaryType:INT,
```

```
                    dStart:MDF, dFinish:MDF)
  PDEB    ==> Record(pde:LEDF, constraints:List PDEC,
                    f:List LEDF, st:ST, tol:DF)
  IFL     ==> List(Record(ifail:INT,instruction:ST))
  Entry   ==> Record(chapter:ST, type:ST, domainName: ST,
                     defaultMin:F, measure:F, failList:IFL, explList:LST)
  Measure ==> Record(measure:F,name:ST, explanations:LST)

  AnnaPartialDifferentialEquationPackage(): with
    solve:(NumericalPDEProblem) -> Result
      ++ solve(PDEProblem) is a top level ANNA function to solve numerically a syst
      ++ of partial differential equations.
      ++
      ++ The method used to perform the numerical
      ++ process will be one of the routines contained in the NAG numerical
      ++ Library.  The function predicts the likely most effective routine
      ++ by checking various attributes of the system of PDE's and calculating
      ++ a measure of compatibility of each routine to these attributes.
      ++
      ++ It then calls the resulting 'best' routine.
      ++
      ++ ** At the moment, only Second Order Elliptic Partial Differential
      ++ Equations are solved **
    solve:(NumericalPDEProblem,RT) -> Result
      ++ solve(PDEProblem,routines) is a top level ANNA function to solve numerical
      ++ of partial differential equations.
      ++
      ++ The method used to perform the numerical
      ++ process will be one of the routines contained in the NAG numerical
      ++ Library.  The function predicts the likely most effective routine
      ++ by checking various attributes of the system of PDE's and calculating
      ++ a measure of compatibility of each routine to these attributes.
      ++
      ++ It then calls the resulting 'best' routine.
      ++
      ++ ** At the moment, only Second Order Elliptic Partial Differential
      ++ Equations are solved **
    solve:(F,F,F,F,NNI,NNI,LEF,List LEF,ST,DF) -> Result
      ++ solve(xmin,ymin,xmax,ymax,ngx,ngy,pde,bounds,st,tol) is a top level
      ++ ANNA function to solve numerically a system of partial differential
      ++ equations.  This is defined as a list of coefficients (\axiom{pde}),
      ++ a grid (\axiom{xmin}, \axiom{ymin}, \axiom{xmax}, \axiom{ymax},
      ++ \axiom{ngx}, \axiom{ngy}), the boundary values (\axiom{bounds}) and a
      ++ tolerance requirement (\axiom{tol}).  There is also a parameter
      ++ (\axiom{st}) which should contain the value "elliptic" if the PDE is
      ++ known to be elliptic, or "unknown" if it is uncertain.  This causes the
```

```
    ++ routine to check whether the PDE is elliptic.
    ++
    ++ The method used to perform the numerical
    ++ process will be one of the routines contained in the NAG numerical
    ++ Library.  The function predicts the likely most effective routine
    ++ by checking various attributes of the system of PDE's and calculating
    ++ a measure of compatibility of each routine to these attributes.
    ++
    ++ It then calls the resulting 'best' routine.
    ++
    ++ ** At the moment, only Second Order Elliptic Partial Differential
    ++ Equations are solved **
  solve:(F,F,F,F,NNI,NNI,LEF,List LEF,ST) -> Result
    ++ solve(xmin,ymin,xmax,ymax,ngx,ngy,pde,bounds,st) is a top level
    ++ ANNA function to solve numerically a system of partial differential
    ++ equations.  This is defined as a list of coefficients (\axiom{pde}),
    ++ a grid (\axiom{xmin}, \axiom{ymin}, \axiom{xmax}, \axiom{ymax},
    ++ \axiom{ngx}, \axiom{ngy}) and the boundary values (\axiom{bounds}).
    ++ A default value for tolerance is used.  There is also a parameter
    ++ (\axiom{st}) which should contain the value "elliptic" if the PDE is
    ++ known to be elliptic, or "unknown" if it is uncertain.  This causes the
    ++ routine to check whether the PDE is elliptic.
    ++
    ++ The method used to perform the numerical
    ++ process will be one of the routines contained in the NAG numerical
    ++ Library.  The function predicts the likely most effective routine
    ++ by checking various attributes of the system of PDE's and calculating
    ++ a measure of compatibility of each routine to these attributes.
    ++
    ++ It then calls the resulting 'best' routine.
    ++
    ++ ** At the moment, only Second Order Elliptic Partial Differential
    ++ Equations are solved **
  measure:(NumericalPDEProblem) -> Measure
    ++ measure(prob) is a top level ANNA function for identifying the most
    ++ appropriate numerical routine from those in the routines table
    ++ provided for solving the numerical PDE
    ++ problem defined by \axiom{prob}.
    ++
    ++ It calls each \axiom{domain} of \axiom{category}
    ++ \axiomType{PartialDifferentialEquationsSolverCategory} in turn to
    ++ calculate all measures and returns the best i.e. the name of
    ++ the most appropriate domain and any other relevant information.
    ++ It predicts the likely most effective NAG numerical
    ++ Library routine to solve the input set of PDEs
    ++ by checking various attributes of the system of PDEs and calculating
```

```
    ++ a measure of compatibility of each routine to these attributes.
  measure:(NumericalPDEProblem,RT) -> Measure
    ++ measure(prob,R) is a top level ANNA function for identifying the most
    ++ appropriate numerical routine from those in the routines table
    ++ provided for solving the numerical PDE
    ++ problem defined by \axiom{prob}.
    ++
    ++ It calls each \axiom{domain} listed in \axiom{R} of \axiom{category}
    ++ \axiomType{PartialDifferentialEquationsSolverCategory} in turn to
    ++ calculate all measures and returns the best i.e. the name of
    ++ the most appropriate domain and any other relevant information.
    ++ It predicts the likely most effective NAG numerical
    ++ Library routine to solve the input set of PDEs
    ++ by checking various attributes of the system of PDEs and calculating
    ++ a measure of compatibility of each routine to these attributes.


  == add

   import PDEB, d03AgentsPackage, ExpertSystemToolsPackage, NumericalPDEProblem

   zeroMeasure:Measure -> Result
   measureSpecific:(ST,RT,PDEB) -> Record(measure:F,explanations:ST)
   solveSpecific:(PDEB,ST) -> Result
   changeName:(Result,ST) -> Result
   recoverAfterFail:(PDEB,RT,Measure,Integer,Result) -> Record(a:Result,b:Measure)

   zeroMeasure(m:Measure):Result ==
     a := coerce(0$F)$AnyFunctions1(F)
     text := coerce("No available routine appears appropriate")$AnyFunctions1(ST)
     r := construct([[result@Symbol,a],[method@Symbol,text]])$Result
     concat(measure2Result m,r)$ExpertSystemToolsPackage

   measureSpecific(name:ST,R:RT,p:PDEB):Record(measure:F,explanations:ST) ==
     name = "d03eefAnnaType" => measure(R,p)$d03eefAnnaType
     --name = "d03fafAnnaType" => measure(R,p)$d03fafAnnaType
     error("measureSpecific","invalid type name: " name)$ErrorFunctions

   measure(P:NumericalPDEProblem,R:RT):Measure ==
     p:PDEB := retract(P)$NumericalPDEProblem
     sofar := 0$F
     best := "none" :: ST
     routs := copy R
     routs := selectPDERoutines(routs)$RT
     empty?(routs)$RT =>
       error("measure", "no routines found")$ErrorFunctions
```

```
    rout := inspect(routs)$RT
    e := retract(rout.entry)$AnyFunctions1(Entry)
    meth := empty()$LST
    for i in 1..# routs repeat
      rout := extract!(routs)$RT
      e := retract(rout.entry)$AnyFunctions1(Entry)
      n := e.domainName
      if e.defaultMin > sofar then
        m := measureSpecific(n,R,p)
        if m.measure > sofar then
          sofar := m.measure
          best := n
        str:LST := [string(rout.key)$Symbol "measure: "
                    outputMeasure(m.measure)$ExpertSystemToolsPackage " - "
                     m.explanations]
      else
        str := [string(rout.key)$Symbol " is no better than other routines"]
      meth := append(meth,str)$LST
    [sofar,best,meth]

  measure(P:NumericalPDEProblem):Measure == measure(P,routines()$RT)

  solveSpecific(p:PDEB,n:ST):Result ==
    n = "d03eefAnnaType" => PDESolve(p)$d03eefAnnaType
    --n = "d03fafAnnaType" => PDESolve(p)$d03fafAnnaType
    error("solveSpecific","invalid type name: " n)$ErrorFunctions

  changeName(ans:Result,name:ST):Result ==
    sy:Symbol := coerce(name "Answer")$Symbol
    anyAns:Any := coerce(ans)$AnyFunctions1(Result)
    construct([[sy,anyAns]])$Result

  recoverAfterFail(p:PDEB,routs:RT,m:Measure,iint:Integer,r:Result):
                                          Record(a:Result,b:Measure) ==
    while positive?(iint) repeat
      routineName := m.name
      s := recoverAfterFail(routs,routineName(1..6),iint)$RT
      s case "failed" => iint := 0
      (s = "no action")@Boolean => iint := 0
      fl := coerce(s)$AnyFunctions1(ST)
      flrec:Record(key:Symbol,entry:Any):=[failure@Symbol,fl]
      m2 := measure(p::NumericalPDEProblem,routs)
      zero?(m2.measure) => iint := 0
      r2:Result := solveSpecific(p,m2.name)
      m := m2
      insert!(flrec,r2)$Result
```

```
      r := concat(r2,changeName(r,routineName))$ExpertSystemToolsPackage
      iany := search(ifail@Symbol,r2)$Result
      iany case "failed" => iint := 0
      iint := retract(iany)$AnyFunctions1(Integer)
    [r,m]

  solve(P:NumericalPDEProblem,t:RT):Result ==
    routs := copy(t)$RT
    m := measure(P,routs)
    p:PDEB := retract(P)$NumericalPDEProblem
    zero?(m.measure) => zeroMeasure m
    r := solveSpecific(p,n := m.name)
    iany := search(ifail@Symbol,r)$Result
    iint := 0$Integer
    if (iany case Any) then
      iint := retract(iany)$AnyFunctions1(Integer)
    if positive?(iint) then
      tu:Record(a:Result,b:Measure) := recoverAfterFail(p,routs,m,iint,r)
      r := tu.a
      m := tu.b
    expl := getExplanations(routs,n(1..6))$RoutinesTable
    expla := coerce(expl)$AnyFunctions1(LST)
    explaa:Record(key:Symbol,entry:Any) := ["explanations"::Symbol,expla]
    r := concat(construct([explaa]),r)
    concat(measure2Result m,r)$ExpertSystemToolsPackage

  solve(P:NumericalPDEProblem):Result == solve(P,routines()$RT)

  solve(xmi:F,xma:F,ymi:F,yma:F,nx:NNI,ny:NNI,pe:LEF,bo:List
                LEF,s:ST,to:DF):Result ==
    cx:PDEC := [f2df xmi, f2df xma, nx, 1, empty()$MDF, empty()$MDF]
    cy:PDEC := [f2df ymi, f2df yma, ny, 1, empty()$MDF, empty()$MDF]
    p:PDEB := [[ef2edf e for e in pe],[cx,cy],
                [[ef2edf u for u in w] for w in bo],s,to]
    solve(p::NumericalPDEProblem,routines()$RT)

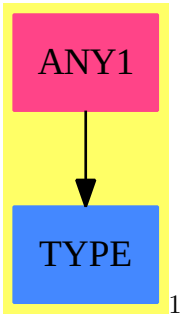
  solve(xmi:F,xma:F,ymi:F,yma:F,nx:NNI,ny:NNI,pe:LEF,bo:List
                LEF,s:ST):Result ==
    solve(xmi,xma,ymi,yma,nx,ny,pe,bo,s,0.0001::DF)
```

⟨*PDEPACK.dotabb*⟩≡

```
"PDEPACK" [color="#FF4488",href="bookvol10.4.pdf#nameddest=PDEPACK"]
"TBAGG" [color="#4488FF",href="bookvol10.2.pdf#nameddest=TBAGG"]
"PDEPACK" -> "TBAGG"
```

2.25 package ANY1 AnyFunctions1

2.26 AnyFunctions1



1.00

Exports:
coerce retract retractable? retractIfCan

⟨*package ANY1 AnyFunctions1*⟩≡
```
)abbrev package ANY1 AnyFunctions1
++ Author:
++ Date Created:
++ Change History:
++ Basic Functions:  coerce, retractIfCan, retractable?, retract
++ Related Constructors: Any
++ Also See:
++ AMS Classification:
++ Keywords:
++ Description:
++   \spadtype{AnyFunctions1} implements several utility functions for
++   working with \spadtype{Any}. These functions are used to go back
++   and forth between objects of \spadtype{Any} and objects of other
++   types.

AnyFunctions1(S:Type): with
        coerce      : S -> Any
          ++ coerce(s) creates an object of \spadtype{Any} from the
          ++ object \spad{s} of type \spad{S}.
        retractIfCan: Any -> Union(S, "failed")
          ++ retractIfCan(a) tries change \spad{a} into an object
          ++ of type \spad{S}. If it can, then such an object is
          ++ returned. Otherwise, "failed" is returned.
        retractable?: Any -> Boolean
          ++ retractable?(a) tests if \spad{a} can be converted
          ++ into an object of type \spad{S}.
        retract     : Any -> S
```

```
            ++ retract(a) tries to convert \spad{a} into an object of
            ++ type \spad{S}. If possible, it returns the object.
            ++ Error: if no such retraction is possible.

        == add
            import NoneFunctions1(S)

            Sexpr:SExpression := devaluate(S)$Lisp

            retractable? a  == dom(a) = Sexpr
            coerce(s:S):Any == any(Sexpr, s::None)

            retractIfCan a ==
                retractable? a => obj(a) pretend S
                "failed"

            retract a ==
                retractable? a => obj(a) pretend S
                error "Cannot retract value."
```

⟨*ANY1.dotabb*⟩≡

```
"ANY1" [color="#FF4488",href="bookvol10.4.pdf#nameddest=ANY1"]
"TYPE" [color="#4488FF",href="bookvol10.2.pdf#nameddest=TYPE"]
"ANY1" -> "TYPE"
```

2.27 package API ApplicationProgramInterface

⟨*ApplicationProgramInterface.input*⟩≡

```
)set break resume
)sys rm -f ApplicationProgramInterface.output
)spool ApplicationProgramInterface.output
)set message test on
)set message auto off
)clear all
--S 1 of 5
getDomains 'Collection
--R
--R   (1)
--R   {AssociationList, Bits, CharacterClass, DataList, EqTable, FlexibleArray,
--R    GeneralPolynomialSet, GeneralSparseTable, GeneralTriangularSet, HashTable,
--R    IndexedBits, IndexedFlexibleArray, IndexedList, IndexedOneDimensionalArray,
--R    IndexedString, IndexedVector, InnerTable, KeyedAccessFile, Library, List,
--R    ListMultiDictionary, Multiset, OneDimensionalArray, Point, PrimitiveArray,
--R    RegularChain, RegularTriangularSet, Result, RoutinesTable, Set,
--R    SparseTable, SquareFreeRegularTriangularSet, Stream, String, StringTable,
--R    Table, Vector, WuWenTsunTriangularSet}
--R                                                          Type: Set Symbol
--E 1

--S 2 of 5
difference(getDomains 'IndexedAggregate,getDomains 'Collection)
--R
--R   (2)
--R   {DirectProduct, DirectProductMatrixModule, DirectProductModule,
--R    HomogeneousDirectProduct, OrderedDirectProduct,
--R    SplitHomogeneousDirectProduct}
--R                                                          Type: Set Symbol
--E 2

--S 3 of 5
credits()
--R
--RAn alphabetical listing of contributors to AXIOM:
--RCyril Alberga          Roy Adler              Christian Aistleitner
--RRichard Anderson       George Andrews         S.J. Atkins
--RHenry Baker            Stephen Balzac         Yurij Baransky
--RDavid R. Barton        Gerald Baumgartner     Gilbert Baumslag
--RMichael Becker         Jay Belanger           David Bindel
--RFred Blair             Vladimir Bondarenko    Mark Botch
--RAlexandre Bouyer       Peter A. Broadbery     Martin Brock
--RManuel Bronstein       Stephen Buchwald       Florian Bundschuh
```

```
--RLuanne Burns           William Burge
--RQuentin Carpent        Robert Caviness        Bruce Char
--ROndrej Certik          Cheekai Chin           David V. Chudnovsky
--RGregory V. Chudnovsky  Josh Cohen             Christophe Conil
--RDon Coppersmith        George Corliss         Robert Corless
--RGary Cornell           Meino Cramer           Claire Di Crescenzo
--RDavid Cyganski
--RTimothy Daly Sr.       Timothy Daly Jr.       James H. Davenport
--RDidier Deshommes       Michael Dewar
--RJean Della Dora        Gabriel Dos Reis       Claire DiCrescendo
--RSam Dooley             Lionel Ducos           Martin Dunstan
--RBrian Dupee            Dominique Duval
--RRobert Edwards         Heow Eide-Goodman      Lars Erickson
--RRichard Fateman        Bertfried Fauser       Stuart Feldman
--RBrian Ford             Albrecht Fortenbacher  George Frances
--RConstantine Frangos    Timothy Freeman        Korrinn Fu
--RMarc Gaetano           Rudiger Gebauer        Kathy Gerber
--RPatricia Gianni        Samantha Goldrich      Holger Gollan
--RTeresa Gomez-Diaz      Laureano Gonzalez-Vega Stephen Gortler
--RJohannes Grabmeier     Matt Grayson           Klaus Ebbe Grue
--RJames Griesmer         Vladimir Grinberg      Oswald Gschnitzer
--RJocelyn Guidry
--RSteve Hague            Satoshi Hamaguchi      Mike Hansen
--RRichard Harke          Vilya Harvey           Martin Hassner
--RArthur S. Hathaway     Dan Hatton             Waldek Hebisch
--RKarl Hegbloom          Ralf Hemmecke          Henderson
--RAntoine Hersen         Gernot Hueber
--RPietro Iglio
--RAlejandro Jakubi       Richard Jenks
--RKai Kaminski           Grant Keady            Tony Kennedy
--RPaul Kosinski          Klaus Kusche           Bernhard Kutzler
--RTim Lahey              Larry Lambe            Franz Lehner
--RFrederic Lehobey       Michel Levaud          Howard Levy
--RLiu Xiaojun            Rudiger Loos           Michael Lucks
--RRichard Luczak
--RCamm Maguire           Francois Maltey        Alasdair McAndrew
--RBob McElrath           Michael McGettrick     Ian Meikle
--RDavid Mentre           Victor S. Miller       Gerard Milmeister
--RMohammed Mobarak       H. Michael Moeller     Michael Monagan
--RMarc Moreno-Maza       Scott Morrison         Joel Moses
--RMark Murray
--RWilliam Naylor         C. Andrew Neff         John Nelder
--RGodfrey Nolan          Arthur Norman          Jinzhong Niu
--RMichael O'Connor       Summat Oemrawsingh     Kostas Oikonomou
--RHumberto Ortiz-Zuazaga
--RJulian A. Padget       Bill Page              Susan Pelzel
```

```
--RMichel Petitot         Didier Pinchon         Ayal Pinkus
--RJose Alfredo Portes
--RClaude Quitte
--RArthur C. Ralfs        Norman Ramsey          Anatoly Raportirenko
--RMichael Richardson     Renaud Rioboo          Jean Rivlin
--RNicolas Robidoux       Simon Robinson         Raymond Rogers
--RMichael Rothstein      Martin Rubey
--RPhilip Santas          Alfred Scheerhorn      William Schelter
--RGerhard Schneider      Martin Schoenert       Marshall Schor
--RFrithjof Schulze       Fritz Schwarz          Nick Simicich
--RWilliam Sit            Elena Smirnova         Jonathan Steinbach
--RFabio Stumbo           Christine Sundaresan   Robert Sutor
--RMoss E. Sweedler       Eugene Surowitz
--RMax Tegmark            James Thatcher         Balbir Thomas
--RMike Thomas            Dylan Thurston         Barry Trager
--RThemos T. Tsikas
--RGregory Vanuxem
--RBernhard Wall          Stephen Watt           Jaap Weel
--RJuergen Weiss          M. Weller              Mark Wegman
--RJames Wen              Thorsten Werther       Michael Wester
--RJohn M. Wiley          Berhard Will           Clifton J. Williamson
--RStephen Wilson         Shmuel Winograd        Robert Wisbauer
--RSandra Wityak          Waldemar Wiwianka      Knut Wolf
--RClifford Yapp          David Yun
--RVadim Zhytnikov        Richard Zippel         Evelyn Zoernack
--RBruno Zuercher         Dan Zwillinger
--R                                                          Type: Void
--E 3
```

The summary command simply types out the contents of the file $AXIOM/lib/summary so no output is captured in the spool file.

⟨*ApplicationProgramInterface.input*⟩+≡

```
--S 4 of 5
summary()
--R
--R                                                                    Type: Void
--E 4

--S 5 of 5
)show API
--R ApplicationProgramInterface  is a package constructor
--R Abbreviation for ApplicationProgramInterface is API
--R This constructor is exposed in this frame.
--R Issue )edit bookvol10.4.spad.pamphlet to see algebra source code for API
--R
--R------------------------------ Operations --------------------------------
--R credits : () -> Void                 getDomains : Symbol -> Set Symbol
--R summary : () -> Void
--R
--E 5

)spool
)lisp (bye)
```

⟨*ApplicationProgramInterface.help*⟩≡

```
==========================================================================
ApplicationProgramInterface examples
==========================================================================

The ApplicationProgramInterface exposes Axiom internal functions
which might be useful for understanding, debugging, or creating
tools.

The getDomains function takes the name of a category and returns
a set of domains which inherit from that category:

  getDomains 'Collection

   {AssociationList, Bits, CharacterClass, DataList, EqTable, FlexibleArray,
    GeneralPolynomialSet, GeneralSparseTable, GeneralTriangularSet, HashTable,
    IndexedBits, IndexedFlexibleArray, IndexedList, IndexedOneDimensionalArray,
    IndexedString, IndexedVector, InnerTable, KeyedAccessFile, Library, List,
    ListMultiDictionary, Multiset, OneDimensionalArray, Point, PrimitiveArray,
    RegularChain, RegularTriangularSet, Result, RoutinesTable, Set,
    SparseTable, SquareFreeRegularTriangularSet, Stream, String, StringTable,
    Table, Vector, WuWenTsunTriangularSet}
                                                          Type: Set Symbol

This can be used to form the set-difference of two categories:

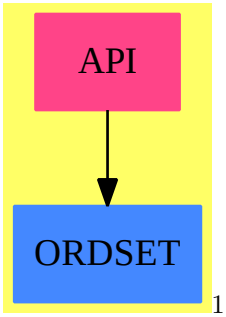
  difference(getDomains 'IndexedAggregate, getDomains 'Collection)

   {DirectProduct, DirectProductMatrixModule, DirectProductModule,
    HomogeneousDirectProduct, OrderedDirectProduct,
    SplitHomogeneousDirectProduct}
                                                          Type: Set Symbol

The credits function prints a list of the people who have contributed
to the development of Axiom. This is equivalent to the )credits command.

The summary function prints a short list of useful console commands.
```

2.28 ApplicationProgramInterface



1.00

Exports:

⟨*package API ApplicationProgramInterface*⟩≡

```
)abbrev package API ApplicationProgramInterface
++ Author: Timothy Daly, Martin Rubey
++ Date Created: 3 March 2009
++ Date Last Updated: 3 March 2009
++ Description: This package contains useful functions that
++ expose Axiom system internals
ApplicationProgramInterface(): Exports == Implementation where
  Exports ==> with
    getDomains : Symbol -> Set Symbol
      ++ The getDomains(s) takes a category and returns the list of domains
      ++ that have that category
      ++
      ++X getDomains 'IndexedAggregate
    credits : () -> Void
      ++ credits() prints a list of people who contributed to Axiom
      ++
      ++X credits()
    summary : () -> Void
      ++ summary() prints a short list of useful console commands
      ++
      ++X summary()
  Implementation ==> add
    getDomains(cat:Symbol):Set(Symbol) ==
      set [symbol car first destruct a _
        for a in (destruct domainsOf(cat,NIL$Lisp)$Lisp)::List(SExpression)]

    credits() == ( credits()$Lisp ; void() )

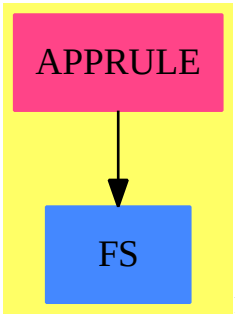
    summary() == ( summary()$Lisp ; void() )
```

⟨*API.dotabb*⟩≡

```
"API" [color="#FF4488",href="bookvol10.4.pdf#nameddest=APPRULE"]
"ORDSET" [color="#4488FF",href="bookvol10.2.pdf#nameddest=ORDSET"]
"API" -> "ORDSET"
```

2.29 package APPRULE ApplyRules

2.30 ApplyRules



1.00

Exports:
applyRules localUnquote

⟨*package APPRULE ApplyRules*⟩≡
```
)abbrev package APPRULE ApplyRules
++ Applications of rules to expressions
++ Author: Manuel Bronstein
++ Date Created: 20 Mar 1990
++ Date Last Updated: 5 Jul 1990
++ Description:
++   This package apply rewrite rules to expressions, calling
++   the pattern matcher.
++ Keywords: pattern, matching, rule.
ApplyRules(Base, R, F): Exports == Implementation where
  Base   : SetCategory
  R      : Join(Ring, PatternMatchable Base, OrderedSet,
                                ConvertibleTo Pattern Base)
  F      : Join(FunctionSpace R, PatternMatchable Base,
                                ConvertibleTo Pattern Base)

  P  ==> Pattern Base
  PR ==> PatternMatchResult(Base, F)
  RR ==> RewriteRule(Base, R, F)
  K  ==> Kernel F

  Exports ==> with
    applyRules  : (List RR, F) -> F
      ++ applyRules([r1,...,rn], expr) applies the rules
      ++ r1,...,rn to f an unlimited number of times, i.e. until
      ++ none of r1,...,rn is applicable to the expression.
    applyRules  : (List RR, F, PositiveInteger) -> F
```

```
      ++ applyRules([r1,...,rn], expr, n) applies the rules
      ++ r1,...,rn to f a most n times.
    localUnquote: (F, List Symbol) -> F
      ++ localUnquote(f,ls) is a local function.

  Implementation ==> add
    import PatternFunctions1(Base, F)

    splitRules: List RR -> Record(lker: List K,lval: List F,rl: List RR)
    localApply  : (List K, List F, List RR, F, PositiveInteger) -> F
    rewrite     : (F, PR, List Symbol) -> F
    app         : (List RR, F) -> F
    applist     : (List RR, List F) -> List F
    isit        : (F, P) -> PR
    isitwithpred: (F, P, List P, List PR) -> PR

    applist(lrule, arglist)  == [app(lrule, arg) for arg in arglist]

    splitRules l ==
      ncr := empty()$List(RR)
      lk  := empty()$List(K)
      lv  := empty()$List(F)
      for r in l repeat
        if (u := retractIfCan(r)@Union(Equation F, "failed"))
          case "failed" then ncr := concat(r, ncr)
          else
            lk := concat(retract(lhs(u::Equation F))@K, lk)
            lv := concat(rhs(u::Equation F), lv)
      [lk, lv, ncr]

    applyRules(l, s) ==
      rec := splitRules l
      repeat
        (new:= localApply(rec.lker,rec.lval,rec.rl,s,1)) = s => return s
        s := new

    applyRules(l, s, n) ==
      rec := splitRules l
      localApply(rec.lker, rec.lval, rec.rl, s, n)

    localApply(lk, lv, lrule, subject, n) ==
      for i in 1..n repeat
        for k in lk for v in lv repeat
          subject := eval(subject, k, v)
        subject := app(lrule, subject)
      subject
```

```
rewrite(f, res, l) ==
  lk := empty()$List(K)
  lv := empty()$List(F)
  for rec in destruct res repeat
    lk := concat(kernel(rec.key), lk)
    lv := concat(rec.entry, lv)
  localUnquote(eval(f, lk, lv), l)

if R has ConvertibleTo InputForm then
  localUnquote(f, l) ==
    empty? l => f
    eval(f, l)
else
  localUnquote(f, l) == f

isitwithpred(subject, pat, vars, bad) ==
  failed?(u := patternMatch(subject, pat, new()$PR)) => u
  satisfy?(u, pat)::Boolean => u
  member?(u, bad) => failed()
  for v in vars repeat addBadValue(v, getMatch(v, u)::F)
  isitwithpred(subject, pat, vars, concat(u, bad))

isit(subject, pat) ==
  hasTopPredicate? pat =>
    for v in (l := variables pat) repeat resetBadValues v
    isitwithpred(subject, pat, l, empty())
  patternMatch(subject, pat, new()$PR)

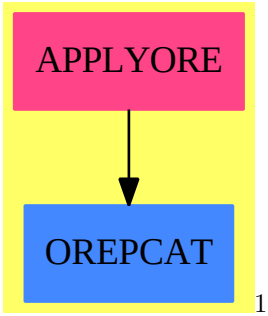
app(lrule, subject) ==
  for r in lrule repeat
    not failed?(u := isit(subject, pattern r)) =>
      return rewrite(rhs r, u, quotedOperators r)
  (k := retractIfCan(subject)@Union(K, "failed")) case K =>
    operator(k::K) applist(lrule, argument(k::K))
  (l := isPlus  subject) case List(F) => +/applist(lrule,l::List(F))
  (l := isTimes subject) case List(F) => */applist(lrule,l::List(F))
  (e := isPower subject) case Record(val:F, exponent:Integer) =>
    ee := e::Record(val:F, exponent:Integer)
    f  := app(lrule, ee.val)
    positive?(ee.exponent) => f ** (ee.exponent)::NonNegativeInteger
    recip(f)::F ** (- ee.exponent)::NonNegativeInteger
  subject
```

⟨*APPRULE.dotabb*⟩≡

```
"APPRULE" [color="#FF4488",href="bookvol10.4.pdf#nameddest=APPRULE"]
"FS" [color="#4488FF",href="bookvol10.2.pdf#nameddest=FS"]
"APPRULE" -> "FS"
```

2.31 package APPLYORE ApplyUnivariateSkewPolynomial

2.32 ApplyUnivariateSkewPolynomial



1.00

Exports:
apply

⟨*package APPLYORE ApplyUnivariateSkewPolynomial*⟩≡

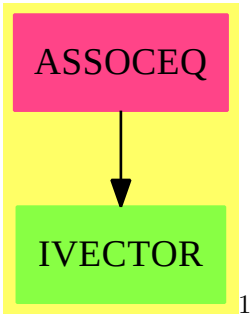
```
)abbrev package APPLYORE ApplyUnivariateSkewPolynomial
++ Author: Manuel Bronstein
++ Date Created: 7 December 1993
++ Date Last Updated: 1 February 1994
++ Description:
++   \spad{ApplyUnivariateSkewPolynomial} (internal) allows univariate
++   skew polynomials to be applied to appropriate modules.
ApplyUnivariateSkewPolynomial(R:Ring, M: LeftModule R,
    P: UnivariateSkewPolynomialCategory R): with
      apply: (P, M -> M, M) -> M
        ++ apply(p, f, m) returns \spad{p(m)} where the action is given
        ++ by \spad{x m = f(m)}.
        ++ \spad{f} must be an R-pseudo linear map on M.
   == add
      apply(p, f, m) ==
        w:M  := 0
        mn:M := m
        for i in 0..degree p repeat
          w  := w + coefficient(p, i) * mn
          mn := f mn
        w
```

⟨*APPLYORE.dotabb*⟩≡

```
"APPLYORE" [color="#FF4488",href="bookvol10.4.pdf#nameddest=APPLYORE"]
"OREPCAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=OREPCAT"]
"APPLYORE" -> "OREPCAT"
```

2.33 package ASSOCEQ AssociatedEquations

2.34 AssociatedEquations



1.00

Exports:
associatedEquations associatedSystem uncouplingMatrices

⟨*package ASSOCEQ AssociatedEquations*⟩≡
```
)abbrev package ASSOCEQ AssociatedEquations
++ Author: Manuel Bronstein
++ Date Created: 10 January 1994
++ Date Last Updated: 3 February 1994
++ Description:
++ \spadtype{AssociatedEquations} provides functions to compute the
++ associated equations needed for factoring operators
AssociatedEquations(R, L):Exports == Implementation where
  R: IntegralDomain
  L: LinearOrdinaryDifferentialOperatorCategory R

  PI  ==> PositiveInteger
  N   ==> NonNegativeInteger
  MAT ==> Matrix R
  REC ==> Record(minor: List PI, eq: L, minors: List List PI, ops: List L)

  Exports ==> with
    associatedSystem: (L, PI) -> Record(mat: MAT, vec:Vector List PI)
      ++ associatedSystem(op, m) returns \spad{[M,w]} such that the
      ++ m-th associated equation system to L is \spad{w' = M w}.
    uncouplingMatrices: MAT -> Vector MAT
      ++ uncouplingMatrices(M) returns \spad{[A_1,...,A_n]} such that if
      ++ \spad{y = [y_1,...,y_n]} is a solution of \spad{y' = M y}, then
      ++ \spad{[$y_j',y_j'',...,y_j^{(n)}$] = $A_j y$} for all j's.
    if R has Field then
        associatedEquations: (L, PI) -> REC
          ++ associatedEquations(op, m) returns \spad{[w, eq, lw, lop]}
```

```
        ++ such that \spad{eq(w) = 0} where w is the given minor, and
        ++ \spad{lw_i = lop_i(w)} for all the other minors.

  Implementation ==> add
    makeMatrix: (Vector MAT, N) -> MAT

    diff:L := D()

    makeMatrix(v, n) == matrix [parts row(v.i, n) for i in 1..#v]

    associatedSystem(op, m) ==
      eq: Vector R
      S := SetOfMIntegersInOneToN(m, n := degree(op)::PI)
      w := enumerate()$S
      s := size()$S
      ww:Vector List PI := new(s, empty())
      M:MAT := new(s, s, 0)
      m1 := (m::Integer - 1)::PI
      an := leadingCoefficient op
      a:Vector(R) := [- (coefficient(op, j) exquo an)::R for j in 0..n - 1]
      for i in 1..s repeat
          eq := new(s, 0)
          wi := w.i
          ww.i := elements wi
          for k in 1..m1 repeat
              u := incrementKthElement(wi, k::PI)$S
              if u case S then eq(lookup(u::S)) := 1
          if member?(n, wi) then
              for j in 1..n | a.j ^= 0 repeat
                  u := replaceKthElement(wi, m, j::PI)
                  if u case S then
                    eq(lookup(u::S)) := (odd? delta(wi, m, j::PI) => -a.j; a.j)
          else
              u := incrementKthElement(wi, m)$S
              if u case S then eq(lookup(u::S)) := 1
          setRow_!(M, i, eq)
      [M, ww]

    uncouplingMatrices m ==
      n := nrows m
      v:Vector MAT := new(n, zero(1, 0)$MAT)
      v.1 := mi := m
      for i in 2..n repeat v.i := mi := map((z1:R):R +-> diff z1, mi) + mi * m
      [makeMatrix(v, i) for i in 1..n]

    if R has Field then
```

```
import PrecomputedAssociatedEquations(R, L)

makeop:    Vector R -> L
makeeq:    (Vector List PI, MAT, N, N) -> REC
computeIt: (L, PI, N) -> REC

makeeq(v, m, i, n) ==
  [v.i, makeop row(m, i) - 1, [v.j for j in 1..n | j ^= i],
                              [makeop row(m, j) for j in 1..n | j ^= i]]

associatedEquations(op, m) ==
  (u := firstUncouplingMatrix(op, m)) case "failed" => computeIt(op,m,1)
  (v := inverse(u::MAT)) case "failed" => computeIt(op, m, 2)
  S := SetOfMIntegersInOneToN(m, degree(op)::PI)
  w := enumerate()$S
  s := size()$S
  ww:Vector List PI := new(s, empty())
  for i in 1..s repeat ww.i := elements(w.i)
  makeeq(ww, v::MAT, 1, s)

computeIt(op, m, k) ==
  rec := associatedSystem(op, m)
  a := uncouplingMatrices(rec.mat)
  n := #a
  for i in k..n repeat
    (u := inverse(a.i)) case MAT => return makeeq(rec.vec,u::MAT,i,n)
  error "associatedEquations: full degenerate case"

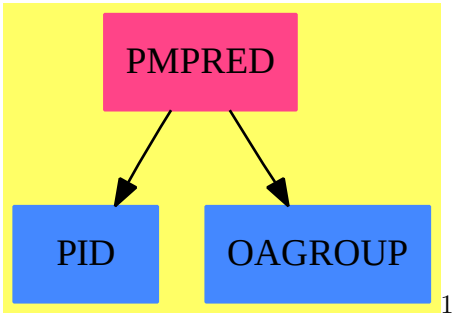
makeop v ==
  op:L := 0
  for i in 1..#v repeat op := op + monomial(v i, i)
  op
```

⟨*ASSOCEQ.dotabb*⟩≡

```
"ASSOCEQ" [color="#FF4488",href="bookvol10.4.pdf#nameddest=ASSOCEQ"]
"IVECTOR" [color="#88FF44",href="bookvol10.3.pdf#nameddest=IVECTOR"]
"ASSOCEQ" -> "IVECTOR"
```

2.35 package PMPRED AttachPredicates

2.36 AttachPredicates



1.00

Exports:
suchThat

⟨*package PMPRED AttachPredicates*⟩≡
```
)abbrev package PMPRED AttachPredicates
++ Predicates for pattern-matching
++ Author: Manuel Bronstein
++ Description: Attaching predicates to symbols for pattern matching.
++ Date Created: 21 Mar 1989
++ Date Last Updated: 23 May 1990
++ Keywords: pattern, matching.
AttachPredicates(D:Type): Exports == Implementation where
  FE ==> Expression Integer

  Exports ==> with
    suchThat: (Symbol, D -> Boolean) -> FE
      ++ suchThat(x, foo) attaches the predicate foo to x.
    suchThat: (Symbol, List(D -> Boolean)) -> FE
      ++ suchThat(x, [f1, f2, ..., fn]) attaches the predicate
      ++ f1 and f2 and ... and fn to x.

  Implementation ==> add
    import FunctionSpaceAttachPredicates(Integer, FE, D)

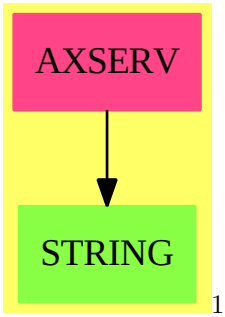
    suchThat(p:Symbol, f:D -> Boolean)       == suchThat(p::FE, f)
    suchThat(p:Symbol, l:List(D -> Boolean)) == suchThat(p::FE, l)
```

⟨*PMPRED.dotabb*⟩≡

```
"PMPRED" [color="#FF4488",href="bookvol10.4.pdf#nameddest=PMPRED"]
"PID" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PID"]
"OAGROUP" [color="#4488FF",href="bookvol10.2.pdf#nameddest=OAGROUP"]
"PMPRED" -> "PID"
"PMPRED" -> "OAGROUP"
```

2.37 package AXSERV AxiomServer

2.38 AxiomServer



1.00

Exports:
multiServ axServer getDatabase

⟨*package AXSERV AxiomServer*⟩≡

```
)abbrev package AXSERV AxiomServer
AxiomServer: public == private where

 public == with

   axServer: (Integer, SExpression->Void) -> Void
   multiServ: SExpression -> Void
   getDatabase: (String,String) -> String

 private == add

   getFile: (SExpression,String) -> Void
   getCommand: (SExpression,String) -> Void
   getDescription: String -> String
   getInterp: (SExpression,String) -> Void
   getLisp:   (SExpression,String) -> Void
   getShow:   (SExpression,String) -> Void
   lastStep: () -> String
   lastType: () -> String
   formatMessages: String -> String
   makeErrorPage: String -> String
   getSourceFile: (String,String,String) -> String
   makeDBPage: String -> String
   getContentType: String -> String
   readTheFile: SExpression -> String
   outputToSocket: (SExpression,String,String) -> Void
```

```
    getDatabase(constructor:String, key:String):String ==
      answer:=string GETDATABASE(INTERN$Lisp constructor,INTERN$Lisp key)$Lisp
--      WriteLine$Lisp concat ["getDatabase: ",constructor," ",key," ",answer]
      answer
```

The axServer function handles the socket connection on the given port. When it gets a input on the socket it calls the server function on the socket input.

⟨*package AXSERV AxiomServer*⟩+≡

```
    axServer(port:Integer,serverfunc:SExpression->Void):Void ==
      WriteLine$Lisp "listening on port 8085"
      s := SiSock(port,serverfunc)$Lisp
      -- To listen for just one connection and then close the socket
      -- uncomment i := 0.
      i:Integer := 1
      while (i > 0) repeat
        if not null?(SiListen(s)$Lisp)$SExpression then
          w := SiAccept(s)$Lisp
          serverfunc(w)
--         i := 0
```

The multiServ function parses the socket input. It expects either a GET or POST request.

A GET request fetches a new page, calling "getFile". A POST request starts with

- "command=" which expects axiom interpreter commands. When this is recognized we call the "getCommand" function.
- "lispcall=" which expects lisp interpreter input When this is recognized we call the "getLisp" function.

⟨*package AXSERV AxiomServer*⟩+≡

```
    multiServ(s:SExpression):Void ==
--     WriteLine("multiServ begin")$Lisp
      headers:String := ""
      char:String
      -- read in the http headers
      while (char := _
        STRING(READ_-CHAR_-NO_-HANG(s,NIL$Lisp,'EOF)$Lisp)$Lisp) ^= "EOF"_
         repeat
          headers := concat [headers,char]
--     sayTeX$Lisp headers
      StringMatch("([^ ]*)", headers)$Lisp
      u:UniversalSegment(Integer)
      u := segment(MatchBeginning(1)$Lisp+1,_
                   MatchEnd(1)$Lisp)$UniversalSegment(Integer)
      reqtype:String := headers.u
--     sayTeX$Lisp  concat ["request type: ",reqtype]
      if  reqtype = "GET" then
          StringMatch("GET ([^ ]*)",headers)$Lisp
          u:UniversalSegment(Integer)
          u := segment(MatchBeginning(1)$Lisp+1,_
                       MatchEnd(1)$Lisp)$UniversalSegment(Integer)
          getFile(s,headers.u)
      if reqtype = "POST" and StringMatch("command=(.*)$",headers)$Lisp > 0
       then
          u:UniversalSegment(Integer)
          u := segment(MatchBeginning(1)$Lisp+1,_
                       MatchEnd(1)$Lisp)$UniversalSegment(Integer)
          getCommand(s,headers.u)
      if reqtype = "POST" and StringMatch("interpcall=(.*)$",headers)$Lisp > 0
       then
          u:UniversalSegment(Integer)
          u := segment(MatchBeginning(1)$Lisp+1,_
                       MatchEnd(1)$Lisp)$UniversalSegment(Integer)
          getInterp(s,headers.u)
```

```
      if reqtype = "POST" and StringMatch("lispcall=(.*)$",headers)$Lisp > 0
       then
          u:UniversalSegment(Integer)
          u := segment(MatchBeginning(1)$Lisp+1,_
                       MatchEnd(1)$Lisp)$UniversalSegment(Integer)
          getLisp(s,headers.u)
      if reqtype = "POST" and StringMatch("showcall=(.*)$",headers)$Lisp > 0
       then
          u:UniversalSegment(Integer)
          u := segment(MatchBeginning(1)$Lisp+1,_
                       MatchEnd(1)$Lisp)$UniversalSegment(Integer)
          getShow(s,headers.u)
--     WriteLine("multiServ end")$Lisp
--     WriteLine("")$Lisp
```

getFile

Given a socket and the URL of the file we create an input stream that contains the file. If the filename contains a question mark then we need to parse the parameters and dynamically construct the file contents.

⟨*package AXSERV AxiomServer*⟩+≡

```
    getFile(s:SExpression,pathvar:String):Void ==
--     WriteLine("")$Lisp
      WriteLine$Lisp concat ["getFile: ",pathvar]
      params:=split(pathvar,char "?")
      if #params = 1
       then if not null? PATHNAME_-NAME(PATHNAME(pathvar)$Lisp)$Lisp
        then
          contentType:String := getContentType(pathvar)
          q:=Open(pathvar)$Lisp
          if null? q
            then
              q := MAKE_-STRING_-INPUT_-STREAM(_
                    makeErrorPage("File doesn't exist"))$Lisp
        else
          q:=MAKE_-STRING_-INPUT_-STREAM(_
              makeErrorPage("Problem with file path"))$Lisp
       else
        q:=MAKE_-STRING_-INPUT_-STREAM(makeDBPage(pathvar))$Lisp
      outputToSocket(s,readTheFile(q),contentType)
```

makeErrorPage

⟨*package AXSERV AxiomServer*⟩+≡

```
    makeErrorPage(msg:String):String ==
      page:String:="<!DOCTYPE html PUBLIC "
      page:=page "_"-//W3C//DTD XHTML 1.0 Strict//EN_" "
      page:=page "_"http://www.w3.org/TR/xthml1/DTD/xhtml1-strict.dtd_">"
      page:=page "<html xmlns=_"http://www.w3.org/1999/xhtml_">"
      page:=page "<head><title>Error</title></head><body>" msg "</body></html>"
--      WriteLine(page)$Lisp
      page
```

getDescription

We need to fish around in the data structure to return the piece of documentation for the domain. We have to call the lisp version of GETDATABASE because the version above returns a string object. The string object is missing quotes and cannot be properly read. So we need to get the lisp object and work with it in native form first.

The doc string also contains spad markup which we need to replace with html.

⟨*package AXSERV AxiomServer*⟩+≡

```
    getDescription(dom:String):String ==
     d:=CADR(CADAR(GETDATABASE(INTERN(dom)$Lisp,'DOCUMENTATION)$Lisp)$Lisp)$Lisp
     string d
```

getSourceFile

During build we construct a hash table that takes the chunk name as the key and returns the filename. We reconstruct the chunk name here and do a lookup for the source file.

⟨*package AXSERV AxiomServer*⟩+≡

```
    getSourceFile(constructorkind:String,_
                  abbreviation:String,_
                  dom:String):String ==
      sourcekey:="<<" constructorkind " " abbreviation " " dom ">>"
--      WriteLine(sourcekey)$Lisp
      sourcefile:=lowerCase last split(getDatabase(dom,"SOURCEFILE"),char "/")
      sourcefile:=sourcefile ".pamphlet"
```

makeDBPage

⟨*package AXSERV AxiomServer*⟩+≡

```
makeDBPage(pathvar:String):String ==
  params:List(String):=split(pathvar,char "?")
  for i in 1..#params repeat WriteLine$Lisp concat ["params: ",params.i]
  pathparts:List(String):=split(params.1,char "/")
  for i in 1..#pathparts repeat
    WriteLine$Lisp concat ["pathparts: ",pathparts.i]
  pagename:=last pathparts
  WriteLine$Lisp concat ["pagename: ",pagename]
  cmd:=first split(pagename,char ".")
  WriteLine$Lisp concat ["cmd: ",cmd]
  args:List(String):=split(params.2, char "&")
  for i in 1..#args repeat WriteLine$Lisp concat ["args: ",args.i]
  page:String:="<!DOCTYPE html PUBLIC "
  page:=page "_"-//W3C//DTD XHTML 1.0 Strict//EN_" "
  page:=page "_"http://www.w3.org/TR/xthml1/DTD/xhtml1-strict.dtd_">"
  page:=page "<html xmlns=_"http://www.w3.org/1999/xhtml_">"
  page:=page "<head>"
  page:=page "<meta http-equiv=_"Content-Type_" content=_"text/html_""
  page:=page " charset=_"us-ascii_"/>"
  page:=page "<title>" cmd " " args.1 "</title></head>"
  page:=page "<style> html { background-color: #FFFF66; } </style>"
  page:=page "<body>"
  cmd = "db" =>
   dom:=args.1
   domi:=INTERN(dom)$Lisp
   -- category, domain, or package?
   constructorkind:=getDatabase(dom,"CONSTRUCTORKIND")
   abbreviation:=getDatabase(dom, "ABBREVIATION")
   sourcefile:=getDatabase(dom, "SOURCEFILE")
   constructorkind.1:=upperCase constructorkind.1
   description:=getDescription(dom)
   page:=page "<div align=_"center_">"
   page:=page "<img align=_"middle_" src=_"doctitle.png_"/></div><hr/>"
   page:=page "<div align=_"center_">" constructorkind " " dom "</div><hr/>"
   page:=page "<table>"
   page:=page "<tr><td valign=_"top_">Description:  </td>"
   page:=page "<td>" description  "</td></tr>"
   page:=page "<tr><td>Abbreviation: </td><td>" abbreviation "</td></tr>"
   page:=page "<tr><td>Source File:  </td><td>" sourcefile   "</td></tr>"
   page:=page "</table><hr/>"
   page:=page "<table>"
   page:=page "<tr>"
   page:=page "<td>"
```

```
 page:=page "<a href=_"?" dom "&lookup=Ancestors_">Ancestors</a>"
 page:=page "</td>"
 page:=page "<td>"
 page:=page "<a href=_"?" dom "&lookup=Dependents_">Dependents</a>"
 page:=page "</td>"
 page:=page "<td>"
 page:=page "<a href=_"?" dom "&lookup=Exports_">Exports</a>"
 page:=page "</td>"
 page:=page "<td>"
 page:=page "<a href=_"?" dom "&lookup=Parents_">Parents</a>"
 page:=page "</td>"
 page:=page "<td>"
 page:=page "<a href=_"?" dom "&lookup=Users_">Users</a>"
 page:=page "</td>"
 page:=page "</tr>"
 page:=page "<tr>"
 page:=page "<td>"
 page:=page "<a href=_"?" dom "&lookup=Attributes_">Attributes</a>"
 page:=page "</td>"
 page:=page "<td>"
 page:=page "<a href=_"?" dom "&lookup=Examples_">Examples</a>"
 page:=page "</td>"
 page:=page "<td>"
 page:=page "<a href=_"?" dom "&lookup=Operations_">Operations</a>"
 page:=page "</td>"
 page:=page "<td>"
 page:=page "<a href=_"?" dom "&lookup=SearchPath_">Search Path</a>"
 page:=page "</td>"
 page:=page "<td>"
 page:=page "<a href=_"?" dom "&lookup=Uses_">Uses</a>"
 page:=page "</td>"
 page:=page "</tr>"
 page:=page "</table>"
cmd = "op" =>
 dom:=args.1
 domi:=INTERN(dom)$Lisp
 -- category, domain, or package?
 constructorkind:=getDatabase(dom,"CONSTRUCTORKIND")
 abbreviation:=getDatabase(dom, "ABBREVIATION")
 sourcefile:=getDatabase(dom, "SOURCEFILE")
 constructorkind.1:=upperCase constructorkind.1
 description:=getDescription(dom)
 page:=page "<div align=_"center_">"
 page:=page "<img align=_"middle_" src=_"doctitle.png_"/></div><hr/>"
 page:=page "<div align=_"center_">" constructorkind " " dom "</div><hr/>"
 page:=page "<table>"
```

```
        page:=page "<tr><td valign=_"top_">Description:  </td>"
        page:=page "<td>" description  "</td></tr>"
        page:=page "<tr><td>Abbreviation: </td><td>" abbreviation "</td></tr>"
        page:=page "<tr><td>Source File:  </td><td>" sourcefile   "</td></tr>"
        page:=page "</table><hr/>"
        page:=page "<table>"
        page:=page "<tr>"
        page:=page "<td>"
        page:=page "<a href=_"?" dom "&lookup=Ancestors_">Ancestors</a>"
        page:=page "</td>"
        page:=page "<td>"
        page:=page "<a href=_"?" dom "&lookup=Dependents_">Dependents</a>"
        page:=page "</td>"
        page:=page "<td>"
        page:=page "<a href=_"?" dom "&lookup=Exports_">Exports</a>"
        page:=page "</td>"
        page:=page "<td>"
        page:=page "<a href=_"?" dom "&lookup=Parents_">Parents</a>"
        page:=page "</td>"
        page:=page "<td>"
        page:=page "<a href=_"?" dom "&lookup=Users_">Users</a>"
        page:=page "</td>"
        page:=page "</tr>"
        page:=page "<tr>"
        page:=page "<td>"
        page:=page "<a href=_"?" dom "&lookup=Attributes_">Attributes</a>"
        page:=page "</td>"
        page:=page "<td>"
        page:=page "<a href=_"?" dom "&lookup=Examples_">Examples</a>"
        page:=page "</td>"
        page:=page "<td>"
        page:=page "<a href=_"?" dom "&lookup=Operations_">Operations</a>"
        page:=page "</td>"
        page:=page "<td>"
        page:=page "<a href=_"?" dom "&lookup=SearchPath_">Search Path</a>"
        page:=page "</td>"
        page:=page "<td>"
        page:=page "<a href=_"?" dom "&lookup=Uses_">Uses</a>"
        page:=page "</td>"
        page:=page "</tr>"
        page:=page "</table>"
--      WriteLine(page)$Lisp
      page:=page "</body></html>"
      page
```

readTheFile

We have q which is a stream which contains the file. We read the file into a string-stream to get it all into one string. We return the string.

⟨*package AXSERV AxiomServer*⟩+≡

```
    readTheFile(q:SExpression):String ==
 --     WriteLine("begin reading file")$Lisp
      r := MAKE_-STRING_-OUTPUT_-STREAM()$Lisp
      SiCopyStream(q,r)$Lisp
      filestream:String := GET_-OUTPUT_-STREAM_-STRING(r)$Lisp
      CLOSE(r)$Lisp
      CLOSE(q)$Lisp
 --     WriteLine("end reading file")$Lisp
      filestream
```

outputToSocket

We have "s" which is the socket, "filestream" which is the text of the file to output, and "contentType" which is the HTML Content-Type. We construct the HTML header information according to the standard and prepend it to the file. The resulting string is output to the socket.

⟨*package AXSERV AxiomServer*⟩+≡

```
    outputToSocket(s:SExpression,filestream:String,contentType:String):Void ==
      filelength:String := string(#filestream)
      file:String := ""
      nl:String:=STRING(NewLine$Lisp)$Lisp
      file := concat ["Content-Length: ",filelength,nl,nl,file]
      file := concat ["Connection: close",nl,file]
      file := concat ["Content-Type: ",contentType,nl,file]
      file := concat ["HTTP/1.1 200 OK",nl,file]
      file := concat [file,filestream]
 --     WriteLine(file)$Lisp
      f:=MAKE_-STRING_-INPUT_-STREAM(file)$Lisp
      SiCopyStream(f,s)$Lisp
      CLOSE(f)$Lisp
      CLOSE(s)$Lisp
```

getCommand

The getCommand function is invoked when the HTTP request is a POST and contains the string "command". Essentially the game here is to rebind the various output streams used by Axiom so we can capture the normal output. This function returns a set of HTML 5 div blocks:

1. stepnum, the value of lastStep()
2. command, the value of the command variable
3. algebra, the value of the algebra variable
4. mathml, the value of the mathml variable
5. type, the value of lastType()

The HTML functions in the hyperdoc browser depend on the order of these variables so do not change this without changing the corresponding functions in the browser HTML.

⟨*package AXSERV AxiomServer*⟩+≡

```
getCommand(s:SExpression,command:String):Void ==
    WriteLine$Lisp concat ["getCommand: ",command]
    SETQ(tmpmathml$Lisp, MAKE_-STRING_-OUTPUT_-STREAM()$Lisp)$Lisp
    SETQ(tmpalgebra$Lisp, MAKE_-STRING_-OUTPUT_-STREAM()$Lisp)$Lisp
    SETQ(savemathml$Lisp, _$texOutputStream$Lisp)$Lisp
    SETQ(savealgebra$Lisp, _$algebraOutputStream$Lisp)$Lisp
    SETQ(_$texOutputStream$Lisp,tmpmathml$Lisp)$Lisp
    SETQ(_$algebraOutputStream$Lisp,tmpalgebra$Lisp)$Lisp
    ans := string parseAndEvalToStringEqNum$Lisp command
    SETQ(resultmathml$Lisp,_
         GET_-OUTPUT_-STREAM_-STRING(_$texOutputStream$Lisp)$Lisp)$Lisp
    SETQ(resultalgebra$Lisp,_
         GET_-OUTPUT_-STREAM_-STRING(_$algebraOutputStream$Lisp)$Lisp)$Lisp
    SETQ(_$texOutputStream$Lisp,savemathml$Lisp)$Lisp
    SETQ(_$algebraOutputStream$Lisp,savealgebra$Lisp)$Lisp
    CLOSE(tmpmathml$Lisp)$Lisp
    CLOSE(tmpalgebra$Lisp)$Lisp
    -- Since strings returned from axiom are going to be displayed in html I
    -- should really check for the characters &,<,> and replace them with
    -- &amp;,&lt;,&gt;.
    -- At present I only check for ampersands in formatMessages.
    mathml:String := string(resultmathml$Lisp)
    algebra:String := string(resultalgebra$Lisp)
    algebra := formatMessages(algebra)
    -- At this point mathml contains the mathml for the output but does not
    -- include step number or type information.
```

```
        -- We should also save the command.
        -- I get the type and step number from the $internalHistoryTable
        axans:String := _
          concat ["<div class=_"stepnum_">", lastStep(), "</div>_
                   <div class=_"command_">", command, "</div>_
                   <div class=_"algebra_">",algebra,"</div>_
                   <div class=_"mathml_">",mathml,"</div>_
                   <div class=_"type_">",lastType(),"</div>"]
--        WriteLine$Lisp concat ["mathml answer: ",mathml]
--        WriteLine$Lisp concat ["algebra answer: ",algebra]
        q:=MAKE_-STRING_-INPUT_-STREAM(axans)$Lisp
        SiCopyStream(q,s)$Lisp
        CLOSE(q)$Lisp
        CLOSE(s)$Lisp
```

getInterp

The getInterp function is invoked when the HTTP request is a POST and contains the string "command". Essentially the game here is to rebind the various output streams used by Axiom so we can capture the normal output. This function returns a set of HTML 5 div blocks:

1. stepnum, the value of lastStep()
2. command, the value of the command variable
3. algebra, the value of the algebra variable
4. mathml, the value of the mathml variable
5. type, the value of lastType()

The HTML functions in the hyperdoc browser depend on the order of these variables so do not change this without changing the corresponding functions in the browser HTML.

⟨*package AXSERV AxiomServer*⟩+≡

```
getInterp(s:SExpression,command:String):Void ==
    WriteLine$Lisp concat ["getInterp: ",command]
    SETQ(tmpmathml$Lisp, MAKE_-STRING_-OUTPUT_-STREAM()$Lisp)$Lisp
    SETQ(tmpalgebra$Lisp, MAKE_-STRING_-OUTPUT_-STREAM()$Lisp)$Lisp
    SETQ(savemathml$Lisp, _$texOutputStream$Lisp)$Lisp
    SETQ(savealgebra$Lisp, _$algebraOutputStream$Lisp)$Lisp
    SETQ(_$texOutputStream$Lisp,tmpmathml$Lisp)$Lisp
    SETQ(_$algebraOutputStream$Lisp,tmpalgebra$Lisp)$Lisp
    ans := string parseAndEvalToStringEqNum$Lisp command
    SETQ(resultmathml$Lisp,_
         GET_-OUTPUT_-STREAM_-STRING(_$texOutputStream$Lisp)$Lisp)$Lisp
    SETQ(resultalgebra$Lisp,_
         GET_-OUTPUT_-STREAM_-STRING(_$algebraOutputStream$Lisp)$Lisp)$Lisp
    SETQ(_$texOutputStream$Lisp,savemathml$Lisp)$Lisp
    SETQ(_$algebraOutputStream$Lisp,savealgebra$Lisp)$Lisp
    CLOSE(tmpmathml$Lisp)$Lisp
    CLOSE(tmpalgebra$Lisp)$Lisp
    -- Since strings returned from axiom are going to be displayed in html I
    -- should really check for the characters &,<,> and replace them with
    -- &amp;,&lt;,&gt;.
    -- At present I only check for ampersands in formatMessages.
    mathml:String := string(resultmathml$Lisp)
    algebra:String := string(resultalgebra$Lisp)
    algebra := formatMessages(algebra)
    -- At this point mathml contains the mathml for the output but does not
    -- include step number or type information.
```

```
        -- We should also save the command.
        -- I get the type and step number from the $internalHistoryTable
        axans:String := _
          concat ["<div class=_"stepnum_">", lastStep(), "</div>_
                   <div class=_"command_">", command, "</div>_
                   <div class=_"algebra_">",algebra,"</div>_
                   <div class=_"mathml_">",mathml,"</div>_
                   <div class=_"type_">",lastType(),"</div>"]
--        WriteLine$Lisp concat ["mathml answer: ",mathml]
--        WriteLine$Lisp concat ["algebra answer: ",algebra]
        q:=MAKE_-STRING_-INPUT_-STREAM(axans)$Lisp
        SiCopyStream(q,s)$Lisp
        CLOSE(q)$Lisp
        CLOSE(s)$Lisp
```

getLisp

The getLisp function is invoked when the HTTP request is a POST and contains the string "lispcall".

⟨*package AXSERV AxiomServer*⟩+≡

```
    getLisp(s:SExpression,command:String):Void ==
        WriteLine$Lisp concat ["getLisp: ",command]
        evalresult:=EVAL(READ_-FROM_-STRING(command)$Lisp)$Lisp
        mathml:String:=string(evalresult)
--        WriteLine$Lisp concat ["getLisp: after ",mathml]
--        WriteLine$Lisp concat ["getLisp output: ",mathml]
        SETQ(tmpalgebra$Lisp, MAKE_-STRING_-OUTPUT_-STREAM()$Lisp)$Lisp
        SETQ(savemathml$Lisp, _$texOutputStream$Lisp)$Lisp
        SETQ(savealgebra$Lisp, _$algebraOutputStream$Lisp)$Lisp
        SETQ(_$texOutputStream$Lisp,tmpmathml$Lisp)$Lisp
        SETQ(_$algebraOutputStream$Lisp,tmpalgebra$Lisp)$Lisp
        SETQ(resultalgebra$Lisp,_
             GET_-OUTPUT_-STREAM_-STRING(_$algebraOutputStream$Lisp)$Lisp)$Lisp
        SETQ(_$texOutputStream$Lisp,savemathml$Lisp)$Lisp
        SETQ(_$algebraOutputStream$Lisp,savealgebra$Lisp)$Lisp
        CLOSE(tmpalgebra$Lisp)$Lisp
        -- Since strings returned from axiom are going to be displayed in html I
        -- should really check for the characters &,<,> and replace them with
        -- &amp;,&lt;,&gt;.
        -- At present I only check for ampersands in formatMessages.
        algebra:String := string(resultalgebra$Lisp)
        algebra := formatMessages(algebra)
        -- At this point mathml contains the mathml for the output but does not
        -- include step number or type information.
        -- We should also save the command.
        -- I get the type and step number from the $internalHistoryTable
        axans:String := _
         concat ["<div class=_"stepnum_">", lastStep(), "</div>_
                  <div class=_"command_">", command, "</div>_
                  <div class=_"algebra_">",algebra,"</div>_
                  <div class=_"mathml_">",mathml,"</div>_
                  <div class=_"type_">",lastType(),"</div>"]
--        WriteLine$Lisp concat ["mathml answer: ",mathml]
--        WriteLine$Lisp concat ["algebra answer: ",algebra]
        q:=MAKE_-STRING_-INPUT_-STREAM(axans)$Lisp
        SiCopyStream(q,s)$Lisp
        CLOSE(q)$Lisp
        CLOSE(s)$Lisp
```

getShow

The getShow function is invoked when the HTTP request is a POST and contains the string "showcall". The)show command generates output to lisp's *standard-output* so we wrap that stream to capture it. The resulting string needs to be transformed into html-friendly form. This is done in the call to replace-entitites (see http.lisp)

⟨*package AXSERV AxiomServer*⟩+≡

```
    getShow(s:SExpression,showarg:String):Void ==
        WriteLine$Lisp concat ["getShow: ",showarg]
        realarg:=SUBSEQ(showarg,6)$Lisp
        show:=_
         "(progn (setq |$options| '((|operations|))) (|show| '|" realarg "|))"
--       WriteLine$Lisp concat ["getShow: ",show]
        SETQ(SAVESTREAM$Lisp,_*STANDARD_-OUTPUT_*$Lisp)$Lisp
        SETQ(_*STANDARD_-OUTPUT_*$Lisp,_
              MAKE_-STRING_-OUTPUT_-STREAM()$Lisp)$Lisp
        evalresult:=EVAL(READ_-FROM_-STRING(show)$Lisp)$Lisp
        SETQ(evalresult,_
              GET_-OUTPUT_-STREAM_-STRING(_*STANDARD_-OUTPUT_*$Lisp)$Lisp)$Lisp
        SETQ(_*STANDARD_-OUTPUT_*$Lisp,SAVESTREAM$Lisp)$Lisp
        mathml:String:=string(REPLACE_-ENTITIES(evalresult)$Lisp)
        SETQ(tmpalgebra$Lisp, MAKE_-STRING_-OUTPUT_-STREAM()$Lisp)$Lisp
        SETQ(savemathml$Lisp, _$texOutputStream$Lisp)$Lisp
        SETQ(savealgebra$Lisp, _$algebraOutputStream$Lisp)$Lisp
        SETQ(_$texOutputStream$Lisp,tmpmathml$Lisp)$Lisp
        SETQ(_$algebraOutputStream$Lisp,tmpalgebra$Lisp)$Lisp
        SETQ(resultalgebra$Lisp,_
             GET_-OUTPUT_-STREAM_-STRING(_$algebraOutputStream$Lisp)$Lisp)$Lisp
        SETQ(_$texOutputStream$Lisp,savemathml$Lisp)$Lisp
        SETQ(_$algebraOutputStream$Lisp,savealgebra$Lisp)$Lisp
        CLOSE(tmpalgebra$Lisp)$Lisp
        -- Since strings returned from axiom are going to be displayed in html I
        -- should really check for the characters &,<,> and replace them with
        -- &amp;,&lt;,&gt;.
        -- At present I only check for ampersands in formatMessages.
        algebra:String := string(resultalgebra$Lisp)
        algebra := formatMessages(algebra)
        -- At this point mathml contains the mathml for the output but does not
        -- include step number or type information.
        -- We should also save the command.
        -- I get the type and step number from the $internalHistoryTable
        axans:String := _
         concat ["<div class=_"stepnum_">", lastStep(), "</div>_
                  <div class=_"command_">", showarg, "</div>_
```

```
                <div class=_"algebra_">",algebra,"</div>_
                <div class=_"mathml_">",mathml,"</div>_
                <div class=_"type_">",lastType(),"</div>"]
--      WriteLine$Lisp concat ["mathml answer: ",mathml]
      q:=MAKE_-STRING_-INPUT_-STREAM(axans)$Lisp
      SiCopyStream(q,s)$Lisp
      CLOSE(q)$Lisp
      CLOSE(s)$Lisp
```

lastType

To examine the $internalHistoryTable use the following line

```
)lisp |$internalHistoryTable|
```

We need to pick out first member of internalHistoryTable and then pick out the element with % as first element. Here is an example showing just the first element of the list, which correponds to the last command.

Note that the last command does not necessarily correspond to the last element of the first element of $internalHistoryTable as it is in this example.

```
(
 (4 NIL
 (x (value (BasicOperator) WRAPPED . #<vector 09a93bd0>))
 (y (value (BasicOperator) WRAPPED . #<vector 09a93bb4>))
 (% (value (Matrix (Polynomial (Integer))) WRAPPED . #<vector 0982e0e0>))
 )
...
)
```

We also need to check for input error in which case the $internalHistoryTable is not changed and the type retrieved would be that for the last correct input.

⟨*package AXSERV AxiomServer*⟩+≡

```
lastType():String ==
  SETQ(first$Lisp,FIRST(_$internalHistoryTable$Lisp)$Lisp)$Lisp
  count:Integer := 0
  hisLength:Integer := LIST_-LENGTH(_$internalHistoryTable$Lisp)$Lisp
  length:Integer := LIST_-LENGTH(first$Lisp)$Lisp
  -- This initializes stepSav.  The test is a bit of a hack, maybe I'll
  -- figure out the right way to do it later.
  if string stepSav$Lisp = "#<OBJNULL>" then SETQ(stepSav$Lisp, 0$Lisp)$Lisp
  -- If hisLength = 0 then the history table has been reset to NIL
  -- and we're starting numbering over
  if hisLength = 0 then SETQ(stepSav$Lisp, 0$Lisp)$Lisp
  if hisLength > 0 and
    car(car(_$internalHistoryTable$Lisp)$Lisp)$Lisp ^= stepSav$Lisp then
     SETQ(stepSav$Lisp,car(car(_$internalHistoryTable$Lisp)$Lisp)$Lisp)$Lisp
     while count < length  repeat
      position(char "%",string FIRST(first$Lisp)$Lisp) = 2 =>
        count := length+1
      count := count +1
      SETQ(first$Lisp,REST(first$Lisp)$Lisp)$Lisp
  count = length + 1 =>
      string SECOND(SECOND(FIRST(first$Lisp)$Lisp)$Lisp)$Lisp
  ""
```

```
    lastStep():String ==
        string car(car(_$internalHistoryTable$Lisp)$Lisp)$Lisp

    formatMessages(str:String):String ==
--        WriteLine("formatMessages")$Lisp
        -- I need to replace any ampersands with &amp; and may also need to
        -- replace < and > with &lt; and &gt;
        strlist:List String
--        WriteLine(str)$Lisp
        strlist := split(str,char "&")
        str := ""
        -- oops, if & is the last character in the string this method
        -- will eliminate it.  Need to redo this.
        for s in strlist repeat
            str := concat [str,s,"&amp;"]
        strlen:Integer := #str
        str := str.(1..(#str - 5))
--        WriteLine(str)$Lisp
        -- Here I split the string into lines and put each line in a "div".
        strlist := split(str, char string NewlineChar$Lisp)
        str := ""
--        WriteLine("formatMessages1")$Lisp
--        WriteLine(concat strlist)$Lisp
        for s in strlist repeat
--            WriteLine(s)$Lisp
            str := concat [str,"<div>",s,"</div>"]
        str

    getContentType(pathvar:String):String ==
--        WriteLine("getContentType begin")$Lisp
        -- set default content type
        contentType:String := "text/plain"
        -- need to test for successful match?
        StringMatch(".*\.(.*)$", pathvar)$Lisp
        u:UniversalSegment(Integer)
        u := segment(MatchBeginning(1)$Lisp+1,_
                     MatchEnd(1)$Lisp)$UniversalSegment(Integer)
        extension:String := pathvar.u
--        WriteLine$Lisp concat ["file extension: ",extension]
        -- test for extensions: html, htm, xml, xhtml, js, css
        if extension = "html" then
            contentType:String := "text/html"
        else if extension = "htm" then
            contentType:String := "text/html"
```

```
        else if extension = "xml" then
            contentType:String := "text/xml"
        else if extension = "xhtml" then
            contentType:String := "application/xhtml+xml"
        else if extension = "js" then
            contentType:String := "text/javascript"
        else if extension = "css" then
            contentType:String := "text/css"
        else if extension = "png" then
            contentType:String := "image/png"
        else if extension = "jpg" then
            contentType:String := "image/jpeg"
        else if extension = "jpeg" then
            contentType:String := "image/jpeg"
--        WriteLine$Lisp concat ["Content-Type: ",contentType]
--        WriteLine("getContentType end")$Lisp
        contentType
```

⟨*AXSERV.dotabb*⟩≡

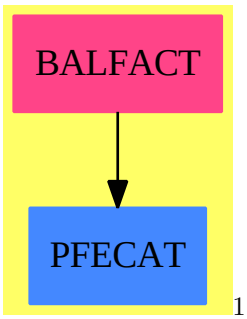
```
"AXSERV" [color="#FF4488",href="bookvol10.4.pdf#nameddest=AXSERV"]
"STRING" [color="#88FF44",href="bookvol10.3.pdf#nameddest=STRING"]
"AXSERV" -> "STRING"
```

Chapter 3

Chapter B

3.1 package BALFACT BalancedFactorisation

3.2 BalancedFactorisation



1.00

Exports:
balancedFactorisation

⟨*package BALFACT BalancedFactorisation*⟩≡
```
)abbrev package BALFACT BalancedFactorisation
++ Author: Manuel Bronstein
++ Date Created: 1 March 1991
++ Date Last Updated: 11 October 1991
++ Description: This package provides balanced factorisations of polynomials.
BalancedFactorisation(R, UP): Exports == Implementation where
  R  : Join(GcdDomain, CharacteristicZero)
  UP : UnivariatePolynomialCategory R

  Exports ==> with
    balancedFactorisation: (UP, UP) -> Factored UP
```

```
      ++ balancedFactorisation(a, b) returns
      ++ a factorisation \spad{a = p1^e1 ... pm^em} such that each
      ++ \spad{pi} is balanced with respect to b.
    balancedFactorisation: (UP, List UP) -> Factored UP
      ++ balancedFactorisation(a, [b1,...,bn]) returns
      ++ a factorisation \spad{a = p1^e1 ... pm^em} such that each
      ++ pi is balanced with respect to \spad{[b1,...,bm]}.

  Implementation ==> add
    balSqfr : (UP, Integer, List UP) -> Factored UP
    balSqfr1: (UP, Integer,      UP) -> Factored UP

    balancedFactorisation(a:UP, b:UP) == balancedFactorisation(a, [b])

    balSqfr1(a, n, b) ==
      g := gcd(a, b)
      fa := sqfrFactor((a exquo g)::UP, n)
      ground? g => fa
      fa * balSqfr1(g, n, (b exquo (g ** order(b, g)))::UP)

    balSqfr(a, n, l) ==
      b := first l
      empty? rest l => balSqfr1(a, n, b)
      */[balSqfr1(f.factor, n, b) for f in factors balSqfr(a,n,rest l)]

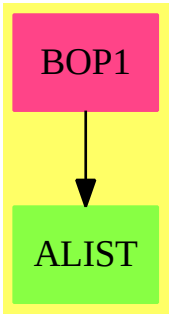
    balancedFactorisation(a:UP, l:List UP) ==
      empty?(ll := select(z1 +-> z1 ^= 0, l)) =>
        error "balancedFactorisation: 2nd argument is empty or all 0"
      sa := squareFree a
      unit(sa) * */[balSqfr(f.factor,f.exponent,ll) for f in factors sa])
```

⟨*BALFACT.dotabb*⟩≡

```
"BALFACT" [color="#FF4488",href="bookvol10.4.pdf#nameddest=BALFACT"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"BALFACT" -> "PFECAT"
```

3.3 package BOP1 BasicOperatorFunctions1

3.4 BasicOperatorFunctions1



1.00

Exports:
constantOpIfCan constantOperator derivative evaluate

⟨*package BOP1 BasicOperatorFunctions1*⟩≡

```
)abbrev package BOP1 BasicOperatorFunctions1
++ Tools to set/get common properties of operators
++ Author: Manuel Bronstein
++ Date Created: 28 Mar 1988
++ Date Last Updated: 15 May 1990
++ Description:
++   This package exports functions to set some commonly used properties
++   of operators, including properties which contain functions.
++ Keywords: operator.
BasicOperatorFunctions1(A:SetCategory): Exports == Implementation where
  OP   ==> BasicOperator
  EVAL    ==> "%eval"
  CONST   ==> "%constant"
  DIFF    ==> "%diff"
  OUT     ==> OutputForm
  IN      ==> InputForm

  Exports ==> with
    evaluate        : (OP, List A)      -> Union(A, "failed")
      ++ evaluate(op, [a1,...,an]) checks if op has an "%eval"
      ++ property f. If it has, then \spad{f(a1,...,an)} is returned, and
      ++ "failed" otherwise.
    evaluate        : (OP, List A -> A) -> OP
      ++ evaluate(op, foo) attaches foo as the "%eval" property
      ++ of op. If op has an "%eval" property f, then applying op
      ++ to \spad{(a1,...,an)} returns the result of \spad{f(a1,...,an)}.
    evaluate        : (OP, A -> A)      -> OP
```

```
    ++ evaluate(op, foo) attaches foo as the "%eval" property
    ++ of op. If op has an "%eval" property f, then applying op
    ++ to a returns the result of \spad{f(a)}. Argument op must be unary.
  evaluate        : OP                -> Union(List A -> A, "failed")
    ++ evaluate(op) returns the value of the "%eval" property of
    ++ op if it has one, and "failed" otherwise.
  derivative      : (OP, List (List A -> A)) -> OP
    ++ derivative(op, [foo1,...,foon]) attaches [foo1,...,foon] as
    ++ the "%diff" property of op. If op has an "%diff" property
    ++ \spad{[f1,...,fn]} then applying a derivation D
    ++ to \spad{op(a1,...,an)}
    ++ returns \spad{f1(a1,...,an) * D(a1) + ... + fn(a1,...,an) * D(an)}.
  derivative      : (OP, A -> A) -> OP
    ++ derivative(op, foo) attaches foo as the "%diff" property
    ++ of op. If op has an "%diff" property f, then applying a
    ++ derivation D to op(a) returns \spad{f(a) * D(a)}.
    ++ Argument op must be unary.
  derivative      : OP -> Union(List(List A -> A), "failed")
    ++ derivative(op) returns the value of the "%diff" property of
    ++ op if it has one, and "failed" otherwise.
  if A has OrderedSet then
    constantOperator: A -> OP
      ++ constantOperator(a) returns a nullary operator op
      ++ such that \spad{op()} always evaluate to \spad{a}.
    constantOpIfCan : OP -> Union(A, "failed")
      ++ constantOpIfCan(op) returns \spad{a} if op is the constant
      ++ nullary operator always returning \spad{a}, "failed" otherwise.

Implementation ==> add
  evaluate(op:OP, func:A -> A) ==
     evaluate(op, (ll:List(A)):A +-> func first ll)

  evaluate op ==
    (func := property(op, EVAL)) case "failed" => "failed"
    (func::None) pretend (List A -> A)

  evaluate(op:OP, args:List A) ==
    (func := property(op, EVAL)) case "failed" => "failed"
    ((func::None) pretend (List A -> A)) args

  evaluate(op:OP, func:List A -> A) ==
    setProperty(op, EVAL, func pretend None)

  derivative op ==
    (func := property(op, DIFF)) case "failed" => "failed"
    ((func::None) pretend List(List A -> A))
```

```
derivative(op:OP, grad:List(List A -> A)) ==
  setProperty(op, DIFF, grad pretend None)

derivative(op:OP, f:A -> A) ==
  unary? op or nary? op =>
    derivative(op, [(ll:List(A)):A +-> f first ll]$List(List A -> A))
  error "Operator is not unary"

if A has OrderedSet then
  cdisp   : (OUT, List OUT) -> OUT
  csex    : (IN,  List IN) -> IN
  eqconst?: (OP, OP) -> Boolean
  ltconst?: (OP, OP) -> Boolean
  constOp : A -> OP

  opconst:OP :=
    comparison(equality(operator("constant"::Symbol, 0), eqconst?),
                                                          ltconst?)

  cdisp(a, l) == a
  csex(a, l)  == a

  eqconst?(a, b) ==
    (va := property(a, CONST)) case "failed" => not has?(b, CONST)
    ((vb := property(b, CONST)) case None) and
       ((va::None) pretend A) = ((vb::None) pretend A)

  ltconst?(a, b) ==
    (va := property(a, CONST)) case "failed" => has?(b, CONST)
    ((vb := property(b, CONST)) case None) and
       ((va::None) pretend A) < ((vb::None) pretend A)

  constOp a ==
    setProperty(
      display(copy opconst, (ll:List(OUT)):OUT +-> cdisp(a::OUT, ll)),
        CONST, a pretend None)

  constantOpIfCan op ==
    is?(op, "constant"::Symbol) and
      ((u := property(op, CONST)) case None) => (u::None) pretend A
    "failed"

  if A has ConvertibleTo IN then
    constantOperator a ==
      input(constOp a, (ll:List(IN)):IN +-> csex(convert a, ll))
```

```
        else
          constantOperator a == constOp a
```

⟨*BOP1.dotabb*⟩≡

```
"BOP1" [color="#FF4488",href="bookvol10.4.pdf#nameddest=BOP1"]
"ALIST" [color="#88FF44",href="bookvol10.3.pdf#nameddest=ALIST"]
"BOP1" -> "ALIST"
```

3.5 package BEZIER Bezier

⟨*Bezier.input*⟩≡

```
)set break resume
)sys rm -f Bezier.output
)spool Bezier.output
)set message test on
)set message auto off
)clear all
--S 1 of 9
n:=linearBezier([2.0,2.0],[4.0,4.0])
--R
--I   (1)  theMap(BEZIER;linearBezier;2LM;1!0,707)
--R                                                Type: (Float -> List Float)
--E 1

--S 2 of 9
[n(t/10.0) for t in 0..10 by 1]
--R
--R   (2)
--R   [[2.0,2.0], [2.2,2.2], [2.4,2.4], [2.6,2.6], [2.8,2.8], [3.0,3.0],
--R    [3.2,3.2], [3.4,3.4], [3.6,3.6], [3.8,3.8], [4.0,4.0]]
--R                                                      Type: List List Float
--E 2

--S 3 of 9
n:=quadraticBezier([2.0,2.0],[4.0,4.0],[6.0,2.0])
--R
--I   (3)  theMap(BEZIER;quadraticBezier;3LM;2!0,291)
--R                                                Type: (Float -> List Float)
--E 3

--S 4 of 9
[n(t/10.0) for t in 0..10 by 1]
--R
--R   (4)
--R   [[2.0,2.0], [2.4,2.36], [2.8,2.64], [3.2,2.84], [3.6,2.96], [4.0,3.0],
--R    [4.4,2.96], [4.8,2.84], [5.2,2.64], [5.6,2.36], [6.0,2.0]]
--R                                                      Type: List List Float
--E 4

--S 5 of 9
n:=cubicBezier([2.0,2.0],[2.0,4.0],[6.0,4.0],[6.0,2.0])
--R
--I   (5)  theMap(BEZIER;cubicBezier;4LM;3!0,915)
--R                                                Type: (Float -> List Float)
```

```
--E 5

--S 6 of 9
[n(t/10.0) for t in 0..10 by 1]
--R
--R   (6)
--R   [[2.0,2.0], [2.112,2.54], [2.416,2.96], [2.864,3.26], [3.408,3.44],
--R    [4.0,3.5], [4.592,3.44], [5.136,3.26], [5.584,2.96], [5.888,2.54],
--R    [6.0,2.0]]
--R                                                    Type: List List Float
--E 6

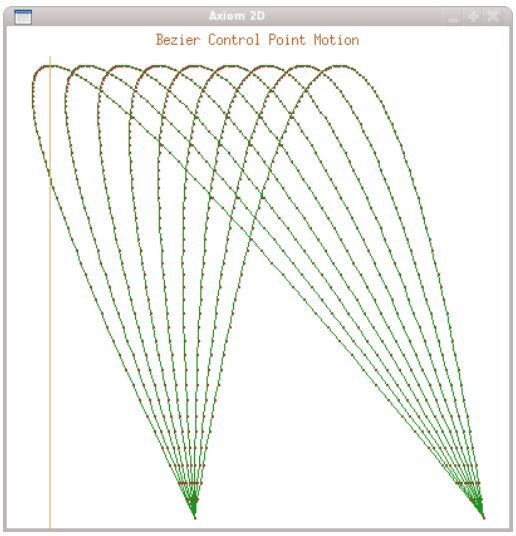
--S 7 of 9
line:=[[i::Float,4.0] for i in -4..4 by 1]
--E 7

--S 8 of 9
functions:=[quadraticBezier([2.0,2.0],m,[6.0,2.0]) for m in line]
--E 8

--S 9 of 9
graphs:=[[point(((functions.i)(j/100.0))::LIST(DFLOAT)) for j in 0..100] for i in
--E 9

-- We do not do these during testing since graphics is not available
-- The resulting image is in the Bezier section of Volume 10.4

--d1:=draw(graphs.1,title=="Bezier Control Point Motion")
--others:=[graphs.i for i in 2..9]
--for i in 2..9 for graph in others repeat putGraph(d1,[graph],i)
--vp:=makeViewport2D(d1)
```



⟨*Bezier.help*⟩≡

```
==========================================================================
Bezier Curve examples
==========================================================================

A linear Bezier curve is a simple interpolation between the
starting point and the ending point based on a parameter t.

Given a start point a=[x1,y1] and an endpoint b=[x2,y2]
f(t) == [(1-t)*x1 + t*x2, (1-t)*y1 + t*y2]

n:=linearBezier([2.0,2.0],[4.0,4.0])
   theMap(BEZIER;linearBezier;2LM;1!0,707)

[n(t/10.0) for t in 0..10 by 1]
   [[2.0,2.0], [2.2,2.2], [2.4,2.4], [2.6,2.6], [2.8,2.8], [3.0,3.0],
    [3.2,3.2], [3.4,3.4], [3.6,3.6], [3.8,3.8], [4.0,4.0]]


A quadratic Bezier curve is a simple interpolation between the
starting point, a middle point, and the ending point based on
a parameter t.

Given a start point a=[x1,y1], a middle point b=[x2,y2],
and an endpoint c=[x3,y3]

f(t) == [(1-t)^2 x1 + 2t(1-t) x2 + t^2 x3,
         (1-t)^2 y1 + 2t(1-t) y2 + t^2 y3]
```

```
n:=quadraticBezier([2.0,2.0],[4.0,4.0],[6.0,2.0])
   theMap(BEZIER;quadraticBezier;3LM;2!0,291)

[n(t/10.0) for t in 0..10 by 1]
   [[2.0,2.0], [2.4,2.36], [2.8,2.64], [3.2,2.84], [3.6,2.96], [4.0,3.0],
    [4.4,2.96], [4.8,2.84], [5.2,2.64], [5.6,2.36], [6.0,2.0]]

A cubic Bezier curve is a simple interpolation between the
starting point, a left-middle point,, a right-middle point,
and the ending point based on a parameter t.

Given a start point a=[x1,y1], the left-middle point b=[x2,y2],
the right-middle point c=[x3,y3] and an endpoint d=[x4,y4]

f(t) == [(1-t)^3 x1 + 3t(1-t)^2 x2 + 3t^2 (1-t) x3 + t^3 x4,
         (1-t)^3 y1 + 3t(1-t)^2 y2 + 3t^2 (1-t) y3 + t^3 y4]

n:=cubicBezier([2.0,2.0],[2.0,4.0],[6.0,4.0],[6.0,2.0])
   theMap(BEZIER;cubicBezier;4LM;3!0,915)

[n(t/10.0) for t in 0..10 by 1]
   [[2.0,2.0], [2.112,2.54], [2.416,2.96], [2.864,3.26], [3.408,3.44],
    [4.0,3.5], [4.592,3.44], [5.136,3.26], [5.584,2.96], [5.888,2.54],
    [6.0,2.0]]

Bezier curves "move" based on moving their control points, which
in the case of the three components of a quadratic Bezier curve
are the three points given. To see this motion we can show what
happens when you "drag" the middle control point along the line
from [-4,4] to [4,4] by increments of 1.

First, we form the line as a list of Floats, 9 in all.

   line:=[[i::Float,4.0] for i in -4..4 by 1]

Next, we form a list of functions, each with a different center control point.
Notice that the endpoints remain fized so we expect that the curve will
begin and end at the same point but that the midpoint is pulled around.

  functions:=[quadraticBezier([2.0,2.0],m,[6.0,2.0]) for m in line]

Then we form a list of the graphs by calling each function in the above
list. Each function call happens 101 times (to include both endpoints).
Thus we get a List of Lists of Points of DoubleFloats

  graphs:=[ [ point( ( (functions.i)(j/100.0) )::LIST(DFLOAT) ) _
```

```
    for j in 0..100] for i in 1..9]
```

We draw the first graph to see if it looks reasonable:

```
  d1:=draw(graphs.1)
```

Since it does we add the other 8 graphs. The 2D graphs can hold up to 9 simultaneous graphs.

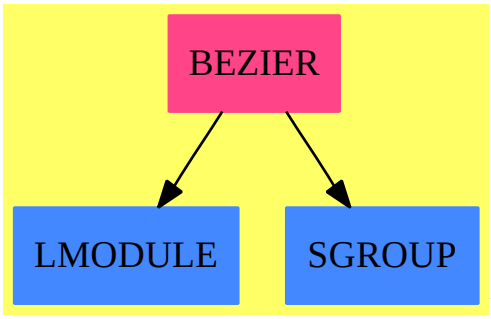
```
  others:=[graphs.i for i in 2..9]
  for i in 2..9 for graph in others repeat putGraph(d1,[graph],i)
```

and now we display them all on one graph.

```
  vp:=makeViewport2D(d1)
```

See Also:
o)show Bezier
o)show TwoDimensionalViewport
o)d op draw
o)d op point
o)d op putGraph
o)d op makeViewport2D

3.6 Bezier



1.00

Exports:
linearBezier quadraticBezier cubicBezier

⟨*package BEZIER Bezier*⟩≡
```
)abbrev package BEZIER Bezier
++ Author: Timothy Daly
++ Date Created: 14 April 2009
++ Description: Provide linear, quadratic, and cubic spline bezier curves
Bezier(R:Ring): with
   linearBezier: (x:List R,y:List R) -> Mapping(List R,R)
    ++ A linear Bezier curve is a simple interpolation between the
    ++ starting point and the ending point based on a parameter t.
    ++ Given a start point a=[x1,y1] and an endpoint b=[x2,y2]
    ++ f(t) == [(1-t)*x1 + t*x2, (1-t)*y1 + t*y2]
    ++
    ++X n:=linearBezier([2.0,2.0],[4.0,4.0])
    ++X [n(t/10.0) for t in 0..10 by 1]
   quadraticBezier: (x:List R,y:List R,z:List R) -> Mapping(List R,R)
    ++ A quadratic Bezier curve is a simple interpolation between the
    ++ starting point, a middle point, and the ending point based on
    ++ a parameter t.
    ++ Given a start point a=[x1,y1], a middle point b=[x2,y2],
    ++ and an endpoint c=[x3,y3]
    ++ f(t) == [(1-t)^2 x1 + 2t(1-t) x2 + t^2 x3,
    ++           (1-t)^2 y1 + 2t(1-t) y2 + t^2 y3]
    ++
    ++X n:=quadraticBezier([2.0,2.0],[4.0,4.0],[6.0,2.0])
    ++X [n(t/10.0) for t in 0..10 by 1]
   cubicBezier: (w:List R,x:List R,y:List R,z:List R) -> Mapping(List R,R)
    ++ A cubic Bezier curve is a simple interpolation between the
    ++ starting point, a left-middle point,, a right-middle point,
    ++ and the ending point based on a parameter t.
    ++ Given a start point a=[x1,y1], the left-middle point b=[x2,y2],
    ++ the right-middle point c=[x3,y3] and an endpoint d=[x4,y4]
```

```
    ++ f(t) == [(1-t)^3 x1 + 3t(1-t)^2 x2 + 3t^2 (1-t) x3 + t^3 x4,
    ++          (1-t)^3 y1 + 3t(1-t)^2 y2 + 3t^2 (1-t) y3 + t^3 y4]
    ++
    ++X n:=cubicBezier([2.0,2.0],[2.0,4.0],[6.0,4.0],[6.0,2.0])
    ++X [n(t/10.0) for t in 0..10 by 1]
  == add
    linearBezier(a,b) ==
     t +-> [(1-t)*(a.1) + t*(b.1), (1-t)*(a.2) + t*(b.2)]

    quadraticBezier(a,b,c) ==
     t +-> [(1-t)**2*(a.1) + 2*t*(1-t)*(b.1) + t**2*(c.1),
            (1-t)**2*(a.2) + 2*t*(1-t)*(b.2) + t**2*(c.2)]

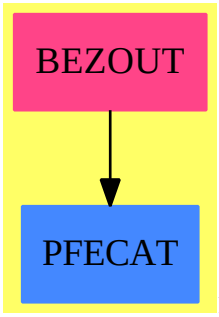
    cubicBezier(a,b,c,d) ==
     t +-> [(1-t)**3*(a.1) + 3*t*(1-t)**2*(b.1)
              + 3*t**2*(1-t)*(c.1) + t**3*(d.1),
            (1-t)**3*(a.2) + 3*t*(1-t)**2*(b.2)
              + 3*t**2*(1-t)*(c.2) + t**3*(d.2)]
```

⟨*BEZIER.dotabb*⟩≡

```
"BEZIER" [color="#FF4488",href="bookvol10.4.pdf#nameddest=BEZIER"]
"LMODULE" [color="#4488FF",href="bookvol10.2.pdf#nameddest=LMODULE"]
"SGROUP" [color="#4488FF",href="bookvol10.2.pdf#nameddest=SGROUP"]
"BEZIER" -> "LMODULE"
"BEZIER" -> "SGROUP"
```

3.7 package BEZOUT BezoutMatrix

3.8 BezoutMatrix



1.00

Exports:
bezoutDiscriminant bezoutMatrix bezoutResultant sylvesterMatrix

⟨*package BEZOUT BezoutMatrix*⟩≡

```
)abbrev package BEZOUT BezoutMatrix
++ Author: Clifton J. Williamson
++ Date Created: 2 August 1988
++ Date Last Updated: 3 November 1993
++ Basic Operations: bezoutMatrix, bezoutResultant, bezoutDiscriminant
++ Related Domains
++ Also See:
++ AMS Classifiactions:
++ Keywords: Bezout matrix, resultant, discriminant
++ Examples:
++ Reference: Knuth, The Art of Computer Programming, 2nd edition,
++            Vol. 2, p. 619, problem 12.
++ Description:
++   \spadtype{BezoutMatrix} contains functions for computing resultants and
++   discriminants using Bezout matrices.

BezoutMatrix(R,UP,M,Row,Col): Exports == Implementation where
  R    : Ring
  UP   : UnivariatePolynomialCategory R
  Row  : FiniteLinearAggregate R
  Col  : FiniteLinearAggregate R
  M    : MatrixCategory(R,Row,Col)
  I  ==> Integer
  lc ==> leadingCoefficient

  Exports ==> with
    sylvesterMatrix: (UP,UP) -> M
```

```
    ++ sylvesterMatrix(p,q) returns the Sylvester matrix for the two
    ++ polynomials p and q.
  bezoutMatrix: (UP,UP) -> M
    ++ bezoutMatrix(p,q) returns the Bezout matrix for the two
    ++ polynomials p and q.

  if R has commutative("*") then
    bezoutResultant: (UP,UP) -> R
     ++ bezoutResultant(p,q) computes the resultant of the two
     ++ polynomials p and q by computing the determinant of a Bezout matrix.

    bezoutDiscriminant: UP -> R
     ++ bezoutDiscriminant(p) computes the discriminant of a polynomial p
     ++ by computing the determinant of a Bezout matrix.

Implementation ==> add

  sylvesterMatrix(p,q) ==
    n1 := degree p; n2 := degree q; n := n1 + n2
    sylmat : M := new(n,n,0)
    minR := minRowIndex sylmat; minC := minColIndex sylmat
    maxR := maxRowIndex sylmat; maxC := maxColIndex sylmat
    p0 := p
    -- fill in coefficients of 'p'
    while not zero? p0 repeat
      coef := lc p0; deg := degree p0; p0 := reductum p0
      -- put bk = coef(p,k) in sylmat(minR + i,minC + i + (n1 - k))
      for i in 0..n2 - 1 repeat
        qsetelt_!(sylmat,minR + i,minC + n1 - deg + i,coef)
    q0 := q
    -- fill in coefficients of 'q'
    while not zero? q0 repeat
      coef := lc q0; deg := degree q0; q0 := reductum q0
      for i in 0..n1-1 repeat
        qsetelt_!(sylmat,minR + n2 + i,minC + n2 - deg + i,coef)
    sylmat

  bezoutMatrix(p,q) ==
  -- This function computes the Bezout matrix for 'p' and 'q'.
  -- See Knuth, The Art of Computer Programming, Vol. 2, p. 619, # 12.
  -- One must have deg(p) >= deg(q), so the arguments are reversed
  -- if this is not the case.
    n1 := degree p; n2 := degree q; n := n1 + n2
    n1 < n2 => bezoutMatrix(q,p)
    m1 : I := n1 - 1; m2 : I := n2 - 1; m : I := n - 1
    -- 'sylmat' will be a matrix consisting of the first n1 columns
```

```
-- of the standard Sylvester matrix for 'p' and 'q'
sylmat : M := new(n,n1,0)
minR := minRowIndex sylmat; minC := minColIndex sylmat
maxR := maxRowIndex sylmat; maxC := maxColIndex sylmat
p0 := p
-- fill in coefficients of 'p'
while not ground? p0 repeat
  coef := lc p0; deg := degree p0; p0 := reductum p0
  -- put bk = coef(p,k) in sylmat(minR + i,minC + i + (n1 - k))
  -- for i = 0...
  -- quit when i > m2 or when i + (n1 - k) > m1, whichever happens first
  for i in 0..min(m2,deg - 1) repeat
    qsetelt_!(sylmat,minR + i,minC + n1 - deg + i,coef)
q0 := q
-- fill in coefficients of 'q'
while not zero? q0 repeat
  coef := lc q0; deg := degree q0; q0 := reductum q0
  -- put ak = coef(q,k) in sylmat(minR + n1 + i,minC + i + (n2 - k))
  -- for i = 0...
  -- quit when i > m1 or when i + (n2 - k) > m1, whichever happens first
  -- since n2 - k >= 0, we quit when i + (n2 - k) > m1
  for i in 0..(deg + n1 - n2 - 1) repeat
    qsetelt_!(sylmat,minR + n2 + i,minC + n2 - deg + i,coef)
-- 'bezmat' will be the 'Bezout matrix' as described in Knuth
bezmat : M := new(n1,n1,0)
for i in 0..m2 repeat
  -- replace A_i by (b_0 A_i + ... + b_{n_2-1-i} A_{n_2 - 1}) -
  -- (a_0 B_i + ... + a_{n_2-1-i} B_{n_2-1}), as in Knuth
  bound : I := n2 - i; q0 := q
  while not zero? q0 repeat
    deg := degree q0
    if (deg < bound) then
      -- add b_deg A_{n_2 - deg} to the new A_i
      coef := lc q0
      for k in minC..maxC repeat
        c := coef * qelt(sylmat,minR + m2 - i - deg,k) +
                    qelt(bezmat,minR + m2 - i,k)
        qsetelt_!(bezmat,minR + m2 - i,k,c)
    q0 := reductum q0
  p0 := p
  while not zero? p0 repeat
    deg := degree p0
    if deg < bound then
      coef := lc p0
      -- subtract a_deg B_{n_2 - deg} from the new A_i
      for k in minC..maxC repeat
```

```
            c := -coef * qelt(sylmat,minR + m - i - deg,k) +
                         qelt(bezmat,minR + m2 - i,k)
            qsetelt_!(bezmat,minR + m2 - i,k,c)
          p0 := reductum p0
        for i in n2..m1 repeat for k in minC..maxC repeat
          qsetelt_!(bezmat,minR + i,k,qelt(sylmat,minR + i,k))
        bezmat

      if R has commutative("*") then

        bezoutResultant(f,g) == determinant bezoutMatrix(f,g)

        if R has IntegralDomain then

          bezoutDiscriminant f ==
            degMod4 := (degree f) rem 4
            (degMod4 = 0) or (degMod4 = 1) =>
              (bezoutResultant(f,differentiate f) exquo (lc f)) :: R
            -((bezoutResultant(f,differentiate f) exquo (lc f)) :: R)

          else

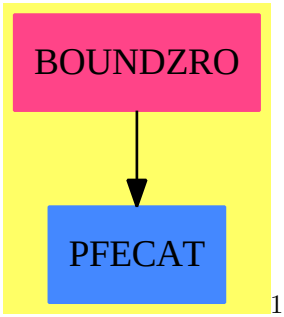
            bezoutDiscriminant f ==
              lc f = 1 =>
                degMod4 := (degree f) rem 4
                (degMod4 = 0) or (degMod4 = 1) =>
                  bezoutResultant(f,differentiate f)
                -bezoutResultant(f,differentiate f)
              error "bezoutDiscriminant: leading coefficient must be 1"
```

⟨*BEZOUT.dotabb*⟩≡

```
"BEZOUT" [color="#FF4488",href="bookvol10.4.pdf#nameddest=BEZOUT"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"BEZOUT" -> "PFECAT"
```

3.9 package BOUNDZRO BoundIntegerRoots

3.10 BoundIntegerRoots



1.00

Exports:
integerBound

⟨*package BOUNDZRO BoundIntegerRoots*⟩≡

```
)abbrev package BOUNDZRO BoundIntegerRoots
++ Author: Manuel Bronstein
++ Date Created: 11 March 1991
++ Date Last Updated: 18 November 1991
++ Description:
++   \spadtype{BoundIntegerRoots} provides functions to
++   find lower bounds on the integer roots of a polynomial.
BoundIntegerRoots(F, UP): Exports == Implementation where
  F  : Join(Field, RetractableTo Fraction Integer)
  UP : UnivariatePolynomialCategory F

  Z   ==> Integer
  Q   ==> Fraction Z
  K   ==> Kernel F
  UPQ ==> SparseUnivariatePolynomial Q
  ALGOP ==> "%alg"

  Exports ==> with
    integerBound: UP -> Z
      ++ integerBound(p) returns a lower bound on the negative integer
      ++ roots of p, and 0 if p has no negative integer roots.

  Implementation ==> add
    import RationalFactorize(UPQ)
    import UnivariatePolynomialCategoryFunctions2(F, UP, Q, UPQ)

    qbound : (UP, UPQ) -> Z
```

```
    zroot1 : UP -> Z
    qzroot1: UPQ -> Z
    negint : Q -> Z

  -- returns 0 if p has no integer root < 0, its negative integer root otherwise
    qzroot1 p == negint(- leadingCoefficient(reductum p) / leadingCoefficient p)

  -- returns 0 if p has no integer root < 0, its negative integer root otherwise
    zroot1 p ==
      z := - leadingCoefficient(reductum p) / leadingCoefficient p
      (r := retractIfCan(z)@Union(Q, "failed")) case Q => negint(r::Q)
      0

  -- returns 0 if r is not a negative integer, r otherwise
    negint r ==
      ((u := retractIfCan(r)@Union(Z, "failed")) case Z) and (u::Z < 0) => u::Z
      0

    if F has ExpressionSpace then
      bringDown: F -> Q

  -- the random substitution used by bringDown is NOT always a ring-homorphism
  -- (because of potential algebraic kernels), but is ALWAYS a Z-linear map.
  -- this guarantees that bringing down the coefficients of (x + n) q(x) for an
  -- integer n yields a polynomial h(x) which is divisible by x + n
  -- the only problem is that evaluating with random numbers can cause a
  -- division by 0. We should really be able to trap this error later and
  -- reevaluate with a new set of random numbers    MB 11/91
      bringDown f ==
        t := tower f
        retract eval(f, t, [random()$Q :: F for k in t])

      integerBound p ==
  --      one? degree p => zroot1 p
        (degree p) = 1 => zroot1 p
        q1 := map(bringDown, p)
        q2 := map(bringDown, p)
        qbound(p, gcd(q1, q2))

    else
      integerBound p ==
  --      one? degree p => zroot1 p
        (degree p) = 1 => zroot1 p
        qbound(p, map((z1:F):Q +-> retract(z1)@Q, p))

  -- we can probably do better here (i.e. without factoring)
```

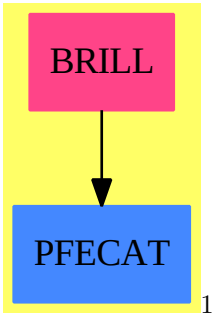
```
      qbound(p, q) ==
        bound:Z := 0
        for rec in factors factor q repeat
  --        if one?(degree(rec.factor)) and ((r := qzroot1(rec.factor)) < bound)
          if ((degree(rec.factor)) = 1) and ((r := qzroot1(rec.factor)) < bound)
             and zero? p(r::Q::F) then bound := r
        bound
```

⟨*BOUNDZRO.dotabb*⟩≡

```
"BOUNDZRO" [color="#FF4488",href="bookvol10.4.pdf#nameddest=BOUNDZRO"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"BOUNDZRO" -> "PFECAT"
```

3.11 package BRILL BrillhartTests

3.12 BrillhartTests



1.00

Exports:
noLinearFactor? brillhartIrreducible? brillhartTrials

⟨*package BRILL BrillhartTests*⟩≡
```
)abbrev package BRILL BrillhartTests
++ Author: Frederic Lehobey, James H. Davenport
++ Date Created: 28 June 1994
++ Date Last Updated: 11 July 1997
++ Basic Operations: brillhartIrreducible?
++ Related Domains:
++ Also See:
++ AMS Classifications:
++ Keywords: factorization
++ Examples:
++ References:
++ [1] John Brillhart, Note on Irreducibility Testing,
++ Mathematics of Computation, vol. 35, num. 35, Oct. 1980, 1379-1381
++ [2] James Davenport, On Brillhart Irreducibility. To appear.
++ [3] John Brillhart, On the Euler and Bernoulli polynomials,
++ J. Reine Angew. Math., v. 234, (1969), pp. 45-64

BrillhartTests(UP): Exports == Implementation where
  N ==> NonNegativeInteger
  Z ==> Integer
  UP: UnivariatePolynomialCategory Z

  Exports ==> with
    brillhartIrreducible?: UP -> Boolean -- See [1]
      ++ brillhartIrreducible?(p) returns \spad{true} if p can be shown to be
      ++ irreducible by a remark of Brillhart, \spad{false} is inconclusive.
    brillhartIrreducible?: (UP,Boolean) -> Boolean -- See [1]
```

```
    ++ brillhartIrreducible?(p,noLinears) returns \spad{true} if p can be
    ++ shown to be irreducible by a remark of Brillhart, \spad{false} else.
    ++ If noLinears is \spad{true}, we are being told p has no linear factors
    ++ \spad{false} does not mean that p is reducible.
  brillhartTrials: () -> N
    ++ brillhartTrials() returns the number of tests in
    ++ \spadfun{brillhartIrreducible?}.
  brillhartTrials: N -> N
    ++ brillhartTrials(n) sets to n the number of tests in
    ++ \spadfun{brillhartIrreducible?} and returns the previous value.
  noLinearFactor?: UP -> Boolean -- See [3] p. 47
    ++ noLinearFactor?(p) returns \spad{true} if p can be shown to have no
    ++ linear factor by a theorem of Lehmer, \spad{false} else. I insist on
    ++ the fact that \spad{false} does not mean that p has a linear factor.

Implementation ==> add

  import GaloisGroupFactorizationUtilities(Z,UP,Float)

  squaredPolynomial(p:UP):Boolean ==
    d := degree p
    d = 0 => true
    odd? d => false
    squaredPolynomial reductum p

  primeEnough?(n:Z,b:Z):Boolean ==
     -- checks if n is prime, with the possible exception of
     -- factors whose product is at most b
     import Float
     bb: Float := b::Float
     for i in 2..b repeat
         while (d:= n exquo i) case Integer repeat
               n:=d::Integer
               bb:=bb / i::Float
               bb < 1$Float => return false
               --- we over-divided, so it can't be prime
     prime? n

  brillharttrials: N := 6
  brillhartTrials():N == brillharttrials

  brillhartTrials(n:N):N ==
    (brillharttrials,n) := (n,brillharttrials)
    n

  brillhartIrreducible?(p:UP):Boolean ==
```

```
      brillhartIrreducible?(p,noLinearFactor? p)

    brillhartIrreducible?(p:UP,noLinears:Boolean):Boolean == -- See [1]
      zero? brillharttrials => false
      origBound := (largeEnough := rootBound(p)+1)
      -- see remarks 2 and 4
      even0 := even? coefficient(p,0)
      even1 := even? p(1)
      polyx2 := squaredPolynomial(p)
      prime? p(largeEnough) => true
      not polyx2 and prime? p(-largeEnough) => true
--      one? brillharttrials => false
      (brillharttrials = 1) => false
      largeEnough := largeEnough+1
      primeEnough?(p(largeEnough),if noLinears then 4 else 2) => true
      not polyx2 and
       primeEnough?(p(-largeEnough),if noLinears then 4 else 2) => true
      if odd? largeEnough then
        if even0 then largeEnough := largeEnough+1
      else
        if even1 then largeEnough := largeEnough+1
      count :=(if polyx2 then 2 else 1)*(brillharttrials-2)+largeEnough
      for i in (largeEnough+1)..count repeat
        small := if noLinears then (i-origBound)**2 else (i-origBound)
        primeEnough?(p(i),small) => return true
        not polyx2 and primeEnough?(p(-i),small) => return true
      false

    noLinearFactor?(p:UP):Boolean ==
      (odd? leadingCoefficient p) and (odd? coefficient(p,0)) and (odd? p(1))
```

⟨*BRILL.dotabb*⟩≡

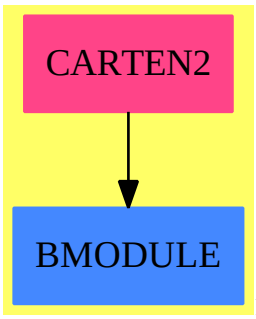
```
"BRILL" [color="#FF4488",href="bookvol10.4.pdf#nameddest=BRILL"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"BRILL" -> "PFECAT"
```

Chapter 4

Chapter C

4.1 package CARTEN2 CartesianTensorFunctions2

4.2 CartesianTensorFunctions2



1.00

Exports:
map reshape

⟨*package CARTEN2 CartesianTensorFunctions2*⟩≡
```
)abbrev package CARTEN2 CartesianTensorFunctions2
++ Author: Stephen M. Watt
++ Date Created: December 1986
++ Date Last Updated: May 30, 1991
++ Basic Operations:  reshape, map
++ Related Domains: CartesianTensor
++ Also See:
++ AMS Classifications:
++ Keywords: tensor
++ Examples:
++ References:
```

```
++ Description:
++   This package provides functions to enable conversion of tensors
++   given conversion of the components.

CartesianTensorFunctions2(minix, dim, S, T): CTPcat == CTPdef where
    minix:  Integer
    dim:    NonNegativeInteger
    S, T:   CommutativeRing
    CS  ==> CartesianTensor(minix, dim, S)
    CT  ==> CartesianTensor(minix, dim, T)

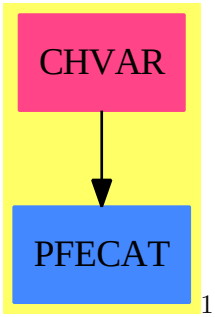
    CTPcat == with
        reshape: (List T, CS) -> CT
            ++ reshape(lt,ts) organizes the list of components lt into
            ++ a tensor with the same shape as ts.
        map: (S->T,   CS) -> CT
            ++ map(f,ts) does a componentwise conversion of the tensor ts
            ++ to a tensor with components of type T.
    CTPdef == add
        reshape(l, s) == unravel l
        map(f, s)     == unravel [f e for e in ravel s]
```

⟨*CARTEN2.dotabb*⟩≡

```
"CARTEN2" [color="#FF4488",href="bookvol10.4.pdf#nameddest=CARTEN2"]
"BMODULE" [color="#4488FF",href="bookvol10.2.pdf#nameddest=BMODULE"]
"CARTEN2" -> "BMODULE"
```

4.3 package CHVAR ChangeOfVariable

4.4 ChangeOfVariable



1.00

Exports:

chvar	eval	goodPoint	mkIntegral
radPoly	rootPoly		

⟨*package CHVAR ChangeOfVariable*⟩≡

```
)abbrev package CHVAR ChangeOfVariable
++ Sends a point to infinity
++ Author: Manuel Bronstein
++ Date Created: 1988
++ Date Last Updated: 22 Feb 1990
++ Description:
++  Tools to send a point to infinity on an algebraic curve.
ChangeOfVariable(F, UP, UPUP): Exports == Implementation where
  F   : UniqueFactorizationDomain
  UP  : UnivariatePolynomialCategory F
  UPUP: UnivariatePolynomialCategory Fraction UP

  N  ==> NonNegativeInteger
  Z  ==> Integer
  Q  ==> Fraction Z
  RF ==> Fraction UP

  Exports ==> with
    mkIntegral: UPUP -> Record(coef:RF, poly:UPUP)
          ++ mkIntegral(p(x,y)) returns \spad{[c(x), q(x,z)]} such that
          ++ \spad{z = c * y} is integral.
          ++ The algebraic relation between x and y is \spad{p(x, y) = 0}.
          ++ The algebraic relation between x and z is \spad{q(x, z) = 0}.
    radPoly   : UPUP -> Union(Record(radicand:RF, deg:N), "failed")
          ++ radPoly(p(x, y)) returns \spad{[c(x), n]} if p is of the form
          ++ \spad{y**n - c(x)}, "failed" otherwise.
```

```
    rootPoly  : (RF, N) -> Record(exponent: N, coef:RF, radicand:UP)
          ++ rootPoly(g, n) returns \spad{[m, c, P]} such that
          ++ \spad{c * g ** (1/n) = P ** (1/m)}
          ++ thus if \spad{y**n = g}, then \spad{z**m = P}
          ++ where \spad{z = c * y}.
    goodPoint : (UPUP,UPUP) -> F
          ++ goodPoint(p, q) returns an integer a such that a is neither
          ++ a pole of \spad{p(x,y)} nor a branch point of \spad{q(x,y) = 0}.
    eval      : (UPUP, RF, RF) -> UPUP
          ++ eval(p(x,y), f(x), g(x)) returns \spad{p(f(x), y * g(x))}.
    chvar : (UPUP,UPUP) -> Record(func:UPUP,poly:UPUP,c1:RF,c2:RF,deg:N)
          ++ chvar(f(x,y), p(x,y)) returns
          ++ \spad{[g(z,t), q(z,t), c1(z), c2(z), n]}
          ++ such that under the change of variable
          ++ \spad{x = c1(z)}, \spad{y = t * c2(z)},
          ++ one gets \spad{f(x,y) = g(z,t)}.
          ++ The algebraic relation between x and y is \spad{p(x, y) = 0}.
          ++ The algebraic relation between z and t is \spad{q(z, t) = 0}.

  Implementation ==> add
    import UnivariatePolynomialCommonDenominator(UP, RF, UPUP)

    algPoly     : UPUP           -> Record(coef:RF, poly:UPUP)
    RPrim       : (UP, UP, UPUP) -> Record(coef:RF, poly:UPUP)
    good?       : (F, UP, UP)    -> Boolean
    infIntegral?: (UPUP, UPUP)   -> Boolean

    eval(p, x, y)  == map(s +-> s(x), p)  monomial(y, 1)
    good?(a, p, q) == p(a) ^= 0 and q(a) ^= 0

    algPoly p ==
      ground?(a:= retract(leadingCoefficient(q:=clearDenominator p))@UP)
        => RPrim(1, a, q)
      c := d := squareFreePart a
      q := clearDenominator q monomial(inv(d::RF), 1)
      while not ground?(a := retract(leadingCoefficient q)@UP) repeat
        c := c * (d := gcd(a, d))
        q := clearDenominator q monomial(inv(d::RF), 1)
      RPrim(c, a, q)

    RPrim(c, a, q) ==
--      one? a => [c::RF, q]
      (a = 1) => [c::RF, q]
      [(a * c)::RF, clearDenominator q monomial(inv(a::RF), 1)]

  -- always makes the algebraic integral, but does not send a point to infinity
```

```
-- if the integrand does not have a pole there (in the case of an nth-root)
    chvar(f, modulus) ==
      r1 := mkIntegral modulus
      f1 := f monomial(r1inv := inv(r1.coef), 1)
      infIntegral?(f1, r1.poly) =>
        [f1, r1.poly, monomial(1,1)$UP :: RF,r1inv,degree(retract(r1.coef)@UP)]
      x  := (a:= goodPoint(f1,r1.poly))::UP::RF + inv(monomial(1,1)::RF)
      r2c:= retract((r2 := mkIntegral map(s+->s(x), r1.poly)).coef)@UP
      t  := inv((monomial(1, 1)$UP - a::UP)::RF)
      [- inv(monomial(1, 2)$UP :: RF) * eval(f1, x, inv(r2.coef)),
                               r2.poly, t, r1.coef * r2c t, degree r2c]

-- returns true if y is an n-th root, and it can be guaranteed that p(x,y)dx
-- is integral at infinity
-- expects y to be integral.
    infIntegral?(p, modulus) ==
      (r := radPoly modulus) case "failed" => false
      ninv := inv(r.deg::Q)
      degy:Q := degree(retract(r.radicand)@UP) * ninv
      degp:Q := 0
      while p ^= 0 repeat
        c := leadingCoefficient p
        degp := max(degp,
            (2 + degree(numer c)::Z - degree(denom c)::Z)::Q + degree(p) * degy)
        p := reductum p
      degp <= ninv

    mkIntegral p ==
      (r := radPoly p) case "failed" => algPoly p
      rp := rootPoly(r.radicand, r.deg)
      [rp.coef, monomial(1, rp.exponent)$UPUP - rp.radicand::RF::UPUP]

    goodPoint(p, modulus) ==
      q :=
        (r := radPoly modulus) case "failed" =>
                   retract(resultant(modulus, differentiate modulus))@UP
        retract(r.radicand)@UP
      d := commonDenominator p
      for i in 0.. repeat
        good?(a := i::F, q, d) => return a
        good?(-a, q, d)        => return -a

    radPoly p ==
      (r := retractIfCan(reductum p)@Union(RF, "failed")) case "failed"
        => "failed"
      [- (r::RF), degree p]
```

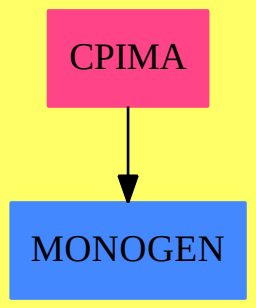
```
  -- we have y**m = g(x) = n(x)/d(x), so if we can write
  -- (n(x) * d(x)**(m-1)) ** (1/m)  =  c(x) * P(x) ** (1/n)
  -- then z**q = P(x) where z = (d(x) / c(x)) * y
      rootPoly(g, m) ==
        zero? g => error "Should not happen"
        pr := nthRoot(squareFree((numer g) * (d := denom g) ** (m-1)::N),
                                                  m)$FactoredFunctions(UP)
        [pr.exponent, d / pr.coef, */(pr.radicand)]
```

⟨*CHVAR.dotabb*⟩≡

```
  "CHVAR" [color="#FF4488",href="bookvol10.4.pdf#nameddest=CHVAR"]
  "PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
  "CHVAR" -> "PFECAT"
```

4.5 package CPIMA CharacteristicPolynomialInMonogenicalAlgebra

4.6 CharacteristicPolynomialInMonogenicalAlgebra



1.00

Exports:
characteristicPolynomial

⟨*package CPIMA CharacteristicPolynomialInMonogenicalAlgebra*⟩≡

```
)abbrev package CPIMA CharacteristicPolynomialInMonogenicalAlgebra
++ Author: Claude Quitte
++ Date Created: 10/12/93
++ Date Last Updated:
++ Basic Functions:
++ Related Constructors:
++ Also See:
++ AMS Classifications:
++ Keywords:
++ References:
++ Description:
++ This package implements characteristicPolynomials for monogenic algebras
++ using resultants
CharacteristicPolynomialInMonogenicalAlgebra(R : CommutativeRing,
    PolR : UnivariatePolynomialCategory(R),
    E : MonogenicAlgebra(R, PolR)): with
     characteristicPolynomial : E -> PolR
        ++ characteristicPolynomial(e) returns the characteristic polynomial
        ++ of e using resultants

  == add
    Pol ==> SparseUnivariatePolynomial

    import UnivariatePolynomialCategoryFunctions2(R, PolR, PolR, Pol(PolR))
    XtoY(Q : PolR) : Pol(PolR) == map(x+->monomial(x, 0), Q)
```

```
P : Pol(PolR) := XtoY(definingPolynomial()$E)
X : Pol(PolR) := monomial(monomial(1, 1)$PolR, 0)

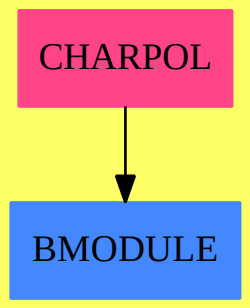
characteristicPolynomial(x : E) : PolR ==
   Qx : PolR := lift(x)
   -- on utilise le fait que resultant_Y (P(Y), X - Qx(Y))
   return resultant(P, X - XtoY(Qx))
```

⟨*CPIMA.dotabb*⟩≡

```
"CPIMA" [color="#FF4488",href="bookvol10.4.pdf#nameddest=CPIMA"]
"MONOGEN" [color="#4488FF",href="bookvol10.2.pdf#nameddest=MONOGEN"]
"CPIMA" -> "MONOGEN"
```

4.7 package CHARPOL CharacteristicPolynomialPackage

4.8 CharacteristicPolynomialPackage



1.00

Exports:
characteristicPolynomial

⟨*package CHARPOL CharacteristicPolynomialPackage*⟩≡

```
)abbrev package CHARPOL CharacteristicPolynomialPackage
++ Author: Barry Trager
++ Date Created:
++ Date Last Updated:
++ Basic Functions:
++ Related Constructors:
++ Also See:
++ AMS Classifications:
++ Keywords:
++ References:
++ Description:
++ This package provides a characteristicPolynomial function
++ for any matrix over a commutative ring.

CharacteristicPolynomialPackage(R:CommutativeRing):C == T where
   PI ==> PositiveInteger
   M ==> Matrix R
   C == with
      characteristicPolynomial: (M, R) -> R
        ++ characteristicPolynomial(m,r) computes the characteristic
        ++ polynomial of the matrix m evaluated at the point r.
        ++ In particular, if r is the polynomial 'x, then it returns
        ++ the characteristic polynomial expressed as a polynomial in 'x.
   T == add

            ---- characteristic polynomial  ----
```

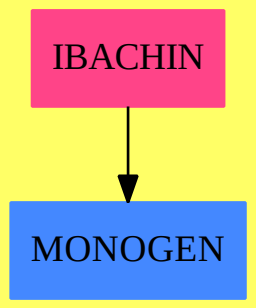
```
characteristicPolynomial(A:M,v:R) : R ==
  dimA :PI := (nrows A):PI
  dimA ^= ncols A => error " The matrix is not square"
  B:M:=zero(dimA,dimA)
  for i in 1..dimA repeat
    for j in 1..dimA repeat  B(i,j):=A(i,j)
    B(i,i) := B(i,i) - v
  determinant B
```

⟨*CHARPOL.dotabb*⟩≡

```
"CHARPOL" [color="#FF4488",href="bookvol10.4.pdf#nameddest=CHARPOL"]
"BMODULE" [color="#4488FF",href="bookvol10.2.pdf#nameddest=BMODULE"]
"CHARPOL" -> "BMODULE"
```

4.9 package IBACHIN ChineseRemainderTools-ForIntegralBases

4.10 ChineseRemainderToolsForIntegralBases



1.00

Exports:
chineseRemainder factorList listConjugateBases

⟨*package IBACHIN ChineseRemainderToolsForIntegralBases*⟩≡

```
)abbrev package IBACHIN ChineseRemainderToolsForIntegralBases
++ Author: Clifton Williamson
++ Date Created: 9 August 1993
++ Date Last Updated: 3 December 1993
++ Basic Operations: chineseRemainder, factorList
++ Related Domains: PAdicWildFunctionFieldIntegralBasis(K,R,UP,F)
++ Also See: WildFunctionFieldIntegralBasis, FunctionFieldIntegralBasis
++ AMS Classifications:
++ Keywords: function field, finite field, integral basis
++ Examples:
++ References:
++ Description:

ChineseRemainderToolsForIntegralBases(K,R,UP): Exports == Implementation where
  K  : FiniteFieldCategory
  R  : UnivariatePolynomialCategory K
  UP : UnivariatePolynomialCategory R

  DDFACT ==> DistinctDegreeFactorize
  I      ==> Integer
  L      ==> List
  L2     ==> ListFunctions2
  Mat    ==> Matrix R
  NNI    ==> NonNegativeInteger
  PI     ==> PositiveInteger
  Q      ==> Fraction R
```

```
SAE    ==> SimpleAlgebraicExtension
SUP    ==> SparseUnivariatePolynomial
SUP2   ==> SparseUnivariatePolynomialFunctions2
Result ==> Record(basis: Mat, basisDen: R, basisInv: Mat)

Exports ==> with
  factorList: (K,NNI,NNI,NNI) -> L SUP K
      ++ factorList(k,n,m,j) \undocumented

  listConjugateBases: (Result,NNI,NNI) -> List Result
    ++ listConjugateBases(bas,q,n) returns the list
    ++ \spad{[bas,bas^Frob,bas^(Frob^2),...bas^(Frob^(n-1))]}, where
    ++ \spad{Frob} raises the coefficients of all polynomials
    ++ appearing in the basis \spad{bas} to the \spad{q}th power.

  chineseRemainder: (List UP, List Result, NNI) -> Result
      ++ chineseRemainder(lu,lr,n) \undocumented

Implementation ==> add
  import ModularHermitianRowReduction(R)
  import TriangularMatrixOperations(R, Vector R, Vector R, Matrix R)

  applyFrobToMatrix: (Matrix R,NNI) -> Matrix R
  applyFrobToMatrix(mat,q) ==
    -- raises the coefficients of the polynomial entries of 'mat'
    -- to the qth power
    m := nrows mat; n := ncols mat
    ans : Matrix R := new(m,n,0)
    for i in 1..m repeat for j in 1..n repeat
      qsetelt_!(ans,i,j,map((k1:K):K +-> k1 ** q,qelt(mat,i,j)))
    ans

  listConjugateBases(bas,q,n) ==
    outList : List Result := list bas
    b := bas.basis; bInv := bas.basisInv; bDen := bas.basisDen
    for i in 1..(n-1) repeat
      b := applyFrobToMatrix(b,q)
      bInv := applyFrobToMatrix(bInv,q)
      bDen := map((k1:K):K +-> k1 ** q,bDen)
      newBasis : Result := [b,bDen,bInv]
      outList := concat(newBasis,outList)
    reverse_! outList

  factorList(a,q,n,k) ==
    coef : SUP K := monomial(a,0); xx : SUP K := monomial(1,1)
    outList : L SUP K := list((xx - coef)**k)
```

```
      for i in 1..(n-1) repeat
        coef := coef ** q
        outList := concat((xx - coef)**k,outList)
      reverse_! outList

    basisInfoToPolys: (Mat,R,R) -> L UP
    basisInfoToPolys(mat,lcm,den) ==
      n := nrows(mat) :: I; n1 := n - 1
      outList : L UP := empty()
      for i in 1..n repeat
        pp : UP := 0
        for j in 0..n1 repeat
          pp := pp + monomial((lcm quo den) * qelt(mat,i,j+1),j)
        outList := concat(pp,outList)
      reverse_! outList

    basesToPolyLists: (L Result,R) -> L L UP
    basesToPolyLists(basisList,lcm) ==
      [basisInfoToPolys(b.basis,lcm,b.basisDen) for b in basisList]

    OUT ==> OutputForm

    approximateExtendedEuclidean: (UP,UP,R,NNI) -> Record(coef1:UP,coef2:UP)
    approximateExtendedEuclidean(f,g,p,n) ==
      -- f and g are monic and relatively prime (mod p)
      -- function returns [coef1,coef2] such that
      -- coef1 * f + coef2 * g = 1 (mod p^n)
      sae := SAE(K,R,p)
      fSUP : SUP R := makeSUP f; gSUP : SUP R := makeSUP g
      fBar : SUP sae := map((r1:R):sae +-> convert(r1)@sae,fSUP)$SUP2(R,sae)
      gBar : SUP sae := map((r1:R):sae +-> convert(r1)@sae,gSUP)$SUP2(R,sae)
      ee := extendedEuclidean(fBar,gBar)
--      not one?(ee.generator) =>
      not (ee.generator = 1) =>
        error "polynomials aren't relatively prime"
      ss1 := ee.coef1; tt1 := ee.coef2
      s1 : SUP R := map((z1:sae):R +-> convert(z1)@R,ss1)$SUP2(sae,R); s := s1
      t1 : SUP R := map((z1:sae):R +-> convert(z1)@R,tt1)$SUP2(sae,R); t := t1
      pPower := p
      for i in 2..n repeat
        num := 1 - s * fSUP - t * gSUP
        rhs := (num exquo pPower) :: SUP R
        sigma := map((r1:R):R +-> r1 rem p,s1*rhs);
        tau   := map((r1:R):R +-> r1 rem p,t1*rhs)
        s := s + pPower * sigma; t := t + pPower * tau
        quorem := monicDivide(s,gSUP)
```

```
    pPower := pPower * p
    s := map((r1:R):R+->r1 rem pPower,quorem.remainder)
    t := map((r1:R):R+->r1 rem pPower,t + fSUP * (quorem.quotient))
  [unmakeSUP s,unmakeSUP t]

--mapChineseToList: (L SUP Q,L SUP Q,I) -> L SUP Q
--mapChineseToList(list,polyList,i) ==
mapChineseToList: (L UP,L UP,I,R) -> L UP
mapChineseToList(list,polyList,i,den) ==
  -- 'polyList' consists of MONIC polynomials
  -- computes a polynomial p such that p = pp (modulo polyList[i])
  -- and p = 0 (modulo polyList[j]) for j ~= i for each 'pp' in 'list'
  -- create polynomials
  q : UP := 1
  for j in 1..(i-1) repeat
    q := q * first polyList
    polyList := rest polyList
  p := first polyList
  polyList := rest polyList
  for j in (i+1).. while not empty? polyList repeat
    q := q * first polyList
    polyList := rest polyList
  --p := map((numer(#1) rem den)/1, p)
  --q := map((numer(#1) rem den)/1, q)
  -- 'den' is a power of an irreducible polynomial
  --!! make this computation more efficient!!
  factoredDen := factor(den)$DDFACT(K,R)
  prime := nthFactor(factoredDen,1)
  n := nthExponent(factoredDen,1) :: NNI
  invPoly := approximateExtendedEuclidean(q,p,prime,n).coef1
  -- monicDivide may be inefficient?
  [monicDivide(pp * invPoly * q,p * q).remainder for pp in list]

polyListToMatrix: (L UP,NNI) -> Mat
polyListToMatrix(polyList,n) ==
  mat : Mat := new(n,n,0)
  for i in 1..n for poly in polyList repeat
    while not zero? poly repeat
      mat(i,degree(poly) + 1) := leadingCoefficient poly
      poly := reductum poly
  mat

chineseRemainder(factors,factorBases,n) ==
  denLCM : R := reduce("lcm",[base.basisDen for base in factorBases])
  denLCM = 1 => [scalarMatrix(n,1),1,scalarMatrix(n,1)]
  -- compute local basis polynomials with denominators cleared
```

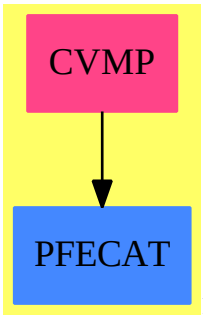
```
factorBasisPolyLists := basesToPolyLists(factorBases,denLCM)
-- use Chinese remainder to compute basis polynomials w/o denominators
basisPolyLists : L L UP := empty()
for i in 1.. for pList in factorBasisPolyLists repeat
  polyList := mapChineseToList(pList,factors,i,denLCM)
  basisPolyLists := concat(polyList,basisPolyLists)
basisPolys := concat reverse_! basisPolyLists
mat := squareTop rowEchelon(polyListToMatrix(basisPolys,n),denLCM)
matInv := UpTriBddDenomInv(mat,denLCM)
[mat,denLCM,matInv]
```

⟨*IBACHIN.dotabb*⟩≡

```
"IBACHIN" [color="#FF4488",href="bookvol10.4.pdf#nameddest=IBACHIN"]
"MONOGEN" [color="#4488FF",href="bookvol10.2.pdf#nameddest=MONOGEN"]
"IBACHIN" -> "MONOGEN"
```

4.11 package CVMP CoerceVectorMatrixPackage

4.12 CoerceVectorMatrixPackage



1.00

Exports:
coerce coerceP

⟨*package CVMP CoerceVectorMatrixPackage*⟩≡

```
)abbrev package CVMP CoerceVectorMatrixPackage
++ Authors: J. Grabmeier
++ Date Created: 26 June 1991
++ Date Last Updated: 26 June 1991
++ Basic Operations: coerceP, coerce
++ Related Constructors: GenericNonAssociativeAlgebra
++ Also See:
++ AMS Classifications:
++ Keywords:
++ Reference:
++ Description:
++  CoerceVectorMatrixPackage: an unexposed, technical package
++  for data conversions
CoerceVectorMatrixPackage(R : CommutativeRing): public == private where
  M2P ==> MatrixCategoryFunctions2(R, Vector R, Vector R, Matrix R, _
    Polynomial R, Vector Polynomial R, Vector Polynomial R, Matrix Polynomial R)
  M2FP ==> MatrixCategoryFunctions2(R, Vector R, Vector R, Matrix R, _
    Fraction Polynomial R, Vector Fraction Polynomial R, _
    Vector Fraction Polynomial R, Matrix Fraction Polynomial R)
  public ==> with
    coerceP: Vector Matrix R -> Vector Matrix Polynomial R
      ++ coerceP(v) coerces a vector v with entries in \spadtype{Matrix R}
      ++ as vector over \spadtype{Matrix Polynomial R}
    coerce: Vector Matrix R -> Vector Matrix Fraction Polynomial R
      ++ coerce(v) coerces a vector v with entries in \spadtype{Matrix R}
      ++ as vector over \spadtype{Matrix Fraction Polynomial R}
```

```
    private ==> add

      imbedFP : R -> Fraction Polynomial R
      imbedFP r == (r:: Polynomial R) :: Fraction Polynomial R

      imbedP : R -> Polynomial R
      imbedP r == (r:: Polynomial R)

      coerceP(g:Vector Matrix R) : Vector Matrix Polynomial R ==
        m2 : Matrix Polynomial R
        lim : List Matrix R := entries g
        l: List Matrix Polynomial R :=  []
        for m in lim repeat
          m2 :=  map(imbedP,m)$M2P
          l := cons(m2,l)
        vector reverse l

      coerce(g:Vector Matrix R) : Vector Matrix Fraction Polynomial R ==
        m3 : Matrix Fraction Polynomial R
        lim : List Matrix R := entries g
        l: List Matrix Fraction Polynomial R :=  []
        for m in lim repeat
          m3 :=  map(imbedFP,m)$M2FP
          l := cons(m3,l)
        vector reverse l
```

⟨*CVMP.dotabb*⟩≡

```
"CVMP" [color="#FF4488",href="bookvol10.4.pdf#nameddest=CVMP"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"CVMP" -> "PFECAT"
```

4.13 package COMBF CombinatorialFunction

⟨*CombinatorialFunction.input*⟩≡

```
)set break resume
)sys rm -f CombinatorialFunction.output
)spool CombinatorialFunction.output
)set message test on
)set message auto off
)clear all
--S 1 of 6
f := operator 'f
--R
--R
--R   (1)  f
--R                                                    Type: BasicOperator
--E 1

--S 2 of 6
D(product(f(i,x),i=1..m),x)
--R
--R
--R          m           m    f  (i,x)
--R        ++-++         --+    ,2
--R   (2)   | |   f(i,x)>     --------
--R         | |         --+    f(i,x)
--R        i= 1         i= 1
--R                                              Type: Expression Integer
--E 2

--S 3 of 6
)set expose add constructor OutputForm
--R
--I   OutputForm is already explicitly exposed in frame frame0
--E 3

--S 4 of 6
pascalRow(n) == [right(binomial(n,i),4) for i in 0..n]
--R
--R                                                                Type: Void
--E 4

--S 5 of 6
displayRow(n)==output center blankSeparate pascalRow(n)
--R
--R                                                                Type: Void
--E 5
```

```
--S 6 of 6
for i in 0..7 repeat displayRow i
--R
--R   Compiling function pascalRow with type NonNegativeInteger -> List
--R      OutputForm
--R   Compiling function displayRow with type NonNegativeInteger -> Void
--R                                       1
--R                                     1    1
--R                                   1    2    1
--R                                 1    3    3    1
--R                               1    4    6    4    1
--R                             1    5   10   10    5    1
--R                           1    6   15   20   15    6    1
--R                         1    7   21   35   35   21    7    1
--R                                                                  Type: Void
--E 6


)spool
)lisp (bye)
```

⟨*CombinatorialFunction.help*⟩≡

```
==========================================================================
CombinatorialFunction examples
==========================================================================

f := operator 'f


   (1)  f
                                                          Type: BasicOperator

D(product(f(i,x),i=1..m),x)


          m           m    f  (i,x)
        ++-++         --+    ,2
   (2)   | |   f(i,x)>     --------
         | |           --+    f(i,x)
        i= 1          i= 1
                                                Type: Expression Integer


The binomial(n, r) returns the number of subsets of r objects
taken among n objects, i.e. n!/(r! * (n-r)!)

The binomial coefficients are the coefficients of the series expansion
of a power of a binomial, that is

    n
   --+    / n \   k             n
   >     |     | x    =  (1 + x)
   --+    \ k /
   k= 0

This leads to the famous pascal triangle. First we expose the OutputForm
domain, which is normally hidden, so we can use it to format the lines.

  )set expose add constructor OutputForm

Next we define a function that will output the list of binomial coefficients
right justified with proper spacing:

  pascalRow(n) == [right(binomial(n,i),4) for i in 0..n]

and now we format the whole line so that it looks centered:
```

```
  displayRow(n)==output center blankSeparate pascalRow(n)

and we compute the triangle

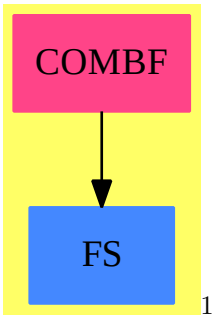
  for i in 0..7 repeat displayRow i

giving the pretty result:

   Compiling function pascalRow with type NonNegativeInteger -> List
      OutputForm
   Compiling function displayRow with type NonNegativeInteger -> Void
                                      1
                                   1     1
                                1     2     1
                             1     3     3     1
                          1     4     6     4     1
                       1     5    10    10     5     1
                    1     6    15    20    15     6     1
                 1     7    21    35    35    21     7     1

See Also:
o )show CombinatorialFunction
o )d op binomial
o )show OutputForm
o )help set
```

4.14 CombinatorialFunction



1.00

Exports:

belong?	binomial	factorial	factorials	iibinom
iidprod	iidsum	iifact	iiperm	iipow
ipow	permutation	product	summation	operator
product	summation	?**?		

binomial

We currently simplify binomial coefficients only for non-negative integral second argument, using the formula

$$\left(\begin{array}{c} n \\ k \end{array} \right) = \frac{1}{k!} \prod_{i=0..k-1} (n-i),$$

except if the second argument is symbolic: in this case binomial(n,n) is simplified to one.

Note that there are at least two different ways to define binomial coefficients for negative integral second argument. One way, particular suitable for combinatorics, is to set the binomial coefficient equal to zero for negative second argument. This is, partially, also the approach taken in `combinat.spad`, where we find

```
binomial(n, m) ==
   n < 0 or m < 0 or m > n => 0
   m = 0 => 1
```

Of course, here n and m are integers. This definition agrees with the recurrence

$$\left(\begin{array}{c} n \\ k \end{array} \right) + \left(\begin{array}{c} n \\ k+1 \end{array} \right) = \left(\begin{array}{c} n+1 \\ k+1 \end{array} \right).$$

Alternatively, one can use the formula

$$\left(\begin{array}{c} n \\ k \end{array} \right) = \frac{\Gamma(n+1)}{\Gamma(k+1)\Gamma(n-k+1)},$$

and leave the case where $k \in \mathbf{Z}$, $n \in \mathbf{Z}$ and $k \leq n < 0$ undefined, since the limit does not exist in this case:

Since we then have that $n-k+1 \geq 1$, $\Gamma(n-k+1)$ is finite. So it is sufficient to consider $\frac{\Gamma(n+1)}{\Gamma(k+1)}$. On the one hand, we have

$$\lim_{n_0 \to n} \lim_{k_0 \to k} \frac{\Gamma(n_0+1)}{\Gamma(k_0+1)} = 0,$$

since for any non-integral n_0, $\Gamma(n_0+1)$ is finite. On the other hand,

$$\lim_{k_0 \to k} \lim_{n_0 \to n} \frac{\Gamma(n_0+1)}{\Gamma(k_0+1)}$$

does not exist, since for non-integral k_0, $\Gamma(k_0+1)$ is finite while $\Gamma(n_0+1)$ is unbounded.

However, since for $k \in \mathbf{Z}$, $n \in \mathbf{Z}$ and $0 < k < n$ both definitions agree, one could also combine them. This is what, for example, Mathematica does. It seems that MuPAD sets binomial(n,n)=1 for all arguments n, and returns binomial(-2, n) unevaluated. Provisos may help here.

dvsum and dvdsum

The dvsum and dvdsum operations implement differentiation of sums with and without bounds. Note that the function

$$n \mapsto \sum_{k=1}^{n} f(k, n)$$

is well defined only for integral values of n greater than or equal to zero. There is not even consensus how to define this function for $n < 0$. Thus, it is not differentiable. Therefore, we need to check whether we erroneously are differentiating with respect to the upper bound or the lower bound, where the same reasoning holds.

Differentiating a sum with respect to its indexing variable correctly gives zero. This is due to the introduction of dummy variables in the internal representation of a sum: the operator `%defsum` takes 5 arguments, namely

1. the summands, where each occurrence of the indexing variable is replaced by
2. the dummy variable,
3. the indexing variable,
4. the lower bound, and
5. the upper bound.

dvprod and dvdprod

The dvprod and dvdprod operations implement differentiation of products with and without bounds. Note again, that we cannot even properly define products with bounds that are not integral.

To differentiate the product, we use Leibniz rule:

$$\frac{d}{dx} \prod_{i=a}^{b} f(i, x) = \sum_{i=a}^{b} \frac{\frac{d}{dx} f(i, x)}{f(i, x)} \prod_{i=a}^{b} f(i, x)$$

There is one situation where this definition might produce wrong results, namely when the product is zero, but axiom failed to recognize it: in this case,

$$\frac{d}{dx} f(i, x)/f(i, x)$$

is undefined for some i. However, I was not able to come up with an example. The alternative definition

$$\frac{d}{dx} \prod_{i=a}^{b} f(i, x) = \sum_{i=a}^{b} \left(\frac{d}{dx} f(i, x) \right) \prod_{j=a, j \neq i}^{b} f(j, x)$$

has the slight (display) problem that we would have to come up with a new index variable, which looks very ugly. Furthermore, it seems to me that more simplifications will occur with the first definition.

```
f := operator 'f
D(product(f(i,x),i=1..m),x)
```

dvpow2

The dvpow2 operation implements the differentiation of the power operator `%power` with respect to its second argument, i.e., the exponent. It uses the formula

$$\frac{d}{dx}g(y)^x = \frac{d}{dx}e^{x\log g(y)} = \log g(y) g(y)^x.$$

If $g(y)$ equals zero, this formula is not valid, since the logarithm is not defined there. Although strictly speaking 0^x is not differentiable at zero, we return zero for convenience.

⟨*package COMBF CombinatorialFunction*⟩≡

```
)abbrev package COMBF CombinatorialFunction
++ Provides the usual combinatorial functions
++ Author: Manuel Bronstein, Martin Rubey
++ Date Created: 2 Aug 1988
++ Date Last Updated: 30 October 2005
++ Description:
++   Provides combinatorial functions over an integral domain.
++ Keywords: combinatorial, function, factorial.
++ Examples:  )r COMBF INPUT

CombinatorialFunction(R, F): Exports == Implementation where
  R: Join(OrderedSet, IntegralDomain)
  F: FunctionSpace R

  OP  ==> BasicOperator
  K   ==> Kernel F
  SE  ==> Symbol
  O   ==> OutputForm
  SMP ==> SparseMultivariatePolynomial(R, K)
  Z   ==> Integer

  POWER        ==> "%power"::Symbol
  OPEXP        ==> "exp"::Symbol
  SPECIALDIFF  ==> "%specialDiff"
  SPECIALDISP  ==> "%specialDisp"
  SPECIALEQUAL ==> "%specialEqual"
```

```
Exports ==> with
  belong?    : OP -> Boolean
    ++ belong?(op) is true if op is a combinatorial operator;
  operator   : OP -> OP
    ++ operator(op) returns a copy of op with the domain-dependent
    ++ properties appropriate for F;
    ++ error if op is not a combinatorial operator;
  "**"       : (F, F) -> F
    ++ a ** b is the formal exponential a**b;
  binomial   : (F, F) -> F
    ++ binomial(n, r) returns the number of subsets of r objects
    ++ taken among n objects, i.e. n!/(r! * (n-r)!);
    ++
    ++X [binomial(5,i) for i in 0..5]
  permutation: (F, F) -> F
    ++ permutation(n, r) returns the number of permutations of
    ++ n objects taken r at a time, i.e. n!/(n-r)!;
  factorial  : F -> F
    ++ factorial(n) returns the factorial of n, i.e. n!;
  factorials : F -> F
    ++ factorials(f) rewrites the permutations and binomials in f
    ++ in terms of factorials;
  factorials : (F, SE) -> F
    ++ factorials(f, x) rewrites the permutations and binomials in f
    ++ involving x in terms of factorials;
  summation  : (F, SE) -> F
    ++ summation(f(n), n) returns the formal sum S(n) which verifies
    ++ S(n+1) - S(n) = f(n);
  summation  : (F, SegmentBinding F)  -> F
    ++ summation(f(n), n = a..b) returns f(a) + ... + f(b) as a
    ++ formal sum;
  product    : (F, SE) -> F
    ++ product(f(n), n) returns the formal product P(n) which verifies
    ++ P(n+1)/P(n) = f(n);
  product    : (F, SegmentBinding  F) -> F
    ++ product(f(n), n = a..b) returns f(a) * ... * f(b) as a
    ++ formal product;
  iifact     : F -> F
    ++ iifact(x) should be local but conditional;
  iibinom    : List F -> F
    ++ iibinom(l) should be local but conditional;
  iiperm     : List F -> F
    ++ iiperm(l) should be local but conditional;
  iipow      : List F -> F
    ++ iipow(l) should be local but conditional;
  iidsum     : List F -> F
```

```
      ++ iidsum(l) should be local but conditional;
    iidprod    : List F -> F
      ++ iidprod(l) should be local but conditional;
    ipow       : List F -> F
      ++ ipow(l) should be local but conditional;

  Implementation ==> add
    ifact     : F -> F
    iiipow    : List F -> F
    iperm     : List F -> F
    ibinom    : List F -> F
    isum      : List F -> F
    idsum     : List F -> F
    iprod     : List F -> F
    idprod    : List F -> F
    dsum      : List F -> O
    ddsum     : List F -> O
    dprod     : List F -> O
    ddprod    : List F -> O
    equalsumprod  : (K, K) -> Boolean
    equaldsumprod : (K, K) -> Boolean
    fourth    : List F -> F
    dvpow1    : List F -> F
    dvpow2    : List F -> F
    summand   : List F -> F
    dvsum     : (List F, SE) -> F
    dvdsum    : (List F, SE) -> F
    dvprod    : (List F, SE) -> F
    dvdprod   : (List F, SE) -> F
    facts     : (F, List SE) -> F
    K2fact    : (K, List SE) -> F
    smpfact   : (SMP, List SE) -> F

-- This macro will be used in product and summation, both the 5 and 3
-- argument forms. It is used to introduce a dummy variable in place of the
-- summation index within the summands. This in turn is necessary to keep the
-- indexing variable local, circumventing problems, for example, with
-- differentiation.

    dummy == new()$SE :: F

    opfact  := operator("factorial"::Symbol)$CommonOperators
    opperm  := operator("permutation"::Symbol)$CommonOperators
    opbinom := operator("binomial"::Symbol)$CommonOperators
    opsum   := operator("summation"::Symbol)$CommonOperators
    opdsum  := operator("%defsum"::Symbol)$CommonOperators
```

```
opprod  := operator("product"::Symbol)$CommonOperators
opdprod := operator("%defprod"::Symbol)$CommonOperators
oppow   := operator(POWER::Symbol)$CommonOperators

factorial x          == opfact x
binomial(x, y)       == opbinom [x, y]
permutation(x, y)    == opperm [x, y]

import F
import Kernel F

number?(x:F):Boolean ==
  if R has RetractableTo(Z) then
    ground?(x) or
     ((retractIfCan(x)@Union(Fraction(Z),"failed")) case Fraction(Z))
  else
    ground?(x)

x ** y               ==
  -- Do some basic simplifications
  is?(x,POWER) =>
    args : List F := argument first kernels x
    not(#args = 2) => error "Too many arguments to **"
    number?(first args) and number?(y) =>
      oppow [first(args)**y, second args]
    oppow [first args, (second args)* y]
  -- Generic case
  exp : Union(Record(val:F,exponent:Z),"failed") := isPower x
  exp case Record(val:F,exponent:Z) =>
    expr := exp::Record(val:F,exponent:Z)
    oppow [expr.val, (expr.exponent)*y]
  oppow [x, y]

belong? op           == has?(op, "comb")
fourth l             == third rest l
dvpow1 l             == second(l) * first(l) ** (second l - 1)
factorials x         == facts(x, variables x)
factorials(x, v)     == facts(x, [v])
facts(x, l)          == smpfact(numer x, l) / smpfact(denom x, l)
summand l            == eval(first l, retract(second l)@K, third l)

product(x:F, i:SE) ==
  dm := dummy
  opprod [eval(x, k := kernel(i)$K, dm), dm, k::F]

summation(x:F, i:SE) ==
```

```
      dm := dummy
      opsum [eval(x, k := kernel(i)$K, dm), dm, k::F]

  -- These two operations return the product or the sum as unevaluated operators
  -- A dummy variable is introduced to make the indexing variable local.

    dvsum(l, x) ==
      opsum [differentiate(first l, x), second l, third l]

    dvdsum(l, x) ==
      x = retract(y := third l)@SE => 0
      if member?(x, variables(h := third rest rest l)) or
         member?(x, variables(g := third rest l)) then
        error "a sum cannot be differentiated with respect to a bound"
      else
        opdsum [differentiate(first l, x), second l, y, g, h]

    dvprod(l, x) ==
      dm := retract(dummy)@SE
      f := eval(first l, retract(second l)@K, dm::F)
      p := product(f, dm)

      opsum [differentiate(first l, x)/first l * p, second l, third l]


    dvdprod(l, x) ==
      x = retract(y := third l)@SE => 0
      if member?(x, variables(h := third rest rest l)) or
         member?(x, variables(g := third rest l)) then
        error "a product cannot be differentiated with respect to a bound"
      else
        opdsum cons(differentiate(first l, x)/first l, rest l) * opdprod l

  -- These four operations handle the conversion of sums and products to
  -- OutputForm

    dprod l ==
      prod(summand(l)::O, third(l)::O)

    ddprod l ==
      prod(summand(l)::O, third(l)::O = fourth(l)::O, fourth(rest l)::O)

    dsum l ==
      sum(summand(l)::O, third(l)::O)

    ddsum l ==
```

```
      sum(summand(l)::O, third(l)::O = fourth(l)::O, fourth(rest l)::O)

  -- The two operations handle the testing for equality of sums and products.
  -- The corresponding property \verb|%specialEqual| set below is checked in
  -- Kernel. Note that we can assume that the operators are equal, since this is
  -- checked in Kernel itself.

    equalsumprod(s1, s2) ==
      l1 := argument s1
      l2 := argument s2
      (eval(first l1, retract(second l1)@K, second l2) = first l2)

    equaldsumprod(s1, s2) ==
      l1 := argument s1
      l2 := argument s2
      ((third rest l1 = third rest l2) and
       (third rest rest l1 = third rest rest l2) and
       (eval(first l1, retract(second l1)@K, second l2) = first l2))

  -- These two operations return the product or the sum as unevaluated operators
  -- A dummy variable is introduced to make the indexing variable local.

    product(x:F, s:SegmentBinding F) ==
      k := kernel(variable s)$K
      dm := dummy
      opdprod [eval(x,k,dm), dm, k::F, lo segment s, hi segment s]

    summation(x:F, s:SegmentBinding F) ==
      k := kernel(variable s)$K
      dm := dummy
      opdsum [eval(x,k,dm), dm, k::F, lo segment s, hi segment s]

    smpfact(p, l) ==
      map(x +-> K2fact(x, l), y+->y::F, p)_
        $PolynomialCategoryLifting(IndexedExponents K, K, R, SMP, F)

    K2fact(k, l) ==
      empty? [v for v in variables(kf := k::F) | member?(v, l)] => kf
      empty?(args:List F := [facts(a, l) for a in argument k]) => kf
      is?(k, opperm) =>
        factorial(n := first args) / factorial(n - second args)
      is?(k, opbinom) =>
        n := first args
        p := second args
        factorial(n) / (factorial(p) * factorial(n-p))
      (operator k) args
```

```
    operator op ==
      is?(op, "factorial"::Symbol)   => opfact
      is?(op, "permutation"::Symbol) => opperm
      is?(op, "binomial"::Symbol)    => opbinom
      is?(op, "summation"::Symbol)   => opsum
      is?(op, "%defsum"::Symbol)     => opdsum
      is?(op, "product"::Symbol)     => opprod
      is?(op, "%defprod"::Symbol)    => opdprod
      is?(op, POWER)                 => oppow
      error "Not a combinatorial operator"

    iprod l ==
      zero? first l => 0
--      one? first l => 1
      (first l = 1) => 1
      kernel(opprod, l)

    isum l ==
      zero? first l => 0
      kernel(opsum, l)

    idprod l ==
      member?(retract(second l)@SE, variables first l) =>
        kernel(opdprod, l)
      first(l) ** (fourth rest l - fourth l + 1)

    idsum l ==
      member?(retract(second l)@SE, variables first l) =>
        kernel(opdsum, l)
      first(l) * (fourth rest l - fourth l + 1)

    ifact x ==
--      zero? x or one? x => 1
      zero? x or (x = 1) => 1
      kernel(opfact, x)

    ibinom l ==
      n := first l
      ((p := second l) = 0) or (p = n) => 1
--      one? p or (p = n - 1) => n
      (p = 1) or (p = n - 1) => n
      kernel(opbinom, l)

    iperm l ==
      zero? second l => 1
```

```
        kernel(opperm, l)

      if R has RetractableTo Z then
        iidsum l ==
          (r1:=retractIfCan(fourth l)@Union(Z,"failed"))
           case "failed" or
            (r2:=retractIfCan(fourth rest l)@Union(Z,"failed"))
              case "failed" or
               (k:=retractIfCan(second l)@Union(K,"failed")) case "failed"
                 => idsum l
          +/[eval(first l,k::K,i::F) for i in r1::Z .. r2::Z]

        iidprod l ==
          (r1:=retractIfCan(fourth l)@Union(Z,"failed"))
           case "failed" or
            (r2:=retractIfCan(fourth rest l)@Union(Z,"failed"))
              case "failed" or
               (k:=retractIfCan(second l)@Union(K,"failed")) case "failed"
                 => idprod l
          */[eval(first l,k::K,i::F) for i in r1::Z .. r2::Z]

        iiipow l ==
            (u := isExpt(x := first l, OPEXP)) case "failed" => kernel(oppow, l)
            rec := u::Record(var: K, exponent: Z)
            y := first argument(rec.var)
            (r := retractIfCan(y)@Union(Fraction Z, "failed")) case
                "failed" => kernel(oppow, l)
            (operator(rec.var)) (rec.exponent * y * second l)

        if F has RadicalCategory then
          ipow l ==
            (r := retractIfCan(second l)@Union(Fraction Z,"failed"))
              case "failed" => iiipow l
            first(l) ** (r::Fraction(Z))
        else
          ipow l ==
            (r := retractIfCan(second l)@Union(Z, "failed"))
              case "failed" => iiipow l
            first(l) ** (r::Z)

      else
        ipow l ==
          zero?(x := first l) =>
            zero? second l => error "0 ** 0"
            0
--          one? x or zero?(n := second l) => 1
```

```
        (x = 1) or zero?(n: F := second l) => 1
--        one? n => x
        (n = 1) => x
        (u := isExpt(x, OPEXP)) case "failed" => kernel(oppow, l)
        rec := u::Record(var: K, exponent: Z)
--        one?(y := first argument(rec.var)) or y = -1 =>
        ((y := first argument(rec.var))=1) or y = -1 =>
            (operator(rec.var)) (rec.exponent * y * n)
        kernel(oppow, l)

    if R has CombinatorialFunctionCategory then
      iifact x ==
        (r:=retractIfCan(x)@Union(R,"failed")) case "failed" => ifact x
        factorial(r::R)::F

      iiperm l ==
        (r1 := retractIfCan(first l)@Union(R,"failed")) case "failed" or
          (r2 := retractIfCan(second l)@Union(R,"failed")) case "failed"
            => iperm l
        permutation(r1::R, r2::R)::F

      if R has RetractableTo(Z) and F has Algebra(Fraction(Z)) then
         iibinom l ==
           (s:=retractIfCan(second l)@Union(R,"failed")) case R and
              (t:=retractIfCan(s)@Union(Z,"failed")) case Z and t>0 =>
                ans:=1::F
                for i in 0..t-1 repeat
                    ans:=ans*(first l - i::R::F)
                (1/factorial t) * ans
           (s:=retractIfCan(first l-second l)@Union(R,"failed")) case R and
             (t:=retractIfCan(s)@Union(Z,"failed")) case Z and t>0 =>
                ans:=1::F
                for i in 1..t repeat
                    ans:=ans*(second l+i::R::F)
                (1/factorial t) * ans
           (r1 := retractIfCan(first l)@Union(R,"failed")) case "failed" or
             (r2 := retractIfCan(second l)@Union(R,"failed")) case "failed"
               => ibinom l
           binomial(r1::R, r2::R)::F

  -- iibinom checks those cases in which the binomial coefficient may
  -- be evaluated explicitly. Currently, the naive iterative algorithm is
  -- used to calculate the coefficient, there is room for improvement here.

      else
         iibinom l ==
```

```
            (r1 := retractIfCan(first l)@Union(R,"failed")) case "failed" or
              (r2 := retractIfCan(second l)@Union(R,"failed")) case "failed"
                => ibinom l
            binomial(r1::R, r2::R)::F

      else
        iifact x  == ifact x
        iibinom l == ibinom l
        iiperm l  == iperm l

      if R has ElementaryFunctionCategory then
        iipow l ==
          (r1:=retractIfCan(first l)@Union(R,"failed")) case "failed" or
            (r2:=retractIfCan(second l)@Union(R,"failed")) case "failed"
              => ipow l
          (r1::R ** r2::R)::F
      else
        iipow l == ipow l

      if F has ElementaryFunctionCategory then
        dvpow2 l == if zero?(first l) then
                      0
                    else
                      log(first l) * first(l) ** second(l)

      evaluate(opfact, iifact)$BasicOperatorFunctions1(F)
      evaluate(oppow, iipow)
      evaluate(opperm, iiperm)
      evaluate(opbinom, iibinom)
      evaluate(opsum, isum)
      evaluate(opdsum, iidsum)
      evaluate(opprod, iprod)
      evaluate(opdprod, iidprod)
      derivative(oppow, [dvpow1, dvpow2])

  -- These four properties define special differentiation rules for sums and
  -- products.

      setProperty(opsum,   SPECIALDIFF, dvsum@((List F, SE) -> F) pretend None)
      setProperty(opdsum,  SPECIALDIFF, dvdsum@((List F, SE)->F) pretend None)
      setProperty(opprod,  SPECIALDIFF, dvprod@((List F, SE)->F) pretend None)
      setProperty(opdprod, SPECIALDIFF, dvdprod@((List F, SE)->F) pretend None)

  -- Set the properties for displaying sums and products and testing for
  -- equality.
```

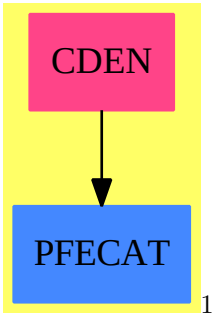
```
setProperty(opsum,   SPECIALDISP, dsum@(List F -> O) pretend None)
setProperty(opdsum,  SPECIALDISP, ddsum@(List F -> O) pretend None)
setProperty(opprod,  SPECIALDISP, dprod@(List F -> O) pretend None)
setProperty(opdprod, SPECIALDISP, ddprod@(List F -> O) pretend None)
setProperty(opsum,   SPECIALEQUAL, equalsumprod@((K,K) -> Boolean) pretend No
setProperty(opdsum,  SPECIALEQUAL, equaldsumprod@((K,K) -> Boolean) pretend N
setProperty(opprod,  SPECIALEQUAL, equalsumprod@((K,K) -> Boolean) pretend No
setProperty(opdprod, SPECIALEQUAL, equaldsumprod@((K,K) -> Boolean) pretend N
```

⟨*COMBF.dotabb*⟩≡

```
"COMBF" [color="#FF4488",href="bookvol10.4.pdf#nameddest=COMBF"]
"FS" [color="#4488FF",href="bookvol10.2.pdf#nameddest=FS"]
"COMBF" -> "FS"
```

4.15 package CDEN CommonDenominator

4.16 CommonDenominator



1.00

Exports:
clearDenominator commonDenominator splitDenominator

⟨*package CDEN CommonDenominator*⟩≡

```
)abbrev package CDEN CommonDenominator
--% CommonDenominator
++ Author: Manuel Bronstein
++ Date Created: 2 May 1988
++ Date Last Updated: 22 Nov 1989
++ Description: CommonDenominator provides functions to compute the
++ common denominator of a finite linear aggregate of elements of
++ the quotient field of an integral domain.
++ Keywords: gcd, quotient, common, denominator.
CommonDenominator(R, Q, A): Exports == Implementation where
  R: IntegralDomain
  Q: QuotientFieldCategory R
  A: FiniteLinearAggregate Q

  Exports ==> with
    commonDenominator: A -> R
      ++ commonDenominator([q1,...,qn]) returns a common denominator
      ++ d for q1,...,qn.
    clearDenominator : A -> A
      ++ clearDenominator([q1,...,qn]) returns \spad{[p1,...,pn]} such that
      ++ \spad{qi = pi/d} where d is a common denominator for the qi's.
    splitDenominator : A -> Record(num: A, den: R)
      ++ splitDenominator([q1,...,qn]) returns
      ++ \spad{[[p1,...,pn], d]} such that
      ++ \spad{qi = pi/d} and d is a common denominator for the qi's.

  Implementation ==> add
```

```
clearDenominator l ==
  d := commonDenominator l
  map(x+->numer(d*x)::Q, l)

splitDenominator l ==
  d := commonDenominator l
  [map(x+->numer(d*x)::Q, l), d]

if R has GcdDomain then
  qlcm: (Q, Q) -> Q

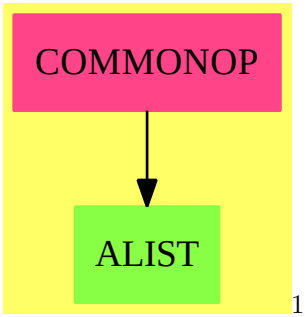
  qlcm(a, b)          == lcm(numer a, numer b)::Q
  commonDenominator l == numer reduce(qlcm, map(x+->denom(x)::Q, l), 1)
else
  commonDenominator l == numer reduce("*", map(x+->denom(x)::Q, l), 1)
```

⟨*CDEN.dotabb*⟩≡

```
"CDEN" [color="#FF4488",href="bookvol10.4.pdf#nameddest=CDEN"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"CDEN" -> "PFECAT"
```

4.17 package COMMONOP CommonOperators

4.18 CommonOperators



1.00

Exports:
operator

⟨*package COMMONOP CommonOperators*⟩≡
```
)abbrev package COMMONOP CommonOperators
++ Provides commonly used operators
++ Author: Manuel Bronstein
++ Date Created: 25 Mar 1988
++ Date Last Updated: 2 December 1994
++ Description:
++ This package exports the elementary operators, with some semantics
++ already attached to them. The semantics that is attached here is not
++ dependent on the set in which the operators will be applied.
++ Keywords: operator.
CommonOperators(): Exports == Implementation where
  OP  ==> BasicOperator
  O   ==> OutputForm
  POWER ==> "%power"::Symbol
  ALGOP ==> "%alg"
  EVEN  ==> "even"
  ODD   ==> "odd"
  DUMMYVAR ==> "%dummyVar"

  Exports ==> with
    operator: Symbol -> OP
        ++ operator(s) returns an operator with name s, with the
        ++ appropriate semantics if s is known. If s is not known,
        ++ the result has no semantics.

  Implementation ==> add
    dpi        : List O -> O
```

```
dgamma     : List O -> O
dquote     : List O -> O
dexp       : O -> O
dfact      : O -> O
startUp    : Boolean -> Void
setDummyVar: (OP, NonNegativeInteger) -> OP

brandNew?:Reference(Boolean) := ref true

opalg   := operator("rootOf"::Symbol, 2)$OP
oproot  := operator("nthRoot"::Symbol, 2)
oppi    := operator("pi"::Symbol, 0)
oplog   := operator("log"::Symbol, 1)
opexp   := operator("exp"::Symbol, 1)
opabs   := operator("abs"::Symbol, 1)
opsin   := operator("sin"::Symbol, 1)
opcos   := operator("cos"::Symbol, 1)
optan   := operator("tan"::Symbol, 1)
opcot   := operator("cot"::Symbol, 1)
opsec   := operator("sec"::Symbol, 1)
opcsc   := operator("csc"::Symbol, 1)
opasin  := operator("asin"::Symbol, 1)
opacos  := operator("acos"::Symbol, 1)
opatan  := operator("atan"::Symbol, 1)
opacot  := operator("acot"::Symbol, 1)
opasec  := operator("asec"::Symbol, 1)
opacsc  := operator("acsc"::Symbol, 1)
opsinh  := operator("sinh"::Symbol, 1)
opcosh  := operator("cosh"::Symbol, 1)
optanh  := operator("tanh"::Symbol, 1)
opcoth  := operator("coth"::Symbol, 1)
opsech  := operator("sech"::Symbol, 1)
opcsch  := operator("csch"::Symbol, 1)
opasinh := operator("asinh"::Symbol, 1)
opacosh := operator("acosh"::Symbol, 1)
opatanh := operator("atanh"::Symbol, 1)
opacoth := operator("acoth"::Symbol, 1)
opasech := operator("asech"::Symbol, 1)
opacsch := operator("acsch"::Symbol, 1)
opbox   := operator("%box"::Symbol)$OP
oppren  := operator("%paren"::Symbol)$OP
opquote := operator("applyQuote"::Symbol)$OP
opdiff  := operator("%diff"::Symbol, 3)
opsi    := operator("Si"::Symbol, 1)
opci    := operator("Ci"::Symbol, 1)
opei    := operator("Ei"::Symbol, 1)
```

```
    opli    := operator("li"::Symbol, 1)
    operf   := operator("erf"::Symbol, 1)
    opli2   := operator("dilog"::Symbol, 1)
    opGamma     := operator("Gamma"::Symbol, 1)
    opGamma2    := operator("Gamma2"::Symbol, 2)
    opBeta      := operator("Beta"::Symbol, 2)
    opdigamma   := operator("digamma"::Symbol, 1)
    oppolygamma := operator("polygamma"::Symbol, 2)
    opBesselJ   := operator("besselJ"::Symbol, 2)
    opBesselY   := operator("besselY"::Symbol, 2)
    opBesselI   := operator("besselI"::Symbol, 2)
    opBesselK   := operator("besselK"::Symbol, 2)
    opAiryAi    := operator("airyAi"::Symbol,  1)
    opAiryBi    := operator("airyBi"::Symbol , 1)
    opint   := operator("integral"::Symbol, 3)
    opdint  := operator("%defint"::Symbol, 5)
    opfact  := operator("factorial"::Symbol, 1)
    opperm  := operator("permutation"::Symbol, 2)
    opbinom := operator("binomial"::Symbol, 2)
    oppow   := operator(POWER, 2)
    opsum   := operator("summation"::Symbol, 3)
    opdsum  := operator("%defsum"::Symbol, 5)
    opprod  := operator("product"::Symbol, 3)
    opdprod := operator("%defprod"::Symbol, 5)

    algop   := [oproot, opalg]$List(OP)
    rtrigop := [opsin, opcos, optan, opcot, opsec, opcsc,
                        opasin, opacos, opatan, opacot, opasec, opacsc]
    htrigop := [opsinh, opcosh, optanh, opcoth, opsech, opcsch,
                    opasinh, opacosh, opatanh, opacoth, opasech, opacsch]
    trigop  := concat(rtrigop, htrigop)
    elemop  := concat(trigop, [oppi, oplog, opexp])
    primop  := [opei, opli, opsi, opci, operf, opli2, opint, opdint]
    combop  := [opfact, opperm, opbinom, oppow,
                                        opsum, opdsum, opprod, opdprod]
    specop  := [opGamma, opGamma2, opBeta, opdigamma, oppolygamma, opabs,
                opBesselJ, opBesselY, opBesselI, opBesselK, opAiryAi,
                 opAiryBi]
    anyop   := [oppren, opdiff, opbox, opquote]
    allop   := concat(concat(concat(concat(concat(
                            algop,elemop),primop),combop),specop),anyop)

-- odd and even operators, must be maintained current!
    evenop := [opcos, opsec, opcosh, opsech, opabs]
    oddop  := [opsin, opcsc, optan, opcot, opasin, opacsc, opatan,
               opsinh, opcsch, optanh, opcoth, opasinh, opacsch,opatanh,opacoth,
```

```
                opsi, operf]

  -- operators whose second argument is a dummy variable
     dummyvarop1 := [opdiff,opalg, opint, opsum, opprod]
  -- operators whose second and third arguments are dummy variables
     dummyvarop2 := [opdint, opdsum, opdprod]

     operator s ==
       if (deref brandNew?) then startUp false
       for op in allop repeat
         is?(op, s) => return copy op
       operator(s)$OP

     dpi l    == "%pi"::Symbol::O
     dfact x  == postfix("!"::Symbol::O, (ATOM(x)$Lisp => x; paren x))
     dquote l == prefix(quote(first(l)::O), rest l)
     dgamma l == prefix(hconcat("|"::Symbol::O, overbar(" "::Symbol::O)), l)
     setDummyVar(op, n) == setProperty(op, DUMMYVAR, n pretend None)

     dexp x ==
       e := "%e"::Symbol::O
       x = 1::O => e
       e ** x

     fsupersub(x:List O):O == supersub("A"::Symbol::O, x)
     fbinomial(x:List O):O == binomial(first x, second x)
     fpower(x:List O):O    == first(x) ** second(x)
     fsum(x:List O):O      == sum(first x, second x, third x)
     fprod(x:List O):O     == prod(first x, second x, third x)

     fint(x:List O):O      ==
        int(first x * hconcat("d"::Symbol::O, second x),empty(), third x)

     fpren(x:List InputForm):InputForm  ==
        convert concat(convert("("::Symbol)@InputForm,
                             concat(x, convert(")"::Symbol)@InputForm))

     fpow(x:List InputForm):InputForm  ==
        convert concat(convert("**"::Symbol)@InputForm, x)

     froot(x:List InputForm):InputForm  ==
        convert [convert("**"::Symbol)@InputForm, first x, 1 / second x]

     startUp b ==
       brandNew?() := b
       display(oppren,   paren)
```

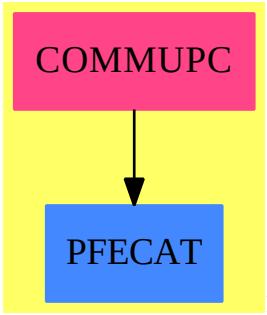
```
        display(opbox,    commaSeparate)
        display(oppi,     dpi)
        display(opexp,    dexp)
        display(opGamma,  dgamma)
        display(opGamma2, dgamma)
        display(opfact,   dfact)
        display(opquote,  dquote)
        display(opperm,   fsupersub)
        display(opbinom,  fbinomial)
        display(oppow,    fpower)
        display(opsum,    fsum)
        display(opprod,   fprod)
        display(opint,    fint)
        input(oppren,     fpren)
        input(oppow,      fpow)
        input(oproot,     froot)
        for op in algop   repeat assert(op, ALGOP)
        for op in rtrigop repeat assert(op, "rtrig")
        for op in htrigop repeat assert(op, "htrig")
        for op in trigop  repeat assert(op, "trig")
        for op in elemop  repeat assert(op, "elem")
        for op in primop  repeat assert(op, "prim")
        for op in combop  repeat assert(op, "comb")
        for op in specop  repeat assert(op, "special")
        for op in anyop   repeat assert(op, "any")
        for op in evenop  repeat assert(op, EVEN)
        for op in oddop   repeat assert(op, ODD)
        for op in dummyvarop1 repeat setDummyVar(op, 1)
        for op in dummyvarop2 repeat setDummyVar(op, 2)
        assert(oppren, "linear")
        void
```

⟨*COMMONOP.dotabb*⟩≡

```
"COMMONOP" [color="#FF4488",href="bookvol10.4.pdf#nameddest=COMMONOP"]
"ALIST" [color="#88FF44",href="bookvol10.3.pdf#nameddest=ALIST"]
"COMMONOP" -> "ALIST"
```

4.19 package COMMUPC CommuteUnivariatePolynomialCategory

4.20 CommuteUnivariatePolynomialCategory



1.00

Exports:
swap

⟨*package COMMUPC CommuteUnivariatePolynomialCategory*⟩≡

```
)abbrev package COMMUPC CommuteUnivariatePolynomialCategory
++ Author: Manuel Bronstein
++ Date Created:
++ Date Last Updated:
++ Basic Functions:
++ Related Constructors:
++ Also See:
++ AMS Classifications:
++ Keywords:
++ References:
++ Description:
++ A package for swapping the order of two variables in a tower of two
++ UnivariatePolynomialCategory extensions.

CommuteUnivariatePolynomialCategory(R, UP, UPUP): Exports == Impl where
  R   : Ring
  UP  : UnivariatePolynomialCategory R
  UPUP: UnivariatePolynomialCategory UP

  N ==> NonNegativeInteger

  Exports ==> with
    swap: UPUP -> UPUP
      ++ swap(p(x,y)) returns p(y,x).

  Impl ==> add
```

```
      makePoly: (UP, N) -> UPUP

  -- converts P(x,y) to P(y,x)
      swap poly ==
        ans:UPUP := 0
        while poly ^= 0 repeat
          ans  := ans + makePoly(leadingCoefficient poly, degree poly)
          poly := reductum poly
        ans

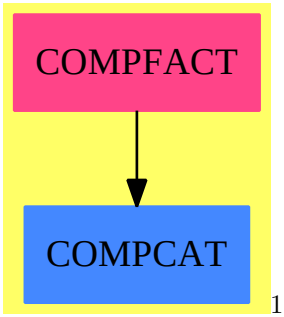
      makePoly(poly, d) ==
        ans:UPUP := 0
        while poly ^= 0 repeat
          ans  := ans +
               monomial(monomial(leadingCoefficient poly, d), degree poly)
          poly := reductum poly
        ans
```

⟨*COMMUPC.dotabb*⟩≡

```
"COMMUPC" [color="#FF4488",href="bookvol10.4.pdf#nameddest=COMMUPC"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"COMMUPC" -> "PFECAT"
```

4.21 package COMPFACT ComplexFactorization

4.22 ComplexFactorization



1.00

Exports:
factor

⟨*package COMPFACT ComplexFactorization*⟩≡

```
)abbrev package COMPFACT ComplexFactorization
++ Author:
++ Date Created:
++ Date Last Updated:
++ Basic Functions:
++ Related Constructors: Complex, UnivariatePolynomial
++ Also See:
++ AMS Classifications:
++ Keywords: complex, polynomial factorization, factor
++ References:
ComplexFactorization(RR,PR) : C == T where
  RR   :    EuclideanDomain   -- R is Z or Q
  PR   :    UnivariatePolynomialCategory Complex RR
  R    ==>  Complex RR
  I    ==>  Integer
  RN   ==>  Fraction I
  GI   ==>  Complex I
  GRN  ==>  Complex RN


  C  == with

     factor        :   PR   ->  Factored PR
       ++ factor(p) factorizes the polynomial p with complex coefficients.

  T  == add
     SUP    ==> SparseUnivariatePolynomial
```

```
fUnion ==> Union("nil", "sqfr", "irred", "prime")
FF     ==> Record(flg:fUnion, fctr:PR, xpnt:Integer)
SAEF   :=  SimpleAlgebraicExtensionAlgFactor(SUP RN,GRN,SUP GRN)
UPCF2  :=  UnivariatePolynomialCategoryFunctions2(R,PR,GRN,SUP GRN)
UPCFB  :=  UnivariatePolynomialCategoryFunctions2(GRN,SUP GRN,R,PR)

myMap(r:R) : GRN ==
  R is GI   =>
    cr :GI := r pretend GI
    complex((real cr)::RN,(imag cr)::RN)
  R is GRN  => r pretend GRN

compND(cc:GRN):Record(cnum:GI,cden:Integer) ==
  ccr:=real cc
  cci:=imag cc
  dccr:=denom ccr
  dcci:=denom cci
  ccd:=lcm(dccr,dcci)
  [complex(((ccd exquo dccr)::Integer)*numer ccr,
           ((ccd exquo dcci)::Integer)*numer cci),ccd]

conv(f:SUP GRN) :Record(convP:SUP GI, convD:RN) ==
  pris:SUP GI :=0
  dris:Integer:=1
  dris1:Integer:=1
  pdris:Integer:=1
  for i in 0..(degree f) repeat
    (cf:= coefficient(f,i)) = 0 => "next i"
    cdf:=compND cf
    dris:=lcm(cdf.cden,dris1)
    pris:=((dris exquo dris1)::Integer)*pris +
          ((dris exquo cdf.cden)::Integer)*
            monomial(cdf.cnum,i)$(SUP GI)
    dris1:=dris
  [pris,dris::RN]

backConv(ffr:Factored SUP GRN) : Factored PR ==
  R is GRN =>
    makeFR((unit ffr) pretend PR,[[f.flg,(f.fctr) pretend PR,f.xpnt]
                                    for f in factorList ffr])
  R is GI  =>
    const:=unit ffr
    ris: List FF :=[]
    for ff in factorList ffr repeat
      fact:=primitivePart(conv(ff.fctr).convP)
      expf:=ff.xpnt
```

```
            ris:=cons([ff.flg,fact pretend PR,expf],ris)
            lc:GRN := myMap leadingCoefficient(fact pretend PR)
            const:= const*(leadingCoefficient(ff.fctr)/lc)**expf
          uconst:GI:= compND(coefficient(const,0)).cnum
          makeFR((uconst pretend R)::PR,ris)

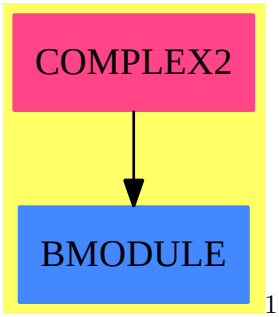

      factor(pol : PR)  : Factored PR ==
        ratPol:SUP GRN := 0
        ratPol:=map(myMap,pol)$UPCF2
        ffr:=factor ratPol
        backConv ffr
```

⟨*COMPFACT.dotabb*⟩≡

```
"COMPFACT" [color="#FF4488",href="bookvol10.4.pdf#nameddest=COMPFACT"]
"COMPCAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=COMPCAT"]
"COMPFACT" -> "COMPCAT"
```

4.23 package COMPLEX2 ComplexFunctions2

4.24 ComplexFunctions2



1.00

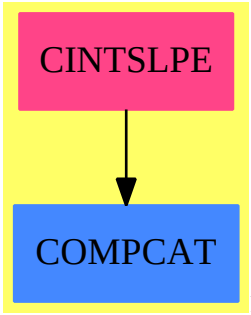
Exports:
map

⟨*package COMPLEX2 ComplexFunctions2*⟩≡
```
)abbrev package COMPLEX2 ComplexFunctions2
++ Author:
++ Date Created:
++ Date Last Updated:
++ Basic Functions:
++ Related Constructors:
++ Also See:
++ AMS Classifications:
++ Keywords:
++ References:
++ Description:
++   This package extends maps from underlying rings to maps between
++   complex over those rings.
ComplexFunctions2(R:CommutativeRing, S:CommutativeRing): with
    map:     (R -> S, Complex R) -> Complex S
      ++ map(f,u) maps f onto real and imaginary parts of u.
 == add
    map(fn, gr) == complex(fn real gr, fn imag gr)
```

⟨*COMPLEX2.dotabb*⟩≡
```
"COMPLEX2" [color="#FF4488",href="bookvol10.4.pdf#nameddest=COMPLEX2"]
"BMODULE" [color="#4488FF",href="bookvol10.2.pdf#nameddest=BMODULE"]
"COMPLEX2" -> "BMODULE"
```

4.25 package CINTSLPE ComplexIntegerSolveLinearPolynomialEquation

4.26 ComplexIntegerSolveLinearPolynomialEquation



1.00

Exports:
solveLinearPolynomialEquation

⟨*package CINTSLPE ComplexIntegerSolveLinearPolynomialEquation*⟩≡

```
)abbrev package CINTSLPE ComplexIntegerSolveLinearPolynomialEquation
++ Author: James Davenport
++ Date Created: 1990
++ Date Last Updated:
++ Basic Functions:
++ Related Constructors:
++ Also See:
++ AMS Classifications:
++ Keywords:
++ References:
++ Description:
++ This package provides the generalized euclidean algorithm which is
++ needed as the basic step for factoring polynomials.
ComplexIntegerSolveLinearPolynomialEquation(R,CR): C == T
 where
  CP ==> SparseUnivariatePolynomial CR
  R:IntegerNumberSystem
  CR:ComplexCategory(R)
  C == with
      solveLinearPolynomialEquation: (List CP,CP) -> Union(List CP,"failed")
                  ++ solveLinearPolynomialEquation([f1, ..., fn], g)
                  ++ where (fi relatively prime to each other)
                  ++ returns a list of ai such that
                  ++ g = sum ai prod fj (j \= i) or
                  ++ equivalently g/prod fj = sum (ai/fi)
                  ++ or returns "failed" if no such list exists
```

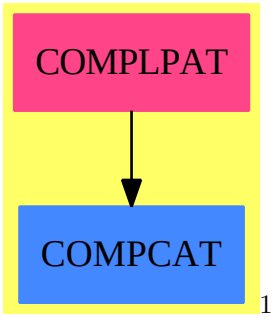
```
T == add
    oldlp:List CP := []
    slpePrime:R:=(2::R)
    oldtable:Vector List CP := empty()
    solveLinearPolynomialEquation(lp,p) ==
       if (oldlp ^= lp) then
          -- we have to generate a new table
          deg:= _+/[degree u for u in lp]
          ans:Union(Vector List CP,"failed"):="failed"
          slpePrime:=67108859::R   -- 2**26 -5 : a prime
               -- a good test case for this package is
               --  (good question?)
          while (ans case "failed") repeat
            ans:=tablePow(deg,complex(slpePrime,0),lp)$GenExEuclid(CR,CP)
            if (ans case "failed") then
               slpePrime:=  slpePrime-4::R
               while not prime?(slpePrime)$IntegerPrimesPackage(R) repeat
                 slpePrime:= slpePrime-4::R
          oldtable:=(ans:: Vector List CP)
       answer:=solveid(p,complex(slpePrime,0),oldtable)
       answer
```

⟨*CINTSLPE.dotabb*⟩≡

```
"CINTSLPE" [color="#FF4488",href="bookvol10.4.pdf#nameddest=CINTSLPE"]
"COMPCAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=COMPCAT"]
"CINTSLPE" -> "COMPCAT"
```

4.27 package COMPLPAT ComplexPattern

4.28 ComplexPattern



1.00

Exports:
convert

⟨*package COMPLPAT ComplexPattern*⟩≡

```
)abbrev package COMPLPAT ComplexPattern
++ Author: Barry Trager
++ Date Created: 30 Nov 1995
++ Date Last Updated:
++ Basic Functions:
++ Related Constructors:
++ Also See:
++ AMS Classifications:
++ Keywords: complex, patterns
++ References:
++ Description:
++ This package supports converting complex expressions to patterns
ComplexPattern(R, S, CS) : C == T where
    R: SetCategory
    S: Join(ConvertibleTo Pattern R, CommutativeRing)
    CS: ComplexCategory S
    C == with
       convert: CS -> Pattern R
          ++ convert(cs) converts the complex expression cs to a pattern

    T == add

       ipat : Pattern R := patternVariable("%i"::Symbol, true, false, false)

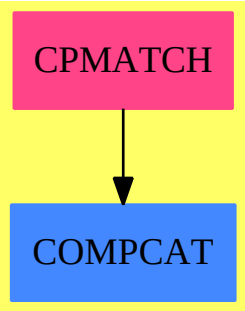
       convert(cs) ==
          zero? imag cs => convert real cs
          convert real cs + ipat * convert imag cs
```

⟨*COMPLPAT.dotabb*⟩≡

```
"COMPLPAT" [color="#FF4488",href="bookvol10.4.pdf#nameddest=COMPLPAT"]
"COMPCAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=COMPCAT"]
"COMPLPAT" -> "COMPCAT"
```

4.29 package CPMATCH ComplexPatternMatch

4.30 ComplexPatternMatch



1.00

Exports:
patternMatch

⟨*package CPMATCH ComplexPatternMatch*⟩≡

```
)abbrev package CPMATCH ComplexPatternMatch
++ Author: Barry Trager
++ Date Created: 30 Nov 1995
++ Date Last Updated:
++ Basic Functions:
++ Related Constructors:
++ Also See:
++ AMS Classifications:
++ Keywords: complex, pattern matching
++ References:
++ Description:
++ This package supports matching patterns involving complex expressions
ComplexPatternMatch(R, S, CS) : C == T where
    R: SetCategory
    S: Join(PatternMatchable R, CommutativeRing)
    CS: ComplexCategory S
    PMRS ==> PatternMatchResult(R, CS)
    PS   ==> Polynomial S
    C == with
       if PS has PatternMatchable(R) then
           patternMatch: (CS, Pattern R, PMRS) -> PMRS
             ++ patternMatch(cexpr, pat, res) matches the pattern pat to the
             ++ complex expression cexpr. res contains the variables of pat
             ++ which are already matched and their matches.

    T == add
```

```
import PatternMatchPushDown(R, S, CS)
import PatternMatchResultFunctions2(R, PS, CS)
import PatternMatchResultFunctions2(R, CS, PS)

ivar : PS := "%i"::Symbol::PS

makeComplex(p:PS):CS ==
   up := univariate p
   degree up > 1 => error "not linear in %i"
   icoef:=leadingCoefficient(up)
   rcoef:=leadingCoefficient(reductum p)
   complex(rcoef,icoef)

makePoly(cs:CS):PS == real(cs)*ivar + imag(cs)::PS

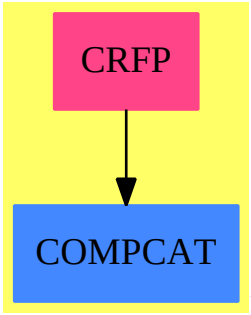
if PS has PatternMatchable(R) then
   patternMatch(cs, pat, result) ==
      zero? imag cs =>
         patternMatch(real cs, pat, result)
      map(makeComplex,
         patternMatch(makePoly cs, pat, map(makePoly, result)))
```

⟨*CPMATCH.dotabb*⟩≡

```
"CPMATCH" [color="#FF4488",href="bookvol10.4.pdf#nameddest=CPMATCH"]
"COMPCAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=COMPCAT"]
"CPMATCH" -> "COMPCAT"
```

4.31 package CRFP ComplexRootFindingPackage

4.32 ComplexRootFindingPackage



1.00

Exports:

complexZeros	divisorCascade	factor	graeffe	norm
pleskenSplit	reciprocalPolynomial	rootRadius	schwerpunkt	setErrorBound
startPolynomial				

⟨*package CRFP ComplexRootFindingPackage*⟩≡

```
)abbrev package CRFP ComplexRootFindingPackage
++ Author: J. Grabmeier
++ Date Created: 31 January 1991
++ Date Last Updated: 12 April 1991
++ Basic Operations: factor, pleskenSplit
++ Related Constructors:
++ Also See:
++ AMS Classifications:
++ Keywords: complex zeros, roots
++ References: J. Grabmeier: On Plesken's root finding algorithm,
++  in preparation
++  A. Schoenhage: The fundamental theorem of algebra in terms of computational
++  complexity, preliminary report, Univ. Tuebingen, 1982
++ Description:
++  \spadtype{ComplexRootFindingPackage} provides functions to
++  find all roots of a polynomial p over the complex number by
++  using Plesken's idea to calculate in the polynomial ring
++  modulo f and employing the Chinese Remainder Theorem.
++  In this first version, the precision (see \spadfunFrom{digits}{Float})
++  is not increased when this is necessary to
++  avoid rounding errors. Hence it is the user's responsibility to
++  increase the precision if necessary.
++  Note also, if this package is called with e.g. \spadtype{Fraction Integer},
++  the precise calculations could require a lot of time.
```

```
++  Also note that evaluating the zeros is not necessarily a good check
++  whether the result is correct: already evaluation can cause
++  rounding errors.
ComplexRootFindingPackage(R, UP): public == private where
   -- R   : Join(Field, OrderedRing, CharacteristicZero)
   -- Float not in CharacteristicZero !|
   R   : Join(Field, OrderedRing)
   UP  : UnivariatePolynomialCategory Complex R

   C      ==> Complex R
   FR     ==> Factored
   I      ==> Integer
   L      ==> List
   FAE    ==> Record(factors : L UP, error : R)
   NNI    ==> NonNegativeInteger
   OF     ==> OutputForm
   ICF    ==> IntegerCombinatoricFunctions(I)

   public ==> with
     complexZeros : UP -> L C
       ++ complexZeros(p) tries to determine all complex zeros
       ++ of the polynomial p with accuracy given by the package
       ++ constant {\em globalEps} which you may change by
       ++ {\em setErrorBound}.
     complexZeros : (UP, R) -> L C
       ++ complexZeros(p, eps) tries to determine all complex zeros
       ++ of the polynomial p with accuracy given by {\em eps}.
     divisorCascade : (UP,UP, Boolean) -> L FAE
       ++ divisorCascade(p,tp) assumes that degree of polynomial {\em tp}
       ++ is smaller than degree of polynomial p, both monic.
       ++ A sequence of divisions are calculated
       ++ using the remainder, made monic, as divisor
       ++ for the the next division. The result contains also the error of the
       ++ factorizations, i.e. the norm of the remainder polynomial.
       ++ If {\em info} is {\em true}, then information messages are issued.
     divisorCascade : (UP,UP) -> L FAE
       ++ divisorCascade(p,tp) assumes that degree of polynomial {\em tp}
       ++ is smaller than degree of polynomial p, both monic.
       ++ A sequence of divisions is calculated
       ++ using the remainder, made monic, as divisor
       ++ for the  the next division. The result contains also the error of the
       ++ factorizations, i.e. the norm of the remainder polynomial.
     factor: (UP,R,Boolean)  ->  FR UP
       ++ factor(p, eps, info) tries to factor p into linear factors
       ++ with error atmost {\em eps}. An overall error bound
       ++ {\em eps0} is determined and iterated tree-like calls
```

```
  ++ to {\em pleskenSplit} are used to get the factorization.
  ++ If {\em info} is {\em true}, then information messages are given.
factor: (UP,R)  ->  FR UP
  ++ factor(p, eps) tries to factor p into linear factors
  ++ with error atmost {\em eps}. An overall error bound
  ++ {\em eps0} is determined and iterated tree-like calls
  ++ to {\em pleskenSplit} are used to get the factorization.
factor: UP  ->  FR UP
  ++ factor(p) tries to factor p into linear factors
  ++ with error atmost {\em globalEps}, the internal error bound,
  ++ which can be set by {\em setErrorBound}. An overall error bound
  ++ {\em eps0} is determined and iterated tree-like calls
  ++ to {\em pleskenSplit} are used to get the factorization.
graeffe : UP -> UP
  ++ graeffe p determines q such that \spad{q(-z**2) = p(z)*p(-z)}.
  ++ Note that the roots of q are the squares of the roots of p.
norm : UP -> R
  ++ norm(p) determines sum of absolute values of coefficients
  ++ Note: this function depends on \spadfunFrom{abs}{Complex}.
pleskenSplit: (UP, R, Boolean)  ->  FR UP
  ++ pleskenSplit(poly,eps,info) determines a start polynomial {\em start}
  ++ by using "startPolynomial then it increases the exponent
  ++ n of {\em start ** n mod poly} to get an approximate factor of
  ++ {\em poly}, in general of degree "degree poly -1". Then a divisor
  ++ cascade is calculated and the best splitting is chosen, as soon
  ++ as the error is small enough.
  --++ In a later version we plan
  --++ to use the whole information to get a split into more than 2
  --++ factors.
  ++ If {\em info} is {\em true}, then information messages are issued.
pleskenSplit: (UP, R)  ->  FR UP
  ++ pleskenSplit(poly, eps)  determines a start polynomial {\em start}\
  ++ by using "startPolynomial then it increases the exponent
  ++ n of {\em start ** n mod poly} to get an approximate factor of
  ++ {\em poly}, in general of degree "degree poly -1". Then a divisor
  ++ cascade is calculated and the best splitting is chosen, as soon
  ++ as the error is small enough.
  --++ In a later version we plan
  --++ to use the whole information to get a split into more than 2
  --++ factors.
reciprocalPolynomial: UP  -> UP
  ++ reciprocalPolynomial(p) calulates a polynomial which has exactly
  ++ the inverses of the non-zero roots of p as roots, and the same
  ++ number of 0-roots.
rootRadius: (UP,R) -> R
  ++ rootRadius(p,errQuot) calculates the root radius of p with a
```

```
    ++ maximal error quotient of {\em errQuot}.
  rootRadius: UP -> R
    ++ rootRadius(p) calculates the root radius of p with a
    ++ maximal error quotient of {\em 1+globalEps}, where
    ++ {\em globalEps} is the internal error bound, which can be
    ++ set by {\em setErrorBound}.
  schwerpunkt: UP ->  C
    ++ schwerpunkt(p) determines the 'Schwerpunkt' of the roots of the
    ++ polynomial p of degree n, i.e. the center of gravity, which is
    ++ {\em coeffient of \spad{x**(n-1)}} divided by
    ++ {\em n times coefficient of \spad{x**n}}.
  setErrorBound : R -> R
    ++ setErrorBound(eps) changes the internal error bound,
    -- by default being {\em 10 ** (-20)} to eps, if R is
    ++ by default being {\em 10 ** (-3)} to eps, if R is
    ++ a member in the category \spadtype{QuotientFieldCategory Integer}.
    ++ The internal {\em globalDigits} is set to
    -- {\em ceiling(1/r)**2*10} being {\em 10**41} by default.
    ++ {\em ceiling(1/r)**2*10} being {\em 10**7} by default.
  startPolynomial: UP  -> Record(start: UP, factors: FR UP)
    ++ startPolynomial(p) uses the ideas of Schoenhage's
    ++ variant of Graeffe's method to construct circles which separate
    ++ roots to get a good start polynomial, i.e. one whose
    ++ image under the Chinese Remainder Isomorphism has both entries
    ++ of norm smaller and greater or equal to 1. In case the
    ++ roots are found during internal calculations.
    ++ The corresponding factors
    ++ are in {\em factors} which are otherwise 1.

private ==> add


  Rep := ModMonic(C, UP)

  -- constants
  c : C
  r : R
  --globalDigits : I := 10 ** 41
  globalDigits : I := 10 ** 7
  globalEps : R :=
    --a : R := (10000000000000000000 :: I) :: R
    a : R := (1000 :: I) :: R
    1/a
  emptyLine : OF := "  "
  dashes : OF := center "------------------------------------------------"
  dots : OF :=   center "................................................"
```

```
one : R := 1$R
two : R := 2 * one
ten : R := 10 * one
eleven : R := 11 * one
weakEps := eleven/ten
--invLog2 : R := 1/log10 (2*one)

-- signatures of local functions

absC : C -> R
  --
absR : R -> R
  --
calculateScale : UP -> R
  --
makeMonic : UP -> UP
  -- 'makeMonic p' divides 'p' by the leading coefficient,
  -- to guarantee new leading coefficient to be 1$R  we cannot
  -- simply divide the leading monomial by the leading coefficient
  -- because of possible rounding errors
min: (FAE, FAE) -> FAE
  -- takes factorization with smaller error
nthRoot : (R, NNI) -> R
  -- nthRoot(r,n) determines an approximation to the n-th
  -- root of r, if \spadtype{R} has {\em ?**?: (R,Fraction Integer)->R}
  -- we use this, otherwise we use {\em approxNthRoot} via
  -- \spadtype{Integer}
shift: (UP,C) ->  UP
  -- shift(p,c) changes p(x) into p(x+c), thereby modifying the
  -- roots u_j of p to the roots (u_j - c)  of shift(p,c)
scale: (UP,C) -> UP
  -- scale(p,c) changes p(x) into p(cx), thereby modifying the
  -- roots u_j of p to the roots ((1/c) u_j)  of scale(p,c)


-- implementation of exported functions


complexZeros(p,eps) ==
  --r1 : R := rootRadius(p,weakEps)
  --eps0 : R = r1 * nthRoot(eps, degree p)
  -- right now we are content with
  eps0 : R := eps/(ten ** degree p)
  facs : FR UP := factor(p,eps0)
  [-coefficient(linfac.factor,0) for linfac in factors facs]
```

```
      complexZeros p == complexZeros(p,globalEps)
      setErrorBound r ==
        r <= 0 => error "setErrorBound: need error bound greater 0"
        globalEps := r
        if R has QuotientFieldCategory Integer then
          rd : Integer := ceiling(1/r)
          globalDigits := rd * rd * 10
          lof : List OF := _
            ["setErrorBound: internal digits set to",globalDigits::OF]
          print hconcat lof
        messagePrint  "setErrorBound: internal error bound set to"
        globalEps

      pleskenSplit(poly,eps,info) ==
        p := makeMonic poly
        fp : FR UP
        if not zero? (md := minimumDegree p) then
          fp : FR UP := irreducibleFactor(monomial(1,1)$UP,md)$(FR UP)
          p := p quo monomial(1,md)$UP
        sP : Record(start: UP, factors: FR UP) := startPolynomial p
        fp : FR UP := sP.factors
--      if not one? fp then
        if not (fp = 1) then
          qr: Record(quotient: UP, remainder: UP):= divide(p,makeMonic expand fp)
          p := qr.quotient
        st := sP.start
        zero? degree st => fp
          -- we calculate in ModMonic(C, UP),
          -- next line defines the polynomial, which is used for reducing
        setPoly p
        nm : R := eps
        split : FAE
        sR : Rep := st :: Rep
        psR : Rep := sR ** (degree poly)

        notFoundSplit : Boolean := true
        while notFoundSplit repeat
        --  if info then
        --    lof : L OF := ["not successfull, new exponent:", nn::OF]
        --    print hconcat lof
          psR := psR * psR * sR   -- exponent (2*d +1)
          -- be careful, too large exponent results in rounding errors
          -- tp is the first approximation of a divisor of poly:
          tp : UP  := lift psR
          zero? degree tp  =>
            if info then print "we leave as we got constant factor"
```

```
          nilFactor(poly,1)$(FR UP)
        -- this was the case where we don't find a non-trivial factorization
        -- we refine tp by repeated polynomial division and hope that
        -- the norm of the remainder gets small  from time to time
        splits : L FAE :=  divisorCascade(p, makeMonic tp, info)
        split := reduce(min,splits)
        notFoundSplit := (eps <=  split.error)

      for fac in split.factors repeat
        fp :=
--          one? degree fac => fp * nilFactor(fac,1)$(FR UP)
          (degree fac = 1) => fp * nilFactor(fac,1)$(FR UP)
          fp * irreducibleFactor(fac,1)$(FR UP)
      fp

    startPolynomial p == -- assume minimumDegree is 0
      --print (p :: OF)
      fp : FR UP := 1
--      one? degree p =>
      (degree p = 1) =>
        p := makeMonic p
        [p,irreducibleFactor(p,1)]
      startPoly : UP := monomial(1,1)$UP
      eps : R := weakEps   -- 10 per cent errors allowed
      r1 : R := rootRadius(p, eps)
      rd : R := 1/rootRadius(reciprocalPolynomial p, eps)
      (r1 > (2::R)) and (rd < 1/(2::R)) => [startPoly,fp] -- unit circle splitti
      -- otherwise the norms of the roots are too closed so we
      -- take the center of gravity as new origin:
      u  : C := schwerpunkt p
      startPoly := startPoly-monomial(u,0)
      p := shift(p,-u)
      -- determine new rootRadius:
      r1 : R := rootRadius(p, eps)
      startPoly := startPoly/(r1::C)
      -- use one of the 4 points r1*zeta, where zeta is a 4th root of unity
      -- as new origin, this could be changed to an arbitrary list
      -- of elements of norm 1.
      listOfCenters : L C := [complex(r1,0), complex(0,r1), _
        complex(-r1,0), complex(0,-r1)]
      lp   : L UP := [shift(p,v) for v in listOfCenters]
      -- next we check if one of these centers is a root
      centerIsRoot : Boolean := false
      for i in 1..maxIndex lp repeat
        if (mD := minimumDegree lp.i) > 0 then
          pp : UP := monomial(1,1)-monomial(listOfCenters.i-u,0)
```

```
      centerIsRoot := true
      fp := fp * irreducibleFactor(pp,mD)
  centerIsRoot =>
    p := shift(p,u) quo expand fp
    --print (p::OF)
    zero? degree p => [p,fp]
    sP:= startPolynomial(p)
    [sP.start,fp]
  -- choose the best one w.r.t. maximal quotient of norm of largest
  -- root and norm of smallest root
  lpr1 : L R := [rootRadius(q,eps) for  q in lp]
  lprd : L R := [1/rootRadius(reciprocalPolynomial q,eps) for  q in lp]
  -- later we should check here of an rd is smaller than globalEps
  lq : L R := []
  for i in 1..maxIndex lpr1 repeat
    lq := cons(lpr1.i/lprd.i, lq)
  --lq : L R := [(l/s)::R for l in lpr1 for s in lprd])
  lq := reverse lq
  po := position(reduce(max,lq),lq)
  --p := lp.po
  --lrr : L R := [rootRadius(p,i,1+eps) for i in 2..(degree(p)-1)]
  --lrr := concat(concat(lpr1.po,lrr),lprd.po)
  --lu : L R := [(lrr.i + lrr.(i+1))/2 for i in 1..(maxIndex(lrr)-1)]
  [startPoly - monomial(listOfCenters.po,0),fp]

norm p ==
 -- reduce(_+$R,map(absC,coefficients p))
 nm : R := 0
 for c in  coefficients p repeat
   nm := nm + absC c
 nm

pleskenSplit(poly,eps) == pleskenSplit(poly,eps,false)

graeffe p ==
  -- If  p = ao x**n + a1 x**(n-1) + ... + a<n-1> x + an
  -- and q = bo x**n + b1 x**(n-1) + ... + b<n-1> x + bn
  -- are such that q(-x**2) = p(x)p(-x), then
  -- bk := ak**2 + 2 * ((-1) * a<k-1>*a<k+1> + ... +
  --                    (-1)**l * a<l>*a<l>) where l = min(k, n-k).
  -- graeffe(p) constructs q using these identities.
  n   : NNI  := degree p
  aForth : L C := []
  for k in 0..n repeat  --  aForth = [a0, a1, ..., a<n-1>, an]
    aForth := cons(coefficient(p, k::NNI), aForth)
  aBack  : L C := [] --  after k steps
```

```
                        --  aBack = [ak, a<k-1>, ..., a1, a0]
  gp : UP := 0$UP
  for k in 0..n repeat
    ak : C := first aForth
    aForth := rest aForth
    aForthCopy : L C := aForth  -- we iterate over aForth and
    aBackCopy  : L C := aBack   -- aBack but do not want to
                                -- destroy them
    sum        :   C := 0
    const : I  := -1  --  after i steps const = (-1)**i
    for aminus in aBack for aplus in aForth repeat
      -- after i steps aminus = a<k-i> and aplus = a<k+i>
      sum := sum + const * aminus * aplus
      aForthCopy := rest aForthCopy
      aBackCopy  := rest aBackCopy
      const := -const
    gp := gp + monomial(ak*ak + 2 * sum, (n-k)::NNI)
    aBack := cons(ak, aBack)
  gp



rootRadius(p,errorQuotient) ==
  errorQuotient <= 1$R =>
    error "rootRadius: second Parameter must be greater than 1"
  pp   : UP  := p
  rho  : R   := calculateScale makeMonic pp
  rR   : R   := rho
  pp := makeMonic scale(pp,complex(rho,0$R))
  expo : NNI := 1
  d    : NNI := degree p
  currentError:  R   := nthRoot(2::R, 2)
  currentError     := d*20*currentError
  while nthRoot(currentError, expo) >= errorQuotient repeat
    -- if info then print (expo :: OF)
    pp := graeffe pp
    rho := calculateScale pp
    expo := 2 * expo
    rR := nthRoot(rho, expo) * rR
    pp :=  makeMonic scale(pp,complex(rho,0$R))
  rR

rootRadius(p) == rootRadius(p, 1+globalEps)

schwerpunkt p ==
  zero? p => 0$C
```

```
    zero? (d := degree p) => error _
    "schwerpunkt: non-zero const. polynomial has no roots and no schwerpunkt"
    -- coeffient of x**d and x**(d-1)
    lC : C :=  coefficient(p,d)  -- ^= 0
    nC : C :=  coefficient(p,(d-1) pretend NNI)
    (denom := recip ((d::I::C)*lC)) case "failed" => error  "schwerpunkt: _
      degree * leadingCoefficient not invertible in ring of coefficients"
    - (nC*(denom::C))

  reciprocalPolynomial p ==
    zero? p => 0
    d : NNI := degree p
    md : NNI := d+minimumDegree p
    lm : L UP := [monomial(coefficient(p,i),(md-i) :: NNI) for i in 0..d]
    sol := reduce(_+, lm)

  divisorCascade(p, tp, info) ==
    lfae : L FAE :=  nil()
    for i in 1..degree tp while (degree tp > 0)  repeat
      -- USE monicDivide !!!
      qr  : Record(quotient: UP, remainder: UP)  :=  divide(p,tp)
      factor1 : UP := tp
      factor2 : UP := makeMonic qr.quotient
      -- refinement of tp:
      tp := qr.remainder
      nm : R := norm tp
      listOfFactors  : L UP := cons(factor2,nil()$(L UP))
      listOfFactors := cons(factor1,listOfFactors)
      lfae := cons( [listOfFactors,nm], lfae)
      if info then
        --lof : L OF :=  [i :: OF,"-th division:"::OF]
        --print center box hconcat lof
        print emptyLine
        lof : L OF :=  ["error polynomial has degree " ::OF,_
          (degree tp)::OF, " and norm " :: OF, nm :: OF]
        print center hconcat lof
        lof : L OF := ["degrees of factors:" ::OF,_
          (degree factor1)::OF,"  ", (degree factor2)::OF]
        print center hconcat lof
    if info then print emptyLine
    reverse lfae

  divisorCascade(p, tp) == divisorCascade(p, tp, false)

  factor(poly,eps) == factor(poly,eps,false)
  factor(p) == factor(p, globalEps)
```

```
      factor(poly,eps,info) ==
        result : FR  UP := coerce monomial(leadingCoefficient poly,0)
        d : NNI := degree poly
        --should be
        --den : R := (d::I)::R * two**(d::Integer) * norm poly
        --eps0 : R := eps / den
        -- for now only
        eps0 : R := eps / (ten*ten)
--        one? d  => irreducibleFactor(poly,1)$(FR UP)
        (d = 1) => irreducibleFactor(poly,1)$(FR UP)
        listOfFactors : L Record(factor: UP,exponent: I) :=_
          list [makeMonic poly,1]
        if info then
          lof : L OF := [dashes,dots,"list of Factors:",dots,listOfFactors::OF, _
            dashes, "list of Linear Factors:", dots, result::OF, _
            dots,dashes]
          print vconcat lof
        while not null listOfFactors  repeat
          p : UP := (first listOfFactors).factor
          exponentOfp : I := (first listOfFactors).exponent
          listOfFactors := rest listOfFactors
          if info then
            lof : L OF := ["just now we try to split the polynomial:",p::OF]
            print vconcat lof
          split : FR UP  := pleskenSplit(p, eps0, info)
--          one? numberOfFactors split =>
          (numberOfFactors split = 1) =>
            -- in a later version we will change error bound and
            -- accuracy here to deal this case as well
            lof : L OF := ["factor: couldn't split factor",_
              center(p :: OF), "with required error bound"]
            print vconcat lof
            result := result * nilFactor(p, exponentOfp)
          -- now we got 2 good factors of p, we drop p and continue
          -- with the factors, if they are not linear, or put a
          -- linear factor to the result
          for rec in factors(split)$(FR UP) repeat
            newFactor : UP := rec.factor
            expOfFactor := exponentOfp * rec.exponent
--            one? degree newFactor =>
            (degree newFactor = 1) =>
              result := result * nilFactor(newFactor,expOfFactor)
            listOfFactors:=cons([newFactor,expOfFactor],_
              listOfFactors)
        result
```

```
-- implementation of local functions

absC c == nthRoot(norm(c)$C,2)
absR r ==
  r < 0 => -r
  r
min(fae1,fae2) ==
  fae2.error <  fae1.error => fae2
  fae1
calculateScale p ==
  d  := degree p
  maxi :R := 0
  for j in 1..d for cof in rest coefficients p repeat
    -- here we need abs: R -> R
    rc :  R := absR real cof
    ic :  R := absR imag cof
    locmax: R := max(rc,ic)
    maxi := max( nthRoot( locmax/(binomial(d,j)$ICF::R), j), maxi)
  -- Maybe I should use some type of logarithm for the following:
  maxi = 0$R => error("Internal Error: scale cannot be 0")
  rho  :R := one
  rho < maxi =>
    while rho < maxi repeat rho := ten * rho
    rho / ten
  while maxi < rho repeat rho := rho / ten
  rho = 0 => one
  rho
makeMonic p  ==
  p = 0 => p
  monomial(1,degree p)$UP + (reductum p)/(leadingCoefficient p)

scale(p, c) ==
  -- eval(p,cx) is missing !!
  eq : Equation UP := equation(monomial(1,1), monomial(c,1))
  eval(p,eq)
  -- improvement?: direct calculation of the new coefficients

shift(p,c) ==
  rhs : UP := monomial(1,1) + monomial(c,0)
  eq : Equation UP := equation(monomial(1,1), rhs)
  eval(p,eq)
  -- improvement?: direct calculation of the new coefficients

nthRoot(r,n) ==
  R has RealNumberSystem =>  r ** (1/n)
```

```
        R has QuotientFieldCategory Integer =>
          den : I := approxNthRoot(globalDigits * denom r ,n)$IntegerRoots(I)
          num : I := approxNthRoot(globalDigits * numer r ,n)$IntegerRoots(I)
          num/den
        -- the following doesn't compile
        --R has coerce: % -> Fraction Integer =>
        --  q : Fraction Integer := coerce(r)@Fraction(Integer)
        --  den : I := approxNthRoot(globalDigits * denom q ,n)$IntegerRoots(I)
        --  num : I := approxNthRoot(globalDigits * numer q ,n)$IntegerRoots(I)
        --  num/den
        r -- this is nonsense, perhaps a Newton iteration for x**n-r here

  )fin
      -- for late use:

      graeffe2 p ==
        -- substitute x by -x :
        eq : Equation UP := equation(monomial(1,1), monomial(-1$C,1))
        pp : UP := p*eval(p,eq)
        gp : UP :=  0$UP
        while pp ^= 0 repeat
           i:NNI := (degree pp) quo (2::NNI)
           coef:C:=
             even? i => leadingCoefficient pp
             - leadingCoefficient pp
           gp    := gp + monomial(coef,i)
           pp    := reductum pp
        gp
      shift2(p,c) ==
        d := degree p
        cc : C := 1
        coef := List C := [cc := c * cc for i in 1..d]
        coef := cons(1,coef)
        coef := [coefficient(p,i)*coef.(1+i) for i in 0..d]
        res : UP := 0
        for j in 0..d repeat
          cc := 0
          for i in j..d repeat
            cc := cc + coef.i * (binomial(i,j)$ICF :: R)
          res := res + monomial(cc,j)$UP
        res
      scale2(p,c) ==
        d := degree p
        cc : C := 1
        coef := List C := [cc := c * cc for i in 1..d]
        coef := cons(1,coef)
```

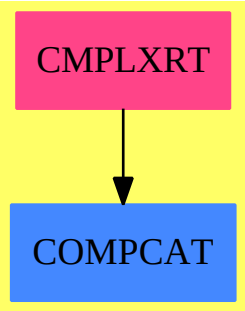
```
    coef := [coefficient(p,i)*coef.(i+1) for i in 0..d]
    res : UP := 0
    for i in 0..d repeat  res := res + monomial(coef.(i+1),i)$UP
    res
  scale2: (UP,C) -> UP
  shift2: (UP,C) ->  UP
  graeffe2 : UP -> UP
    ++ graeffe2 p determines q such that \spad{q(-z**2) = p(z)*p(-z)}.
    ++ Note that the roots of q are the squares of the roots of p.
```

⟨*CRFP.dotabb*⟩≡

```
"CRFP" [color="#FF4488",href="bookvol10.4.pdf#nameddest=CRFP"]
"COMPCAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=COMPCAT"]
"CRFP" -> "COMPCAT"
```

4.33 package CMPLXRT ComplexRootPackage

4.34 ComplexRootPackage



1.00

Exports:
complexZeros

⟨*package CMPLXRT ComplexRootPackage*⟩≡

```
)abbrev package CMPLXRT ComplexRootPackage
++ Author: P. Gianni
++ Date Created:
++ Date Last Updated:
++ Basic Functions:
++ Related Constructors: Complex, Float, Fraction, UnivariatePolynomial
++ Also See:
++ AMS Classifications:
++ Keywords:
++ References:
++ Description:
++   This package provides functions complexZeros
++ for finding the complex zeros
++ of univariate polynomials with complex rational number coefficients.
++ The results are to any user specified precision and are returned
++ as either complex rational number or complex floating point numbers
++ depending on the type of the second argument which specifies the
++ precision.

-- Packages for the computation of complex roots of
-- univariate polynomials with rational or gaussian coefficients.

-- Simplified version, the old original based on Gebauer's solver is
-- in ocmplxrt spad
RN     ==> Fraction Integer
I      ==> Integer
NF     ==> Float
```

```
ComplexRootPackage(UP,Par) : T == C where
   UP   :   UnivariatePolynomialCategory Complex Integer
   Par  :   Join(Field, OrderedRing) -- will be Float or RN
   CP   ==> Complex Par
   PCI  ==> Polynomial Complex Integer

   T == with
        complexZeros:(UP,Par)  -> List CP
          ++ complexZeros(poly, eps) finds the complex zeros of the
          ++ univariate polynomial poly to precision eps with
          ++ solutions returned as complex floats or rationals
          ++ depending on the type of eps.

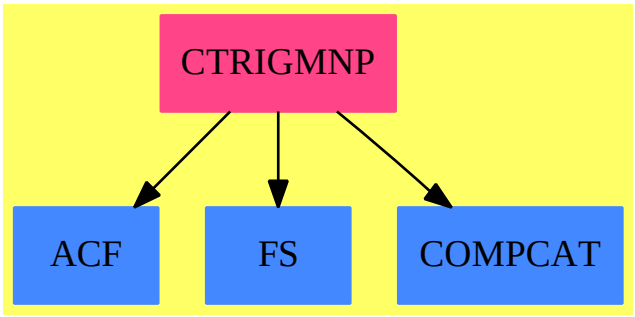
   C == add
    complexZeros(p:UP,eps:Par):List CP ==
      x1:Symbol():=new()
      x2:Symbol():=new()
      vv:Symbol():=new()
      lpf:=factors factor(p)$ComplexFactorization(I,UP)
      ris:List CP:=empty()
      for pf in lpf repeat
          pp:=pf.factor pretend SparseUnivariatePolynomial Complex Integer
          q:PCI :=multivariate(pp,vv)
          q:=eval(q,vv,x1::PCI+complex(0,1)*(x2::PCI))
          p1:=map(real,q)$PolynomialFunctions2(Complex I,I)
          p2:=map(imag,q)$PolynomialFunctions2(Complex I,I)
          lz:=innerSolve([p1,p2],[],[x1,x2],
                         eps)$InnerNumericFloatSolvePackage(I,Par,Par)
          ris:=append([complex(first z,second z) for z in lz],ris)
      ris
```

⟨*CMPLXRT.dotabb*⟩≡

```
"CMPLXRT" [color="#FF4488",href="bookvol10.4.pdf#nameddest=CMPLXRT"]
"COMPCAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=COMPCAT"]
"CMPLXRT" -> "COMPCAT"
```

4.35 package CTRIGMNP ComplexTrigonometricManipulations

4.36 ComplexTrigonometricManipulations



1.00

Exports:

complexElementary	complexForm	complexNormalize	imag
real	trigs		

⟨*package CTRIGMNP ComplexTrigonometricManipulations*⟩≡

```
)abbrev package CTRIGMNP ComplexTrigonometricManipulations
++ Real and Imaginary parts of complex functions
++ Author: Manuel Bronstein
++ Date Created: 11 June 1993
++ Date Last Updated: 14 June 1993
++ Description:
++   \spadtype{ComplexTrigonometricManipulations} provides function that
++   compute the real and imaginary parts of complex functions.
++ Keywords: complex, function, manipulation.
ComplexTrigonometricManipulations(R, F): Exports == Implementation where
  R : Join(IntegralDomain, OrderedSet, RetractableTo Integer)
  F : Join(AlgebraicallyClosedField, TranscendentalFunctionCategory,
           FunctionSpace Complex R)


  SY  ==> Symbol
  FR  ==> Expression R
  K   ==> Kernel F


  Exports ==> with
    complexNormalize: F -> F
      ++ complexNormalize(f) rewrites \spad{f} using the least possible number
      ++ of complex independent kernels.
    complexNormalize: (F, SY) -> F
```

```
    ++ complexNormalize(f, x) rewrites \spad{f} using the least possible
    ++ number of complex independent kernels involving \spad{x}.
  complexElementary: F -> F
    ++ complexElementary(f) rewrites \spad{f} in terms of the 2 fundamental
    ++ complex transcendental elementary functions: \spad{log, exp}.
  complexElementary: (F, SY) -> F
    ++ complexElementary(f, x) rewrites the kernels of \spad{f} involving
    ++ \spad{x} in terms of the 2 fundamental complex
    ++ transcendental elementary functions: \spad{log, exp}.
  real   : F -> FR
    ++ real(f) returns the real part of \spad{f} where \spad{f} is a complex
    ++ function.
  imag   : F -> FR
    ++ imag(f) returns the imaginary part of \spad{f} where \spad{f}
    ++ is a complex function.
  real?  : F -> Boolean
    ++ real?(f) returns \spad{true} if \spad{f = real f}.
  trigs  : F -> F
    ++ trigs(f) rewrites all the complex logs and exponentials
    ++ appearing in \spad{f} in terms of trigonometric functions.
  complexForm: F -> Complex FR
    ++ complexForm(f) returns \spad{[real f, imag f]}.

Implementation ==> add
  import InnerTrigonometricManipulations(R, FR, F)
  import ElementaryFunctionStructurePackage(Complex R, F)

  rreal?: Complex R -> Boolean
  kreal?: Kernel F -> Boolean
  localexplogs  : (F, F, List SY) -> F

  real f        == real complexForm f
  imag f        == imag complexForm f
  rreal? r      == zero? imag r
  kreal? k      == every?(real?, argument k)$List(F)
  complexForm f == explogs2trigs f

  trigs f ==
    GF2FG explogs2trigs f

  real? f ==
    every?(rreal?, coefficients numer f)
      and every?(rreal?, coefficients denom f) and every?(kreal?, kernels f)

  localexplogs(f, g, lx) ==
    trigs2explogs(g, [k for k in tower f
```

```
                          | is?(k, "tan"::SY) or is?(k, "cot"::SY)], lx)

    complexElementary f ==
      any?(x +-> has?(x, "rtrig"),
        operators(g := realElementary f))$List(BasicOperator) =>
          localexplogs(f, g, variables g)
      g

    complexElementary(f, x) ==
      any?(y +-> has?(operator y, "rtrig"),
       [k for k in tower(g := realElementary(f, x))
                 | member?(x, variables(k::F))]$List(K))$List(K) =>
                     localexplogs(f, g, [x])
      g

    complexNormalize(f, x) ==
      any?(y +-> has?(operator y, "rtrig"),
       [k for k in tower(g := realElementary(f, x))
                | member?(x, variables(k::F))]$List(K))$List(K) =>
                   (rischNormalize(localexplogs(f, g, [x]), x).func)
      rischNormalize(g, x).func

    complexNormalize f ==
      l := variables(g := realElementary f)
      any?(y +-> has?(y, "rtrig"), operators g)$List(BasicOperator) =>
        h := localexplogs(f, g, l)
        for x in l repeat h := rischNormalize(h, x).func
        h
      for x in l repeat g := rischNormalize(g, x).func
      g
```

⟨*CTRIGMNP.dotabb*⟩≡

```
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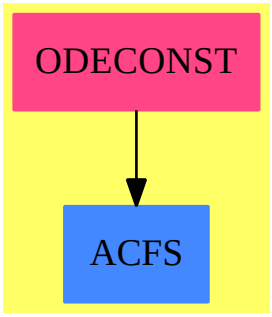
"ACF" [color="#4488FF",href="bookvol10.2.pdf#nameddest=ACF"]
"FS" [color="#4488FF",href="bookvol10.2.pdf#nameddest=FS"]
"COMPCAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=COMPCAT"]
"CTRIGMNP" -> "ACF"
"CTRIGMNP" -> "FS"
"CTRIGMNP" -> "COMPCAT"
```

4.37 package ODECONST ConstantLODE

4.38 ConstantLODE



1.00

Exports:
constDsolve

⟨*package ODECONST ConstantLODE*⟩≡

```
)abbrev package ODECONST ConstantLODE
++ Author: Manuel Bronstein
++ Date Created: 18 March 1991
++ Date Last Updated: 3 February 1994
++ Description: Solution of linear ordinary differential equations, constant coefficient cas
ConstantLODE(R, F, L): Exports == Implementation where
  R: Join(OrderedSet, EuclideanDomain, RetractableTo Integer,
          LinearlyExplicitRingOver Integer, CharacteristicZero)
  F: Join(AlgebraicallyClosedFunctionSpace R,
          TranscendentalFunctionCategory, PrimitiveFunctionCategory)
  L: LinearOrdinaryDifferentialOperatorCategory F

  Z   ==> Integer
  SY  ==> Symbol
  K   ==> Kernel F
  V   ==> Vector F
  M   ==> Matrix F
  SUP ==> SparseUnivariatePolynomial F

  Exports ==> with
    constDsolve: (L, F, SY) -> Record(particular:F, basis:List F)
      ++ constDsolve(op, g, x) returns \spad{[f, [y1,...,ym]]}
      ++ where f is a particular solution of the equation \spad{op y = g},
      ++ and the \spad{yi}'s form a basis for the solutions of \spad{op y = 0}.

  Implementation ==> add
    import ODETools(F, L)
```

```
    import ODEIntegration(R, F)
    import ElementaryFunctionSign(R, F)
    import AlgebraicManipulations(R, F)
    import FunctionSpaceIntegration(R, F)
    import FunctionSpaceUnivariatePolynomialFactor(R, F, SUP)

    homoBasis: (L, F) -> List F
    quadSol  : (SUP, F) -> List F
    basisSqfr: (SUP, F) -> List F
    basisSol : (SUP, Z, F) -> List F

    constDsolve(op, g, x) ==
      b := homoBasis(op, x::F)
      [particularSolution(op, g, b, (f1:F):F +-> int(f1, x))::F, b]

    homoBasis(op, x) ==
      p:SUP := 0
      while op ^= 0 repeat
          p  := p + monomial(leadingCoefficient op, degree op)
          op := reductum op
      b:List(F) := empty()
      for ff in factors ffactor p repeat
        b := concat_!(b, basisSol(ff.factor, dec(ff.exponent), x))
      b

    basisSol(p, n, x) ==
      l := basisSqfr(p, x)
      zero? n => l
      ll := copy l
      xn := x::F
      for i in 1..n repeat
        l := concat_!(l, [xn * f for f in ll])
        xn := x * xn
      l

    basisSqfr(p, x) ==
--      one?(d := degree p) =>
      ((d := degree p) = 1) =>
        [exp(- coefficient(p, 0) * x / leadingCoefficient p)]
      d = 2 => quadSol(p, x)
      [exp(a * x) for a in rootsOf p]

    quadSol(p, x) ==
      (u := sign(delta := (b := coefficient(p, 1))**2 - 4 *
        (a := leadingCoefficient p) * (c := coefficient(p, 0))))
          case Z and negative?(u::Z) =>
```

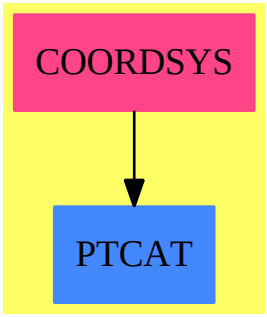
```
            y := x / (2 * a)
            r := - b * y
            i := rootSimp(sqrt(-delta)) * y
            [exp(r) * cos(i), exp(r) * sin(i)]
        [exp(a * x) for a in zerosOf p]
```

⟨*ODECONST.dotabb*⟩≡

```
"ODECONST" [color="#FF4488",href="bookvol10.4.pdf#nameddest=ODECONST"]
"ACFS" [color="#4488FF",href="bookvol10.2.pdf#nameddest=ACFS"]
"ODECONST" -> "ACFS"
```

4.39 package COORDSYS CoordinateSystems

4.40 CoordinateSystems



1.00

Exports:

bipolar	bipolarCylindrical	cartesian	conical	cylindrical
elliptic	ellipticCylindrical	oblateSpheroidal	parabolic	parabolicCylindrical
paraboloidal	polar	prolateSpheroidal	spherical	toroidal

⟨*package COORDSYS CoordinateSystems*⟩≡

```
)abbrev package COORDSYS CoordinateSystems
++ Author: Jim Wen
++ Date Created: 12 March 1990
++ Date Last Updated: 19 June 1990, Clifton J. Williamson
++ Basic Operations: cartesian, polar, cylindrical, spherical, parabolic, ellipti
++ parabolicCylindrical, paraboloidal, ellipticCylindrical, prolateSpheroidal,
++ oblateSpheroidal, bipolar, bipolarCylindrical, toroidal, conical
++ Related Constructors:
++ Also See:
++ AMS Classifications:
++ Keywords:
++ References:
++ Description: CoordinateSystems provides coordinate transformation functions
++ for plotting.  Functions in this package return conversion functions
++ which take points expressed in other coordinate systems and return points
++ with the corresponding Cartesian coordinates.

CoordinateSystems(R): Exports == Implementation where

  R : Join(Field,TranscendentalFunctionCategory,RadicalCategory)
  Pt ==> Point R

  Exports ==> with
    cartesian : Pt -> Pt
      ++ cartesian(pt) returns the Cartesian coordinates of point pt.
```

```
polar: Pt -> Pt
  ++ polar(pt) transforms pt from polar coordinates to Cartesian
  ++ coordinates: the function produced will map the point \spad{(r,theta)}
  ++ to \spad{x = r * cos(theta)} , \spad{y = r * sin(theta)}.
cylindrical: Pt -> Pt
  ++ cylindrical(pt) transforms pt from polar coordinates to Cartesian
  ++ coordinates: the function produced will map the point \spad{(r,theta,z)}
  ++ to \spad{x = r * cos(theta)}, \spad{y = r * sin(theta)}, \spad{z}.
spherical: Pt -> Pt
  ++ spherical(pt) transforms pt from spherical coordinates to Cartesian
  ++ coordinates: the function produced will map the point \spad{(r,theta,phi)}
  ++ to \spad{x = r*sin(phi)*cos(theta)}, \spad{y = r*sin(phi)*sin(theta)},
  ++ \spad{z = r*cos(phi)}.
parabolic: Pt -> Pt
  ++ parabolic(pt) transforms pt from parabolic coordinates to Cartesian
  ++ coordinates: the function produced will map the point \spad{(u,v)} to
  ++ \spad{x = 1/2*(u**2 - v**2)}, \spad{y = u*v}.
parabolicCylindrical: Pt -> Pt
  ++ parabolicCylindrical(pt) transforms pt from parabolic cylindrical
  ++ coordinates to Cartesian coordinates: the function produced will
  ++ map the point \spad{(u,v,z)} to \spad{x = 1/2*(u**2 - v**2)},
  ++ \spad{y = u*v}, \spad{z}.
paraboloidal: Pt -> Pt
  ++ paraboloidal(pt) transforms pt from paraboloidal coordinates to
  ++ Cartesian coordinates: the function produced will map the point
  ++ \spad{(u,v,phi)} to \spad{x = u*v*cos(phi)}, \spad{y = u*v*sin(phi)},
  ++ \spad{z = 1/2 * (u**2 - v**2)}.
elliptic: R -> (Pt -> Pt)
  ++ elliptic(a) transforms from elliptic coordinates to Cartesian
  ++ coordinates: \spad{elliptic(a)} is a function which will map the
  ++ point \spad{(u,v)} to \spad{x = a*cosh(u)*cos(v)}, \spad{y = a*sinh(u)*sin(v)}.
ellipticCylindrical: R -> (Pt -> Pt)
  ++ ellipticCylindrical(a) transforms from elliptic cylindrical coordinates
  ++ to Cartesian coordinates: \spad{ellipticCylindrical(a)} is a function
  ++ which will map the point \spad{(u,v,z)} to \spad{x = a*cosh(u)*cos(v)},
  ++ \spad{y = a*sinh(u)*sin(v)}, \spad{z}.
prolateSpheroidal: R -> (Pt -> Pt)
  ++ prolateSpheroidal(a) transforms from prolate spheroidal coordinates to
  ++ Cartesian coordinates: \spad{prolateSpheroidal(a)} is a function
  ++ which will map the point \spad{(xi,eta,phi)} to
  ++ \spad{x = a*sinh(xi)*sin(eta)*cos(phi)}, \spad{y = a*sinh(xi)*sin(eta)*sin(phi)},
  ++ \spad{z = a*cosh(xi)*cos(eta)}.
oblateSpheroidal: R -> (Pt -> Pt)
  ++ oblateSpheroidal(a) transforms from oblate spheroidal coordinates to
  ++ Cartesian coordinates: \spad{oblateSpheroidal(a)} is a function which
  ++ will map the point \spad{(xi,eta,phi)} to \spad{x = a*sinh(xi)*sin(eta)*cos(phi)},
```

```
    ++ \spad{y = a*sinh(xi)*sin(eta)*sin(phi)}, \spad{z = a*cosh(xi)*cos(eta)}.
  bipolar: R -> (Pt -> Pt)
    ++ bipolar(a) transforms from bipolar coordinates to Cartesian coordinates:
    ++ \spad{bipolar(a)} is a function which will map the point \spad{(u,v)} to
    ++ \spad{x = a*sinh(v)/(cosh(v)-cos(u))}, \spad{y = a*sin(u)/(cosh(v)-cos(u
  bipolarCylindrical: R -> (Pt -> Pt)
    ++ bipolarCylindrical(a) transforms from bipolar cylindrical coordinates
    ++ to Cartesian coordinates: \spad{bipolarCylindrical(a)} is a function whi
    ++ will map the point \spad{(u,v,z)} to \spad{x = a*sinh(v)/(cosh(v)-cos(u)
    ++ \spad{y = a*sin(u)/(cosh(v)-cos(u))}, \spad{z}.
  toroidal: R -> (Pt -> Pt)
    ++ toroidal(a) transforms from toroidal coordinates to Cartesian
    ++ coordinates: \spad{toroidal(a)} is a function which will map the point
    ++ \spad{(u,v,phi)} to \spad{x = a*sinh(v)*cos(phi)/(cosh(v)-cos(u))},
    ++ \spad{y = a*sinh(v)*sin(phi)/(cosh(v)-cos(u))}, \spad{z = a*sin(u)/(cosh
  conical: (R,R) -> (Pt -> Pt)
    ++ conical(a,b) transforms from conical coordinates to Cartesian coordinate
    ++ \spad{conical(a,b)} is a function which will map the point \spad{(lambda
    ++ \spad{x = lambda*mu*nu/(a*b)},
    ++ \spad{y = lambda/a*sqrt((mu**2-a**2)*(nu**2-a**2)/(a**2-b**2))},
    ++ \spad{z = lambda/b*sqrt((mu**2-b**2)*(nu**2-b**2)/(b**2-a**2))}.

Implementation ==> add

  cartesian pt ==
    -- we just want to interpret the cartesian coordinates
    -- from the first N elements of the point - so the
    -- identity function will do
    pt

  polar pt0 ==
    pt := copy pt0
    r := elt(pt0,1); theta := elt(pt0,2)
    pt.1 := r * cos(theta); pt.2 := r * sin(theta)
    pt

  cylindrical pt0 == polar pt0
  -- apply polar transformation to first 2 coordinates

  spherical pt0 ==
    pt := copy pt0
    r := elt(pt0,1); theta := elt(pt0,2); phi := elt(pt0,3)
    pt.1 := r * sin(phi) * cos(theta); pt.2 := r * sin(phi) * sin(theta)
    pt.3 := r * cos(phi)
    pt
```

```
parabolic pt0 ==
  pt := copy pt0
  u := elt(pt0,1); v := elt(pt0,2)
  pt.1 := (u*u - v*v)/(2::R) ; pt.2 := u*v
  pt

parabolicCylindrical pt0 == parabolic pt0
-- apply parabolic transformation to first 2 coordinates

paraboloidal pt0 ==
  pt := copy pt0
  u := elt(pt0,1); v := elt(pt0,2); phi := elt(pt0,3)
  pt.1 := u*v*cos(phi); pt.2 := u*v*sin(phi); pt.3 := (u*u - v*v)/(2::R)
  pt

elliptic a ==
 x+->
  pt := copy(x)
  u := elt(x,1); v := elt(x,2)
  pt.1 := a*cosh(u)*cos(v); pt.2 := a*sinh(u)*sin(v)
  pt

ellipticCylindrical a == elliptic a
-- apply elliptic transformation to first 2 coordinates

prolateSpheroidal a ==
 x+->
  pt := copy(x)
  xi := elt(x,1); eta := elt(x,2); phi := elt(x,3)
  pt.1 := a*sinh(xi)*sin(eta)*cos(phi)
  pt.2 := a*sinh(xi)*sin(eta)*sin(phi)
  pt.3 := a*cosh(xi)*cos(eta)
  pt

oblateSpheroidal a ==
 x+->
  pt := copy(x)
  xi := elt(x,1); eta := elt(x,2); phi := elt(x,3)
  pt.1 := a*sinh(xi)*sin(eta)*cos(phi)
  pt.2 := a*cosh(xi)*cos(eta)*sin(phi)
  pt.3 := a*sinh(xi)*sin(eta)
  pt

bipolar a ==
 x+->
  pt := copy(x)
```

```
    u := elt(x,1); v := elt(x,2)
    pt.1 := a*sinh(v)/(cosh(v)-cos(u))
    pt.2 := a*sin(u)/(cosh(v)-cos(u))
    pt

  bipolarCylindrical a == bipolar a
  -- apply bipolar transformation to first 2 coordinates

  toroidal a ==
   x+->
    pt := copy(x)
    u := elt(x,1); v := elt(x,2); phi := elt(x,3)
    pt.1 := a*sinh(v)*cos(phi)/(cosh(v)-cos(u))
    pt.2 := a*sinh(v)*sin(phi)/(cosh(v)-cos(u))
    pt.3 := a*sin(u)/(cosh(v)-cos(u))
    pt

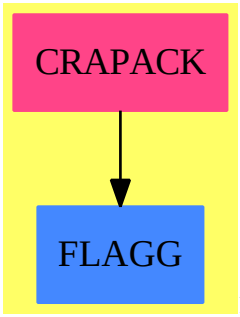
  conical(a,b) ==
   x+->
    pt := copy(x)
    lambda := elt(x,1); mu := elt(x,2); nu := elt(x,3)
    pt.1 := lambda*mu*nu/(a*b)
    pt.2 := lambda/a*sqrt((mu**2-a**2)*(nu**2-a**2)/(a**2-b**2))
    pt.3 := lambda/b*sqrt((mu**2-b**2)*(nu**2-b**2)/(b**2-a**2))
    pt
```

⟨*COORDSYS.dotabb*⟩≡

```
"COORDSYS" [color="#FF4488",href="bookvol10.4.pdf#nameddest=COORDSYS"]
"PTCAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PTCAT"]
"COORDSYS" -> "PTCAT"
```

4.41 package CRAPACK CRApackage

4.42 CRApackage



1.00

Exports:
modTree chineseRemainder multiEuclideanTree

⟨*package CRAPACK CRApackage*⟩≡
```
)abbrev package CRAPACK CRApackage

++ This package \undocumented{}
CRApackage(R:EuclideanDomain): Exports == Implementation where
  Exports == with
    modTree: (R,List R) -> List R
        ++ modTree(r,l) \undocumented{}
    chineseRemainder: (List R, List R) -> R
    ++ chineseRemainder(lv,lm) returns a value \axiom{v} such that, if
    ++ x is \axiom{lv.i} modulo \axiom{lm.i} for all \axiom{i}, then
    ++ x is \axiom{v} modulo \axiom{lm(1)*lm(2)*...*lm(n)}.
    chineseRemainder: (List List R, List R) -> List R
    ++ chineseRemainder(llv,lm) returns a list of values, each of which
    ++ corresponds to the Chinese remainder of the associated element of
    ++ \axiom{llv} and axiom{lm}.  This is more efficient than applying
    ++ chineseRemainder several times.
    multiEuclideanTree: (List R, R) -> List R
        ++ multiEuclideanTree(l,r) \undocumented{}
  Implementation == add

    BB:=BalancedBinaryTree(R)
    x:BB

    -- Definition for modular reduction mapping with several moduli
    modTree(a,lm) ==
      t := balancedBinaryTree(#lm, 0$R)
      setleaves_!(t,lm)
```

```
  mapUp_!(t,"*")
  leaves mapDown_!(t, a, "rem")

chineseRemainder(lv:List(R), lm:List(R)):R ==
  #lm ^= #lv => error "lists of moduli and values not of same length"
  x := balancedBinaryTree(#lm, 0$R)
  x := setleaves_!(x, lm)
  mapUp_!(x,"*")
  y := balancedBinaryTree(#lm, 1$R)
  y := mapUp_!(copy y,x,(a,b,c,d)+->a*d + b*c)
  (u := extendedEuclidean(value y, value x,1)) case "failed" =>
    error "moduli not relatively prime"
  inv := u . coef1
  linv := modTree(inv, lm)
  l := [(u*v) rem m for v in lv for u in linv for m in lm]
  y := setleaves_!(y,l)
  value(mapUp_!(y, x, (a,b,c,d)+->a*d + b*c)) rem value(x)

chineseRemainder(llv:List List(R), lm:List(R)):List(R) ==
  x := balancedBinaryTree(#lm, 0$R)
  x := setleaves_!(x, lm)
  mapUp_!(x,"*")
  y := balancedBinaryTree(#lm, 1$R)
  y := mapUp_!(copy y,x,(a,b,c,d)+->a*d + b*c)
  (u := extendedEuclidean(value y, value x,1)) case "failed" =>
    error "moduli not relatively prime"
  inv := u . coef1
  linv := modTree(inv, lm)
  retVal:List(R) := []
  for lv in llv repeat
    l := [(u3*v) rem m for v in lv for u3 in linv for m in lm]
    y := setleaves!(y,l)
    retVal :=
      cons(value(mapUp!(y, x, (a,b,c,d)+->a*d+b*c)) rem value(x),retVal)
  reverse retVal

extEuclidean: (R, R, R) -> List R
extEuclidean(a, b, c) ==
  u := extendedEuclidean(a, b, c)
  u case "failed" => error [c, " not spanned by ", a, " and ",b]
  [u.coef2, u.coef1]

multiEuclideanTree(fl, rhs) ==
  x := balancedBinaryTree(#fl, rhs)
  x := setleaves_!(x, fl)
  mapUp_!(x,"*")
```

```
        leaves mapDown_!(x, rhs, extEuclidean)
```

⟨*CRAPACK.dotabb*⟩≡

```
  "CRAPACK" [color="#FF4488",href="bookvol10.4.pdf#nameddest=CRAPACK"]
  "FLAGG" [color="#4488FF",href="bookvol10.2.pdf#nameddest=FLAGG"]
  "CRAPACK" -> "FLAGG"
```

4.43 package CYCLES CycleIndicators

⟨*CycleIndicators.input*⟩≡

```
)set break resume
)sys rm -f CycleIndicators.output
)spool CycleIndicators.output
)set message test on
)set message auto off
)clear all
--S 1 of 47
complete 1
--R
--R
--R   (1)  (1)
--R                                    Type: SymmetricPolynomial Fraction Integer
--E 1

--S 2 of 47
complete 2
--R
--R
--R        1       1   2
--R   (2)  - (2) + - (1 )
--R        2       2
--R                                    Type: SymmetricPolynomial Fraction Integer
--E 2

--S 3 of 47
complete 3
--R
--R
--R        1       1         1   3
--R   (3)  - (3) + - (2 1) + - (1 )
--R        3       2         6
--R                                    Type: SymmetricPolynomial Fraction Integer
--E 3

--S 4 of 47
complete 7
--R
--R
--R   (4)
--R     1       1          1          1     2     1         1            1     3
--R     - (7) + - (6 1) + -- (5 2) + -- (5 1 ) + -- (4 3) + - (4 2 1) + -- (4 1 )
--R     7       6         10         10          12         8           24
--R   +
```

```
--R      1   2      1     2     1       2     1     4     1   3      1   2 3
--R     -- (3 1) + -- (3 2 ) + -- (3 2 1 ) + -- (3 1 ) + -- (2 1) + -- (2 1 )
--R     18         24          12            72          48         48
--R   +
--R      1      5      1    7
--R     --- (2 1 ) + ---- (1 )
--R     240          5040
--R                                    Type: SymmetricPolynomial Fraction Integer
--E 4

--S 5 of 47
elementary 7
--R
--R
--R   (5)
--R     1       1          1          1     2     1         1            1     3
--R     - (7) - - (6 1) - -- (5 2) + -- (5 1 ) - -- (4 3) + - (4 2 1) - -- (4 1 )
--R     7       6         10         10          12         8           24
--R   +
--R      1   2      1     2     1       2     1     4     1   3      1   2 3
--R     -- (3 1) + -- (3 2 ) - -- (3 2 1 ) + -- (3 1 ) - -- (2 1) + -- (2 1 )
--R     18         24          12            72          48         48
--R   +
--R        1      5      1    7
--R     - --- (2 1 ) + ---- (1 )
--R       240          5040
--R                                    Type: SymmetricPolynomial Fraction Integer
--E 5

--S 6 of 47
alternating 7
--R
--R
--R   (6)
--R     2       1     2    1          1   2      1     2     1     4     1   2 3
--R     - (7) + - (5 1 ) + - (4 2 1) + - (3 1) + -- (3 2 ) + -- (3 1 ) + -- (2 1 )
--R     7       5          4          9         12          36          24
--R   +
--R       1    7
--R     ---- (1 )
--R     2520
--R                                    Type: SymmetricPolynomial Fraction Integer
--E 6

--S 7 of 47
cyclic 7
```

```
--R
--R
--R        6       1   7
--R   (7)  - (7) + - (1 )
--R        7       7
--R                                    Type: SymmetricPolynomial Fraction Integer
--E 7

--S 8 of 47
dihedral 7
--R
--R
--R        3       1   3      1   7
--R   (8)  - (7) + - (2 1) + -- (1 )
--R        7       2         14
--R                                    Type: SymmetricPolynomial Fraction Integer
--E 8

--S 9 of 47
graphs 5
--R
--R
--R   (9)
--R   1           1   2    1   2     1   3     1   4 2     1   3 4     1    10
--R   - (6 3 1) + - (5 ) + - (4 2) + - (3 1) + - (2 1 ) + -- (2 1 ) + --- (1  )
--R   6           5        4         6         8          12          120
--R                                    Type: SymmetricPolynomial Fraction Integer
--E 9

--S 10 of 47
cap(complete 2**2, complete 2*complete 1**2)
--R
--R
--R   (10)  4
--R                                                  Type: Fraction Integer
--E 10

--S 11 of 47
cap(elementary 2**2, complete 2*complete 1**2)
--R
--R
--R   (11)  2
--R                                                  Type: Fraction Integer
--E 11

--S 12 of 47
```

```
cap(complete 3*complete 2*complete 1,complete 2**2*complete 1**2)
--R
--R
--R   (12)  24
--R                                                    Type: Fraction Integer
--E 12

--S 13 of 47
cap(elementary 3*elementary 2*elementary 1,complete 2**2*complete 1**2)
--R
--R
--R   (13)  8
--R                                                    Type: Fraction Integer
--E 13

--S 14 of 47
cap(complete 3*complete 2*complete 1,elementary 2**2*elementary 1**2)
--R
--R
--R   (14)  8
--R                                                    Type: Fraction Integer
--E 14

--S 15 of 47
eval(cup(complete 3*complete 2*complete 1, cup(complete 2**2*complete 1**2,complete 2**3)))
--R
--R
--R   (15)  1500
--R                                                    Type: Fraction Integer
--E 15

--S 16 of 47
square:=dihedral 4
--R
--R
--R         1       3   2    1     2    1   4
--R   (16)  - (4) + - (2 ) + - (2 1 ) + - (1 )
--R         4       8        4          8
--R                                  Type: SymmetricPolynomial Fraction Integer
--E 16

--S 17 of 47
cap(complete 2**2,square)
--R
--R
--R   (17)  2
```

```
--R                                                          Type: Fraction Integer
--E 17

--S 18 of 47
cap(complete 3*complete 2**2,dihedral 7)
--R
--R
--R   (18)  18
--R                                                          Type: Fraction Integer
--E 18

--S 19 of 47
cap(graphs 5,complete 7*complete 3)
--R
--R
--R   (19)  4
--R                                                          Type: Fraction Integer
--E 19

--S 20 of 47
s(x) == powerSum(x)
--R
--R                                                                   Type: Void
--E 20

--S 21 of 47
cube:=(1/24)*(s 1**8+9*s 2**4 + 8*s 3**2*s 1**2+6*s 4**2)
--R
--R   Compiling function s with type PositiveInteger ->
--R      SymmetricPolynomial Fraction Integer
--R
--R         1   2    1   2 2    3   4     1   8
--R   (21)  - (4 ) + - (3 1 ) + - (2 ) + -- (1 )
--R         4        3          8        24
--R                                  Type: SymmetricPolynomial Fraction Integer
--E 21

--S 22 of 47
cap(complete 4**2,cube)
--R
--R
--R   (22)  7
--R                                                          Type: Fraction Integer
--E 22

--S 23 of 47
```

```
cap(complete 2**3*complete 1**2,wreath(elementary 4,elementary 2))
--R
--R
--R   (23)  7
--R                                                    Type: Fraction Integer
--E 23

--S 24 of 47
cap(complete 2**3*complete 1**2,wreath(elementary 4,complete 2))
--R
--R
--R   (24)  17
--R                                                    Type: Fraction Integer
--E 24

--S 25 of 47
cap(complete 2**3*complete 1**2,wreath(complete 4,elementary 2))
--R
--R
--R   (25)  10
--R                                                    Type: Fraction Integer
--E 25

--S 26 of 47
cap(complete 2**3*complete 1**2,wreath(complete 4,complete 2))
--R
--R
--R   (26)  23
--R                                                    Type: Fraction Integer
--E 26

--S 27 of 47
x: ULS(FRAC INT,'x,0) := 'x
--R
--R
--R   (27)  x
--R                          Type: UnivariateLaurentSeries(Fraction Integer,x,0)
--E 27

--S 28 of 47
ZeroOrOne: INT -> ULS(FRAC INT, 'x, 0)
--R
--R                                                                  Type: Void
--E 28

--S 29 of 47
```

```
Integers: INT -> ULS(FRAC INT, 'x, 0)
--R
--R                                                                    Type: Void
--E 29

--S 30 of 47
ZeroOrOne n == 1+x**n
--R
--R                                                                    Type: Void
--E 30

--S 31 of 47
ZeroOrOne 5
--R
--R   Compiling function ZeroOrOne with type Integer ->
--R      UnivariateLaurentSeries(Fraction Integer,x,0)
--R
--R              5
--R   (31)  1 + x
--R                           Type: UnivariateLaurentSeries(Fraction Integer,x,0)
--E 31

--S 32 of 47
Integers n == 1/(1-x**n)
--R
--R                                                                    Type: Void
--E 32

--S 33 of 47
Integers 5
--R
--R   Compiling function Integers with type Integer ->
--R      UnivariateLaurentSeries(Fraction Integer,x,0)
--R
--R              5    10      11
--R   (33)  1 + x  + x   + O(x  )
--R                           Type: UnivariateLaurentSeries(Fraction Integer,x,0)
--E 33

--S 34 of 47
)expose EVALCYC
--R
--I   EvaluateCycleIndicators is now explicitly exposed in frame frame0
--E 34

--S 35 of 47
```

```
eval(ZeroOrOne, graphs 5)
--R
--R
--R                   2     3     4     5     6     7     8    9    10      11
--R   (34)  1 + x + 2x  + 4x  + 6x  + 6x  + 6x  + 4x  + 2x  + x  + x   + O(x  )
--R                           Type: UnivariateLaurentSeries(Fraction Integer,x,0)
--E 35

--S 36 of 47
eval(ZeroOrOne,dihedral 8)
--R
--R
--R                   2     3     4     5     6    7    8
--R   (35)  1 + x + 4x  + 5x  + 8x  + 5x  + 4x  + x  + x
--R                           Type: UnivariateLaurentSeries(Fraction Integer,x,0)
--E 36

--S 37 of 47
eval(Integers,complete 4)
--R
--R
--R   (36)
--R             2     3     4     5     6      7      8      9      10      11
--R   1 + x + 2x  + 3x  + 5x  + 6x  + 9x  + 11x  + 15x  + 18x  + 23x   + O(x  )
--R                           Type: UnivariateLaurentSeries(Fraction Integer,x,0)
--E 37

--S 38 of 47
eval(Integers,elementary 4)
--R
--R
--R   (37)
--R      6    7     8     9     10     11     12      13      14      15      16
--R     x  + x  + 2x  + 3x  + 5x   + 6x   + 9x   + 11x   + 15x   + 18x   + 23x
--R   +
--R        17
--R     O(x  )
--R                           Type: UnivariateLaurentSeries(Fraction Integer,x,0)
--E 38

--S 39 of 47
eval(ZeroOrOne,cube)
--R
--R
--R                   2     3     4     5     6    7    8
--R   (38)  1 + x + 3x  + 3x  + 7x  + 3x  + 3x  + x  + x
```

```
--R                          Type: UnivariateLaurentSeries(Fraction Integer,x,0)
--E 39

--S 40 of 47
eval(Integers,cube)
--R
--R
--R   (39)
--R               2     3      4      5      6       7       8       9       10
--R     1 + x + 4x  + 7x  + 21x  + 37x  + 85x  + 151x  + 292x  + 490x  + 848x
--R   +
--R         11
--R     O(x  )
--R                          Type: UnivariateLaurentSeries(Fraction Integer,x,0)
--E 40

--S 41 of 47
eval(Integers,graphs 5)
--R
--R
--R   (40)
--R               2     3      4      5      6       7       8       9       10
--R     1 + x + 3x  + 7x  + 17x  + 35x  + 76x  + 149x  + 291x  + 539x  + 974x
--R   +
--R         11
--R     O(x  )
--R                          Type: UnivariateLaurentSeries(Fraction Integer,x,0)
--E 41

--S 42 of 47
eval(ZeroOrOne ,graphs 15)
--R
--R
--R   (41)
--R               2     3      4      5      6       7       8        9        10
--R     1 + x + 2x  + 5x  + 11x  + 26x  + 68x  + 177x  + 496x  + 1471x  + 4583x
--R   +
--R         11
--R     O(x  )
--R                          Type: UnivariateLaurentSeries(Fraction Integer,x,0)
--E 42

--S 43 of 47
cap(dihedral 30,complete 7*complete 8*complete 5*complete 10)
--R
--R
```

```
--R   (42)  49958972383320
--R                                                    Type: Fraction Integer
--E 43

--S 44 of 47
sf3221:= SFunction [3,2,2,1]
--R
--R
--R   (43)
--R      1          1     2     1   2     1            1     4     1   2
--R     -- (6 2) - -- (6 1 ) - -- (4 ) + -- (4 3 1) + -- (4 1 ) - -- (3 2)
--R     12         12          16        12           24          36
--R   +
--R      1   2 2     1     2      1       3     1     5     1    4     1   3 2
--R     -- (3 1 ) - -- (3 2 1) - -- (3 2 1 ) - -- (3 1 ) - --- (2 ) + -- (2 1 )
--R     36          24           36            72          192        48
--R   +
--R      1   2 4     1      6     1    8
--R     -- (2 1 ) - --- (2 1 ) + --- (1 )
--R     96          144          576
--R                                   Type: SymmetricPolynomial Fraction Integer
--E 44

--S 45 of 47
cap(sf3221,complete 2**4)
--R
--R
--R   (44)  3
--R                                                    Type: Fraction Integer
--E 45

--S 46 of 47
cap(sf3221, powerSum 1**8)
--R
--R
--R   (45)  70
--R                                                    Type: Fraction Integer
--E 46

--S 47 of 47
eval(Integers, sf3221)
--R
--R
--R   (46)
--R      9     10     11      12      13      14      15       16       17       18
--R     x  + 3x   + 7x   + 14x   + 27x   + 47x   + 79x   + 126x   + 196x   + 294x
```

```
--R   +
--R         19      20
--R     432x   + O(x  )
--R                          Type: UnivariateLaurentSeries(Fraction Integer,x,0)
--E 47
)spool
)lisp (bye)
```

⟨*CycleIndicators.help*⟩≡

```
==========================================================================
CycleIndicators examples
==========================================================================

This section is based upon the paper J. H. Redfield, ``The Theory of
Group-Reduced Distributions'', American J. Math.,49 (1927) 433-455,
and is an application of group theory to enumeration problems.  It is
a development of the work by P. A. MacMahon on the application of
symmetric functions and Hammond operators to combinatorial theory.

The theory is based upon the power sum symmetric functions s(i) which
are the sum of the i-th powers of the variables.  The cycle index of a
permutation is an expression that specifies the sizes of the cycles of
a permutation, and may be represented as a partition.  A partition of
a non-negative integer n is a collection of positive integers called
its parts whose sum is n.  For example, the partition (3^2 2 1^2) will
be used to represent s^2_3 s_2 s^2_1 and will indicate that the
permutation has two cycles of length 3, one of length 2 and two of
length 1.  The cycle index of a permutation group is the sum of the
cycle indices of its permutations divided by the number of
permutations.  The cycle indices of certain groups are provided.

The operation complete returns the cycle index of the symmetric group
of order n for argument n.  Alternatively, it is the n-th complete
homogeneous symmetric function expressed in terms of power sum
symmetric functions.

  complete 1
   (1)
                        Type: SymmetricPolynomial Fraction Integer

  complete 2
   1       1   2
   - (2) + - (1 )
   2       2
                        Type: SymmetricPolynomial Fraction Integer

  complete 3
   1       1         1   3
   - (3) + - (2 1) + - (1 )
   3       2         6
                        Type: SymmetricPolynomial Fraction Integer

  complete 7
     1       1          1          1     2     1         1           1     3
```

```
   - (7) + - (6 1) + -- (5 2) + -- (5 1 ) + -- (4 3) + - (4 2 1) + -- (4 1 )
     7       6         10         10          12         8           24
  +
     1   2      1     2     1       2     1     4     1   3      1   2 3
   -- (3 1) + -- (3 2 ) + -- (3 2 1 ) + -- (3 1 ) + -- (2 1) + -- (2 1 )
   18         24          12            72          48         48
  +
     1      5      1    7
   --- (2 1 ) + ---- (1 )
   240          5040
                 Type: SymmetricPolynomial Fraction Integer
```

The operation elementary computes the n-th elementary symmetric function for argument n.

```
  elementary 7
     1       1          1          1     2     1         1            1     3
     - (7) - - (6 1) - -- (5 2) + -- (5 1 ) - -- (4 3) + - (4 2 1) - -- (4 1 )
     7       6         10         10          12         8           24
  +
     1   2      1     2     1       2     1     4     1   3      1   2 3
   -- (3 1) + -- (3 2 ) - -- (3 2 1 ) + -- (3 1 ) - -- (2 1) + -- (2 1 )
   18         24          12            72          48         48
  +
       1      5      1    7
   - --- (2 1 ) + ---- (1 )
     240          5040
                 Type: SymmetricPolynomial Fraction Integer
```

The operation alternating returns the cycle index of the alternating group having an even number of even parts in each cycle partition.

```
  alternating 7
     2       1     2    1           1   2      1     2     1     4     1   2 3
   - (7) + - (5 1 ) + - (4 2 1) + - (3 1) + -- (3 2 ) + -- (3 1 ) + -- (2 1 )
     7       5          4           9         12          36          24
  +
     1    7
   ---- (1 )
   2520
                 Type: SymmetricPolynomial Fraction Integer
```

The operation cyclic returns the cycle index of the cyclic group.

```
  cyclic 7
   6       1   7
```

```
   - (7) + - (1 )
   7       7
                  Type: SymmetricPolynomial Fraction Integer
```

The operation dihedral is the cycle index of the dihedral group.

```
  dihedral 7
   3       1   3      1   7
   - (7) + - (2 1) + -- (1 )
   7       2         14
                  Type: SymmetricPolynomial Fraction Integer
```

The operation graphs for argument n returns the cycle index of the group of permutations on the edges of the complete graph with n nodes induced by applying the symmetric group to the nodes.

```
  graphs 5
   1           1   2    1   2     1   3     1   4 2     1   3 4     1    10
   - (6 3 1) + - (5 ) + - (4 2) + - (3 1) + - (2 1 ) + -- (2 1 ) + --- (1  )
   6           5        4         6         8          12          120
                  Type: SymmetricPolynomial Fraction Integer
```

The cycle index of a direct product of two groups is the product of the cycle indices of the groups. Redfield provided two operations on two cycle indices which will be called "cup" and "cap" here. The cup of two cycle indices is a kind of scalar product that combines monomials for permutations with the same cycles. The cap operation provides the sum of the coefficients of the result of the cup operation which will be an integer that enumerates what Redfield called group-reduced distributions.

We can, for example, represent complete 2 * complete 2 as the set of objects a a b b and complete 2 * complete 1 * complete 1 as c c d e.

This integer is the number of different sets of four pairs.

```
  cap(complete 2**2, complete 2*complete 1**2)
    4
                  Type: Fraction Integer
```

For example,

```
  a a b b     a a b b    a a b b   a a b b
  c c d e     c d c e    c e c d   d e c c
```

This integer is the number of different sets of four pairs no two pairs being equal.

```
  cap(elementary 2**2, complete 2*complete 1**2)
    2
                  Type: Fraction Integer
```

For example,

```
  a a b b    a a b b
  c d c e    c e c d
```

In this case the configurations enumerated are easily constructed, however the theory merely enumerates them providing little help in actually constructing them.

Here are the number of 6-pairs, first from a a a b b c, second from d d e e f g.

```
  cap(complete 3*complete 2*complete 1,complete 2**2*complete 1**2)
    24
                      Type: Fraction Integer
```

Here it is again, but with no equal pairs.

```
  cap(elementary 3*elementary 2*elementary 1,complete 2**2*complete 1**2)
    8
                      Type: Fraction Integer

  cap(complete 3*complete 2*complete 1,elementary 2**2*elementary 1**2)
    8
                      Type: Fraction Integer
```

The number of 6-triples, first from a a a b b c, second from d d e e f g, third from h h i i j j.

```
  eval(cup(complete 3*complete 2*complete 1, cup(complete 2**2*complete 1**2,comp
    1500
                      Type: Fraction Integer
```

The cycle index of vertices of a square is dihedral 4.

```
  square:=dihedral 4
   1       3   2    1     2    1   4
   - (4) + - (2 ) + - (2 1 ) + - (1 )
   4       8        4          8
                      Type: SymmetricPolynomial Fraction Integer
```

```
The number of different squares with 2 red vertices and 2 blue vertices.

  cap(complete 2**2,square)
    2
                    Type: Fraction Integer

The number of necklaces with 3 red beads, 2 blue beads and 2 green beads.

  cap(complete 3*complete 2**2,dihedral 7)
    18
                    Type: Fraction Integer

The number of graphs with 5 nodes and 7 edges.

  cap(graphs 5,complete 7*complete 3)
    4
                    Type: Fraction Integer

The cycle index of rotations of vertices of a cube.

  s(x) == powerSum(x)
                    Type: Void

  cube:=(1/24)*(s 1**8+9*s 2**4 + 8*s 3**2*s 1**2+6*s 4**2)
   1   2    1   2 2    3   4     1   8
   - (4 ) + - (3 1 ) + - (2 ) + -- (1 )
   4        3          8        24
                    Type: SymmetricPolynomial Fraction Integer

The number of cubes with 4 red vertices and 4 blue vertices.

  cap(complete 4**2,cube)
    7
                    Type: Fraction Integer

The number of labeled graphs with degree sequence 2 2 2 1 1 with no
loops or multiple edges.

  cap(complete 2**3*complete 1**2,wreath(elementary 4,elementary 2))
    7
                    Type: Fraction Integer

Again, but with loops allowed but not multiple edges.

  cap(complete 2**3*complete 1**2,wreath(elementary 4,complete 2))
    17
```

```
                    Type: Fraction Integer
```

Again, but with multiple edges allowed, but not loops

```
  cap(complete 2**3*complete 1**2,wreath(complete 4,elementary 2))
    10
                    Type: Fraction Integer
```

Again, but with both multiple edges and loops allowed

```
  cap(complete 2**3*complete 1**2,wreath(complete 4,complete 2))
    23
                    Type: Fraction Integer
```

Having constructed a cycle index for a configuration we are at liberty to evaluate the s_i components any way we please. For example we can produce enumerating generating functions. This is done by providing a function f on an integer i to the value required of s_i, and then evaluating eval(f, cycleindex).

```
  x: ULS(FRAC INT,'x,0) := 'x
   x
                    Type: UnivariateLaurentSeries(Fraction Integer,x,0)

  ZeroOrOne: INT -> ULS(FRAC INT, 'x, 0)
                    Type: Void

  Integers: INT -> ULS(FRAC INT, 'x, 0)
                    Type: Void
```

For the integers 0 and 1, or two colors.

```
  ZeroOrOne n == 1+x**n
                    Type: Void

  ZeroOrOne 5
         5
    1 + x
                    Type: UnivariateLaurentSeries(Fraction Integer,x,0)
```

For the integers 0, 1, 2, ... we have this.

```
  Integers n == 1/(1-x**n)
                    Type: Void

  Integers 5
```

```
          5    10      11
    1 + x  + x   + O(x  )
                    Type: UnivariateLaurentSeries(Fraction Integer,x,0)
```

The coefficient of x^n is the number of graphs with 5 nodes and n edges.

Note that there is an eval function that takes two arguments. It has the signature:

```
  ((Integer -> D1),SymmetricPolynomial Fraction Integer) -> D1
    from EvaluateCycleIndicators D1 if D1 has ALGEBRA FRAC INT
```

This function is not normally exposed (it will not normally be considered in the list of eval functions) as it is only useful for this particular domain. To use it we ask that it be considered thus:

```
  )expose EVALCYC
```

and now we can use it:

```
  eval(ZeroOrOne, graphs 5)
              2     3     4     5     6     7     8    9    10      11
    1 + x + 2x  + 4x  + 6x  + 6x  + 6x  + 4x  + 2x  + x  + x   + O(x  )
                    Type: UnivariateLaurentSeries(Fraction Integer,x,0)
```

The coefficient of x^n is the number of necklaces with n red beads and n-8 green beads.

```
  eval(ZeroOrOne,dihedral 8)
              2     3     4     5     6    7    8
    1 + x + 4x  + 5x  + 8x  + 5x  + 4x  + x  + x
                    Type: UnivariateLaurentSeries(Fraction Integer,x,0)
```

The coefficient of x^n is the number of partitions of n into 4 or fewer parts.

```
  eval(Integers,complete 4)
             2     3     4     5     6      7      8      9      10      11
   1 + x + 2x  + 3x  + 5x  + 6x  + 9x  + 11x  + 15x  + 18x  + 23x   + O(x  )
                    Type: UnivariateLaurentSeries(Fraction Integer,x,0)
```

The coefficient of x^n is the number of partitions of n into 4 boxes containing ordered distinct parts.

```
  eval(Integers,elementary 4)
      6    7     8     9     10     11     12      13      14      15      16
     x  + x  + 2x  + 3x  + 5x   + 6x   + 9x   + 11x   + 15x   + 18x   + 23x
```

```
   +
        17
     O(x  )
                    Type: UnivariateLaurentSeries(Fraction Integer,x,0)
```

The coefficient of x^n is the number of different cubes with n red vertices and 8-n green ones.

```
  eval(ZeroOrOne,cube)
              2     3     4     5     6    7    8
    1 + x + 3x  + 3x  + 7x  + 3x  + 3x  + x  + x
                    Type: UnivariateLaurentSeries(Fraction Integer,x,0)
```

The coefficient of x^n is the number of different cubes with integers on the vertices whose sum is n.

```
  eval(Integers,cube)
              2     3      4      5      6       7       8       9       10
    1 + x + 4x  + 7x  + 21x  + 37x  + 85x  + 151x  + 292x  + 490x  + 848x
   +
        11
     O(x  )
                          Type: UnivariateLaurentSeries(Fraction Integer,x,0)
```

The coefficient of x^n is the number of graphs with 5 nodes and with integers on the edges whose sum is n. In other words, the enumeration is of multigraphs with 5 nodes and n edges.

```
  eval(Integers,graphs 5)
              2     3      4      5      6       7       8       9       10
    1 + x + 3x  + 7x  + 17x  + 35x  + 76x  + 149x  + 291x  + 539x  + 974x
   +
        11
     O(x  )
                    Type: UnivariateLaurentSeries(Fraction Integer,x,0)
```

Graphs with 15 nodes enumerated with respect to number of edges.

```
  eval(ZeroOrOne ,graphs 15)
              2     3      4      5      6       7       8        9        10
    1 + x + 2x  + 5x  + 11x  + 26x  + 68x  + 177x  + 496x  + 1471x  + 4583x
   +
        11
     O(x  )
                    Type: UnivariateLaurentSeries(Fraction Integer,x,0)
```

Necklaces with 7 green beads, 8 white beads, 5 yellow beads and 10 red beads.

```
  cap(dihedral 30,complete 7*complete 8*complete 5*complete 10)
    49958972383320
                    Type: Fraction Integer
```

The operation SFunction is the S-function or Schur function of a partition written as a descending list of integers expressed in terms of power sum symmetric functions.

In this case the argument partition represents a tableau shape. For example 3,2,2,1 represents a tableau with three boxes in the first row, two boxes in the second and third rows, and one box in the fourth row. SFunction [3,2,2,1] counts the number of different tableaux of shape 3, 2, 2, 1 filled with objects with an ascending order in the columns and a non-descending order in the rows.

```
  sf3221:= SFunction [3,2,2,1]
      1          1     2     1   2     1             1     4     1   2
     -- (6 2) - -- (6 1 ) - -- (4 ) + -- (4 3 1) + -- (4 1 ) - -- (3 2)
     12         12          16        12           24          36
   +
      1   2 2     1     2      1       3     1     5     1    4     1   3 2
     -- (3 1 ) - -- (3 2 1) - -- (3 2 1 ) - -- (3 1 ) - --- (2 ) + -- (2 1 )
     36          24          36            72          192        48
   +
      1   2 4     1      6     1    8
     -- (2 1 ) - --- (2 1 ) + --- (1 )
     96          144          576
                     Type: SymmetricPolynomial Fraction Integer
```

This is the number filled with a a b b c c d d.

```
  cap(sf3221,complete 2**4)
    3
                    Type: Fraction Integer
```

The configurations enumerated above are:

```
  a a b    a a c    a a d
  b c      b b      b b
  c d      c d      c c
  d        d        d
```

This is the number of tableaux filled with 1..8.

```
  cap(sf3221, powerSum 1**8)
    70
                    Type: Fraction Integer
```

The coefficient of x^n is the number of column strict reverse plane partitions of n of shape 3 2 2 1.

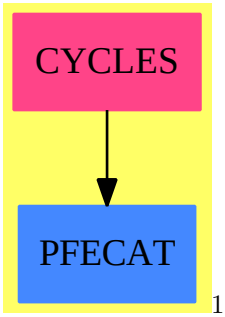
```
  eval(Integers, sf3221)
      9     10     11      12      13      14      15       16       17
     x  + 3x   + 7x   + 14x   + 27x   + 47x   + 79x   + 126x   + 196x
   +
        18      19      20
    294x  + 432x   + O(x  )
                          Type: UnivariateLaurentSeries(Fraction Integer,x,0)
```

The smallest is

```
  0 0 0
  1 1
  2 2
  3
```

See Also:
o)show CycleIndicators

4.44 CycleIndicators



1.00

Exports:

alternating	cap	complete	cup	cyclic
dihedral	elementary	eval	graphs	powerSum
SFunction	skewSFunction	wreath		

⟨*package CYCLES CycleIndicators*⟩≡

```
)abbrev package CYCLES CycleIndicators
++ Polya-Redfield enumeration by cycle indices.
++ Author: William H. Burge
++ Date Created: 1986
++ Date Last Updated: 11 Feb 1992
++ Keywords:Polya, Redfield, enumeration
++ Examples:
++ References: J.H.Redfield, 'The Theory of Group-Reduced Distributions',
++             American J. Math., 49 (1927) 433-455.
++             G.Polya, 'Kombinatorische Anzahlbestimmungen fur Gruppen,
++               Graphen und chemische Verbindungen', Acta Math. 68
++                (1937) 145-254.
++ Description: Enumeration by cycle indices.
CycleIndicators: Exports == Implementation where
  I    ==> Integer
  L    ==> List
  B    ==> Boolean
  SPOL ==> SymmetricPolynomial
  PTN  ==> Partition
  RN   ==> Fraction Integer
  FR   ==> Factored Integer
  h ==> complete
  s ==> powerSum
  --a ==> elementary
  alt ==> alternating
  cyc ==> cyclic
  dih ==> dihedral
  ev == eval
```

```
    Exports ==> with

      complete: I -> SPOL RN
        ++\spad{complete n} is the \spad{n} th complete homogeneous
        ++ symmetric function expressed in terms of power sums.
        ++ Alternatively it is the cycle index of the symmetric
        ++ group of degree n.

      powerSum: I -> SPOL RN
        ++\spad{powerSum n} is the \spad{n} th power sum symmetric
        ++ function.

      elementary: I -> SPOL RN
        ++\spad{elementary n} is the \spad{n} th elementary symmetric
        ++ function expressed in terms of power sums.

    -- s2h: I -> SPOL RN--s to h

      alternating: I -> SPOL RN
        ++\spad{alternating n} is the cycle index of the
        ++ alternating group of degree n.

      cyclic: I -> SPOL RN    --cyclic group
        ++\spad{cyclic n} is the cycle index of the
        ++ cyclic group of degree n.

      dihedral: I -> SPOL RN    --dihedral group
        ++\spad{dihedral n} is the cycle index of the
        ++ dihedral group of degree n.

      graphs: I -> SPOL RN
        ++\spad{graphs n} is the cycle index of the group induced on
        ++ the edges of a graph by applying the symmetric function to the
        ++ n nodes.

      cap: (SPOL RN,SPOL RN) -> RN
        ++\spad{cap(s1,s2)}, introduced by Redfield,
        ++ is the scalar product of two cycle indices.

      cup: (SPOL RN,SPOL RN) -> SPOL RN
        ++\spad{cup(s1,s2)}, introduced by Redfield,
        ++ is the scalar product of two cycle indices, in which the
        ++ power sums are retained to produce a cycle index.

      eval: SPOL RN -> RN
        ++\spad{eval s} is the sum of the coefficients of a cycle index.
```

```
    wreath: (SPOL RN,SPOL RN) -> SPOL RN
      ++\spad{wreath(s1,s2)} is the cycle index of the wreath product
      ++ of the two groups whose cycle indices are \spad{s1} and
      ++ \spad{s2}.

    SFunction:L I -> SPOL RN
      ++\spad{SFunction(li)} is the S-function of the partition \spad{li}
      ++ expressed in terms of power sum symmetric functions.

    skewSFunction:(L I,L I) -> SPOL RN
      ++\spad{skewSFunction(li1,li2)} is the S-function
      ++ of the partition difference \spad{li1 - li2}
      ++ expressed in terms of power sum symmetric functions.

  Implementation ==> add
    import PartitionsAndPermutations
    import IntegerNumberTheoryFunctions

    trm: PTN -> SPOL RN
    trm pt == monomial(inv(pdct(pt) :: RN),pt)

    list: Stream L I -> L L I
    list st == entries complete st

    complete i ==
           if i=0
           then 1
           else if i<0
                then 0
                else
                   _+/[trm(partition pt) for pt in list(partitions i)]


    even?: L I -> B
    even? li == even?( #([i for i in li | even? i]))

    alt i ==
      2 * _+/[trm(partition li) for li in list(partitions i) | even? li]
    elementary i ==
           if i=0
           then 1
           else if i<0
                then 0
                else
                  _+/[(spol := trm(partition pt); even? pt => spol; -spol)
```

```
                        for pt in list(partitions i)]

divisors: I -> L I
divisors n ==
  b := factors(n :: FR)
  c := concat(1,"append"/
             [[a.factor**j for j in 1..a.exponent] for a in b]);
  if #(b) = 1 then c else concat(n,c)

ss: (I,I) -> SPOL RN
ss(n,m) ==
  li : L I := [n for j in 1..m]
  monomial(1,partition li)

s n == ss(n,1)

cyc n ==
  n = 1 => s 1
  _+/[(eulerPhi(i) / n) * ss(i,numer(n/i)) for i in divisors n]

dih n ==
  k := n quo 2
  odd? n => (1/2) * cyc n + (1/2) * ss(2,k) * s 1
  (1/2) * cyc n + (1/4) * ss(2,k) + (1/4) * ss(2,k-1) * ss(1,2)

trm2: L I -> SPOL RN
trm2 li ==
  lli := powers(li)$PTN
  xx := 1/(pdct partition li)
  prod : SPOL RN := 1
  for ll in lli repeat
    ll0 := first ll; ll1 := second ll
    k := ll0 quo 2
    c :=
      odd? ll0 => ss(ll0,ll1 * k)
      ss(k,ll1) * ss(ll0,ll1 * (k - 1))
    c := c * ss(ll0,ll0 * ((ll1*(ll1 - 1)) quo 2))
    prod2 : SPOL RN := 1
    for r in lli | first(r) < ll0 repeat
      r0 := first r; r1 := second r
      prod2 := ss(lcm(r0,ll0),gcd(r0,ll0) * r1 * ll1) * prod2
    prod := c * prod2 * prod
  xx * prod

graphs n == _+/[trm2 li for li in list(partitions n)]
```

```
cupp: (PTN,SPOL RN) -> SPOL RN
cupp(pt,spol) ==
  zero? spol => 0
  (dg := degree spol) < pt => 0
  dg = pt => (pdct pt) * monomial(leadingCoefficient spol,dg)
  cupp(pt,reductum spol)

cup(spol1,spol2) ==
  zero? spol1 => 0
  p := leadingCoefficient(spol1) * cupp(degree spol1,spol2)
  p + cup(reductum spol1,spol2)

ev spol ==
  zero? spol => 0
  leadingCoefficient(spol) + ev(reductum spol)

cap(spol1,spol2) == ev cup(spol1,spol2)

mtpol: (I,SPOL RN) -> SPOL RN
mtpol(n,spol)==
  zero? spol => 0
  deg := partition [n*k for k in (degree spol)::L(I)]
  monomial(leadingCoefficient spol,deg) + mtpol(n,reductum spol)

fn2: I -> SPOL RN
evspol: ((I -> SPOL RN),SPOL RN) -> SPOL RN
evspol(fn2,spol) ==
  zero? spol => 0
  lc := leadingCoefficient spol
  prod := _*/[fn2 i for i in (degree spol)::L(I)]
  lc * prod + evspol(fn2,reductum spol)

wreath(spol1,spol2) == evspol(x+->mtpol(x,spol2),spol1)

hh: I -> SPOL RN      --symmetric group
hh n == if n=0 then 1 else if n<0 then 0 else h n
SFunction li==
  a:Matrix SPOL RN:=matrix [[hh(k -j+i) for k in li for j in 1..#li]
                for i in 1..#li]
  determinant a

roundup:(L I,L I)-> L I
roundup(li1,li2)==
          #li1 > #li2 => roundup(li1,concat(li2,0))
          li2
```

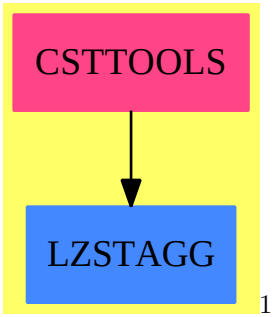
```
skewSFunction(li1,li2)==
  #li1 < #li2 =>
    error "skewSFunction: partition1 does not include partition2"
  li2:=roundup (li1,li2)
  a:Matrix SPOL RN:=matrix [[hh(k-li2.i-j+i)
           for k in li1 for j in 1..#li1]  for i in 1..#li1]
  determinant a
```

⟨*CYCLES.dotabb*⟩≡

```
"CYCLES" [color="#FF4488",href="bookvol10.4.pdf#nameddest=CYCLES"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"CYCLES" -> "PFECAT"
```

4.45 package CSTTOOLS CyclicStreamTools

4.46 CyclicStreamTools



1.00

Exports:
computeCycleEntry computeCycleLength cycleElt

⟨*package CSTTOOLS CyclicStreamTools*⟩≡

```
)abbrev package CSTTOOLS CyclicStreamTools
++ Functions for dealing with cyclic streams
++ Author: Clifton J. Williamson
++ Date Created: 5 December 1989
++ Date Last Updated: 5 December 1989
++ Keywords: stream, cyclic
++ Description:
++ This package provides tools for working with cyclic streams.
CyclicStreamTools(S,ST): Exports == Implementation where
  S  : Type
  ST : LazyStreamAggregate S

  Exports ==> with

    cycleElt: ST -> Union(ST,"failed")
      ++ cycleElt(s) returns a pointer to a node in the cycle if the stream
      ++ s is cyclic and returns "failed" if s is not cyclic
      ++
      ++X p:=repeating([1,2,3])
      ++X q:=cons(4,p)
      ++X cycleElt q
      ++X r:=[1,2,3]::Stream(Integer)
      ++X cycleElt r

    computeCycleLength: ST -> NonNegativeInteger
      ++ computeCycleLength(s) returns the length of the cycle of a
      ++ cyclic stream t, where s is a pointer to a node in the
```

```
      ++ cyclic part of t.
      ++
      ++X p:=repeating([1,2,3])
      ++X q:=cons(4,p)
      ++X computeCycleLength(cycleElt(q))

    computeCycleEntry: (ST,ST) -> ST
      ++ computeCycleEntry(x,cycElt), where cycElt is a pointer to a
      ++ node in the cyclic part of the cyclic stream x, returns a
      ++ pointer to the first node in the cycle
      ++
      ++X p:=repeating([1,2,3])
      ++X q:=cons(4,p)
      ++X computeCycleEntry(q,cycleElt(q))

  Implementation ==> add

    cycleElt x ==
      y := x
      for i in 0.. repeat
        (explicitlyEmpty? y) or (lazy? y) => return "failed"
        y := rst y
        if odd? i then x := rst x
        eq?(x,y) => return y

    computeCycleLength cycElt ==
      i : NonNegativeInteger
      y := cycElt
      for i in 1.. repeat
        y := rst y
        eq?(y,cycElt) => return i

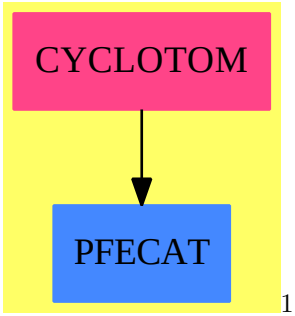
    computeCycleEntry(x,cycElt) ==
      y := rest(x, computeCycleLength cycElt)
      repeat
        eq?(x,y) => return x
        x := rst x ; y := rst y
```

⟨*CSTTOOLS.dotabb*⟩≡

```
"CSTTOOLS" [color="#FF4488",href="bookvol10.4.pdf#nameddest=CSTTOOLS"]
"LZSTAGG" [color="#4488FF",href="bookvol10.2.pdf#nameddest=LZSTAGG"]
"CSTTOOLS" -> "LZSTAGG"
```

4.47 package CYCLOTOM CyclotomicPolynomialPackage

4.48 CyclotomicPolynomialPackage



1.00

Exports:
cyclotomic cyclotomicDecomposition cyclotomicFactorization

⟨*package CYCLOTOM CyclotomicPolynomialPackage*⟩≡

```
)abbrev package CYCLOTOM CyclotomicPolynomialPackage
++ This package \undocumented{}
CyclotomicPolynomialPackage: public == private where
  SUP  ==> SparseUnivariatePolynomial(Integer)
  LSUP ==> List(SUP)
  NNI  ==> NonNegativeInteger
  FR   ==> Factored SUP
  IFP  ==> IntegerFactorizationPackage Integer

  public == with
    cyclotomicDecomposition: Integer -> LSUP
        ++ cyclotomicDecomposition(n) \undocumented{}
    cyclotomic:              Integer -> SUP
        ++ cyclotomic(n) \undocumented{}
    cyclotomicFactorization: Integer -> FR
        ++ cyclotomicFactorization(n) \undocumented{}

  private == add
    cyclotomic(n:Integer): SUP ==
      x,y,z,l: SUP
      g := factors factor(n)$IFP
      --Now, for each prime in the factorization apply recursion
      l := monomial(1,1) - monomial(1,0)
      for u in g repeat
         l := (monicDivide(multiplyExponents(l,u.factor::NNI),l)).quotient
         if u.exponent>1 then
```

```
            l := multiplyExponents(l,((u.factor)**((u.exponent-1)::NNI))::NNI)
      l

    cyclotomicDecomposition(n:Integer):LSUP ==
      x,y,z: SUP
      l,ll,m: LSUP
      rr: Integer
      g := factors factor(n)$IFP
      l := [monomial(1,1) - monomial(1,0)]
      --Now, for each prime in the factorization apply recursion
      for u in g repeat
         m := [(monicDivide(
            multiplyExponents(z,u.factor::NNI),z)).quotient for z in l]
         for rr in 1..(u.exponent-1) repeat
            l := append(l,m)
            m := [multiplyExponents(z,u.factor::NNI) for z in m]
         l := append(l,m)
      l

    cyclotomicFactorization(n:Integer):FR ==
      f :  SUP
      fr : FR := 1$FR
      for f in cyclotomicDecomposition(n) repeat
        fr := fr * primeFactor(f,1$Integer)
      fr
```

⟨*CYCLOTOM.dotabb*⟩≡

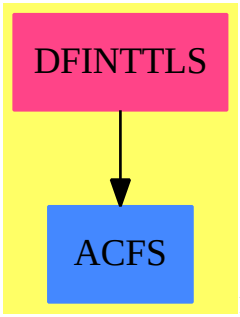
```
"CYCLOTOM" [color="#FF4488",href="bookvol10.4.pdf#nameddest=CYCLOTOM"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"CYCLOTOM" -> "PFECAT"
```

Chapter 5

Chapter D

5.1 package DFINTTLS DefiniteIntegrationTools

5.2 DefiniteIntegrationTools



1.00

Exports:
ignore? checkForZero computeInt

⟨*package DFINTTLS DefiniteIntegrationTools*⟩≡
```
)abbrev package DFINTTLS DefiniteIntegrationTools
++ Tools for definite integration
++ Author: Manuel Bronstein
++ Date Created: 15 April 1992
++ Date Last Updated: 24 February 1993
++ Description:
++   \spadtype{DefiniteIntegrationTools} provides common tools used
++   by the definite integration of both rational and elementary functions.
DefiniteIntegrationTools(R, F): Exports == Implementation where
  R : Join(GcdDomain, OrderedSet, RetractableTo Integer,
           LinearlyExplicitRingOver Integer)
```

```
F : Join(TranscendentalFunctionCategory,
         AlgebraicallyClosedFunctionSpace R)

B   ==> Boolean
Z   ==> Integer
Q   ==> Fraction Z
SE  ==> Symbol
P   ==> Polynomial R
RF  ==> Fraction P
UP  ==> SparseUnivariatePolynomial F
K   ==> Kernel F
OFE ==> OrderedCompletion F
UPZ ==> SparseUnivariatePolynomial Z
UPQ ==> SparseUnivariatePolynomial Q
REC ==> Record(left:Q, right:Q)
REC2==> Record(endpoint:Q, dir:Z)
U   ==> Union(fin:REC, halfinf:REC2, all:"all", failed:"failed")
IGNOR ==> "noPole"

Exports ==> with
  ignore?: String -> B
    ++ ignore?(s) is true if s is the string that tells the integrator
    ++ to assume that the function has no pole in the integration interval.
  computeInt: (K, F, OFE, OFE, B) -> Union(OFE, "failed")
    ++ computeInt(x, g, a, b, eval?) returns the integral of \spad{f} for x
    ++ between a and b, assuming that g is an indefinite integral of
    ++ \spad{f} and \spad{f} has no pole between a and b.
    ++ If \spad{eval?} is true, then \spad{g} can be evaluated safely
    ++ at \spad{a} and \spad{b}, provided that they are finite values.
    ++ Otherwise, limits must be computed.
  checkForZero: (P,  SE, OFE, OFE, B) -> Union(B, "failed")
    ++ checkForZero(p, x, a, b, incl?) is true if p has a zero for x between
    ++ a and b, false otherwise, "failed" if this cannot be determined.
    ++ Check for a and b inclusive if incl? is true, exclusive otherwise.
  checkForZero: (UP, OFE, OFE, B) -> Union(B, "failed")
    ++ checkForZero(p, a, b, incl?) is true if p has a zero between
    ++ a and b, false otherwise, "failed" if this cannot be determined.
    ++ Check for a and b inclusive if incl? is true, exclusive otherwise.

Implementation ==> add
  import RealZeroPackage UPZ
  import InnerPolySign(F, UP)
  import ElementaryFunctionSign(R, F)
  import PowerSeriesLimitPackage(R, F)
  import UnivariatePolynomialCommonDenominator(Z, Q, UPQ)
```

```
mkLogPos    : F -> F
keeprec?    : (Q, REC) -> B
negative    : F -> Union(B, "failed")
mkKerPos    : K -> Union(F, "positive")
posRoot     : (UP, B) -> Union(B, "failed")
realRoot    : UP -> Union(B, "failed")
var         : UP -> Union(Z, "failed")
maprat      : UP -> Union(UPZ, "failed")
variation   : (UP, F) -> Union(Z, "failed")
infeval     : (UP, OFE) -> Union(F, "failed")
checkHalfAx : (UP, F, Z, B) -> Union(B, "failed")
findLimit   : (F, K, OFE, String, B) -> Union(OFE, "failed")
checkBudan  : (UP, OFE, OFE, B) -> Union(B, "failed")
checkDeriv  : (UP, OFE, OFE) -> Union(B, "failed")
sameSign    : (UP, OFE, OFE) -> Union(B, "failed")
intrat      : (OFE, OFE) -> U
findRealZero: (UPZ, U, B) -> List REC

variation(p, a)      == var p(monomial(1, 1)$UP - a::UP)
keeprec?(a, rec)     == (a > rec.right) or (a < rec.left)

checkHalfAx(p, a, d, incl?) ==
  posRoot(p(d * (monomial(1, 1)$UP - a::UP)), incl?)

ignore? str ==
  str = IGNOR => true
  error "integrate: last argument must be 'noPole'"

computeInt(k, f, a, b, eval?) ==
  is?(f, "integral"::SE) => "failed"
  if not eval? then f := mkLogPos f
  ((ib := findLimit(f, k, b, "left", eval?)) case "failed") or
      ((ia := findLimit(f, k, a, "right", eval?)) case "failed") => "failed"
  infinite?(ia::OFE) and (ia::OFE = ib::OFE) => "failed"
  ib::OFE - ia::OFE

findLimit(f, k, a, dir, eval?) ==
  r := retractIfCan(a)@Union(F, "failed")
  r case F =>
    eval? => mkLogPos(eval(f, k, r::F))::OFE
    (u := limit(f, equation(k::F, r::F), dir)) case OFE => u::OFE
    "failed"
  (u := limit(f, equation(k::F::OFE, a))) case OFE => u::OFE
  "failed"

mkLogPos f ==
```

```
    lk := empty()$List(K)
    lv := empty()$List(F)
    for k in kernels f | is?(k, "log"::SE) repeat
      if (v := mkKerPos k) case F then
        lk := concat(k, lk)
        lv := concat(v::F, lv)
    eval(f, lk, lv)

  mkKerPos k ==
    (u := negative(f := first argument k)) case "failed" =>
                                                log(f**2) / (2::F)
    u::B => log(-f)
    "positive"

  negative f ==
    (u := sign f) case "failed" => "failed"
    u::Z < 0

  checkForZero(p, x, a, b, incl?) ==
    checkForZero(
      map(s+->s::F, univariate(p, x))_
       $SparseUnivariatePolynomialFunctions2(P, F),
          a, b, incl?)

  checkForZero(q, a, b, incl?) ==
    ground? q => false
    (d := maprat q) case UPZ and not((i := intrat(a, b)) case failed) =>
        not empty? findRealZero(d::UPZ, i, incl?)
    (u := checkBudan(q, a, b, incl?)) case "failed" =>
       incl? => checkDeriv(q, a, b)
       "failed"
    u::B

  maprat p ==
    ans:UPQ := 0
    while p ^= 0 repeat
      (r := retractIfCan(c := leadingCoefficient p)@Union(Q,"failed"))
        case "failed"  => return "failed"
      ans := ans + monomial(r::Q, degree p)
      p   := reductum p
    map(numer,(splitDenominator ans).num
       )$SparseUnivariatePolynomialFunctions2(Q, Z)

  intrat(a, b) ==
    (n := whatInfinity a) ^= 0 =>
      (r := retractIfCan(b)@Union(F,"failed")) case "failed" => ["all"]
```

```
        (q := retractIfCan(r::F)@Union(Q, "failed")) case "failed" =>
          ["failed"]
        [[q::Q, n]]
      (q := retractIfCan(retract(a)@F)@Union(Q,"failed")) case "failed"
        => ["failed"]
      (n := whatInfinity b) ^= 0 => [[q::Q, n]]
      (t := retractIfCan(retract(b)@F)@Union(Q,"failed")) case "failed"
        => ["failed"]
      [[q::Q, t::Q]]

    findRealZero(p, i, incl?) ==
      i case fin =>
        l := realZeros(p, r := i.fin)
        incl? => l
        select_!(s+->keeprec?(r.left, s) and keeprec?(r.right, s), l)
      i case all => realZeros p
      i case halfinf =>
        empty?(l := realZeros p) => empty()
        bounds:REC :=
          i.halfinf.dir > 0 => [i.halfinf.endpoint, "max"/[t.right for t in l]]
          ["min"/[t.left for t in l], i.halfinf.endpoint]
        l := [u::REC for t in l | (u := refine(p, t, bounds)) case REC]
        incl? => l
        ep := i.halfinf.endpoint
        select_!(s+->keeprec?(ep, s), l)
      error "findRealZero: should not happpen"

    checkBudan(p, a, b, incl?) ==
      r := retractIfCan(b)@Union(F, "failed")
      (n := whatInfinity a) ^= 0 =>
        r case "failed" => realRoot p
        checkHalfAx(p, r::F, n, incl?)
      (za? := zero? p(aa := retract(a)@F)) and incl? => true
      (n := whatInfinity b) ^= 0 => checkHalfAx(p, aa, n, incl?)
      (zb? := zero? p(bb := r::F)) and incl? => true
      (va := variation(p, aa)) case "failed" or
                  (vb := variation(p, bb)) case "failed" => "failed"
      m:Z := 0
      if za? then m := inc m
      if zb? then m := inc m
      odd?(v := va::Z - vb::Z) =>          -- p has an odd number of roots
        incl? or even? m => true
--        one? v => false
        (v = 1) => false
        "failed"
      zero? v => false                     -- p has no roots
```

```
--      one? m => true                    -- p has an even number > 0 of roots
      (m = 1) => true                     -- p has an even number > 0 of roots
      "failed"

    checkDeriv(p, a, b) ==
      (r := retractIfCan(p)@Union(F, "failed")) case F => zero?(r::F)
      (s := sameSign(p, a, b)) case "failed" => "failed"
      s::B =>                   -- p has the same nonzero sign at a and b
        (u := checkDeriv(differentiate p,a,b)) case "failed" => "failed"
        u::B => "failed"
        false
      true

    realRoot p ==
      (b := posRoot(p, true)) case "failed" => "failed"
      b::B => true
      posRoot(p(p - monomial(1, 1)$UP), true)

    sameSign(p, a, b) ==
      (ea := infeval(p, a)) case "failed" => "failed"
      (eb := infeval(p, b)) case "failed" => "failed"
      (s := sign(ea::F * eb::F)) case "failed" => "failed"
      s::Z > 0

-- returns true if p has a positive root. Include 0 is incl0? is true
    posRoot(p, incl0?) ==
      (z0? := zero?(coefficient(p, 0))) and incl0? => true
      (v := var p) case "failed" => "failed"
      odd?(v::Z) =>             -- p has an odd number of positive roots
        incl0? or not(z0?) => true
--        one?(v::Z) => false
        (v::Z) = 1 => false
        "failed"
      zero?(v::Z) => false      -- p has no positive roots
      z0? => true               -- p has an even number > 0 of positive roots
      "failed"

    infeval(p, a) ==
      zero?(n := whatInfinity a) => p(retract(a)@F)
      (u := signAround(p, n, sign)) case "failed" => "failed"
      u::Z::F

    var q ==
      i:Z := 0
      (lastCoef := negative leadingCoefficient q) case "failed" =>
        "failed"
```

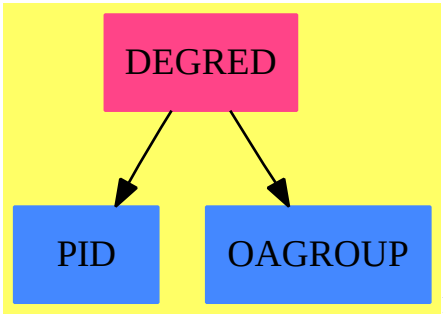
```
        while ((q := reductum q) ^= 0) repeat
          (next := negative leadingCoefficient q) case "failed" =>
            return "failed"
          if ((not(lastCoef::B)) and next::B) or
                         ((not(next::B)) and lastCoef::B) then i := i + 1
          lastCoef := next
        i
```

⟨*DFINTTLS.dotabb*⟩≡

```
"DFINTTLS" [color="#FF4488",href="bookvol10.4.pdf#nameddest=DFINTTLS"]
"ACFS" [color="#4488FF",href="bookvol10.2.pdf#nameddest=ACFS"]
"DFINTTLS" -> "ACFS"
```

5.3 package DEGRED DegreeReductionPackage

5.4 DegreeReductionPackage



1.00

Exports:
expand reduce

⟨*package DEGRED DegreeReductionPackage*⟩≡

```
)abbrev package DEGRED DegreeReductionPackage
++ This package \undocumented{}
DegreeReductionPackage(R1, R2): Cat == Capsule where
    R1: Ring
    R2: Join(IntegralDomain,OrderedSet)

    I    ==> Integer
    PI   ==> PositiveInteger
    UP   ==> SparseUnivariatePolynomial
    RE   ==> Expression R2

    Cat == with
        reduce:  UP R1    ->  Record(pol: UP R1, deg: PI)
                ++ reduce(p) \undocumented{}
        expand:  (RE, PI) ->  List RE
                ++ expand(f,n) \undocumented{}

    Capsule == add


        degrees(u: UP R1): List Integer ==
            l: List Integer := []
            while u ^= 0 repeat
              l := concat(degree u,l)
              u := reductum u
            l
        reduce(u: UP R1) ==
```

```
            g := "gcd"/[d for d in degrees u]
            u := divideExponents(u, g:PI)::(UP R1)
            [u, g:PI]

        import Fraction Integer

        rootOfUnity(j:I,n:I):RE ==
            j = 0 => 1
            arg:RE := 2*j*pi()/(n::RE)
            cos arg + (-1)**(1/2) * sin arg

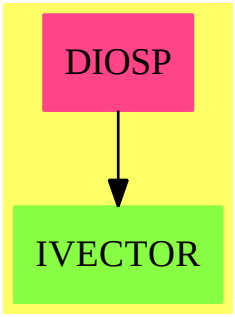
        expand(s, g) ==
            g = 1 => [s]
            [rootOfUnity(i,g)*s**(1/g) for i in 0..g-1]
```

⟨*DEGRED.dotabb*⟩≡

```
"DEGRED" [color="#FF4488",href="bookvol10.4.pdf#nameddest=DEGRED"]
"PID" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PID"]
"OAGROUP" [color="#4488FF",href="bookvol10.2.pdf#nameddest=OAGROUP"]
"DEGRED" -> "PID"
"DEGRED" -> "OAGROUP"
```

5.5 package DIOSP DiophantineSolutionPackage

5.6 DiophantineSolutionPackage



1.00

Exports:
dioSolve

⟨*package DIOSP DiophantineSolutionPackage*⟩≡

```
)abbrev package DIOSP DiophantineSolutionPackage
++ Author: A. Fortenbacher
++ Date Created: 29 March 1991
++ Date Last Updated: 29 March 1991
++ Basic Operations: dioSolve
++ Related Constructors: Equation, Vector
++ Also See:
++ AMS Classifications:
++ Keywords: Diophantine equation, nonnegative solutions,
++   basis, depth-first-search
++ Reference:
++   M. Clausen, A. Fortenbacher: Efficient Solution of
++   Linear Diophantine Equations. in JSC (1989) 8, 201-216
++ Description:
++   any solution of a homogeneous linear Diophantine equation
++   can be represented as a sum of minimal solutions, which
++   form a "basis" (a minimal solution cannot be represented
++   as a nontrivial sum of solutions)
++   in the case of an inhomogeneous linear Diophantine equation,
++   each solution is the sum of a inhomogeneous solution and
++   any number of homogeneous solutions
++   therefore, it suffices to compute two sets:
++      1. all minimal inhomogeneous solutions
++      2. all minimal homogeneous solutions
++   the algorithm implemented is a completion procedure, which
++   enumerates all solutions in a recursive depth-first-search
++   it can be seen as finding monotone paths in a graph
```

```
++   for more details see Reference

DiophantineSolutionPackage(): Cat == Capsule where

  B ==> Boolean
  I ==> Integer
  NI ==> NonNegativeInteger

  LI ==> List(I)
  VI ==> Vector(I)
  VNI ==> Vector(NI)

  POLI ==> Polynomial(I)
  EPOLI ==> Equation(POLI)
  LPOLI ==> List(POLI)

  S ==> Symbol
  LS ==> List(S)

  ListSol ==> List(VNI)
  Solutions ==> Record(varOrder: LS, inhom: Union(ListSol,"failed"),
                       hom: ListSol)

  Node ==> Record(vert: VI, free: B)
  Graph ==> Record(vn: Vector(Node), dim : NI, zeroNode: I)

  Cat ==> with

    dioSolve: EPOLI -> Solutions
      ++ dioSolve(u) computes a basis of all minimal solutions for
      ++ linear homogeneous Diophantine equation u,
      ++ then all minimal solutions of inhomogeneous equation

  Capsule ==> add

    import I
    import POLI

    -- local function specifications

    initializeGraph: (LPOLI, I) -> Graph
    createNode: (I, VI, NI, I) -> Node
    findSolutions: (VNI, I, I, I, Graph, B) -> ListSol
    verifyMinimality: (VNI, Graph, B) -> B
    verifySolution: (VNI, I, I, I, Graph) -> B
```

```
-- exported functions

dioSolve(eq) ==
  p := lhs(eq) - rhs(eq)
  n := totalDegree(p)
  n = 0 or n > 1 =>
    error "a linear Diophantine equation is expected"
  mon := empty()$LPOLI
  c : I := 0
  for x in monomials(p) repeat
    ground?(x) =>
      c := ground(x) :: I
    mon := cons(x, mon)$LPOLI
  graph := initializeGraph(mon, c)
  sol := zero(graph.dim)$VNI
  hs := findSolutions(sol, graph.zeroNode, 1, 1, graph, true)
  ihs : ListSol :=
    c = 0 => [sol]
    findSolutions(sol, graph.zeroNode + c, 1, 1, graph, false)
  vars := [first(variables(x))$LS for x in mon]
  [vars, if empty?(ihs)$ListSol then "failed" else ihs, hs]

-- local functions

initializeGraph(mon, c) ==
  coeffs := vector([first(coefficients(x))$LI for x in mon])$VI
  k := #coeffs
  m := min(c, reduce(min, coeffs)$VI)
  n := max(c, reduce(max, coeffs)$VI)
  [[createNode(i, coeffs, k, 1 - m) for i in m..n], k, 1 - m]

createNode(ind, coeffs, k, zeroNode) ==
  -- create vertices from node ind to other nodes
  v := zero(k)$VI
  for i in 1..k repeat
    ind > 0 =>
      coeffs.i < 0 =>
        v.i := zeroNode + ind + coeffs.i
    coeffs.i > 0 =>
      v.i := zeroNode + ind + coeffs.i
  [v, true]

findSolutions(sol, ind, m, n, graph, flag) ==
  -- return all solutions (paths) from node ind to node zeroNode
  sols := empty()$ListSol
  node := graph.vn.ind
```

```
    node.free =>
      node.free := false
      v := node.vert
      k := if ind < graph.zeroNode then m else n
      for i in k..graph.dim repeat
        x := sol.i
        v.i > 0 =>  -- vertex exists to other node
          sol.i := x + 1
          v.i = graph.zeroNode =>  -- solution found
            verifyMinimality(sol, graph, flag) =>
              sols := cons(copy(sol)$VNI, sols)$ListSol
              sol.i := x
            sol.i := x
          s :=
            ind < graph.zeroNode =>
              findSolutions(sol, v.i, i, n, graph, flag)
            findSolutions(sol, v.i, m, i, graph, flag)
          sols := append(s, sols)$ListSol
          sol.i := x
      node.free := true
      sols
    sols

  verifyMinimality(sol, graph, flag) ==
    -- test whether sol contains a minimal homogeneous solution
    flag =>  -- sol is a homogeneous solution
      i := 1
      while sol.i = 0 repeat
        i := i + 1
      x := sol.i
      sol.i := (x - 1) :: NI
      flag := verifySolution(sol, graph.zeroNode, 1, 1, graph)
      sol.i := x
      flag
    verifySolution(sol, graph.zeroNode, 1, 1, graph)

  verifySolution(sol, ind, m, n, graph) ==
    -- test whether sol contains a path from ind to zeroNode
    flag := true
    node := graph.vn.ind
    v := node.vert
    k := if ind < graph.zeroNode then m else n
    for i in k..graph.dim while flag repeat
      x := sol.i
      x > 0 and v.i > 0 =>  -- vertex exists to other node
        sol.i := (x - 1) :: NI
```

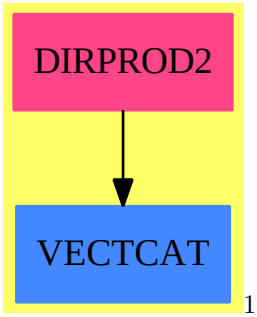
```
            v.i = graph.zeroNode =>  -- solution found
              flag := false
              sol.i := x
            flag :=
              ind < graph.zeroNode =>
                verifySolution(sol, v.i, i, n, graph)
              verifySolution(sol, v.i, m, i, graph)
            sol.i := x
          flag
```

⟨*DIOSP.dotabb*⟩≡

```
"DIOSP" [color="#FF4488",href="bookvol10.4.pdf#nameddest=DIOSP"]
"IVECTOR" [color="#88FF44",href="bookvol10.3.pdf#nameddest=IVECTOR"]
"DIOSP" -> "IVECTOR"
```

5.7 package DIRPROD2 DirectProductFunctions2

5.8 DirectProductFunctions2



1.00

Exports:
map reduce scan

⟨*package DIRPROD2 DirectProductFunctions2*⟩≡

```
)abbrev package DIRPROD2 DirectProductFunctions2
++ Author:
++ Date Created:
++ Date Last Updated:
++ Basic Functions:
++ Related Constructors:
++ Also See:
++ AMS Classifications:
++ Keywords:
++ References:
++ Description:
++   This package provides operations which all take as arguments
++ direct products of elements of some type \spad{A} and functions from \spad{A} to another
++ type B. The operations all iterate over their vector argument
++ and either return a value of type B or a direct product over B.

DirectProductFunctions2(dim, A, B): Exports == Implementation where
  dim : NonNegativeInteger
  A, B: Type

  DA ==> DirectProduct(dim, A)
  DB ==> DirectProduct(dim, B)
  VA ==> Vector A
  VB ==> Vector B
  O2 ==> FiniteLinearAggregateFunctions2(A, VA, B, VB)

  Exports ==> with
```

```
    scan   : ((A, B) -> B, DA, B) -> DB
      ++ scan(func,vec,ident) creates a new vector whose elements are
      ++ the result of applying reduce to the binary function func,
      ++ increasing initial subsequences of the vector vec,
      ++ and the element ident.
    reduce : ((A, B) -> B, DA, B) -> B
      ++ reduce(func,vec,ident) combines the elements in vec using the
      ++ binary function func. Argument ident is returned if the vector is empty.
    map    : (A -> B, DA) -> DB
      ++ map(f, v) applies the function f to every element of the vector v
      ++ producing a new vector containing the values.

  Implementation ==> add
    import FiniteLinearAggregateFunctions2(A, VA, B, VB)

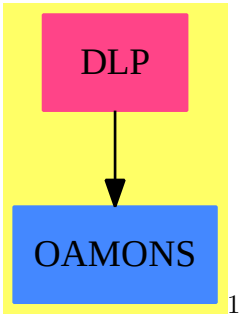
    map(f, v)       == directProduct map(f, v::VA)
    scan(f, v, b)   == directProduct scan(f, v::VA, b)
    reduce(f, v, b) == reduce(f, v::VA, b)
```

⟨*DIRPROD2.dotabb*⟩≡

```
"DIRPROD2" [color="#FF4488",href="bookvol10.4.pdf#nameddest=DIRPROD2"]
"VECTCAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=VECTCAT"]
"DIRPROD2" -> "VECTCAT"
```

5.9 package DLP DiscreteLogarithmPackage

5.10 DiscreteLogarithmPackage



1.00

Exports:
shanksDiscLogAlgorithm

⟨*package DLP DiscreteLogarithmPackage*⟩≡

```
)abbrev package DLP DiscreteLogarithmPackage
++ Author: J. Grabmeier, A. Scheerhorn
++ Date Created: 12 March 1991
++ Date Last Updated: 31 March 1991
++ Basic Operations:
++ Related Constructors:
++ Also See:
++ AMS Classifications:
++ Keywords: discrete logarithm
++ References:
++  J. Grabmeier, A. Scheerhorn: Finite Fields in AXIOM.
++  AXIOM Technical Report Series, ATR/5 NP2522.
++ Description:
++  DiscreteLogarithmPackage implements help functions for discrete logarithms
++  in monoids using small cyclic groups.

DiscreteLogarithmPackage(M): public == private where
  M : Join(Monoid,Finite) with
   "**": (M,Integer) -> M
        ++ x ** n returns x raised to the integer power n
  public ==> with
    shanksDiscLogAlgorithm:(M,M,NonNegativeInteger)->  _
        Union(NonNegativeInteger,"failed")
      ++ shanksDiscLogAlgorithm(b,a,p) computes s with \spad{b**s = a} for
      ++ assuming that \spad{a} and b are elements in a 'small' cyclic group of
      ++ order p by Shank's algorithm.
      ++ Note: this is a subroutine of the function \spadfun{discreteLog}.
```

```
I   ==> Integer
PI  ==> PositiveInteger
NNI ==> NonNegativeInteger
SUP ==> SparseUnivariatePolynomial
DLP ==> DiscreteLogarithmPackage

private ==> add
  shanksDiscLogAlgorithm(logbase,c,p) ==
    limit:Integer:= 30
    -- for logarithms up to cyclic groups of order limit a full
    -- logarithm table is computed
    p < limit =>
      a:M:=1
      disclog:Integer:=0
      found:Boolean:=false
      for i in 0..p-1 while not found repeat
        a = c =>
          disclog:=i
          found:=true
        a:=a*logbase
      not found =>
        messagePrint("discreteLog: second argument not in cyclic group_
generated by first argument")$OutputForm
        "failed"
      disclog pretend NonNegativeInteger
    l:Integer:=length(p)$Integer
    if odd?(l)$Integer then n:Integer:= shift(p,-(l quo 2))
                       else n:Integer:= shift(1,(l quo 2))
    a:M:=1
    exptable : Table(PI,NNI) :=table()$Table(PI,NNI)
    for i in (0::NNI)..(n-1)::NNI repeat
      insert_!([lookup(a),i::NNI]$Record(key:PI,entry:NNI),_
                exptable)$Table(PI,NNI)
      a:=a*logbase
    found := false
    end := (p-1) quo n
    disclog:Integer:=0
    a := c
    b := logbase ** (-n)
    for i in 0..end while not found repeat
      rho:= search(lookup(a),exptable)_
            $Table(PositiveInteger,NNI)
      rho case NNI =>
        found := true
        disclog:= n * i + rho pretend Integer
      a := a * b
```

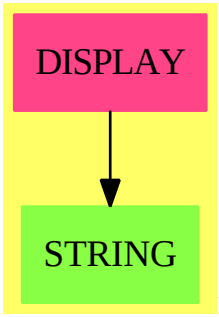
```
        not found =>
          messagePrint("discreteLog: second argument not in cyclic group_
   generated by first argument")$OutputForm
          "failed"
        disclog pretend NonNegativeInteger
```

⟨*DLP.dotabb*⟩≡

```
  "DLP" [color="#FF4488",href="bookvol10.4.pdf#nameddest=DLP"]
  "OAMONS" [color="#4488FF",href="bookvol10.2.pdf#nameddest=OAMONS"]
  "DLP" -> "OAMONS"
```

5.11 package DISPLAY DisplayPackage

5.12 DisplayPackage



1.00

Exports:
bright center copies newLine say sayLength

⟨*package DISPLAY DisplayPackage*⟩≡

```
)abbrev package DISPLAY DisplayPackage
++ Author: Robert S. Sutor
++ Date Created: September 1986
++ Date Last Updated:
++ Basic Operations: bright, newLine, copies, center, say, sayLength
++ Related Constructors:
++ Also See:
++ AMS Classifications:
++ Keywords:
++ References:
++ Description: DisplayPackage allows one to print strings in a nice manner,
++ including highlighting substrings.

DisplayPackage: public == private where
  I       ==> Integer
  L       ==> List
  S       ==> String
  RECLR   ==> Record(lhs : S, rhs : S)

  public  == with
    bright:       S           -> L S
      ++ bright(s) sets the font property of the string s to bold-face type.
    bright:       L S         -> L S
      ++ bright(l) sets the font property of a list of strings, l, to
      ++ bold-face type.
    newLine:      ()          -> S
      ++ newLine() sends a new line command to output.
```

```
    copies:       (I,S)       -> S
      ++ copies(i,s) will take a string s and create a new string composed of
      ++ i copies of s.
    center:       (S,I,S)     -> S
      ++ center(s,i,s) takes the first string s, and centers it within a string
      ++ of length i, in which the other elements of the string are composed
      ++ of as many replications as possible of the second indicated string, s
      ++ which must have a length greater than that of an empty string.
    center:       (L S,I,S)   -> L S
      ++ center(l,i,s) takes a list of strings l, and centers them within a
      ++ list of strings which is i characters long, in which the remaining
      ++ spaces are filled with strings composed of as many repetitions as
      ++ possible of the last string parameter s.

    say:          S           -> Void
      ++ say(s) sends a string s to output.
    say:          L S         -> Void
      ++ say(l) sends a list of strings l to output.
    sayLength:    S           -> I
      ++ sayLength(s) returns the length of a string s as an integer.
    sayLength:    L S         -> I
      ++ sayLength(l) returns the length of a list of strings l as an integer.

  private == add
    --StringManipulations()

    center0:  (I,I,S) -> RECLR

    s : S
    l : L S

    HION    : S := "%b"
    HIOFF   : S := "%d"
    NEWLINE : S := "%l"

    bright s == [HION,s,HIOFF]$(L S)
    bright l == cons(HION,append(l,list HIOFF))
    newLine() == NEWLINE

    copies(n : I, s : S) ==
      n < 1 => ""
      n = 1 => s
      t : S := copies(n quo 2, s)
      odd? n => concat [s,t,t]
      concat [t,t]
```

```
    center0(len : I, wid : I, fill : S) : RECLR ==
      (wid < 1) or (len >= wid) => ["",""]$RECLR
      m : I := (wid - len) quo 2
      t : S := copies(1 + (m quo (sayLength fill)),fill)
      [t(1..m),t(1..wid-len-m)]$RECLR

    center(s, wid, fill) ==
      wid < 1 => ""
      len : I := sayLength s
      len = wid => s
      len > wid => s(1..wid)
      rec : RECLR := center0(len,wid,fill)
      concat [rec.lhs,s,rec.rhs]

    center(l, wid, fill) ==
      wid < 1 => [""]$(L S)
      len : I := sayLength l
      len = wid => l
--    len > wid => s(1..wid)
      rec : RECLR := center0(len,wid,fill)
      cons(rec.lhs,append(l,list rec.rhs))

    say s ==
      sayBrightly$Lisp s
      void()$Void

    say l ==
      sayBrightly$Lisp l
      void()$Void

    sayLength s == #s

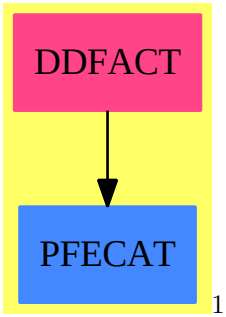
    sayLength l ==
      sum : I := 0
      for s in l repeat
        s = HION      => sum := sum + 1
        s = HIOFF     => sum := sum + 1
        s = NEWLINE   => sum
        sum := sum + sayLength s
      sum
```

⟨*DISPLAY.dotabb*⟩≡

```
"DISPLAY" [color="#FF4488",href="bookvol10.4.pdf#nameddest=DISPLAY"]
"STRING" [color="#88FF44",href="bookvol10.3.pdf#nameddest=STRING"]
"DISPLAY" -> "STRING"
```

5.13 package DDFACT DistinctDegreeFactorize

5.14 DistinctDegreeFactorize



1.00

Exports:

distdfact	exptMod	factor	factorSquareFree	irreducible?
separateDegrees	separateFactors	trace2PowMod	tracePowMod	

⟨*package DDFACT DistinctDegreeFactorize*⟩≡

```
)abbrev package DDFACT DistinctDegreeFactorize
++ Author: P. Gianni, B.Trager
++ Date Created: 1983
++ Date Last Updated: 22 November 1993
++ Basic Functions: factor, irreducible?
++ Related Constructors: PrimeField, FiniteField
++ Also See:
++ AMS Classifications:
++ Keywords:
++ References:
++ Description:
++   Package for the factorization of a univariate polynomial with
++   coefficients in a finite field. The algorithm used is the
++   "distinct degree" algorithm of Cantor-Zassenhaus, modified
++   to use trace instead of the norm and a table for computing
++   Frobenius as suggested by Naudin and Quitte .

DistinctDegreeFactorize(F,FP): C == T
  where
   F  : FiniteFieldCategory
   FP : UnivariatePolynomialCategory(F)

   fUnion ==> Union("nil", "sqfr", "irred", "prime")
   FFE    ==> Record(flg:fUnion, fctr:FP, xpnt:Integer)
   NNI       == NonNegativeInteger
   Z         == Integer
```

```
fact      == Record(deg : NNI,prod : FP)
ParFact   == Record(irr:FP,pow:Z)
FinalFact == Record(cont:F,factors:List(ParFact))

C == with
   factor        :         FP      -> Factored FP
     ++ factor(p) produces the complete factorization of the polynomial p.
   factorSquareFree :         FP      -> Factored FP
     ++ factorSquareFree(p) produces the complete factorization of the
     ++ square free polynomial p.
   distdfact       :   (FP,Boolean)  -> FinalFact
     ++ distdfact(p,sqfrflag) produces the complete factorization
     ++ of the polynomial p returning an internal data structure.
     ++ If argument sqfrflag is true, the polynomial is assumed square free.
   separateDegrees :         FP      -> List fact
     ++ separateDegrees(p) splits the square free polynomial p into
     ++ factors each of which is a product of irreducibles of the same degree.
   separateFactors :  List fact  -> List FP
     ++ separateFactors(lfact) takes the list produced by
     ++ \spadfunFrom{separateDegrees}{DistinctDegreeFactorization}
     ++ and produces the complete list of factors.
   exptMod         :   (FP,NNI,FP)   -> FP
     ++ exptMod(u,k,v) raises the polynomial u to the kth power
     ++ modulo the polynomial v.
   trace2PowMod     :   (FP,NNI,FP)   -> FP
     ++ trace2PowMod(u,k,v) produces the sum of \spad{u**(2**i)} for i running
     ++ from 1 to k all computed modulo the polynomial v.
   tracePowMod     :   (FP,NNI,FP)   -> FP
     ++ tracePowMod(u,k,v) produces the sum of \spad{u**(q**i)}
     ++ for i running and q= size F
   irreducible?    :         FP      -> Boolean
     ++ irreducible?(p) tests whether the polynomial p is irreducible.


T == add
   --declarations
   D:=ModMonic(F,FP)
   import UnivariatePolynomialSquareFree(F,FP)

   --local functions
   notSqFr : (FP,FP -> List(FP)) -> List(ParFact)
   ddffact : FP -> List(FP)
   ddffact1 : (FP,Boolean) -> List fact
   ranpol :         NNI       -> FP

   charF : Boolean := characteristic()$F = 2
```

```
--construct a random polynomial of random degree < d
ranpol(d:NNI):FP ==
  k1: NNI := 0
  while k1 = 0 repeat k1 := random d
  -- characteristic F = 2
  charF =>
     u:=0$FP
     for j in 1..k1 repeat u:=u+monomial(random()$F,j)
     u
  u := monomial(1,k1)
  for j in 0..k1-1 repeat u:=u+monomial(random()$F,j)
  u

notSqFr(m:FP,appl: FP->List(FP)):List(ParFact) ==
  factlist : List(ParFact) :=empty()
  llf : List FFE
  fln :List(FP) := empty()
  if (lcm:=leadingCoefficient m)^=1 then m:=(inv lcm)*m
  llf:= factorList(squareFree(m))
  for lf in llf repeat
    d1:= lf.xpnt
    pol := lf.fctr
    if (lcp:=leadingCoefficient pol)^=1 then pol := (inv lcp)*pol
    degree pol=1 => factlist:=cons([pol,d1]$ParFact,factlist)
    fln := appl(pol)
    factlist :=append([[pf,d1]$ParFact for pf in fln],factlist)
  factlist

-- compute u**k mod v (requires call to setPoly of multiple of v)
-- characteristic not equal 2
exptMod(u:FP,k:NNI,v:FP):FP == (reduce(u)$D**k):FP rem v

-- compute u**k mod v (requires call to setPoly of multiple of v)
-- characteristic equal 2
trace2PowMod(u:FP,k:NNI,v:FP):FP ==
  uu:=u
  for i in 1..k repeat uu:=(u+uu*uu) rem v
  uu

-- compute u+u**q+..+u**(q**k) mod v
-- (requires call to setPoly of multiple of v) where q=size< F
tracePowMod(u:FP,k:NNI,v:FP):FP ==
  u1 :D :=reduce(u)$D
  uu : D := u1
  for i in 1..k repeat uu:=(u1+frobenius uu)
```

```
    (lift uu) rem v

  -- compute u**(1+q+..+q**k) rem v where q=#F
  -- (requires call to setPoly of multiple of v)
  -- frobenius map is used
  normPowMod(u:FP,k:NNI,v:FP):FP ==
    u1 :D :=reduce(u)$D
    uu : D := u1
    for i in 1..k repeat uu:=(u1*frobenius uu)
    (lift uu) rem v

  --find the factorization of m as product of factors each containing
  --terms of equal degree .
  -- if testirr=true the function returns the first factor found
  ddffact1(m:FP,testirr:Boolean):List(fact) ==
    p:=size$F
    dg:NNI :=0
    ddfact:List(fact):=empty()
    --evaluation of x**p mod m
    k1:NNI
    u:= m
    du := degree u
    setPoly u
    mon: FP := monomial(1,1)
    v := mon
    for k1 in 1.. while k1 <= (du quo 2) repeat
        v := lift frobenius reduce(v)$D
        g := gcd(v-mon,u)
        dg := degree g
        dg =0  => "next k1"
        if leadingCoefficient g ^=1 then g := (inv leadingCoefficient g)*g
        ddfact := cons([k1,g]$fact,ddfact)
        testirr => return ddfact
        u := u quo g
        du := degree u
        du = 0 => return ddfact
        setPoly u
    cons([du,u]$fact,ddfact)

  -- test irreducibility
  irreducible?(m:FP):Boolean ==
    mf:fact:=first ddffact1(m,true)
    degree m = mf.deg

  --export ddfact1
  separateDegrees(m:FP):List(fact) == ddffact1(m,false)
```

```
--find the complete factorization of m, using the result of ddfact1
separateFactors(distf : List fact) :List FP ==
  ddfact := distf
  n1:Integer
  p1:=size()$F
  if charF then n1:=length(p1)-1
  newaux,aux,ris : List FP
  ris := empty()
  t,fprod : FP
  for ffprod in ddfact repeat
    fprod := ffprod.prod
    d := ffprod.deg
    degree fprod = d => ris := cons(fprod,ris)
    aux:=[fprod]
    setPoly fprod
    while ^(empty? aux) repeat
      t := ranpol(2*d)
      if charF then t:=trace2PowMod(t,(n1*d-1)::NNI,fprod)
      else t:=exptMod(tracePowMod(t,(d-1)::NNI,fprod),
                                (p1 quo 2)::NNI,fprod)-1$FP
      newaux:=empty()
      for u in aux repeat
          g := gcd(u,t)
          dg:= degree g
          dg=0 or dg = degree u => newaux:=cons(u,newaux)
          v := u quo g
          if dg=d then ris := cons(inv(leadingCoefficient g)*g,ris)
                  else newaux := cons(g,newaux)
          if degree v=d then ris := cons(inv(leadingCoefficient v)*v,ris)
                        else newaux := cons(v,newaux)
      aux:=newaux
  ris

--distinct degree algorithm for monic ,square-free polynomial
ddffact(m:FP):List(FP)==
  ddfact:=ddffact1(m,false)
  empty? ddfact => [m]
  separateFactors ddfact

--factorize a general polynomial with distinct degree algorithm
--if test=true no check is executed on square-free
distdfact(m:FP,test:Boolean):FinalFact ==
  factlist: List(ParFact):= empty()
  fln : List(FP) :=empty()
```

```
      --make m monic
      if (lcm := leadingCoefficient m) ^=1 then m := (inv lcm)*m

      --is x**d factor of m?
      if (d := minimumDegree m)>0 then
        m := (monicDivide (m,monomial(1,d))).quotient
        factlist := [[monomial(1,1),d]$ParFact]
      d:=degree m

      --is m constant?
      d=0 => [lcm,factlist]$FinalFact

      --is m linear?
      d=1 => [lcm,cons([m,d]$ParFact,factlist)]$FinalFact

      --m is square-free
      test =>
        fln := ddffact m
        factlist := append([[pol,1]$ParFact for pol in fln],factlist)
        [lcm,factlist]$FinalFact

      --factorize the monic,square-free terms
      factlist:= append(notSqFr(m,ddffact),factlist)
      [lcm,factlist]$FinalFact

    --factorize the polynomial m
    factor(m:FP) ==
      m = 0 => 0
      flist := distdfact(m,false)
      makeFR(flist.cont::FP,[["prime",u.irr,u.pow]$FFE
                             for u in flist.factors])

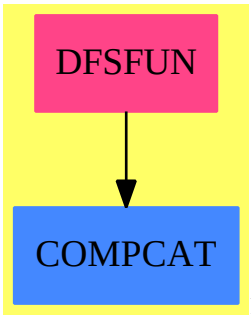

    --factorize the square free polynomial m
    factorSquareFree(m:FP) ==
      m = 0 => 0
      flist := distdfact(m,true)
      makeFR(flist.cont::FP,[["prime",u.irr,u.pow]$FFE
                             for u in flist.factors])
```

⟨*DDFACT.dotabb*⟩≡

```
"DDFACT" [color="#FF4488",href="bookvol10.4.pdf#nameddest=DDFACT"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"DDFACT" -> "PFECAT"
```

5.15 package DFSFUN DoubleFloatSpecialFunctions

5.16 DoubleFloatSpecialFunctions



1.00

Exports:

airyAi	airyBi	besselI	besselJ	besselK
besselY	Beta	digamma	E1	Ei
Ei1	Ei2	Ei3	Ei4	Ei5
Ei6	En	Gamma	hypergeometric0F1	logGamma
polygamma				

⟨*package DFSFUN DoubleFloatSpecialFunctions*⟩≡

```
)abbrev package DFSFUN DoubleFloatSpecialFunctions
++ Author: Bruce W. Char, Timothy Daly, Stephen M. Watt
++ Date Created:  1990
++ Date Last Updated: Jan 19, 2008
++ Basic Operations:
++ Related Domains:
++ Also See:
++ AMS Classifications:
++ Keywords:
++ Examples:
++ References:
++ Description:
++   This package provides special functions for double precision
++   real and complex floating point.

DoubleFloatSpecialFunctions(): Exports == Impl where
    NNI ==> NonNegativeInteger
    PI  ==> Integer
    R   ==> DoubleFloat
    C   ==> Complex DoubleFloat
    OPR ==> OnePointCompletion R
```

```
Exports ==> with
    Gamma: R -> R
     ++ Gamma(x) is the Euler gamma function, \spad{Gamma(x)}, defined by
     ++   \spad{Gamma(x) = integrate(t^(x-1)*exp(-t), t=0..%infinity)}.
    Gamma: C -> C
     ++ Gamma(x) is the Euler gamma function, \spad{Gamma(x)}, defined by
     ++   \spad{Gamma(x) = integrate(t^(x-1)*exp(-t), t=0..%infinity)}.

    E1: R -> OPR
     ++ E1(x) is the Exponential Integral function
     ++ The current implementation is a piecewise approximation
     ++ involving one poly from -4..4 and a second poly for x > 4

    En: (PI,R) -> OPR
     ++ En(n,x) is the nth Exponential Integral Function

    Ei: (OPR) -> OPR
     ++ Ei is the Exponential Integral function
     ++ This is computed using a 6 part piecewise approximation.
     ++ DoubleFloat can only preserve about 16 digits but the
     ++ Chebyshev approximation used can give 30 digits.

    Ei1: (OPR) -> OPR
     ++ Ei1 is the first approximation of Ei where the result is
     ++ x*%e^-x*Ei(x) from -infinity to -10 (preserves digits)

    Ei2: (OPR) -> OPR
     ++ Ei2 is the first approximation of Ei where the result is
     ++ x*%e^-x*Ei(x) from -10 to -4 (preserves digits)

    Ei3: (OPR) -> OPR
     ++ Ei3 is the first approximation of Ei where the result is
     ++ (Ei(x)-log |x| - gamma)/x from -4 to 4 (preserves digits)

    Ei4: (OPR) -> OPR
     ++ Ei4 is the first approximation of Ei where the result is
     ++ x*%e^-x*Ei(x) from 4 to 12 (preserves digits)

    Ei5: (OPR) -> OPR
     ++ Ei5 is the first approximation of Ei where the result is
     ++ x*%e^-x*Ei(x) from 12 to 32 (preserves digits)

    Ei6: (OPR) -> OPR
     ++ Ei6 is the first approximation of Ei where the result is
     ++ x*%e^-x*Ei(x) from 32 to infinity (preserves digits)
```

```
Beta: (R, R) -> R
 ++ Beta(x, y) is the Euler beta function, \spad{B(x,y)}, defined by
 ++   \spad{Beta(x,y) = integrate(t^(x-1)*(1-t)^(y-1), t=0..1)}.
 ++ This is related to \spad{Gamma(x)} by
 ++   \spad{Beta(x,y) = Gamma(x)*Gamma(y) / Gamma(x + y)}.
Beta: (C, C) -> C
 ++ Beta(x, y) is the Euler beta function, \spad{B(x,y)}, defined by
 ++   \spad{Beta(x,y) = integrate(t^(x-1)*(1-t)^(y-1), t=0..1)}.
 ++ This is related to \spad{Gamma(x)} by
 ++   \spad{Beta(x,y) = Gamma(x)*Gamma(y) / Gamma(x + y)}.

logGamma: R -> R
 ++ logGamma(x) is the natural log of \spad{Gamma(x)}.
 ++ This can often be computed even if \spad{Gamma(x)} cannot.
logGamma: C -> C
 ++ logGamma(x) is the natural log of \spad{Gamma(x)}.
 ++ This can often be computed even if \spad{Gamma(x)} cannot.

digamma: R -> R
 ++ digamma(x) is the function, \spad{psi(x)}, defined by
 ++   \spad{psi(x) = Gamma'(x)/Gamma(x)}.
digamma: C -> C
 ++ digamma(x) is the function, \spad{psi(x)}, defined by
 ++   \spad{psi(x) = Gamma'(x)/Gamma(x)}.

polygamma: (NNI, R) -> R
 ++ polygamma(n, x) is the n-th derivative of \spad{digamma(x)}.
polygamma: (NNI, C) -> C
 ++ polygamma(n, x) is the n-th derivative of \spad{digamma(x)}.

besselJ: (R,R) -> R
 ++ besselJ(v,x) is the Bessel function of the first kind,
 ++ \spad{J(v,x)}.
 ++ This function satisfies the differential equation:
 ++   \spad{x^2 w''(x) + x w'(x) + (x^2-v^2)w(x) = 0}.
besselJ: (C,C) -> C
 ++ besselJ(v,x) is the Bessel function of the first kind,
 ++ \spad{J(v,x)}.
 ++ This function satisfies the differential equation:
 ++   \spad{x^2 w''(x) + x w'(x) + (x^2-v^2)w(x) = 0}.

besselY: (R, R) -> R
 ++ besselY(v,x) is the Bessel function of the second kind,
 ++ \spad{Y(v,x)}.
 ++ This function satisfies the differential equation:
 ++   \spad{x^2 w''(x) + x w'(x) + (x^2-v^2)w(x) = 0}.
```

```
 ++ Note: The default implementation uses the relation
 ++   \spad{Y(v,x) = (J(v,x) cos(v*%pi) - J(-v,x))/sin(v*%pi)}
 ++ so is not valid for integer values of v.
besselY: (C, C) -> C
 ++ besselY(v,x) is the Bessel function of the second kind,
 ++ \spad{Y(v,x)}.
 ++ This function satisfies the differential equation:
 ++   \spad{x^2 w''(x) + x w'(x) + (x^2-v^2)w(x) = 0}.
 ++ Note: The default implementation uses the relation
 ++   \spad{Y(v,x) = (J(v,x) cos(v*%pi) - J(-v,x))/sin(v*%pi)}
 ++ so is not valid for integer values of v.

besselI: (R,R) -> R
 ++ besselI(v,x) is the modified Bessel function of the first kind,
 ++ \spad{I(v,x)}.
 ++ This function satisfies the differential equation:
 ++   \spad{x^2 w''(x) + x w'(x) - (x^2+v^2)w(x) = 0}.
besselI: (C,C) -> C
 ++ besselI(v,x) is the modified Bessel function of the first kind,
 ++ \spad{I(v,x)}.
 ++ This function satisfies the differential equation:
 ++   \spad{x^2 w''(x) + x w'(x) - (x^2+v^2)w(x) = 0}.

besselK: (R, R) -> R
 ++ besselK(v,x) is the modified Bessel function of the second kind,
 ++ \spad{K(v,x)}.
 ++ This function satisfies the differential equation:
 ++   \spad{x^2 w''(x) + x w'(x) - (x^2+v^2)w(x) = 0}.
 ++ Note: The default implementation uses the relation
 ++   \spad{K(v,x) = %pi/2*(I(-v,x) - I(v,x))/sin(v*%pi)}.
 ++ so is not valid for integer values of v.
besselK: (C, C) -> C
 ++ besselK(v,x) is the modified Bessel function of the second kind,
 ++ \spad{K(v,x)}.
 ++ This function satisfies the differential equation:
 ++   \spad{x^2 w''(x) + x w'(x) - (x^2+v^2)w(x) = 0}.
 ++ Note: The default implementation uses the relation
 ++   \spad{K(v,x) = %pi/2*(I(-v,x) - I(v,x))/sin(v*%pi)}
 ++ so is not valid for integer values of v.

airyAi:   C -> C
 ++ airyAi(x) is the Airy function \spad{Ai(x)}.
 ++ This function satisfies the differential equation:
 ++   \spad{Ai''(x) - x * Ai(x) = 0}.
airyAi:   R -> R
 ++ airyAi(x) is the Airy function \spad{Ai(x)}.
```

```
     ++ This function satisfies the differential equation:
     ++   \spad{Ai''(x) - x * Ai(x) = 0}.

    airyBi:   R -> R
     ++ airyBi(x) is the Airy function \spad{Bi(x)}.
     ++ This function satisfies the differential equation:
     ++   \spad{Bi''(x) - x * Bi(x) = 0}.
    airyBi:   C -> C
     ++ airyBi(x) is the Airy function \spad{Bi(x)}.
     ++ This function satisfies the differential equation:
     ++   \spad{Bi''(x) - x * Bi(x) = 0}.

    hypergeometricOF1: (R, R) -> R
     ++ hypergeometricOF1(c,z) is the hypergeometric function
     ++ \spad{0F1(; c; z)}.
    hypergeometricOF1: (C, C) -> C
     ++ hypergeometricOF1(c,z) is the hypergeometric function
     ++ \spad{0F1(; c; z)}.


Impl ==> add
    a, v, w, z: C
    n, x, y: R

    -- These are hooks to Bruce's boot code.
    Gamma z         == CGAMMA(z)$Lisp
    Gamma x         == RGAMMA(x)$Lisp
```

5.16.1 The Exponential Integral

5.16.2 The E1 function

(Quoted from Segletes[**?**]):

A number of useful integrals exist for which no exact solutions have been found. In other cases, an exact solution, if found, may be impractical to utilize over the complete domain of the function because of precision limitations associated with what usually ends up as a series solution to the challenging integral. For many of these integrals, tabulated values may be published in various mathematical handbooks and articles. In some handbooks, fits (usually piecewise) also are offered. In some cases, an application may be forced to resort to numerical integration in order to acquire the integrated function. In this context, compact (*i.e.* not piecewise) analytical fits to some of these problematic integrals, accurate to within a small fraction of the numerically integrated value, serve as a useful tool to applications requiring the results of the integration, especially when the integration is required numerous times throughout the course of the application. Furthermore, the ability and methodology to develop intelligent fits, in contract to the more traditional "brute force" fits, provide the means to minimize parameters and maximize accuracy when tackling some of these difficult functions. The exponential integral will be used as an opportunity to both demonstrate a methodology for intelligent fitting as well as for providing an accurate, compact, analytical fit to the exponential integral.

The exponential integral is a useful class of functions that arise in a variety of applications [...]. The real branch of the family of exponential integrals may be defined as

$$E_n(x) = x^{n-1} \int_x^\infty \frac{e^{-t}}{t^n}\, dt \tag{5.1}$$

where n, a positive integer, denotes the specific member of the exponential integral family. The argument of the exponential integral, rather than expressing a lower limit of integration as in (1), may be thought of as describing the exponential decay constant, as given in this equivalent (and perhaps more popular) definition of the integral:

$$E_n(x) = \int_1^\infty \frac{e^{-xt}}{t^n}\, dt \tag{5.2}$$

Integration by parts permits any member of the exponential integral family to be converted to an adjacent member of the family, by way of

$$\int_x^\infty \frac{e^{-t}}{t^{n+1}}\, dt = \frac{1}{n}\left(\frac{e^{-x}}{x^n} - \int_x^\infty \frac{e^{-t}}{t^n}\, dt\right) \tag{5.3}$$

expressable in terms of E_n as

$$E_{n+1}(x) = \frac{1}{n}\left[e^{-x} - xE_n(x)\right] \;\; (n = 1, 2, 3) \tag{5.4}$$

Through recursive employment of this equation, all members of the exponential integral family may be analytically related. However, this technique only allows for the transformation of one integral into another. There remains the problem of evaluating $E_1(x)$. There is an exact solution to the integral of (e^{-t}/t), appearing in a number of mathematical references [**?**, **?**] which is obtainable by expanding the exponential into a power series and integrating term by term. That exact solution, which is convergent, may be used to specify $E_1(x)$ as

$$E_1(x) = -\gamma - ln(x) + \frac{x}{1!} - \frac{x^2}{2 \cdot 2!} + \frac{x^3}{3 \cdot 3!} - \dots \tag{5.5}$$

Euler's constant, γ, equal to $0.57721\dots$, arises when the power series expansion for (e^{-t}/t) is integrated and evaluated at its upper limit, as $x \to \infty$[**?**].

Employing eqn (5), however, to evaluate $E_1(x)$ is problematic for finite x significantly larger than unity. One may well ask of the need to evaluate the exponential integral for large x, since the function to be integrated drops off so rapidly that the integral is surely a very flat function. Such reasoning is true when comparing the integrand at large x to that at small x. However, the definition of eqn (1) has as its upper limit not a small value of x, but rather that of ∞. Therefore, the actual values for $E_n(x)$ are extremely small numbers for large values of x. Thus, it is not sufficient merely to select enough terms of eqn (5) to evaluate the integral to within a value of, for example ± 0.0001 because the actual integral value for large x would be smaller than this arbitrary tolerance. To draw an analogy, it would be like saying that it is good enough to approximate e^{-x} as 0.0 for $x > 10$, since its actual value is within 0.0001 of zero. For some applications, such an approximation may be warranted. In general, though, such an approximation is mathematically unacceptable. Worse yet, as seen from eqns (1) and (2), the need to evaluate the exponential integral for large arguments can arise in real-world problems from either a large integraion limit or a large value of an exponential decay constant. Thus, the need to evaluate exponential integrals for large values of the argument is established. It is here that the practical problems with the evaluation of eqn (5) become manifest.

First, the number of terms, N, required to achieve convergence rises rapidly with increasing x, making the summation an inefficient tool, even when expressed as a recursion relation (for three digits of accuracy, N is observed to vary roughly as $9 + 1.6x$, for $1 < x < 7$). More important, however, is the fact that, for calculations of finite precision, the accuracy of the complete summation will be governed by the individual term of greatest magnitude. The source of the problem is that as x is increased, the total summation decreases in magnitude more rapidly than a decaying exponential, while at the same time, the largest individual term in the series is observed to grow rapidly with increasing x ($\urcorner 10^1$ for $x = 7, \urcorner 10^2$ for $x = 10, \urcorner 10^3$ for $x = 13$, *etc.*). The magnitude of this largest individual term consumes the available precision and, as a result, leaves little or none left for the ever-diminishing net sum that constitutes the desired integral.

Literally, the use of eqn (5), even with (32-bit) double precision, does not permit the exponential integral to be evaluated to three places for $x > 14$ in any case, and with the situation worsening for lesser precision. For these reasons, the use of eqn (5) to evaluate the exponential integral numerically for large x is wholly unsuitable.

$$E_1(x) = e^{-x} \cdot \cfrac{1}{x + \cfrac{1}{1 + \cfrac{1}{x + \cfrac{2}{1 + \cfrac{2}{x + \ldots}}}}} \tag{5.6}$$

But as x becomes smaller, the number of terms required for convergence rises quickly. Similar arguments apply for the use of an asymptotic expansion for E_1, which also converges for large x. As such, the more typical approach employed by handbooks is that of a fit. While some steps are taken to make the fits intelligent (*e.g.*, transformation of variables), the fits are all piecewise over the domain of the integral.

Cody and Thatcher [**?**] performed what is perhaps the definitive work, with the use of Chebyshev[**?**, **?**] approximations to the exponential integral E_1. Like others, they fit the integral over a piecewise series of subdomains (three in their case) and provide the fitting parameters necessary to evaluate the function to various required precisions, down to relative errors of 10^-20. One of the problems with piecewise fitting over two or more subdomains is that functional value and derivatives of the spliced fits will not, in general, match at the domain transition point, unless special accomodations are made. This sort of discontinuity in functional value and/or slope, curvature, *etc.*, may cause difficulties for some numerical algorithms operating upon the fitted function. Numerical splicing/smoothing algorithms aimed at eliminating discontinuities in the value and/or derivatives of a piecewise fit are not, in general, computationally insignificant. Problems associated with piecewise splicing of fits may also be obviated by obtaining an accurate enough fit, such that the error is on the order of magnitude of the limiting machine precision. This alternative, however, requires the use of additional fitting parameters to acquire the improved precision. Thus, regardless of approach, the desire to eliminate discontinuities in the function and its derivatives, between piecewise splices, requires extra computational effort. One final benefit to be had by avoiding the use of piecewise fits is the concomitant avoidance of conditional (*i.e.*, IF...THEN) programming statements in the coding of the routine. The use of conditional statements can preclude maximum computing efficiency on certain parallel computing architectures.

Segletes constructs an analytic, non-piecewise fit to the Exponential Integral but the precision is on the order of 4 decimal places and is not sufficient to compare against the Abramowitz and Stegun Handbook.

Instead we have chosen to use a two piece fitting function based on the Cheby-

shev polynomial for computing E_1. This agrees with the handbook values to almost the last published digit. See the `e1.input` pamphlet for regression testing against the handbook tables.

5.16.3 E1:R→OPR

The special function E1 below was originally derived from a function written by T.Haavie as the `expint.c` function in the Numlibc library by Lars Erik Lund. Haavie approximates the E1 function by two Chebyshev polynomials. For the range $-4 < x < 4$ the Chebyshev coefficients are:

```
 7.8737715392882774, -8.0314874286705335,  3.8797325768522250,
-1.6042971072992259,  0.5630905453891458, -0.1704423017433357,
 0.0452099390015415, -0.0106538986439085,  0.0022562638123478,
-0.0004335700473221,  0.0000762166811878, -0.0000123417443064,
 0.0000018519745698, -0.0000002588698662,  0.0000000338604319,
-0.0000000041611418,  0.0000000004821606, -0.0000000000528465,
 0.0000000000054945, -0.0000000000005433,  0.0000000000000512,
-0.0000000000000046,  0.0000000000000004
```

and for the range $x > 4$ the Chebyshev coefficients are:

```
 0.2155283776715125,  0.1028106215227030, -0.0045526707131788,
 0.0003571613122851, -0.0000379341616932,  0.0000049143944914,
-0.0000007355024922,  0.0000001230603606, -0.0000000225236907,
 0.0000000044412375, -0.0000000009328509,  0.0000000002069297,
-0.0000000000481502,  0.0000000000116891, -0.0000000000029474,
 0.0000000000007691, -0.0000000000002070,  0.0000000000000573,
-0.0000000000000163,  0.0000000000000047, -0.0000000000000014,
 0.0000000000000004, -0.0000000000000001
```

I've rewritten the polynomial to use precomputed coefficients that take into account the scaling used by Haavie. I've also rewritten the polynomial using Horner's method so the large powers of x are only computed once.

The result can be either a double float or, or if the argument is zero, infinity. Thus we need to extend the result to be a one-point completion to include infinity.

⟨*package DFSFUN DoubleFloatSpecialFunctions*⟩+≡

```
E1(x:R):OPR ==
 x = 0.0::R => infinity()
 x > 4.0::R =>
  t1:R:=0.14999948967737774608E-15::R
  t2:R:=0.9999999999993112::R
  ta:R:=(t1*x+t2)
  t3:R:=0.99999999953685760001::R
  tb:R:=(ta*x-t3)
```

```
t4:R:=1.9999998808293376::R
tc:R:=(tb*x+t4)
t5:R:=5.999983407661056::R
td:R:=(tc*x-t5)
t6:R:=23.9985380938481664::R
te:R:=(td*x+t6)
t7:R:=119.9108830382784512::R
tf:R:=(te*x-t7)
t8:R:=716.01351020920176641::R
tg:R:=(tf*x+t8)
t9:R:=4903.3466623370985473::R
th:R:=(tg*x-t9)
t10:R:=36601.25841454446674::R
ti:R:=(th*x+t10)
t11:R:=279913.28608482691646::R
tj:R:=(ti*x-t11)
t12:R:=2060518.7020296525186::R
tk:R:=(tj*x+t12)
t13:R:=13859772.093039815059::R
tl:R:=(tk*x-t13)
t14:R:=81945572.630072918857::R
tm:R:=(tl*x+t14)
t15:R:=413965714.82128317479::R
tn:R:=(tm*x-t15)
t16:R:=1747209536.2595547568::R
to:R:=(tn*x+t16)
t17:R:=6036182333.96179427::R
tp:R:=(to*x-t17)
t18:R:=16693683576.106267572::R
tq:R:=(tp*x+t18)
t19:R:=35938625644.58286097::R
tr:R:=(tq*x-t19)
t20:R:=57888657293.609258888::R
ts:R:=(tr*x+t20)
t21:R:=65523779423.11290127::R
tt:R:=(ts*x-t21)
t22:R:=46422751473.201760309::R
tu:R:=(tt*x+t22)
t23:R:=15474250491.067253436::R
tv:R:=(tu*x-t23)
tw:R:=(-1.0::R*x)
tx:R:=exp(tw)
ty:R:=tv*tx
tz:R:=x**22
taz:R:=ty/tz
taz::OPR
```

```
x > -4.0::R =>
 a1:R:=0.476837158203125E-22::R
 a2:R:=0.10967254638671875E-20::R
 aa:R:=(-a1*x+a2)
 a3:R:=0.2021789550781250001E-19::R
 ab:R:=(aa*x-a3)
 a4:R:=0.4260063171386718750 1E-18::R
 ac:R:=(ab*x+a4)
 a5:R:=0.868625640869140625E-17::R
 ad:R:=(ac*x-a5)
 a6:R:=0.16553192138671875E-15::R
 ae:R:=(ad*x+a6)
 a7:R:=0.29870208740234375E-14::R
 af:R:=(ae*x-a7)
 a8:R:=0.5097890777587890625E-13::R
 ag:R:=(af*x+a8)
 a9:R:=0.8193406921386718749 9E-12::R
 ah:R:=(ag*x-a9)
 a10:R:=0.1235313123779296875E-10::R
 ai:R:=(ah*x+a10)
 a11:R:=0.173972962084960937 5E-9::R
 aj:R:=(ai*x-a11)
 a12:R:=0.22774642697021484375E-8::R
 ak:R:=(aj*x+a12)
 a13:R:=0.275573192853515625E-7::R
 al:R:=(ak*x-a13)
 a14:R:=0.3061924363508789062 5E-6::R
 am:R:=(al*x+a14)
 a15:R:=0.0000031001984125191406 25::R
 an:R:=(am*x-a15)
 a16:R:=0.0000283446712004554687 5::R
 ao:R:=(an*x+a16)
 a17:R:=0.0002314814814817695312 5::R
 ap:R:=(ao*x-a17)
 a18:R:=0.0016666666666686093 75::R
 aq:R:=(ap*x+a18)
 a19:R:=0.0104166666666664687 5::R
 ar:R:=(aq*x-a19)
 a20:R:=0.0555555555555416875 1::R
 as:R:=(ar*x+a20)
 a21:R:=0.2500000000000000375::R
 at:R:=(as*x-a21)
 a22:R:=1.0000000000000003 25::R
 au:R:=(at*x+a22)
 a23:R:=0.5772156649015328::R
 av:R:=au*x-a23
```

```
    aw:R:=- 1.0::R*log(abs(x)) + av
    aw::OPR
   error "E1: no approximation available"
```

5.16.4 En:(PI,R)→OPR

The E_n function is computed using the recurrence relation:

$$E_{n+1}(z) = \frac{1}{n}\left(e^{-z} - zE_n(z)\right) \quad (n = 1, 2, 3, \ldots)$$

The base case of the recursion depends on E1 above.

The formula is 5.1.14 in Abramowitz and Stegun, 1965, p229[**?**].

⟨*package DFSFUN DoubleFloatSpecialFunctions*⟩+≡

```
En(n:PI,x:R):OPR ==
  n=1 => E1(x)
  v:R:=retract(En((n-1)::PI,x))
  w:R:=1/(n-1)*(exp(-x)-x*v)
  w::OPR
```

5.16.5 The Ei Function

This function is based on Kin L. Lee's work[**?**]. See also [**?**].

5.16.6 Abstract

The exponential integral Ei(x) is evaluated via Chebyshev series expansion of its associated functions to achieve high relative accuracy throughout the entire real line. The Chebyshev coefficients for these functions are given to 30 significant digits. Clenshaw's[**?**] method is modified to furnish an efficient procedure for the accurate solution of linear systems having near-triangular coefficient matrices.

5.16.7 Introduction

The evaulation of the exponential integral

$$Ei(x) = \int_{-\infty}^{X} \frac{e^u}{u}\, du = -E_1(-x), x \neq 0 \tag{5.7}$$

is usually based on the value of its associated functions, for example, $xe^{-x}Ei(x)$. High accuracy tabulations of integral (1) by means of Taylor series techniques are given by Harris [**?**] and Miller and Hurst [**?**]. The evaluation of $Ei(x)$ for $-4 \leq x \leq \infty$ by means of Chebyshev series is provided by Clenshaw [**?**] to have the absolute accuracy of 20 decimal places. The evaluation of the same integral (1) by rational approximation of its associated functions is furnished by Cody and Thacher [**?**, **?**] for $-\infty < x < \infty$, and has the relative accuracy of 17 significant figures.

The approximation of Cody and Thacher from the point of view of efficient function evaluation are preferable to those of Clenshaw. However, the accuracy of the latter's procedure, unlike those of the former, is not limited by the accuracy or the availability of a master function, which is a means of explicitly evaluating the function in question.

In this paper $Ei(x)$ (or equivalently $-E_1(-x)$) for the entire real line is evaluted via Chebyshev series expansion of its associated functions that are accurate to 30 significant figures by a modification of Clenshaw's procedure. To verify the accuracy of the several Chebyshev series, values of the associated functions were checked against those computed by Taylor series and those of Murnaghan and Wrench [**?**] (see Remarks on Convergence and Accuracy).

Although for most purposes fewer than 30 figures of accuracy are required, such high accuracy is desirable for the following reasons. In order to further reduce the number of arithmetical operations in the evaluation of a function, the Chebyshev series in question can either be converted into a rational function or rearranged into an ordinary polynomial. Since several figures may be lost in either of these procedures, it is necessary to provide the Chebyshev series

with a sufficient number of figures to achieve the desired accuracy. Furthermore, general function approximation routines, such as those used for minimax rational function approximations, require the explicit evaluation of the function to be approximated. To take account of the errors commited by these routines, the function values must have an accuracy higher than the approximation to be determined. Consequently, high-precision results are useful as a master function for finding approximations for (or involving) $Ei(x)$ (e.g. [**?**, **?**]) where prescribed accuracy is less than 30 figures.

5.16.8 Discussion

It is proposed here to provide for the evaluation of $Ei(x)$ by obtaining Chebyshev coefficients for the associated functions given by table 1.

Table 1: Associated Functions of $Ei(x)$ and their ranges of Chebyshev Series Expansions

	Associated function	Range of expansion
Ei1	$xe^{-x}Ei(x)$	$-\infty < x \leq -10$
Ei2	$xe^{-x}Ei(x)$	$-10 \leq x \leq -4$
Ei3	$\frac{Ei(x)-log\|x\|-\gamma}{x}$	$-4 \leq x \leq d42$
Ei4	$xe^{-x}Ei(x)$	$4 \leq x \leq 12$
Ei5	$xe^{-x}Ei(x)$	$12 \leq x \leq 32$
Ei6	$xe^{-x}Ei(x)$	$32 \leq x < \infty$

($\gamma = 0.5772156649...$ is Euler's constant.)

⟨*package DFSFUN DoubleFloatSpecialFunctions*⟩+≡

```
Ei(y:OPR):OPR ==
  infinite? y => 1
  x:R:=retract(y)
  x < -10.0::R =>
    ei:R:=retract(Ei1(y))
    (ei/(x*exp(-x)))::OPR
  x <  -4.0::R =>
    ei:R:=retract(Ei2(y))
    (ei/(x*exp(-x)))::OPR
  x <   4.0::R =>
    ei3:R:=retract(Ei3(y))
    gamma:R:=0.57721566490153286060651209008 2::R
    (ei3*x+log(abs(x))+gamma)::OPR
  x <  12.0::R =>
    ei:R:=retract(Ei4(y))
    (ei/(x*exp(-x)))::OPR
  x <  32.0::R =>
    ei:R:=retract(Ei5(y))
    (ei/(x*exp(-x)))::OPR
```

```
ei:R:=retract(Ei6(y))
(ei/(x*exp(-x)))::OPR
```

Note that the functions $[Ei(x) - log|x| - \gamma]/x$ and $xe^{-x}Ei(x)$ have the limiting values of unity at the origin and at infinity, respectively, and that the range of the associated function values is close to unity (see table 4). This makes for the evaluation of the associated functions over the indicated ranges in table 1 (and thus $Ei(x)$ over the entire real line) with high relative accuracy by means of the Chebyshev series. The reason for this will become apparent later.

Some remarks about the choice of the intervals of expansion for the serveral Chebyshev series are in order here. The partition of the real line indicated by table 1 is chosen to allow for the approximation of the associated functions with a maximum error of 0.5×10^{-30} by polynomials of degress < 50. The real line has also been partitioned with the objective of providing the interval about zero with the lowest degree of polynomial approximation of the six intervals. This should compensate for the computation of $log|x|$ required in the evaluation of $Ei(x)$ over that interval. The ranges $-\infty < x \leq -4$ and $4 \leq x < \infty$ are partitioned into 2 and 3 intervals, respectively, to provide approximations to $xe^{-x}Ei(x)$ by polynomials of about the same degree.

5.16.9 Expansions in Chebyshev Series

Let $\phi(t)$ be a differentiable function defined on [-1,1]. To facilitate discussion, denote its Chebyshev series and that of its derivative by

$$\phi(t) = \sum_{k=0}^{\infty}{}' A_k^{(0)} T_k(t) \quad \phi'(t) = \sum_{k=0}^{\infty}{}' A_k^{(1)} T_k(t) \tag{5.8}$$

where $T_k(t)$ are Chebyshev polynomials defined by

$$T_k(t) = cos(k \ \arccos \ t), \quad -1 \leq t \leq 1 \tag{5.9}$$

(A prime over a summation sign indicates that the first term is to be halved.)

If $\phi(t)$ and $\phi'(t)$ are continuous, the Chebyshev coefficients $A_k^{(0)}$ and $A_k^{(1)}$ can be obtained analytically (if possible) or by numerical quadrature. However, since each function in table 1 satisfies a linear differential equation with polynomial coefficients, the Chebyshev coefficients can be more readily evaluated by the method of Clenshaw [**?**].

There are several variations of Clenshaw's procedure (see, e.g. [**?**]), but for high-precision computation, where multiple precision arithmetic is employed, we find his original procedure easiest to implement. However, straightforward application of it may result in a loss of accuracy if the trial solutions selected are not sufficiently independent. How the difficulty is overcome will be pointed out subsequently.

5.16.10 The function $xe^{-x}Ei(x)$ on the Finite Interval

We consider first the Chebyshev series expansion of

$$f(x) = xe^{-x}Ei(x), \quad (a \leq x \leq b) \tag{5.10}$$

with $x \neq 0$. One can easily verify that after the change of variables

$$x = [(b-a)T + a + b]/2, \quad (-1 \leq t \leq 1) \tag{5.11}$$

the function

$$\phi(t) = f\left[\frac{(b-a)t + a + b}{2}\right] = f(x) \tag{5.12}$$

satisfies the differential equation

$$2(pt+q)\phi^{'}(t) + p(pt+q-2)\phi(t) = p(pt+q) \tag{5.13}$$

with[1]

$$\phi(-1) = ae^{-a}Ei(a) \tag{5.14}$$

where $p = b - a$ and $q = b + a$. Replacing $\phi(t)$ and $\phi^{'}(t)$ in equations 7 by their Chebyshev series, we obtain

$$\sum_{k=0}^{\infty}{}^{'}(-1)^k A_k^{(0)} = \phi(-1) \tag{5.15}$$

$$2\sum_{k=0}^{\infty}{}^{'}A_k^{(1)}(pt+q)T_k(t) + p\sum_{k=0}^{\infty}{}^{'}A_k^{(0)}(pt+q-2)T_k(t) = p(pt+q) \tag{5.16}$$

It can be demonstrated that if B_k are the Chebyshev coefficients of a function $\Psi(t)$, then C_k, the Chebyshev coefficients of $t^r\Psi(t)$ for positive integers r, are given by [**?**]

$$C_k = 2^{-r}\sum_{i=0}^{r}\binom{r}{i}B_{|k-r+2i|} \tag{5.17}$$

Consequently, the left member of equation 15 can be rearranged into a single series involving $T_k(t)$. The comparison of the coefficients of $T_k(t)$ that yields the infinite system of equations

$$\left.\begin{aligned} &\sum_{k=0}^{\infty}{}^{'}(-1)^k A_k^{(0)} = \phi(-1) \\ &2pA_{|k-1|}^{(1)} + 4qA_k^{(1)} + 2pA_{k+1}^{(1)} + p^2A_{|k-1|}^{(0)} + 2p(q-2)A_k^{(0)} + p^2A_{k+1}^{(0)} \\ &\qquad = \begin{cases} 4pq & , \quad k = 0 \\ 2p^2 & , \quad k = 1 \\ 0 & , \quad k = 2, 3, \ldots \end{cases} \end{aligned}\right\} \tag{5.18}$$

[1] The value of $Ei(a)$ may be evaluated by means of the Taylor series. In this report $Ei(a)$ is computed by first finding the Chebyshev series approximation to $[Ei(x) - log|x| - \gamma]/x$ to get $Ei(a)$. The quantities e^a and $\log|a|$ for integral values of a may be found in existing tables

The relation [**?**]

$$2kA_k^{(0)} = A_{k-1}^{(1)} - A_{k+1}^{(1)} \tag{5.19}$$

can be used to reduce equation 18 to a system of equations involving only $A_k^{(0)}$. Thus, replacing k of equations 18 by $k+2$ and subtracting the resulting equation from equations 18, we have, by means of equation 19, the system of equations

$$\left.\begin{array}{c} {\sum_{k=0}^{\infty}}' (-1)^k A_k^{(0)} = \phi(-1) \\ 2p(q-2)A_0 + (8q+p^2)A_1 + 2p(6-q)A_2 - p^2 A_3 = 4pq \\ p^2 A_{k-1} + 2p(2k+q-2)A_k + 8q(k+1)A_{k+1} + 2p(2k-q+6)A_{k+2} - p^2 A_{k+3} \\ = \left\{ \begin{array}{rcl} 2p^2 & , & k=1 \\ 0 & , & k=2,3,\ldots \end{array} \right. \end{array}\right\} \tag{5.20}$$

The superscript of $A_k^{(0)}$ is dropped for simplicity. In order to solve the infinite system 20, Clenshaw [**?**] essentially considered the required solution as the limiting solution of the sequence of truncated systems consisting of the first $M+1$ equations of the same system, that is, the solution of the system

$${\sum_{k=0}^{M}}' (-1)^k A_k = \phi(-1) \tag{5.21}$$

$$2p(q-2)A_0 + (8q+p^2)A_1 + 2p(q-6)A_2 - p^2 A_3 = 4pq \tag{5.22}$$

$$\left.\begin{array}{c} p^2 A_{k-1} + 2p(2k+q-2)A_k + 8q(k+1)A_{k+1} + 2p(2k-q+6)A_{k+2} - p^2 A_{k+3} \\ = \left\{ \begin{array}{rcl} 2p^2 & , & k=1 \\ 0 & , & k=2,3,\ldots,M-3 \end{array} \right. \\ p^2 A_{M-3} + 2p(2M+q-6)A_{M-2} + 8q(M-1)A_{M-1} + 2p(2M+4-q)A_M = 0 \\ p^2 A_{M-2} + 2p(2M+q-4)A_{M-1} + 8qMA_M = 0 \end{array}\right\} \tag{5.23}$$

where A_k is assumed to vanish for $K \geq M+1$. To solve system (21,22,23) consider first the subsystem 23 consisting of $M-2$ equations in M unknowns. Here use is made of the fact that the subsystem 23 is satisfied by

$$A_k = c_1\alpha_k + c_2\beta_k + \gamma_k \quad (k = 0,1,2,\ldots) \tag{5.24}$$

for arbitrary constants c_1 and c_2, where γ_k is a particular solution of 23 and where α_k and β_k are two independent solutions of the homogeneous equations (23 with $2p^2$ deleted) of the same subsystem. Hence, if α_k, β_k, and γ_k are available, the solution of system (21,22,23) reduces to the determinant of c_1 and c_2 from equations 21 and 22.

To solve equations (21,22,23), we note that

$$\gamma_0 = 2, \quad \gamma_k = 0, \quad for\ k = 1(1)M \tag{5.25}$$

is obviously a particular solution of equation 23. The two independent solutions γ_k and β_k of the homogeneous equations of the same subsystem can be generated in turn by backward recurrence if we set

$$\left.\begin{array}{ll} & \alpha_{M-1} = 0, \quad \alpha_M = 1 \\ \text{and} & \\ & \beta_{M-1} = 1, \quad \beta_M = 0 \end{array}\right\} \tag{5.26}$$

or choose any $\alpha M - 1$, α_M, and β_{M-1}, β_M for which $\alpha_{M-1}\beta_M - \alpha_M\beta_{M-1} \neq 0$. The arbitrary constants c_1 and c_2 are determined, and consequently the solution of equations (21,22,23) if equation 24 is substituted into equation 21 and 22 and the resulting equations

$$c_1 R(\alpha) + c_2 R(\beta) = \phi(-1) - 1 \tag{5.27}$$

$$c_1 S(\alpha) + c_2 S(\beta) = 8p \tag{5.28}$$

are solved as two equations in two unknowns. The terms $R(\alpha)$ and $S(\alpha)$ are equal, respectively, to the left members of equations 21 and 22 corresponding to solution α_k. (The identical designation holds for $R(\beta)$ and $S(\beta)$.)

The quantities α_k and β_k are known as trial solutions in reference [**?**]. Clenshaw has pointed out that if α_k and β_k are not sufficiently independent, loss of significance will occur in the formation of the linear combination 24, with consequent loss of accuracy. Clenshaw suggested the Gauss-Seidel iteration procedure to improve the accuracy of the solution. However, this requires the application of an additional computing procedure and may prove to be extremely slow. A simpler procedure which does not alter the basic computing scheme given above is proposed here. The loss of accuracy can effectively be regained if we first generate a third trial solution δ_k (k=0,1,...,M), where δ_{M-1} and δ_M are equal to $c_1\alpha_{M-1} + c_2\beta_{M-1}$ and $c_1\alpha_M + c_2\beta_M$, respectively, and where δ_k (k=M-2,M-3,...,0) is determined using backward recurrence as before by means of equation 23. Then either α_k or β_k is replaced by δ_k and a new set of c_1 and c_2 is determined by equations 27 and 28. Such a procedure can be repeated until the required accuracy is reached. However, only one application of it was necessary in the computation of the coefficients of this report.

As an example, consider the case for $4 \leq x \leq 12$ with $M = 15$. The right member of equation 27 and of equation 28 assume, respectively, the values of 0.43820800 and 64. The trial solutions α_k and β_k generated with $\alpha_{14} = 8$, $\alpha_{15} = 9$ and $\beta_{14} = 7$, $\beta_{15} = 8$ are certainly independent, since $\alpha_{14}\beta_{15} - \alpha_{15}\beta_{14} = 1 \neq 0$. A check of table 2 shows that equations 27 and 28 have, respectively, the residuals of $-0.137 \times 10^{-}4$ and -0.976×10^{-3}. The same table also shows that $c_1\alpha_k$ is opposite in sign but nearly equal in magnitude to $c_2\beta_k$. Cancellations in the formation of the linear combination 24 causes a loss of significance of

2 to 6 figures in the computed A_k. In the second iteration, where a new set of β_k is generated replacing β_{14} and β_{15}, respectively, by $c_1\alpha_{14} + c_2\beta_{14}$ and $c_1\alpha_{15} + c_2\beta_{15}$ of the first iteration, the new $c_1\alpha_k$ and $c_2\beta_k$ differed from 2 to 5 orders of magnitude. Consequently, no cancellation of significant figures in the computation of A_k occurred. Notice that equations 27 and 28 are now satisfied exactly. Further note that the new c_1 and c_2 are near zero and unity, respectively, for the reason that if equations 21, 22, and 23 are satisfied by equation 24 exactly in the first iteration, the new c_1 and c_2 should have the precise values zero and 1, respectively. The results of the third iteration show that the A_k of the second iteration are already accurate to eight decimal places, since the A_k in the two iterations differ in less that 0.5×10^{-8}. Notice that for the third iteration, equations 27 and 28 are also satisfied exactly and that $c_1 = 1$ and $c_2 = 0$ (relative to 8 places of accuracy).

Table 2: Computation of Chebyshev Coefficients for $xe^{-x}Ei(x)$

First iteration: $\alpha_{14} = 8$, $\alpha_{15} = 9$; $\beta_{14} = 7$, $\beta_{15} = 8$

k	$c_1\alpha_k$	$c_2\beta_k$	A_k
0	0.71690285E 03	-0.71644773E 03	0.24551200E 01
1	-0.33302683E 03	0.33286440E 03	-0.16243000E 00
2	0.13469341E 03	-0.13464845E 03	0.44960000E-01
3	-0.43211869E 02	0.43205127E 02	-0.67420000E-02
4	0.99929173E 01	-0.99942238E 01	-0.13065000E-02
5	-0.11670764E 01	0.11684574E 01	0.13810000E-02
6	-0.25552137E 00	0.25493635E 00	-0.58502000E-02
7	0.20617247E 00	-0.20599754E 00	0.17493000E-03
8	-0.75797238E-01	0.75756767E-01	-0.40471000E-04
9	0.20550680E-01	-0.20543463E-01	0.72170000E-05
10	-0.45192333E-02	0.45183721E-02	-0.86120000E-06
11	0.82656562E-03	-0.82656589E-03	-0.27000000E-09
12	-0.12333571E-03	0.12337366E-03	0.37950000E-07
13	0.13300910E-04	-0.13315328E-04	-0.14418000E-07
14	-0.29699001E-06	0.30091136E-06	0.39213500E-08
15	-0.33941716E-06	0.33852528E-06	-0.89188000E-09

$$c_1 = 0.37613920E - 07$$
$$c_2 = -0.42427144E - 07$$
$$c_1R(\alpha) + c_2R(\beta) - 0.43820800E\ 00 = -0.13700000E - 04$$
$$c_1S(\alpha) + c_2S(\beta) - 0.64000000E\ 00 = -0.97600000E - 03$$

Second iteration: $\alpha_{14} = 8$, $\alpha_{15} = 9$;
$\beta_{14} = 0.39213500E - 08$, $\beta_{15} = -0.89188000E - 09$

k	$c_1\alpha_k$	$c_2\beta_k$	A_k
0	0.36701576E-05	0.45512986E 00	0.24551335E 01
1	-0.17051695E-05	-0.16243666E 00	-0.16243837E 00
2	0.68976566E-06	0.44956834E-01	0.44957523E-01
3	-0.22132756E-06	-0.67413538E-02	-0.67415751E-02
4	0.51197561E-07	-0.13067496E-02	-0.13066984E-02
5	-0.59856744E-08	0.13810895E-02	0.13810835E-02
6	-0.13059663E-08	-0.58502164E-03	-0.58502294E-03
7	0.10552667E-08	0.17492889E-03	0.17492994E-03
8	-0.38808033E-09	-0.40472426E-04	-0.40472814E-04
9	0.10523831E-09	0.72169965E-05	0.72171017E-05
10	-0.23146333E-10	-0.86125438E-06	-0.86127752E-06
11	0.42342615E-11	-0.25542252E-09	-0.25118825E-09
12	-0.63200810E-12	0.37946968E-07	0.37946336E-07
13	0.68210630E-13	-0.14417584E-07	-0.14417516E-07
14	-0.15414832E-14	0.39212981E-08	0.39212965E-08
15	-0.17341686E-14	-0.89186818E-09	-0.89186991E-09

$c_1 = -0.19268540E - 15$

$c_2 = 0.99998675E\ 00$

$c_1R(\alpha) + c_2R(\beta) - 0.43820800E\ 00 = 0.0$

$c_1S(\alpha) + c_2S(\beta) - 0.64000000E\ 00 = 0.0$

Table 2: Computation of Chebyshev Coefficients for $xe^{-x}Ei(x)$ - Concluded
$[4 \leq x \leq 12 with M = 15; \gamma_0 = 2, \gamma_k = 0 \text{for } k = 1(1)15]$

Third iteration: $\alpha_{14} = 8$, $\alpha_{15} = 9$;
$\beta_{14} = 0.39212965E - 08$, $\beta_{15} = -0.89186991E - 09$

k	$c_1\alpha_k$	$c_2\beta_k$	A_k
0	-0.23083059E-07	0.45513355E 00	0.24551335E 01
1	0.10724479E-07	-0.16243838E 00	-0.16243837E 00
2	-0.43382065E-08	0.44957526E-01	0.44957522E-01
3	0.13920157E-08	-0.67415759E-02	-0.67415745E-02
4	-0.32200152E-09	-0.13066983E-02	-0.13066986E-02
5	0.37646251E-10	0.13810835E-02	0.13810836E-02
6	0.82137336E-11	-0.58502297E-03	-0.58502296E-03
7	-0.66369857E-11	0.17492995E-03	0.17492994E-03
8	0.24407892E-11	-0.40472817E-04	-0.40472814E-04
9	-0.66188494E-12	0.72171023E-05	0.72171017E-05
10	0.14557636E-12	-0.86127766E-06	-0.86127751E-06
11	-0.26630930E-13	-0.25116620E-09	-0.25119283E-09
12	0.39749465E-14	0.37946334E-07	0.37946337E-07
13	-0.42900337E-15	-0.14417516E-07	-0.14417516E-07
14	0.96949915E-17	0.39212966E-08	0.39212966E-08
15	0.10906865E-16	-0.89186992E-09	-0.89186990E-09

$c_1 = 0.12118739E - 17$

$$c_2 = 0.10000000E\ 01$$
$$c_1 R(\alpha) + c_2 R(\beta) - 0.43820800E\ 00 = 0.0$$
$$c_1 S(\alpha) + c_2 S(\beta) - 0.64000000E\ 00 = 0.0$$

It is worth noting that the coefficient matrix of system (21,22,23) yields an upper triangular matrix of order $M-1$ after the deletion of the first two rows and the last two columns. Consequently, the procedure of this section is applicable to any linear system having this property. As a matter of fact, the same procedure can be generalized to solve linear systems having coefficient matrices of order N, the deletion of whose first r $(r < N)$ rows and last r columns yields upper triangular matrices of order $N - r$.

5.16.11 The Function $(1/x)[Ei(x) - log|x| - \gamma]$

Let

$$f(x) = (1/x)[Ei(x) - log|x| - \gamma], \quad g(x) = e^x, \quad |x| \le b \tag{5.29}$$

These functions, with the change of variable $x = bt$, simultaneously satisfy the differential equations

$$bt^2\phi'(t) + bt\phi(t) - \psi(t) = -1 \tag{5.30}$$

$$\psi'(t) - b\psi(t) = 0, \quad -1 \le t \le 1 \tag{5.31}$$

Conversely,[2] any solution of equations 30 and 31 is equal to the functions given by equations 29 for the change of variable $x = bt$. Therefore, boundary conditions need not be imposed for the solution of the differential equations.

A procedure similar to that of the previous section gives the coupled infinite recurrence relations

$$bA_1 + bA_3 - B_0 + B_2 = -2 \tag{5.32}$$

$$\left.\begin{aligned} kbA_{k-1} + 2(k+1)bA_{k+1} + (k+2)bA_{k+3} - 2B_k + 2B_{k+2} = 0 \\ bB_{k-1} - 2kB_k - bB_{k+1} = 0, \quad k = 1, 2, \ldots \end{aligned}\right\} \tag{5.33}$$

where A_k and B_k are the Chebyshev coefficients of $\phi(t)$ and $\psi(t)$, respectively.

Consider first the subsystem 33. If $A_k = \alpha_k$ and $B_k = \beta_k$ are a simultaneous solution of the system, which is homogeneous, then

$$\text{and} \qquad \left.\begin{aligned} A_k = c\alpha_k \\ B_k = c\beta_k \end{aligned}\right\} \tag{5.34}$$

---

[2] The general solution of the differential equations has the form

$$\phi(t) = (c_1/t) + [Ei(bt) - log|bt| - \gamma]/bt$$
$$\psi(t) = c_2 e^{bt}$$

where the first and second terms of $\phi(t)$ are, respectively, the complementary solution and a particular integral of equation 30. The requirement that $\phi(t)$ is bounded makes the constant $c_1 = 0$. The fact that $\psi(0) = 1$ is implicit in equation 30.

are also a solution for an arbitrary constant c. Thus based on considerations analogous to the solution of equations 21, 22, and 23, one can initiate an approximate solution of equations 32 and 33 by setting

$$\left.\begin{array}{l} \alpha_M = 0, \quad \alpha_k = 0 \quad \text{for } k \geq M+1 \\ \beta_M = 1, \quad \beta_k = 0 \quad \text{for } k \geq M+1 \end{array}\right\} \tag{5.35}$$

and then determining α_k and β_k $(k = M-1, M-2, \ldots, 0)$ by backward recurrence by means of equation 33. The arbitrary constant c is determined by substituting 34 into 32.

5.16.12 The Function $xe^{-x}Ei(x)$ on the Infinite Interval

Let

$$f(x) = xe^{-x}Ei(x), \quad -\infty < x \leq b < 0, \quad or 0 < b \leq x < \infty \tag{5.36}$$

By making the change of variables,

$$x = 2b/(t+1) \tag{5.37}$$

we can easily demonstrate that

$$f(x) = f[2b/(t+1)] = \phi(t) \tag{5.38}$$

satisfies the differential equation

$$(t+1)^2\phi'(t) + (t+1-2b)\phi(t) = -2b \tag{5.39}$$

with

$$\phi(1) = be^{-b}Ei(b) \tag{5.40}$$

An infinite system of equations involving the Chebyshev coefficients A_k of $\phi(t)$ is deducible from equations 39 and 40 by the same procedure as applied to equations 13 and 14 to obtain the infinite system 20; it is given as follows.

$$\sum_{k=0}^{\infty}{}' A_k = \phi(1) = be^{-b}Ei(b) \tag{5.41}$$

$$(1-2b)A_0 + 3A_1 + (3+2b)A_2 + A_3 = -4b \tag{5.42}$$

$$\begin{array}{r} kA_{k-1} + 2[(2k+1)-2b]A_k + 6(k+1)A_{k+1} + 2(2k+3+2b)A_{k+2} \\ + (k+2)A_{k+3} = 0, \quad k = 1, 2, \ldots \end{array} \tag{5.43}$$

As in the case of equations 21, 22 and 23, the solution of 41, 42 and 43 can be assumed to be

$$A_k = c_1\alpha_k + c_2\beta_k \tag{5.44}$$

with A_k vanishing for a $k \geq M$. Thus, we can set, say

$$\left.\begin{array}{lll} \alpha_{M-1} = 0 & , & \alpha_M = 1 \\ \beta_{M-1} = 1 & , & \beta_M = 0 \end{array}\right\} \tag{5.45}$$

and determine the trial solutions α_k and β_k (k=M-1,M-2,...,0) by means of equation 43 by backward recurrence. The required solution of equations 41,42,adn 43 is then determined by substituting equation 44 in equations 41 and 42 and solving the resulting equations for c_1 and c_2.

Loss of accuracy in the computation of A_k can also occur here, as in the solution of equations 21, 22 and 23, if the trial solutions are not sufficiently independent. The process used to improve the accuracy of A_k of the system 21, 22 and 23 can also be applied here.

For efficiency in computation, it is worth noting that for $b < 0$ ($-\infty < x \leq b < 0$) the boundary condition 40 is not required for the solution of equation 39 and 40. This follows from the fact that any solution[3] of the differential equation 39 is equal to $xe^{-x}Ei(x)$ ($x = 2b/(t+1)$). Hence the A_k of $xe^{-x}Ei(x)$ for $-\infty < x \leq b < 0$ can be obtained without the use of equation 39 and can be assumed to have the form

$$A_k = c\alpha_k, \quad (k = 0, 1, \ldots, M) \tag{5.46}$$

The M+1 values of α_k can be generated by setting $\alpha_M = 1$ and computing α_k (k=0,1,...,M-1) by means of equation 43 by backward recurrence. The substitution of equation 46 into 42 then enables one to determine c from the resulting equation.

5.16.13 Remarks on Convergence and Accuracy

The Chebyshev coefficients of table 3 were computed on the IBM 7094 with 50-digit normalized floating-point arithmetic. In order to assure that the sequence of approximate solutions (see Discusion) converged to the limiting solution of the differential equation in question, a trial M was incremented by 4 until the approximate Chebyshev coefficients showed no change greter than or equal to 0.5×10^{-35}. Hence the maximum error is bounded by

$$0.5(M+1) \times 10^{-35} + \sum_{M+1}^{\infty} |A_k| \tag{5.47}$$

where the first term is the maximum error of the M+1 approximate Chebyshev coefficients, and the sum is the maximum error of the truncated Chebyshev series

[3]The general solution of the differential equation 39. Since equation 39 has no bounded complementary solution for $-\infty < x \leq b < 0$, every solution of it is equal to the particular integral $xe^{-x}Ei(x)$. On the other hand, a solution of equation 39 for $0 < x \leq b < \infty$ would, in general, involve the complementary function. Hence, boundary condition 40 is required to guarantee that the solution of equation 39 is equal to $xe^{-x}Ei(x)$.

of M+1 terms. If the Chebyshev series is rapidly convergent, the maximum error of the approximate Chebyshev series should be of the order of 10^{-30}. The coefficients of table 3 have been rounded to 30 digits, and higher terms for $k > N$ giving the maximum residual

$$\sum_{k=N+1}^{M} |A_k| < 0.5 \times 10^{-30} \tag{5.48}$$

have been dropped. This should allow for evaluation of the relevant function that is accurate to 30 decimal places. Since the range of values of each function is bounded between 2/5 and 5, the evaluated function should be good to 30 significant digits. Taylor series evaluation also checks with that of the function values of table 4 (computed with 30-digit floating-point arithmetic using the coefficients of table 3) for at least 28-1/2 significant digits. Evaluation of Ei(x) using the coefficients of table 3 also checked with Murnaghan and Wrench [**?**] for 28-1/2 significant figures.

Table 3: Chebyshev Coefficients (a)

$$xe^{-x}Ei(x) = \sum_{k-0}^{40}{}' A_k T_k(t), \quad t = (-20/x) - 1, \quad (-\infty < x \leq -10)$$

k	A_k
0	0.1912173225 8605534539 1519326510E 01
1	-0.4208355052 8684843755 0974986680E-01
2	0.1722819627 2843267833 7118157835E-02
3	-0.9915782173 4445636455 9842322973E-04
4	0.7176093168 0227750526 5590665592E-05
5	-0.6152733145 0951269682 7956791331E-06
6	0.6024857106 5627583129 3999701610E-07
7	-0.6573848845 2883048229 5894189637E-08
8	0.7853167541 8323998199 4810079871E-09
9	-0.1013730288 0038789855 4202774257E-09
10	0.1399770413 2267686027 7823488623E-10
11	-0.2051008376 7838189961 8962318711E-11
12	0.3168388726 0024778181 4907985818E-12
13	-0.5132760082 8391806541 5984751899E-13
14	0.8680933040 7665493418 7433687383E-14
15	-0.1527015040 9030849719 8572355351E-14
16	0.2784686251 6493573965 0105251453E-15
17	-0.5249890437 4217669680 8472933696E-16
18	0.1020717991 2485612924 7455787226E-16
19	-0.2042264679 8997184130 8462421876E-17
20	0.4197064172 7264847440 8827228562E-18
21	-0.8844508176 1728105081 6483737536E-19
22	0.1908272629 5947174199 5060168262E-19
23	-0.4209746222 9351995033 6450865676E-20
24	0.9483904058 1983732764 1500214512E-21
25	-0.2179467860 1366743199 4032574014E-21
26	0.5103936869 0714509499 3452562741E-22
27	-0.1216883113 3344150908 9746779693E-22
28	0.2951289166 4478751929 4773757144E-23
29	-0.7275353763 7728468971 4438950920E-24
30	0.1821639048 6230739612 1667115976E-24
31	-0.4629629963 1633171661 2753482064E-25
32	0.1193539790 9715779152 3052371292E-25
33	-0.3119493285 2201424493 1062147473E-26
34	0.8261419734 5334664228 4170028518E-27
35	-0.2215803373 6609829830 2591177697E-27
36	0.6016031671 6542638904 5303124429E-28
37	-0.1652725098 3821265964 9744302314E-28
38	0.4592230358 7730270279 5636377166E-29
39	-0.1290062767 2132638473 7453212670E-29
40	0.3662718481 0320025908 1177078922E-30

⟨*package DFSFUN DoubleFloatSpecialFunctions*⟩+≡

```
Ei1(y:OPR):OPR ==
  infinite? y => 1
  x:R:=retract(y)
  t:R:=acos((-20.0::R/x)-1.0::R)::R
  t01:=    0.19121732258605534539151932651OE1::R*cos(0.0::R)/2.0::R
  t02:=t01-0.42083550528684843755097498668OE-01::R*cos(t::R)::R
  t03:=t02+0.17228196272843267833711815783 5E-02::R*cos( 2.0::R*t)
  t04:=t03-0.99157821734445636455984232297 3E-04::R*cos( 3.0::R*t)
  t05:=t04+0.71760931680227750526559066559 2E-05::R*cos( 4.0::R*t)
  t06:=t05-0.61527331450951269682795679133 1E-06::R*cos( 5.0::R*t)
  t07:=t06+0.60248571065627583129399970161 0E-07::R*cos( 6.0::R*t)
  t08:=t07-0.65738488452883048229589418963 7E-08::R*cos( 7.0::R*t)
  t09:=t08+0.78531675418323998199481007987 1E-09::R*cos( 8.0::R*t)
  t10:=t09-0.10137302880038789855420277425 7E-09::R*cos( 9.0::R*t)
  t11:=t10+0.13997704132267686027782348862 3E-10::R*cos(10.0::R*t)
  t12:=t11-0.20510083767838189961896231871 1E-11::R*cos(11.0::R*t)
  t13:=t12+0.31683887260024778181490798581 8E-12::R*cos(12.0::R*t)
  t14:=t13-0.51327600828391806541598475189 9E-13::R*cos(13.0::R*t)
  t15:=t14+0.86809330407665493418743368738 3E-14::R*cos(14.0::R*t)
  t16:=t15-0.15270150409030849719857235535 1E-14::R*cos(15.0::R*t)
  t17:=t16+0.27846862516493573965010525145 3E-15::R*cos(16.0::R*t)
  t18:=t17-0.52498904374217669680847293369 6E-16::R*cos(17.0::R*t)
  t19:=t18+0.10207179912485612924745578722 6E-16::R*cos(18.0::R*t)
  t20:=t19-0.20422646798997184130846242187 6E-17::R*cos(19.0::R*t)
  t21:=t20+0.41970641727264847440882722856 2E-18::R*cos(20.0::R*t)
  t22:=t21-0.88445081761728105081648373753 6E-19::R*cos(21.0::R*t)
  t23:=t22+0.19082726295947174199506016826 2E-19::R*cos(22.0::R*t)
  t24:=t23-0.42097462229351995033645086567 6E-20::R*cos(23.0::R*t)
  t25:=t24+0.94839040581983732764150021451 2E-21::R*cos(24.0::R*t)
  t26:=t25-0.21794678601366743199403257401 4E-21::R*cos(25.0::R*t)
  t27:=t26+0.51039368690714509499345256274 1E-22::R*cos(26.0::R*t)
  t28:=t27-0.12168831133344150908974677969 3E-22::R*cos(27.0::R*t)
  t29:=t28+0.29512891664478751929477375714 4E-23::R*cos(28.0::R*t)
  t30:=t29-0.72753537637728468971443895092 0E-24::R*cos(29.0::R*t)
  t31:=t30+0.18216390486230739612166711597 6E-24::R*cos(30.0::R*t)
  t32:=t31-0.46296299631633171661275348206 4E-25::R*cos(31.0::R*t)
  t33:=t32+0.11935397909715779152305237129 2E-25::R*cos(32.0::R*t)
  t34:=t33-0.31194932852201424493106214747 3E-26::R*cos(33.0::R*t)
  t35:=t34+0.82614197345334664228417002851 8E-27::R*cos(34.0::R*t)
  t36:=t35-0.22158033736609829830259117769 7E-27::R*cos(35.0::R*t)
  t37:=t36+0.60160316716542638904530312442 9E-28::R*cos(36.0::R*t)
  t38:=t37-0.16527250983821265964974430231 4E-28::R*cos(37.0::R*t)
  t39:=t38+0.45922303587730270279563637716 6E-29::R*cos(38.0::R*t)
  t40:=t39-0.12900627672132638473745321267 0E-29::R*cos(39.0::R*t)
```

```
t41:=t40+0.3662718481032002590811770789 22E-30::R*cos(40.0::R*t)
t41::OPR
```

Table 3: Chebyshev Coefficients - Continued (b)

$$xe^{-x}Ei(x) = \sum_{k-0}^{40}{}' A_k T_k(t), \quad t = (x+7)/3, \quad (-10 \le x \le -4)$$

k	A_k	21	-0.4322776783 3833850564 5764394579E-1
		22	-0.9063014799 6650172551 4905603356E-1
0	0.1757556496 0612937384 8762834691E 011	23	-0.1904669979 5816613974 4015963342E-1
1	-0.4358541517 7361661170 5001867964E-01	24	-0.4011792326 3502786634 6744227520E-1
2	-0.7979507139 5584254013 3217027492E-02	25	-0.8467772130 0168322313 4166334685E-1
3	-0.1484372327 3037121385 0970210001E-02	26	-0.1790842733 6586966555 5826492204E-1
4	-0.2800301984 3775145748 6203954948E-03	27	-0.3794490638 1714782440 1106175166E-1
5	-0.5348648512 8657932303 9177361553E-04	28	-0.8053999236 7982798526 0999654058E-2
6	-0.1032867243 5735548661 0233266460E-04	29	-0.1712339011 2362012974 3228671244E-2
7	-0.2014083313 0055368773 2226198639E-05	30	-0.3646274058 7749686208 6576562816E-2
8	-0.3961758434 2738664582 2338443500E-06	31	-0.7775969638 8939479435 3098157647E-2
9	-0.7853872767 0966316306 7607656069E-07	32	-0.1660628498 4484020566 2531950966E-2
10	-0.1567925981 0074698262 4616270279E-07	33	-0.3551178625 7882509300 5927145352E-2
11	-0.3150055939 3763998825 0007372851E-08	34	-0.7603722685 9413580929 5734653294E-2
12	-0.6365096822 5242037304 0380263972E-09	35	-0.1630074137 2584900288 9638374755E-2
13	-0.1292888113 2805631835 6593121259E-09	36	-0.3498575202 7286322350 7538497255E-2
14	-0.2638690999 6592557613 2149942808E-10	37	-0.7517179627 8900988246 0645145143E-2
15	-0.5408958287 0450687349 1922207896E-11	38	-0.1616877440 0527227629 8777317918E-2
16	-0.1113222784 6010898999 7676692708E-11	39	-0.3481270085 7247569174 8202271565E-2
17	-0.2299624726 0744624618 4338864145E-12	40	-0.7502707775 5024654701 0642233720E-2
18	-0.4766682389 4951902622 3913482091E-13	41	-0.1618454364 4959102680 7612330206E-2
19	-0.9911756747 3352709450 6246643371E-14	42	-0.3494366771 7051616674 9482836452E-2
20	-0.2067103580 4957072400 0900805021E-14	43	-0.7551036906 1261678585 6037026797E-3

⟨*package DFSFUN DoubleFloatSpecialFunctions*⟩+≡

```
Ei2(y:OPR):OPR ==
  x:R:=retract(y)
  t:R:=acos((x+7.0::R)/3.0::R)::R
  t01:=    0.17575564960612937384876283469 1E1::R*cos(0.0::R)/2.0::R
  t02:=t01-0.43585415177361661170500186796 4E-01::R*cos(t)
  t03:=t02-0.79795071395584254013321702749 2E-02::R*cos( 2.0::R*t)
  t04:=t03-0.14843723273037121385097021000 1E-02::R*cos( 3.0::R*t)
  t05:=t04-0.28003019843775145748620395494 8E-03::R*cos( 4.0::R*t)
  t06:=t05-0.53486485128657932303917736155 3E-04::R*cos( 5.0::R*t)
  t07:=t06-0.10328672435735548661023326646 0E-04::R*cos( 6.0::R*t)
  t08:=t07-0.20140833130055368773222619863 9E-05::R*cos( 7.0::R*t)
  t09:=t08-0.39617584342738664582233844350 0E-06::R*cos( 8.0::R*t)
```

```
t10:=t09-0.78538727670966316306760765606 9E-07::R*cos( 9.0::R*t)
t11:=t10-0.15679259810074698262461627027 9E-07::R*cos(10.0::R*t)
t12:=t11-0.31500559393763998825000737285 1E-08::R*cos(11.0::R*t)
t13:=t12-0.63650968225242037304038026397 2E-09::R*cos(12.0::R*t)
t14:=t13-0.12928881132805631835659312125 9E-09::R*cos(13.0::R*t)
t15:=t14-0.26386909996592557613214994280 8E-10::R*cos(14.0::R*t)
t16:=t15-0.54089582870450687349192220789 6E-11::R*cos(15.0::R*t)
t17:=t16-0.11132227846010898999767669270 8E-11::R*cos(16.0::R*t)
t18:=t17-0.22996247260744624618433886414 5E-12::R*cos(17.0::R*t)
t19:=t18-0.47666823894951902622391348209 1E-13::R*cos(18.0::R*t)
t20:=t19-0.99117567473352709450624664337 1E-14::R*cos(19.0::R*t)
t21:=t20-0.20671035804957072400090080502 1E-14::R*cos(20.0::R*t)
t22:=t21-0.43227767833833850564576439457 9E-15::R*cos(21.0::R*t)
t23:=t22-0.90630147996650172551490560335 6E-16::R*cos(22.0::R*t)
t24:=t23-0.19046699795816613974401596334 2E-16::R*cos(23.0::R*t)
t25:=t24-0.40117923263502786634674422752 0E-17::R*cos(24.0::R*t)
t26:=t25-0.84677721300168322313416633468 5E-18::R*cos(25.0::R*t)
t27:=t26-0.17908427336586966555582649220 4E-18::R*cos(26.0::R*t)
t28:=t27-0.37944906381714782440110617516 6E-19::R*cos(27.0::R*t)
t29:=t28-0.80539992367982798526099965405 8E-20::R*cos(28.0::R*t)
t30:=t29-0.17123390112362012974322867124 4E-20::R*cos(29.0::R*t)
t31:=t30-0.36462740587749686208657656281 6E-21::R*cos(30.0::R*t)
t32:=t31-0.77759696388939479435309815764 7E-22::R*cos(31.0::R*t)
t33:=t32-0.16606284984484020566253195096 6E-22::R*cos(32.0::R*t)
t34:=t33-0.35511786257882509300592714535 2E-23::R*cos(33.0::R*t)
t35:=t34-0.76037226859413580929573465329 4E-24::R*cos(34.0::R*t)
t36:=t35-0.16300741372584900288963837475 5E-24::R*cos(35.0::R*t)
t37:=t36-0.34985752027286322350753849725 5E-25::R*cos(36.0::R*t)
t38:=t37-0.75171796278900988246064514514 3E-26::R*cos(37.0::R*t)
t39:=t38-0.16168774400527227629877731791 8E-26::R*cos(38.0::R*t)
t40:=t39-0.34812700857247569174820227156 5E-27::R*cos(39.0::R*t)
t41:=t40-0.75027077755024654701064223372 0E-28::R*cos(40.0::R*t)
t42:=t41-0.16184543644959102680761233020 6E-28::R*cos(41.0::R*t)
t43:=t42-0.34943667717051616674948283645 2E-29::R*cos(42.0::R*t)
t44:=t43-0.75510369061261678585603702679 7E-30::R*cos(43.0::R*t)
t44::OPR
```

Table 3: Chebyshev Coefficients - Continued (c)

$$[Ei - log|x| - \gamma]/x = \sum_{k-0}^{33}{}' A_k T_k(t), \quad t = x/4, \quad (-4 \le x \le 4)$$

k	A_k	16	0.2615386378 8854429666 9068664148E-10
		17	0.2721858622 8541670644 6550268995E-11
0	0.3293700103 7673912939 3905231421E 01	18	0.2693750031 9835792992 5326427442E-12
1	0.1679835052 3713029156 5505796064E 01	19	0.2541220946 7072635546 7884089307E-13
2	0.7220436105 6787543524 0299679644E 00	20	0.2290130406 8650370941 8510620516E-14
3	0.2600312360 5480956171 3740181192E 00	21	0.1975465739 0746229940 1057650412E-15
4	0.8010494308 1737502239 4742889237E-01	22	0.1634024551 9289317406 8635419984E-16
5	0.2151403663 9763337548 0552483005E-01	23	0.1298235437 0796376099 1961293204E-17
6	0.5116207789 9303312062 1968910894E-02	24	0.9922587925 0737105964 4632581302E-19
7	0.1090932861 0073913560 5066199014E-02	25	0.7306252806 7221032944 7230880087E-20
8	0.2107415320 2393891631 8348675226E-03	26	0.5189676834 6043451272 0780080019E-21
9	0.3719904516 6518885709 5940815956E-04	27	0.3560409454 0997068112 8043162227E-22
10	0.6043491637 1238787570 4767032866E-05	28	0.2361979432 5793864237 0187203948E-23
11	0.9092954273 9626095264 9596541772E-06	29	0.1516837767 7214529754 9624516819E-24
12	0.1273805160 6592647886 5567184969E-06	30	0.9439089722 2448744292 5310405245E-26
13	0.1669185748 4109890739 0896143814E-07	31	0.5697227559 5036921198 9581737831E-27
14	0.2054417026 4010479254 7612484551E-08	32	0.3338333627 7954330315 6597939562E-28
15	0.2383584444 4668176591 4052321417E-09	33	0.1900626012 8161914852 6680482237E-29

(γ=0.5772156649 0153286060 6512090082 E 00)

⟨*package DFSFUN DoubleFloatSpecialFunctions*⟩+≡

```
Ei3(y:OPR):OPR ==
  x:R:=retract(y)
  x = 0.0::R => 1
  t:R:=acos(x/4.0::R)::R
  t01:=    0.32937001037673912939390523142 1E1::R*cos(0.0::R)/2.0::R
  t02:=t01+0.16798350523713029156550579606 4E1::R*cos(t)
  t03:=t02+0.72204361056787543524029967964 4E0::R*cos( 2.0::R*t)
  t04:=t03+0.26003123605480956171374018119 2E0::R*cos( 3.0::R*t)
  t05:=t04+0.80104943081737502239474288923 7E-01::R*cos( 4.0::R*t)
  t06:=t05+0.21514036639763337548055248300 5E-01::R*cos( 5.0::R*t)
  t07:=t06+0.51162077899303312062196891089 4E-02::R*cos( 6.0::R*t)
  t08:=t07+0.10909328610073913560506619901 4E-02::R*cos( 7.0::R*t)
  t09:=t08+0.21074153202393891631834867522 6E-03::R*cos( 8.0::R*t)
  t10:=t09+0.37199045166518885709594081595 6E-04::R*cos( 9.0::R*t)
  t11:=t10+0.60434916371238787570476703286 6E-05::R*cos(10.0::R*t)
  t12:=t11+0.90929542739626095264959654177 2E-06::R*cos(11.0::R*t)
```

```
t13:=t12+0.12738051606592647886556718496 9E-06::R*cos(12.0::R*t)
t14:=t13+0.16691857484109890739089614381 4E-07::R*cos(13.0::R*t)
t15:=t14+0.20544170264010479254761248455 1E-08::R*cos(14.0::R*t)
t16:=t15+0.23835844444668176591405232141 7E-09::R*cos(15.0::R*t)
t17:=t16+0.26153863788854429666906866414 8E-10::R*cos(16.0::R*t)
t18:=t17+0.27218586228541670644655026899 5E-11::R*cos(17.0::R*t)
t19:=t18+0.26937500319835792992532642744 2E-12::R*cos(18.0::R*t)
t20:=t19+0.25412209467072635546788408930 7E-13::R*cos(19.0::R*t)
t21:=t20+0.22901304068650370941851062051 6E-14::R*cos(20.0::R*t)
t22:=t21+0.19754657390746229940105765041 2E-15::R*cos(21.0::R*t)
t23:=t22+0.16340245519289317406863541998 4E-16::R*cos(22.0::R*t)
t24:=t23+0.12982354370796376099196129320 4E-17::R*cos(23.0::R*t)
t25:=t24+0.99225879250737105964463258130 2E-19::R*cos(24.0::R*t)
t26:=t25+0.73062528067221032944723088008 7E-20::R*cos(25.0::R*t)
t27:=t26+0.51896768346043451272078008001 9E-21::R*cos(26.0::R*t)
t28:=t27+0.35604094540997068112804316222 7E-22::R*cos(27.0::R*t)
t29:=t28+0.23619794325793864237018720394 8E-23::R*cos(28.0::R*t)
t30:=t29+0.15168377677214529754962451681 9E-24::R*cos(29.0::R*t)
t31:=t30+0.94390897222448744292531040524 5E-26::R*cos(30.0::R*t)
t32:=t31+0.56972275595036921198958173783 1E-27::R*cos(31.0::R*t)
t33:=t32+0.33383336277954330315659793956 2E-28::R*cos(32.0::R*t)
t34:=t33+0.19006260128161914852668048223 7E-29::R*cos(33.0::R*t)
t34::OPR
```

Table 3: Chebyshev Coefficients - Continued (d)

$$xe^{-x}Ei(x) = \sum_{k-0}^{49}{}' A_k T_k(t), \quad t = (x-8)/4, \quad (4 \le x \le 12)$$

k	A_k	k	A_k
0	0.2455133538 7812952867 3420457043E 01	24	0.1069023072 9386369566 8857256409E-1
1	-0.1624383791 3037652439 6002276856E 00	25	-0.2507030070 5700729569 2572254042E-1
2	0.4495753080 9357264148 0785417193E-01	26	0.5937322503 7915516070 6073763509E-1
3	-0.6741578679 9892299884 8718835050E-02	27	-0.1417734582 4376625234 4732005648E-1
4	-0.1306697142 8032942805 1599341387E-02	28	0.3409203754 3608089342 6806402093E-1
5	0.1381083146 0007257602 0202089820E-02	29	-0.8248290269 5054937928 8702529656E-1
6	-0.5850228790 1596579868 7368242394E-03	30	0.2006369712 6214423139 8824095937E-1
7	0.1749299341 0789197003 8740976432E-03	31	-0.4903851667 9674222440 3498152027E-1
8	-0.4047281499 0529303552 2869333800E-04	32	0.1203734482 3483321716 6664609324E-1
9	0.7217102412 1709975003 5752600049E-05	33	-0.2966282447 1413682538 1453572575E-2
10	-0.8612776970 1986775241 4815450193E-06	34	0.7335512384 2880759924 2142328436E-2
11	-0.2514475296 5322559777 9084739054E-09	35	-0.1819924142 9085112734 4263485604E-2
12	0.3794747138 2014951081 4074505574E-07	36	0.4528629374 2957606021 7359526404E-2
13	-0.1442117969 5211980616 0265640172E-07	37	-0.1129980043 7506096133 8906717853E-2
14	0.3935049295 9761013108 7190848042E-08	38	0.2826681251 2901165692 3764408445E-2
15	-0.9284689401 0633175304 7289210353E-09	39	-0.7087717977 1690496166 6732640699E-2
16	0.2031789568 0065461336 6090995698E-09	40	0.1781104524 0187095153 4401530034E-2
17	-0.4292498504 9923683142 7918026902E-10	41	-0.4485004076 6189635731 2006142358E-2
18	0.8992647177 7812393526 8001544182E-11	42	0.1131540292 5754766224 5053090840E-2
19	-0.1900869118 4121097524 2396635722E-11	43	-0.2859957899 7793216379 0414326136E-2
20	0.4092198912 2237383452 6121178338E-12	44	0.7240775806 9226736175 8172726753E-2
21	-0.8999253437 2931901982 5435824585E-13	45	-0.1836132234 1257789805 0666710105E-2
22	0.2019654670 8242638335 4948543451E-13	46	0.4663128735 2273048658 2600122073E-2
23	-0.4612930261 3830820719 4950531726E-14	47	-0.1185959588 9190288794 6724005478E-2
		48	0.3020290590 5567131073 1137614875E-2
		49	-0.7701650548 1663660609 8827057102E-3

⟨*package DFSFUN DoubleFloatSpecialFunctions*⟩+≡

```
Ei4(y:OPR):OPR ==
  x:R:=retract(y)
  t:R:=acos((x-8.0::R)/4.0::R)::R
  t01:=    0.24551335387812952867342045704 3E1::R*cos(0.0::R)/2.0::R
  t02:=t01-0.16243837913037652439600227685 6E0::R*cos(t)
  t03:=t02+0.44957530809357264148078541719 3E-01::R*cos( 2.0::R*t)
  t04:=t03-0.67415786799892299884871883505 0E-02::R*cos( 3.0::R*t)
  t05:=t04-0.13066971428032942805159934138 7E-02::R*cos( 4.0::R*t)
  t06:=t05+0.13810831460007257602020208982 0E-02::R*cos( 5.0::R*t)
```

```
t07:=t06-0.58502287901596579868736824239 4E-03::R*cos( 6.0::R*t)
t08:=t07+0.17492993410789197003874097643 2E-03::R*cos( 7.0::R*t)
t09:=t08-0.40472814990529303552286933380 0E-04::R*cos( 8.0::R*t)
t10:=t09+0.72171024121709975003575260004 9E-05::R*cos( 9.0::R*t)
t11:=t10-0.86127769701986775241481545019 3E-06::R*cos(10.0::R*t)
t12:=t11-0.25144752965322559777908473905 4E-09::R*cos(11.0::R*t)
t13:=t12+0.37947471382014951081407450557 4E-07::R*cos(12.0::R*t)
t14:=t13-0.14421179695211980616026564017 2E-07::R*cos(13.0::R*t)
t15:=t14+0.39350492959761013108719084804 2E-08::R*cos(14.0::R*t)
t16:=t15-0.92846894010633175304728921035 3E-09::R*cos(15.0::R*t)
t17:=t16+0.20317895680065461336609099569 8E-09::R*cos(16.0::R*t)
t18:=t17-0.42924985049923683142791802690 2E-10::R*cos(17.0::R*t)
t19:=t18+0.89926471777812393526800154418 2E-11::R*cos(18.0::R*t)
t20:=t19-0.19008691184121097524239663572 2E-11::R*cos(19.0::R*t)
t21:=t20+0.40921989122237383452612117833 8E-12::R*cos(20.0::R*t)
t22:=t21-0.89992534372931901982543582458 5E-13::R*cos(21.0::R*t)
t23:=t22+0.20196546708242638335494854345 1E-13::R*cos(22.0::R*t)
t24:=t23-0.46129302613830820719495053172 6E-14::R*cos(23.0::R*t)
t25:=t24+0.10690230729386369566885725640 9E-14::R*cos(24.0::R*t)
t26:=t25-0.25070300705700729569257225404 2E-15::R*cos(25.0::R*t)
t27:=t26+0.59373225037915516070607376350 9E-16::R*cos(26.0::R*t)
t28:=t27-0.14177345824376625234473200564 8E-16::R*cos(27.0::R*t)
t29:=t28+0.34092037543608089342680640209 3E-17::R*cos(28.0::R*t)
t30:=t29-0.82482902695054937928870252965 6E-18::R*cos(29.0::R*t)
t31:=t30+0.20063697126214423139882409593 7E-18::R*cos(30.0::R*t)
t32:=t31-0.49038516679674222440349815202 7E-19::R*cos(31.0::R*t)
t33:=t32+0.12037344823483321716666460932 4E-19::R*cos(32.0::R*t)
t34:=t33-0.29662824471413682538145357257 5E-20::R*cos(33.0::R*t)
t35:=t34+0.73355123842880759924214232843 6E-21::R*cos(34.0::R*t)
t36:=t35-0.18199241429085112734426348560 4E-21::R*cos(35.0::R*t)
t37:=t36+0.45286293742957606021735952640 4E-22::R*cos(36.0::R*t)
t38:=t37-0.11299800437506096133890671785 3E-22::R*cos(37.0::R*t)
t39:=t38+0.28266812512901165692376440844 5E-23::R*cos(38.0::R*t)
t40:=t39-0.70877179771690496166673264069 9E-24::R*cos(39.0::R*t)
t41:=t40+0.17811045240187095153440153003 4E-24::R*cos(40.0::R*t)
t42:=t41-0.44850040766189635731200614235 8E-25::R*cos(41.0::R*t)
t43:=t42+0.11315402925754766224505309084 0E-25::R*cos(42.0::R*t)
t44:=t43-0.28599578997793216379041432613 6E-26::R*cos(43.0::R*t)
t45:=t44+0.72407758069226736175817272675 3E-27::R*cos(44.0::R*t)
t46:=t45-0.18361322341257789805066671010 5E-27::R*cos(45.0::R*t)
t47:=t46+0.46631287352273048658260012207 3E-28::R*cos(46.0::R*t)
t48:=t47-0.11859595889190288794672400547 8E-28::R*cos(47.0::R*t)
t49:=t48+0.30202905905567131073113761487 5E-29::R*cos(48.0::R*t)
t50:=t49-0.77016505481663660609882705710 2E-30::R*cos(49.0::R*t)
t50::OPR
```

Table 3: Chebyshev Coefficients - Continued (e)

$$xe^{-x}Ei(x) = \sum_{k-0}^{47} {}'A_k T_k(t), \quad t = (x-22)/10, \quad (12 \le x \le 32)$$

k	A_k		
0	0.2117028640 4369866832 9789991614E 01	23	0.5898114347 0713196171 1164283918E-1
1	-0.3204237273 7548579499 0618303177E-01	24	-0.9099707635 9564920464 3554720718E-1
2	0.8891732077 3531683589 0182400335E-02	25	0.1040752382 6695538658 5405697541E-1
3	-0.2507952805 1892993708 8352442063E-02	26	-0.1809815426 0592279322 7163355935E-1
4	0.7202789465 9598754887 5760902487E-03	27	-0.3777098842 5639477336 9593494417E-1
5	-0.2103490058 5011305342 3531441256E-03	28	0.1580332901 0284795713 6759888420E-1
6	0.6205732318 2769321658 8857730842E-04	29	-0.4684291758 8088273064 8433752957E-1
7	-0.1826566749 8167026544 9155689733E-04	30	0.1199516852 5919809370 7533478542E-1
8	0.5270651575 2893637580 7788296811E-05	31	-0.2823594749 8418651767 9349931117E-1
9	-0.1459666547 6199457532 3066719367E-05	32	0.6293738065 6446352262 7520190349E-2
10	0.3781719973 5896367198 0484193981E-06	33	-0.1352410249 5047975630 5343973177E-2
11	-0.8842581282 8407192007 7971589012E-07	34	0.2837106053 8552914159 0980426210E-2
12	0.1741749198 5383936137 7350309156E-07	35	-0.5867007420 2463832353 1936371015E-2
13	-0.2313517747 0436906350 6474480152E-08	36	0.1205247636 0954731111 2449686917E-2
14	-0.1228609819 1808623883 2104835230E-09	37	-0.2474446616 9988486972 8416011246E-2
15	0.2349966236 3228637047 8311381926E-09	38	0.5099962585 8378500814 2986465688E-2
16	-0.1100719401 0272628769 0738963049E-09	39	-0.1058382578 7754224088 7093294733E-2
17	0.3848275157 8612071114 9705563369E-10	40	0.2215276245 0704827856 6429387155E-2
18	-0.1148440967 4900158965 8439301603E-10	41	-0.4679278754 7569625867 1852546231E-2
19	0.3056876293 0885208263 0893626200E-11	42	0.9972872990 6020770482 4269828079E-2
20	-0.7388278729 2847356645 4163131431E-12	43	-0.2143267945 2167880459 1907805844E-2
21	0.1630933094 1659411056 4148013749E-12	44	0.4640656908 8381811433 8414829515E-2
22	-0.3276989373 3127124965 7111774748E-13	45	-0.1011447349 2115139094 8461800780E-2
		46	0.2217211522 7100771109 3046878345E-2
		47	-0.4884890469 2437855322 4914645512E-3

⟨*package DFSFUN DoubleFloatSpecialFunctions*⟩+≡

```
Ei5(y:OPR):OPR ==
  x:R:=retract(y)
  t:R:=acos((x-22.0::R)/10.0::R)::R
  t01:=    0.21170286404369866832929789991614E1::R*cos(0.0::R)::R/2.0::R
  t02:=t01-0.32042372737548579499061830317 7E-01::R*cos(t)
  t03:=t02+0.88917320773531683589018240033 5E-02::R*cos( 2.0::R*t)
  t04:=t03-0.25079528051892993708835244206 3E-02::R*cos( 3.0::R*t)
  t05:=t04+0.72027894659598754887576090248 7E-03::R*cos( 4.0::R*t)
  t06:=t05-0.21034900585011305342353144125 6E-03::R*cos( 5.0::R*t)
  t07:=t06+0.62057323182769321658885773084 2E-04::R*cos( 6.0::R*t)
```

```
t08:=t07-0.18265667498167026544915568973 3E-04::R*cos( 7.0::R*t)
t09:=t08+0.52706515752893637580778829681 1E-05::R*cos( 8.0::R*t)
t10:=t09-0.14596665476199457532306671936 7E-05::R*cos( 9.0::R*t)
t11:=t10+0.37817199735896367198048419398 1E-06::R*cos(10.0::R*t)
t12:=t11-0.88425812828407192007797158901 2E-07::R*cos(11.0::R*t)
t13:=t12+0.17417491985383936137735030915 6E-07::R*cos(12.0::R*t)
t14:=t13-0.23135177470436906350647448015 2E-08::R*cos(13.0::R*t)
t15:=t14-0.12286098191808623883210483523 0E-09::R*cos(14.0::R*t)
t16:=t15+0.23499662363228637047831138192 6E-09::R*cos(15.0::R*t)
t17:=t16-0.11007194010272628769073896304 9E-09::R*cos(16.0::R*t)
t18:=t17+0.38482751578612071114970556336 9E-10::R*cos(17.0::R*t)
t19:=t18-0.11484409674900158965843930160 3E-10::R*cos(18.0::R*t)
t20:=t19+0.30568762930885208263089362620 0E-11::R*cos(19.0::R*t)
t21:=t20-0.73882787292847356645416313143 1E-12::R*cos(20.0::R*t)
t22:=t21+0.16309330941659411056414801374 9E-12::R*cos(21.0::R*t)
t23:=t22-0.32769893733127124965711177474 8E-13::R*cos(22.0::R*t)
t24:=t23+0.58981143470713196171116428391 8E-14::R*cos(23.0::R*t)
t25:=t24-0.90997076359564920464355472071 8E-15::R*cos(24.0::R*t)
t26:=t25+0.10407523826695538658540569754 1E-15::R*cos(25.0::R*t)
t27:=t26-0.18098154260592279322716335593 5E-17::R*cos(26.0::R*t)
t28:=t27-0.37770988425639477336959349441 7E-17::R*cos(27.0::R*t)
t29:=t28+0.15803329010284795713675988842 0E-17::R*cos(28.0::R*t)
t30:=t29-0.46842917588088273064843375295 7E-18::R*cos(29.0::R*t)
t31:=t30+0.11995168525919809370753347854 2E-18::R*cos(30.0::R*t)
t32:=t31-0.28235947498418651767934993111 7E-19::R*cos(31.0::R*t)
t33:=t32+0.62937380656446352262752019034 9E-20::R*cos(32.0::R*t)
t34:=t33-0.13524102495047975630534397317 7E-20::R*cos(33.0::R*t)
t35:=t34+0.28371060538552914159098042621 0E-21::R*cos(34.0::R*t)
t36:=t35-0.58670074202463832353193637101 5E-22::R*cos(35.0::R*t)
t37:=t36+0.12052476360954731111244968691 7E-22::R*cos(36.0::R*t)
t38:=t37-0.24744466169988486972841601124 6E-23::R*cos(37.0::R*t)
t39:=t38+0.50999625858378500814298646568 8E-24::R*cos(38.0::R*t)
t40:=t39-0.10583825787754224088709329473 3E-24::R*cos(39.0::R*t)
t41:=t40+0.22152762450704827856642938715 5E-25::R*cos(40.0::R*t)
t42:=t41-0.46792787547569625867185254623 1E-26::R*cos(41.0::R*t)
t43:=t42+0.99728729906020770482426982807 9E-27::R*cos(42.0::R*t)
t44:=t42-0.21432679452167880459190780584 4E-27::R*cos(43.0::R*t)
t45:=t42+0.46406569088381811433841482951 5E-28::R*cos(44.0::R*t)
t46:=t42-0.10114473492115139094846180078 0E-28::R*cos(45.0::R*t)
t47:=t42+0.22172115227100771109304687834 5E-29::R*cos(46.0::R*t)
t48:=t42-0.48848904692437855322491464551 2E-30::R*cos(47.0::R*t)
t48::OPR
```

Table 3: Chebyshev Coefficients - Continued (f)

$$xe^{-x}Ei(x) = \sum_{k-0}^{46}{}' A_k T_k(t), \quad t = (64/x) - 1, \quad (32 \le x < \infty)$$

k	A_k
0	0.2032843945 7961669908 7873844202E 01
1	0.1669920452 0313628514 7618434339E-01
2	0.2845284724 3613468074 2489985325E-03
3	0.7563944358 5162064894 8786693854E-05
4	0.2798971289 4508591575 0484318090E-06
5	0.1357901828 5345310695 2556392593E-07
6	0.8343596202 0404692558 5610289412E-09
7	0.6370971727 6402484382 7524337306E-10
8	0.6007247608 8118612357 6083084850E-11
9	0.7022876174 6797735907 5059216588E-12
10	0.1018302673 7036876930 9667322152E-12
11	0.1761812903 4308800404 0656741554E-13
12	0.3250828614 2353606942 4072007647E-14
13	0.5071770025 5058186788 1479300685E-15
14	0.1665177387 0432942985 3520036957E-16
15	-0.3166753890 7975144007 2410018963E-16
16	-0.1588403763 6641415154 8423134074E-16
17	-0.4175513256 1380188308 9626455063E-17
18	-0.2892347749 7071418820 2868862358E-18
19	0.2800625903 3966080728 9978777339E-18
20	0.1322938639 5392708914 0532005364E-18
21	0.1804447444 1773019958 5334811191E-19
22	-0.7905384086 5226165620 2021080364E-20
23	-0.4435711366 3695734471 8167314045E-20
24	-0.4264103994 9781026176 0579779746E-2
25	0.3920101766 9371439072 5625388636E-2
26	0.1527378051 3439636447 2804486402E-2
27	-0.1024849527 0494906078 6953149788E-2
28	-0.2134907874 7710893794 8904287231E-2
29	-0.3239139475 1602368761 4279789345E-2
30	0.2142183762 2964597029 6249355934E-2
31	0.8234609419 6189955316 9207838151E-2
32	-0.1524652829 6206721081 1495038147E-2
33	-0.1378208282 4882440129 0438126477E-2
34	0.2131311201 4287370679 1513005998E-2
35	0.2012649651 8713266585 9213006507E-2
36	0.1995535662 0563740232 0607178286E-2
37	-0.2798995812 2017971142 6020884464E-2
38	-0.5534511830 5070025094 9784942560E-2
39	0.3884995422 6845525312 9749000696E-2
40	0.1121304407 2330701254 0043264712E-2
41	-0.5566568286 7445948805 7823816866E-2
42	-0.2045482612 4651357628 8865878722E-2
43	0.8453814064 4893808943 7361193598E-2
44	0.3565755151 2015152659 0791715785E-2
45	-0.1383652423 4779775181 0195772006E-2
46	-0.6062142653 2093450576 7865286306E-3

⟨*package DFSFUN DoubleFloatSpecialFunctions*⟩+≡

```
Ei6(y:OPR):OPR ==
  infinite? y => 1
  x:R:=retract(y)
  m:R:=64.0::R/x-1.0::R
  t:R:=acos(m::R)::R
  t01:=    0.20328439457961669908787384420 2E1::R*cos(0.0::R)::R/2.0::R
  t02:=t01+0.16699204520313628514761843433 9E-01::R*cos(t)
  t03:=t02+0.28452847243613468074248998532 5E-03::R*cos( 2.0::R*t)
  t04:=t03+0.75639443585162064894878669385 4E-05::R*cos( 3.0::R*t)
```

```
t05:=t04+0.27989712894508591575048431809OE-06::R*cos( 4.0::R*t)
t06:=t05+0.13579018285345310695255639259 3E-07::R*cos( 5.0::R*t)
t07:=t06+0.83435962020404692558561028941 2E-09::R*cos( 6.0::R*t)
t08:=t07+0.63709717276402484382752433730 6E-10::R*cos( 7.0::R*t)
t09:=t08+0.60072476088118612357608308485 0E-11::R*cos( 8.0::R*t)
t10:=t09+0.70228761746797735907505921658 8E-12::R*cos( 9.0::R*t)
t11:=t10+0.10183026737036876930966732215 2E-12::R*cos(10.0::R*t)
t12:=t11+0.17618129034308800404065674155 4E-13::R*cos(11.0::R*t)
t13:=t12+0.32508286142353606942407200764 7E-14::R*cos(12.0::R*t)
t14:=t13+0.50717700255058186788147930068 5E-15::R*cos(13.0::R*t)
t15:=t14+0.16651773870432942985352003695 7E-16::R*cos(14.0::R*t)
t16:=t15-0.31667538907975144007241001896 3E-16::R*cos(15.0::R*t)
t17:=t16-0.15884037636641415154842313407 4E-16::R*cos(16.0::R*t)
t18:=t17-0.41755132561380188308962645506 3E-17::R*cos(17.0::R*t)
t19:=t18-0.28923477497071418820286886235 8E-18::R*cos(18.0::R*t)
t20:=t19+0.28006259033966080728997877733 9E-18::R*cos(19.0::R*t)
t21:=t20+0.13229386395392708914053200536 4E-18::R*cos(20.0::R*t)
t22:=t21+0.18044474441773019958533481119 1E-19::R*cos(21.0::R*t)
t23:=t22-0.79053840865226165620202108036 4E-20::R*cos(22.0::R*t)
t24:=t23-0.44357113663695734471816731404 5E-20::R*cos(23.0::R*t)
t25:=t24-0.42641039949781026176057977974 6E-21::R*cos(24.0::R*t)
t26:=t25+0.39201017669371439072562538863 6E-21::R*cos(25.0::R*t)
t27:=t26+0.15273780513439636447280448640 2E-21::R*cos(26.0::R*t)
t28:=t27-0.10248495270494906078695314978 8E-22::R*cos(27.0::R*t)
t29:=t28-0.21349078747710893794890428723 1E-22::R*cos(28.0::R*t)
t30:=t29-0.32391394751602368761427978934 5E-23::R*cos(29.0::R*t)
t31:=t30+0.21421837622964597029624935593 4E-23::R*cos(30.0::R*t)
t32:=t31+0.82346094196189955316920783815 1E-24::R*cos(31.0::R*t)
t33:=t32-0.15246528296206721081149503814 7E-24::R*cos(32.0::R*t)
t34:=t33-0.13782082824882440129043812647 7E-24::R*cos(33.0::R*t)
t35:=t34+0.21313112014287370679151300599 8E-26::R*cos(34.0::R*t)
t36:=t35+0.20126496518713266585921300650 7E-25::R*cos(35.0::R*t)
t37:=t36+0.19955356620563740232060717828 6E-26::R*cos(36.0::R*t)
t38:=t37-0.27989958122017971142602088446 4E-26::R*cos(37.0::R*t)
t39:=t38-0.55345118305070025094978494256 0E-27::R*cos(38.0::R*t)
t40:=t39+0.38849954226845525312974900069 6E-27::R*cos(39.0::R*t)
t41:=t40+0.11213044072330701254004326471 2E-27::R*cos(40.0::R*t)
t42:=t41-0.55665682867445948805782381686 6E-28::R*cos(41.0::R*t)
t43:=t42-0.20454826124651357628886587872 2E-28::R*cos(42.0::R*t)
t44:=t43+0.84538140644893808943736119359 8E-29::R*cos(43.0::R*t)
t45:=t44+0.35657551512015152659079171578 5E-29::R*cos(44.0::R*t)
t46:=t45-0.13836524234779775181019577200 6E-29::R*cos(45.0::R*t)
t47:=t46-0.60621426532093450576786528630 6E-30::R*cos(46.0::R*t)
t47::OPR
```

Table 4: Function Values of the Associated Functions

x	$t = -(20/x) - 1$	$xe^{-x}Ei(x)$
$-\infty$	-1.000	0.1000000000 0000000000 0000000000 E 01
-160	-0.875	0.9938266956 7406127387 8797850088 E 00
-80	-0.750	0.9878013330 9428877356 4522608410 E 00
-53 1/3	-0.625	0.9819162901 4319443961 7735426105 E 00
-40	-0.500	0.9761646031 8514305080 8000604060 E 00
-32	-0.375	0.9705398840 7466392046 2584664361 E 00
-26 2/3	-0.250	0.9650362511 2337703576 3536593528 E 00
-22 6/7	-0.125	0.9596482710 7936727616 5478970820 E 00
-20	-0.000	0.9543709099 1921683397 5195829433 E 00
-17 7/9	0.125	0.9491994907 7974574460 6445346803 E 00
-16	0.250	0.9441296577 3690297898 4149471583 E 00
-14 6/11	0.375	0.9391573444 1928424124 0422409988 E 00
-13 1/3	0.500	0.9342787466 5341046480 9375801650 E 00
-12 4/13	0.625	0.9294902984 9721403772 5319679042 E 00
-11 3/7	0.750	0.9247886511 4084169605 5993585492 E 00
-10 2/3	0.875	0.9201706542 4944567620 2148012149 E 00
-10	1.000	0.9156333393 9788081876 0698157666 E 00

x	$t = -(x+7)/3$	$xe^{-x}Ei(x)$
-10.000	-1.000	0.9156333393 9788081876 0698157661 E 01
-9.625	-0.875	0.9128444614 6799341885 6575662217 E 00
-9.250	-0.750	0.9098627515 2542413937 8954274597 E 00
-8.875	-0.625	0.9066672706 5475388033 4995756418 E 00
-8.500	-0.500	0.9032339019 7320784414 4682926135 E 00
-8.125	-0.375	0.8995347176 8847383630 1415777697 E 00
-7.750	-0.250	0.8955371870 8753915717 9475513219 E 00
-7.375	-0.125	0.8912031763 2125431626 7087476258 E 00
-7.000	-0.000	0.8864876725 3642935289 3993846569 E 00
-6.625	0.125	0.8813371384 6821020039 4305706270 E 00
-6.250	0.250	0.8756873647 8846593227 6462155532 E 00
-5.875	0.375	0.8694606294 5411341030 2047153364 E 00
-5.500	0.500	0.8625618846 9070142209 0918986586 E 00
-5.125	0.625	0.8548735538 9019954239 2425567234 E 00
-4.750	0.750	0.8462482991 0358736117 1665798810 E 00
-4.375	0.875	0.8364987545 5629874174 2152267582 E 00
-4.000	1.000	0.8253825996 0422333240 8183035504 E 00

x	$t = x/4$	$[Ei(x) - log\|x\| - \gamma]/x$
-4.0	-1.000	0.4918223446 0781809647 9962798267 E 00
-3.5	-0.875	0.5248425066 4412835691 8258753311 E 00
-3.0	-0.750	0.5629587782 2127986313 8086024270 E 00
-2.5	-0.625	0.6073685258 5838306451 4266925640 E 00
-2.0	-0.500	0.6596316780 8476964479 5492023380 E 00
-1.5	-0.375	0.7218002369 4421992965 7623030310 E 00
-1.0	-0.250	0.7965995992 9705313428 3675865540 E 00
-0.5	-0.125	0.8876841582 3549672587 2151815870 E 00
0.0	-0.000	0.1000000000 0000000000 0000000000 E 01
0.5	0.125	0.1140302841 0431720574 6248768807 E 01
1.0	0.250	0.1317902151 4544038948 6000884424 E 01
1.5	0.375	0.1545736450 7467337302 4859074039 E 01
2.0	0.500	0.1841935755 2702059966 7788045934 E 01
2.5	0.625	0.2232103799 1211651144 5340506423 E 01
3.0	0.750	0.2752668205 6852580020 0219289740 E 01
3.5	0.875	0.3455821531 9301241243 7300898811 E 01
4.0	1.000	0.4416841111 0086991358 0118598668 E 01

x	$t = (x-8)/4$	$xe^{-x}Ei(x)$
4.0	-1.000	0.1438208031 4544827847 0968670330 E 01
4.5	-0.875	0.1396419029 6297460710 0674523183 E 01
5.0	-0.750	0.1353831277 4552859779 0189174047 E 01
5.5	-0.625	0.1314143565 7421192454 1219816991 E 01
6.0	-0.500	0.1278883860 4895616189 2314099578 E 01
6.5	-0.375	0.1248391155 0017014864 0741941387 E 01
7.0	-0.250	0.1222408052 3605310590 3656846622 E 01
7.5	-0.125	0.1200421499 5996307864 3879158950 E 01
8.0	-0.000	0.1181847986 9872079731 7739362644 E 01
8.5	0.125	0.1166126525 8117484943 9918142965 E 01
9.0	0.250	0.1152759208 7089248132 2396814952 E 01
9.5	0.375	0.1141323475 9526242015 5338560641 E 01
10.0	0.500	0.1131470204 7341077803 4051681355 E 01
10.5	0.625	0.1122915570 0177606064 2888630755 E 01
11.0	0.750	0.1115430938 9980384416 4779434229 E 01
11.5	0.875	0.1108832926 3050773058 6855234934 E 01
12.0	1.000	0.1102974544 9067590726 7241234953 E 01

x	$t = (x-22)/10$	$xe^{-x}Ei(x)$
12.00	-1.000	0.1102974544 9067590726 7241234952 E 01
13.25	-0.875	0.1090844898 2154756926 6468614954 E 01
14.50	-0.750	0.1081351395 7351912850 6346643795 E 01
15.75	-0.625	0.1073701384 1997572371 2157900374 E 01
17.00	-0.500	0.1067393691 9585378312 9572196197 E 01
18.25	-0.375	0.1062096608 6221502426 8372647556 E 01
19.50	-0.250	0.1057581342 1587250319 5393949410 E 01
20.75	-0.125	0.1053684451 2894094408 2102194964 E 01
22.00	-0.000	0.1050285719 6851897941 1780664532 E 01
23.25	0.125	0.1047294551 7053248581 1492365591 E 01
24.50	0.250	0.1044641267 9046436368 9761075289 E 01
25.75	0.375	0.1042271337 2023202388 5710928048 E 01
27.00	0.500	0.1040141438 3230104381 3713899754 E 01
28.25	0.625	0.1038216700 3601458768 0056548394 E 01
29.50	0.750	0.1036468726 2924118457 5154685419 E 01
30.75	0.875	0.1034874149 8964796947 2990938990 E 01
32.00	1.000	0.1033413564 2162410494 3493552567 E 01

x	$t = (64/x) - 1$	$xe^{-x}Ei(x)$
∞	-1.000	0.100000000 0000000000 00000000001 E 01
512	-0.875	0.100196079 9450711925 31337468473 E 01
256	-0.750	0.100393713 0905698627 88009078297 E 01
170 2/3	-0.625	0.100592927 5692929112 94663030932 E 01
128	-0.500	0.100793752 4408140182 81776821694 E 01
102 2/5	-0.375	0.100996217 7406449755 74367545570 E 01
85 1/3	-0.250	0.101200354 5332988482 01864466702 E 01
73 1/7	-0.125	0.101406194 9696971331 45942329335 E 01
64	-0.000	0.101613772 3494325321 70357100831 E 01
56 8/9	0.125	0.101823121 1884832696 82337017143 E 01
51 1/5	0.250	0.102034277 2930783774 87217829808 E 01
46 6/11	0.375	0.102247277 8405420595 91275364791 E 01
42 2/3	0.500	0.102462161 4681078391 01187804247 E 01
39 5/13	0.625	0.102678968 3709028524 50984510823 E 01
36 4/7	0.750	0.102897740 4105808008 63378435059 E 01
34 2/15	0.875	0.103118521 2364659263 55875784663 E 01
32	1.000	0.103341356 4216241049 43493552567 E 01

⟨*package DFSFUN DoubleFloatSpecialFunctions*⟩+≡

```
polygamma(k,z)  == CPSI(k, z)$Lisp
polygamma(k,x)  == RPSI(k, x)$Lisp

logGamma z      == CLNGAMMA(z)$Lisp
logGamma x      == RLNGAMMA(x)$Lisp

besselJ(v,z)    == CBESSELJ(v,z)$Lisp
besselJ(n,x)    == RBESSELJ(n,x)$Lisp

besselI(v,z)    == CBESSELI(v,z)$Lisp
besselI(n,x)    == RBESSELI(n,x)$Lisp

hypergeometric0F1(a,z) == CHYPER0F1(a, z)$Lisp
hypergeometric0F1(n,x) == retract hypergeometric0F1(n::C, x::C)


-- All others are defined in terms of these.
digamma x == polygamma(0, x)
digamma z == polygamma(0, z)

Beta(x,y) == Gamma(x)*Gamma(y)/Gamma(x+y)
Beta(w,z) == Gamma(w)*Gamma(z)/Gamma(w+z)

fuzz := (10::R)**(-7)
```

```
import IntegerRetractions(R)
import IntegerRetractions(C)

besselY(n,x) ==
    if integer? n then n := n + fuzz
    vp := n * pi()$R
    (cos(vp) * besselJ(n,x) - besselJ(-n,x) )/sin(vp)
besselY(v,z) ==
    if integer? v then v := v + fuzz::C
    vp := v * pi()$C
    (cos(vp) * besselJ(v,z) - besselJ(-v,z) )/sin(vp)

besselK(n,x) ==
    if integer? n then n := n + fuzz
    p    := pi()$R
    vp   := n*p
    ahalf:= 1/(2::R)
    p * ahalf * ( besselI(-n,x) - besselI(n,x) )/sin(vp)
besselK(v,z) ==
    if integer? v then v := v + fuzz::C
    p    := pi()$C
    vp   := v*p
    ahalf:= 1/(2::C)
    p * ahalf * ( besselI(-v,z) - besselI(v,z) )/sin(vp)

airyAi x ==
    ahalf  := recip(2::R)::R
    athird := recip(3::R)::R
    eta := 2 * athird * (-x) ** (3*ahalf)
    (-x)**ahalf * athird * (besselJ(-athird,eta) + besselJ(athird,eta))
airyAi z ==
    ahalf  := recip(2::C)::C
    athird := recip(3::C)::C
    eta := 2 * athird * (-z) ** (3*ahalf)
    (-z)**ahalf * athird * (besselJ(-athird,eta) + besselJ(athird,eta))

airyBi x ==
    ahalf  := recip(2::R)::R
    athird := recip(3::R)::R
    eta := 2 * athird * (-x) ** (3*ahalf)
    (-x*athird)**ahalf * ( besselJ(-athird,eta) - besselJ(athird,eta) )

airyBi z ==
    ahalf  := recip(2::C)::C
    athird := recip(3::C)::C
```

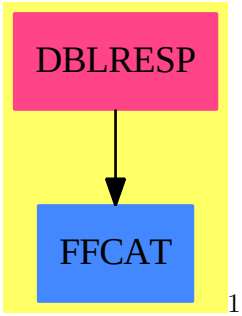
```
            eta := 2 * athird * (-z) ** (3*ahalf)
            (-z*athird)**ahalf * ( besselJ(-athird,eta) - besselJ(athird,eta) )
```

⟨*DFSFUN.dotabb*⟩≡

```
"DFSFUN" [color="#FF4488",href="bookvol10.4.pdf#nameddest=DFSFUN"]
"COMPCAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=COMPCAT"]
"DFSFUN" -> "COMPCAT"
```

5.17 package DBLRESP DoubleResultantPackage

5.18 DoubleResultantPackage



1.00

Exports:
doubleResultant

⟨*package DBLRESP DoubleResultantPackage*⟩≡
```
)abbrev package DBLRESP DoubleResultantPackage
++ Residue resultant
++ Author: Manuel Bronstein
++ Date Created: 1987
++ Date Last Updated: 12 July 1990
++ Description:
++ This package provides functions for computing the residues
++ of a function on an algebraic curve.
DoubleResultantPackage(F, UP, UPUP, R): Exports == Implementation where
  F   : Field
  UP  : UnivariatePolynomialCategory F
  UPUP: UnivariatePolynomialCategory Fraction UP
  R   : FunctionFieldCategory(F, UP, UPUP)

  RF  ==> Fraction UP
  UP2 ==> SparseUnivariatePolynomial UP
  UP3 ==> SparseUnivariatePolynomial UP2

  Exports ==> with
    doubleResultant: (R, UP -> UP) -> UP
      ++ doubleResultant(f, ') returns p(x) whose roots are
      ++ rational multiples of the residues of f at all its
      ++ finite poles. Argument ' is the derivation to use.

  Implementation ==> add
    import CommuteUnivariatePolynomialCategory(F, UP, UP2)
```

```
import UnivariatePolynomialCommonDenominator(UP, RF, UPUP)

UP22   : UP   -> UP2
UP23   : UPUP -> UP3
remove0: UP   -> UP             -- removes the power of x dividing p

remove0 p ==
  primitivePart((p exquo monomial(1, minimumDegree p))::UP)

UP22 p ==
  map(x+->x::UP, p)$UnivariatePolynomialCategoryFunctions2(F,UP,UP,UP2)

UP23 p ==
  map(x+->UP22(retract(x)@UP),p)_
     $UnivariatePolynomialCategoryFunctions2(RF, UPUP, UP2, UP3)

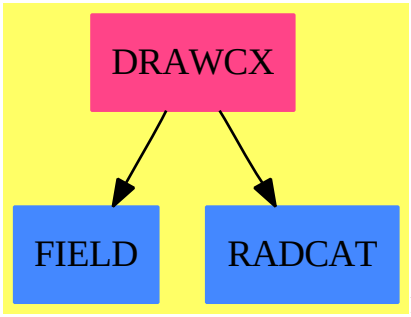
doubleResultant(h, derivation) ==
  cd := splitDenominator lift h
  d  := (cd.den exquo (g := gcd(cd.den, derivation(cd.den))))::UP
  r  := swap primitivePart swap resultant(UP23(cd.num)
      - ((monomial(1, 1)$UP :: UP2) * UP22(g * derivation d))::UP3,
                                        UP23 definingPolynomial())
  remove0 resultant(r, UP22 d)
```

⟨*DBLRESP.dotabb*⟩≡

```
"DBLRESP" [color="#FF4488",href="bookvol10.4.pdf#nameddest=DBLRESP"]
"FFCAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=FFCAT"]
"DBLRESP" -> "FFCAT"
```

5.19 package DRAWCX DrawComplex

5.20 DrawComplex



1.00

Exports:
setImagSteps setRealSteps drawComplex drawComplexVectorField setClipValue

⟨*package DRAWCX DrawComplex*⟩≡

```
)abbrev package DRAWCX DrawComplex
++ Description: \axiomType{DrawComplex} provides some facilities
++ for drawing complex functions.
C ==> Complex DoubleFloat
S ==> Segment DoubleFloat
PC ==> Record(rr:SF, th:SF)
INT ==> Integer
SF ==> DoubleFloat
NNI ==> NonNegativeInteger
VIEW3D ==> ThreeDimensionalViewport
ARRAY2 ==> TwoDimensionalArray

DrawComplex(): Exports == Implementation where
  Exports == with
    drawComplex: (C -> C,S,S,Boolean) -> VIEW3D
      ++ drawComplex(f,rRange,iRange,arrows?)
      ++ draws a complex function as a height field.
      ++ It uses the complex norm as the height and the complex
      ++ argument as the color.
      ++ It will optionally draw arrows on the surface indicating the direction
      ++ of the complex value.\newline
      ++ Sample call:
      ++   \spad{f z == exp(1/z)}
      ++   \spad{drawComplex(f, 0.3..3, 0..2*%pi, false)}
      ++ Parameter descriptions:
      ++   f:  the function to draw
      ++   rRange : the range of the real values
```

```
    ++   iRange : the range of imaginary values
    ++   arrows? : a flag indicating whether to draw the phase arrows for f
    ++ Call the functions \axiomFunFrom{setRealSteps}{DrawComplex} and
    ++ \axiomFunFrom{setImagSteps}{DrawComplex} to change the
    ++ number of steps used in each direction.
  drawComplexVectorField: (C -> C,S,S) -> VIEW3D
    ++ drawComplexVectorField(f,rRange,iRange)
    ++ draws a complex vector field using arrows on the \spad{x--y} plane.
    ++ These vector fields should be viewed from the top by pressing the
    ++ "XY" translate button on the 3-d viewport control panel.\newline
    ++ Sample call:
    ++    \spad{f z == sin z}
    ++    \spad{drawComplexVectorField(f, -2..2, -2..2)}
    ++ Parameter descriptions:
    ++   f : the function to draw
    ++   rRange : the range of the real values
    ++   iRange : the range of the imaginary values
    ++ Call the functions \axiomFunFrom{setRealSteps}{DrawComplex} and
    ++ \axiomFunFrom{setImagSteps}{DrawComplex} to change the
    ++ number of steps used in each direction.
  setRealSteps: INT -> INT
    ++ setRealSteps(i)
    ++ sets to i the number of steps to use in the real direction
    ++ when drawing complex functions. Returns i.
  setImagSteps: INT -> INT
    ++ setImagSteps(i)
    ++ sets to i  the number of steps to use in the imaginary direction
    ++ when drawing complex functions. Returns i.
  setClipValue: SF-> SF
    ++ setClipValue(x)
    ++ sets to x the maximum value to plot when drawing complex functions. Returns x.
Implementation == add
  -- relative size of the arrow head compared to the length of the arrow
  arrowScale : SF := (0.125)::SF
  arrowAngle: SF := pi()-pi()/(20::SF)    -- angle of the arrow head
  realSteps: INT  := 11     -- the number of steps in the real direction
  imagSteps: INT  := 11     -- the number of steps in the imaginary direction
  clipValue: SF  := 10::SF -- the maximum length of a vector to draw


  -- Add an arrow head to a line segment, which starts at 'p1', ends at 'p2',
  -- has length 'len', and and angle 'arg'.  We pass 'len' and 'arg' as
  -- arguments since thet were already computed by the calling program
  makeArrow(p1:Point SF, p2:Point SF, len: SF, arg:SF):List List Point SF ==
    c1 := cos(arg + arrowAngle)
    s1 := sin(arg + arrowAngle)
```

```
    c2 := cos(arg - arrowAngle)
    s2 := sin(arg - arrowAngle)
    p3 := point [p2.1 + c1*arrowScale*len, p2.2 + s1*arrowScale*len,
                 p2.3, p2.4]
    p4 := point [p2.1 + c2*arrowScale*len, p2.2 + s2*arrowScale*len,
                 p2.3, p2.4]
    [[p1, p2, p3], [p2, p4]]

  -- clip a value in the interval (-clip...clip)
  clipFun(x:SF):SF ==
    min(max(x, -clipValue), clipValue)

  drawComplex(f, realRange, imagRange, arrows?) ==
    delReal := (hi(realRange) - lo(realRange))/realSteps::SF
    delImag := (hi(imagRange) - lo(imagRange))/imagSteps::SF
    funTable: ARRAY2(PC) :=
       new((realSteps::NNI)+1, (imagSteps::NNI)+1, [0,0]$PC)
    real := lo(realRange)
    for i in 1..realSteps+1 repeat
      imag := lo(imagRange)
      for j in 1..imagSteps+1 repeat
        z := f complex(real, imag)
        funTable(i,j) := [clipFun(sqrt norm z), argument(z)]$PC
        imag := imag + delImag
      real := real + delReal
    llp := empty()$(List List Point SF)
    real := lo(realRange)
    for i in 1..realSteps+1 repeat
      imag := lo(imagRange)
      lp := empty()$(List Point SF)
      for j in 1..imagSteps+1 repeat
        p := point [real, imag, funTable(i,j).rr, funTable(i,j).th]
        lp := cons(p, lp)
        imag := imag + delImag
      real := real + delReal
      llp := cons(lp, llp)
    space := mesh(llp)$(ThreeSpace SF)
    if arrows? then
      real := lo(realRange)
      for i in 1..realSteps+1 repeat
        imag := lo(imagRange)
        for j in 1..imagSteps+1 repeat
          arg := funTable(i,j).th
          p1 := point [real,imag, funTable(i,j).rr, arg]
          len := delReal*2.0::SF
          p2 := point [p1.1 + len*cos(arg), p1.2 + len*sin(arg),
```

```
                    p1.3, p1.4]
        arrow := makeArrow(p1, p2, len, arg)
        for a in arrow repeat curve(space, a)$(ThreeSpace SF)
        imag := imag + delImag
      real := real + delReal
  makeViewport3D(space, "Complex Function")$VIEW3D

drawComplexVectorField(f, realRange, imagRange): VIEW3D ==
  -- compute the steps size of the grid
  delReal := (hi(realRange) - lo(realRange))/realSteps::SF
  delImag := (hi(imagRange) - lo(imagRange))/imagSteps::SF
  -- create the space to hold the arrows
  space := create3Space()$(ThreeSpace SF)
  real := lo(realRange)
  for i in 1..realSteps+1 repeat
    imag := lo(imagRange)
    for j in 1..imagSteps+1 repeat
      -- compute the function
      z := f complex(real, imag)
      -- get the direction of the arrow
      arg := argument z
      -- get the length of the arrow
      len := clipFun(sqrt norm z)
      -- create point at the base of the arrow
      p1 :=  point [real, imag, 0::SF, arg]
      -- scale the arrow length so it isn't too long
      scaleLen := delReal * len
      -- create the point at the top of the arrow
      p2 := point [p1.1 + scaleLen*cos(arg), p1.2 + scaleLen*sin(arg),
                  0::SF, arg]
      -- make the pointer at the top of the arrow
      arrow := makeArrow(p1, p2, scaleLen, arg)
      -- add the line segments in the arrow to the space
      for a in arrow repeat curve(space, a)$(ThreeSpace SF)
      imag := imag + delImag
    real := real + delReal
  -- draw the vector feild
  makeViewport3D(space, "Complex Vector Field")$VIEW3D

-- set the number of steps to use in the real direction
setRealSteps(n) ==
  realSteps := n

-- set the number of steps to use in the imaginary direction
setImagSteps(n) ==
  imagSteps := n
```

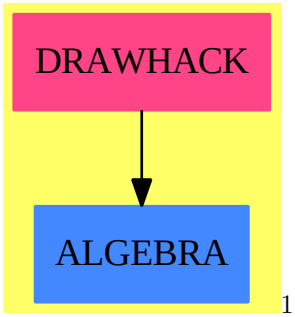
```
      -- set the maximum value to plot
      setClipValue clip ==
        clipValue := clip
```

⟨*DRAWCX.dotabb*⟩≡

```
  "DRAWCX" [color="#FF4488",href="bookvol10.4.pdf#nameddest=DRAWCX"]
  "FIELD"  [color="#4488FF",href="bookvol10.2.pdf#nameddest=FIELD"]
  "RADCAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=RADCAT"]
  "DRAWCX" -> "FIELD"
  "DRAWCX" -> "RADCAT"
```

5.21 package DRAWHACK DrawNumericHack

5.22 DrawNumericHack



1.00

Exports:
coerce

⟨*package DRAWHACK DrawNumericHack*⟩≡
```
)abbrev package DRAWHACK DrawNumericHack
++ Author: Manuel Bronstein
++ Date Created: 21 Feb 1990
++ Date Last Updated: 21 Feb 1990
++ Basic Operations: coerce
++ Related Constructors:
++ Also See:
++ AMS Classifications:
++ Keywords:
++ References:
++ Description: Hack for the draw interface. DrawNumericHack provides
++ a "coercion" from something of the form \spad{x = a..b} where \spad{a}
++ and b are
++ formal expressions to a binding of the form \spad{x = c..d} where c and d
++ are the numerical values of \spad{a} and b. This "coercion" fails if
++ \spad{a} and b contains symbolic variables, but is meant for expressions
++ involving %pi.
++ NOTE:  This is meant for internal use only.

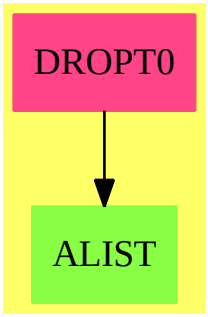
DrawNumericHack(R:Join(OrderedSet,IntegralDomain,ConvertibleTo Float)):
 with coerce: SegmentBinding Expression R -> SegmentBinding Float
        ++ coerce(x = a..b) returns \spad{x = c..d} where c and d are the
        ++ numerical values of \spad{a} and b.
  == add
   coerce s ==
     map(numeric$Numeric(R),s)$SegmentBindingFunctions2(Expression R, Float)
```

⟨*DRAWHACK.dotabb*⟩≡

```
"DRAWHACK" [color="#FF4488",href="bookvol10.4.pdf#nameddest=DRAWHACK"]
"ALGEBRA" [color="#4488FF",href="bookvol10.2.pdf#nameddest=ALGEBRA"]
"DRAWHACK" -> "ALGEBRA"
```

5.23 package DROPT0 DrawOptionFunctions0

5.24 DrawOptionFunctions0



1.00

Exports:

adaptive	clipBoolean	coord	curveColorPalette	pointColorPalette
ranges	space	style	title	toScale
tubePoints	tubeRadius	units	var1Steps	var2Steps
viewpoint				

⟨*package DROPT0 DrawOptionFunctions0*⟩≡

```
)abbrev package DROPT0 DrawOptionFunctions0
-- The functions here are not in DrawOptions since they are not
-- visible to the interpreter.
++ This package \undocumented{}
DrawOptionFunctions0(): Exports == Implementation where
 RANGE ==> List Segment Float
 UNIT  ==> List Float
 PAL   ==> Palette
 POINT ==> Point(DoubleFloat)
 SEG   ==> Segment Float
 SF     ==> DoubleFloat
 SPACE3 ==> ThreeSpace(DoubleFloat)
 VIEWPT ==> Record( theta:SF, phi:SF, scale:SF, scaleX:SF, scaleY:SF, scaleZ:SF, deltaX:SF,

 Exports ==> with
  adaptive: (List DrawOption, Boolean) -> Boolean
    ++ adaptive(l,b) takes the list of draw options, l, and checks
    ++ the list to see if it contains the option \spad{adaptive}.
    ++ If the option does not exist the value, b is returned.
  clipBoolean: (List DrawOption, Boolean) -> Boolean
    ++ clipBoolean(l,b) takes the list of draw options, l, and checks
    ++ the list to see if it contains the option \spad{clipBoolean}.
    ++ If the option does not exist the value, b is returned.
  viewpoint: (List DrawOption, VIEWPT) -> VIEWPT
```

```
    ++ viewpoint(l,ls) takes the list of draw options, l, and checks
    ++ the list to see if it contains the option \spad{viewpoint}.
    ++ IF the option does not exist, the value ls is returned.
  title: (List DrawOption, String) -> String
    ++ title(l,s) takes the list of draw options, l, and checks
    ++ the list to see if it contains the option \spad{title}.
    ++ If the option does not exist the value, s is returned.
  style: (List DrawOption, String) -> String
    ++ style(l,s) takes the list of draw options, l, and checks
    ++ the list to see if it contains the option \spad{style}.
    ++ If the option does not exist the value, s is returned.
  toScale: (List DrawOption, Boolean) -> Boolean
    ++ toScale(l,b) takes the list of draw options, l, and checks
    ++ the list to see if it contains the option \spad{toScale}.
    ++ If the option does not exist the value, b is returned.

  pointColorPalette: (List DrawOption,PAL) -> PAL
    ++ pointColorPalette(l,p) takes the list of draw options, l, and checks
    ++ the list to see if it contains the option \spad{pointColorPalette}.
    ++ If the option does not exist the value, p is returned.
  curveColorPalette: (List DrawOption,PAL) -> PAL
    ++ curveColorPalette(l,p) takes the list of draw options, l, and checks
    ++ the list to see if it contains the option \spad{curveColorPalette}.
    ++ If the option does not exist the value, p is returned.

  ranges: (List DrawOption, RANGE) -> RANGE
    ++ ranges(l,r) takes the list of draw options, l, and checks
    ++ the list to see if it contains the option \spad{ranges}.
    ++ If the option does not exist the value, r is returned.
  var1Steps: (List DrawOption, PositiveInteger) -> PositiveInteger
    ++ var1Steps(l,n) takes the list of draw options, l, and checks
    ++ the list to see if it contains the option \spad{var1Steps}.
    ++ If the option does not exist the value, n is returned.
  var2Steps: (List DrawOption, PositiveInteger) -> PositiveInteger
    ++ var2Steps(l,n) takes the list of draw options, l, and checks
    ++ the list to see if it contains the option \spad{var2Steps}.
    ++ If the option does not exist the value, n is returned.
  space: (List DrawOption) -> SPACE3
    ++ space(l) takes a list of draw options, l, and checks to see
    ++ if it contains the option \spad{space}.  If the the option
    ++ doesn't exist, then an empty space is returned.
  tubePoints : (List DrawOption, PositiveInteger) -> PositiveInteger
    ++ tubePoints(l,n) takes the list of draw options, l, and checks
    ++ the list to see if it contains the option \spad{tubePoints}.
    ++ If the option does not exist the value, n is returned.
  tubeRadius : (List DrawOption, Float) -> Float
```

```
    ++ tubeRadius(l,n) takes the list of draw options, l, and checks
    ++ the list to see if it contains the option \spad{tubeRadius}.
    ++ If the option does not exist the value, n is returned.
  coord: (List DrawOption, (POINT->POINT)) -> (POINT->POINT)
    ++ coord(l,p) takes the list of draw options, l, and checks
    ++ the list to see if it contains the option \spad{coord}.
    ++ If the option does not exist the value, p is returned.
  units: (List DrawOption, UNIT) -> UNIT
    ++ units(l,u) takes the list of draw options, l, and checks
    ++ the list to see if it contains the option \spad{unit}.
    ++ If the option does not exist the value, u is returned.

Implementation ==> add
  adaptive(l,s) ==
    (u := option(l, "adaptive"::Symbol)$DrawOptionFunctions1(Boolean))
      case "failed" => s
    u::Boolean

  clipBoolean(l,s) ==
    (u := option(l, "clipBoolean"::Symbol)$DrawOptionFunctions1(Boolean))
      case "failed" => s
    u::Boolean

  title(l, s) ==
    (u := option(l, "title"::Symbol)$DrawOptionFunctions1(String))
      case "failed" => s
    u::String

  viewpoint(l, vp) ==
    (u := option(l, "viewpoint"::Symbol)$DrawOptionFunctions1(VIEWPT))
      case "failed" => vp
    u::VIEWPT

  style(l, s) ==
    (u := option(l, "style"::Symbol)$DrawOptionFunctions1(String))
      case "failed" => s
    u::String

  toScale(l,s) ==
    (u := option(l, "toScale"::Symbol)$DrawOptionFunctions1(Boolean))
      case "failed" => s
    u::Boolean

  pointColorPalette(l,s) ==
    (u := option(l, "pointColorPalette"::Symbol)$DrawOptionFunctions1(PAL))
      case "failed" => s
```

```
    u::PAL

  curveColorPalette(l,s) ==
    (u := option(l, "curveColorPalette"::Symbol)$DrawOptionFunctions1(PAL))
      case "failed" => s
    u::PAL



  ranges(l, s) ==
    (u := option(l, "ranges"::Symbol)$DrawOptionFunctions1(RANGE))
      case "failed" => s
    u::RANGE

  space(l) ==
    (u := option(l, "space"::Symbol)$DrawOptionFunctions1(SPACE3))
      case "failed" => create3Space()$SPACE3
    u::SPACE3

  var1Steps(l,s) ==
    (u := option(l, "var1Steps"::Symbol)$DrawOptionFunctions1(PositiveInteger))
      case "failed" => s
    u::PositiveInteger

  var2Steps(l,s) ==
    (u := option(l, "var2Steps"::Symbol)$DrawOptionFunctions1(PositiveInteger))
      case "failed" => s
    u::PositiveInteger

  tubePoints(l,s) ==
    (u := option(l, "tubePoints"::Symbol)$DrawOptionFunctions1(PositiveInteger))
      case "failed" => s
    u::PositiveInteger

  tubeRadius(l,s) ==
    (u := option(l, "tubeRadius"::Symbol)$DrawOptionFunctions1(Float))
      case "failed" => s
    u::Float

  coord(l,s) ==
    (u := option(l, "coord"::Symbol)$DrawOptionFunctions1(POINT->POINT))
      case "failed" => s
    u::(POINT->POINT)

  units(l,s) ==
    (u := option(l, "unit"::Symbol)$DrawOptionFunctions1(UNIT))
```

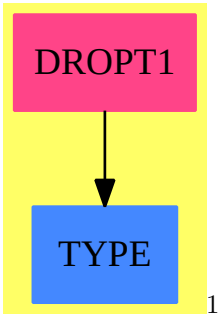
```
        case "failed" => s
      u::UNIT
```

⟨*DROPT0.dotabb*⟩≡

```
"DROPT0" [color="#FF4488",href="bookvol10.4.pdf#nameddest=DROPT0"]
"ALIST" [color="#88FF44",href="bookvol10.3.pdf#nameddest=ALIST"]
"DROPT0" -> "ALIST"
```

5.25 package DROPT1 DrawOptionFunctions1

5.26 DrawOptionFunctions1



1.00

Exports:
option

⟨*package DROPT1 DrawOptionFunctions1*⟩≡

```
)abbrev package DROPT1 DrawOptionFunctions1
++ This package \undocumented{}
DrawOptionFunctions1(S:Type): Exports == Implementation where
 RANGE ==> List Segment Float
 UNIT  ==> List Float
 PAL   ==> Palette
 POINT ==> Point(DoubleFloat)
 SEG   ==> Segment Float
 SF     ==> DoubleFloat
 SPACE3 ==> ThreeSpace(DoubleFloat)
 VIEWPT ==> Record( theta:SF, phi:SF, scale:SF, scaleX:SF, scaleY:SF, scaleZ:SF,

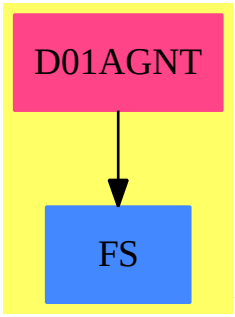
 Exports ==> with
  option: (List DrawOption, Symbol) -> Union(S, "failed")
    ++ option(l,s) determines whether the indicated drawing option, s,
    ++ is contained in the list of drawing options, l, which is defined
    ++ by the draw command.
 Implementation ==> add
  option(l, s) ==
    (u := option(l, s)@Union(Any, "failed")) case "failed" => "failed"
    retract(u::Any)$AnyFunctions1(S)
```

⟨*DROPT1.dotabb*⟩≡

```
"DROPT1" [color="#FF4488",href="bookvol10.4.pdf#nameddest=DROPT1"]
"TYPE" [color="#4488FF",href="bookvol10.2.pdf#nameddest=TYPE"]
"DROPT1" -> "TYPE"
```

5.27 package D01AGNT d01AgentsPackage

5.28 d01AgentsPackage



1.00

Exports:

changeName	commaSeparate	df2st	functionIsContinuousAtEndPoints	functionIsOscillato
gethi	getlo	ldf2lst	problemPoints	rangeIsFinite
sdf2lst	singularitiesOf			

⟨*package D01AGNT d01AgentsPackage*⟩≡

```
)abbrev package D01AGNT d01AgentsPackage
++ Author: Brian Dupee
++ Date Created: March 1994
++ Date Last Updated: December 1997
++ Basic Operations: rangeIsFinite, functionIsContinuousAtEndPoints,
++ functionIsOscillatory
++ Description:
++ \axiomType{d01AgentsPackage} is a package of numerical agents to be used
++ to investigate attributes of an input function so as to decide the
++ \axiomFun{measure} of an appropriate numerical integration routine.
++ It contains functions \axiomFun{rangeIsFinite} to test the input range and
++ \axiomFun{functionIsContinuousAtEndPoints} to check for continuity at
++ the end points of the range.


d01AgentsPackage(): E == I where
  EF2   ==> ExpressionFunctions2
  EFI   ==> Expression Fraction Integer
  FI    ==> Fraction Integer
  LEDF  ==> List Expression DoubleFloat
  KEDF  ==> Kernel Expression DoubleFloat
  EEDF  ==> Equation Expression DoubleFloat
  EDF   ==> Expression DoubleFloat
  PDF   ==> Polynomial DoubleFloat
  LDF   ==> List DoubleFloat
```

```
SDF   ==> Stream DoubleFloat
DF    ==> DoubleFloat
F     ==> Float
ST    ==> String
LST   ==> List String
SI    ==> SingleInteger
SOCDF ==> Segment OrderedCompletion DoubleFloat
OCDF  ==> OrderedCompletion DoubleFloat
OCEDF ==> OrderedCompletion Expression DoubleFloat
EOCEFI  ==> Equation OrderedCompletion Expression Fraction Integer
OCEFI   ==> OrderedCompletion Expression Fraction Integer
OCFI    ==> OrderedCompletion Fraction Integer
NIA   ==> Record(var:Symbol,fn:EDF,range:SOCDF,abserr:DF,relerr:DF)
INT   ==> Integer
CTYPE ==> Union(continuous: "Continuous at the end points",
           lowerSingular: "There is a singularity at the lower end point",
            upperSingular: "There is a singularity at the upper end point",
             bothSingular: "There are singularities at both end points",
              notEvaluated: "End point continuity not yet evaluated")
RTYPE ==> Union(finite: "The range is finite",
            lowerInfinite: "The bottom of range is infinite",
              upperInfinite: "The top of range is infinite",
                bothInfinite: "Both top and bottom points are infinite",
                  notEvaluated: "Range not yet evaluated")
STYPE ==> Union(str:SDF,
                 notEvaluated:"Internal singularities not yet evaluated")
ATT   ==> Record(endPointContinuity:CTYPE,
                  singularitiesStream:STYPE,range:RTYPE)
ROA   ==> Record(key:NIA,entry:ATT)

E ==> with

  rangeIsFinite : NIA -> RTYPE
    ++ rangeIsFinite(args) tests the endpoints of \spad{args.range} for
    ++ infinite end points.
  functionIsContinuousAtEndPoints: NIA -> CTYPE
    ++ functionIsContinuousAtEndPoints(args) uses power series limits
    ++ to check for problems at the end points of the range of \spad{args}.
  getlo : SOCDF -> DF
    ++ getlo(x) gets the \axiomType{DoubleFloat} equivalent of
    ++ the first endpoint of the range \axiom{x}
  gethi : SOCDF -> DF
    ++ gethi(x) gets the \axiomType{DoubleFloat} equivalent of
    ++ the second endpoint of the range \axiom{x}
  functionIsOscillatory:NIA -> F
    ++ functionIsOscillatory(a) tests whether the function \spad{a.fn}
```

```
      ++ has many zeros of its derivative.
    problemPoints: (EDF, Symbol, SOCDF) -> List DF
      ++ problemPoints(f,var,range) returns a list of possible problem points
      ++ by looking at the zeros of the denominator of the function if it
      ++ can be retracted to \axiomType{Polynomial DoubleFloat}.
    singularitiesOf:NIA -> SDF
      ++ singularitiesOf(args) returns a list of potential
      ++ singularities of the function within the given range
    df2st:DF -> String
      ++ df2st(n) coerces a \axiomType{DoubleFloat} to \axiomType{String}
    ldf2lst:LDF -> LST
      ++ ldf2lst(ln) coerces a List of \axiomType{DoubleFloat} to
      ++ \axiomType{List String}
    sdf2lst:SDF -> LST
      ++ sdf2lst(ln) coerces a Stream of \axiomType{DoubleFloat} to
      ++ \axiomType{List String}
    commaSeparate:LST -> ST
      ++ commaSeparate(l) produces a comma separated string from a
      ++ list of strings.
    changeName:(Symbol,Symbol,Result) -> Result
      ++ changeName(s,t,r) changes the name of item \axiom{s} in \axiom{r}
      ++ to \axiom{t}.

  I ==> ExpertSystemContinuityPackage add

    import ExpertSystemToolsPackage
    import ExpertSystemContinuityPackage

    -- local functions
    ocdf2ocefi : OCDF -> OCEFI
    rangeOfArgument : (KEDF, NIA) -> DF
    continuousAtPoint? : (EFI,EOCEFI) -> Boolean
    rand:(SOCDF,INT) -> LDF
    eval:(EDF,Symbol,LDF) -> LDF
    numberOfSignChanges:LDF -> INT
    rangeIsFiniteFunction:NIA -> RTYPE
    functionIsContinuousAtEndPointsFunction:NIA -> CTYPE

    changeName(s:Symbol,t:Symbol,r:Result):Result ==
      a := remove!(s,r)$Result
      a case Any =>
        insert!([t,a],r)$Result
        r
      r

    commaSeparate(l:LST):ST ==
```

```
      empty?(l)$LST => ""
--      one?(#(l)) => concat(l)$ST
      (#(l) = 1) => concat(l)$ST
      f := first(l)$LST
      t := [concat([", ",l.i])$ST for i in 2..#(l)]
      concat(f,concat(t)$ST)$ST

    rand(seg:SOCDF,n:INT):LDF ==
      -- produced a sorted list of random numbers in the given range
      l:DF := getlo seg
      s:DF := (gethi seg) - l
      seed:INT := random()$INT
      dseed:DF := seed :: DF
      r:LDF := [(((random(seed)$INT) :: DF)*s/dseed + l) for i in 1..n]
      sort(r)$LDF

    eval(f:EDF,var:Symbol,l:LDF):LDF ==
      empty?(l)$LDF => [0$DF]
      ve := var::EDF
      [retract(eval(f,equation(ve,u::EDF)$EEDF)$EDF)@DF for u in l]

    numberOfSignChanges(l:LDF):INT ==
      -- calculates the number of sign changes in a list
      a := 0$INT
      empty?(l)$LDF => 0
      for i in 2..# l repeat
        if negative?(l.i*l.(i-1))  then
          a := a + 1
      a

    rangeOfArgument(k: KEDF, args:NIA): DF ==
      Args := copy args
      Args.fn := arg := first(argument(k)$KEDF)$LEDF
      functionIsContinuousAtEndPoints(Args) case continuous =>
        r:SOCDF := args.range
        low:EDF := (getlo r) :: EDF
        high:EDF := (gethi r) :: EDF
        eql := equation(a := args.var :: EDF, low)$EEDF
        eqh := equation(a, high)$EEDF
        e1 := (numeric(eval(arg,eql)$EDF)$Numeric(DF)) :: DF
        e2 := (numeric(eval(arg,eqh)$EDF)$Numeric(DF)) :: DF
        e2-e1
      0$DF

    ocdf2ocefi(r:OCDF):OCEFI ==
      finite?(r)$OCDF => (edf2efi(((retract(r)@DF)$OCDF)::EDF))::OCEFI
```

```
  r pretend OCEFI

continuousAtPoint?(f:EFI,e:EOCEFI):Boolean ==
  (l := limit(f,e)$PowerSeriesLimitPackage(FI,EFI)) case OCEFI =>
                  finite?(l :: OCEFI)
  -- if the left hand limit equals the right hand limit, or if neither
  -- side has a limit at this point, the return type of  limit() is
  -- Union(Ordered Completion Expression Fraction Integer,"failed")
  false

-- exported functions

rangeIsFiniteFunction(args:NIA): RTYPE ==
  -- rangeIsFinite(x) tests the endpoints of x.range for infinite
  -- end points.
  --             [-inf,  inf]  =>  4
  --             [ x  ,  inf]  =>  3
  --             [-inf,  x  ]  =>  1
  --             [ x  ,  y  ]  =>  0
  fr:SI := (3::SI * whatInfinity(hi(args.range))$OCDF
                  - whatInfinity(lo(args.range))$OCDF)
  fr = 0 => ["The range is finite"]
  fr = 1 => ["The bottom of range is infinite"]
  fr = 3 => ["The top of range is infinite"]
  fr = 4 => ["Both top and bottom points are infinite"]
  error("rangeIsFinite",["this is not a valid range"])$ErrorFunctions

rangeIsFinite(args:NIA): RTYPE ==
  nia := copy args
  (t := showAttributes(nia)$IntegrationFunctionsTable) case ATT =>
    s := coerce(t)@ATT
    s.range case notEvaluated =>
      s.range := rangeIsFiniteFunction(nia)
      r:ROA := [nia,s]
      insert!(r)$IntegrationFunctionsTable
      s.range
    s.range
  a:ATT := [["End point continuity not yet evaluated"],
              ["Internal singularities not yet evaluated"],
                  e:=rangeIsFiniteFunction(nia)]
  r:ROA := [nia,a]
  insert!(r)$IntegrationFunctionsTable
  e

functionIsContinuousAtEndPointsFunction(args:NIA):CTYPE ==
```

```
    v := args.var :: EFI :: OCEFI
    high:OCEFI := ocdf2ocefi(hi(args.range))
    low:OCEFI := ocdf2ocefi(lo(args.range))
    f := edf2efi(args.fn)
    l:Boolean := continuousAtPoint?(f,equation(v,low)$EOCEFI)
    h:Boolean := continuousAtPoint?(f,equation(v,high)$EOCEFI)
    l and h => ["Continuous at the end points"]
    l => ["There is a singularity at the upper end point"]
    h => ["There is a singularity at the lower end point"]
    ["There are singularities at both end points"]

  functionIsContinuousAtEndPoints(args:NIA): CTYPE ==
    nia := copy args
    (t := showAttributes(nia)$IntegrationFunctionsTable) case ATT =>
      s := coerce(t)@ATT
      s.endPointContinuity case notEvaluated =>
        s.endPointContinuity := functionIsContinuousAtEndPointsFunction(nia)
        r:ROA := [nia,s]
        insert!(r)$IntegrationFunctionsTable
        s.endPointContinuity
      s.endPointContinuity
    a:ATT := [e:=functionIsContinuousAtEndPointsFunction(nia),
              ["Internal singularities not yet evaluated"],
                ["Range not yet evaluated"]]
    r:ROA := [nia,a]
    insert!(r)$IntegrationFunctionsTable
    e

  functionIsOscillatory(a:NIA):F ==

    args := copy a
    k := tower(numerator args.fn)$EDF
    p:F := pi()$F
    for i in 1..# k repeat
      is?(ker := k.i, sin :: Symbol) =>
        ra := convert(rangeOfArgument(ker,args))@F
        ra > 2*p => return (ra/p)
      is?(ker, cos :: Symbol) =>
        ra := convert(rangeOfArgument(ker,args))@F
        ra > 2*p => return (ra/p)
    l:LDF := rand(args.range,30)
    l := eval(args.fn,args.var,l)
    numberOfSignChanges(l) :: F

  singularitiesOf(args:NIA):SDF ==
    nia := copy args
```

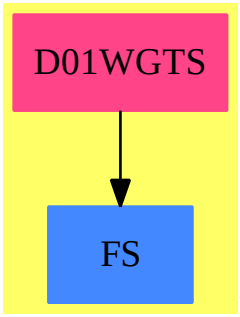
```
(t := showAttributes(nia)$IntegrationFunctionsTable) case ATT =>
  s:ATT := coerce(t)@ATT
  p:STYPE := s.singularitiesStream
  p case str => p.str
  e:SDF := singularitiesOf(nia.fn,[nia.var],nia.range)
  if not empty?(e) then
    if less?(e,10)$SDF then extend(e,10)$SDF
  s.singularitiesStream := [e]
  r:ROA := [nia,s]
  insert!(r)$IntegrationFunctionsTable
  e
e:=singularitiesOf(nia.fn,[nia.var],nia.range)
if not empty?(e) then
  if less?(e,10)$SDF then extend(e,10)$SDF
a:ATT := [["End point continuity not yet evaluated"],[e],
                    ["Range not yet evaluated"]]
r:ROA := [nia,a]
insert!(r)$IntegrationFunctionsTable
e
```

⟨*D01AGNT.dotabb*⟩≡

```
"D01AGNT" [color="#FF4488",href="bookvol10.4.pdf#nameddest=D01AGNT"]
"FS" [color="#4488FF",href="bookvol10.2.pdf#nameddest=FS"]
"D01AGNT" -> "FS"
```

5.29 package D01WGTS d01WeightsPackage

5.30 d01WeightsPackage



1.00

Exports:
exprHasAlgebraicWeight exprHasLogarithmicWeights exprHasWeightCosWXorSinWX

⟨*package D01WGTS d01WeightsPackage*⟩≡

```
)abbrev package D01WGTS d01WeightsPackage
++ Author: Brian Dupee
++ Date Created: July 1994
++ Date Last Updated: January 1998 (Bug fix - exprHasListOfWeightsCosWXorSinWX)
++ Basic Operations: exprHasWeightCosWXorSinWX, exprHasAlgebraicWeight,
++ exprHasLogarithmicWeights
++ Description:
++ \axiom{d01WeightsPackage} is a package for functions used to investigate
++ whether a function can be divided into a simpler function and a weight
++ function.  The types of weights investigated are those giving rise to
++ end-point singularities of the algebraico-logarithmic type, and
++ trigonometric weights.
d01WeightsPackage(): E == I where
  LEDF  ==> List Expression DoubleFloat
  KEDF  ==> Kernel Expression DoubleFloat
  LKEDF ==> List Kernel Expression DoubleFloat
  EDF   ==> Expression DoubleFloat
  PDF   ==> Polynomial DoubleFloat
  FI    ==> Fraction Integer
  LDF   ==> List DoubleFloat
  DF    ==> DoubleFloat
  SOCDF ==> Segment OrderedCompletion DoubleFloat
  OCDF  ==> OrderedCompletion DoubleFloat
  NIA   ==> Record(var:Symbol,fn:EDF,range:SOCDF,abserr:DF,relerr:DF)
  INT   ==> Integer
  BOP   ==> BasicOperator
  URBODF        ==> Union(Record(op:BasicOperator,w:DF),"failed")
```

```
LURBODF       ==> List(Union(Record(op:BasicOperator,w:DF), "failed"))

E ==> with
  exprHasWeightCosWXorSinWX:NIA -> URBODF
    ++ \axiom{exprHasWeightCosWXorSinWX} looks for trigonometric
    ++ weights in an expression of the form \axiom{cos \omega x} or
    ++ \axiom{sin \omega x}, returning the value of \omega
    ++ (\notequal 1) and the operator.
  exprHasAlgebraicWeight:NIA -> Union(LDF,"failed")
    ++ \axiom{exprHasAlgebraicWeight} looks for algebraic weights
    ++ giving rise to singularities of the function at the end-points.
  exprHasLogarithmicWeights:NIA -> INT
    ++ \axiom{exprHasLogarithmicWeights} looks for logarithmic weights
    ++ giving rise to singularities of the function at the end-points.



I ==> add
  score:(EDF,EDF) -> FI
  kernelIsLog:KEDF -> Boolean
  functionIsPolynomial?:EDF -> Boolean
  functionIsNthRoot?:(EDF,EDF) -> Boolean
  functionIsQuotient:EDF -> Union(EDF,"failed")
  findCommonFactor:LEDF -> Union(LEDF,"failed")
  findAlgebraicWeight:(NIA,EDF) -> Union(DF,"failed")
  exprHasListOfWeightsCosWXorSinWX:(EDF,Symbol) -> LURBODF
  exprOfFormCosWXorSinWX:(EDF,Symbol) -> URBODF
  bestWeight:LURBODF -> URBODF
  weightIn?:(URBODF,LURBODF) -> Boolean
  inRest?:(EDF,LEDF)->Boolean
  factorIn?:(EDF,LEDF)->Boolean
  voo?:(EDF,EDF)->Boolean

  kernelIsLog(k:KEDF):Boolean ==
    (name k = (log :: Symbol))@Boolean

  factorIn?(a:EDF,l:LEDF):Boolean ==
    for i in 1..# l repeat
      (a = l.i)@Boolean => return true
    false

  voo?(b:EDF,a:EDF):Boolean ==
     (voo:=isTimes(b)) case LEDF and factorIn?(a,voo)

  inRest?(a:EDF,l:LEDF):Boolean ==
    every?(x+->voo?(x,a) ,l)
```

```
    findCommonFactor(l:LEDF):Union(LEDF,"failed") ==
      empty?(l)$LEDF => "failed"
      f := first(l)$LEDF
      r := rest(l)$LEDF
      (t := isTimes(f)$EDF) case LEDF =>
        pos:=select(x+->inRest?(x,r),t)
        empty?(pos) => "failed"
        pos
      "failed"

    exprIsLogarithmicWeight(f:EDF,Var:EDF,a:EDF,b:EDF):INT ==
      ans := 0$INT
      k := tower(f)$EDF
      lf := select(kernelIsLog,k)$LKEDF
      empty?(lf)$LKEDF => ans
      for i in 1..# lf repeat
        arg := argument lf.i
        if (arg.1 = (Var - a)) then
          ans := ans + 1
        else if (arg.1 = (b - Var)) then
          ans := ans + 2
      ans

    exprHasLogarithmicWeights(args:NIA):INT ==
      ans := 1$INT
      a := getlo(args.range)$d01AgentsPackage :: EDF
      b := gethi(args.range)$d01AgentsPackage :: EDF
      Var := args.var :: EDF
      (l := isPlus numerator args.fn) case LEDF =>
        (cf := findCommonFactor l) case LEDF =>
          for j in 1..# cf repeat
            ans := ans + exprIsLogarithmicWeight(cf.j,Var,a,b)
          ans
        ans
      ans := ans + exprIsLogarithmicWeight(args.fn,Var,a,b)

    functionIsQuotient(expr:EDF):Union(EDF,"failed") ==
      (k := mainKernel expr) case KEDF =>
        expr = inv(f := k :: KEDF :: EDF)$EDF => f
--        one?(numerator expr) => denominator expr
        (numerator expr = 1) => denominator expr
        "failed"
      "failed"

    functionIsPolynomial?(f:EDF):Boolean ==
```

```
      (retractIfCan(f)@Union(PDF,"failed"))$EDF case PDF

    functionIsNthRoot?(f:EDF,e:EDF):Boolean ==
      (m := mainKernel f) case "failed" => false
--      (one?(# (kernels f)))
      ((# (kernels f)) = 1)
        and (name operator m = (nthRoot :: Symbol))@Boolean
          and (((argument m).1 = e)@Boolean)

    score(f:EDF,e:EDF):FI ==
      ans := 0$FI
      (t := isTimes f) case LEDF =>
        for i in 1..# t repeat
          ans := ans + score(t.i,e)
        ans
      (q := functionIsQuotient f) case EDF =>
        ans := ans - score(q,e)
      functionIsPolynomial? f =>
        g:EDF := f/e
        if functionIsPolynomial? g then
          ans := 1+score(g,e)
        else
          ans
      (l := isPlus f) case LEDF =>
        (cf := findCommonFactor l) case LEDF =>
          factor := 1$EDF
          for i in 1..# cf repeat
            factor := factor*cf.i
          ans := ans + score(f/factor,e) + score(factor,e)
        ans
      functionIsNthRoot?(f,e) =>
        (p := isPower f) case "failed" => ans
        exp := p.exponent
        m := mainKernel f
        m case KEDF =>
          arg := argument m
          a:INT := (retract(arg.2)@INT)$EDF
          exp / a
        ans
      ans

    findAlgebraicWeight(args:NIA,e:EDF):Union(DF,"failed") ==
      zero?(s := score(args.fn,e)) => "failed"
      s :: DF

    exprHasAlgebraicWeight(args:NIA):Union(LDF,"failed") ==
```

```
      (f := functionIsContinuousAtEndPoints(args)$d01AgentsPackage)
                                          case continuous =>"failed"
      Var := args.var :: EDF
      a := getlo(args.range)$d01AgentsPackage :: EDF
      b := gethi(args.range)$d01AgentsPackage :: EDF
      A := Var - a
      B := b - Var
      f case lowerSingular =>
        (h := findAlgebraicWeight(args,A)) case "failed" => "failed"
        [h,0]
      f case upperSingular =>
        (g := findAlgebraicWeight(args,B)) case "failed" => "failed"
        [0,g]
      h := findAlgebraicWeight(args,A)
      g := findAlgebraicWeight(args,B)
      r := (h case "failed")
      s := (g case "failed")
      (r) and (s) => "failed"
      r => [0,coerce(g)@DF]
      s => [coerce(h)@DF,0]
      [coerce(h)@DF,coerce(g)@DF]

    exprOfFormCosWXorSinWX(f:EDF,var:Symbol): URBODF ==
      l:LKEDF := kernels(f)$EDF
--      one?((# l)$LKEDF)$INT =>
      # l = 1 =>
        a:LEDF := argument(e:KEDF := first(l)$LKEDF)$KEDF
        empty?(a) => "failed"
        m:Union(LEDF,"failed") := isTimes(first(a)$LEDF)$EDF
        m case LEDF => -- if it is a list, it will have at least two elements
          is?(second(m)$LEDF,var)$EDF =>
            omega:DF := retract(first(m)$LEDF)@DF
            o:BOP := operator(n:Symbol:=name(e)$KEDF)$BOP
            (n = cos@Symbol)@Boolean => [o,omega]
            (n = sin@Symbol)@Boolean => [o,omega]
            "failed"
          "failed"
        "failed"
      "failed"

    exprHasListOfWeightsCosWXorSinWX(f:EDF,var:Symbol): LURBODF ==
      (e := isTimes(f)$EDF) case LEDF =>
        [exprOfFormCosWXorSinWX(u,var) for u in e]
      empty?(k := kernels f) => ["failed"]
      ((first(k)::EDF) = f) =>
        [exprOfFormCosWXorSinWX(f,var)]
```

```
  ["failed"]

bestWeight(l:LURBODF): URBODF ==
  empty?(l)$LURBODF => "failed"
  best := first(l)$LURBODF        --  best is first in list
  empty?(rest(l)$LURBODF) => best
  for i in 2..# l repeat          --  unless next is better
    r:URBODF := l.i
    if r case "failed" then leave
    else if best case "failed" then
      best := r
    else if r.w > best.w then
      best := r
  best

weightIn?(weight:URBODF,listOfWeights:LURBODF):Boolean ==
  n := # listOfWeights
  for i in 1..n repeat                              -- cycle through list
    (weight = listOfWeights.i)@Boolean => return true -- return when found
  false

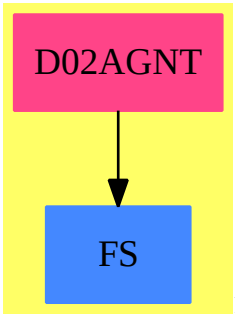
exprHasWeightCosWXorSinWX(args:NIA):URBODF ==
  ans := empty()$LURBODF
  f:EDF := numerator(args.fn)$EDF
  (t:Union(LEDF,"failed") := isPlus(f)) case "failed" =>
    bestWeight(exprHasListOfWeightsCosWXorSinWX(f,args.var))
  if t case LEDF then
    e1 := first(t)$LEDF
    le1:LURBODF := exprHasListOfWeightsCosWXorSinWX(e1,args.var)
    le1 := [u for u in le1 | (not (u case "failed"))]
    empty?(le1)$LURBODF => "failed"
    test := true
    for i in 1..# le1 repeat
      le1i:URBODF := le1.i
      for j in 2..# t repeat
        if test then
          tj:LURBODF := exprHasListOfWeightsCosWXorSinWX(t.j,args.var)
          test := weightIn?(le1i,tj)
      if test then
        ans := concat([le1i],ans)
    bestWeight ans
  else "failed"
```

⟨*D01WGTS.dotabb*⟩≡

```
"D01WGTS" [color="#FF4488",href="bookvol10.4.pdf#nameddest=D01WGTS"]
"FS" [color="#4488FF",href="bookvol10.2.pdf#nameddest=FS"]
"D01WGTS" -> "FS"
```

5.31 package D02AGNT d02AgentsPackage

5.32 d02AgentsPackage



1.00

Exports:

accuracyIF	combineFeatureCompatibility	eval	expenseOfEvaluationIF
jacobian	sparsityIF	stiffnessAndStabilityFactor	stiffnessAndStabilityOF

⟨*package D02AGNT d02AgentsPackage*⟩≡

```
)abbrev package D02AGNT d02AgentsPackage
++ Author: Brian Dupee
++ Date Created: May 1994
++ Date Last Updated: January 1997
++ Basic Operations: stiffnessFactor, jacobian
++ Description:
++ \axiom{d02AgentsPackage} contains a set of computational agents
++ for use with Ordinary Differential Equation solvers.
d02AgentsPackage(): E == I where
  LEDF  ==> List Expression DoubleFloat
  LEEDF ==> List Equation Expression DoubleFloat
  EEDF  ==> Equation Expression DoubleFloat
  VEDF  ==> Vector Expression DoubleFloat
  MEDF  ==> Matrix Expression DoubleFloat
  MDF   ==> Matrix DoubleFloat
  EDF   ==> Expression DoubleFloat
  DF    ==> DoubleFloat
  F     ==> Float
  INT   ==> Integer
  CDF   ==> Complex DoubleFloat
  LDF   ==> List DoubleFloat
  LF    ==> List Float
  S     ==> Symbol
  LS    ==> List Symbol
  MFI   ==> Matrix Fraction Integer
  LFI   ==> List Fraction Integer
```

```
FI    ==> Fraction Integer
ODEA  ==> Record(xinit:DF,xend:DF,fn:VEDF,yinit:LDF,intvals:LDF,g:EDF,abserr:DF,relerr:DF)
ON    ==> Record(additions:INT,multiplications:INT,exponentiations:INT,functionCalls:INT)
RVE   ==> Record(val:EDF,exponent:INT)
RSS   ==> Record(stiffnessFactor:F,stabilityFactor:F)
ATT   ==> Record(stiffness:F,stability:F,expense:F,accuracy:F,intermediateResults:F)
ROA   ==> Record(key:ODEA,entry:ATT)

E ==>  with
  combineFeatureCompatibility: (F,F) -> F
    ++ combineFeatureCompatibility(C1,C2) is for interacting attributes
  combineFeatureCompatibility: (F,LF) -> F
    ++ combineFeatureCompatibility(C1,L) is for interacting attributes
  sparsityIF: MEDF -> F
    ++ sparsityIF(m) calculates the sparsity of a jacobian matrix
  jacobian: (VEDF,LS) -> MEDF
    ++ jacobian(v,w) is a local function to make a jacobian matrix
  eval: (MEDF,LS,VEDF) -> MEDF
    ++ eval(mat,symbols,values) evaluates a multivariable matrix at given values
    ++ for each of a list of variables
  stiffnessAndStabilityFactor: MEDF -> RSS
    ++ stiffnessAndStabilityFactor(me) calculates the stability and
    ++ stiffness factor of a system of first-order differential equations
    ++ (by evaluating the maximum difference in the real parts of the
    ++ negative eigenvalues of the jacobian of the system for which O(10)
    ++ equates to mildly stiff wheras stiffness ratios of O(10^6) are not
    ++ uncommon) and whether the system is likely to show any oscillations
    ++ (identified by the closeness to the imaginary axis of the complex
    ++ eigenvalues of the jacobian).
  stiffnessAndStabilityOfODEIF:ODEA -> RSS
    ++ stiffnessAndStabilityOfODEIF(ode) calculates the intensity values
    ++ of stiffness of a system of first-order differential equations
    ++ (by evaluating the maximum difference in the real parts of the
    ++ negative eigenvalues of the jacobian of the system for which O(10)
    ++ equates to mildly stiff wheras stiffness ratios of O(10^6) are not
    ++ uncommon) and whether the system is likely to show any oscillations
    ++ (identified by the closeness to the imaginary axis of the complex
    ++ eigenvalues of the jacobian).
    ++
    ++ It returns two values in the range [0,1].
  systemSizeIF:ODEA -> F
    ++ systemSizeIF(ode) returns the intensity value of the size of
    ++ the system of ODEs.  20 equations corresponds to the neutral
    ++ value.  It returns a value in the range [0,1].
  expenseOfEvaluationIF:ODEA -> F
    ++ expenseOfEvaluationIF(o) returns the intensity value of the
```

```
      ++ cost of evaluating the input ODE.  This is in terms of the number
      ++ of ``operational units''.  It returns a value in the range
      ++ [0,1].\newline\indent{20}
      ++ 400 ``operation units'' -> 0.75 \newline
      ++ 200 ``operation units'' -> 0.5 \newline
      ++ 83 ``operation units'' -> 0.25 \newline\indent{15}
      ++ exponentiation = 4 units , function calls = 10 units.
    accuracyIF:ODEA -> F
      ++ accuracyIF(o) returns the intensity value of the accuracy
      ++ requirements of the input ODE.  A request of accuracy of 10^-6
      ++ corresponds to the neutral intensity.  It returns a value
      ++ in the range [0,1].
    intermediateResultsIF:ODEA -> F
      ++ intermediateResultsIF(o) returns a value corresponding to the
      ++ required number of intermediate results required and, therefore,
      ++ an indication of how much this would affect the step-length of the
      ++ calculation.  It returns a value in the range [0,1].

  I ==> add

    import ExpertSystemToolsPackage

    accuracyFactor:ODEA -> F
    expenseOfEvaluation:ODEA -> F
    eval1:(LEDF,LEEDF) -> LEDF
    stiffnessAndStabilityOfODE:ODEA -> RSS
    intermediateResultsFactor:ODEA -> F
    leastStabilityAngle:(LDF,LDF) -> F

    intermediateResultsFactor(ode:ODEA):F ==
      resultsRequirement := #(ode.intvals)
      (1.0-exp(-(resultsRequirement::F)/50.0)$F)

    intermediateResultsIF(o:ODEA):F ==
      ode := copy o
      (t := showIntensityFunctions(ode)$ODEIntensityFunctionsTable) case ATT =>
        s := coerce(t)@ATT
        negative?(s.intermediateResults)$F =>
          s.intermediateResults := intermediateResultsFactor(ode)
          r:ROA := [ode,s]
          insert!(r)$ODEIntensityFunctionsTable
          s.intermediateResults
        s.intermediateResults
      a:ATT := [-1.0,-1.0,-1.0,-1.0,e:=intermediateResultsFactor(ode)]
      r:ROA := [ode,a]
      insert!(r)$ODEIntensityFunctionsTable
```

```
    e

  accuracyFactor(ode:ODEA):F ==
    accuracyRequirements := convert(ode.abserr)@F
    if zero?(accuracyRequirements) then
      accuracyRequirements := convert(ode.relerr)@F
    val := inv(accuracyRequirements)$F
    n := log10(val)$F
    (1.0-exp(-(n/(2.0))**2/(15.0))$F)

  accuracyIF(o:ODEA):F ==
    ode := copy o
    (t := showIntensityFunctions(ode)$ODEIntensityFunctionsTable) case ATT =>
      s := coerce(t)@ATT
      negative?(s.accuracy)$F =>
        s.accuracy := accuracyFactor(ode)
        r:ROA := [ode,s]
        insert!(r)$ODEIntensityFunctionsTable
        s.accuracy
      s.accuracy
    a:ATT := [-1.0,-1.0,-1.0,e:=accuracyFactor(ode),-1.0]
    r:ROA := [ode,a]
    insert!(r)$ODEIntensityFunctionsTable
    e

  systemSizeIF(ode:ODEA):F ==
    n := #(ode.fn)
    (1.0-exp((-n::F/75.0))$F)

  expenseOfEvaluation(o:ODEA):F ==
    -- expense of evaluation of an ODE -- <0.3 inexpensive - 0.5 neutral - >0.7 very exper
    -- 400 'operation units' -> 0.75
    -- 200 'operation units' -> 0.5
    -- 83 'operation units' -> 0.25
    -- ** = 4 units , function calls = 10 units.
    ode := copy o.fn
    expenseOfEvaluation(ode)

  expenseOfEvaluationIF(o:ODEA):F ==
    ode := copy o
    (t := showIntensityFunctions(ode)$ODEIntensityFunctionsTable) case ATT =>
      s := coerce(t)@ATT
      negative?(s.expense)$F =>
        s.expense := expenseOfEvaluation(ode)
        r:ROA := [ode,s]
        insert!(r)$ODEIntensityFunctionsTable
```

```
      s.expense
    s.expense
  a:ATT := [-1.0,-1.0,e:=expenseOfEvaluation(ode),-1.0,-1.0]
  r:ROA := [ode,a]
  insert!(r)$ODEIntensityFunctionsTable
  e

leastStabilityAngle(realPartsList:LDF,imagPartsList:LDF):F ==
  complexList := [complex(u,v)$CDF for u in realPartsList for v in imagPartsL
  argumentList := [abs((abs(argument(u)$CDF)$DF)-(pi()$DF)/2)$DF for u in com
  sortedArgumentList := sort(argumentList)$LDF
  list := [u for u in sortedArgumentList | not zero?(u) ]
  empty?(list)$LDF => 0$F
  convert(first(list)$LDF)@F

stiffnessAndStabilityFactor(me:MEDF):RSS ==

  -- search first for real eigenvalues of the jacobian (symbolically)
  -- if the system isn't too big
  r:INT := ncols(me)$MEDF
  b:Boolean := ((# me) < 150)
  if b then
    mc:MFI := map(edf2fi,me)$ExpertSystemToolsPackage2(EDF,FI)
    e:LFI := realEigenvalues(mc,1/100)$NumericRealEigenPackage(FI)
    b := ((# e) >= r-1)@Boolean
  b =>
    -- if all the eigenvalues are real, find negative ones
    e := sort(neglist(e)$ExpertSystemToolsPackage1(FI))
    -- if there are two or more, calculate stiffness ratio
    ((n:=#e)>1)@Boolean => [coerce(e.1/e.n)@F,0$F]
    -- otherwise stiffness not present
    [0$F,0$F]

  md:MDF := map(edf2df,me)$ExpertSystemToolsPackage2(EDF,DF)

  -- otherwise calculate numerically the complex eigenvalues
  -- using NAG routine f02aff.

  res:Result := f02aff(r,r,md,-1)$NagEigenPackage
  realParts:Union(Any,"failed") := search(rr::Symbol,res)$Result
  realParts case "failed" => [0$F,0$F]
  realPartsMatrix:MDF := retract(realParts)$AnyFunctions1(MDF) -- array === m
  imagParts:Union(Any,"failed") := search(ri::Symbol,res)$Result
  imagParts case "failed" => [0$F,0$F]
  imagPartsMatrix:MDF := retract(imagParts)$AnyFunctions1(MDF) -- array === m
  imagPartsList:LDF := members(imagPartsMatrix)$MDF
```

```
  realPartsList:LDF := members(realPartsMatrix)$MDF
  stabilityAngle := leastStabilityAngle(realPartsList,imagPartsList)
  negRealPartsList := sort(neglist(realPartsList)$ExpertSystemToolsPackage1(DF))
  empty?(negRealPartsList)$LDF => [0$F,stabilityAngle]
  ((n:=#negRealPartsList)>1)@Boolean =>
    out := convert(negRealPartsList.1/negRealPartsList.n)@F
    [out,stabilityAngle]    -- calculate stiffness ratio
  [-convert(negRealPartsList.1)@F,stabilityAngle]

eval1(l:LEDF,e:LEEDF):LEDF ==
  [eval(u,e)$EDF for u in l]

eval(mat:MEDF,symbols:LS,values:VEDF):MEDF ==
  l := listOfLists(mat)
  ledf := entries(values)$VEDF
  e := [equation(u::EDF,v)$EEDF for u in symbols for v in ledf]
  l := [eval1(w,e) for w in l]
  matrix l

combineFeatureCompatibility(C1:F,C2:F):F ==

  --                          C1 C2
  --    s(C1,C2) = ----------------------
  --               C1 C2 + (1 - C1)(1 - C2)

  C1*C2/((C1*C2)+(1$F-C1)*(1$F-C2))

combineFeatureCompatibility(C1:F,L:LF):F ==

  empty?(L)$LF => C1
  C2 := combineFeatureCompatibility(C1,first(L)$LF)
  combineFeatureCompatibility(C2,rest(L)$LF)

jacobian(v:VEDF,w:LS):Matrix EDF ==
  jacobian(v,w)$MultiVariableCalculusFunctions(S,EDF,VEDF,LS)

sparsityIF(m:Matrix EDF):F ==
  l:LEDF :=parts m
  z:LEDF := [u for u in l | zero?(u)$EDF]
  ((#z)::F/(#l)::F)

sum(a:EDF,b:EDF):EDF == a+b

stiffnessAndStabilityOfODE(ode:ODEA):RSS ==
  odefns := copy ode.fn
  ls:LS := [subscript(Y,[coerce(n)])$Symbol for n in 1..# odefns]
```

```
    yvals := copy ode.yinit
    for i in 1..#yvals repeat
      zero?(yvals.i) => yvals.i := 0.1::DF
    yexpr := [coerce(v)@EDF for v in yvals]
    yv:VEDF := vector(yexpr)
    j1:MEDF := jacobian(odefns,ls)
    ej1:MEDF := eval(j1,ls,yv)
    ej1 := eval(ej1,variables(reduce(sum,members(ej1)$MEDF)),vector([(ode.xinit
    ssf := stiffnessAndStabilityFactor(ej1)
    stability := 1.0-sqrt((ssf.stabilityFactor)*(2.0)/(pi()$F))
    stiffness := (1.0)-exp(-(ssf.stiffnessFactor)/(500.0))
    [stiffness,stability]

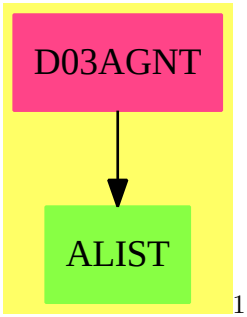
  stiffnessAndStabilityOfODEIF(ode:ODEA):RSS ==
    odefn := copy ode
    (t := showIntensityFunctions(odefn)$ODEIntensityFunctionsTable) case ATT =>
      s:ATT := coerce(t)@ATT
      negative?(s.stiffness)$F =>
        ssf:RSS := stiffnessAndStabilityOfODE(odefn)
        s := [ssf.stiffnessFactor,ssf.stabilityFactor,s.expense,
                s.accuracy,s.intermediateResults]
        r:ROA := [odefn,s]
        insert!(r)$ODEIntensityFunctionsTable
        ssf
      [s.stiffness,s.stability]
    ssf:RSS := stiffnessAndStabilityOfODE(odefn)
    s:ATT := [ssf.stiffnessFactor,ssf.stabilityFactor,-1.0,-1.0,-1.0]
    r:ROA := [odefn,s]
    insert!(r)$ODEIntensityFunctionsTable
    ssf
```

⟨*D02AGNT.dotabb*⟩≡

```
"D02AGNT" [color="#FF4488",href="bookvol10.4.pdf#nameddest=D02AGNT"]
"FS" [color="#4488FF",href="bookvol10.2.pdf#nameddest=FS"]
"D02AGNT" -> "FS"
```

5.33 package D03AGNT d03AgentsPackage

5.34 d03AgentsPackage



1.00

Exports:
central? elliptic? subscriptedVariables varList

⟨*package D03AGNT d03AgentsPackage*⟩≡
```
)abbrev package D03AGNT d03AgentsPackage
++ Author: Brian Dupee
++ Date Created: May 1994
++ Date Last Updated: December 1997
++ Basic Operations:
++ Description:
++ \axiom{d03AgentsPackage} contains a set of computational agents
++ for use with Partial Differential Equation solvers.
LEDF    ==> List Expression DoubleFloat
EDF     ==> Expression DoubleFloat
MDF     ==> Matrix DoubleFloat
DF      ==> DoubleFloat
F       ==> Float
INT     ==> Integer
NNI     ==> NonNegativeInteger
EEDF    ==> Equation Expression DoubleFloat
LEEDF   ==> List Equation Expression DoubleFloat
LDF     ==> List DoubleFloat
LOCDF   ==> List OrderedCompletion DoubleFloat
OCDF    ==> OrderedCompletion DoubleFloat
LS      ==> List Symbol
PDEC    ==> Record(start:DF, finish:DF, grid:NNI, boundaryType:INT,
                   dStart:MDF, dFinish:MDF)
PDEB    ==> Record(pde:LEDF, constraints:List PDEC,
                   f:List LEDF, st:String, tol:DF)
NOA     ==> Record(fn:EDF, init:LDF, lb:LOCDF, cf:LEDF, ub:LOCDF)
```

```
d03AgentsPackage(): E == I where
  E ==>  with
    varList:(Symbol,NonNegativeInteger) -> LS
      ++ varList(s,n) \undocumented{}
    subscriptedVariables:EDF -> EDF
      ++ subscriptedVariables(e) \undocumented{}
    central?:(DF,DF,LEDF) -> Boolean
      ++ central?(f,g,l) \undocumented{}
    elliptic?:PDEB -> Boolean
      ++ elliptic?(r) \undocumented{}

  I ==> add

    import ExpertSystemToolsPackage

    sum(a:EDF,b:EDF):EDF == a+b

    varList(s:Symbol,n:NonNegativeInteger):LS ==
      [subscript(s,[t::OutputForm]) for t in expand([1..n])$Segment(Integer)]

    subscriptedVariables(e:EDF):EDF ==
      oldVars:List Symbol := variables(e)
      o := [a :: EDF for a in oldVars]
      newVars := varList(X::Symbol,# oldVars)
      n := [b :: EDF for b in newVars]
      subst(e,[a=b for a in o for b in n])

    central?(x:DF,y:DF,p:LEDF):Boolean ==
      ls := variables(reduce(sum,p))
      le := [equation(u::EDF,v)$EEDF for u in ls for v in [x::EDF,y::EDF]]
      l := [eval(u,le)$EDF for u in p]
      max(l.4,l.5) < 20 * max(l.1,max(l.2,l.3))

    elliptic?(args:PDEB):Boolean ==
      (args.st)="elliptic" => true
      p := args.pde
      xcon:PDEC := first(args.constraints)
      ycon:PDEC := second(args.constraints)
      xs := xcon.start
      ys := ycon.start
      xf := xcon.finish
      yf := ycon.finish
      xstart:DF := ((xf-xs)/2)$DF
      ystart:DF := ((yf-ys)/2)$DF
      optStart:LDF := [xstart,ystart]
      lower:LOCDF := [xs::OCDF,ys::OCDF]
```

```
        upper:LOCDF := [xf::OCDF,yf::OCDF]
        v := variables(e := 4*first(p)*third(p)-(second(p))**2)
        eq := subscriptedVariables(e)
        noa:NOA :=
  --        one?(# v) =>
          (# v) = 1 =>
            ((first v) = X@Symbol) =>
              [eq,[xstart],[xs::OCDF],empty()$LEDF,[xf::OCDF]]
            [eq,[ystart],[ys::OCDF],empty()$LEDF,[yf::OCDF]]
          [eq,optStart,lower,empty()$LEDF,upper]
        ell := optimize(noa::NumericalOptimizationProblem)$AnnaNumericalOptimizationPackage
        o:Union(Any,"failed") := search(objf::Symbol,ell)$Result
        o case "failed" => false
        ob := o :: Any
        obj:DF := retract(ob)$AnyFunctions1(DF)
        positive?(obj)
```

⟨*D03AGNT.dotabb*⟩≡

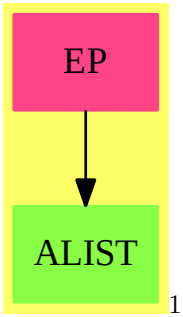
```
"D03AGNT" [color="#FF4488",href="bookvol10.4.pdf#nameddest=D03AGNT"]
"ALIST" [color="#88FF44",href="bookvol10.3.pdf#nameddest=ALIST"]
"D03AGNT" -> "ALIST"
```

Chapter 6

Chapter E

6.1 package EP EigenPackage

6.2 EigenPackage



1.00

Exports:
characteristicPolynomial eigenvalues eigenvector eigenvectors generalizedEigenvector generalizedEige

⟨*package EP EigenPackage*⟩≡

```
)abbrev package EP EigenPackage
++ Author: P. Gianni
++ Date Created: summer 1986
++ Date Last Updated: October 1992
++ Basic Functions:
++ Related Constructors: NumericRealEigenPackage,  NumericComplexEigenPackage,
++ RadicalEigenPackage
++ Also See:
++ AMS Classifications:
++ Keywords:
++ References:
```

```
++ Description:
++ This is a package for the exact computation of eigenvalues and eigenvectors.
++ This package can be made to work for matrices with coefficients which are
++ rational functions over a ring where we can factor polynomials.
++ Rational eigenvalues are always explicitly computed while the
++ non-rational ones are expressed in terms of their minimal
++ polynomial.
-- Functions for the numeric computation of eigenvalues and eigenvectors
-- are in numeigen spad.
EigenPackage(R) : C == T
 where
   R     : GcdDomain
   P     ==> Polynomial R
   F     ==> Fraction P
   SE    ==> Symbol()
   SUP   ==> SparseUnivariatePolynomial(P)
   SUF   ==> SparseUnivariatePolynomial(F)
   M     ==> Matrix(F)
   NNI   ==> NonNegativeInteger
   ST    ==> SuchThat(SE,P)

   Eigenvalue  ==> Union(F,ST)
   EigenForm   ==> Record(eigval:Eigenvalue,eigmult:NNI,eigvec : List M)
   GenEigen    ==> Record(eigval:Eigenvalue,geneigvec:List M)

   C == with
     characteristicPolynomial :  (M,Symbol)  ->  P
       ++ characteristicPolynomial(m,var) returns the
       ++ characteristicPolynomial of the matrix m using
       ++ the symbol var as the main variable.

     characteristicPolynomial :      M       ->  P
       ++ characteristicPolynomial(m) returns the
       ++ characteristicPolynomial of the matrix m using
       ++ a new generated symbol symbol as the main variable.

     eigenvalues       :    M        ->  List Eigenvalue
       ++ eigenvalues(m) returns the
       ++ eigenvalues of the matrix m which are expressible
       ++ as rational functions over the rational numbers.

     eigenvector       :   (Eigenvalue,M)  ->  List M
       ++ eigenvector(eigval,m) returns the
       ++ eigenvectors belonging to the eigenvalue eigval
       ++ for the matrix m.
```

```
    generalizedEigenvector  : (Eigenvalue,M,NNI,NNI) -> List M
      ++ generalizedEigenvector(alpha,m,k,g)
      ++ returns the generalized eigenvectors
      ++ of the matrix relative to the eigenvalue alpha.
      ++ The integers k and g are respectively the algebraic and the
      ++ geometric multiplicity of tye eigenvalue alpha.
      ++ alpha can be either rational or not.
      ++ In the seconda case apha is the minimal polynomial of the
      ++ eigenvalue.

    generalizedEigenvector  : (EigenForm,M) -> List M
      ++ generalizedEigenvector(eigen,m)
      ++ returns the generalized eigenvectors
      ++ of the matrix relative to the eigenvalue eigen, as
      ++ returned by the function eigenvectors.

    generalizedEigenvectors  : M -> List GenEigen
      ++ generalizedEigenvectors(m)
      ++ returns the generalized eigenvectors
      ++ of the matrix m.

    eigenvectors      :    M        ->  List(EigenForm)
      ++ eigenvectors(m) returns the eigenvalues and eigenvectors
      ++ for the matrix m.
      ++ The rational eigenvalues and the correspondent eigenvectors
      ++ are explicitely computed, while the non rational ones
      ++ are given via their minimal polynomial and the corresponding
      ++ eigenvectors are expressed in terms of a "generic" root of
      ++ such a polynomial.

  T == add
    PI       ==> PositiveInteger


    MF  := GeneralizedMultivariateFactorize(SE,IndexedExponents SE,R,R,P)
    UPCF2:= UnivariatePolynomialCategoryFunctions2(P,SUP,F,SUF)


                ----  Local  Functions  ----
    tff               :  (SUF,SE)      ->  F
    fft               :  (SUF,SE)      ->  F
    charpol           :   (M,SE)       ->   F
    intRatEig         :  (F,M,NNI)    ->   List M
    intAlgEig         :  (ST,M,NNI)    ->   List M
    genEigForm        : (EigenForm,M)  ->   GenEigen
```

```
---- next functions needed for defining  ModularField ----
reduction(u:SUF,p:SUF):SUF == u rem p

merge(p:SUF,q:SUF):Union(SUF,"failed") ==
    p = q => p
    p = 0 => q
    q = 0 => p
    "failed"

exactquo(u:SUF,v:SUF,p:SUF):Union(SUF,"failed") ==
   val:=extendedEuclidean(v,p,u)
   val case "failed" => "failed"
   val.coef1

          ----  functions for conversions  ----
fft(t:SUF,x:SE):F ==
  n:=degree(t)
  cf:=monomial(1,x,n)$P :: F
  cf * leadingCoefficient t

tff(p:SUF,x:SE) : F ==
  degree p=0 => leadingCoefficient p
  r:F:=0$F
  while p^=0 repeat
    r:=r+fft(p,x)
    p := reductum p
  r

 ---- generalized eigenvectors associated to a given eigenvalue ---
genEigForm(eigen : EigenForm,A:M) : GenEigen ==
  alpha:=eigen.eigval
  k:=eigen.eigmult
  g:=#(eigen.eigvec)
  k = g  => [alpha,eigen.eigvec]
  [alpha,generalizedEigenvector(alpha,A,k,g)]

      ---- characteristic polynomial  ----
charpol(A:M,x:SE) : F ==
  dimA :PI := (nrows A):PI
  dimA ^= ncols A => error " The matrix is not square"
  B:M:=zero(dimA,dimA)
  for i in 1..dimA repeat
    for j in 1..dimA repeat  B(i,j):=A(i,j)
    B(i,i) := B(i,i) - monomial(1$P,x,1)::F
  determinant B
```

```
        --------  EXPORTED  FUNCTIONS  --------

          ----  characteristic polynomial of a matrix A ----
  characteristicPolynomial(A:M):P ==
    x:SE:=new()$SE
    numer charpol(A,x)

          ----  characteristic polynomial of a matrix A ----
  characteristicPolynomial(A:M,x:SE) : P == numer charpol(A,x)

             ----  Eigenvalues of the matrix A  ----
  eigenvalues(A:M): List Eigenvalue  ==
    x:=new()$SE
    pol:= charpol(A,x)
    lrat:List F :=empty()
    lsym:List ST :=empty()
    for eq in solve(pol,x)$SystemSolvePackage(R) repeat
      alg:=numer lhs eq
      degree(alg, x)=1 => lrat:=cons(rhs eq,lrat)
      lsym:=cons([x,alg],lsym)
    append([lr::Eigenvalue for lr in lrat],
           [ls::Eigenvalue for ls in lsym])

       ----  Eigenvectors belonging to a given eigenvalue  ----
             ----  the eigenvalue must be exact  ----
  eigenvector(alpha:Eigenvalue,A:M) : List M  ==
    alpha case F => intRatEig(alpha::F,A,1$NNI)
    intAlgEig(alpha::ST,A,1$NNI)

----  Eigenvectors belonging to a given rational eigenvalue  ----
             ---- Internal function -----
  intRatEig(alpha:F,A:M,m:NNI) : List M  ==
    n:=nrows A
    B:M := zero(n,n)$M
    for i in 1..n repeat
      for j in 1..n repeat B(i,j):=A(i,j)
      B(i,i):= B(i,i) - alpha
    [v::M for v in nullSpace(B**m)]

----  Eigenvectors belonging to a given algebraic eigenvalue  ----
      ------   Internal  Function  -----
  intAlgEig(alpha:ST,A:M,m:NNI) : List M  ==
    n:=nrows A
    MM := ModularField(SUF,SUF,reduction,merge,exactquo)
    AM:=Matrix MM
    x:SE:=lhs alpha
```

```
  pol:SUF:=unitCanonical map(coerce,univariate(rhs alpha,x))$UPCF2
  alg:MM:=reduce(monomial(1,1),pol)
  B:AM := zero(n,n)
  for i in 1..n repeat
    for j in 1..n repeat B(i,j):=reduce(A(i,j)::SUF,pol)
    B(i,i):= B(i,i) - alg
  sol: List M :=empty()
  for vec in nullSpace(B**m) repeat
    w:M:=zero(n,1)
    for i in 1..n repeat w(i,1):=tff((vec.i)::SUF,x)
    sol:=cons(w,sol)
  sol

----  Generalized Eigenvectors belonging to a given eigenvalue  ----
generalizedEigenvector(alpha:Eigenvalue,A:M,k:NNI,g:NNI) : List M  ==
  alpha case F => intRatEig(alpha::F,A,(1+k-g)::NNI)
  intAlgEig(alpha::ST,A,(1+k-g)::NNI)

----  Generalized Eigenvectors belonging to a given eigenvalue  ----
generalizedEigenvector(eigen :EigenForm,A:M) : List M  ==
  generalizedEigenvector(eigen.eigval,A,eigen.eigmult,# eigen.eigvec)

    ----  Generalized Eigenvectors -----
generalizedEigenvectors(A:M) : List GenEigen  ==
  n:= nrows A
  leig:=eigenvectors A
  [genEigForm(leg,A) for leg in leig]

           ----  eigenvectors and eigenvalues  ----
eigenvectors(A:M):List(EigenForm) ==
  n:=nrows A
  x:=new()$SE
  p:=numer charpol(A,x)
  MM := ModularField(SUF,SUF,reduction,merge,exactquo)
  AM:=Matrix(MM)
  ratSol : List EigenForm := empty()
  algSol : List EigenForm := empty()
  lff:=factors factor  p
  for fact in lff repeat
    pol:=fact.factor
    degree(pol,x)=1 =>
      vec:F :=-coefficient(pol,x,0)/coefficient(pol,x,degree(pol,x))
      ratSol:=cons([vec,fact.exponent :: NNI,
                    intRatEig(vec,A,1$NNI)]$EigenForm,ratSol)
    alpha:ST:=[x,pol]
    algSol:=cons([alpha,fact.exponent :: NNI,
```

```
                        intAlgEig(alpha,A,1$NNI)]$EigenForm,algSol)
        append(ratSol,algSol)
```

⟨*EP.dotabb*⟩≡

```
  "EP" [color="#FF4488",href="bookvol10.4.pdf#nameddest=EP"]
  "ALIST" [color="#88FF44",href="bookvol10.3.pdf#nameddest=ALIST"]
  "EP" -> "ALIST"
```

6.3 package EF ElementaryFunction

⟨*ElementaryFunction.input*⟩≡

```
)set break resume
)sys rm -f ElementaryFunction.output
)spool ElementaryFunction.output
)set message test on
)set message auto off
)clear all

--S 1 of 32
)trace EF
--R
--R
--R   Parameterized constructors traced:
--R      EF
--E 1

--S 2 of 32
D(cos(3*x+6*y),x)
--I
--I1<enter ElementaryFunction.cos,64 : ((1 #<vector 0941ef18> (1 0 . 6) (0 1 #<ve
--I 1<enter ElementaryFunction.iicos,154 : ((1 #<vector 0941ef18> (1 0 . 6) (0 1 
--I  1<enter ElementaryFunction.iisqrt2,58 :
--I  1>exit  ElementaryFunction.iisqrt2,58 : ((1 #<vector 0917aab8> (1 0 . 1)) 0 
--I  1<enter ElementaryFunction.iisqrt3,59 :
--I  1>exit  ElementaryFunction.iisqrt3,59 : ((1 #<vector 0917a1dc> (1 0 . 1)) 0 
--I  1<enter ElementaryFunction.specialTrigs,116 : ((1 #<vector 0941ef18> (1 0 . 
--I   1<enter ElementaryFunction.pi,46 :
--I   1>exit  ElementaryFunction.pi,46 : ((1 #<vector 090c3a64> (1 0 . 1)) 0 . 1)
--I  1>exit  ElementaryFunction.specialTrigs,116 : (1 . "failed")
--I 1>exit  ElementaryFunction.iicos,154 : ((1 #<vector 0941ed74> (1 0 . 1)) 0 .
--I1>exit  ElementaryFunction.cos,64 : ((1 #<vector 0941ed74> (1 0 . 1)) 0 . 1)
--I1<enter ElementaryFunction.sin,63 : ((1 #<vector 0941ef18> (1 0 . 6) (0 1 #<ve
--I 1<enter ElementaryFunction.iisin,152 : ((1 #<vector 0941ef18> (1 0 . 6) (0 1 
--I  1<enter ElementaryFunction.iisqrt2,58 :
--I  1>exit  ElementaryFunction.iisqrt2,58 : ((1 #<vector 0917aab8> (1 0 . 1)) 0 
--I  1<enter ElementaryFunction.iisqrt3,59 :
--I  1>exit  ElementaryFunction.iisqrt3,59 : ((1 #<vector 0917a1dc> (1 0 . 1)) 0 
--I  1<enter ElementaryFunction.specialTrigs,116 : ((1 #<vector 0941ef18> (1 0 . 
--I   1<enter ElementaryFunction.pi,46 :
--I   1>exit  ElementaryFunction.pi,46 : ((1 #<vector 090c3a64> (1 0 . 1)) 0 . 1)
--I  1>exit  ElementaryFunction.specialTrigs,116 : (1 . "failed")
--I 1>exit  ElementaryFunction.iisin,152 : ((1 #<vector 0941eb60> (1 0 . 1)) 0 .
--I1>exit  ElementaryFunction.sin,63 : ((1 #<vector 0941eb60> (1 0 . 1)) 0 . 1)
--R
```

```
--R   (1)  - 3sin(6y + 3x)
--R                                                    Type: Expression Integer
--E 2

--S 3 of 32
)trace )off
--R
--R
--R   Nothing is traced now.
--R
--E 3


--S 4 of 32
D(sin(3*x+6*y),x)
--R
--R
--R   (2)  3cos(6y + 3x)
--R                                                    Type: Expression Integer
--E 4

--S 5 of 32
D(cos(3*x+6*y),x)
--R
--R
--R   (3)  - 3sin(6y + 3x)
--R                                                    Type: Expression Integer
--E 5
```

Note that both Mathematica and Maxima return this result as $3sec(3*x+6*y)^2$ but Maple returns the same result as Axiom. They are equivalent results.

⟨*ElementaryFunction.input*⟩+≡

```
--S 6 of 32
D(tan(3*x+6*y),x)
--R
--R
--R                      2
--R   (4)  3tan(6y + 3x)  + 3
--R                                                    Type: Expression Integer
--E 6

--S 7 of 32
simplify ((3*tan(6*y+3*x)^2+3) - (3*sec(3*x+6*y)^2))
--R
--R
--R   (5)  0
--R                                                    Type: Expression Integer
--E 7
```

Note that Mathematica and Maxima return $-3csc(3x+6y)^2$ and Maple returns the same form as Axiom. They are equivalent.

⟨*ElementaryFunction.input*⟩+≡

```
--S 8 of 32
D(cot(3*x+6*y),x)
--R
--R
--R                          2
--R   (6)  - 3cot(6y + 3x)  - 3
--R                                                    Type: Expression Integer
--E 8

--S 9 of 32
simplify ((-3*cot(6*y+3*x)^2-3) -(-3*csc(3*x+6*y)^2))
--R
--R
--R   (7)  0
--R                                                    Type: Expression Integer
--E 9

--S 10 of 32
D(sec(3*x+6*y),x)
--R
--R
--R   (8)  3sec(6y + 3x)tan(6y + 3x)
--R                                                    Type: Expression Integer
--E 10

--S 11 of 32
D(csc(3*x+6*y),x)
--R
--R
--R   (9)  - 3cot(6y + 3x)csc(6y + 3x)
--R                                                    Type: Expression Integer
--E 11

--S 12 of 32
D(asin(3*x+6*y),x)
--R
--R
--R                       3
--R   (10)  --------------------------
--R          +-----------------------+
--R          |     2             2
```

```
--R         \|- 36y  - 36x y - 9x  + 1
--R                                                    Type: Expression Integer
--E 12

--S 13 of 32
D(acos(3*x+6*y),x)
--R
--R
--R                         3
--R   (11)  - ---------------------------
--R             +-------------------------+
--R             |     2             2
--R           \|- 36y  - 36x y - 9x  + 1
--R                                                    Type: Expression Integer
--E 13

--S 14 of 32
D(atan(3*x+6*y),x)
--R
--R
--R                      3
--R   (12)  ----------------------
--R             2             2
--R          36y  + 36x y + 9x  + 1
--R                                                    Type: Expression Integer
--E 14

--S 15 of 32
D(acot(3*x+6*y),x)
--R
--R
--R                        3
--R   (13)  - ----------------------
--R              2             2
--R           36y  + 36x y + 9x  + 1
--R                                                    Type: Expression Integer
--E 15
```

Mathematica, Maple, and Maxima give:

$$\frac{3}{(3*x+6*y)^2\sqrt{1-\frac{1}{(3*x+6*y)^2}}}$$

which proceeds directly from the formula for the derivative of asec:

$$\frac{d}{dx}arcsec(x) = \frac{1}{x\sqrt{x^2-1}}$$

⟨*ElementaryFunction.input*⟩+≡

```
--S 16 of 32
D(asec(3*x+6*y),x)
--R
--R
--R                            1
--R   (14)  ---------------------------------
--R                  +----------------------+
--R                  |   2              2
--R         (2y + x)\|36y  + 36x y + 9x  - 1
--R                                                    Type: Expression Integer
--E 16
```

If we use the same formula for this example:

$$\frac{1}{(3*x+6*y)\sqrt{(3*x+6*y)^2-1}}d(3*x+6*y)/dx$$

⟨*ElementaryFunction.input*⟩+≡

```
--S 17 of 32
3/((3*x+6*y)*sqrt((3*x+6*y)^2-1))
--R
--R
--R                            1
--R   (15)  ---------------------------------
--R                  +----------------------+
--R                  |   2              2
--R         (2y + x)\|36y  + 36x y + 9x  - 1
--R                                                    Type: Expression Integer
--E 17
```

Mathematica, Maple, and Maxima give

$$-\frac{3}{(3*x+6*y)^2\sqrt{1-\frac{1}{(3*x+6*y)^2}}}$$

which is just the negative of the above result and we can see that the same analysis applies to explain the results.

⟨*ElementaryFunction.input*⟩+≡

```
--S 18 of 32
D(acsc(3*x+6*y),x)
--R
--R
--R                              1
--R   (16)  - ----------------------------------
--R                     +----------------------+
--R                     |   2             2
--R           (2y + x)\|36y  + 36x y + 9x  - 1
--R                                                    Type: Expression Integer
--E 18

--S 19 of 32
D(sinh(3*x+6*y),x)
--R
--R
--R   (17)  3cosh(6y + 3x)
--R                                                    Type: Expression Integer
--E 19

--S 20 of 32
D(cosh(3*x+6*y),x)
--R
--R
--R   (18)  3sinh(6y + 3x)
--R                                                    Type: Expression Integer
--E 20
```

Mathematica and Maxima return

$$3sech(3x+6y)^2$$

Maple returns Axiom's answer. Both are equivalent.
⟨*ElementaryFunction.input*⟩+≡

```
--S 21 of 32
D(tanh(3*x+6*y),x)
--R
--R
--R                          2
--R   (19)  - 3tanh(6y + 3x)  + 3
--R                                                    Type: Expression Integer
--E 21

--S 22 of 32
simplify ((-3*tanh(6*y+3*x)^2+3)-(3*sech(3*x+6*y)^2))
--R
--R
--R   (20)  0
--R                                                    Type: Expression Integer
--E 22

Mathematica and Maxima return
\[-3 csch(3*x+6*y)^2\]
Maple returns Axiom's answer. Both are equivalent.
```

⟨*ElementaryFunction.input*⟩+≡

```
--S 23 of 32
D(coth(3*x+6*y),x)
--R
--R
--R                          2
--R   (21)  - 3coth(6y + 3x)  + 3
--R                                                    Type: Expression Integer
--E 23

--S 24 of 32
simplify ((-3*coth(6*y+3*x)^2+3) - (-3*csch(3*x+6*y)^2))
--R
--R
--R   (22)  0
--R                                                    Type: Expression Integer
--E 24

--S 25 of 32
D(sech(3*x+6*y),x)
--R
--R
--R   (23)  - 3sech(6y + 3x)tanh(6y + 3x)
--R                                                    Type: Expression Integer
--E 25

--S 26 of 32
D(csch(3*x+6*y),x)
--R
--R
--R   (24)  - 3coth(6y + 3x)csch(6y + 3x)
--R                                                    Type: Expression Integer
--E 26

--S 27 of 32
D(asinh(3*x+6*y),x)
--R
--R
--R                     3
--R   (25)  -------------------------
--R          +----------------------+
--R          |   2             2
--R         \|36y  + 36x y + 9x  + 1
--R                                                    Type: Expression Integer
--E 27
```

Mathematica and Maple show

$$\frac{3}{\sqrt{-1+3*x+6*y}\sqrt{1+3*x+6*y}}$$

Maxima gives Axiom's answer. Both are equivalent, just factored forms.
⟨*ElementaryFunction.input*⟩+≡

```
--S 28 of 32
D(acosh(3*x+6*y),x)
--R
--R
--R                      3
--R   (26)  -------------------------
--R          +---------------------+
--R          |   2             2
--R         \|36y  + 36x y + 9x  - 1
--R                                                  Type: Expression Integer
--E 28

--S 29 of 32
D(atanh(3*x+6*y),x)
--R
--R
--R                       3
--R   (27)  - ----------------------
--R               2             2
--R            36y  + 36x y + 9x  - 1
--R                                                  Type: Expression Integer
--E 29

--S 30 of 32
D(acoth(3*x+6*y),x)
--R
--R
--R                       3
--R   (28)  - ----------------------
--R               2             2
--R            36y  + 36x y + 9x  - 1
--R                                                  Type: Expression Integer
--E 30
```

Mathematica gives

$$-\frac{3}{(3x+6y)\sqrt{\frac{1-3x-6y}{1_3x+6y}}(1+3x+6y)}$$

Maxima gives

$$-\frac{3}{(6y+3x)^2\sqrt{\frac{1}{(6y+3x)^2}-1}}$$

Maple gives

$$-\frac{3}{(3x+6y)^2\sqrt{\frac{1}{3x+6y}-1}\sqrt{\frac{1}{3x+6y}+1}}$$

Axiom cannot simplify these differences to zero but Maxima does which shows they are all equivalent answers.

⟨*ElementaryFunction.input*⟩+≡

```
--S 31 of 32
D(asech(3*x+6*y),x)
--R
--R
--R                              1
--R   (29)  - ------------------------------------
--R                     +-------------------------+
--R                     |     2              2
--R           (2y + x)\|- 36y  - 36x y - 9x  + 1
--R                                                  Type: Expression Integer
--E 31
```

Mathematica, Maple, and Maxima all generate the answer

$$-\frac{3}{(3x+6y)^2\sqrt{1+\frac{1}{(3x+6y)^2}}}$$

Axiom cannot simplify these differences to zero but Maxima does which shows they are all equivalent answers.

⟨*ElementaryFunction.input*⟩+≡

```
--S 32 of 32
D(acsch(3*x+6*y),x)
--R
--R
--R                              1
--R   (30)  - ---------------------------------
--R                      +----------------------+
--R                      |   2              2
--R            (2y + x)\|36y  + 36x y + 9x  + 1
--R                                                    Type: Expression Integer
--E 32

)spool
)lisp (bye)
```

⟨*ElementaryFunction.help*⟩≡

```
==========================================================================
ElementaryFunction examples
==========================================================================

D(sin(3*x+6*y),x)
   3cos(6y + 3x)

D(cos(3*x+6*y),x)
   - 3sin(6y + 3x)

D(tan(3*x+6*y),x)
                2
   3tan(6y + 3x)  + 3

D(cot(3*x+6*y),x)
                  2
   - 3cot(6y + 3x)  - 3

D(sec(3*x+6*y),x)
   3sec(6y + 3x)tan(6y + 3x)

D(csc(3*x+6*y),x)
   - 3cot(6y + 3x)csc(6y + 3x)

D(asin(3*x+6*y),x)
                3
   --------------------------
    +-----------------------+
    |     2             2
   \|- 36y  - 36x y - 9x  + 1

D(acos(3*x+6*y),x)
                  3
   - --------------------------
      +-----------------------+
      |     2             2
     \|- 36y  - 36x y - 9x  + 1

D(atan(3*x+6*y),x)
               3
    ---------------------
       2             2
    36y  + 36x y + 9x  + 1

D(acot(3*x+6*y),x)
```

```
                3
    - ----------------------
         2             2
      36y  + 36x y + 9x  + 1

D(asec(3*x+6*y),x)
                    1
    --------------------------------
             +----------------------+
             |   2             2
    (2y + x)\|36y  + 36x y + 9x  - 1

D(acsc(3*x+6*y),x)
                     1
    - --------------------------------
               +----------------------+
               |   2             2
      (2y + x)\|36y  + 36x y + 9x  - 1


D(sinh(3*x+6*y),x)
    3cosh(6y + 3x)

D(cosh(3*x+6*y),x)
    3sinh(6y + 3x)

D(tanh(3*x+6*y),x)
                    2
    - 3tanh(6y + 3x)  + 3

D(coth(3*x+6*y),x)
                    2
    - 3coth(6y + 3x)  + 3

D(sech(3*x+6*y),x)
    - 3sech(6y + 3x)tanh(6y + 3x)

D(csch(3*x+6*y),x)
    - 3coth(6y + 3x)csch(6y + 3x)

D(asinh(3*x+6*y),x)
                3
    ------------------------
     +----------------------+
     |   2             2
    \|36y  + 36x y + 9x  + 1
```

```
D(acosh(3*x+6*y),x)
               3
    -------------------------
     +---------------------+
     |   2             2
    \|36y  + 36x y + 9x  - 1

D(atanh(3*x+6*y),x)
               3
    - ----------------------
         2             2
      36y  + 36x y + 9x  - 1

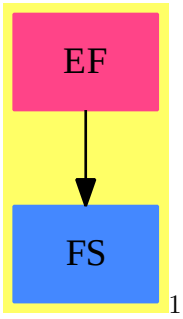
D(acoth(3*x+6*y),x)
               3
    - ----------------------
         2             2
      36y  + 36x y + 9x  - 1

D(asech(3*x+6*y),x)
                     1
    - ----------------------------------
              +-----------------------+
              |     2             2
      (2y + x)\|- 36y  - 36x y - 9x  + 1

D(acsch(3*x+6*y),x)
                    1
    - ---------------------------------
              +----------------------+
              |   2             2
      (2y + x)\|36y  + 36x y + 9x  + 1

See Also:
o )show ElementaryFunction
```

6.4 ElementaryFunction



1.00

Exports:

acos	acosh	acot	acoth	acsc
acsch	asec	asech	asin	asinh
atan	atanh	belong?	cos	cosh
cot	coth	csc	csch	exp
iiacos	iiacosh	iiacot	iiacoth	iiacsc
iiacsch	iiasec	iiasech	iiasin	iiasinh
iiatan	iiatanh	iicos	iicosh	iicot
iicoth	iicsc	iicsch	iiexp	iilog
iisec	iisech	iisin	iisinh	iisqrt2
iisqrt3	iitan	iitanh	localReal?	log
operator	pi	sec	sech	sin
sinh	specialTrigs	tan	tanh	

⟨*package EF ElementaryFunction*⟩≡

```
)abbrev package EF ElementaryFunction
++ Author: Manuel Bronstein
++ Date Created: 1987
++ Date Last Updated: 10 April 1995
++ Keywords: elementary, function, logarithm, exponential.
++ Examples:  )r EF INPUT
++ Description: Provides elementary functions over an integral domain.
ElementaryFunction(R, F): Exports == Implementation where
  R: Join(OrderedSet, IntegralDomain)
  F: Join(FunctionSpace R, RadicalCategory)

  B   ==> Boolean
  L   ==> List
  Z   ==> Integer
  OP  ==> BasicOperator
  K   ==> Kernel F
  INV ==> error "Invalid argument"

  Exports ==> with
```

```
exp     : F -> F
    ++ exp(x) applies the exponential operator to x
log     : F -> F
    ++ log(x) applies the logarithm operator to x
sin     : F -> F
    ++ sin(x) applies the sine operator to x
cos     : F -> F
    ++ cos(x) applies the cosine operator to x
tan     : F -> F
    ++ tan(x) applies the tangent operator to x
cot     : F -> F
    ++ cot(x) applies the cotangent operator to x
sec     : F -> F
    ++ sec(x) applies the secant operator to x
csc     : F -> F
    ++ csc(x) applies the cosecant operator to x
asin    : F -> F
    ++ asin(x) applies the inverse sine operator to x
acos    : F -> F
    ++ acos(x) applies the inverse cosine operator to x
atan    : F -> F
    ++ atan(x) applies the inverse tangent operator to x
acot    : F -> F
    ++ acot(x) applies the inverse cotangent operator to x
asec    : F -> F
    ++ asec(x) applies the inverse secant operator to x
acsc    : F -> F
    ++ acsc(x) applies the inverse cosecant operator to x
sinh    : F -> F
    ++ sinh(x) applies the hyperbolic sine operator to x
cosh    : F -> F
    ++ cosh(x) applies the hyperbolic cosine operator to x
tanh    : F -> F
    ++ tanh(x) applies the hyperbolic tangent operator to x
coth    : F -> F
    ++ coth(x) applies the hyperbolic cotangent operator to x
sech    : F -> F
    ++ sech(x) applies the hyperbolic secant operator to x
csch    : F -> F
    ++ csch(x) applies the hyperbolic cosecant operator to x
asinh   : F -> F
    ++ asinh(x) applies the inverse hyperbolic sine operator to x
acosh   : F -> F
    ++ acosh(x) applies the inverse hyperbolic cosine operator to x
atanh   : F -> F
    ++ atanh(x) applies the inverse hyperbolic tangent operator to x
```

```
acoth   : F -> F
    ++ acoth(x) applies the inverse hyperbolic cotangent operator to x
asech   : F -> F
    ++ asech(x) applies the inverse hyperbolic secant operator to x
acsch   : F -> F
    ++ acsch(x) applies the inverse hyperbolic cosecant operator to x
pi      : () -> F
    ++ pi() returns the pi operator
belong? : OP -> Boolean
    ++ belong?(p) returns true if operator p is elementary
operator: OP -> OP
    ++ operator(p) returns an elementary operator with the same symbol as p
-- the following should be local, but are conditional
iisqrt2   : () -> F
    ++ iisqrt2() should be local but conditional
iisqrt3   : () -> F
    ++ iisqrt3() should be local but conditional
iiexp     : F -> F
    ++ iiexp(x) should be local but conditional
iilog     : F -> F
    ++ iilog(x) should be local but conditional
iisin     : F -> F
    ++ iisin(x) should be local but conditional
iicos     : F -> F
    ++ iicos(x) should be local but conditional
iitan     : F -> F
    ++ iitan(x) should be local but conditional
iicot     : F -> F
    ++ iicot(x) should be local but conditional
iisec     : F -> F
    ++ iisec(x) should be local but conditional
iicsc     : F -> F
    ++ iicsc(x) should be local but conditional
iiasin    : F -> F
    ++ iiasin(x) should be local but conditional
iiacos    : F -> F
    ++ iiacos(x) should be local but conditional
iiatan    : F -> F
    ++ iiatan(x) should be local but conditional
iiacot    : F -> F
    ++ iiacot(x) should be local but conditional
iiasec    : F -> F
    ++ iiasec(x) should be local but conditional
iiacsc    : F -> F
    ++ iiacsc(x) should be local but conditional
iisinh    : F -> F
```

```
        ++ iisinh(x) should be local but conditional
    iicosh    : F -> F
        ++ iicosh(x) should be local but conditional
    iitanh    : F -> F
        ++ iitanh(x) should be local but conditional
    iicoth    : F -> F
        ++ iicoth(x) should be local but conditional
    iisech    : F -> F
        ++ iisech(x) should be local but conditional
    iicsch    : F -> F
        ++ iicsch(x) should be local but conditional
    iiasinh   : F -> F
        ++ iiasinh(x) should be local but conditional
    iiacosh   : F -> F
        ++ iiacosh(x) should be local but conditional
    iiatanh   : F -> F
        ++ iiatanh(x) should be local but conditional
    iiacoth   : F -> F
        ++ iiacoth(x) should be local but conditional
    iiasech   : F -> F
        ++ iiasech(x) should be local but conditional
    iiacsch   : F -> F
        ++ iiacsch(x) should be local but conditional
    specialTrigs:(F, L Record(func:F,pole:B)) -> Union(F, "failed")
        ++ specialTrigs(x,l) should be local but conditional
    localReal?: F -> Boolean
        ++ localReal?(x) should be local but conditional

  Implementation ==> add
    ipi      : List F -> F
    iexp     : F -> F
    ilog     : F -> F
    iiilog   : F -> F
    isin     : F -> F
    icos     : F -> F
    itan     : F -> F
    icot     : F -> F
    isec     : F -> F
    icsc     : F -> F
    iasin    : F -> F
    iacos    : F -> F
    iatan    : F -> F
    iacot    : F -> F
    iasec    : F -> F
    iacsc    : F -> F
    isinh    : F -> F
```

```
icosh    : F -> F
itanh    : F -> F
icoth    : F -> F
isech    : F -> F
icsch    : F -> F
iasinh   : F -> F
iacosh   : F -> F
iatanh   : F -> F
iacoth   : F -> F
iasech   : F -> F
iacsch   : F -> F
dropfun  : F -> F
kernel   : F -> K
posrem   :(Z, Z) -> Z
iisqrt1  : () -> F
valueOrPole : Record(func:F, pole:B) -> F

oppi  := operator("pi"::Symbol)$CommonOperators
oplog := operator("log"::Symbol)$CommonOperators
opexp := operator("exp"::Symbol)$CommonOperators
opsin := operator("sin"::Symbol)$CommonOperators
opcos := operator("cos"::Symbol)$CommonOperators
optan := operator("tan"::Symbol)$CommonOperators
opcot := operator("cot"::Symbol)$CommonOperators
opsec := operator("sec"::Symbol)$CommonOperators
opcsc := operator("csc"::Symbol)$CommonOperators
opasin := operator("asin"::Symbol)$CommonOperators
opacos := operator("acos"::Symbol)$CommonOperators
opatan := operator("atan"::Symbol)$CommonOperators
opacot := operator("acot"::Symbol)$CommonOperators
opasec := operator("asec"::Symbol)$CommonOperators
opacsc := operator("acsc"::Symbol)$CommonOperators
opsinh := operator("sinh"::Symbol)$CommonOperators
opcosh := operator("cosh"::Symbol)$CommonOperators
optanh := operator("tanh"::Symbol)$CommonOperators
opcoth := operator("coth"::Symbol)$CommonOperators
opsech := operator("sech"::Symbol)$CommonOperators
opcsch := operator("csch"::Symbol)$CommonOperators
opasinh := operator("asinh"::Symbol)$CommonOperators
opacosh := operator("acosh"::Symbol)$CommonOperators
opatanh := operator("atanh"::Symbol)$CommonOperators
opacoth := operator("acoth"::Symbol)$CommonOperators
opasech := operator("asech"::Symbol)$CommonOperators
opacsch := operator("acsch"::Symbol)$CommonOperators

-- Pi is a domain...
```

```
Pie, isqrt1, isqrt2, isqrt3: F

-- following code is conditionalized on arbitraryPrecesion to recompute in
-- case user changes the precision

if R has TranscendentalFunctionCategory then
  Pie := pi()$R :: F
else
  Pie := kernel(oppi, nil()$List(F))

if R has TranscendentalFunctionCategory and R has arbitraryPrecision then
  pi() == pi()$R :: F
else
  pi() == Pie

if R has imaginary: () -> R then
  isqrt1 := imaginary()$R :: F
else isqrt1 := sqrt(-1::F)

if R has RadicalCategory then
  isqrt2 := sqrt(2::R)::F
  isqrt3 := sqrt(3::R)::F
else
  isqrt2 := sqrt(2::F)
  isqrt3 := sqrt(3::F)

iisqrt1() == isqrt1
if R has RadicalCategory and R has arbitraryPrecision then
  iisqrt2() == sqrt(2::R)::F
  iisqrt3() == sqrt(3::R)::F
else
  iisqrt2() == isqrt2
  iisqrt3() == isqrt3

ipi l == pi()
log x == oplog x
exp x == opexp x
sin x == opsin x
cos x == opcos x
tan x == optan x
cot x == opcot x
sec x == opsec x
csc x == opcsc x
asin x == opasin x
acos x == opacos x
atan x == opatan x
```

```
acot x == opacot x
asec x == opasec x
acsc x == opacsc x
sinh x == opsinh x
cosh x == opcosh x
tanh x == optanh x
coth x == opcoth x
sech x == opsech x
csch x == opcsch x
asinh x == opasinh x
acosh x == opacosh x
atanh x == opatanh x
acoth x == opacoth x
asech x == opasech x
acsch x == opacsch x
kernel x == retract(x)@K

posrem(n, m)    == ((r := n rem m) < 0 => r + m; r)
valueOrPole rec == (rec.pole => INV; rec.func)
belong? op      == has?(op, "elem")

operator op ==
  is?(op, "pi"::Symbol)    => oppi
  is?(op, "log"::Symbol)   => oplog
  is?(op, "exp"::Symbol)   => opexp
  is?(op, "sin"::Symbol)   => opsin
  is?(op, "cos"::Symbol)   => opcos
  is?(op, "tan"::Symbol)   => optan
  is?(op, "cot"::Symbol)   => opcot
  is?(op, "sec"::Symbol)   => opsec
  is?(op, "csc"::Symbol)   => opcsc
  is?(op, "asin"::Symbol)  => opasin
  is?(op, "acos"::Symbol)  => opacos
  is?(op, "atan"::Symbol)  => opatan
  is?(op, "acot"::Symbol)  => opacot
  is?(op, "asec"::Symbol)  => opasec
  is?(op, "acsc"::Symbol)  => opacsc
  is?(op, "sinh"::Symbol)  => opsinh
  is?(op, "cosh"::Symbol)  => opcosh
  is?(op, "tanh"::Symbol)  => optanh
  is?(op, "coth"::Symbol)  => opcoth
  is?(op, "sech"::Symbol)  => opsech
  is?(op, "csch"::Symbol)  => opcsch
  is?(op, "asinh"::Symbol) => opasinh
  is?(op, "acosh"::Symbol) => opacosh
  is?(op, "atanh"::Symbol) => opatanh
```

```
      is?(op, "acoth"::Symbol) => opacoth
      is?(op, "asech"::Symbol) => opasech
      is?(op, "acsch"::Symbol) => opacsch
      error "Not an elementary operator"

    dropfun x ==
      ((k := retractIfCan(x)@Union(K, "failed")) case "failed") or
        empty?(argument(k::K)) => 0
      first argument(k::K)

    if R has RetractableTo Z then
      specialTrigs(x, values) ==
        (r := retractIfCan(y := x/pi())@Union(Fraction Z, "failed"))
          case "failed" => "failed"
        q := r::Fraction(Integer)
        m := minIndex values
        (n := retractIfCan(q)@Union(Z, "failed")) case Z =>
          even?(n::Z) => valueOrPole(values.m)
          valueOrPole(values.(m+1))
        (n := retractIfCan(2*q)@Union(Z, "failed")) case Z =>
--          one?(s := posrem(n::Z, 4)) => valueOrPole(values.(m+2))
          (s := posrem(n::Z, 4)) = 1 => valueOrPole(values.(m+2))
          valueOrPole(values.(m+3))
        (n := retractIfCan(3*q)@Union(Z, "failed")) case Z =>
--          one?(s := posrem(n::Z, 6)) => valueOrPole(values.(m+4))
          (s := posrem(n::Z, 6)) = 1 => valueOrPole(values.(m+4))
          s = 2 => valueOrPole(values.(m+5))
          s = 4 => valueOrPole(values.(m+6))
          valueOrPole(values.(m+7))
        (n := retractIfCan(4*q)@Union(Z, "failed")) case Z =>
--          one?(s := posrem(n::Z, 8)) => valueOrPole(values.(m+8))
          (s := posrem(n::Z, 8)) = 1 => valueOrPole(values.(m+8))
          s = 3 => valueOrPole(values.(m+9))
          s = 5 => valueOrPole(values.(m+10))
          valueOrPole(values.(m+11))
        (n := retractIfCan(6*q)@Union(Z, "failed")) case Z =>
--          one?(s := posrem(n::Z, 12)) => valueOrPole(values.(m+12))
          (s := posrem(n::Z, 12)) = 1 => valueOrPole(values.(m+12))
          s = 5 => valueOrPole(values.(m+13))
          s = 7 => valueOrPole(values.(m+14))
          valueOrPole(values.(m+15))
        "failed"

    else specialTrigs(x, values) == "failed"

    isin x ==
```

```
    zero? x => 0
    y := dropfun x
    is?(x, opasin) => y
    is?(x, opacos) => sqrt(1 - y**2)
    is?(x, opatan) => y / sqrt(1 + y**2)
    is?(x, opacot) => inv sqrt(1 + y**2)
    is?(x, opasec) => sqrt(y**2 - 1) / y
    is?(x, opacsc) => inv y
    h  := inv(2::F)
    s2 := h * iisqrt2()
    s3 := h * iisqrt3()
    u  := specialTrigs(x, [[0,false], [0,false], [1,false], [-1,false],
                        [s3,false], [s3,false], [-s3,false], [-s3,false],
                         [s2,false], [s2,false], [-s2,false], [-s2,false],
                          [h,false], [h,false], [-h,false], [-h,false]])
    u case F => u :: F
    kernel(opsin, x)

  icos x ==
    zero? x => 1
    y := dropfun x
    is?(x, opasin) => sqrt(1 - y**2)
    is?(x, opacos) => y
    is?(x, opatan) => inv sqrt(1 + y**2)
    is?(x, opacot) => y / sqrt(1 + y**2)
    is?(x, opasec) => inv y
    is?(x, opacsc) => sqrt(y**2 - 1) / y
    h  := inv(2::F)
    s2 := h * iisqrt2()
    s3 := h * iisqrt3()
    u  := specialTrigs(x, [[1,false],[-1,false], [0,false], [0,false],
                           [h,false],[-h,false],[-h,false],[h,false],
                            [s2,false],[-s2,false],[-s2,false],[s2,false],
                             [s3,false], [-s3,false],[-s3,false],[s3,false]])
    u case F => u :: F
    kernel(opcos, x)

  itan x ==
    zero? x => 0
    y := dropfun x
    is?(x, opasin) => y / sqrt(1 - y**2)
    is?(x, opacos) => sqrt(1 - y**2) / y
    is?(x, opatan) => y
    is?(x, opacot) => inv y
    is?(x, opasec) => sqrt(y**2 - 1)
    is?(x, opacsc) => inv sqrt(y**2 - 1)
```

```
    s33 := (s3 := iisqrt3()) / (3::F)
    u := specialTrigs(x, [[0,false], [0,false], [0,true], [0,true],
                    [s3,false], [-s3,false], [s3,false], [-s3,false],
                     [1,false], [-1,false], [1,false], [-1,false],
                      [s33,false], [-s33, false], [s33,false], [-s33, false]])
    u case F => u :: F
    kernel(optan, x)

  icot x ==
    zero? x => INV
    y := dropfun x
    is?(x, opasin) => sqrt(1 - y**2) / y
    is?(x, opacos) => y / sqrt(1 - y**2)
    is?(x, opatan) => inv y
    is?(x, opacot) => y
    is?(x, opasec) => inv sqrt(y**2 - 1)
    is?(x, opacsc) => sqrt(y**2 - 1)
    s33 := (s3 := iisqrt3()) / (3::F)
    u := specialTrigs(x, [[0,true], [0,true], [0,false], [0,false],
                      [s33,false], [-s33,false], [s33,false], [-s33,false],
                       [1,false], [-1,false], [1,false], [-1,false],
                        [s3,false], [-s3, false], [s3,false], [-s3, false]])
    u case F => u :: F
    kernel(opcot, x)

  isec x ==
    zero? x => 1
    y := dropfun x
    is?(x, opasin) => inv sqrt(1 - y**2)
    is?(x, opacos) => inv y
    is?(x, opatan) => sqrt(1 + y**2)
    is?(x, opacot) => sqrt(1 + y**2) / y
    is?(x, opasec) => y
    is?(x, opacsc) => y / sqrt(y**2 - 1)
    s2 := iisqrt2()
    s3 := 2 * iisqrt3() / (3::F)
    h  := 2::F
    u  := specialTrigs(x, [[1,false],[-1,false],[0,true],[0,true],
                        [h,false], [-h,false], [-h,false], [h,false],
                         [s2,false], [-s2,false], [-s2,false], [s2,false],
                          [s3,false], [-s3,false], [-s3,false], [s3,false]])
    u case F => u :: F
    kernel(opsec, x)

  icsc x ==
    zero? x => INV
```

```
      y := dropfun x
      is?(x, opasin) => inv y
      is?(x, opacos) => inv sqrt(1 - y**2)
      is?(x, opatan) => sqrt(1 + y**2) / y
      is?(x, opacot) => sqrt(1 + y**2)
      is?(x, opasec) => y / sqrt(y**2 - 1)
      is?(x, opacsc) => y
      s2 := iisqrt2()
      s3 := 2 * iisqrt3() / (3::F)
      h  := 2::F
      u  := specialTrigs(x, [[0,true], [0,true], [1,false], [-1,false],
                            [s3,false], [s3,false], [-s3,false], [-s3,false],
                              [s2,false], [s2,false], [-s2,false], [-s2,false],
                                 [h,false], [h,false], [-h,false], [-h,false]])
      u case F => u :: F
      kernel(opcsc, x)

    iasin x ==
      zero? x => 0
--      one? x =>   pi() / (2::F)
      (x = 1) =>   pi() / (2::F)
      x = -1 => - pi() / (2::F)
      y := dropfun x
      is?(x, opsin) => y
      is?(x, opcos) => pi() / (2::F) - y
      kernel(opasin, x)

    iacos x ==
      zero? x => pi() / (2::F)
--      one? x => 0
      (x = 1) => 0
      x = -1 => pi()
      y := dropfun x
      is?(x, opsin) => pi() / (2::F) - y
      is?(x, opcos) => y
      kernel(opacos, x)

    iatan x ==
      zero? x => 0
--      one? x =>   pi() / (4::F)
      (x = 1) =>   pi() / (4::F)
      x = -1 => - pi() / (4::F)
      x = (r3:=iisqrt3()) => pi() / (3::F)
--      one?(x*r3)          => pi() / (6::F)
      (x*r3) = 1          => pi() / (6::F)
      y := dropfun x
```

```
      is?(x, optan) => y
      is?(x, opcot) => pi() / (2::F) - y
      kernel(opatan, x)

    iacot x ==
      zero? x =>   pi() / (2::F)
--      one? x  =>   pi() / (4::F)
      (x = 1)  =>   pi() / (4::F)
      x = -1  =>   3 * pi() / (4::F)
      x = (r3:=iisqrt3())  =>  pi() / (6::F)
      x = -r3              =>  5 * pi() / (6::F)
--      one?(xx:=x*r3)       =>  pi() / (3::F)
      (xx:=x*r3) = 1      =>  pi() / (3::F)
      xx = -1           =>     2* pi() / (3::F)
      y := dropfun x
      is?(x, optan) => pi() / (2::F) - y
      is?(x, opcot) => y
      kernel(opacot, x)

    iasec x ==
      zero? x => INV
--      one? x => 0
      (x = 1) => 0
      x = -1 => pi()
      y := dropfun x
      is?(x, opsec) => y
      is?(x, opcsc) => pi() / (2::F) - y
      kernel(opasec, x)

    iacsc x ==
      zero? x => INV
--      one? x =>   pi() / (2::F)
      (x = 1) =>   pi() / (2::F)
      x = -1 => - pi() / (2::F)
      y := dropfun x
      is?(x, opsec) => pi() / (2::F) - y
      is?(x, opcsc) => y
      kernel(opacsc, x)

    isinh x ==
      zero? x => 0
      y := dropfun x
      is?(x, opasinh) => y
      is?(x, opacosh) => sqrt(y**2 - 1)
      is?(x, opatanh) => y / sqrt(1 - y**2)
      is?(x, opacoth) => - inv sqrt(y**2 - 1)
```

```
    is?(x, opasech) => sqrt(1 - y**2) / y
    is?(x, opacsch) => inv y
    kernel(opsinh, x)

  icosh x ==
    zero? x => 1
    y := dropfun x
    is?(x, opasinh) => sqrt(y**2 + 1)
    is?(x, opacosh) => y
    is?(x, opatanh) => inv sqrt(1 - y**2)
    is?(x, opacoth) => y / sqrt(y**2 - 1)
    is?(x, opasech) => inv y
    is?(x, opacsch) => sqrt(y**2 + 1) / y
    kernel(opcosh, x)

  itanh x ==
    zero? x => 0
    y := dropfun x
    is?(x, opasinh) => y / sqrt(y**2 + 1)
    is?(x, opacosh) => sqrt(y**2 - 1) / y
    is?(x, opatanh) => y
    is?(x, opacoth) => inv y
    is?(x, opasech) => sqrt(1 - y**2)
    is?(x, opacsch) => inv sqrt(y**2 + 1)
    kernel(optanh, x)

  icoth x ==
    zero? x => INV
    y := dropfun x
    is?(x, opasinh) => sqrt(y**2 + 1) / y
    is?(x, opacosh) => y / sqrt(y**2 - 1)
    is?(x, opatanh) => inv y
    is?(x, opacoth) => y
    is?(x, opasech) => inv sqrt(1 - y**2)
    is?(x, opacsch) => sqrt(y**2 + 1)
    kernel(opcoth, x)

  isech x ==
    zero? x => 1
    y := dropfun x
    is?(x, opasinh) => inv sqrt(y**2 + 1)
    is?(x, opacosh) => inv y
    is?(x, opatanh) => sqrt(1 - y**2)
    is?(x, opacoth) => sqrt(y**2 - 1) / y
    is?(x, opasech) => y
    is?(x, opacsch) => y / sqrt(y**2 + 1)
```

```
  kernel(opsech, x)

icsch x ==
  zero? x => INV
  y := dropfun x
  is?(x, opasinh) => inv y
  is?(x, opacosh) => inv sqrt(y**2 - 1)
  is?(x, opatanh) => sqrt(1 - y**2) / y
  is?(x, opacoth) => - sqrt(y**2 - 1)
  is?(x, opasech) => y / sqrt(1 - y**2)
  is?(x, opacsch) => y
  kernel(opcsch, x)

iasinh x ==
  is?(x, opsinh) => first argument kernel x
  kernel(opasinh, x)

iacosh x ==
  is?(x, opcosh) => first argument kernel x
  kernel(opacosh, x)

iatanh x ==
  is?(x, optanh) => first argument kernel x
  kernel(opatanh, x)

iacoth x ==
  is?(x, opcoth) => first argument kernel x
  kernel(opacoth, x)

iasech x ==
  is?(x, opsech) => first argument kernel x
  kernel(opasech, x)

iacsch x ==
  is?(x, opcsch) => first argument kernel x
  kernel(opacsch, x)

iexp x ==
  zero? x => 1
  is?(x, oplog) => first argument kernel x
  x < 0 and empty? variables x => inv iexp(-x)
  h  := inv(2::F)
  i  := iisqrt1()
  s2 := h * iisqrt2()
  s3 := h * iisqrt3()
  u  := specialTrigs(x / i, [[1,false],[-1,false], [i,false], [-i,false],
```

```
            [h + i * s3,false], [-h + i * s3, false], [-h - i * s3, false],
             [h - i * s3, false], [s2 + i * s2, false], [-s2 + i * s2, false],
              [-s2 - i * s2, false], [s2 - i * s2, false], [s3 + i * h, false],
               [-s3 + i * h, false], [-s3 - i * h, false], [s3 - i * h, false]])
      u case F => u :: F
      kernel(opexp, x)

  -- THIS DETERMINES WHEN TO PERFORM THE log exp f -> f SIMPLIFICATION
  -- CURRENT BEHAVIOR:
  --     IF R IS COMPLEX(S) THEN ONLY ELEMENTS WHICH ARE RETRACTABLE TO R
  --     AND EQUAL TO THEIR CONJUGATES ARE DEEMED REAL (OVERRESTRICTIVE FOR NOW)
  --     OTHERWISE (e.g. R = INT OR FRAC INT), ALL THE ELEMENTS ARE DEEMED REAL

      if (R has imaginary:() -> R) and (R has conjugate: R -> R) then
           localReal? x ==
              (u := retractIfCan(x)@Union(R, "failed")) case R
                 and (u::R) = conjugate(u::R)

      else localReal? x == true

      iiilog x ==
        zero? x => INV
  --      one? x => 0
        (x = 1) => 0
        (u := isExpt(x, opexp)) case Record(var:K, exponent:Integer) =>
             rec := u::Record(var:K, exponent:Integer)
             arg := first argument(rec.var);
             localReal? arg => rec.exponent * first argument(rec.var);
             ilog x
        ilog x

      ilog x ==
  --      ((num1 := one?(num := numer x)) or num = -1) and (den := denom x) ^= 1
        ((num1 := ((num := numer x) = 1)) or num = -1) and (den := denom x) ^= 1
          and empty? variables x => - kernel(oplog, (num1 => den; -den)::F)
        kernel(oplog, x)

      if R has ElementaryFunctionCategory then
        iilog x ==
          (r:=retractIfCan(x)@Union(R,"failed")) case "failed" => iiilog x
          log(r::R)::F

        iiexp x ==
          (r:=retractIfCan(x)@Union(R,"failed")) case "failed" => iexp x
          exp(r::R)::F
```

```
    else
      iilog x == iiilog x
      iiexp x == iexp x

    if R has TrigonometricFunctionCategory then
      iisin x ==
        (r:=retractIfCan(x)@Union(R,"failed")) case "failed" => isin x
        sin(r::R)::F

      iicos x ==
        (r:=retractIfCan(x)@Union(R,"failed")) case "failed" => icos x
        cos(r::R)::F

      iitan x ==
        (r:=retractIfCan(x)@Union(R,"failed")) case "failed" => itan x
        tan(r::R)::F

      iicot x ==
        (r:=retractIfCan(x)@Union(R,"failed")) case "failed" => icot x
        cot(r::R)::F

      iisec x ==
        (r:=retractIfCan(x)@Union(R,"failed")) case "failed" => isec x
        sec(r::R)::F

      iicsc x ==
        (r:=retractIfCan(x)@Union(R,"failed")) case "failed" => icsc x
        csc(r::R)::F

    else
      iisin x == isin x
      iicos x == icos x
      iitan x == itan x
      iicot x == icot x
      iisec x == isec x
      iicsc x == icsc x

    if R has ArcTrigonometricFunctionCategory then
      iiasin x ==
        (r:=retractIfCan(x)@Union(R,"failed")) case "failed" => iasin x
        asin(r::R)::F

      iiacos x ==
        (r:=retractIfCan(x)@Union(R,"failed")) case "failed" => iacos x
        acos(r::R)::F
```

```
    iiatan x ==
      (r:=retractIfCan(x)@Union(R,"failed")) case "failed" => iatan x
      atan(r::R)::F

    iiacot x ==
      (r:=retractIfCan(x)@Union(R,"failed")) case "failed" => iacot x
      acot(r::R)::F

    iiasec x ==
      (r:=retractIfCan(x)@Union(R,"failed")) case "failed" => iasec x
      asec(r::R)::F

    iiacsc x ==
      (r:=retractIfCan(x)@Union(R,"failed")) case "failed" => iacsc x
      acsc(r::R)::F

  else
    iiasin x == iasin x
    iiacos x == iacos x
    iiatan x == iatan x
    iiacot x == iacot x
    iiasec x == iasec x
    iiacsc x == iacsc x

  if R has HyperbolicFunctionCategory then
    iisinh x ==
      (r:=retractIfCan(x)@Union(R,"failed")) case "failed" => isinh x
      sinh(r::R)::F

    iicosh x ==
      (r:=retractIfCan(x)@Union(R,"failed")) case "failed" => icosh x
      cosh(r::R)::F

    iitanh x ==
      (r:=retractIfCan(x)@Union(R,"failed")) case "failed" => itanh x
      tanh(r::R)::F

    iicoth x ==
      (r:=retractIfCan(x)@Union(R,"failed")) case "failed" => icoth x
      coth(r::R)::F

    iisech x ==
      (r:=retractIfCan(x)@Union(R,"failed")) case "failed" => isech x
      sech(r::R)::F

    iicsch x ==
```

```
    (r:=retractIfCan(x)@Union(R,"failed")) case "failed" => icsch x
    csch(r::R)::F

else
  iisinh x == isinh x
  iicosh x == icosh x
  iitanh x == itanh x
  iicoth x == icoth x
  iisech x == isech x
  iicsch x == icsch x

if R has ArcHyperbolicFunctionCategory then
  iiasinh x ==
    (r:=retractIfCan(x)@Union(R,"failed")) case "failed" => iasinh x
    asinh(r::R)::F

  iiacosh x ==
    (r:=retractIfCan(x)@Union(R,"failed")) case "failed" => iacosh x
    acosh(r::R)::F

  iiatanh x ==
    (r:=retractIfCan(x)@Union(R,"failed")) case "failed" => iatanh x
    atanh(r::R)::F

  iiacoth x ==
    (r:=retractIfCan(x)@Union(R,"failed")) case "failed" => iacoth x
    acoth(r::R)::F

  iiasech x ==
    (r:=retractIfCan(x)@Union(R,"failed")) case "failed" => iasech x
    asech(r::R)::F

  iiacsch x ==
    (r:=retractIfCan(x)@Union(R,"failed")) case "failed" => iacsch x
    acsch(r::R)::F

else
  iiasinh x == iasinh x
  iiacosh x == iacosh x
  iiatanh x == iatanh x
  iiacoth x == iacoth x
  iiasech x == iasech x
  iiacsch x == iacsch x

import BasicOperatorFunctions1(F)
```

```
evaluate(oppi, ipi)
evaluate(oplog, iilog)
evaluate(opexp, iiexp)
evaluate(opsin, iisin)
evaluate(opcos, iicos)
evaluate(optan, iitan)
evaluate(opcot, iicot)
evaluate(opsec, iisec)
evaluate(opcsc, iicsc)
evaluate(opasin, iiasin)
evaluate(opacos, iiacos)
evaluate(opatan, iiatan)
evaluate(opacot, iiacot)
evaluate(opasec, iiasec)
evaluate(opacsc, iiacsc)
evaluate(opsinh, iisinh)
evaluate(opcosh, iicosh)
evaluate(optanh, iitanh)
evaluate(opcoth, iicoth)
evaluate(opsech, iisech)
evaluate(opcsch, iicsch)
evaluate(opasinh, iiasinh)
evaluate(opacosh, iiacosh)
evaluate(opatanh, iiatanh)
evaluate(opacoth, iiacoth)
evaluate(opasech, iiasech)
evaluate(opacsch, iiacsch)
derivative(opexp, exp)
derivative(oplog, inv)
derivative(opsin, cos)
derivative(opcos,(x:F):F +-> - sin x)
derivative(optan,(x:F):F +-> 1 + tan(x)**2)
derivative(opcot,(x:F):F +-> - 1 - cot(x)**2)
derivative(opsec,(x:F):F +-> tan(x) * sec(x))
derivative(opcsc,(x:F):F +-> - cot(x) * csc(x))
derivative(opasin,(x:F):F +-> inv sqrt(1 - x**2))
derivative(opacos,(x:F):F +-> - inv sqrt(1 - x**2))
derivative(opatan,(x:F):F +-> inv(1 + x**2))
derivative(opacot,(x:F):F +-> - inv(1 + x**2))
derivative(opasec,(x:F):F +-> inv(x * sqrt(x**2 - 1)))
derivative(opacsc,(x:F):F +-> - inv(x * sqrt(x**2 - 1)))
derivative(opsinh, cosh)
derivative(opcosh, sinh)
derivative(optanh,(x:F):F +-> 1 - tanh(x)**2)
derivative(opcoth,(x:F):F +-> 1 - coth(x)**2)
derivative(opsech,(x:F):F +-> - tanh(x) * sech(x))
```

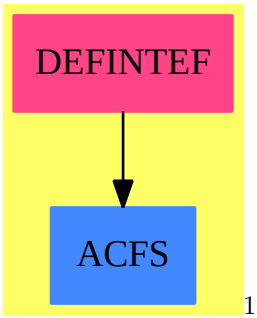
```
derivative(opcsch,(x:F):F +-> - coth(x) * csch(x))
derivative(opasinh,(x:F):F +-> inv sqrt(1 + x**2))
derivative(opacosh,(x:F):F +-> inv sqrt(x**2 - 1))
derivative(opatanh,(x:F):F +-> inv(1 - x**2))
derivative(opacoth,(x:F):F +-> inv(1 - x**2))
derivative(opasech,(x:F):F +-> - inv(x * sqrt(1 - x**2)))
derivative(opacsch,(x:F):F +-> - inv(x * sqrt(1 + x**2)))
```

⟨*EF.dotabb*⟩≡

```
"EF" [color="#FF4488",href="bookvol10.4.pdf#nameddest=EF"]
"FS" [color="#4488FF",href="bookvol10.2.pdf#nameddest=FS"]
"EF" -> "FS"
```

6.5 package DEFINTEF ElementaryFunctionDefiniteIntegration

6.6 ElementaryFunctionDefiniteIntegration



1.00

Exports:
innerint integrate integrate

⟨*package DEFINTEF ElementaryFunctionDefiniteIntegration*⟩≡
```
)abbrev package DEFINTEF ElementaryFunctionDefiniteIntegration
++ Definite integration of elementary functions.
++ Author: Manuel Bronstein
++ Date Created: 14 April 1992
++ Date Last Updated: 2 February 1993
++ Description:
++   \spadtype{ElementaryFunctionDefiniteIntegration}
++   provides functions to compute definite
++   integrals of elementary functions.
ElementaryFunctionDefiniteIntegration(R, F): Exports == Implementation where
  R : Join(EuclideanDomain, OrderedSet, CharacteristicZero,
           RetractableTo Integer, LinearlyExplicitRingOver Integer)
  F : Join(TranscendentalFunctionCategory, PrimitiveFunctionCategory,
           AlgebraicallyClosedFunctionSpace R)

  B   ==> Boolean
  SE  ==> Symbol
  Z   ==> Integer
  P   ==> SparseMultivariatePolynomial(R, K)
  K   ==> Kernel F
  UP  ==> SparseUnivariatePolynomial F
  OFE ==> OrderedCompletion F
  U   ==> Union(f1:OFE, f2:List OFE, fail:"failed", pole:"potentialPole")

  Exports ==> with
    integrate: (F, SegmentBinding OFE) -> U
```

```
    ++ integrate(f, x = a..b) returns the integral of
    ++ \spad{f(x)dx} from a to b.
    ++ Error: if f has a pole for x between a and b.
  integrate: (F, SegmentBinding OFE, String) -> U
    ++ integrate(f, x = a..b, "noPole") returns the
    ++ integral of \spad{f(x)dx} from a to b.
    ++ If it is not possible to check whether f has a pole for x
    ++ between a and b (because of parameters), then this function
    ++ will assume that f has no such pole.
    ++ Error: if f has a pole for x between a and b or
    ++ if the last argument is not "noPole".
  innerint: (F, SE, OFE, OFE, B) -> U
    ++ innerint(f, x, a, b, ignore?) should be local but conditional

Implementation ==> add
  import ElementaryFunctionSign(R, F)
  import DefiniteIntegrationTools(R, F)
  import FunctionSpaceIntegration(R, F)

  polyIfCan   : (P, K) -> Union(UP, "failed")
  int         : (F, SE, OFE, OFE, B) -> U
  nopole      : (F, SE, K, OFE, OFE) -> U
  checkFor0   : (P, K, OFE, OFE) -> Union(B, "failed")
  checkSMP    : (P, SE, K, OFE, OFE) -> Union(B, "failed")
  checkForPole: (F, SE, K, OFE, OFE) -> Union(B, "failed")
  posit       : (F, SE, K, OFE, OFE) -> Union(B, "failed")
  negat       : (F, SE, K, OFE, OFE) -> Union(B, "failed")
  moreThan    : (OFE, Fraction Z) -> Union(B, "failed")

  if R has Join(ConvertibleTo Pattern Integer, PatternMatchable Integer)
    and F has SpecialFunctionCategory then
      import PatternMatchIntegration(R, F)

      innerint(f, x, a, b, ignor?) ==
        ((u := int(f, x, a, b, ignor?)) case f1) or (u case f2)
          or ((v := pmintegrate(f, x, a, b)) case "failed") => u
        [v::F::OFE]

  else
    innerint(f, x, a, b, ignor?) == int(f, x, a, b, ignor?)

  integrate(f:F, s:SegmentBinding OFE) ==
    innerint(f, variable s, lo segment s, hi segment s, false)

  integrate(f:F, s:SegmentBinding OFE, str:String) ==
    innerint(f, variable s, lo segment s, hi segment s, ignore? str)
```

```
    int(f, x, a, b, ignor?) ==
      a = b => [0::OFE]
      k := kernel(x)@Kernel(F)
      (z := checkForPole(f, x, k, a, b)) case "failed" =>
        ignor? => nopole(f, x, k, a, b)
        ["potentialPole"]
      z::B => error "integrate: pole in path of integration"
      nopole(f, x, k, a, b)

    checkForPole(f, x, k, a, b) ==
      ((u := checkFor0(d := denom f, k, a, b)) case "failed") or (u::B) => u
      ((u := checkSMP(d, x, k, a, b)) case "failed") or (u::B) => u
      checkSMP(numer f, x, k, a, b)

-- true if p has a zero between a and b exclusive
    checkFor0(p, x, a, b) ==
      (u := polyIfCan(p, x)) case UP => checkForZero(u::UP, a, b, false)
      (v := isTimes p) case List(P) =>
         for t in v::List(P) repeat
           ((w := checkFor0(t, x, a, b)) case "failed") or (w::B) => return w
         false
      (r := retractIfCan(p)@Union(K, "failed")) case "failed" => "failed"
      k := r::K
-- functions with no real zeros
      is?(k, "exp"::SE) or is?(k, "acot"::SE) or is?(k, "cosh"::SE) => false
-- special case for log
      is?(k, "log"::SE) =>
        (w := moreThan(b, 1)) case "failed" or not(w::B) => w
        moreThan(-a, -1)
      "failed"

-- returns true if a > b, false if a < b, "failed" if can't decide
    moreThan(a, b) ==
      (r := retractIfCan(a)@Union(F, "failed")) case "failed" =>  -- infinite
        whatInfinity(a) > 0
      (u := retractIfCan(r::F)@Union(Fraction Z, "failed")) case "failed" =>
        "failed"
      u::Fraction(Z) > b

-- true if p has a pole between a and b
    checkSMP(p, x, k, a, b) ==
      (u := polyIfCan(p, k)) case UP => false
      (v := isTimes p) case List(P) =>
         for t in v::List(P) repeat
           ((w := checkSMP(t, x, k, a, b)) case "failed") or (w::B) => return w
```

```
          false
        (v := isPlus p) case List(P) =>
          n := 0              -- number of summand having a pole
          for t in v::List(P) repeat
            (w := checkSMP(t, x, k, a, b)) case "failed" => return w
            if w::B then n := n + 1
          zero? n => false    -- no summand has a pole
--          one? n => true      -- only one summand has a pole
          (n = 1) => true     -- only one summand has a pole
          "failed"            -- at least 2 summands have a pole
        (r := retractIfCan(p)@Union(K, "failed")) case "failed" => "failed"
        kk := r::K
        -- nullary operators have no poles
        nullary? operator kk => false
        f := first argument kk
        -- functions which are defined over all the reals:
        is?(kk, "exp"::SE) or is?(kk, "sin"::SE) or is?(kk, "cos"::SE)
          or is?(kk, "sinh"::SE) or is?(kk, "cosh"::SE) or is?(kk, "tanh"::SE)
            or is?(kk, "sech"::SE) or is?(kk, "atan"::SE) or is?(kk, "acot"::SE)
              or is?(kk, "asinh"::SE) => checkForPole(f, x, k, a, b)
        -- functions which are defined on (-1,+1):
        is?(kk, "asin"::SE) or is?(kk, "acos"::SE) or is?(kk, "atanh"::SE) =>
          ((w := checkForPole(f, x, k, a, b)) case "failed") or (w::B) => w
          ((w := posit(f - 1, x, k, a, b)) case "failed") or (w::B) => w
          negat(f + 1, x, k, a, b)
        -- functions which are defined on (+1, +infty):
        is?(kk, "acosh"::SE) =>
          ((w := checkForPole(f, x, k, a, b)) case "failed") or (w::B) => w
          negat(f - 1, x, k, a, b)
        -- functions which are defined on (0, +infty):
        is?(kk, "log"::SE) =>
          ((w := checkForPole(f, x, k, a, b)) case "failed") or (w::B) => w
          negat(f, x, k, a, b)
        "failed"

    -- returns true if it is certain that f takes at least one strictly positive
    -- value for x in (a,b), false if it is certain that f takes no strictly
    -- positive value in (a,b), "failed" otherwise
    -- f must be known to have no poles in (a,b)
     posit(f, x, k, a, b) ==
        z :=
          (r := retractIfCan(a)@Union(F, "failed")) case "failed" => sign(f, x, a)
          sign(f, x, r::F, "right")
        (b1 := z case Z) and z::Z > 0 => true
        z :=
          (r := retractIfCan(b)@Union(F, "failed")) case "failed" => sign(f, x, b)
```

```
        sign(f, x, r::F, "left")
      (b2 := z case Z) and z::Z > 0 => true
      b1 and b2 =>
        ((w := checkFor0(numer f, k, a, b)) case "failed") or (w::B) => "failed"
        false
      "failed"

-- returns true if it is certain that f takes at least one strictly negative
-- value for x in (a,b), false if it is certain that f takes no strictly
-- negative value in (a,b), "failed" otherwise
-- f must be known to have no poles in (a,b)
    negat(f, x, k, a, b) ==
      z :=
        (r := retractIfCan(a)@Union(F, "failed")) case "failed" => sign(f, x, a)
        sign(f, x, r::F, "right")
      (b1 := z case Z) and z::Z < 0 => true
      z :=
        (r := retractIfCan(b)@Union(F, "failed")) case "failed" => sign(f, x, b)
        sign(f, x, r::F, "left")
      (b2 := z case Z) and z::Z < 0 => true
      b1 and b2 =>
        ((w := checkFor0(numer f, k, a, b)) case "failed") or (w::B) => "failed"
        false
      "failed"

-- returns a UP if p is only a poly w.r.t. the kernel x
    polyIfCan(p, x) ==
      q := univariate(p, x)
      ans:UP := 0
      while q ^= 0 repeat
        member?(x, tower(c := leadingCoefficient(q)::F)) => return "failed"
        ans := ans + monomial(c, degree q)
        q := reductum q
      ans

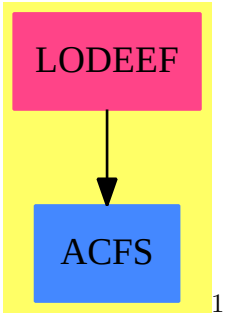
-- integrate f for x between a and b assuming that f has no pole in between
    nopole(f, x, k, a, b) ==
      (u := integrate(f, x)) case F =>
        (v := computeInt(k, u::F, a, b, false)) case "failed" => ["failed"]
        [v::OFE]
      ans := empty()$List(OFE)
      for g in u::List(F) repeat
        (v := computeInt(k, g, a, b, false)) case "failed" => return ["failed"]
        ans := concat_!(ans, [v::OFE])
      [ans]
```

⟨*DEFINTEF.dotabb*⟩≡

```
"DEFINTEF" [color="#FF4488",href="bookvol10.4.pdf#nameddest=DEFINTEF"]
"ACFS" [color="#4488FF",href="bookvol10.2.pdf#nameddest=ACFS"]
"DEFINTEF" -> "ACFS"
```

6.7 package LODEEF ElementaryFunctionLODE-Solver

6.8 ElementaryFunctionLODESolver



1.00

Exports:
solve

⟨*package LODEEF ElementaryFunctionLODESolver*⟩≡

```
)abbrev package LODEEF ElementaryFunctionLODESolver
++ Author: Manuel Bronstein
++ Date Created: 3 February 1994
++ Date Last Updated: 9 March 1994
++ Description:
++ \spad{ElementaryFunctionLODESolver} provides the top-level
++ functions for finding closed form solutions of linear ordinary
++ differential equations and initial value problems.
++ Keywords: differential equation, ODE
ElementaryFunctionLODESolver(R, F, L): Exports == Implementation where
  R: Join(OrderedSet, EuclideanDomain, RetractableTo Integer,
          LinearlyExplicitRingOver Integer, CharacteristicZero)
  F: Join(AlgebraicallyClosedFunctionSpace R, TranscendentalFunctionCategory,
          PrimitiveFunctionCategory)
  L: LinearOrdinaryDifferentialOperatorCategory F

  SY  ==> Symbol
  N   ==> NonNegativeInteger
  K   ==> Kernel F
  V   ==> Vector F
  M   ==> Matrix F
  UP  ==> SparseUnivariatePolynomial F
  RF  ==> Fraction UP
  UPUP==> SparseUnivariatePolynomial RF
  P   ==> SparseMultivariatePolynomial(R, K)
  P2  ==> SparseMultivariatePolynomial(P, K)
```

```
LQ  ==> LinearOrdinaryDifferentialOperator1 RF
REC ==> Record(particular: F, basis: List F)
U   ==> Union(REC, "failed")
ALGOP  ==> "%alg"

Exports ==> with
  solve: (L, F, SY) -> U
    ++ solve(op, g, x) returns either a solution of the ordinary differential
    ++ equation \spad{op y = g} or "failed" if no non-trivial solution can be
    ++ found; When found, the solution is returned in the form
    ++ \spad{[h, [b1,...,bm]]} where \spad{h} is a particular solution and
    ++ and \spad{[b1,...bm]} are linearly independent solutions of the
    ++ associated homogenuous equation \spad{op y = 0}.
    ++ A full basis for the solutions of the homogenuous equation
    ++ is not always returned, only the solutions which were found;
    ++ \spad{x} is the dependent variable.
  solve: (L, F, SY, F, List F) -> Union(F, "failed")
    ++ solve(op, g, x, a, [y0,...,ym]) returns either the solution
    ++ of the initial value problem \spad{op y = g, y(a) = y0, y'(a) = y1,...}
    ++ or "failed" if the solution cannot be found;
    ++ \spad{x} is the dependent variable.

Implementation ==> add
  import Kovacic(F, UP)
  import ODETools(F, L)
  import RationalLODE(F, UP)
  import RationalRicDE(F, UP)
  import ODEIntegration(R, F)
  import ConstantLODE(R, F, L)
  import IntegrationTools(R, F)
  import ReductionOfOrder(F, L)
  import ReductionOfOrder(RF, LQ)
  import PureAlgebraicIntegration(R, F, L)
  import FunctionSpacePrimitiveElement(R, F)
  import LinearSystemMatrixPackage(F, V, V, M)
  import SparseUnivariatePolynomialFunctions2(RF, F)
  import FunctionSpaceUnivariatePolynomialFactor(R, F, UP)
  import LinearOrdinaryDifferentialOperatorFactorizer(F, UP)
  import PolynomialCategoryQuotientFunctions(IndexedExponents K,
                                                        K, R, P, F)

  upmp       : (P, List K) -> P2
  downmp     : (P2, List K, List P) -> P
  xpart      : (F, SY) -> F
  smpxpart   : (P, SY, List K, List P) -> P
  multint    : (F, List F, SY) -> F
```

```
    ulodo      : (L, K) -> LQ
    firstOrder : (F, F, F, SY) -> REC
    rfSolve    : (L, F, K, SY) -> U
    ratlogsol  : (LQ, List RF, K, SY) -> List F
    expsols    : (LQ, K, SY) -> List F
    homosolve  : (L, LQ, List RF, K, SY) -> List F
    homosolve1 : (L, List F, K, SY) -> List F
    norf1      : (L, K, SY, N) -> List F
    kovode     : (LQ, K, SY) -> List F
    doVarParams: (L, F, List F, SY) -> U
    localmap   : (F -> F, L) -> L
    algSolve   : (L, F, K, List K, SY) -> U
    palgSolve  : (L, F, K, K, SY) -> U
    lastChance : (L, F, SY) -> U

    diff := D()$L

    smpxpart(p, x, l, lp) == downmp(primitivePart upmp(p, l), l, lp)
    downmp(p, l, lp)      == ground eval(p, l, lp)
    homosolve(lf, op, sols, k, x) == homosolve1(lf, ratlogsol(op,sols,k,x),k,x)

-- left hand side has algebraic (not necessarily pure) coefficients
    algSolve(op, g, k, l, x) ==
      symbolIfCan(kx := ksec(k, l, x)) case SY => palgSolve(op, g, kx, k, x)
      has?(operator kx, ALGOP) =>
        rec := primitiveElement(kx::F, k::F)
        z   := rootOf(rec.prim)
        lk:List K := [kx, k]
        lv:List F := [(rec.pol1) z, (rec.pol2) z]
        (u := solve(localmap((f1:F):F +-> eval(f1, lk, lv), op), _
                              eval(g, lk, lv), x))
            case "failed" => "failed"
        rc := u::REC
        kz := retract(z)@K
        [eval(rc.particular, kz, rec.primelt),
            [eval(f, kz, rec.primelt) for f in rc.basis]]
      lastChance(op, g, x)

    doVarParams(eq, g, bas, x) ==
      (u := particularSolution(eq, g, bas, (f1:F):F +-> int(f1, x)))
         case "failed" => lastChance(eq, g, x)
      [u::F, bas]

    lastChance(op, g, x) ==
--      one? degree op => firstOrder(coefficient(op,0), leadingCoefficient op,g,x)
      (degree op) = 1 => firstOrder(coefficient(op,0), leadingCoefficient op,g,x)
```

```
      "failed"

  -- solves a0 y + a1 y' = g
  -- does not check whether there is a solution in the field generated by
  -- a0, a1 and g
    firstOrder(a0, a1, g, x) ==
      h := xpart(expint(- a0 / a1, x), x)
      [h * int((g / h) / a1, x), [h]]

  -- xpart(f,x) removes any constant not involving x from f
    xpart(f, x) ==
      l  := reverse_! varselect(tower f, x)
      lp := [k::P for k in l]
      smpxpart(numer f, x, l, lp) / smpxpart(denom f, x, l, lp)

    upmp(p, l) ==
      empty? l => p::P2
      up := univariate(p, k := first l)
      l := rest l
      ans:P2 := 0
      while up ^= 0 repeat
        ans := ans + monomial(upmp(leadingCoefficient up, l), k, degree up)
        up  := reductum up
      ans

  -- multint(a, [g1,...,gk], x) returns gk \int(g(k-1) \int(....g1 \int(a))...)
    multint(a, l, x) ==
       for g in l repeat a := g * xpart(int(a, x), x)
       a

    expsols(op, k, x) ==
  --      one? degree op =>
      (degree op) = 1 =>
          firstOrder(multivariate(coefficient(op, 0), k),
                     multivariate(leadingCoefficient op, k), 0, x).basis
      [xpart(expint(multivariate(h, k), x), x) for h in ricDsolve(op, ffactor)]

  -- Finds solutions with rational logarithmic derivative
    ratlogsol(oper, sols, k, x) ==
      bas := [xpart(multivariate(h, k), x) for h in sols]
      degree(oper) = #bas => bas            -- all solutions are found already
      rec := ReduceOrder(oper, sols)
      le := expsols(rec.eq, k, x)
      int:List(F) := [xpart(multivariate(h, k), x) for h in rec.op]
      concat_!([xpart(multivariate(h, k), x) for h in sols],
               [multint(e, int, x) for e in le])
```

```
    homosolve1(oper, sols, k, x) ==
      zero?(n := (degree(oper) - #sols)::N) => sols   -- all solutions found
      rec := ReduceOrder(oper, sols)
      int:List(F) := [xpart(h, x) for h in rec.op]
      concat_!(sols, [multint(e, int, x) for e in norf1(rec.eq, k, x, n::N)])

-- if the coefficients are rational functions, then the equation does not
-- not have a proper 1st-order right factor over the rational functions
    norf1(op, k, x, n) ==
--      one? n => firstOrder(coefficient(op, 0), leadingCoefficient op,0,x).basis
      (n = 1) => firstOrder(coefficient(op, 0), leadingCoefficient op,0,x).basis
-- for order > 2, we check that the coeffs are still rational functions
      symbolIfCan(kmax vark(coefficients op, x)) case SY =>
        eq := ulodo(op, k)
        n = 2 => kovode(eq, k, x)
        eq := last factor1 eq        -- eq cannot have order 1
        degree(eq) = 2 =>
          empty?(bas := kovode(eq, k, x)) => empty()
          homosolve1(op, bas, k, x)
        empty()
      empty()

    kovode(op, k, x) ==
      b := coefficient(op, 1)
      a := coefficient(op, 2)
      (u := kovacic(coefficient(op, 0), b, a, ffactor)) case "failed" => empty()
      p := map(z1+->multivariate(z1, k), u::UPUP)
      ba := multivariate(- b / a, k)
-- if p has degree 2 (case 2), then it must be squarefree since the
-- ode is irreducible over the rational functions, so the 2 roots of p
-- are distinct and must yield 2 independent solutions.
      degree(p) = 2 => [xpart(expint(ba/(2::F) + e, x), x) for e in zerosOf p]
-- otherwise take 1 root of p and find the 2nd solution by reduction of order
      y1 := xpart(expint(ba / (2::F) + zeroOf p, x), x)
      [y1, y1 * xpart(int(expint(ba, x) / y1**2, x), x)]

    solve(op:L, g:F, x:SY) ==
      empty?(l := vark(coefficients op, x)) => constDsolve(op, g, x)
      symbolIfCan(k := kmax l) case SY => rfSolve(op, g, k, x)
      has?(operator k, ALGOP) => algSolve(op, g, k, l, x)
      lastChance(op, g, x)

    ulodo(eq, k) ==
        op:LQ := 0
        while eq ^= 0 repeat
```

```
            op := op + monomial(univariate(leadingCoefficient eq, k), degree eq)
            eq := reductum eq
        op

  -- left hand side has rational coefficients
    rfSolve(eq, g, k, x) ==
      op := ulodo(eq, k)
      empty? remove_!(k, varselect(kernels g, x)) =>  -- i.e. rhs is rational
        rc := ratDsolve(op, univariate(g, k))
        rc.particular case "failed" =>                -- this implies g ^= 0
          doVarParams(eq, g, homosolve(eq, op, rc.basis, k, x), x)
        [multivariate(rc.particular::RF, k), homosolve(eq, op, rc.basis, k, x)]
      doVarParams(eq, g, homosolve(eq, op, ratDsolve(op, 0).basis, k, x), x)

    solve(op, g, x, a, y0) ==
      (u := solve(op, g, x)) case "failed" => "failed"
      hp := h := (u::REC).particular
      b := (u::REC).basis
      v:V := new(n := #y0, 0)
      kx:K := kernel x
      for i in minIndex v .. maxIndex v for yy in y0 repeat
        v.i := yy - eval(h, kx, a)
        h := diff h
      (sol := particularSolution(
         map_!((f1:F):F+->eval(f1,kx,a),wronskianMatrix(b,n)), v))
           case "failed" => "failed"
      for f in b for i in minIndex(s := sol::V) .. repeat
        hp := hp + s.i * f
      hp

    localmap(f, op) ==
        ans:L := 0
        while op ^= 0 repeat
            ans := ans + monomial(f leadingCoefficient op, degree op)
            op  := reductum op
        ans

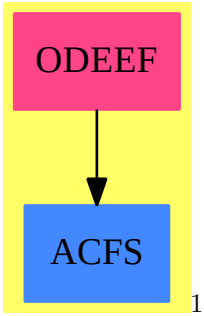
  -- left hand side has pure algebraic coefficients
    palgSolve(op, g, kx, k, x) ==
      rec := palgLODE(op, g, kx, k, x)   -- finds solutions in the coef. field
      rec.particular case "failed" =>
        doVarParams(op, g, homosolve1(op, rec.basis, k, x), x)
      [(rec.particular)::F, homosolve1(op, rec.basis, k, x)]
```

⟨*LODEEF.dotabb*⟩≡

```
"LODEEF" [color="#FF4488",href="bookvol10.4.pdf#nameddest=LODEEF"]
"ACFS" [color="#4488FF",href="bookvol10.2.pdf#nameddest=ACFS"]
"LODEEF" -> "ACFS"
```

6.9 package ODEEF ElementaryFunctionODE-Solver

6.10 ElementaryFunctionODESolver



1.00

Exports:
solve

⟨*package ODEEF ElementaryFunctionODESolver*⟩≡

```
)abbrev package ODEEF ElementaryFunctionODESolver
++ Author: Manuel Bronstein
++ Date Created: 18 March 1991
++ Date Last Updated: 8 March 1994
++ Description:
++ \spad{ElementaryFunctionODESolver} provides the top-level
++ functions for finding closed form solutions of ordinary
++ differential equations and initial value problems.
++ Keywords: differential equation, ODE
ElementaryFunctionODESolver(R, F): Exports == Implementation where
  R: Join(OrderedSet, EuclideanDomain, RetractableTo Integer,
          LinearlyExplicitRingOver Integer, CharacteristicZero)
  F: Join(AlgebraicallyClosedFunctionSpace R, TranscendentalFunctionCategory,
          PrimitiveFunctionCategory)

  N   ==> NonNegativeInteger
  OP  ==> BasicOperator
  SY  ==> Symbol
  K   ==> Kernel F
  EQ  ==> Equation F
  V   ==> Vector F
  M   ==> Matrix F
  UP  ==> SparseUnivariatePolynomial F
  P   ==> SparseMultivariatePolynomial(R, K)
  LEQ ==> Record(left:UP, right:F)
  NLQ ==> Record(dx:F, dy:F)
```

```
REC ==> Record(particular: F, basis: List F)
VEC ==> Record(particular: V, basis: List V)
ROW ==> Record(index: Integer, row: V, rh: F)
SYS ==> Record(mat:M, vec: V)
U   ==> Union(REC, F, "failed")
UU  ==> Union(F, "failed")
OPDIFF ==> "%diff"::SY

Exports ==> with
  solve: (M, V, SY) -> Union(VEC, "failed")
    ++ solve(m, v, x) returns \spad{[v_p, [v_1,...,v_m]]} such that
    ++ the solutions of the system \spad{D y = m y + v} are
    ++ \spad{v_p + c_1 v_1 + ... + c_m v_m} where the \spad{c_i's} are
    ++ constants, and the \spad{v_i's} form a basis for the solutions of
    ++ \spad{D y = m y}.
    ++ \spad{x} is the dependent variable.
  solve: (M, SY) -> Union(List V, "failed")
    ++ solve(m, x) returns a basis for the solutions of \spad{D y = m y}.
    ++ \spad{x} is the dependent variable.
  solve: (List EQ, List OP, SY) -> Union(VEC, "failed")
    ++ solve([eq_1,...,eq_n], [y_1,...,y_n], x) returns either "failed"
    ++ or, if the equations form a fist order linear system, a solution
    ++ of the form \spad{[y_p, [b_1,...,b_n]]} where \spad{h_p} is a
    ++ particular solution and \spad{[b_1,...b_m]} are linearly independent
    ++ solutions of the associated homogenuous system.
    ++ error if the equations do not form a first order linear system
  solve: (List F, List OP, SY) -> Union(VEC, "failed")
    ++ solve([eq_1,...,eq_n], [y_1,...,y_n], x) returns either "failed"
    ++ or, if the equations form a fist order linear system, a solution
    ++ of the form \spad{[y_p, [b_1,...,b_n]]} where \spad{h_p} is a
    ++ particular solution and \spad{[b_1,...b_m]} are linearly independent
    ++ solutions of the associated homogenuous system.
    ++ error if the equations do not form a first order linear system
  solve: (EQ, OP, SY) -> U
    ++ solve(eq, y, x) returns either a solution of the ordinary differential
    ++ equation \spad{eq} or "failed" if no non-trivial solution can be found;
    ++ If the equation is linear ordinary, a solution is of the form
    ++ \spad{[h, [b1,...,bm]]} where \spad{h} is a particular solution
    ++ and \spad{[b1,...bm]} are linearly independent solutions of the
    ++ associated homogenuous equation \spad{f(x,y) = 0};
    ++ A full basis for the solutions of the homogenuous equation
    ++ is not always returned, only the solutions which were found;
    ++ If the equation is of the form {dy/dx = f(x,y)}, a solution is of
    ++ the form \spad{h(x,y)} where \spad{h(x,y) = c} is a first integral
    ++ of the equation for any constant \spad{c};
    ++ error if the equation is not one of those 2 forms;
```

```
    solve: (F, OP, SY) -> U
      ++ solve(eq, y, x) returns either a solution of the ordinary differential
      ++ equation \spad{eq} or "failed" if no non-trivial solution can be found;
      ++ If the equation is linear ordinary, a solution is of the form
      ++ \spad{[h, [b1,...,bm]]} where \spad{h} is a particular solution and
      ++ and \spad{[b1,...bm]} are linearly independent solutions of the
      ++ associated homogenuous equation \spad{f(x,y) = 0};
      ++ A full basis for the solutions of the homogenuous equation
      ++ is not always returned, only the solutions which were found;
      ++ If the equation is of the form {dy/dx = f(x,y)}, a solution is of
      ++ the form \spad{h(x,y)} where \spad{h(x,y) = c} is a first integral
      ++ of the equation for any constant \spad{c};
    solve: (EQ, OP, EQ, List F) -> UU
      ++ solve(eq, y, x = a, [y0,...,ym]) returns either the solution
      ++ of the initial value problem \spad{eq, y(a) = y0, y'(a) = y1,...}
      ++ or "failed" if the solution cannot be found;
      ++ error if the equation is not one linear ordinary or of the form
      ++ \spad{dy/dx = f(x,y)};
    solve: (F, OP, EQ, List F) -> UU
      ++ solve(eq, y, x = a, [y0,...,ym]) returns either the solution
      ++ of the initial value problem \spad{eq, y(a) = y0, y'(a) = y1,...}
      ++ or "failed" if the solution cannot be found;
      ++ error if the equation is not one linear ordinary or of the form
      ++ \spad{dy/dx = f(x,y)};

  Implementation ==> add
    import ODEIntegration(R, F)
    import IntegrationTools(R, F)
    import NonLinearFirstOrderODESolver(R, F)

    getfreelincoeff : (F, K, SY) -> F
    getfreelincoeff1: (F, K, List F) -> F
    getlincoeff     : (F, K) -> F
    getcoeff        : (F, K) -> UU
    parseODE        : (F, OP, SY) -> Union(LEQ, NLQ)
    parseLODE       : (F, List K, UP, SY) -> LEQ
    parseSYS        : (List F, List OP, SY) -> Union(SYS, "failed")
    parseSYSeq      : (F, List K, List K, List F, SY) -> Union(ROW, "failed")

    solve(diffeq:EQ, y:OP, x:SY) == solve(lhs diffeq - rhs diffeq, y, x)

    solve(leq: List EQ, lop: List OP, x:SY) ==
        solve([lhs eq - rhs eq for eq in leq], lop, x)

    solve(diffeq:EQ, y:OP, center:EQ, y0:List F) ==
      solve(lhs diffeq - rhs diffeq, y, center, y0)
```

```
    solve(m:M, x:SY) ==
        (u := solve(m, new(nrows m, 0), x)) case "failed" => "failed"
        u.basis

    solve(m:M, v:V, x:SY) ==
        Lx := LinearOrdinaryDifferentialOperator(F, diff x)
        uu := solve(m, v, (z1,z2) +-> solve(z1, z2, x)_
          $ElementaryFunctionLODESolver(R, F, Lx))$SystemODESolver(F, Lx)
        uu case "failed" => "failed"
        rec := uu::Record(particular: V, basis: M)
        [rec.particular, [column(rec.basis, i) for i in 1..ncols(rec.basis)]]

    solve(diffeq:F, y:OP, center:EQ, y0:List F) ==
      a := rhs center
      kx:K := kernel(x := retract(lhs(center))@SY)
      (ur := parseODE(diffeq, y, x)) case NLQ =>
--        not one?(#y0) => error "solve: more than one initial condition!"
        not ((#y0) = 1) => error "solve: more than one initial condition!"
        rc := ur::NLQ
        (u := solve(rc.dx, rc.dy, y, x)) case "failed" => "failed"
        u::F - eval(u::F,  [kx, retract(y(x::F))@K], [a, first y0])
      rec := ur::LEQ
      p := rec.left
      Lx := LinearOrdinaryDifferentialOperator(F, diff x)
      op:Lx := 0
      while p ^= 0 repeat
        op := op + monomial(leadingCoefficient p, degree p)
        p  := reductum p
      solve(op, rec.right, x, a, y0)$ElementaryFunctionLODESolver(R, F, Lx)

    solve(leq: List F, lop: List OP, x:SY) ==
        (u := parseSYS(leq, lop, x)) case SYS =>
            rec := u::SYS
            solve(rec.mat, rec.vec, x)
        error "solve: not a first order linear system"

    solve(diffeq:F, y:OP, x:SY) ==
      (u := parseODE(diffeq, y, x)) case NLQ =>
        rc := u::NLQ
        (uu := solve(rc.dx, rc.dy, y, x)) case "failed" => "failed"
        uu::F
      rec := u::LEQ
      p := rec.left
      Lx := LinearOrdinaryDifferentialOperator(F, diff x)
      op:Lx := 0
```

```
      while p ^= 0 repeat
        op := op + monomial(leadingCoefficient p, degree p)
        p  := reductum p
      (uuu := solve(op, rec.right, x)$ElementaryFunctionLODESolver(R, F, Lx))
         case "failed" => "failed"
      uuu::REC

-- returns [M, v] s.t. the equations are D x = M x + v
    parseSYS(eqs, ly, x) ==
      (n := #eqs) ^= #ly => "failed"
      m:M := new(n, n, 0)
      v:V := new(n, 0)
      xx := x::F
      lf := [y xx for y in ly]
      lk0:List(K) := [retract(f)@K for f in lf]
      lk1:List(K) := [retract(differentiate(f, x))@K for f in lf]
      for eq in eqs repeat
          (u := parseSYSeq(eq,lk0,lk1,lf,x)) case "failed" => return "failed"
          rec := u::ROW
          setRow_!(m, rec.index, rec.row)
          v(rec.index) := rec.rh
      [m, v]

    parseSYSeq(eq, l0, l1, lf, x) ==
      l := [k for k in varselect(kernels eq, x) | is?(k, OPDIFF)]
      empty? l or not empty? rest l or zero?(n := position(k := first l,l1)) =>
         "failed"
      c := getfreelincoeff1(eq, k, lf)
      eq := eq - c * k::F
      v:V := new(#l0, 0)
      for y in l0 for i in 1.. repeat
          ci := getfreelincoeff1(eq, y, lf)
          v.i := - ci / c
          eq := eq - ci * y::F
      [n, v, -eq]

-- returns either [p, g] where the equation (diffeq) is of the form p(D)(y) = g
-- or [p, q] such that the equation (diffeq) is of the form p dx + q dy = 0
    parseODE(diffeq, y, x) ==
      f := y(x::F)
      l:List(K) := [retract(f)@K]
      n:N := 2
      for k in varselect(kernels diffeq, x) | is?(k, OPDIFF) repeat
        if (m := height k) > n then n := m
      n := (n - 2)::N
-- build a list of kernels in the order [y^(n)(x),...,y''(x),y'(x),y(x)]
```

```
      for i in 1..n repeat
        l := concat(retract(f := differentiate(f, x))@K, l)
      k:K   -- #$^#& compiler requires this line and the next one too...
      c:F
      while not(empty? l) and zero?(c := getlincoeff(diffeq, k := first l))
        repeat l := rest l
      empty? l or empty? rest l => error "parseODE: equation has order 0"
      diffeq := diffeq - c * (k::F)
      ny := name y
      l := rest l
      height(k) > 3 => parseLODE(diffeq, l, monomial(c, #l), ny)
      (u := getcoeff(diffeq, k := first l)) case "failed" => [diffeq, c]
      eqrhs := (d := u::F) * (k::F) - diffeq
      freeOf?(eqrhs, ny) and freeOf?(c, ny) and freeOf?(d, ny) =>
        [monomial(c, 1) + d::UP, eqrhs]
      [diffeq, c]

  -- returns [p, g] where the equation (diffeq) is of the form p(D)(y) = g
    parseLODE(diffeq, l, p, y) ==
      not freeOf?(leadingCoefficient p, y) =>
        error "parseLODE: not a linear ordinary differential equation"
      d := degree(p)::Integer - 1
      for k in l repeat
        p := p + monomial(c := getfreelincoeff(diffeq, k, y), d::N)
        d := d - 1
        diffeq := diffeq - c * (k::F)
      freeOf?(diffeq, y) => [p, - diffeq]
      error "parseLODE: not a linear ordinary differential equation"

    getfreelincoeff(f, k, y) ==
      freeOf?(c := getlincoeff(f, k), y) => c
      error "getfreelincoeff: not a linear ordinary differential equation"

    getfreelincoeff1(f, k, ly) ==
      c := getlincoeff(f, k)
      for y in ly repeat
         not freeOf?(c, y) =>
           error "getfreelincoeff: not a linear ordinary differential equation"
      c

    getlincoeff(f, k) ==
      (u := getcoeff(f, k)) case "failed" =>
        error "getlincoeff: not an appropriate ordinary differential equation"
      u::F

    getcoeff(f, k) ==
```

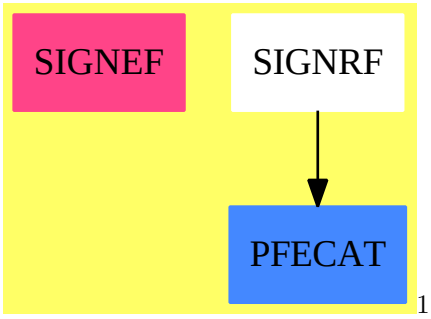
```
        (r := retractIfCan(univariate(denom f, k))@Union(P, "failed"))
          case "failed" or degree(p := univariate(numer f, k)) > 1 => "failed"
        coefficient(p, 1) / (r::P)
```

⟨*ODEEF.dotabb*⟩≡

```
"ODEEF" [color="#FF4488",href="bookvol10.4.pdf#nameddest=ODEEF"]
"ACFS" [color="#4488FF",href="bookvol10.2.pdf#nameddest=ACFS"]
"ODEEF" -> "ACFS"
```

6.11 package SIGNEF ElementaryFunctionSign

6.12 ElementaryFunctionSign



1.00

Exports:
sign

⟨*package SIGNEF ElementaryFunctionSign*⟩≡

```
)abbrev package SIGNEF ElementaryFunctionSign
++ Author: Manuel Bronstein
++ Date Created: 25 Aug 1989
++ Date Last Updated: 4 May 1992
++ Basic Operations:
++ Related Domains:
++ Also See:
++ AMS Classifications:
++ Keywords: elementary function, sign
++ Examples:
++ References:
++ Description:
++   This package provides functions to determine the sign of an
++   elementary function around a point or infinity.
ElementaryFunctionSign(R,F): Exports == Implementation where
  R : Join(IntegralDomain,OrderedSet,RetractableTo Integer,_
           LinearlyExplicitRingOver Integer,GcdDomain)
  F : Join(AlgebraicallyClosedField,TranscendentalFunctionCategory,_
           FunctionSpace R)

  N  ==> NonNegativeInteger
  Z  ==> Integer
  SY ==> Symbol
  RF ==> Fraction Polynomial R
  ORF ==> OrderedCompletion RF
  OFE ==> OrderedCompletion F
  K  ==> Kernel F
```

```
P  ==> SparseMultivariatePolynomial(R, K)
U  ==> Union(Z, "failed")
FS2 ==> FunctionSpaceFunctions2
POSIT ==> "positive"
NEGAT ==> "negative"

Exports ==> with
  sign: F -> U
    ++ sign(f) returns the sign of f if it is constant everywhere.
  sign: (F, SY, OFE) -> U
    ++ sign(f, x, a) returns the sign of f as x nears \spad{a}, from both
    ++ sides if \spad{a} is finite.
  sign: (F, SY, F, String) -> U
    ++ sign(f, x, a, s) returns the sign of f as x nears \spad{a} from below
    ++ if s is "left", or above if s is "right".

Implementation ==> add
  import ToolsForSign R
  import RationalFunctionSign(R)
  import PowerSeriesLimitPackage(R, F)
  import TrigonometricManipulations(R, F)

  smpsign : P -> U
  sqfrSign: P -> U
  termSign: P -> U
  kerSign : K -> U
  listSign: (List P,Z) -> U
  insign  : (F,SY,OFE, N) -> U
  psign   : (F,SY,F,String, N) -> U
  ofesign : OFE -> U
  overRF  : OFE -> Union(ORF, "failed")

  sign(f, x, a) ==
    not real? f => "failed"
    insign(f, x, a, 0)

  sign(f, x, a, st) ==
    not real? f => "failed"
    psign(f, x, a, st, 0)

  sign f ==
    not real? f => "failed"
    (u := retractIfCan(f)@Union(RF,"failed")) case RF => sign(u::RF)
    (un := smpsign numer f) case Z and (ud := smpsign denom f) case Z =>
      un::Z * ud::Z
    --abort if there are any variables
```

```
  not empty? variables f => "failed"
  -- abort in the presence of algebraic numbers
  member?(coerce("rootOf")::Symbol,
    map(name,operators f)$ListFunctions2(BasicOperator,Symbol)) => "failed"
  -- In the last resort try interval evaluation where feasible.
  if R has ConvertibleTo Float then
    import Interval(Float)
    import Expression(Interval Float)
    mapfun : (R -> Interval(Float)) := z +-> interval(convert(z)$R)
    f2 : Expression(Interval Float) :=
        map(mapfun,f)$FS2(R,F,Interval(Float),Expression(Interval Float))
    r : Union(Interval(Float),"failed") := retractIfCan f2
    if r case "failed" then  return "failed"
    negative? r => return(-1)
    positive? r => return 1
    zero? r => return 0
    "failed"
  "failed"

overRF a ==
  (n := whatInfinity a) = 0 =>
    (u := retractIfCan(retract(a)@F)@Union(RF,"failed")) _
           case "failed" => "failed"
    u::RF::ORF
  n * plusInfinity()$ORF

ofesign a ==
  (n := whatInfinity a) ^= 0 => convert(n)@Z
  sign(retract(a)@F)

insign(f, x, a, m) ==
  m > 10 => "failed"                  -- avoid infinite loops for now
  (uf := retractIfCan(f)@Union(RF,"failed")) case RF and
               (ua := overRF a) case ORF => sign(uf::RF, x, ua::ORF)
  eq : Equation OFE := equation(x :: F :: OFE,a)
  (u := limit(f,eq)) case "failed" => "failed"
  u case OFE =>
    (n := whatInfinity(u::OFE)) ^= 0 => convert(n)@Z
    (v := retract(u::OFE)@F) = 0 =>
      (s := insign(differentiate(f, x), x, a, m + 1)) case "failed"
                                                        => "failed"
      - s::Z * n
    sign v
  (u.leftHandLimit case "failed") or
     (u.rightHandLimit case "failed") => "failed"
  (ul := ofesign(u.leftHandLimit::OFE))  case "failed" => "failed"
```

```
  (ur := ofesign(u.rightHandLimit::OFE)) case "failed" => "failed"
  (ul::Z) = (ur::Z) => ul
  "failed"

psign(f, x, a, st, m) ==
  m > 10 => "failed"                  -- avoid infinite loops for now
  f = 0 => 0
  (uf := retractIfCan(f)@Union(RF,"failed")) case RF and
       (ua := retractIfCan(a)@Union(RF,"failed")) case RF =>
        sign(uf::RF, x, ua::RF, st)
  eq : Equation F := equation(x :: F,a)
  (u := limit(f,eq,st)) case "failed" => "failed"
  u case OFE =>
    (n := whatInfinity(u::OFE)) ^= 0 => convert(n)@Z
    (v := retract(u::OFE)@F) = 0 =>
      (s := psign(differentiate(f,x),x,a,st,m + 1)) case "failed"=>
        "failed"
      direction(st) * s::Z
    sign v

smpsign p ==
  (r := retractIfCan(p)@Union(R,"failed")) case R => sign(r::R)
  (u := sign(retract(unit(s := squareFree p))@R)) case "failed" =>
    "failed"
  ans := u::Z
  for term in factorList s | odd?(term.xpnt) repeat
    (u := sqfrSign(term.fctr)) case "failed" => return "failed"
    ans := ans * u::Z
  ans

sqfrSign p ==
  (u := termSign first(l := monomials p)) case "failed" => "failed"
  listSign(rest l, u::Z)

listSign(l, s) ==
  for term in l repeat
    (u := termSign term) case "failed" => return "failed"
    not(s = u::Z) => return "failed"
  s

termSign term ==
  (us := sign leadingCoefficient term) case "failed" => "failed"
  for var in (lv := variables term) repeat
    odd? degree(term, var) =>
      empty? rest lv and (vs := kerSign first lv) case Z =>
                                              return(us::Z * vs::Z)
```

```
        return "failed"
      us::Z

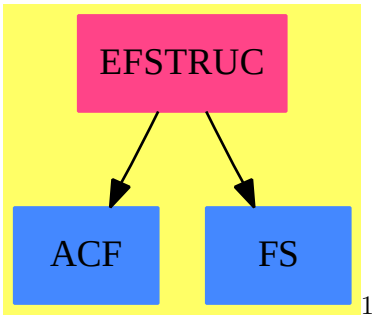
    kerSign k ==
      has?(op := operator k, "NEGAT") => -1
      has?(op, "POSIT") or is?(op,  "pi"::SY) or is?(op,"exp"::SY) or
                           is?(op,"cosh"::SY) or is?(op,"sech"::SY) => 1
      empty?(arg := argument k) => "failed"
      (s := sign first arg) case "failed" =>
        is?(op,"nthRoot" :: SY) =>
          even?(retract(second arg)@Z) => 1
          "failed"
        "failed"
      is?(op,"log" :: SY) =>
        s::Z < 0 => "failed"
        sign(first arg - 1)
      is?(op,"tanh" :: SY) or is?(op,"sinh" :: SY) or
                      is?(op,"csch" :: SY) or is?(op,"coth" :: SY) => s
      is?(op,"nthRoot" :: SY) =>
        even?(retract(second arg)@Z) =>
          s::Z < 0 => "failed"
          s
        s
      "failed"
```

⟨*SIGNEF.dotabb*⟩≡

```
"SIGNEF" [color="#FF4488",href="bookvol10.4.pdf#nameddest=SIGNEF"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"SIGNRF" -> "PFECAT"
```

6.13 package EFSTRUC ElementaryFunctionStructurePackage

6.14 ElementaryFunctionStructurePackage



1.00

Exports:
normalize realElementary rootNormalize rischNormalize tanQ validExponential

⟨*package EFSTRUC ElementaryFunctionStructurePackage*⟩≡

```
)abbrev package EFSTRUC ElementaryFunctionStructurePackage
++ Risch structure theorem
++ Author: Manuel Bronstein
++ Date Created: 1987
++ Date Last Updated: 16 August 1995
++ Description:
++   ElementaryFunctionStructurePackage provides functions to test the
++   algebraic independence of various elementary functions, using the
++   Risch structure theorem (real and complex versions).
++   It also provides transformations on elementary functions
++   which are not considered simplifications.
++ Keywords: elementary, function, structure.
ElementaryFunctionStructurePackage(R,F): Exports == Implementation where
  R : Join(IntegralDomain, OrderedSet, RetractableTo Integer,
           LinearlyExplicitRingOver Integer)
  F : Join(AlgebraicallyClosedField, TranscendentalFunctionCategory,
           FunctionSpace R)

  B   ==> Boolean
  N   ==> NonNegativeInteger
  Z   ==> Integer
  Q   ==> Fraction Z
  SY  ==> Symbol
  K   ==> Kernel F
  UP  ==> SparseUnivariatePolynomial F
  SMP ==> SparseMultivariatePolynomial(R, K)
```

```
REC ==> Record(func:F, kers: List K, vals:List F)
U   ==> Union(vec:Vector Q, func:F, fail: Boolean)
POWER ==> "%power"::SY
NTHR  ==> "nthRoot"::SY

Exports ==> with
  normalize: F -> F
    ++ normalize(f) rewrites \spad{f} using the least possible number of
    ++ real algebraically independent kernels.
  normalize: (F, SY) -> F
    ++ normalize(f, x) rewrites \spad{f} using the least possible number of
    ++ real algebraically independent kernels involving \spad{x}.
  rischNormalize: (F, SY) -> REC
    ++ rischNormalize(f, x) returns \spad{[g, [k1,...,kn], [h1,...,hn]]}
    ++ such that \spad{g = normalize(f, x)} and each \spad{ki} was
    ++ rewritten as \spad{hi} during the normalization.
  realElementary: F -> F
    ++ realElementary(f) rewrites \spad{f} in terms of the 4 fundamental real
    ++ transcendental elementary functions: \spad{log, exp, tan, atan}.
  realElementary: (F, SY) -> F
    ++ realElementary(f,x) rewrites the kernels of \spad{f} involving
    ++ \spad{x} in terms of the 4 fundamental real
    ++ transcendental elementary functions: \spad{log, exp, tan, atan}.
  validExponential: (List K, F, SY) -> Union(F, "failed")
    ++ validExponential([k1,...,kn],f,x) returns \spad{g} if \spad{exp(f)=g}
    ++ and \spad{g} involves only \spad{k1...kn}, and "failed" otherwise.
  rootNormalize: (F, K) -> F
    ++ rootNormalize(f, k) returns \spad{f} rewriting either \spad{k} which
    ++ must be an nth-root in terms of radicals already in \spad{f}, or some
    ++ radicals in \spad{f} in terms of \spad{k}.
  tanQ: (Q, F) -> F
    ++ tanQ(q,a) is a local function with a conditional implementation.

Implementation ==> add
  import TangentExpansions F
  import IntegrationTools(R, F)
  import IntegerLinearDependence F
  import AlgebraicManipulations(R, F)
  import InnerCommonDenominator(Z, Q, Vector Z, Vector Q)

  k2Elem             : (K, List SY) -> F
  realElem           : (F, List SY) -> F
  smpElem            : (SMP, List SY) -> F
  deprel             : (List K, K, SY) -> U
  rootDep            : (List K, K)     -> U
  qdeprel            : (List F, F)     -> U
```

```
factdeprel         : (List K, K)     -> U
toR                : (List K, F) -> List K
toY                : List K -> List F
toZ                : List K -> List F
toU                : List K -> List F
toV                : List K -> List F
ktoY               : K  -> F
ktoZ               : K  -> F
ktoU               : K  -> F
ktoV               : K  -> F
gdCoef?            : (Q, Vector Q) -> Boolean
goodCoef           : (Vector Q, List K, SY) ->
                              Union(Record(index:Z, ker:K), "failed")
tanRN              : (Q, K) -> F
localnorm          : F -> F
rooteval           : (F, List K, K, Q) -> REC
logeval            : (F, List K, K, Vector Q) -> REC
expeval            : (F, List K, K, Vector Q) -> REC
taneval            : (F, List K, K, Vector Q) -> REC
ataneval           : (F, List K, K, Vector Q) -> REC
depeval            : (F, List K, K, Vector Q) -> REC
expnosimp          : (F, List K, K, Vector Q, List F, F) -> REC
tannosimp          : (F, List K, K, Vector Q, List F, F) -> REC
rtNormalize        : F -> F
rootNormalize0     : F -> REC
rootKernelNormalize: (F, List K, K) -> Union(REC, "failed")
tanSum             : (F, List F) -> F

comb?     := F has CombinatorialOpsCategory
mpiover2:F := pi()$F / (-2::F)

realElem(f, l)       == smpElem(numer f, l) / smpElem(denom f, l)
realElementary(f, x) == realElem(f, [x])
realElementary f     == realElem(f, variables f)
toY ker              == [func for k in ker | (func := ktoY k) ^= 0]
toZ ker              == [func for k in ker | (func := ktoZ k) ^= 0]
toU ker              == [func for k in ker | (func := ktoU k) ^= 0]
toV ker              == [func for k in ker | (func := ktoV k) ^= 0]
rtNormalize f        == rootNormalize0(f).func
toR(ker, x) == select(s+->is?(s, NTHR) and first argument(s) = x, ker)

if R has GcdDomain then
  tanQ(c, x) ==
    tanNa(rootSimp zeroOf tanAn(x, denom(c)::PositiveInteger), numer c)
else
  tanQ(c, x) ==
```

```
    tanNa(zeroOf tanAn(x, denom(c)::PositiveInteger), numer c)

-- tanSum(c, [a1,...,an]) returns f(c, a1,...,an) such that
-- if ai = tan(ui) then f(c, a1,...,an) = tan(c + u1 + ... + un).
-- MUST BE CAREFUL FOR WHEN c IS AN ODD MULTIPLE of pi/2
tanSum(c, l) ==
  k := c / mpiover2        -- k = - 2 c / pi, check for odd integer
                           -- tan((2n+1) pi/2 x) = - 1 / tan x
  (r := retractIfCan(k)@Union(Z, "failed")) case Z and odd?(r::Z) =>
       - inv tanSum l
  tanSum concat(tan c, l)

rootNormalize0 f ==
  ker := select_!(s+->is?(s, NTHR) and empty? variables first argument s,
                  tower f)$List(K)
  empty? ker => [f, empty(), empty()]
  (n := (#ker)::Z - 1) < 1 => [f, empty(), empty()]
  for i in 1..n for kk in rest ker repeat
    (u := rootKernelNormalize(f, first(ker, i), kk)) case REC =>
      rec := u::REC
      rn  := rootNormalize0(rec.func)
      return [rn.func, concat(rec.kers,rn.kers), concat(rec.vals, rn.vals)]
  [f, empty(), empty()]

deprel(ker, k, x) ==
  is?(k, "log"::SY) or is?(k, "exp"::SY) =>
    qdeprel([differentiate(g, x) for g in toY ker],
             differentiate(ktoY k, x))
  is?(k, "atan"::SY) or is?(k, "tan"::SY) =>
    qdeprel([differentiate(g, x) for g in toU ker],
             differentiate(ktoU k, x))
  is?(k, NTHR) => rootDep(ker, k)
  comb? and is?(k, "factorial"::SY) =>
    factdeprel([x for x in ker | is?(x,"factorial"::SY) and x^=k],k)
  [true]

ktoY k ==
  is?(k, "log"::SY) => k::F
  is?(k, "exp"::SY) => first argument k
  0

ktoZ k ==
  is?(k, "log"::SY) => first argument k
  is?(k, "exp"::SY) => k::F
  0
```

```
ktoU k ==
  is?(k, "atan"::SY) => k::F
  is?(k,  "tan"::SY) => first argument k
  0

ktoV k ==
  is?(k,  "tan"::SY) => k::F
  is?(k, "atan"::SY) => first argument k
  0

smpElem(p, l) ==
  map(x+->k2Elem(x, l), y+->y::F, p)_
   $PolynomialCategoryLifting(IndexedExponents K, K, R, SMP, F)

k2Elem(k, l) ==
  ez, iez, tz2: F
  kf := k::F
  not(empty? l) and empty? [v for v in variables kf | member?(v, l)] => kf
  empty?(args :List F := [realElem(a, l) for a in argument k]) => kf
  z := first args
  is?(k, POWER)       => (zero? z => 0; exp(last(args) * log z))
  is?(k, "cot"::SY)   => inv tan z
  is?(k, "acot"::SY)  => atan inv z
  is?(k, "asin"::SY)  => atan(z / sqrt(1 - z**2))
  is?(k, "acos"::SY)  => atan(sqrt(1 - z**2) / z)
  is?(k, "asec"::SY)  => atan sqrt(1 - z**2)
  is?(k, "acsc"::SY)  => atan inv sqrt(1 - z**2)
  is?(k, "asinh"::SY) => log(sqrt(1 + z**2) + z)
  is?(k, "acosh"::SY) => log(sqrt(z**2 - 1) + z)
  is?(k, "atanh"::SY) => log((z + 1) / (1 - z)) / (2::F)
  is?(k, "acoth"::SY) => log((z + 1) / (z - 1)) / (2::F)
  is?(k, "asech"::SY) => log((inv z) + sqrt(inv(z**2) - 1))
  is?(k, "acsch"::SY) => log((inv z) + sqrt(1 + inv(z**2)))
  is?(k, "%paren"::SY) or is?(k, "%box"::SY) =>
    empty? rest args => z
    kf
  if has?(op := operator k, "htrig") then iez  := inv(ez  := exp z)
  is?(k, "sinh"::SY)  => (ez - iez) / (2::F)
  is?(k, "cosh"::SY)  => (ez + iez) / (2::F)
  is?(k, "tanh"::SY)  => (ez - iez) / (ez + iez)
  is?(k, "coth"::SY)  => (ez + iez) / (ez - iez)
  is?(k, "sech"::SY)  => 2 * inv(ez + iez)
  is?(k, "csch"::SY)  => 2 * inv(ez - iez)
  if has?(op, "trig") then tz2  := tan(z / (2::F))
  is?(k, "sin"::SY)   => 2 * tz2 / (1 + tz2**2)
  is?(k, "cos"::SY)   => (1 - tz2**2) / (1 + tz2**2)
```

```
        is?(k, "sec"::SY)   => (1 + tz2**2) / (1 - tz2**2)
        is?(k, "csc"::SY)   => (1 + tz2**2) / (2 * tz2)
        op args

  --The next 5 functions are used by normalize, once a relation is found
      depeval(f, lk, k, v) ==
        is?(k, "log"::SY)  => logeval(f, lk, k, v)
        is?(k, "exp"::SY)  => expeval(f, lk, k, v)
        is?(k, "tan"::SY)  => taneval(f, lk, k, v)
        is?(k, "atan"::SY) => ataneval(f, lk, k, v)
        is?(k, NTHR) => rooteval(f, lk, k, v(minIndex v))
        [f, empty(), empty()]

      rooteval(f, lk, k, n) ==
        nv := nthRoot(x := first argument k, m := retract(n)@Z)
        l  := [r for r in concat(k, toR(lk, x)) |
               retract(second argument r)@Z ^= m]
        lv := [nv ** (n / (retract(second argument r)@Z::Q)) for r in l]
        [eval(f, l, lv), l, lv]

      ataneval(f, lk, k, v) ==
        w := first argument k
        s := tanSum [tanQ(qelt(v,i), x)
                     for i in minIndex v .. maxIndex v for x in toV lk]
        g := +/[qelt(v, i) * x for i in minIndex v .. maxIndex v for x in toU lk]
        h:F :=
          zero?(d := 1 + s * w) => mpiover2
          atan((w - s) / d)
        g := g + h
        [eval(f, [k], [g]), [k], [g]]

      gdCoef?(c, v) ==
        for i in minIndex v .. maxIndex v repeat
          retractIfCan(qelt(v, i) / c)@Union(Z, "failed") case "failed" =>
            return false
        true

      goodCoef(v, l, s) ==
        for i in minIndex v .. maxIndex v for k in l repeat
          is?(k, s) and
             ((r:=recip(qelt(v,i))) case Q) and
              (retractIfCan(r::Q)@Union(Z, "failed") case Z)
                and gdCoef?(qelt(v, i), v) => return([i, k])
        "failed"

      taneval(f, lk, k, v) ==
```

```
      u := first argument k
      fns := toU lk
      c := u - +/[qelt(v, i)*x for i in minIndex v .. maxIndex v for x in fns]
      (rec := goodCoef(v, lk, "tan"::SY)) case "failed" =>
          tannosimp(f, lk, k, v, fns, c)
      v0 := retract(inv qelt(v, rec.index))@Z
      lv := [qelt(v, i) for i in minIndex v .. maxIndex v |
                                              i ^= rec.index]$List(Q)
      l  := [kk for kk in lk | kk ^= rec.ker]
      g := tanSum(-v0 * c, concat(tanNa(k::F, v0),
           [tanNa(x, - retract(a * v0)@Z) for a in lv for x in toV l]))
      [eval(f, [rec.ker], [g]), [rec.ker], [g]]

    tannosimp(f, lk, k, v, fns, c) ==
      every?(x+->is?(x, "tan"::SY), lk) =>
        dd := (d := (cd := splitDenominator v).den)::F
        newt := [tan(u / dd) for u in fns]$List(F)
        newtan := [tanNa(t, d) for t in newt]$List(F)
        h := tanSum(c, [tanNa(t, qelt(cd.num, i))
                        for i in minIndex v .. maxIndex v for t in newt])
        lk := concat(k, lk)
        newtan := concat(h, newtan)
        [eval(f, lk, newtan), lk, newtan]
      h := tanSum(c, [tanQ(qelt(v, i), x)
                      for i in minIndex v .. maxIndex v for x in toV lk])
      [eval(f, [k], [h]), [k], [h]]

    expnosimp(f, lk, k, v, fns, g) ==
      every?(x+->is?(x, "exp"::SY), lk) =>
        dd := (d := (cd := splitDenominator v).den)::F
        newe := [exp(y / dd) for y in fns]$List(F)
        newexp := [e ** d for e in newe]$List(F)
        h := */[e ** qelt(cd.num, i)
                for i in minIndex v .. maxIndex v for e in newe] * g
        lk := concat(k, lk)
        newexp := concat(h, newexp)
        [eval(f, lk, newexp), lk, newexp]
      h := */[exp(y) ** qelt(v, i)
              for i in minIndex v .. maxIndex v for y in fns] * g
      [eval(f, [k], [h]), [k], [h]]

    logeval(f, lk, k, v) ==
      z := first argument k
      c := z / (*/[x**qelt(v, i)
                   for x in toZ lk for i in minIndex v .. maxIndex v])
-- CHANGED log ktoZ x TO ktoY x SINCE WE WANT log exp f TO BE REPLACED BY f.
```

```
    g := +/[qelt(v, i) * x
            for i in minIndex v .. maxIndex v for x in toY lk] + log c
    [eval(f, [k], [g]), [k], [g]]

  rischNormalize(f, v) ==
    empty?(ker := varselect(tower f, v)) => [f, empty(), empty()]
    first(ker) ^= kernel(v)@K => error "Cannot happen"
    ker := rest ker
    (n := (#ker)::Z - 1) < 1 => [f, empty(), empty()]
    for i in 1..n for kk in rest ker repeat
      klist := first(ker, i)
      -- NO EVALUATION ON AN EMPTY VECTOR, WILL CAUSE INFINITE LOOP
      (c := deprel(klist, kk, v)) case vec and not empty?(c.vec) =>
        rec := depeval(f, klist, kk, c.vec)
        rn  := rischNormalize(rec.func, v)
        return [rn.func,
                  concat(rec.kers, rn.kers), concat(rec.vals, rn.vals)]
      c case func =>
        rn := rischNormalize(eval(f, [kk], [c.func]), v)
        return [rn.func, concat(kk, rn.kers), concat(c.func, rn.vals)]
    [f, empty(), empty()]

  rootNormalize(f, k) ==
    (u := rootKernelNormalize(f, toR(tower f, first argument k), k))
       case "failed" => f
    (u::REC).func

  rootKernelNormalize(f, l, k) ==
    (c := rootDep(l, k)) case vec =>
      rooteval(f, l, k, (c.vec)(minIndex(c.vec)))
    "failed"

  localnorm f ==
    for x in variables f repeat
      f := rischNormalize(f, x).func
    f

  validExponential(twr, eta, x) ==
    (c := solveLinearlyOverQ(construct([differentiate(g, x)
       for g in (fns := toY twr)]$List(F))@Vector(F),
         differentiate(eta, x))) case "failed" => "failed"
    v := c::Vector(Q)
    g := eta - +/[qelt(v, i) * yy
                    for i in minIndex v .. maxIndex v for yy in fns]
    */[exp(yy) ** qelt(v, i)
              for i in minIndex v .. maxIndex v for yy in fns] * exp g
```

```
rootDep(ker, k) ==
  empty?(ker := toR(ker, first argument k)) => [true]
  [new(1,lcm(retract(second argument k)@Z,
   "lcm"/[retract(second argument r)@Z for r in ker])::Q)$Vector(Q)]

qdeprel(l, v) ==
  (u := solveLinearlyOverQ(construct(l)@Vector(F), v))
    case Vector(Q) => [u::Vector(Q)]
  [true]

expeval(f, lk, k, v) ==
  y   := first argument k
  fns := toY lk
  g := y - +/[qelt(v, i) * z for i in minIndex v .. maxIndex v for z in fns]
  (rec := goodCoef(v, lk, "exp"::SY)) case "failed" =>
    expnosimp(f, lk, k, v, fns, exp g)
  v0 := retract(inv qelt(v, rec.index))@Z
  lv := [qelt(v, i) for i in minIndex v .. maxIndex v |
                                          i ^= rec.index]$List(Q)
  l  := [kk for kk in lk | kk ^= rec.ker]
  h :F := */[exp(z) ** (- retract(a * v0)@Z) for a in lv for z in toY l]
  h := h * exp(-v0 * g) * (k::F) ** v0
  [eval(f, [rec.ker], [h]), [rec.ker], [h]]

if F has CombinatorialOpsCategory then
  normalize f == rtNormalize localnorm factorials realElementary f

  normalize(f, x) ==
    rtNormalize(rischNormalize(factorials(realElementary(f,x),x),x).func)

  factdeprel(l, k) ==
    ((r := retractIfCan(n := first argument k)@Union(Z, "failed"))
      case Z) and (r::Z > 0) => [factorial(r::Z)::F]
    for x in l repeat
      m := first argument x
      ((r := retractIfCan(n - m)@Union(Z, "failed")) case Z) and
        (r::Z > 0) => return([*/[(m + i::F) for i in 1..r] * x::F])
    [true]

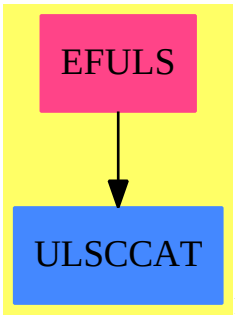
else
  normalize f     == rtNormalize localnorm realElementary f
  normalize(f, x) == rtNormalize(rischNormalize(realElementary(f,x),x).func)
```

⟨*EFSTRUC.dotabb*⟩≡

```
"EFSTRUC" [color="#FF4488",href="bookvol10.4.pdf#nameddest=EFSTRUC"]
"ACF" [color="#4488FF",href="bookvol10.2.pdf#nameddest=ACF"]
"FS" [color="#4488FF",href="bookvol10.2.pdf#nameddest=FS"]
"EFSTRUC" -> "ACF"
"EFSTRUC" -> "FS"
```

6.15 package EFULS ElementaryFunctionsUnivariateLaurentSeries

6.16 ElementaryFunctionsUnivariateLaurentSeries



1.00

Exports:

acos	acosIfCan	acosh	acoshIfCan	acot
acotIfCan	acoth	acothIfCan	acsc	acscIfCan
acsch	acschIfCan	asec	asecIfCan	asech
asechIfCan	asin	asinIfCan	asinh	asinhIfCan
atan	atanIfCan	atanh	atanhIfCan	cos
cosIfCan	cosh	coshIfCan	cot	cotIfCan
coth	cothIfCan	csc	cscIfCan	csch
cschIfCan	exp	expIfCan	log	logIfCan
nthRootIfCan	sec	secIfCan	sech	sechIfCan
sin	sinIfCan	sinh	sinhIfCan	tan
tanIfCan	tanh	tanhIfCan	?**?	

⟨*package EFULS ElementaryFunctionsUnivariateLaurentSeries*⟩≡

```
)abbrev package EFULS ElementaryFunctionsUnivariateLaurentSeries
++ This package provides elementary functions on Laurent series.
++ Author: Clifton J. Williamson
++ Date Created: 6 February 1990
++ Date Last Updated: 25 February 1990
++ Keywords: elementary function, Laurent series
++ Examples:
++ References:
ElementaryFunctionsUnivariateLaurentSeries(Coef,UTS,ULS):_
 Exports == Implementation where
  ++ This package provides elementary functions on any Laurent series
  ++ domain over a field which was constructed from a Taylor series
  ++ domain.  These functions are implemented by calling the
  ++ corresponding functions on the Taylor series domain.  We also
  ++ provide 'partial functions' which compute transcendental
  ++ functions of Laurent series when possible and return "failed"
```

```
    ++ when this is not possible.
    Coef   : Algebra Fraction Integer
    UTS    : UnivariateTaylorSeriesCategory Coef
    ULS    : UnivariateLaurentSeriesConstructorCategory(Coef,UTS)
    I    ==> Integer
    NNI  ==> NonNegativeInteger
    RN   ==> Fraction Integer
    S    ==> String
    STTF ==> StreamTranscendentalFunctions(Coef)

    Exports ==> PartialTranscendentalFunctions(ULS) with

      if Coef has Field then
        "**": (ULS,RN) -> ULS
          ++ s ** r raises a Laurent series s to a rational power r

  --% Exponentials and Logarithms

      exp: ULS -> ULS
        ++ exp(z) returns the exponential of Laurent series z.
      log: ULS -> ULS
        ++ log(z) returns the logarithm of Laurent series z.

  --% TrigonometricFunctionCategory

      sin: ULS -> ULS
        ++ sin(z) returns the sine of Laurent series z.
      cos: ULS -> ULS
        ++ cos(z) returns the cosine of Laurent series z.
      tan: ULS -> ULS
        ++ tan(z) returns the tangent of Laurent series z.
      cot: ULS -> ULS
        ++ cot(z) returns the cotangent of Laurent series z.
      sec: ULS -> ULS
        ++ sec(z) returns the secant of Laurent series z.
      csc: ULS -> ULS
        ++ csc(z) returns the cosecant of Laurent series z.

  --% ArcTrigonometricFunctionCategory

      asin: ULS -> ULS
        ++ asin(z) returns the arc-sine of Laurent series z.
      acos: ULS -> ULS
        ++ acos(z) returns the arc-cosine of Laurent series z.
      atan: ULS -> ULS
        ++ atan(z) returns the arc-tangent of Laurent series z.
```

```
    acot: ULS -> ULS
      ++ acot(z) returns the arc-cotangent of Laurent series z.
    asec: ULS -> ULS
      ++ asec(z) returns the arc-secant of Laurent series z.
    acsc: ULS -> ULS
      ++ acsc(z) returns the arc-cosecant of Laurent series z.

  --% HyperbolicFunctionCategory

    sinh: ULS -> ULS
      ++ sinh(z) returns the hyperbolic sine of Laurent series z.
    cosh: ULS -> ULS
      ++ cosh(z) returns the hyperbolic cosine of Laurent series z.
    tanh: ULS -> ULS
      ++ tanh(z) returns the hyperbolic tangent of Laurent series z.
    coth: ULS -> ULS
      ++ coth(z) returns the hyperbolic cotangent of Laurent series z.
    sech: ULS -> ULS
      ++ sech(z) returns the hyperbolic secant of Laurent series z.
    csch: ULS -> ULS
      ++ csch(z) returns the hyperbolic cosecant of Laurent series z.

  --% ArcHyperbolicFunctionCategory

    asinh: ULS -> ULS
      ++ asinh(z) returns the inverse hyperbolic sine of Laurent series z.
    acosh: ULS -> ULS
      ++ acosh(z) returns the inverse hyperbolic cosine of Laurent series z.
    atanh: ULS -> ULS
      ++ atanh(z) returns the inverse hyperbolic tangent of Laurent series z.
    acoth: ULS -> ULS
      ++ acoth(z) returns the inverse hyperbolic cotangent of Laurent series z.
    asech: ULS -> ULS
      ++ asech(z) returns the inverse hyperbolic secant of Laurent series z.
    acsch: ULS -> ULS
      ++ acsch(z) returns the inverse hyperbolic cosecant of Laurent series z.

  Implementation ==> add

  --% roots

    RATPOWERS : Boolean := Coef has "**":(Coef,RN) -> Coef
    TRANSFCN  : Boolean := Coef has TranscendentalFunctionCategory
    RATS      : Boolean := Coef has retractIfCan: Coef -> Union(RN,"failed")

    nthRootUTS:(UTS,I) -> Union(UTS,"failed")
```

```
    nthRootUTS(uts,n) ==
      -- assumed: n > 1, uts has non-zero constant term
--      one? coefficient(uts,0) => uts ** inv(n::RN)
      coefficient(uts,0) = 1 => uts ** inv(n::RN)
      RATPOWERS => uts ** inv(n::RN)
      "failed"

    nthRootIfCan(uls,nn) ==
      (n := nn :: I) < 1 => error "nthRootIfCan: n must be positive"
      n = 1 => uls
      deg := degree uls
      if zero? (coef := coefficient(uls,deg)) then
        uls := removeZeroes(1000,uls); deg := degree uls
        zero? (coef := coefficient(uls,deg)) =>
          error "root of series with many leading zero coefficients"
      (k := deg exquo n) case "failed" => "failed"
      uts := taylor(uls * monomial(1,-deg))
      (root := nthRootUTS(uts,n)) case "failed" => "failed"
      monomial(1,k :: I) * (root :: UTS :: ULS)

    if Coef has Field then
       (uls:ULS) ** (r:RN) ==
         num := numer r; den := denom r
--         one? den => uls ** num
         den = 1 => uls ** num
         deg := degree uls
         if zero? (coef := coefficient(uls,deg)) then
           uls := removeZeroes(1000,uls); deg := degree uls
           zero? (coef := coefficient(uls,deg)) =>
             error "power of series with many leading zero coefficients"
         (k := deg exquo den) case "failed" =>
           error "**: rational power does not exist"
         uts := taylor(uls * monomial(1,-deg)) ** r
         monomial(1,(k :: I) * num) * (uts :: ULS)

--% transcendental functions

    applyIfCan: (UTS -> UTS,ULS) -> Union(ULS,"failed")
    applyIfCan(fcn,uls) ==
      uts := taylorIfCan uls
      uts case "failed" => "failed"
      fcn(uts :: UTS) :: ULS

    expIfCan   uls == applyIfCan(exp,uls)
    sinIfCan   uls == applyIfCan(sin,uls)
    cosIfCan   uls == applyIfCan(cos,uls)
```

```
asinIfCan  uls == applyIfCan(asin,uls)
acosIfCan  uls == applyIfCan(acos,uls)
asecIfCan  uls == applyIfCan(asec,uls)
acscIfCan  uls == applyIfCan(acsc,uls)
sinhIfCan  uls == applyIfCan(sinh,uls)
coshIfCan  uls == applyIfCan(cosh,uls)
asinhIfCan uls == applyIfCan(asinh,uls)
acoshIfCan uls == applyIfCan(acosh,uls)
atanhIfCan uls == applyIfCan(atanh,uls)
acothIfCan uls == applyIfCan(acoth,uls)
asechIfCan uls == applyIfCan(asech,uls)
acschIfCan uls == applyIfCan(acsch,uls)

logIfCan uls ==
  uts := taylorIfCan uls
  uts case "failed" => "failed"
  zero? coefficient(ts := uts :: UTS,0) => "failed"
  log(ts) :: ULS

tanIfCan uls ==
  -- don't call 'tan' on a UTS (tan(uls) may have a singularity)
  uts := taylorIfCan uls
  uts case "failed" => "failed"
  sc := sincos(coefficients(uts :: UTS))$STTF
  (cosInv := recip(series(sc.cos) :: ULS)) case "failed" => "failed"
  (series(sc.sin) :: ULS) * (cosInv :: ULS)

cotIfCan uls ==
  -- don't call 'cot' on a UTS (cot(uls) may have a singularity)
  uts := taylorIfCan uls
  uts case "failed" => "failed"
  sc := sincos(coefficients(uts :: UTS))$STTF
  (sinInv := recip(series(sc.sin) :: ULS)) case "failed" => "failed"
  (series(sc.cos) :: ULS) * (sinInv :: ULS)

secIfCan uls ==
  cos := cosIfCan uls
  cos case "failed" => "failed"
  (cosInv := recip(cos :: ULS)) case "failed" => "failed"
  cosInv :: ULS

cscIfCan uls ==
  sin := sinIfCan uls
  sin case "failed" => "failed"
  (sinInv := recip(sin :: ULS)) case "failed" => "failed"
  sinInv :: ULS
```

```
atanIfCan uls ==
  coef := coefficient(uls,0)
  (ord := order(uls,0)) = 0 and coef * coef = -1 => "failed"
  cc : Coef :=
    ord < 0 =>
      TRANSFCN =>
        RATS =>
          lc := coefficient(uls,ord)
          (rat := retractIfCan(lc)@Union(RN,"failed")) case "failed" =>
            (1/2) * pi()
          (rat :: RN) > 0 => (1/2) * pi()
          (-1/2) * pi()
        (1/2) * pi()
      return "failed"
    coef = 0 => 0
    TRANSFCN => atan coef
    return "failed"
  (z := recip(1 + uls*uls)) case "failed" => "failed"
  (cc :: ULS) + integrate(differentiate(uls) * (z :: ULS))

acotIfCan uls ==
  coef := coefficient(uls,0)
  (ord := order(uls,0)) = 0 and coef * coef = -1 => "failed"
  cc : Coef :=
    ord < 0 =>
      RATS =>
        lc := coefficient(uls,ord)
        (rat := retractIfCan(lc)@Union(RN,"failed")) case "failed" => 0
        (rat :: RN) > 0 => 0
        TRANSFCN => pi()
        return "failed"
      0
    TRANSFCN => acot coef
    return "failed"
  (z := recip(1 + uls*uls)) case "failed" => "failed"
  (cc :: ULS) - integrate(differentiate(uls) * (z :: ULS))

tanhIfCan uls ==
  -- don't call 'tanh' on a UTS (tanh(uls) may have a singularity)
  uts := taylorIfCan uls
  uts case "failed" => "failed"
  sc := sinhcosh(coefficients(uts :: UTS))$STTF
  (coshInv := recip(series(sc.cosh) :: ULS)) case "failed" =>
    "failed"
  (series(sc.sinh) :: ULS) * (coshInv :: ULS)
```

```
cothIfCan uls ==
  -- don't call 'coth' on a UTS (coth(uls) may have a singularity)
  uts := taylorIfCan uls
  uts case "failed" => "failed"
  sc := sinhcosh(coefficients(uts :: UTS))$STTF
  (sinhInv := recip(series(sc.sinh) :: ULS)) case "failed" =>
    "failed"
  (series(sc.cosh) :: ULS) * (sinhInv :: ULS)

sechIfCan uls ==
  cosh := coshIfCan uls
  cosh case "failed" => "failed"
  (coshInv := recip(cosh :: ULS)) case "failed" => "failed"
  coshInv :: ULS

cschIfCan uls ==
  sinh := sinhIfCan uls
  sinh case "failed" => "failed"
  (sinhInv := recip(sinh :: ULS)) case "failed" => "failed"
  sinhInv :: ULS

applyOrError:(ULS -> Union(ULS,"failed"),S,ULS) -> ULS
applyOrError(fcn,name,uls) ==
  ans := fcn uls
  ans case "failed" =>
    error concat(name," of function with singularity")
  ans :: ULS

exp uls   == applyOrError(expIfCan,"exp",uls)
log uls   == applyOrError(logIfCan,"log",uls)
sin uls   == applyOrError(sinIfCan,"sin",uls)
cos uls   == applyOrError(cosIfCan,"cos",uls)
tan uls   == applyOrError(tanIfCan,"tan",uls)
cot uls   == applyOrError(cotIfCan,"cot",uls)
sec uls   == applyOrError(secIfCan,"sec",uls)
csc uls   == applyOrError(cscIfCan,"csc",uls)
asin uls  == applyOrError(asinIfCan,"asin",uls)
acos uls  == applyOrError(acosIfCan,"acos",uls)
asec uls  == applyOrError(asecIfCan,"asec",uls)
acsc uls  == applyOrError(acscIfCan,"acsc",uls)
sinh uls  == applyOrError(sinhIfCan,"sinh",uls)
cosh uls  == applyOrError(coshIfCan,"cosh",uls)
tanh uls  == applyOrError(tanhIfCan,"tanh",uls)
coth uls  == applyOrError(cothIfCan,"coth",uls)
sech uls  == applyOrError(sechIfCan,"sech",uls)
```

```
csch uls  == applyOrError(cschIfCan,"csch",uls)
asinh uls == applyOrError(asinhIfCan,"asinh",uls)
acosh uls == applyOrError(acoshIfCan,"acosh",uls)
atanh uls == applyOrError(atanhIfCan,"atanh",uls)
acoth uls == applyOrError(acothIfCan,"acoth",uls)
asech uls == applyOrError(asechIfCan,"asech",uls)
acsch uls == applyOrError(acschIfCan,"acsch",uls)

atan uls ==
-- code is duplicated so that correct error messages will be returned
  coef := coefficient(uls,0)
  (ord := order(uls,0)) = 0 and coef * coef = -1 =>
    error "atan: series expansion has logarithmic term"
  cc : Coef :=
    ord < 0 =>
      TRANSFCN =>
        RATS =>
          lc := coefficient(uls,ord)
          (rat := retractIfCan(lc)@Union(RN,"failed")) case "failed" =>
            (1/2) * pi()
          (rat :: RN) > 0 => (1/2) * pi()
          (-1/2) * pi()
        (1/2) * pi()
      error "atan: series expansion involves transcendental constants"
    coef = 0 => 0
    TRANSFCN => atan coef
    error "atan: series expansion involves transcendental constants"
  (z := recip(1 + uls*uls)) case "failed" =>
    error "atan: leading coefficient not invertible"
  (cc :: ULS) + integrate(differentiate(uls) * (z :: ULS))

acot uls ==
-- code is duplicated so that correct error messages will be returned
  coef := coefficient(uls,0)
  (ord := order(uls,0)) = 0 and coef * coef = -1 =>
    error "acot: series expansion has logarithmic term"
  cc : Coef :=
    ord < 0 =>
      RATS =>
        lc := coefficient(uls,ord)
        (rat := retractIfCan(lc)@Union(RN,"failed")) case "failed" => 0
        (rat :: RN) > 0 => 0
        TRANSFCN => pi()
        error "acot: series expansion involves transcendental constants"
      0
    TRANSFCN => acot coef
```

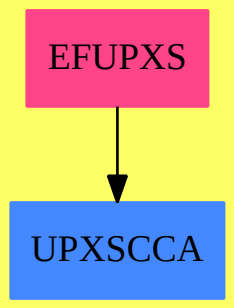
```
        error "acot: series expansion involves transcendental constants"
      (z := recip(1 + uls*uls)) case "failed" =>
        error "acot: leading coefficient not invertible"
      (cc :: ULS) - integrate(differentiate(uls) * (z :: ULS))
```

⟨*EFULS.dotabb*⟩≡

```
"EFULS" [color="#FF4488",href="bookvol10.4.pdf#nameddest=EFULS"]
"ULSCCAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=ULSCCAT"]
"EFULS" -> "ULSCCAT"
```

6.17 package EFUPXS ElementaryFunctionsUnivariatePuiseuxSeries

6.18 ElementaryFunctionsUnivariatePuiseuxSeries



1.00

Exports:

acos	acosIfCan	acosh	acoshIfCan	acot
acotIfCan	acoth	acothIfCan	acsc	acscIfCan
acsch	acschIfCan	asec	asecIfCan	asech
asechIfCan	asin	asinIfCan	asinh	asinhIfCan
atan	atanIfCan	atanh	atanhIfCan	cos
cosIfCan	cosh	coshIfCan	cot	cotIfCan
coth	cothIfCan	csc	cscIfCan	csch
cschIfCan	exp	expIfCan	log	logIfCan
nthRootIfCan	sec	secIfCan	sech	sechIfCan
sin	sinIfCan	sinh	sinhIfCan	tan
tanIfCan	tanh	tanhIfCan	?**?	

⟨*package EFUPXS ElementaryFunctionsUnivariatePuiseuxSeries*⟩≡

```
)abbrev package EFUPXS ElementaryFunctionsUnivariatePuiseuxSeries
++ This package provides elementary functions on Puiseux series.
++ Author: Clifton J. Williamson
++ Date Created: 20 February 1990
++ Date Last Updated: 20 February 1990
++ Keywords: elementary function, Laurent series
++ Examples:
++ References:
ElementaryFunctionsUnivariatePuiseuxSeries(Coef,ULS,UPXS,EFULS):_
 Exports == Implementation where
  ++ This package provides elementary functions on any Laurent series
  ++ domain over a field which was constructed from a Taylor series
  ++ domain.  These functions are implemented by calling the
  ++ corresponding functions on the Taylor series domain.  We also
  ++ provide 'partial functions' which compute transcendental
  ++ functions of Laurent series when possible and return "failed"
```

```
  ++ when this is not possible.
  Coef   : Algebra Fraction Integer
  ULS    : UnivariateLaurentSeriesCategory Coef
  UPXS   : UnivariatePuiseuxSeriesConstructorCategory(Coef,ULS)
  EFULS  : PartialTranscendentalFunctions(ULS)
  I    ==> Integer
  NNI  ==> NonNegativeInteger
  RN   ==> Fraction Integer

  Exports ==> PartialTranscendentalFunctions(UPXS) with

    if Coef has Field then
      "**": (UPXS,RN) -> UPXS
        ++ z ** r raises a Puiseaux series z to a rational power r

--% Exponentials and Logarithms

    exp: UPXS -> UPXS
      ++ exp(z) returns the exponential of a Puiseux series z.
    log: UPXS -> UPXS
      ++ log(z) returns the logarithm of a Puiseux series z.

--% TrigonometricFunctionCategory

    sin: UPXS -> UPXS
      ++ sin(z) returns the sine of a Puiseux series z.
    cos: UPXS -> UPXS
      ++ cos(z) returns the cosine of a Puiseux series z.
    tan: UPXS -> UPXS
      ++ tan(z) returns the tangent of a Puiseux series z.
    cot: UPXS -> UPXS
      ++ cot(z) returns the cotangent of a Puiseux series z.
    sec: UPXS -> UPXS
      ++ sec(z) returns the secant of a Puiseux series z.
    csc: UPXS -> UPXS
      ++ csc(z) returns the cosecant of a Puiseux series z.

--% ArcTrigonometricFunctionCategory

    asin: UPXS -> UPXS
      ++ asin(z) returns the arc-sine of a Puiseux series z.
    acos: UPXS -> UPXS
      ++ acos(z) returns the arc-cosine of a Puiseux series z.
    atan: UPXS -> UPXS
      ++ atan(z) returns the arc-tangent of a Puiseux series z.
    acot: UPXS -> UPXS
```

```
      ++ acot(z) returns the arc-cotangent of a Puiseux series z.
    asec: UPXS -> UPXS
      ++ asec(z) returns the arc-secant of a Puiseux series z.
    acsc: UPXS -> UPXS
      ++ acsc(z) returns the arc-cosecant of a Puiseux series z.

--% HyperbolicFunctionCategory

    sinh: UPXS -> UPXS
      ++ sinh(z) returns the hyperbolic sine of a Puiseux series z.
    cosh: UPXS -> UPXS
      ++ cosh(z) returns the hyperbolic cosine of a Puiseux series z.
    tanh: UPXS -> UPXS
      ++ tanh(z) returns the hyperbolic tangent of a Puiseux series z.
    coth: UPXS -> UPXS
      ++ coth(z) returns the hyperbolic cotangent of a Puiseux series z.
    sech: UPXS -> UPXS
      ++ sech(z) returns the hyperbolic secant of a Puiseux series z.
    csch: UPXS -> UPXS
      ++ csch(z) returns the hyperbolic cosecant of a Puiseux series z.

--% ArcHyperbolicFunctionCategory

    asinh: UPXS -> UPXS
      ++ asinh(z) returns the inverse hyperbolic sine of a Puiseux series z.
    acosh: UPXS -> UPXS
      ++ acosh(z) returns the inverse hyperbolic cosine of a Puiseux series z.
    atanh: UPXS -> UPXS
      ++ atanh(z) returns the inverse hyperbolic tangent of a Puiseux series z.
    acoth: UPXS -> UPXS
      ++ acoth(z) returns the inverse hyperbolic cotangent
      ++ of a Puiseux series z.
    asech: UPXS -> UPXS
      ++ asech(z) returns the inverse hyperbolic secant of a Puiseux series z.
    acsch: UPXS -> UPXS
      ++ acsch(z) returns the inverse hyperbolic cosecant
      ++ of a Puiseux series z.

  Implementation ==> add

    TRANSFCN : Boolean := Coef has TranscendentalFunctionCategory

--% roots

    nthRootIfCan(upxs,n) ==
--      one? n => upxs
```

```
      n = 1 => upxs
      r := rationalPower upxs; uls := laurentRep upxs
      deg := degree uls
      if zero?(coef := coefficient(uls,deg)) then
        deg := order(uls,deg + 1000)
        zero?(coef := coefficient(uls,deg)) =>
          error "root of series with many leading zero coefficients"
      uls := uls * monomial(1,-deg)$ULS
      (ulsRoot := nthRootIfCan(uls,n)) case "failed" => "failed"
      puiseux(r,ulsRoot :: ULS) * monomial(1,deg * r * inv(n :: RN))

    if Coef has Field then
       (upxs:UPXS) ** (q:RN) ==
        num := numer q; den := denom q
--        one? den => upxs ** num
        den = 1 => upxs ** num
        r := rationalPower upxs; uls := laurentRep upxs
        deg := degree uls
        if zero?(coef := coefficient(uls,deg)) then
          deg := order(uls,deg + 1000)
          zero?(coef := coefficient(uls,deg)) =>
            error "power of series with many leading zero coefficients"
        ulsPow := (uls * monomial(1,-deg)$ULS) ** q
        puiseux(r,ulsPow) * monomial(1,deg*q*r)

--% transcendental functions

    applyIfCan: (ULS -> Union(ULS,"failed"),UPXS) -> Union(UPXS,"failed")
    applyIfCan(fcn,upxs) ==
      uls := fcn laurentRep upxs
      uls case "failed" => "failed"
      puiseux(rationalPower upxs,uls :: ULS)

    expIfCan   upxs == applyIfCan(expIfCan,upxs)
    logIfCan   upxs == applyIfCan(logIfCan,upxs)
    sinIfCan   upxs == applyIfCan(sinIfCan,upxs)
    cosIfCan   upxs == applyIfCan(cosIfCan,upxs)
    tanIfCan   upxs == applyIfCan(tanIfCan,upxs)
    cotIfCan   upxs == applyIfCan(cotIfCan,upxs)
    secIfCan   upxs == applyIfCan(secIfCan,upxs)
    cscIfCan   upxs == applyIfCan(cscIfCan,upxs)
    atanIfCan  upxs == applyIfCan(atanIfCan,upxs)
    acotIfCan  upxs == applyIfCan(acotIfCan,upxs)
    sinhIfCan  upxs == applyIfCan(sinhIfCan,upxs)
    coshIfCan  upxs == applyIfCan(coshIfCan,upxs)
    tanhIfCan  upxs == applyIfCan(tanhIfCan,upxs)
```

```
cothIfCan  upxs == applyIfCan(cothIfCan,upxs)
sechIfCan  upxs == applyIfCan(sechIfCan,upxs)
cschIfCan  upxs == applyIfCan(cschIfCan,upxs)
asinhIfCan upxs == applyIfCan(asinhIfCan,upxs)
acoshIfCan upxs == applyIfCan(acoshIfCan,upxs)
atanhIfCan upxs == applyIfCan(atanhIfCan,upxs)
acothIfCan upxs == applyIfCan(acothIfCan,upxs)
asechIfCan upxs == applyIfCan(asechIfCan,upxs)
acschIfCan upxs == applyIfCan(acschIfCan,upxs)

asinIfCan upxs ==
  order(upxs,0) < 0 => "failed"
  (coef := coefficient(upxs,0)) = 0 =>
    integrate((1 - upxs*upxs)**(-1/2) * (differentiate upxs))
  TRANSFCN =>
    cc := asin(coef) :: UPXS
    cc + integrate((1 - upxs*upxs)**(-1/2) * (differentiate upxs))
  "failed"

acosIfCan upxs ==
  order(upxs,0) < 0 => "failed"
  TRANSFCN =>
    cc := acos(coefficient(upxs,0)) :: UPXS
    cc + integrate(-(1 - upxs*upxs)**(-1/2) * (differentiate upxs))
  "failed"

asecIfCan upxs ==
  order(upxs,0) < 0 => "failed"
  TRANSFCN =>
    cc := asec(coefficient(upxs,0)) :: UPXS
    f := (upxs*upxs - 1)**(-1/2) * (differentiate upxs)
    (rec := recip upxs) case "failed" => "failed"
    cc + integrate(f * (rec :: UPXS))
  "failed"

acscIfCan upxs ==
  order(upxs,0) < 0 => "failed"
  TRANSFCN =>
    cc := acsc(coefficient(upxs,0)) :: UPXS
    f := -(upxs*upxs - 1)**(-1/2) * (differentiate upxs)
    (rec := recip upxs) case "failed" => "failed"
    cc + integrate(f * (rec :: UPXS))
  "failed"

asinhIfCan upxs ==
  order(upxs,0) < 0 => "failed"
```

```
  TRANSFCN or (coefficient(upxs,0) = 0) =>
    log(upxs + (1 + upxs*upxs)**(1/2))
  "failed"

acoshIfCan upxs ==
  TRANSFCN =>
    order(upxs,0) < 0 => "failed"
    log(upxs + (upxs*upxs - 1)**(1/2))
  "failed"

asechIfCan upxs ==
  TRANSFCN =>
    order(upxs,0) < 0 => "failed"
    (rec := recip upxs) case "failed" => "failed"
    log((1 + (1 - upxs*upxs)*(1/2)) * (rec :: UPXS))
  "failed"

acschIfCan upxs ==
  TRANSFCN =>
    order(upxs,0) < 0 => "failed"
    (rec := recip upxs) case "failed" => "failed"
    log((1 + (1 + upxs*upxs)*(1/2)) * (rec :: UPXS))
  "failed"

applyOrError:(UPXS -> Union(UPXS,"failed"),String,UPXS) -> UPXS
applyOrError(fcn,name,upxs) ==
  ans := fcn upxs
  ans case "failed" =>
    error concat(name," of function with singularity")
  ans :: UPXS

exp upxs   == applyOrError(expIfCan,"exp",upxs)
log upxs   == applyOrError(logIfCan,"log",upxs)
sin upxs   == applyOrError(sinIfCan,"sin",upxs)
cos upxs   == applyOrError(cosIfCan,"cos",upxs)
tan upxs   == applyOrError(tanIfCan,"tan",upxs)
cot upxs   == applyOrError(cotIfCan,"cot",upxs)
sec upxs   == applyOrError(secIfCan,"sec",upxs)
csc upxs   == applyOrError(cscIfCan,"csc",upxs)
asin upxs  == applyOrError(asinIfCan,"asin",upxs)
acos upxs  == applyOrError(acosIfCan,"acos",upxs)
atan upxs  == applyOrError(atanIfCan,"atan",upxs)
acot upxs  == applyOrError(acotIfCan,"acot",upxs)
asec upxs  == applyOrError(asecIfCan,"asec",upxs)
acsc upxs  == applyOrError(acscIfCan,"acsc",upxs)
sinh upxs  == applyOrError(sinhIfCan,"sinh",upxs)
```

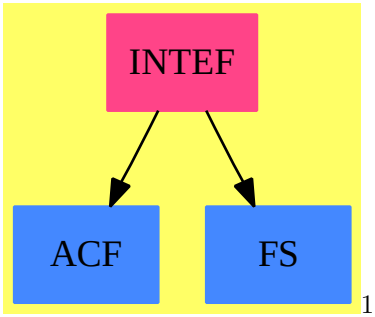
```
cosh upxs  == applyOrError(coshIfCan,"cosh",upxs)
tanh upxs  == applyOrError(tanhIfCan,"tanh",upxs)
coth upxs  == applyOrError(cothIfCan,"coth",upxs)
sech upxs  == applyOrError(sechIfCan,"sech",upxs)
csch upxs  == applyOrError(cschIfCan,"csch",upxs)
asinh upxs == applyOrError(asinhIfCan,"asinh",upxs)
acosh upxs == applyOrError(acoshIfCan,"acosh",upxs)
atanh upxs == applyOrError(atanhIfCan,"atanh",upxs)
acoth upxs == applyOrError(acothIfCan,"acoth",upxs)
asech upxs == applyOrError(asechIfCan,"asech",upxs)
acsch upxs == applyOrError(acschIfCan,"acsch",upxs)
```

⟨*EFUPXS.dotabb*⟩≡

```
"EFUPXS" [color="#FF4488",href="bookvol10.4.pdf#nameddest=EFUPXS"]
"UPXSCCA" [color="#4488FF",href="bookvol10.2.pdf#nameddest=UPXSCCA"]
"EFUPXS" -> "UPXSCCA"
```

6.19 package INTEF ElementaryIntegration

6.20 ElementaryIntegration



1.00

Exports:
lfextendedint lfextlimint lfinfieldint lfintegrate lflimitedint

⟨*package INTEF ElementaryIntegration*⟩≡
```
)abbrev package INTEF ElementaryIntegration
++ Integration of elementary functions
++ Author: Manuel Bronstein
++ Date Created: 1 February 1988
++ Date Last Updated: 24 October 1995
++ Description:
++ This package provides functions for integration, limited integration,
++ extended integration and the risch differential equation for
++ elemntary functions.
++ Keywords: elementary, function, integration.
++ Examples: )r INTEF INPUT
ElementaryIntegration(R, F): Exports == Implementation where
  R : Join(GcdDomain, OrderedSet, CharacteristicZero,
           RetractableTo Integer, LinearlyExplicitRingOver Integer)
  F : Join(AlgebraicallyClosedField, TranscendentalFunctionCategory,
           FunctionSpace R)

  SE     ==> Symbol
  K      ==> Kernel F
  P      ==> SparseMultivariatePolynomial(R, K)
  UP     ==> SparseUnivariatePolynomial F
  RF     ==> Fraction UP
  IR     ==> IntegrationResult F
  FF     ==> Record(ratpart:RF, coeff:RF)
  LLG    ==> List Record(coeff:F, logand:F)
  U2     ==> Union(Record(ratpart:F, coeff:F), "failed")
  U3     ==> Union(Record(mainpart:F, limitedlogs:LLG), "failed")
```

```
ANS    ==> Record(special:F, integrand:F)
PSOL   ==> Record(ans:F, right:F, sol?:Boolean)
FAIL   ==> error "failed - cannot handle that integrand"
ALGOP  ==> "%alg"
OPDIFF ==> "%diff"::SE

Exports ==> with
  lfextendedint: (F, SE, F) -> U2
     ++ lfextendedint(f, x, g) returns functions \spad{[h, c]} such that
     ++ \spad{dh/dx = f - cg}, if (h, c) exist, "failed" otherwise.
  lflimitedint : (F, SE, List F) -> U3
     ++ lflimitedint(f,x,[g1,...,gn]) returns functions \spad{[h,[[ci, gi]]]}
     ++ such that the gi's are among \spad{[g1,...,gn]}, and
     ++ \spad{d(h+sum(ci log(gi)))/dx = f}, if possible, "failed" otherwise.
  lfinfieldint : (F, SE) -> Union(F, "failed")
     ++ lfinfieldint(f, x) returns a function g such that \spad{dg/dx = f}
     ++ if g exists, "failed" otherwise.
  lfintegrate  : (F, SE) -> IR
     ++ lfintegrate(f, x) = g such that \spad{dg/dx = f}.
  lfextlimint  : (F, SE, K, List K) -> U2
     ++ lfextlimint(f,x,k,[k1,...,kn]) returns functions \spad{[h, c]}
     ++ such that \spad{dh/dx = f - c dk/dx}. Value h is looked for in a field
     ++ containing f and k1,...,kn (the ki's must be logs).

Implementation ==> add
  import IntegrationTools(R, F)
  import ElementaryRischDE(R, F)
  import RationalIntegration(F, UP)
  import AlgebraicIntegration(R, F)
  import AlgebraicManipulations(R, F)
  import ElementaryRischDESystem(R, F)
  import TranscendentalIntegration(F, UP)
  import PureAlgebraicIntegration(R, F, F)
  import IntegrationResultFunctions2(F, F)
  import IntegrationResultFunctions2(RF, F)
  import FunctionSpacePrimitiveElement(R, F)
  import PolynomialCategoryQuotientFunctions(IndexedExponents K,
                                                        K, R, P, F)

  alglfint    : (F, K, List K, SE) -> IR
  alglfextint : (F, K, List K, SE, F) -> U2
  alglflimint : (F, K, List K, SE, List F) -> U3
  primextint  : (F, SE, K, F) -> U2
  expextint   : (F, SE, K, F) -> U2
  primlimint  : (F, SE, K, List F) -> U3
  explimint   : (F, SE, K, List F) -> U3
```

```
    algprimint  : (F, K, K, SE) -> IR
    algexpint   : (F, K, K, SE) -> IR
    primint     : (F, SE, K) -> IR
    expint      : (F, SE, K) -> IR
    tanint      : (F, SE, K) -> IR
    prim?       : (K, SE)  -> Boolean
    isx?        : (F, SE)  -> Boolean
    addx        : (IR, F) -> IR
    cfind       : (F, LLG) -> F
    lfintegrate0: (F, SE) -> IR
    unknownint  : (F, SE) -> IR
    unkextint   : (F, SE, F) -> U2
    unklimint   : (F, SE, List F) -> U3
    tryChangeVar: (F, K, SE) -> Union(IR, "failed")
    droponex    : (F, F, K, F) -> Union(F, "failed")

    prim?(k, x)      == is?(k, "log"::SE) or has?(operator k, "prim")

    tanint(f, x, k) ==
      eta' := differentiate(eta := first argument k, x)
      r1  :=
       tanintegrate(univariate(f, k),
        (x1:UP):UP +-> differentiate(x1,
         (x2:F):F +-> differentiate(x2, x),
          monomial(eta', 2) + eta'::UP),
           (x3:Integer,x4:F,x5:F):Union(List F,"failed") +->
             rischDEsys(x3, 2 * eta, x4, x5, x,
              (x6:F,x7:List F):U3 +-> lflimitedint(x6, x, x7),
               (x8:F,x9:F):U2 +-> lfextendedint(x8, x, x9)))
      map((x1:RF):F+->multivariate(x1, k), r1.answer) + lfintegrate(r1.a0, x)

  -- tries various tricks since the integrand contains something not elementary
    unknownint(f, x) ==
      ((r := retractIfCan(f)@Union(K, "failed")) case K) and
        is?(k := r::K, OPDIFF) and
         ((ka:=retractIfCan(a:=second(l:=argument k))@Union(K,"failed"))case K)
           and ((z := retractIfCan(zz := third l)@Union(SE, "failed")) case SE)
              and (z::SE = x)
                and ((u := droponex(first l, a, ka, zz)) case F) => u::F::IR
      (da := differentiate(a := denom(f)::F, x)) ^= 0 and
        zero? differentiate(c := numer(f)::F / da, x) => (c * log a)::IR
      mkAnswer(0, empty(), [[f, x::F]])

    droponex(f, a, ka, x) ==
      (r := retractIfCan(f)@Union(K, "failed")) case "failed" => "failed"
      is?(op := operator(k := r::K), OPDIFF) =>
```

```
      (z := third(arg := argument k)) = a => op [first arg, second arg, x]
      (u := droponex(first arg, a, ka, x)) case "failed" => "failed"
      op [u::F, second arg, z]
    eval(f, [ka], [x])

  unklimint(f, x, lu) ==
    for u in lu | u ^= 0 repeat
      zero? differentiate(c := f * u / differentiate(u, x), x) => [0,[[c,u]]]
    "failed"

  unkextint(f, x, g) ==
    zero?(g' := differentiate(g, x)) => "failed"
    zero? differentiate(c := f / g', x) => [0, c]
    "failed"

  isx?(f, x) ==
    (k := retractIfCan(f)@Union(K, "failed")) case "failed" => false
    (r := symbolIfCan(k::K)) case "failed" => false
    r::SE = x

  alglfint(f, k, l, x) ==
    xf := x::F
    symbolIfCan(kx := ksec(k,l,x)) case SE => addx(palgint(f, kx, k), xf)
    is?(kx, "exp"::SE) => addx(algexpint(f, kx, k, x), xf)
    prim?(kx, x)       => addx(algprimint(f, kx, k, x), xf)
    has?(operator kx, ALGOP) =>
      rec := primitiveElement(kx::F, k::F)
      y   := rootOf(rec.prim)
      map((x1:F):F +-> eval(x1, retract(y)@K, rec.primelt),
        lfintegrate(eval(f, [kx,k], [(rec.pol1) y, (rec.pol2) y]), x))
    unknownint(f, x)

  alglfextint(f, k, l, x, g) ==
    symbolIfCan(kx := ksec(k,l,x)) case SE => palgextint(f, kx, k, g)
    has?(operator kx, ALGOP) =>
      rec := primitiveElement(kx::F, k::F)
      y   := rootOf(rec.prim)
      lrhs := [(rec.pol1) y, (rec.pol2) y]$List(F)
      (u := lfextendedint(eval(f, [kx, k], lrhs), x,
                  eval(g, [kx, k], lrhs))) case "failed" => "failed"
      ky := retract(y)@K
      r := u::Record(ratpart:F, coeff:F)
      [eval(r.ratpart,ky,rec.primelt), eval(r.coeff,ky,rec.primelt)]
    is?(kx, "exp"::SE) or is?(kx, "log"::SE) => FAIL
    unkextint(f, x, g)
```

```
alglflimint(f, k, l, x, lu) ==
  symbolIfCan(kx := ksec(k,l,x)) case SE => palglimint(f, kx, k, lu)
  has?(operator kx, ALGOP) =>
    rec := primitiveElement(kx::F, k::F)
    y   := rootOf(rec.prim)
    lrhs := [(rec.pol1) y, (rec.pol2) y]$List(F)
    (u := lflimitedint(eval(f, [kx, k], lrhs), x,
      map((x1:F):F+->eval(x1,[kx, k],lrhs), lu))) case "failed" => "failed"
    ky := retract(y)@K
    r := u::Record(mainpart:F, limitedlogs:LLG)
    [eval(r.mainpart, ky, rec.primelt),
      [[eval(rc.coeff, ky, rec.primelt),
        eval(rc.logand,ky, rec.primelt)] for rc in r.limitedlogs]]
  is?(kx, "exp"::SE) or is?(kx, "log"::SE) => FAIL
  unklimint(f, x, lu)

if R has Join(ConvertibleTo Pattern Integer, PatternMatchable Integer)
  and F has Join(LiouvillianFunctionCategory, RetractableTo SE) then
    import PatternMatchIntegration(R, F)
    lfintegrate(f, x) == intPatternMatch(f, x, lfintegrate0, pmintegrate)

else lfintegrate(f, x) == lfintegrate0(f, x)

lfintegrate0(f, x) ==
  zero? f => 0
  xf := x::F
  empty?(l := varselect(kernels f, x)) => (xf * f)::IR
  symbolIfCan(k := kmax l) case SE =>
    map((x1:RF):F +-> multivariate(x1, k), integrate univariate(f, k))
  is?(k, "tan"::SE)  => addx(tanint(f, x, k), xf)
  is?(k, "exp"::SE)  => addx(expint(f, x, k), xf)
  prim?(k, x)        => addx(primint(f, x, k), xf)
  has?(operator k, ALGOP) => alglfint(f, k, l, x)
  unknownint(f, x)

addx(i, x) ==
  elem? i => i
  mkAnswer(ratpart i, logpart i,
                            [[ne.integrand, x] for ne in notelem i])

tryChangeVar(f, t, x) ==
    z := new()$Symbol
    g := subst(f / differentiate(t::F, x), [t], [z::F])
    freeOf?(g, x) =>               -- can we do change of variables?
        map((x1:F):F+->eval(x1, kernel z, t::F), lfintegrate(g, z))
    "failed"
```

```
algexpint(f, t, y, x) ==
    (u := tryChangeVar(f, t, x)) case IR => u::IR
    algint(f, t, y,
           (x1:UP):UP +-> differentiate(x1,
            (x2:F):F +-> differentiate(x2, x),
             monomial(differentiate(first argument t, x), 1)))

algprimint(f, t, y, x) ==
    (u := tryChangeVar(f, t, x)) case IR => u::IR
    algint(f, t, y,
           (x1:UP):UP +-> differentiate(x1,
            (x2:F):F +-> differentiate(x2, x),
             differentiate(t::F, x)::UP))
```

Bug #100 is an infinite loop that eventually kills Axiom from the input

```
integrate((z^a+1)^b,z)
```

Line 2 of this function used to read:

```
symbolIfCan(k := kmax(l := union(l, varselect(kernels g, x))))
```

The loop occurs when the call to union causes

```
                a log(z)
               %e
```

to get added to the list every time. This gives the argument to kmax

```
            a log(z)
arg1= [z,%e        ]
```

and the result being

```
  a log(z)
%e
```

We keep coming back to process this term, which ends up putting the same term back on the list and we loop. Waldek's solution is to remove the union call.

The original patch fixed the infinite regression mentioned above but caused Axiom to return a closed form of the integral:

$$integrate(asech(x)/x, x)$$

which should not have a closed form. This is referenced in the FriCAS SVN revision 279.

Essentially this new patch uses only logarithms of rational functions when integrating rational functions. It is unclear whether this is the correct fix.

⟨*package INTEF ElementaryIntegration*⟩+≡

```
lfextendedint(f, x, g) ==
  empty?(l := varselect(kernels f, x)) => [x::F * f, 0]
  symbolIfCan(k := kmax(l))
    case SE =>
     g1 :=
       empty?(l1 := varselect(kernels g,x)) => 0::F
       kmax(l1) = k => g
       0::F
     map((x1:RF):F +-> multivariate(x1, k),
           extendedint(univariate(f, k),
```

```
            univariate(g1, k)))
    is?(k, "exp"::SE) => expextint(f, x, k, g)
    prim?(k, x)       => primextint(f, x, k, g)
    has?(operator k, ALGOP) => alglfextint(f, k, l, x, g)
    unkextint(f, x, g)
```

This is part of the fix for bug 100. Line 2 of this function used to read:

```
symbolIfCan(k := kmax(l := union(l, vark(lu, x)))) case SE =>
```

See the above discussion for why this causes an infinite loop.

⟨*package INTEF ElementaryIntegration*⟩+≡

```
lflimitedint(f, x, lu) ==
  empty?(l := varselect(kernels f, x)) => [x::F * f, empty()]
  symbolIfCan(k := kmax(l)) case SE =>
   map((x1:RF):F +-> multivariate(x1, k),
        limitedint(univariate(f, k),
          [univariate(u, k) for u in lu]))
  is?(k, "exp"::SE) => explimint(f, x, k, lu)
  prim?(k, x)       => primlimint(f, x, k, lu)
  has?(operator k, ALGOP) => alglflimint(f, k, l, x, lu)
  unklimint(f, x, lu)

lfinfieldint(f, x) ==
  (u := lfextendedint(f, x, 0)) case "failed" => "failed"
  u.ratpart

primextint(f, x, k, g) ==
  lk := varselect([a for a in tower f | k ^= a and is?(a, "log"::SE)], x)
  (u1 := primextendedint(univariate(f, k),
   (x1:UP):UP +-> differentiate(x1,
    (x2:F):F +-> differentiate(x2, x), differentiate(k::F, x)::UP),
     (x3:F):U2+->lfextlimint(x3,x,k,lk), univariate(g, k))) case "failed"
           => "failed"
  u1 case FF =>
    [multivariate(u1.ratpart, k), multivariate(u1.coeff, k)]
  (u2 := lfextendedint(u1.a0, x, g)) case "failed" => "failed"
  [multivariate(u1.answer, k) + u2.ratpart, u2.coeff]

expextint(f, x, k, g) ==
  (u1 := expextendedint(univariate(f, k),
   (x1:UP):UP +-> differentiate(x1,
    (x2:F):F +->  differentiate(x2, x),
     monomial(differentiate(first argument k, x), 1)),
      (x3:Integer,x4:F):PSOL+->rischDE(x3, first argument k, x4, x,
       (x5:F,x6:List F):U3 +-> lflimitedint(x5, x, x6),
        (x7:F,x8:F):U2+->lfextendedint(x7, x, x8)), univariate(g, k)))
           case "failed" => "failed"
  u1 case FF =>
    [multivariate(u1.ratpart, k), multivariate(u1.coeff, k)]
  (u2 := lfextendedint(u1.a0, x, g)) case "failed" => "failed"
  [multivariate(u1.answer, k) + u2.ratpart, u2.coeff]
```

```
primint(f, x, k) ==
  lk := varselect([a for a in tower f | k ^= a and is?(a, "log"::SE)], x)
  r1 := primintegrate(univariate(f, k),
         (x1:UP):UP +-> differentiate(x1,
          (x2:F):F +-> differentiate(x2, x), differentiate(k::F, x)::UP),
           (x3:F):U2 +-> lfextlimint(x3, x, k, lk))
  map((x1:RF):F+->multivariate(x1, k), r1.answer) + lfintegrate(r1.a0, x)

lfextlimint(f, x, k, lk) ==
  not((u1 := lfextendedint(f, x, differentiate(k::F, x)))
    case "failed") => u1
  twr := tower f
  empty?(lg := [kk for kk in lk | not member?(kk, twr)]) => "failed"
  is?(k, "log"::SE) =>
    (u2 := lflimitedint(f, x,
      [first argument u for u in union(lg, [k])])) case "failed"
                                                    => "failed"
    cf := cfind(first argument k, u2.limitedlogs)
    [u2.mainpart - cf * k::F +
            +/[c.coeff * log(c.logand) for c in u2.limitedlogs], cf]
  "failed"

cfind(f, l) ==
  for u in l repeat
    f = u.logand => return u.coeff
  0

expint(f, x, k) ==
  eta := first argument k
  r1  :=
   expintegrate(univariate(f, k),
    (x1:UP):UP +-> differentiate(x1,
     (x2:F):F +-> differentiate(x2, x),
      monomial(differentiate(eta, x), 1)),
       (x3:Integer,x4:F):PSOL+->rischDE(x3, eta, x4, x,
        (x5:F,x6:List F):U3 +-> lflimitedint(x5, x, x6),
         (x7:F,x8:F):U2+->lfextendedint(x7, x, x8)))
  map((x1:RF):F+->multivariate(x1, k), r1.answer) + lfintegrate(r1.a0, x)

primlimint(f, x, k, lu) ==
  lk := varselect([a for a in tower f | k ^= a and is?(a, "log"::SE)], x)
  (u1 :=
    primlimitedint(univariate(f, k),
     (x1:UP):UP+->differentiate(x1,
       (x2:F):F+->differentiate(x2, x), differentiate(k::F, x)::UP),
```

```
            (x3:F):U2+->lfextlimint(x3,x,k,lk),
             [univariate(u, k) for u in lu]))
               case "failed" => "failed"
      l := [[multivariate(lg.coeff, k),multivariate(lg.logand, k)]
                                    for lg in u1.answer.limitedlogs]$LLG
      (u2 := lflimitedint(u1.a0, x, lu)) case "failed" => "failed"
      [multivariate(u1.answer.mainpart, k) + u2.mainpart,
                                          concat(u2.limitedlogs, l)]

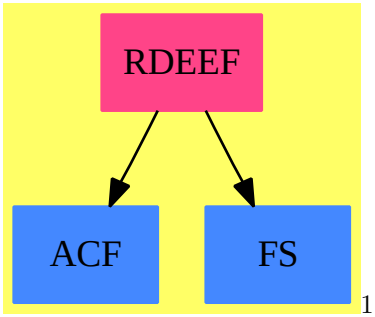
    explimint(f, x, k, lu) ==
      eta := first argument k
      (u1 :=
        explimitedint(univariate(f, k),
         (x1:UP):UP+->differentiate(x1,
          (x2:F):F+->differentiate(x2,x), monomial(differentiate(eta,x), 1)),
           (x3:Integer,x4:F):PSOL+->rischDE(x3, eta, x4, x,
             (x5:F,x6:List F):U3+->lflimitedint(x5, x, x6),
              (x7:F,x8:F):U2+->lfextendedint(x7, x, x8)),
               [univariate(u, k) for u in lu])) case "failed" => "failed"
      l := [[multivariate(lg.coeff, k),multivariate(lg.logand, k)]
                                    for lg in u1.answer.limitedlogs]$LLG
      (u2 := lflimitedint(u1.a0, x, lu)) case "failed" => "failed"
      [multivariate(u1.answer.mainpart, k) + u2.mainpart,
                                          concat(u2.limitedlogs, l)]
```

⟨*INTEF.dotabb*⟩≡

```
"INTEF" [color="#FF4488",href="bookvol10.4.pdf#nameddest=INTEF"]
"ACF" [color="#4488FF",href="bookvol10.2.pdf#nameddest=ACF"]
"FS" [color="#4488FF",href="bookvol10.2.pdf#nameddest=FS"]
"INTEF" -> "ACF"
"INTEF" -> "FS"
```

6.21 package RDEEF ElementaryRischDE

6.22 ElementaryRischDE



1.00

Exports:
rischDE

⟨*package RDEEF ElementaryRischDE*⟩≡
```
)abbrev package RDEEF ElementaryRischDE
++ Risch differential equation, elementary case.
++ Author: Manuel Bronstein
++ Date Created: 1 February 1988
++ Date Last Updated: 2 November 1995
++ Keywords: elementary, function, integration.
ElementaryRischDE(R, F): Exports == Implementation where
  R : Join(GcdDomain, OrderedSet, CharacteristicZero,
           RetractableTo Integer, LinearlyExplicitRingOver Integer)
  F : Join(TranscendentalFunctionCategory, AlgebraicallyClosedField,
           FunctionSpace R)

  N   ==> NonNegativeInteger
  Z   ==> Integer
  SE  ==> Symbol
  LF  ==> List F
  K   ==> Kernel F
  LK  ==> List K
  P   ==> SparseMultivariatePolynomial(R, K)
  UP  ==> SparseUnivariatePolynomial F
  RF  ==> Fraction UP
  GP  ==> LaurentPolynomial(F, UP)
  Data ==> List Record(coeff:Z, argument:P)
  RRF ==> Record(mainpart:F,limitedlogs:List NL)
  NL  ==> Record(coeff:F,logand:F)
  U   ==> Union(RRF, "failed")
  UF  ==> Union(F, "failed")
```

```
UUP ==> Union(UP, "failed")
UGP ==> Union(GP, "failed")
URF ==> Union(RF, "failed")
UEX ==> Union(Record(ratpart:F,  coeff:F), "failed")
PSOL==> Record(ans:F, right:F, sol?:Boolean)
FAIL==> error("Function not supported by Risch d.e.")
ALGOP ==> "%alg"

Exports ==> with
  rischDE: (Z, F, F, SE, (F, LF) -> U, (F, F) -> UEX) -> PSOL
       ++ rischDE(n, f, g, x, lim, ext) returns \spad{[y, h, b]} such that
       ++ \spad{dy/dx + n df/dx y = h} and \spad{b := h = g}.
       ++ The equation \spad{dy/dx + n df/dx y = g} has no solution
       ++ if \spad{h \~~= g} (y is a partial solution in that case).
       ++ Notes: lim is a limited integration function, and
       ++ ext is an extended integration function.

Implementation ==> add
  import IntegrationTools(R, F)
  import TranscendentalRischDE(F, UP)
  import TranscendentalIntegration(F, UP)
  import PureAlgebraicIntegration(R, F, F)
  import FunctionSpacePrimitiveElement(R, F)
  import ElementaryFunctionStructurePackage(R, F)
  import PolynomialCategoryQuotientFunctions(IndexedExponents K,
                                                        K, R, P, F)

  RF2GP:     RF -> GP
  makeData  : (F, SE, K)    -> Data
  normal0   : (Z, F, F, SE) -> UF
  normalise0: (Z, F, F, SE) -> PSOL
  normalise : (Z, F, F, F, SE, K, (F, LF) -> U, (F, F) -> UEX) -> PSOL
  rischDEalg: (Z, F, F, F, K, LK, SE, (F, LF) -> U, (F, F) -> UEX) -> PSOL
  rischDElog: (LK, RF, RF, SE, K, UP->UP,(F,LF)->U,(F,F)->UEX) -> URF
  rischDEexp: (LK, RF, RF, SE, K, UP->UP,(F,LF)->U,(F,F)->UEX) -> URF
  polyDElog : (LK, UP, UP,UP,SE,K,UP->UP,(F,LF)->U,(F,F)->UEX) -> UUP
  polyDEexp : (LK, UP, UP,UP,SE,K,UP->UP,(F,LF)->U,(F,F)->UEX) -> UUP
  gpolDEexp : (LK, UP, GP,GP,SE,K,UP->UP,(F,LF)->U,(F,F)->UEX) -> UGP
  boundAt0  : (LK, F, Z,  Z,    SE, K, (F, LF) -> U) -> Z
  boundInf  : (LK, F, Z,  Z, Z, SE, K, (F, LF) -> U) -> Z
  logdegrad : (LK, F, UP, Z, SE, K,(F,LF)->U, (F,F) -> UEX) -> UUP
  expdegrad : (LK, F, UP, Z, SE, K,(F,LF)->U, (F,F) -> UEX) -> UUP
  logdeg    : (UP, F, Z, SE, F, (F, LF) -> U, (F, F) -> UEX) -> UUP
  expdeg    : (UP, F, Z, SE, F, (F, LF) -> U, (F, F) -> UEX) -> UUP
  exppolyint: (UP, (Z, F) -> PSOL) -> UUP
  RRF2F     : RRF -> F
```

```
      logdiff   : (List K, List K) -> List K

      tab:AssociationList(F, Data) := table()

      RF2GP f == (numer(f)::GP exquo denom(f)::GP)::GP

      logdiff(twr, bad) ==
        [u for u in twr | is?(u, "log"::SE) and not member?(u, bad)]

      rischDEalg(n, nfp, f, g, k, l, x, limint, extint) ==
        symbolIfCan(kx := ksec(k, l, x)) case SE =>
          (u := palgRDE(nfp, f, g, kx, k,
            (z1,z2,z3) +-> normal0(n, z1, z2, z3))) case "failed"
              => [0, 0, false]
          [u::F, g, true]
        has?(operator kx, ALGOP) =>
          rec := primitiveElement(kx::F, k::F)
          y   := rootOf(rec.prim)
          lk:LK := [kx, k]
          lv:LF := [(rec.pol1) y, (rec.pol2) y]
          rc := rischDE(n, eval(f, lk, lv), eval(g, lk, lv), x, limint, extint)
          rc.sol? => [eval(rc.ans, retract(y)@K, rec.primelt), rc.right, true]
          [0, 0, false]
        FAIL

  -- solve y' + n f'y = g for a rational function y
      rischDE(n, f, g, x, limitedint, extendedint) ==
        zero? g => [0, g, true]
        zero?(nfp := n * differentiate(f, x)) =>
          (u := limitedint(g, empty())) case "failed" => [0, 0, false]
          [u.mainpart, g, true]
        freeOf?(y := g / nfp, x) => [y, g, true]
        vl := varselect(union(kernels nfp, kernels g), x)
        symbolIfCan(k := kmax vl) case SE => normalise0(n, f, g, x)
        is?(k, "log"::SE) or is?(k, "exp"::SE) =>
          normalise(n, nfp, f, g, x, k, limitedint, extendedint)
        has?(operator k, ALGOP) =>
          rischDEalg(n, nfp, f, g, k, vl, x, limitedint, extendedint)
        FAIL

      normal0(n, f, g, x) ==
        rec := normalise0(n, f, g, x)
        rec.sol? => rec.ans
        "failed"

  -- solve y' + n f' y = g
```

```
-- when f' and g are rational functions over a constant field
    normalise0(n, f, g, x) ==
      k := kernel(x)@K
      if (data1 := search(f, tab)) case "failed" then
        tab.f := data := makeData(f, x, k)
      else data := data1::Data
      f'  := nfprime := n * differentiate(f, x)
      p:P := 1
      for v in data | (m := n * v.coeff) > 0 repeat
        p  := p * v.argument ** (m::N)
        f' := f' - m * differentiate(v.argument::F, x) / (v.argument::F)
      rec := baseRDE(univariate(f', k), univariate(p::F * g, k))
      y := multivariate(rec.ans, k) / p::F
      rec.nosol => [y, differentiate(y, x) + nfprime * y, false]
      [y, g, true]

-- make f weakly normalized, and solve y' + n f' y = g
    normalise(n, nfp, f, g, x, k, limitedint, extendedint) ==
      if (data1:= search(f, tab)) case "failed" then
        tab.f := data := makeData(f, x, k)
      else data := data1::Data
      p:P := 1
      for v in data | (m := n * v.coeff) > 0 repeat
        p  := p * v.argument ** (m::N)
        f  := f - v.coeff * log(v.argument::F)
        nfp := nfp - m * differentiate(v.argument::F, x) / (v.argument::F)
      newf := univariate(nfp, k)
      newg := univariate(p::F * g, k)
      twr := union(logdiff(tower f, empty()), logdiff(tower g, empty()))
      ans1 :=
        is?(k, "log"::SE) =>
          rischDElog(twr, newf, newg, x, k,
            z1 +-> differentiate(z1,(z2:F):F +-> differentiate(z2, x),
                            differentiate(k::F, x)::UP),
                                        limitedint, extendedint)
        is?(k, "exp"::SE) =>
          rischDEexp(twr, newf, newg, x, k,
            z1 +-> differentiate(z1, (z2:F):F +-> differentiate(z2, x),
                      monomial(differentiate(first argument k, x), 1)),
                                            limitedint, extendedint)
      ans1 case "failed" => [0, 0, false]
      [multivariate(ans1::RF, k) / p::F, g, true]

-- find the n * log(P) appearing in f, where P is in P, n in Z
    makeData(f, x, k) ==
      disasters := empty()$Data
```

```
      fnum := numer f
      fden := denom f
      for u in varselect(kernels f, x) | is?(u, "log"::SE) repeat
        logand := first argument u
        if zero?(degree univariate(fden, u)) and
--           one?(degree(num := univariate(fnum, u))) then
           (degree(num := univariate(fnum, u)) = 1) then
            cf := (leadingCoefficient num) / fden
            if (n := retractIfCan(cf)@Union(Z, "failed")) case Z then
              if degree(numer logand, k) > 0 then
                disasters := concat([n::Z, numer logand], disasters)
              if degree(denom logand, k) > 0 then
                disasters := concat([-(n::Z), denom logand], disasters)
      disasters

    rischDElog(twr, f, g, x, theta, driv, limint, extint) ==
      (u := monomRDE(f, g, driv)) case "failed" => "failed"
      (v := polyDElog(twr, u.a, retract(u.b), retract(u.c), x, theta, driv,
                      limint, extint)) case "failed" => "failed"
      v::UP / u.t

    rischDEexp(twr, f, g, x, theta, driv, limint, extint) ==
      (u := monomRDE(f, g, driv)) case "failed" => "failed"
      (v := gpolDEexp(twr, u.a, RF2GP(u.b), RF2GP(u.c), x, theta, driv,
                      limint, extint)) case "failed" => "failed"
      convert(v::GP)@RF / u.t::RF

    polyDElog(twr, aa, bb, cc, x, t, driv, limint, extint) ==
      zero? cc => 0
      t' := differentiate(t::F, x)
      zero? bb =>
        (u := cc exquo aa) case "failed" => "failed"
        primintfldpoly(u::UP, z1 +-> extint(z1, t'), t')
      n := degree(cc)::Z - (db := degree(bb)::Z)
      if ((da := degree(aa)::Z) = db) and (da > 0) then
        lk0 := tower(f0 :=
                      - (leadingCoefficient bb) / (leadingCoefficient aa))
        lk1 := logdiff(twr, lk0)
        (if0 := limint(f0, [first argument u for u in lk1]))
                       case "failed" => error "Risch's theorem violated"
        (alph := validExponential(lk0, RRF2F(if0::RRF), x)) case F =>
          return
            (ans := polyDElog(twr, alph::F * aa,
              differentiate(alph::F, x) * aa + alph::F * bb,
               cc, x, t, driv, limint, extint)) case "failed" => "failed"
            alph::F * ans::UP
```

```
  if (da > db + 1) then n := max(0, degree(cc)::Z - da + 1)
  if (da = db + 1) then
    i := limint(- (leadingCoefficient bb) / (leadingCoefficient aa),
                [first argument t])
    if not(i case "failed") then
      r :=
        null(i.limitedlogs) => 0$F
        i.limitedlogs.first.coeff
      if (nn := retractIfCan(r)@Union(Z, "failed")) case Z then
        n := max(nn::Z, n)
  (v := polyRDE(aa, bb, cc, n, driv)) case ans =>
       v.ans.nosol => "failed"
       v.ans.ans
  w := v.eq
  zero?(w.b) =>
    degree(w.c) > w.m => "failed"
    (u := primintfldpoly(w.c, z1+->extint(1,t'), t'))
       case "failed" => "failed"
    degree(u::UP) > w.m => "failed"
    w.alpha * u::UP + w.beta
  (u := logdegrad(twr, retract(w.b), w.c, w.m, x, t, limint, extint))
    case "failed" => "failed"
  w.alpha * u::UP + w.beta

gpolDEexp(twr, a, b, c, x, t, driv, limint, extint) ==
  zero? c => 0
  zero? b =>
    (u := c exquo (a::GP)) case "failed" => "failed"
    expintfldpoly(u::GP,
       (z1,z2) +-> rischDE(z1, first argument t, z2, x, limint, extint))
  lb := boundAt0(twr, - coefficient(b, 0) / coefficient(a, 0),
                 nb := order b, nc := order c, x, t, limint)
  tm := monomial(1, (m := max(0, max(-nb, lb - nc)))::N)$UP
  (v := polyDEexp(twr,a * tm,lb * differentiate(first argument t, x)
       * a * tm + retract(b * tm::GP)@UP,
           retract(c * monomial(1, m - lb))@UP,
              x, t, driv, limint, extint)) case "failed" => "failed"
  v::UP::GP * monomial(1, lb)

polyDEexp(twr, aa, bb, cc, x, t, driv, limint, extint) ==
  zero? cc => 0
  zero? bb =>
    (u := cc exquo aa) case "failed" => "failed"
    exppolyint(u::UP,
      (z1,z2) +-> rischDE(z1, first argument t, z2, x, limint, extint))
  n := boundInf(twr,-leadingCoefficient(bb) / (leadingCoefficient aa),
```

```
            degree(aa)::Z, degree(bb)::Z, degree(cc)::Z, x, t, limint)
  (v := polyRDE(aa, bb, cc, n, driv)) case ans =>
       v.ans.nosol => "failed"
       v.ans.ans
  w := v.eq
  zero?(w.b) =>
    degree(w.c) > w.m => "failed"
    (u := exppolyint(w.c,
      (z1,z2) +-> rischDE(z1, first argument t, z2, x, limint, extint)))
                    case "failed" => "failed"
    w.alpha * u::UP + w.beta
  (u := expdegrad(twr, retract(w.b), w.c, w.m, x, t, limint, extint))
    case "failed" => "failed"
  w.alpha * u::UP + w.beta

exppolyint(p, rischdiffeq) ==
  (u := expintfldpoly(p::GP, rischdiffeq)) case "failed" => "failed"
  retractIfCan(u::GP)@Union(UP, "failed")

boundInf(twr, f0, da, db, dc, x, t, limitedint) ==
  da < db => dc - db
  da > db => max(0, dc - da)
  l1 := logdiff(twr, l0 := tower f0)
  (if0 := limitedint(f0, [first argument u for u in l1]))
                  case "failed" => error "Risch's theorem violated"
  (alpha := validExponential(concat(t, l0), RRF2F(if0::RRF), x))
   case F =>
    al := separate(univariate(alpha::F, t))$GP
    zero?(al.fracPart) and monomial?(al.polyPart) =>
                          max(0, max(degree(al.polyPart), dc - db))
    dc - db
  dc - db

boundAt0(twr, f0, nb, nc, x, t, limitedint) ==
  nb ^= 0 => min(0, nc - min(0, nb))
  l1 := logdiff(twr, l0 := tower f0)
  (if0 := limitedint(f0, [first argument u for u in l1]))
                  case "failed" => error "Risch's theorem violated"
  (alpha := validExponential(concat(t, l0), RRF2F(if0::RRF), x))
   case F =>
    al := separate(univariate(alpha::F, t))$GP
    zero?(al.fracPart) and monomial?(al.polyPart) =>
                               min(0, min(degree(al.polyPart), nc))
    min(0, nc)
  min(0, nc)
```

```
-- case a = 1, deg(B) = 0, B <> 0
-- cancellation at infinity is possible
    logdegrad(twr, b, c, n, x, t, limitedint, extint) ==
      t'  := differentiate(t::F, x)
      lk1 := logdiff(twr, lk0 := tower(f0 := - b))
      (if0 := limitedint(f0, [first argument u for u in lk1]))
                      case "failed" => error "Risch's theorem violated"
      (alpha := validExponential(lk0, RRF2F(if0::RRF), x)) case F =>
        (u1 := primintfldpoly(inv(alpha::F) * c, z1+->extint(z1, t'), t'))
                                                case "failed" => "failed"
        degree(u1::UP)::Z > n => "failed"
        alpha::F * u1::UP
      logdeg(c, - if0.mainpart -
              +/[v.coeff * log(v.logand) for v in if0.limitedlogs],
                                          n, x, t', limitedint, extint)

-- case a = 1, degree(b) = 0, and (exp integrate b) is not in F
-- this implies no cancellation at infinity
    logdeg(c, f, n, x, t', limitedint, extint) ==
      answr:UP := 0
      repeat
        zero? c => return answr
        (n < 0) or ((m := degree c)::Z > n) => return "failed"
        u := rischDE(1, f, leadingCoefficient c, x, limitedint, extint)
        ~u.sol? => return "failed"
        zero? m => return(answr + u.ans::UP)
        n   := m::Z - 1
        c   := (reductum c) - monomial(m::Z * t' * u.ans, (m - 1)::N)
        answr := answr + monomial(u.ans, m)

-- case a = 1, deg(B) = 0, B <> 0
-- cancellation at infinity is possible
    expdegrad(twr, b, c, n, x, t, limint, extint) ==
      lk1 := logdiff(twr, lk0 := tower(f0 := - b))
      (if0 := limint(f0, [first argument u for u in lk1]))
                      case "failed" => error "Risch's theorem violated"
      intf0 := - if0.mainpart -
                   +/[v.coeff * log(v.logand) for v in if0.limitedlogs]
      (alpha := validExponential(concat(t, lk0), RRF2F(if0::RRF), x))
       case F =>
        al := separate(univariate(alpha::F, t))$GP
        zero?(al.fracPart) and monomial?(al.polyPart) and
         (degree(al.polyPart) >= 0) =>
          (u1 := expintfldpoly(c::GP * recip(al.polyPart)::GP,
            (z1,z2) +-> rischDE(z1, first argument t, z2, x, limint, extint)))
                                                case "failed" => "failed"
```

```
            degree(u1::GP) > n => "failed"
            retractIfCan(al.polyPart * u1::GP)@Union(UP, "failed")
          expdeg(c, intf0, n, x, first argument t, limint,extint)
        expdeg(c, intf0, n, x, first argument t, limint, extint)

  -- case a = 1, degree(b) = 0, and (exp integrate b) is not a monomial
  -- this implies no cancellation at infinity
      expdeg(c, f, n, x, eta, limitedint, extint) ==
        answr:UP := 0
        repeat
          zero? c => return answr
          (n < 0) or ((m := degree c)::Z > n) => return "failed"
          u := rischDE(1, f + m * eta, leadingCoefficient c, x,limitedint,extint)
          ~u.sol? => return "failed"
          zero? m => return(answr + u.ans::UP)
          n   := m::Z - 1
          c   := reductum c
          answr := answr + monomial(u.ans, m)

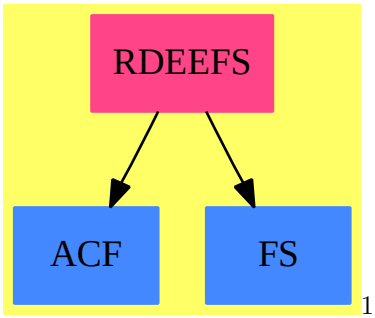
      RRF2F rrf ==
        rrf.mainpart + +/[v.coeff*log(v.logand) for v in rrf.limitedlogs]
```

⟨*RDEEF.dotabb*⟩≡

```
  "RDEEF" [color="#FF4488",href="bookvol10.4.pdf#nameddest=RDEEF"]
  "ACF" [color="#4488FF",href="bookvol10.2.pdf#nameddest=ACF"]
  "FS" [color="#4488FF",href="bookvol10.2.pdf#nameddest=FS"]
  "RDEEF" -> "ACF"
  "RDEEF" -> "FS"
```

6.23 package RDEEFS ElementaryRischDESystem

6.24 ElementaryRischDESystem



1.00

Exports:
rischDEsys

⟨*package RDEEFS ElementaryRischDESystem*⟩≡

```
)abbrev package RDEEFS ElementaryRischDESystem
++ Risch differential equation, elementary case.
++ Author: Manuel Bronstein
++ Date Created: 12 August 1992
++ Date Last Updated: 17 August 1992
++ Keywords: elementary, function, integration.
ElementaryRischDESystem(R, F): Exports == Implementation where
  R : Join(GcdDomain, OrderedSet, CharacteristicZero,
           RetractableTo Integer, LinearlyExplicitRingOver Integer)
  F : Join(TranscendentalFunctionCategory, AlgebraicallyClosedField,
           FunctionSpace R)

  Z   ==> Integer
  SE  ==> Symbol
  K   ==> Kernel F
  P   ==> SparseMultivariatePolynomial(R, K)
  UP  ==> SparseUnivariatePolynomial F
  RF  ==> Fraction UP
  NL  ==> Record(coeff:F,logand:F)
  RRF ==> Record(mainpart:F,limitedlogs:List NL)
  U   ==> Union(RRF, "failed")
  ULF ==> Union(List F, "failed")
  UEX ==> Union(Record(ratpart:F, coeff:F), "failed")

  Exports ==> with
    rischDEsys: (Z, F, F, F, SE, (F, List F) -> U, (F, F) -> UEX) -> ULF
```

```
      ++ rischDEsys(n, f, g_1, g_2, x,lim,ext) returns \spad{y_1.y_2} such that
      ++ \spad{(dy1/dx,dy2/dx) + ((0, - n df/dx),(n df/dx,0)) (y1,y2) = (g1,g2)}
      ++ if \spad{y_1,y_2} exist, "failed" otherwise.
      ++ lim is a limited integration function,
      ++ ext is an extended integration function.

  Implementation ==> add
    import IntegrationTools(R, F)
    import ElementaryRischDE(R, F)
    import TranscendentalRischDESystem(F, UP)
    import PolynomialCategoryQuotientFunctions(IndexedExponents K,
                                                        K, R, P, F)

--  sm1 := sqrt(-1::F)
--  ks1 := retract(sm1)@K

--  gcoeffs    : P -> ULF
--  gets1coeffs: F -> ULF
--  cheat      : (Z, F, F, F, SE, (F, List F) -> U, (F, F) -> UEX) -> ULF
    basecase   : (F, F, F, K) -> ULF

-- solve (y1',y2') + ((0, -nfp), (nfp, 0)) (y1,y2) = (g1, g2), base case
    basecase(nfp, g1, g2, k) ==
      (ans := baseRDEsys(univariate(nfp, k), univariate(g1, k),
                         univariate(g2, k))) case "failed" => "failed"
      l := ans::List(RF)
      [multivariate(first l, k), multivariate(second l, k)]

-- returns [x,y] s.t. f = x + y %i
-- f can be of the form (a + b %i) / (c + d %i)
--  gets1coeffs f ==
--    (lnum := gcoeffs(numer f)) case "failed" => "failed"
--    (lden := gcoeffs(denom f)) case "failed" => "failed"
--    a := first(lnum::List F)
--    b := second(lnum::List F)
--    c := first(lden::List F)
--    zero?(d := second(lden::List F)) => [a/c, b/c]
--    cd := c * c + d * d
--    [(a * c + b * d) / cd, (b * c - a * d) / cd]

--  gcoeffs p ==
--    degree(q := univariate(p, ks1)) > 1 => "failed"
--    [coefficient(q, 0)::F, coefficient(q, 1)::F]

--  cheat(n, f, g1, g2, x, limint, extint) ==
--    (u := rischDE(n, sm1 * f, g1 + sm1 * g2, x, limint, extint))
```

```
--      case "failed" => "failed"
--    (l := gets1coeffs(u::F)) case "failed" =>
--      error "rischDEsys: expect linear result in sqrt(-1)"
--    l::List F

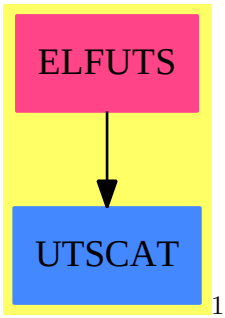
-- solve (y1',y2') + ((0, -n f'), (n f', 0)) (y1,y2) = (g1, g2)
    rischDEsys(n, f, g1, g2, x, limint, extint) ==
      zero? g1 and zero? g2 => [0, 0]
      zero?(nfp := n * differentiate(f, x)) =>
        ((u1 := limint(g1, empty())) case "failed") or
          ((u2 := limint(g1, empty())) case "failed") => "failed"
        [u1.mainpart, u2.mainpart]
      freeOf?(y1 := g2 / nfp, x) and freeOf?(y2 := - g1 / nfp, x) => [y1, y2]
      vl := varselect(union(kernels nfp, union(kernels g1, kernels g2)), x)
      symbolIfCan(k := kmax vl) case SE => basecase(nfp, g1, g2, k)
--    cheat(n, f, g1, g2, x, limint, extint)
      error "rischDEsys: can only handle rational functions for now"
```

⟨*RDEEFS.dotabb*⟩≡

```
"RDEEFS" [color="#FF4488",href="bookvol10.4.pdf#nameddest=RDEEFS"]
"ACF" [color="#4488FF",href="bookvol10.2.pdf#nameddest=ACF"]
"FS" [color="#4488FF",href="bookvol10.2.pdf#nameddest=FS"]
"RDEEFS" -> "ACF"
"RDEEFS" -> "FS"
```

6.25 package ELFUTS EllipticFunctionsUnivariateTaylorSeries

6.26 EllipticFunctionsUnivariateTaylorSeries



1.00

Exports:
cn dn sn sncndn

⟨*package ELFUTS EllipticFunctionsUnivariateTaylorSeries*⟩≡

```
)abbrev package ELFUTS EllipticFunctionsUnivariateTaylorSeries
++ Elliptic functions expanded as Taylor series
++ Author: Bill Burge, Clifton J. Williamson
++ Date Created: 1986
++ Date Last Updated: 17 February 1992
++ Keywords: elliptic function, Taylor series
++ Examples:
++ References:
++ Description: The elliptic functions sn, sc and dn are expanded as
++ Taylor series.
EllipticFunctionsUnivariateTaylorSeries(Coef,UTS):
 Exports == Implementation where
  Coef : Field
  UTS  : UnivariateTaylorSeriesCategory Coef

  L   ==> List
  I   ==> Integer
  RN  ==> Fraction Integer
  ST  ==> Stream Coef
  STT ==> StreamTaylorSeriesOperations Coef
  YS  ==> Y$ParadoxicalCombinatorsForStreams(Coef)

  Exports ==> with
    sn     : (UTS,Coef) -> UTS
      ++\spad{sn(x,k)} expands the elliptic function sn as a Taylor
      ++ series.
```

```
    cn     : (UTS,Coef) -> UTS
      ++\spad{cn(x,k)} expands the elliptic function cn as a Taylor
      ++ series.
    dn     : (UTS,Coef) -> UTS
      ++\spad{dn(x,k)} expands the elliptic function dn as a Taylor
      ++ series.
    sncndn: (ST,Coef) -> L ST
       ++\spad{sncndn(s,c)} is used internally.

  Implementation ==> add
    import StreamTaylorSeriesOperations Coef
    UPS==> StreamTaylorSeriesOperations Coef
    integrate ==> lazyIntegrate
    sncndnre:(Coef,L ST,ST,Coef) -> L ST
    sncndnre(k,scd,dx,sign) ==
            [integrate(0,      scd.2*$UPS scd.3*$UPS dx),  _
             integrate(1,  sign*scd.1*$UPS scd.3*$UPS dx),  _
             integrate(1,sign*k**2*$UPS scd.1*$UPS scd.2*$UPS dx)]

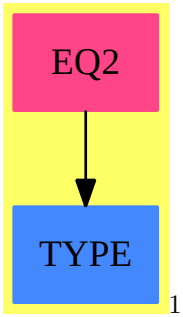
    sncndn(z,k) ==
      empty? z => [0 :: ST,1 :: ST,1::ST]
      frst z = 0 => YS(x +-> sncndnre(k,x,deriv z,-1),3)
      error "ELFUTS:sncndn: constant coefficient should be 0"
    sn(x,k)  == series sncndn.(coefficients x,k).1
    cn(x,k)  == series sncndn.(coefficients x,k).2
    dn(x,k)  == series sncndn.(coefficients x,k).3
```

⟨*ELFUTS.dotabb*⟩≡

```
"ELFUTS" [color="#FF4488",href="bookvol10.4.pdf#nameddest=ELFUTS"]
"UTSCAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=UTSCAT"]
"ELFUTS" -> "UTSCAT"
```

6.27 package EQ2 EquationFunctions2

6.28 EquationFunctions2



1.00

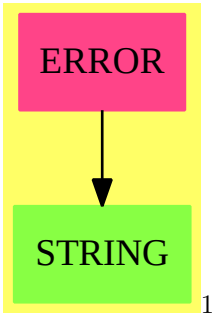
Exports:
map

⟨*package EQ2 EquationFunctions2*⟩≡
```
)abbrev package EQ2 EquationFunctions2
++ Author:
++ Date Created:
++ Date Last Updated: June 3, 1991
++ Basic Operations:
++ Related Domains: Equation
++ Also See:
++ AMS Classifications:
++ Keywords: equation
++ Examples:
++ References:
++ Description:
++   This package provides operations for mapping the sides of equations.
EquationFunctions2(S: Type, R: Type): with
    map: (S ->R ,Equation S) -> Equation R
        ++ map(f,eq) returns an equation where f is applied to the sides of eq
 == add
    map(fn, eqn) == equation(fn lhs eqn, fn rhs eqn)
```

⟨*EQ2.dotabb*⟩≡
```
"EQ2" [color="#FF4488",href="bookvol10.4.pdf#nameddest=EQ2"]
"TYPE" [color="#4488FF",href="bookvol10.2.pdf#nameddest=TYPE"]
"EQ2" -> "TYPE"
```

6.29 package ERROR ErrorFunctions

6.30 ErrorFunctions



1.00

Exports:
error

⟨*package ERROR ErrorFunctions*⟩≡
```
)abbrev package ERROR ErrorFunctions
++ Author: Robert S. Sutor
++ Date Created: 29 May 1990
++ Date Last Updated: 29 May 1990
++ Description:
++ ErrorFunctions implements error functions callable from the system
++ interpreter.  Typically, these functions would be called in user
++ functions.  The simple forms of the functions take one argument
++ which is either a string (an error message) or a list of strings
++ which all together make up a message.  The list can contain
++ formatting codes (see below).  The more sophisticated versions takes
++ two arguments where the first argument is the name of the function
++ from which the error was invoked and the second argument is either a
++ string or a list of strings, as above.  When you use the one
++ argument version in an interpreter function, the system will
++ automatically insert the name of the function as the new first
++ argument.  Thus in the user interpreter function
++   \spad{f x == if x < 0 then error "negative argument" else x}
++ the call to error will actually be of the form
++   \spad{error("f","negative argument")}
++ because the interpreter will have created a new first argument.
++
++ Formatting codes:  error messages may contain the following
++ formatting codes (they should either start or end a string or
++ else have blanks around them):
++    \spad{%l}      start a new line
++    \spad{%b}      start printing in a bold font (where available)
```

```
++    \spad{%d}      stop  printing in a bold font (where available)
++    \spad{ %ceon}  start centering message lines
++    \spad{%ceoff}  stop  centering message lines
++    \spad{%rjon}   start displaying lines "ragged left"
++    \spad{%rjoff}  stop  displaying lines "ragged left"
++    \spad{%i}      indent   following lines 3 additional spaces
++    \spad{%u}      unindent following lines 3 additional spaces
++    \spad{%xN}     insert N blanks (eg, \spad{%x10} inserts 10 blanks)
++
++ Examples:
++   1.  \spad{error "Whoops, you made a %l %ceon %b big %d %ceoff %l mistake!"}
++   2.  \spad{error ["Whoops, you made a","%l %ceon %b","big",
++               "%d %ceoff %l","mistake!"]}

ErrorFunctions() : Exports == Implementation where
  Exports ==> with
    error: String -> Exit
      ++ error(msg) displays error message msg and terminates.
    error: List String -> Exit
      ++ error(lmsg) displays error message lmsg and terminates.
    error: (String,String) -> Exit
      ++ error(nam,msg) displays error message msg preceded by a
      ++ message containing the name nam of the function in which
      ++ the error is contained.
    error: (String,List String) -> Exit
      ++ error(nam,lmsg) displays error messages lmsg preceded by a
      ++ message containing the name nam of the function in which
      ++ the error is contained.
  Implementation ==> add

    prefix1 : String := "Error signalled from user code: %l "
    prefix2 : String := "Error signalled from user code in function %b "

    doit(s : String) : Exit ==
      throwPatternMsg(s,nil$(List String))$Lisp
      -- there are no objects of type Exit, so we'll fake one,
      -- knowing we will never get to this step anyway.
      "exit" pretend Exit

    error(s : String) : Exit ==
      doit concat [prefix1,s]

    error(l : List String) : Exit ==
      s : String := prefix1
      for x in l repeat s := concat [s," ",x]
      doit s
```

```
error(fn : String,s : String) : Exit ==
  doit concat [prefix2,fn,": %d %l ",s]

error(fn : String, l : List String) : Exit ==
  s : String := concat [prefix2,fn,": %d %l"]
  for x in l repeat s := concat [s," ",x]
  doit s
```

⟨*ERROR.dotabb*⟩≡

```
"ERROR" [color="#FF4488",href="bookvol10.4.pdf#nameddest=ERROR"]
"STRING" [color="#88FF44",href="bookvol10.3.pdf#nameddest=STRING"]
"ERROR" -> "STRING"
```

6.31 package GBEUCLID EuclideanGroebnerBasisPackage

⟨*EuclideanGroebnerBasisPackage.input*⟩≡

```
)set break resume
)sys rm -f EuclideanGroebnerBasisPackage.output
)spool EuclideanGroebnerBasisPackage.output
)set message test on
)set message auto off
)clear all
--S 1 of 24
a1:DMP([y,x],INT):= (9*x**2 + 5*x - 3)+ y*(3*x**2 + 2*x + 1)
--R
--R            2                2
--R   (1)  3y x  + 2y x + y + 9x  + 5x - 3
--R                       Type: DistributedMultivariatePolynomial([y,x],Integer)
--E 1

--S 2 of 24
a2:DMP([y,x],INT):= (6*x**3 - 2*x**2 - 3*x +3) + y*(2*x**3 - x - 1)
--R
--R            3               3     2
--R   (2)  2y x  - y x - y + 6x  - 2x  - 3x + 3
--R                       Type: DistributedMultivariatePolynomial([y,x],Integer)
--E 2

--S 3 of 24
a3:DMP([y,x],INT):= (3*x**3 + 2*x**2) + y*(x**3 + x**2)
--R
--R           3      2     3     2
--R   (3)  y x  + y x  + 3x  + 2x
--R                       Type: DistributedMultivariatePolynomial([y,x],Integer)
--E 3

--S 4 of 24
an:=[a1,a2,a3]
--R
--R   (4)
--R        2                2               3               3     2
--R   [3y x  + 2y x + y + 9x  + 5x - 3, 2y x  - y x - y + 6x  - 2x  - 3x + 3,
--R       3      2     3     2
--R    y x  + y x  + 3x  + 2x ]
--R                  Type: List DistributedMultivariatePolynomial([y,x],Integer)
--E 4
```

```
--S 5 of 24
euclideanGroebner(an)
--R
--R                                    2            3     2
--R   (5)  [y x - y + x + 3,2y + 2x  - 3x - 6,2x  - 5x  - 5x]
--R                  Type: List DistributedMultivariatePolynomial([y,x],Integer)
--E 5

--S 6 of 24
euclideanGroebner(an,"redcrit")
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R         2               2
--R   - 2y x  - y x - y - 6x  - 3x + 3
--R
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R   y x - y + x + 3
--R
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R          2
--R   4y + 4x  - 6x - 12
--R
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R       3      2
--R   - 4x  + 10x  + 10x
--R
--R
--R
```

```
--R
--R    reduced Critpair - Polynom :
--R
--R
--R          2
--R   2y + 2x  - 3x - 6
--R
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R   0
--R
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R       3     2
--R   - 2x  + 5x  + 5x
--R
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R   0
--R
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R   0
--R
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
```

```
--R   0
--R
--R
--R       THE GROEBNER BASIS over EUCLIDEAN DOMAIN
--R
--R                                  2            3     2
--R   (6)  [y x - y + x + 3,2y + 2x  - 3x - 6,2x  - 5x  - 5x]
--R                  Type: List DistributedMultivariatePolynomial([y,x],Integer)
--E 6

--S 7 of 24
euclideanGroebner(an,"info")
--R
--R   you choose option  -info-
--R   abbrev. for the following information strings are
--R     ci  =>  Leading monomial  for critpair calculation
--R     tci =>  Number of terms of polynomial i
--R     cj  =>  Leading monomial  for critpair calculation
--R     tcj =>  Number of terms of polynomial j
--R     c   =>  Leading monomial of critpair polynomial
--R     tc  =>  Number of terms of critpair polynomial
--R     rc  =>  Leading monomial of redcritpair polynomial
--R     trc =>  Number of terms of redcritpair polynomial
--R     tF  =>  Number of polynomials in reduction list F
--R     tD  =>  Number of critpairs still to do
--R
--R
--R
--R
--R
--R            3               3              2              2
--R   [[ci= y x ,tci= 7,cj= y x ,tcj= 4,c= y x ,tc= 6,rc= y x ,trc= 6,tH= 3,tD= 3
--R
--R
--R            2               2
--R   [[ci= y x ,tci= 6,cj= y x ,tcj= 6,c= y x,tc= 4,rc= y x,trc= 4,tH= 1,tD= 3]]
--R
--R
--R            2
--R   [[ci= y x ,tci= 6,cj= y x,tcj= 4,c= y x,tc= 5,rc= y,trc= 4,tH= 2,tD= 3]]
--R
--R
--R                                                    3
--R   [[ci= y x,tci= 4,cj= y,tcj= 4,c= y,tc= 5,rc= x ,trc= 3,tH= 3,tD= 3]]
--R
--R
```

```
--R            2
--R   [[ci= y x ,tci= 6,cj= y x,tcj= 4,c= y x,tc= 5,rc= y,trc= 4,tH= 3,tD= 4]]
--R
--R
--R   [[ci= y,tci= 4,cj= y,tcj= 4,c= 0,tc= 0,rc= 0,trc= 0,tH= 3,tD= 3]]
--R
--R
--R                                                    3
--R   [[ci= y x,tci= 4,cj= y,tcj= 4,c= y,tc= 5,rc= x ,trc= 3,tH= 3,tD= 3]]
--R
--R
--R          3             3
--R   [[ci= x ,tci= 3,cj= x ,tcj= 3,c= 0,tc= 0,rc= 0,trc= 0,tH= 3,tD= 2]]
--R
--R
--R            3                               2
--R   [[ci= y x ,tci= 4,cj= y x,tcj= 4,c= y x ,tc= 3,rc= 0,trc= 0,tH= 3,tD= 1]]
--R
--R
--R                       3              2
--R   [[ci= y,tci= 4,cj= x ,tcj= 3,c= y x ,tc= 5,rc= 0,trc= 0,tH= 3,tD= 0]]
--R
--R
--R     There are
--R
--R   3
--R
--R     Groebner Basis Polynomials.
--R
--R
--R       THE GROEBNER BASIS over EUCLIDEAN DOMAIN
--R
--R                                  2            3     2
--R   (7)  [y x - y + x + 3,2y + 2x  - 3x - 6,2x  - 5x  - 5x]
--R                  Type: List DistributedMultivariatePolynomial([y,x],Integer)
--E 7

--S 8 of 24
euclideanGroebner(an,"info","redcrit")
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R         2               2
--R   - 2y x  - y x - y - 6x  - 3x + 3
```

```
--R
--R
--R
--R   you choose option  -info-
--R   abbrev. for the following information strings are
--R     ci  =>  Leading monomial  for critpair calculation
--R     tci =>  Number of terms of polynomial i
--R     cj  =>  Leading monomial  for critpair calculation
--R     tcj =>  Number of terms of polynomial j
--R     c   =>  Leading monomial of critpair polynomial
--R     tc  =>  Number of terms of critpair polynomial
--R     rc  =>  Leading monomial of redcritpair polynomial
--R     trc =>  Number of terms of redcritpair polynomial
--R     tF  =>  Number of polynomials in reduction list F
--R     tD  =>  Number of critpairs still to do
--R
--R
--R
--R
--R
--R           3               3              2              2
--R   [[ci= y x ,tci= 7,cj= y x ,tcj= 4,c= y x ,tc= 6,rc= y x ,trc= 6,tH= 3,tD= 3
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R   y x - y + x + 3
--R
--R
--R
--R           2               2
--R   [[ci= y x ,tci= 6,cj= y x ,tcj= 6,c= y x,tc= 4,rc= y x,trc= 4,tH= 1,tD= 3]]
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R          2
--R   4y + 4x  - 6x - 12
--R
--R
--R
--R           2
```

```
--R   [[ci= y x ,tci= 6,cj= y x,tcj= 4,c= y x,tc= 5,rc= y,trc= 4,tH= 2,tD= 3]]
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R       3      2
--R   - 4x  + 10x  + 10x
--R
--R
--R
--R                                                    3
--R   [[ci= y x,tci= 4,cj= y,tcj= 4,c= y,tc= 5,rc= x ,trc= 3,tH= 3,tD= 3]]
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R          2
--R   2y + 2x  - 3x - 6
--R
--R
--R
--R            2
--R   [[ci= y x ,tci= 6,cj= y x,tcj= 4,c= y x,tc= 5,rc= y,trc= 4,tH= 3,tD= 4]]
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R   0
--R
--R
--R
--R   [[ci= y,tci= 4,cj= y,tcj= 4,c= 0,tc= 0,rc= 0,trc= 0,tH= 3,tD= 3]]
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R       3     2
--R   - 2x  + 5x  + 5x
```

```
--R
--R
--R
--R                                                    3
--R   [[ci= y x,tci= 4,cj= y,tcj= 4,c= y,tc= 5,rc= x ,trc= 3,tH= 3,tD= 3]]
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R   0
--R
--R
--R
--R          3             3
--R   [[ci= x ,tci= 3,cj= x ,tcj= 3,c= 0,tc= 0,rc= 0,trc= 0,tH= 3,tD= 2]]
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R   0
--R
--R
--R
--R            3                               2
--R   [[ci= y x ,tci= 4,cj= y x,tcj= 4,c= y x ,tc= 3,rc= 0,trc= 0,tH= 3,tD= 1]]
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R   0
--R
--R
--R
--R                        3              2
--R   [[ci= y,tci= 4,cj= x ,tcj= 3,c= y x ,tc= 5,rc= 0,trc= 0,tH= 3,tD= 0]]
--R
--R
--R     There are
--R
--R   3
```

```
--R
--R     Groebner Basis Polynomials.
--R
--R
--R       THE GROEBNER BASIS over EUCLIDEAN DOMAIN
--R
--R                                  2            3     2
--R   (8)  [y x - y + x + 3,2y + 2x  - 3x - 6,2x  - 5x  - 5x]
--R                  Type: List DistributedMultivariatePolynomial([y,x],Integer)
--E 8

--S 9 of 24
b1:HDMP([y,x],INT):= (9*x**2 + 5*x - 3)+ y*(3*x**2 + 2*x + 1)
--R
--R            2            2
--R   (9)  3y x  + 2y x + 9x  + y + 5x - 3
--R            Type: HomogeneousDistributedMultivariatePolynomial([y,x],Integer)
--E 9

--S 10 of 24
b2:HDMP([y,x],INT):= (6*x**3 - 2*x**2 - 3*x +3) + y*(2*x**3 - x - 1)
--R
--R             3     3           2
--R   (10)  2y x  + 6x  - y x - 2x  - y - 3x + 3
--R            Type: HomogeneousDistributedMultivariatePolynomial([y,x],Integer)
--E 10

--S 11 of 24
b3:HDMP([y,x],INT):= (3*x**3 + 2*x**2) + y*(x**3 + x**2)
--R
--R            3      2     3     2
--R   (11)  y x  + y x  + 3x  + 2x
--R            Type: HomogeneousDistributedMultivariatePolynomial([y,x],Integer)
--E 11

--S 12 of 24
bn:=[b1,b2,b3]
--R
--R   (12)
--R        2            2                   3     3           2
--R   [3y x  + 2y x + 9x  + y + 5x - 3, 2y x  + 6x  - y x - 2x  - y - 3x + 3,
--R       3      2     3     2
--R    y x  + y x  + 3x  + 2x ]
--R       Type: List HomogeneousDistributedMultivariatePolynomial([y,x],Integer)
--E 12
```

```
--S 13 of 24
euclideanGroebner(bn)
--R
--R            2                                    2
--R   (13)  [2y  - 5y - 8x - 3,y x - y + x + 3,2x  + 2y - 3x - 6]
--R       Type: List HomogeneousDistributedMultivariatePolynomial([y,x],Integer)
--E 13

--S 14 of 24
euclideanGroebner(bn,"redcrit")
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R         2           2
--R   - 2y x  - y x - 6x  - y - 3x + 3
--R
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R   y x - y + x + 3
--R
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R     2
--R   4x  + 4y - 6x - 12
--R
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R     2
--R   2x  + 2y - 3x - 6
--R
--R
--R
```

```
--R
--R    reduced Critpair - Polynom :
--R
--R
--R   0
--R
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R       2
--R   - 2y  + 5y + 8x + 3
--R
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R   0
--R
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R   0
--R
--R
--R       THE GROEBNER BASIS over EUCLIDEAN DOMAIN
--R
--R            2                                  2
--R   (14)  [2y  - 5y - 8x - 3,y x - y + x + 3,2x  + 2y - 3x - 6]
--R       Type: List HomogeneousDistributedMultivariatePolynomial([y,x],Integer)
--E 14

--S 15 of 24
euclideanGroebner(bn,"info")
--R
--R   you choose option  -info-
--R   abbrev. for the following information strings are
--R     ci  =>  Leading monomial  for critpair calculation
--R     tci =>  Number of terms of polynomial i
```

```
--R     cj  =>  Leading monomial  for critpair calculation
--R     tcj =>  Number of terms of polynomial j
--R     c   =>  Leading monomial of critpair polynomial
--R     tc  =>  Number of terms of critpair polynomial
--R     rc  =>  Leading monomial of redcritpair polynomial
--R     trc =>  Number of terms of redcritpair polynomial
--R     tF  =>  Number of polynomials in reduction list F
--R     tD  =>  Number of critpairs still to do
--R
--R
--R
--R
--R
--R            3               3              2              2
--R   [[ci= y x ,tci= 7,cj= y x ,tcj= 4,c= y x ,tc= 6,rc= y x ,trc= 6,tH= 3,tD= 3
--R
--R
--R            2               2
--R   [[ci= y x ,tci= 6,cj= y x ,tcj= 6,c= y x,tc= 4,rc= y x,trc= 4,tH= 1,tD= 3]]
--R
--R
--R            2                                          2
--R   [[ci= y x ,tci= 6,cj= y x,tcj= 4,c= y x,tc= 5,rc= x ,trc= 4,tH= 2,tD= 3]]
--R
--R
--R            2                                          2
--R   [[ci= y x ,tci= 6,cj= y x,tcj= 4,c= y x,tc= 5,rc= x ,trc= 4,tH= 2,tD= 3]]
--R
--R
--R          2             2
--R   [[ci= x ,tci= 4,cj= x ,tcj= 4,c= 0,tc= 0,rc= 0,trc= 0,tH= 2,tD= 2]]
--R
--R
--R                         2            2            2
--R   [[ci= y x,tci= 4,cj= x ,tcj= 4,c= y ,tc= 5,rc= y ,trc= 4,tH= 3,tD= 2]]
--R
--R
--R                         2            2
--R   [[ci= y x,tci= 4,cj= y ,tcj= 4,c= y ,tc= 5,rc= 0,trc= 0,tH= 3,tD= 1]]
--R
--R
--R            3                               2
--R   [[ci= y x ,tci= 4,cj= y x,tcj= 4,c= y x ,tc= 3,rc= 0,trc= 0,tH= 3,tD= 0]]
--R
--R
--R     There are
```

```
--R
--R   3
--R
--R     Groebner Basis Polynomials.
--R
--R
--R       THE GROEBNER BASIS over EUCLIDEAN DOMAIN
--R
--R            2                                  2
--R   (15)  [2y  - 5y - 8x - 3,y x - y + x + 3,2x  + 2y - 3x - 6]
--R       Type: List HomogeneousDistributedMultivariatePolynomial([y,x],Integer)
--E 15

--S 16 of 24
euclideanGroebner(bn,"info","redcrit")
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R         2           2
--R   - 2y x  - y x - 6x  - y - 3x + 3
--R
--R
--R
--R   you choose option  -info-
--R   abbrev. for the following information strings are
--R     ci  =>  Leading monomial  for critpair calculation
--R     tci =>  Number of terms of polynomial i
--R     cj  =>  Leading monomial  for critpair calculation
--R     tcj =>  Number of terms of polynomial j
--R     c   =>  Leading monomial of critpair polynomial
--R     tc  =>  Number of terms of critpair polynomial
--R     rc  =>  Leading monomial of redcritpair polynomial
--R     trc =>  Number of terms of redcritpair polynomial
--R     tF  =>  Number of polynomials in reduction list F
--R     tD  =>  Number of critpairs still to do
--R
--R
--R
--R
--R
--R            3              3              2              2
--R   [[ci= y x ,tci= 7,cj= y x ,tcj= 4,c= y x ,tc= 6,rc= y x ,trc= 6,tH= 3,tD= 3]]
--R
--R
```

```
--R
--R    reduced Critpair - Polynom :
--R
--R
--R   y x - y + x + 3
--R
--R
--R
--R            2               2
--R   [[ci= y x ,tci= 6,cj= y x ,tcj= 6,c= y x,tc= 4,rc= y x,trc= 4,tH= 1,tD= 3]]
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R     2
--R   4x  + 4y - 6x - 12
--R
--R
--R
--R            2                                          2
--R   [[ci= y x ,tci= 6,cj= y x,tcj= 4,c= y x,tc= 5,rc= x ,trc= 4,tH= 2,tD= 3]]
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R     2
--R   2x  + 2y - 3x - 6
--R
--R
--R
--R            2                                          2
--R   [[ci= y x ,tci= 6,cj= y x,tcj= 4,c= y x,tc= 5,rc= x ,trc= 4,tH= 2,tD= 3]]
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R   0
--R
--R
--R
```

```
--R          2             2
--R   [[ci= x ,tci= 4,cj= x ,tcj= 4,c= 0,tc= 0,rc= 0,trc= 0,tH= 2,tD= 2]]
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R       2
--R   - 2y  + 5y + 8x + 3
--R
--R
--R
--R                          2            2            2
--R   [[ci= y x,tci= 4,cj= x ,tcj= 4,c= y ,tc= 5,rc= y ,trc= 4,tH= 3,tD= 2]]
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R   0
--R
--R
--R
--R                          2            2
--R   [[ci= y x,tci= 4,cj= y ,tcj= 4,c= y ,tc= 5,rc= 0,trc= 0,tH= 3,tD= 1]]
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R   0
--R
--R
--R
--R            3                               2
--R   [[ci= y x ,tci= 4,cj= y x,tcj= 4,c= y x ,tc= 3,rc= 0,trc= 0,tH= 3,tD= 0]]
--R
--R
--R     There are
--R
--R   3
--R
--R     Groebner Basis Polynomials.
```

```
--R
--R
--R       THE GROEBNER BASIS over EUCLIDEAN DOMAIN
--R
--R            2                                    2
--R   (16)  [2y  - 5y - 8x - 3,y x - y + x + 3,2x  + 2y - 3x - 6]
--R       Type: List HomogeneousDistributedMultivariatePolynomial([y,x],Integer)
--E 16

--S 17 of 24
c1:GDMP([y,x],INT,DIRPROD(2,NNI)):= (9*x**2 + 5*x - 3)+ y*(3*x**2 + 2*x + 1)
--R
--R             2                2
--R   (17)  3y x  + 2y x + y + 9x  + 5x - 3
--RType: GeneralDistributedMultivariatePolynomial([y,x],Integer,DirectProduct(2,N
--E 17

--S 18 of 24
c2:GDMP([y,x],INT,DIRPROD(2,NNI)):= (6*x**3 - 2*x**2 - 3*x +3) + y*(2*x**3 - x -
--R
--R             3               3     2
--R   (18)  2y x  - y x - y + 6x  - 2x  - 3x + 3
--RType: GeneralDistributedMultivariatePolynomial([y,x],Integer,DirectProduct(2,N
--E 18

--S 19 of 24
c3:GDMP([y,x],INT,DIRPROD(2,NNI)):= (3*x**3 + 2*x**2) + y*(x**3 + x**2)
--R
--R            3      2     3     2
--R   (19)  y x  + y x  + 3x  + 2x
--RType: GeneralDistributedMultivariatePolynomial([y,x],Integer,DirectProduct(2,N
--E 19

--S 20 of 24
cn:=[c1,c2,c3]
--R
--R   (20)
--R        2                2               3               3     2
--R   [3y x  + 2y x + y + 9x  + 5x - 3, 2y x  - y x - y + 6x  - 2x  - 3x + 3,
--R       3      2     3     2
--R    y x  + y x  + 3x  + 2x ]
--RType: List GeneralDistributedMultivariatePolynomial([y,x],Integer,DirectProduc
--E 20

--S 21 of 24
euclideanGroebner(cn)
```

```
--R
--R                                  2            3     2
--R   (21)  [y x - y + x + 3,2y + 2x  - 3x - 6,2x  - 5x  - 5x]
--RType: List GeneralDistributedMultivariatePolynomial([y,x],Integer,DirectProduct(2,NonNega
--E 21

--S 22 of 24
euclideanGroebner(cn,"redcrit")
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R         2               2
--R   - 2y x  - y x - y - 6x  - 3x + 3
--R
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R   y x - y + x + 3
--R
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R          2
--R   4y + 4x  - 6x - 12
--R
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R       3      2
--R   - 4x  + 10x  + 10x
--R
--R
--R
--R
--R    reduced Critpair - Polynom :
```

```
--R
--R
--R          2
--R   2y + 2x  - 3x - 6
--R
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R   0
--R
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R       3     2
--R   - 2x  + 5x  + 5x
--R
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R   0
--R
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R   0
--R
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R   0
--R
```

```
--R
--R       THE GROEBNER BASIS over EUCLIDEAN DOMAIN
--R
--R                                    2            3     2
--R   (22)  [y x - y + x + 3,2y + 2x  - 3x - 6,2x  - 5x  - 5x]
--RType: List GeneralDistributedMultivariatePolynomial([y,x],Integer,DirectProduct(2,NonNega
--E 22

--S 23 of 24
euclideanGroebner(cn,"info")
--R
--R   you choose option  -info-
--R   abbrev. for the following information strings are
--R     ci  =>  Leading monomial  for critpair calculation
--R     tci =>  Number of terms of polynomial i
--R     cj  =>  Leading monomial  for critpair calculation
--R     tcj =>  Number of terms of polynomial j
--R     c   =>  Leading monomial of critpair polynomial
--R     tc  =>  Number of terms of critpair polynomial
--R     rc  =>  Leading monomial of redcritpair polynomial
--R     trc =>  Number of terms of redcritpair polynomial
--R     tF  =>  Number of polynomials in reduction list F
--R     tD  =>  Number of critpairs still to do
--R
--R
--R
--R
--R
--R            3               3              2              2
--R   [[ci= y x ,tci= 7,cj= y x ,tcj= 4,c= y x ,tc= 6,rc= y x ,trc= 6,tH= 3,tD= 3]]
--R
--R
--R            2               2
--R   [[ci= y x ,tci= 6,cj= y x ,tcj= 6,c= y x,tc= 4,rc= y x,trc= 4,tH= 1,tD= 3]]
--R
--R
--R            2
--R   [[ci= y x ,tci= 6,cj= y x,tcj= 4,c= y x,tc= 5,rc= y,trc= 4,tH= 2,tD= 3]]
--R
--R
--R                                                  3
--R   [[ci= y x,tci= 4,cj= y,tcj= 4,c= y,tc= 5,rc= x ,trc= 3,tH= 3,tD= 3]]
--R
--R
--R            2
--R   [[ci= y x ,tci= 6,cj= y x,tcj= 4,c= y x,tc= 5,rc= y,trc= 4,tH= 3,tD= 4]]
```

```
--R
--R
--R   [[ci= y,tci= 4,cj= y,tcj= 4,c= 0,tc= 0,rc= 0,trc= 0,tH= 3,tD= 3]]
--R
--R
--R                                                    3
--R   [[ci= y x,tci= 4,cj= y,tcj= 4,c= y,tc= 5,rc= x ,trc= 3,tH= 3,tD= 3]]
--R
--R
--R          3             3
--R   [[ci= x ,tci= 3,cj= x ,tcj= 3,c= 0,tc= 0,rc= 0,trc= 0,tH= 3,tD= 2]]
--R
--R
--R            3                               2
--R   [[ci= y x ,tci= 4,cj= y x,tcj= 4,c= y x ,tc= 3,rc= 0,trc= 0,tH= 3,tD= 1]]
--R
--R
--R                        3              2
--R   [[ci= y,tci= 4,cj= x ,tcj= 3,c= y x ,tc= 5,rc= 0,trc= 0,tH= 3,tD= 0]]
--R
--R
--R     There are
--R
--R   3
--R
--R     Groebner Basis Polynomials.
--R
--R
--R       THE GROEBNER BASIS over EUCLIDEAN DOMAIN
--R
--R                                    2            3     2
--R   (23)  [y x - y + x + 3,2y + 2x  - 3x - 6,2x  - 5x  - 5x]
--RType: List GeneralDistributedMultivariatePolynomial([y,x],Integer,DirectProduc
--E 23

--S 24 of 24
euclideanGroebner(cn,"info","redcrit")
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R         2               2
--R   - 2y x  - y x - y - 6x  - 3x + 3
--R
--R
```

```
--R
--R   you choose option  -info-
--R   abbrev. for the following information strings are
--R     ci  =>  Leading monomial  for critpair calculation
--R     tci =>  Number of terms of polynomial i
--R     cj  =>  Leading monomial  for critpair calculation
--R     tcj =>  Number of terms of polynomial j
--R     c   =>  Leading monomial of critpair polynomial
--R     tc  =>  Number of terms of critpair polynomial
--R     rc  =>  Leading monomial of redcritpair polynomial
--R     trc =>  Number of terms of redcritpair polynomial
--R     tF  =>  Number of polynomials in reduction list F
--R     tD  =>  Number of critpairs still to do
--R
--R
--R
--R
--R            3               3              2              2
--R   [[ci= y x ,tci= 7,cj= y x ,tcj= 4,c= y x ,tc= 6,rc= y x ,trc= 6,tH= 3,tD= 3]]
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R   y x - y + x + 3
--R
--R
--R
--R            2               2
--R   [[ci= y x ,tci= 6,cj= y x ,tcj= 6,c= y x,tc= 4,rc= y x,trc= 4,tH= 1,tD= 3]]
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R          2
--R   4y + 4x  - 6x - 12
--R
--R
--R
--R            2
--R   [[ci= y x ,tci= 6,cj= y x,tcj= 4,c= y x,tc= 5,rc= y,trc= 4,tH= 2,tD= 3]]
--R
```

```
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R       3      2
--R   - 4x  + 10x  + 10x
--R
--R
--R
--R                                                    3
--R   [[ci= y x,tci= 4,cj= y,tcj= 4,c= y,tc= 5,rc= x ,trc= 3,tH= 3,tD= 3]]
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R          2
--R   2y + 2x  - 3x - 6
--R
--R
--R
--R            2
--R   [[ci= y x ,tci= 6,cj= y x,tcj= 4,c= y x,tc= 5,rc= y,trc= 4,tH= 3,tD= 4]]
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R   0
--R
--R
--R
--R   [[ci= y,tci= 4,cj= y,tcj= 4,c= 0,tc= 0,rc= 0,trc= 0,tH= 3,tD= 3]]
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R       3     2
--R   - 2x  + 5x  + 5x
--R
--R
```

```
--R
--R                                                    3
--R   [[ci= y x,tci= 4,cj= y,tcj= 4,c= y,tc= 5,rc= x ,trc= 3,tH= 3,tD= 3]]
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R   0
--R
--R
--R
--R          3             3
--R   [[ci= x ,tci= 3,cj= x ,tcj= 3,c= 0,tc= 0,rc= 0,trc= 0,tH= 3,tD= 2]]
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R   0
--R
--R
--R
--R            3                              2
--R   [[ci= y x ,tci= 4,cj= y x,tcj= 4,c= y x ,tc= 3,rc= 0,trc= 0,tH= 3,tD= 1]]
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R   0
--R
--R
--R
--R                        3              2
--R   [[ci= y,tci= 4,cj= x ,tcj= 3,c= y x ,tc= 5,rc= 0,trc= 0,tH= 3,tD= 0]]
--R
--R
--R     There are
--R
--R   3
--R
--R     Groebner Basis Polynomials.
```

```
--R
--R
--R       THE GROEBNER BASIS over EUCLIDEAN DOMAIN
--R
--R                                    2            3     2
--R   (24)  [y x - y + x + 3,2y + 2x  - 3x - 6,2x  - 5x  - 5x]
--RType: List GeneralDistributedMultivariatePolynomial([y,x],Integer,DirectProduc
--E 24

)spool
)lisp (bye)
```

⟨*EuclideanGroebnerBasisPackage.help*⟩≡

```
==========================================================================
euclideanGroebner examples
==========================================================================

Example to call euclideanGroebner:

  a1:DMP([y,x],INT):= (9*x**2 + 5*x - 3)+ y*(3*x**2 + 2*x + 1)
  a2:DMP([y,x],INT):= (6*x**3 - 2*x**2 - 3*x +3) + y*(2*x**3 - x - 1)
  a3:DMP([y,x],INT):= (3*x**3 + 2*x**2) + y*(x**3 + x**2)
  an:=[a1,a2,a3]
  euclideanGroebner(an)

This will return the weak euclidean Groebner basis set.
All reductions are total reductions.

You can get more information by providing a second argument.
To get the reduced critical pairs do:

  euclideanGroebner(an,"redcrit")

You can get other information by calling:

  euclideanGroebner(an,"info")

which returns:
   ci  =>  Leading monomial  for critpair calculation
   tci =>  Number of terms of polynomial i
   cj  =>  Leading monomial  for critpair calculation
   tcj =>  Number of terms of polynomial j
   c   =>  Leading monomial of critpair polynomial
   tc  =>  Number of terms of critpair polynomial
   rc  =>  Leading monomial of redcritpair polynomial
   trc =>  Number of terms of redcritpair polynomial
   tH  =>  Number of polynomials in reduction list H
   tD  =>  Number of critpairs still to do

The three argument form returns all of the information:

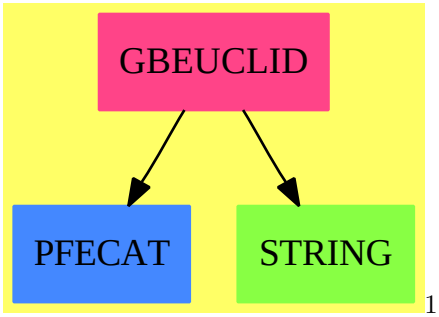
  euclideanGroebner(an,"info","redcrit")


The term ordering is determined by the polynomial type used.
Suggested types include
   DistributedMultivariatePolynomial
   HomogeneousDistributedMultivariatePolynomial
```

```
    GeneralDistributedMultivariatePolynomial

See Also:
o )display operations euclideanGroebner
o )show EuclideanGroebnerBasisPackage
o )show DistributedMultivariatePolynomial
o )show HomogeneousDistributedMultivariatePolynomial
o )show GeneralDistributedMultivariatePolynomial
o )show GroebnerPackage
```

6.32 EuclideanGroebnerBasisPackage



1.00

Exports:
euclideanGroebner euclideanNormalForm

⟨*package GBEUCLID EuclideanGroebnerBasisPackage*⟩≡

```
)abbrev package GBEUCLID EuclideanGroebnerBasisPackage
++ Authors: Gebauer, Moeller
++ Date Created: 12-1-86
++ Date Last Updated: 2-28-91
++ Basic Functions:
++ Related Constructors: Ideal, IdealDecompositionPackage, GroebnerPackage
++ Also See:
++ AMS Classifications:
++ Keywords: groebner basis, polynomial ideal, euclidean domain
++ References:
++ Description: \spadtype{EuclideanGroebnerBasisPackage} computes groebner
++ bases for polynomial ideals over euclidean domains.
++ The basic computation provides
++ a distinguished set of generators for these ideals.
++ This basis allows an easy test for membership: the operation
++ \spadfun{euclideanNormalForm} returns zero on ideal members. The string
++ "info" and "redcrit" can be given as additional args to provide
++ incremental information during the computation. If "info" is given,
++  a computational summary is given for each s-polynomial. If "redcrit"
++ is given, the reduced critical pairs are printed. The term ordering
++ is determined by the polynomial type used. Suggested types include
++ \spadtype{DistributedMultivariatePolynomial},
++ \spadtype{HomogeneousDistributedMultivariatePolynomial},
++ \spadtype{GeneralDistributedMultivariatePolynomial}.

EuclideanGroebnerBasisPackage(Dom, Expon, VarSet, Dpol): T == C where

 Dom: EuclideanDomain
 Expon: OrderedAbelianMonoidSup
 VarSet: OrderedSet
```

```
Dpol: PolynomialCategory(Dom, Expon, VarSet)

T== with

    euclideanNormalForm: (Dpol, List(Dpol) )  ->  Dpol
      ++ euclideanNormalForm(poly,gb) reduces the polynomial poly modulo the
      ++ precomputed groebner basis gb giving a canonical representative
      ++ of the residue class.
    euclideanGroebner: List(Dpol) -> List(Dpol)
      ++ euclideanGroebner(lp) computes a groebner basis for a polynomial
      ++ ideal over a euclidean domain generated by the list of polys lp.
      ++
      ++X a1:DMP([y,x],INT):= (9*x**2 + 5*x - 3)+ y*(3*x**2 + 2*x + 1)
      ++X a2:DMP([y,x],INT):= (6*x**3 - 2*x**2 - 3*x +3) + y*(2*x**3 - x - 1)
      ++X a3:DMP([y,x],INT):= (3*x**3 + 2*x**2) + y*(x**3 + x**2)
      ++X an:=[a1,a2,a3]
      ++X euclideanGroebner(an)

    euclideanGroebner: (List(Dpol), String) -> List(Dpol)
      ++ euclideanGroebner(lp, infoflag) computes a groebner basis
      ++ for a polynomial ideal over a euclidean domain
      ++ generated by the list of polynomials lp.
      ++ During computation, additional information is printed out
      ++ if infoflag is given as
      ++ either "info" (for summary information) or
      ++ "redcrit" (for reduced critical pairs)
      ++
      ++X a1:DMP([y,x],INT):= (9*x**2 + 5*x - 3)+ y*(3*x**2 + 2*x + 1)
      ++X a2:DMP([y,x],INT):= (6*x**3 - 2*x**2 - 3*x +3) + y*(2*x**3 - x - 1)
      ++X a3:DMP([y,x],INT):= (3*x**3 + 2*x**2) + y*(x**3 + x**2)
      ++X an:=[a1,a2,a3]
      ++X euclideanGroebner(an,"redcrit")
      ++X euclideanGroebner(an,"info")

    euclideanGroebner: (List(Dpol), String, String ) -> List(Dpol)
      ++ euclideanGroebner(lp, "info", "redcrit") computes a groebner basis
      ++ for a polynomial ideal generated by the list of polynomials lp.
      ++ If the second argument is "info",
      ++ a summary is given of the critical pairs.
      ++ If the third argument is "redcrit", critical pairs are printed.
      ++
      ++X a1:DMP([y,x],INT):= (9*x**2 + 5*x - 3)+ y*(3*x**2 + 2*x + 1)
      ++X a2:DMP([y,x],INT):= (6*x**3 - 2*x**2 - 3*x +3) + y*(2*x**3 - x - 1)
      ++X a3:DMP([y,x],INT):= (3*x**3 + 2*x**2) + y*(x**3 + x**2)
      ++X an:=[a1,a2,a3]
      ++X euclideanGroebner(an,"info","redcrit")
```

```
C== add
  Ex ==> OutputForm
  lc ==> leadingCoefficient
  red ==> reductum

  import OutputForm

  ------  Definition list of critPair
  ------  lcmfij is now lcm of headterm of poli and polj
  ------  lcmcij is now lcm of of lc poli and lc polj

  critPair ==>Record(lcmfij: Expon, lcmcij: Dom, poli:Dpol, polj: Dpol )
  Prinp    ==> Record( ci:Dpol,tci:Integer,cj:Dpol,tcj:Integer,c:Dpol,
              tc:Integer,rc:Dpol,trc:Integer,tH:Integer,tD:Integer)

  ------  Definition of intermediate functions

  strongGbasis: (List(Dpol), Integer, Integer) -> List(Dpol)
  eminGbasis: List(Dpol) -> List(Dpol)
  ecritT: (critPair ) -> Boolean
  ecritM: (Expon, Dom, Expon, Dom) -> Boolean
  ecritB: (Expon, Dom, Expon, Dom, Expon, Dom) -> Boolean
  ecrithinH: (Dpol, List(Dpol)) -> Boolean
  ecritBonD: (Dpol, List(critPair)) -> List(critPair)
  ecritMTondd1:(List(critPair)) -> List(critPair)
  ecritMondd1:(Expon, Dom, List(critPair)) -> List(critPair)
  crithdelH: (Dpol, List(Dpol)) -> List(Dpol)
  eupdatF: (Dpol, List(Dpol) ) -> List(Dpol)
  updatH: (Dpol, List(Dpol), List(Dpol), List(Dpol) ) -> List(Dpol)
  sortin: (Dpol, List(Dpol) ) -> List(Dpol)
  eRed: (Dpol, List(Dpol), List(Dpol) )  ->  Dpol
  ecredPol: (Dpol, List(Dpol) ) -> Dpol
  esPol: (critPair) -> Dpol
  updatD: (List(critPair), List(critPair)) -> List(critPair)
  lepol: Dpol -> Integer
  prinshINFO : Dpol -> Void
  prindINFO: (critPair, Dpol, Dpol,Integer,Integer,Integer) -> Integer
  prinpolINFO: List(Dpol) -> Void
  prinb: Integer -> Void

  ------    MAIN ALGORITHM GROEBNER ------------------------
  euclideanGroebner( Pol: List(Dpol) ) ==
    eminGbasis(strongGbasis(Pol,0,0))

  euclideanGroebner( Pol: List(Dpol), xx1: String) ==
```

```
    xx1 = "redcrit" =>
      eminGbasis(strongGbasis(Pol,1,0))
    xx1 = "info" =>
      eminGbasis(strongGbasis(Pol,2,1))
    print("   "::Ex)
    print("WARNING: options are - redcrit and/or info - "::Ex)
    print("         you didn't type them correct"::Ex)
    print("         please try again"::Ex)
    print("   "::Ex)
    []

  euclideanGroebner( Pol: List(Dpol), xx1: String, xx2: String) ==
    (xx1 = "redcrit" and xx2 = "info") or
     (xx1 = "info" and xx2 = "redcrit")   =>
      eminGbasis(strongGbasis(Pol,1,1))
    xx1 = "redcrit" and xx2 = "redcrit" =>
      eminGbasis(strongGbasis(Pol,1,0))
    xx1 = "info" and xx2 = "info" =>
      eminGbasis(strongGbasis(Pol,2,1))
    print("   "::Ex)
    print("WARNING:  options are - redcrit and/or info - "::Ex)
    print("          you didn't type them correct"::Ex)
    print("          please try again "::Ex)
    print("   "::Ex)
    []

  ------    calculate basis

  strongGbasis(Pol: List(Dpol),xx1: Integer, xx2: Integer ) ==
    dd1, D : List(critPair)

    ---------   create D and Pol

    Pol1:= sort((z1:Dpol,z2:Dpol):Boolean +-> (degree z1 > degree z2) or
                  ((degree z1 = degree z2 ) and
                     sizeLess?(leadingCoefficient z2,leadingCoefficient z1)),
                Pol)
    Pol:= [first(Pol1)]
    H:= Pol
    Pol1:= rest(Pol1)
    D:= nil
    while ^null Pol1 repeat
       h:= first(Pol1)
       Pol1:= rest(Pol1)
       en:= degree(h)
       lch:= lc h
```

```
    dd1:=
     [[sup(degree(x), en), lcm(leadingCoefficient x, lch), x, h]$critPair
        for x in Pol]
    D:= updatD(
         ecritMTondd1(
          sort(
           (z1:critPair,z2:critPair):Boolean+->
            (z1.lcmfij < z2.lcmfij) or
             (( z1.lcmfij = z2.lcmfij ) and
               ( sizeLess?(z1.lcmcij,z2.lcmcij)) ), dd1)),
                ecritBonD(h,D))
    Pol:= cons(h, eupdatF(h, Pol))
    ((en = degree(first(H))) and
      (leadingCoefficient(h) = leadingCoefficient(first(H)) ) ) =>
          " go to top of while "
    H:= updatH(h,H,crithdelH(h,H),[h])
    H:= sort((z1,z2) +-> (degree z1 > degree z2) or
             ((degree z1 = degree z2 ) and
              sizeLess?(leadingCoefficient z2,leadingCoefficient z1)), H)
D:= sort((z1,z2) +-> (z1.lcmfij < z2.lcmfij) or
          (( z1.lcmfij = z2.lcmfij ) and
           ( sizeLess?(z1.lcmcij,z2.lcmcij)) ) ,D)
xx:= xx2

--------  loop

while ^null D repeat
    D0:= first D
    ep:=esPol(D0)
    D:= rest(D)
    eh:= ecredPol(eRed(ep,H,H),H)
    if xx1 = 1 then
          prinshINFO(eh)
    eh = 0 =>
         if xx2 = 1 then
             ala:= prindINFO(D0,ep,eh,#H, #D, xx)
             xx:= 2
         " go to top of while "
    eh := unitCanonical eh
    e:= degree(eh)
    leh:= lc eh
    dd1:=
     [[sup(degree(x), e), lcm(leadingCoefficient x, leh), x, eh]$critPair
       for x in Pol]
    D:= updatD(
         ecritMTondd1(
```

```
              sort((z1,z2) +-> (z1.lcmfij < z2.lcmfij) or
                   (( z1.lcmfij = z2.lcmfij ) and
                    ( sizeLess?(z1.lcmcij,z2.lcmcij)) ), dd1)),
                      ecritBonD(eh,D))
        Pol:= cons(eh,eupdatF(eh,Pol))
        ^ecrithinH(eh,H) or
          ((e = degree(first(H))) and
            (leadingCoefficient(eh) = leadingCoefficient(first(H)) ) ) =>
             if xx2 = 1 then
                 ala:= prindINFO(D0,ep,eh,#H, #D, xx)
                 xx:= 2
             " go to top of while "
        H:= updatH(eh,H,crithdelH(eh,H),[eh])
        H:= sort((z1,z2)+-> (degree z1 > degree z2) or
            ((degree z1 = degree z2 ) and
               sizeLess?(leadingCoefficient z2,leadingCoefficient z1)), H)
        if xx2 = 1 then
          ala:= prindINFO(D0,ep,eh,#H, #D, xx)
          xx:= 2
          " go to top of while "
    if xx2 = 1 then
      prinpolINFO(Pol)
      print("    THE GROEBNER BASIS over EUCLIDEAN DOMAIN"::Ex)
    if xx1 = 1 and xx2 ^= 1 then
      print("    THE GROEBNER BASIS over EUCLIDEAN DOMAIN"::Ex)
    H

            ---------------------------------------

            --- erase multiple of e in D2 using crit M

ecritMondd1(e: Expon, c: Dom, D2: List(critPair))==
   null D2 => nil
   x:= first(D2)
   ecritM(e,c, x.lcmfij, lcm(leadingCoefficient(x.poli),
          leadingCoefficient(x.polj)))
      => ecritMondd1(e, c, rest(D2))
   cons(x, ecritMondd1(e, c, rest(D2)))

         -------------------------------

ecredPol(h: Dpol, F: List(Dpol) ) ==
     h0:Dpol:= 0
     null F => h
     while h ^= 0 repeat
        h0:= h0 + monomial(leadingCoefficient(h),degree(h))
```

```
          h:= eRed(red(h), F, F)
      h0
           ----------------------------

           --- reduce dd1 using crit T and crit M

ecritMTondd1(dd1: List(critPair))==
        null dd1 => nil
        f1:= first(dd1)
        s1:= #(dd1)
        cT1:= ecritT(f1)
        s1= 1 and cT1 => nil
        s1= 1 => dd1
        e1:= f1.lcmfij
        r1:= rest(dd1)
        f2:= first(r1)
        e1 = f2.lcmfij and f1.lcmcij = f2.lcmcij =>
           cT1 =>   ecritMTondd1(cons(f1, rest(r1)))
           ecritMTondd1(r1)
        dd1 := ecritMondd1(e1, f1.lcmcij, r1)
        cT1 => ecritMTondd1(dd1)
        cons(f1, ecritMTondd1(dd1))

           ----------------------------

           --- erase elements in D fullfilling crit B

ecritBonD(h:Dpol, D: List(critPair))==
      null D => nil
      x:= first(D)
      x1:= x.poli
      x2:= x.polj
      ecritB(degree(h), leadingCoefficient(h),
             degree(x1),leadingCoefficient(x1),
             degree(x2),leadingCoefficient(x2)) =>
        ecritBonD(h, rest(D))
      cons(x, ecritBonD(h, rest(D)))

           ----------------------------

           --- concat F and h and erase multiples of h in F

eupdatF(h: Dpol, F: List(Dpol)) ==
    null F => nil
    f1:= first(F)
    ecritM(degree h,leadingCoefficient(h), degree f1,leadingCoefficient(f1))
```

```
        => eupdatF(h, rest(F))
    cons(f1, eupdatF(h, rest(F)))

          ------------------------------
          --- concat H and h and erase multiples of h in H

updatH(h: Dpol, H: List(Dpol), Hh: List(Dpol), Hhh: List(Dpol)) ==
    null H => append(Hh,Hhh)
    h1:= first(H)
    hlcm:= sup(degree(h1), degree(h))
    plc:= extendedEuclidean(leadingCoefficient(h), leadingCoefficient(h1))
    hp:= monomial(plc.coef1,subtractIfCan(hlcm, degree(h))::Expon)*h +
         monomial(plc.coef2,subtractIfCan(hlcm, degree(h1))::Expon)*h1
    (ecrithinH(hp, Hh) and ecrithinH(hp, Hhh)) =>
      hpp:= append(rest(H),Hh)
      hp:= ecredPol(eRed(hp,hpp,hpp),hpp)
      updatH(h, rest(H), crithdelH(hp,Hh),cons(hp,crithdelH(hp,Hhh)))
    updatH(h, rest(H), Hh,Hhh)

          ------------------------------------------------
          ---- delete elements in cons(h,H)

crithdelH(h: Dpol, H: List(Dpol))==
     null H => nil
     h1:= first(H)
     dh1:= degree h1
     dh:= degree h
     ecritM(dh, lc h, dh1, lc h1) => crithdelH(h, rest(H))
     dh1 = sup(dh,dh1) =>
      plc:= extendedEuclidean( lc h1, lc h)
      cons(plc.coef1*h1+monomial(plc.coef2,subtractIfCan(dh1,dh)::Expon)*h,
            crithdelH(h,rest(H)))
     cons(h1, crithdelH(h,rest(H)))

eminGbasis(F: List(Dpol)) ==
     null F => nil
     newbas := eminGbasis rest F
     cons(ecredPol( first(F), newbas),newbas)

          ------------------------------------------------
          --- does h belong to H

ecrithinH(h: Dpol, H: List(Dpol))==
     null H  => true
     h1:= first(H)
     ecritM(degree h1, lc h1, degree h, lc h) => false
```

```
      ecrithinH(h, rest(H))

          ----------------------------
          --- calculate  euclidean S-polynomial of a critical pair

    esPol(p:critPair)==
       Tij := p.lcmfij
       fi := p.poli
       fj := p.polj
       lij:= lcm(leadingCoefficient(fi), leadingCoefficient(fj))
       red(fi)*monomial((lij exquo leadingCoefficient(fi))::Dom,
                        subtractIfCan(Tij, degree fi)::Expon) -
         red(fj)*monomial((lij exquo leadingCoefficient(fj))::Dom,
                          subtractIfCan(Tij, degree fj)::Expon)

          ---------------------------

          --- euclidean reduction mod F

    eRed(s: Dpol, H: List(Dpol), Hh: List(Dpol)) ==
      ( s = 0 or null H ) => s
      f1:= first(H)
      ds:= degree s
      lf1:= leadingCoefficient(f1)
      ls:= leadingCoefficient(s)
      e: Union(Expon, "failed")
      (((e:= subtractIfCan(ds, degree f1))  case "failed" ) or sizeLess?(ls, lf1) ) =>
         eRed(s, rest(H), Hh)
      sdf1:= divide(ls, lf1)
      q1:= sdf1.quotient
      sdf1.remainder = 0 =>
         eRed(red(s) - monomial(q1,e)*reductum(f1), Hh, Hh)
      eRed(s -(monomial(q1, e)*f1), rest(H), Hh)

          ---------------------------

          --- crit T  true, if e1 and e2 are disjoint

    ecritT(p: critPair) ==
           pi:= p.poli
           pj:= p.polj
           ci:= lc pi
           cj:= lc pj
           (p.lcmfij = degree pi + degree pj) and  (p.lcmcij = ci*cj)

          ---------------------------
```

```
        --- crit M - true, if lcm#2 multiple of lcm#1

ecritM(e1: Expon, c1: Dom, e2: Expon, c2: Dom) ==
  en: Union(Expon, "failed")
  ((en:=subtractIfCan(e2, e1)) case "failed") or
    ((c2 exquo c1) case "failed") => false
  true
        ----------------------------

        --- crit B - true, if eik is a multiple of eh and eik ^equal
        ---          lcm(eh,ei) and eik ^equal lcm(eh,ek)

ecritB(eh:Expon, ch: Dom, ei:Expon, ci: Dom, ek:Expon, ck: Dom) ==
    eik:= sup(ei, ek)
    cik:= lcm(ci, ck)
    ecritM(eh, ch, eik, cik) and
          ^ecritM(eik, cik, sup(ei, eh), lcm(ci, ch)) and
             ^ecritM(eik, cik, sup(ek, eh), lcm(ck, ch))

        --------------------------------

        --- reduce p1 mod lp

euclideanNormalForm(p1: Dpol, lp: List(Dpol))==
    eRed(p1, lp, lp)

        ---------------------------------

        ---  insert element in sorted list

sortin(p1: Dpol, lp: List(Dpol))==
   null lp => [p1]
   f1:= first(lp)
   elf1:= degree(f1)
   ep1:= degree(p1)
   ((elf1 < ep1) or ((elf1 = ep1) and
     sizeLess?(leadingCoefficient(f1),leadingCoefficient(p1)))) =>
      cons(f1,sortin(p1, rest(lp)))
   cons(p1,lp)

updatD(D1: List(critPair), D2: List(critPair)) ==
   null D1 => D2
   null D2 => D1
   dl1:= first(D1)
   dl2:= first(D2)
```

```
      (dl1.lcmfij  <  dl2.lcmfij) => cons(dl1, updatD(D1.rest, D2))
      cons(dl2, updatD(D1, D2.rest))

            ----  calculate number of terms of polynomial

   lepol(p1:Dpol)==
      n: Integer
      n:= 0
      while p1 ^= 0 repeat
         n:= n + 1
         p1:= red(p1)
      n

            ----  print blanc lines

   prinb(n: Integer)==
        for i in 1..n repeat messagePrint("    ")

            ----  print reduced critpair polynom

   prinshINFO(h: Dpol)==
           prinb(2)
           messagePrint(" reduced Critpair - Polynom :")
           prinb(2)
           print(h::Ex)
           prinb(2)

            --------------------------------

            ----  print info string

   prindINFO(cp: critPair, ps: Dpol, ph: Dpol, i1:Integer,
             i2:Integer, n:Integer) ==
       ll: List Prinp
       a: Dom
       cpi:= cp.poli
       cpj:= cp.polj
       if n = 1 then
        prinb(1)
        messagePrint("you choose option  -info-  ")
        messagePrint("abbrev. for the following information strings are")
        messagePrint("  ci  =>  Leading monomial  for critpair calculation")
        messagePrint("  tci =>  Number of terms of polynomial i")
        messagePrint("  cj  =>  Leading monomial  for critpair calculation")
        messagePrint("  tcj =>  Number of terms of polynomial j")
        messagePrint("  c   =>  Leading monomial of critpair polynomial")
```

```
        messagePrint("  tc  =>  Number of terms of critpair polynomial")
        messagePrint("  rc  =>  Leading monomial of redcritpair polynomial")
        messagePrint("  trc =>  Number of terms of redcritpair polynomial")
        messagePrint("  tF  =>  Number of polynomials in reduction list F")
        messagePrint("  tD  =>  Number of critpairs still to do")
        prinb(4)
        n:= 2
      prinb(1)
      a:= 1
      ph = 0  =>
         ps = 0 =>
           ll:= [[monomial(a,degree(cpi)),lepol(cpi),monomial(a,degree(cpj)),
            lepol(cpj),ps,0,ph,0,i1,i2]$Prinp]
           print(ll::Ex)
           prinb(1)
           n
         ll:= [[monomial(a,degree(cpi)),lepol(cpi),
           monomial(a,degree(cpj)),lepol(cpj),monomial(a,degree(ps)),
            lepol(ps), ph,0,i1,i2]$Prinp]
         print(ll::Ex)
         prinb(1)
         n
      ll:= [[monomial(a,degree(cpi)),lepol(cpi),
           monomial(a,degree(cpj)),lepol(cpj),monomial(a,degree(ps)),
            lepol(ps),monomial(a,degree(ph)),lepol(ph),i1,i2]$Prinp]
      print(ll::Ex)
      prinb(1)
      n

           ------------------------------

           ----  print the groebner basis polynomials

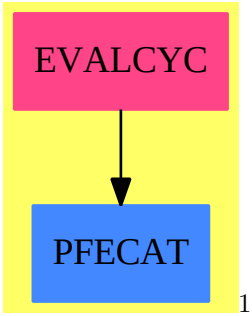
  prinpolINFO(pl: List(Dpol))==
      n:Integer
      n:= #pl
      prinb(1)
      n = 1 =>
        print("  There is 1  Groebner Basis Polynomial "::Ex)
        prinb(2)
      print("  There are "::Ex)
      prinb(1)
      print(n::Ex)
      prinb(1)
      print("  Groebner Basis Polynomials. "::Ex)
      prinb(2)
```

⟨*GBEUCLID.dotabb*⟩≡

```
"GBEUCLID" [color="#FF4488",href="bookvol10.4.pdf#nameddest=GBEUCLID"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"STRING" [color="#88FF44",href="bookvol10.3.pdf#nameddest=STRING"]
"GBEUCLID" -> "PFECAT"
"GBEUCLID" -> "STRING"
```

6.33 package EVALCYC EvaluateCycleIndicators

6.34 EvaluateCycleIndicators



1.00

Exports:
eval

⟨*package EVALCYC EvaluateCycleIndicators*⟩≡

```
)abbrev package EVALCYC EvaluateCycleIndicators
++ Author: William H. Burge
++ Date Created: 1986
++ Date Last Updated: Feb 1992
++ Basic Operations:
++ Related Domains:
++ Also See:
++ AMS Classifications:
++ Keywords:
++ Examples:
++ References:
++ Description: This package is to be used in conjuction with
++              the CycleIndicators package. It provides an evaluation
++              function for SymmetricPolynomials.
EvaluateCycleIndicators(F):T==C where
    F:Algebra Fraction Integer
    I==>Integer
    L==>List
    SPOL==SymmetricPolynomial
    RN==>Fraction Integer
    PR==>Polynomial(RN)
    PTN==>Partition()
    lc ==> leadingCoefficient
    red ==> reductum
    T== with
       eval:((I->F),SPOL RN)->F
```

```
      ++\spad{eval(f,s)} evaluates the cycle index s by applying
      ++ the function f to each integer in a monomial partition,
      ++ forms their product and sums the results over all monomials.
   C== add
      evp:((I->F),PTN)->F
      fn:I->F
      pt:PTN
      spol:SPOL RN
      i:I
      evp(fn, pt)== _*/[fn i for i in pt::(L I)]

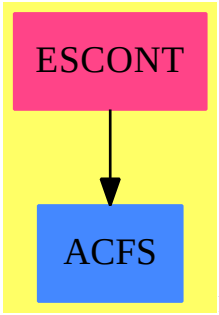
      eval(fn,spol)==
       if spol=0
       then 0
       else ((lc spol)* evp(fn,degree spol)) + eval(fn,red spol)
```

⟨*EVALCYC.dotabb*⟩≡

```
"EVALCYC" [color="#FF4488",href="bookvol10.4.pdf#nameddest=EVALCYC"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"EVALCYC" -> "PFECAT"
```

6.35 package ESCONT ExpertSystemContinuityPackage

6.36 ExpertSystemContinuityPackage



1.00

Exports:

df2st	functionIsFracPolynomial?	gethi	getlo	ldf2lst
polynomialZeros	problemPoints	sdf2lst	singularitiesOf	singularitiesOf
zerosOf				

⟨*package ESCONT ExpertSystemContinuityPackage*⟩≡

```
)abbrev package ESCONT ExpertSystemContinuityPackage
++ Author: Brian Dupee
++ Date Created: May 1994
++ Date Last Updated: June 1995
++ Basic Operations: problemPoints, singularitiesOf, zerosOf
++ Related Constructors:
++ Description:
++ ExpertSystemContinuityPackage is a package of functions for the use of domains
++ belonging to the category \axiomType{NumericalIntegration}.

ExpertSystemContinuityPackage(): E == I where
  EF2   ==> ExpressionFunctions2
  FI    ==> Fraction Integer
  EFI   ==> Expression Fraction Integer
  PFI   ==> Polynomial Fraction Integer
  DF    ==> DoubleFloat
  LDF   ==> List DoubleFloat
  EDF   ==> Expression DoubleFloat
  VEDF  ==> Vector Expression DoubleFloat
  SDF   ==> Stream DoubleFloat
  SS    ==> Stream String
  EEDF  ==> Equation Expression DoubleFloat
  LEDF  ==> List Expression DoubleFloat
  KEDF  ==> Kernel Expression DoubleFloat
```

```
LKEDF ==> List Kernel Expression DoubleFloat
PDF   ==> Polynomial DoubleFloat
FPDF  ==> Fraction Polynomial DoubleFloat
OCDF  ==> OrderedCompletion DoubleFloat
SOCDF ==> Segment OrderedCompletion DoubleFloat
NIA   ==> Record(var:Symbol,fn:EDF,range:SOCDF,abserr:DF,relerr:DF)
UP    ==> UnivariatePolynomial
BO    ==> BasicOperator
RS    ==> Record(zeros: SDF,ones: SDF,singularities: SDF)

E ==> with

  getlo : SOCDF -> DF
    ++ getlo(u) gets the \axiomType{DoubleFloat} equivalent of
    ++ the first endpoint of the range \axiom{u}
  gethi : SOCDF -> DF
    ++ gethi(u) gets the \axiomType{DoubleFloat} equivalent of
    ++ the second endpoint of the range \axiom{u}
  functionIsFracPolynomial?: NIA -> Boolean
    ++ functionIsFracPolynomial?(args) tests whether the function
    ++ can be retracted to \axiomType{Fraction(Polynomial(DoubleFloat))}
  problemPoints:(EDF,Symbol,SOCDF) -> List DF
    ++ problemPoints(f,var,range) returns a list of possible problem points
    ++ by looking at the zeros of the denominator of the function \spad{f}
    ++ if it can be retracted to \axiomType{Polynomial(DoubleFloat)}.
  zerosOf:(EDF,List Symbol,SOCDF) -> SDF
    ++ zerosOf(e,vars,range) returns a list of points
    ++ (\axiomType{Doublefloat}) at which a NAG fortran version of \spad{e}
    ++ will most likely produce an error.
  singularitiesOf: (EDF,List Symbol,SOCDF) -> SDF
    ++ singularitiesOf(e,vars,range) returns a list of points
    ++ (\axiomType{Doublefloat}) at which a NAG fortran
    ++ version of \spad{e} will most likely produce
    ++ an error.  This includes those points which evaluate to 0/0.
  singularitiesOf: (Vector EDF,List Symbol,SOCDF) -> SDF
    ++ singularitiesOf(v,vars,range) returns a list of points
    ++ (\axiomType{Doublefloat}) at which a NAG fortran
    ++ version of \spad{v} will most likely produce
    ++ an error.  This includes those points which evaluate to 0/0.
  polynomialZeros:(PFI,Symbol,SOCDF) -> LDF
    ++ polynomialZeros(fn,var,range) calculates the real zeros of the
    ++ polynomial which are contained in the given interval. It returns
    ++ a list of points (\axiomType{Doublefloat}) for which the univariate
    ++ polynomial \spad{fn} is zero.
  df2st:DF -> String
    ++ df2st(n) coerces a \axiomType{DoubleFloat} to \axiomType{String}
```

```
    ldf2lst:LDF -> List String
      ++ ldf2lst(ln) coerces a List of \axiomType{DoubleFloat} to
      ++ \axiomType{List}(\axiomType{String})
    sdf2lst:SDF -> List String
      ++ sdf2lst(ln) coerces a Stream of \axiomType{DoubleFloat} to
      ++ \axiomType{List}(\axiomType{String})

I ==> ExpertSystemToolsPackage add

    import ExpertSystemToolsPackage

    functionIsPolynomial?(args:NIA):Boolean ==
      -- tests whether the function can be retracted to a polynomial
      (retractIfCan(args.fn)@Union(PDF,"failed"))$EDF case PDF

    isPolynomial?(f:EDF):Boolean ==
      -- tests whether the function can be retracted to a polynomial
      (retractIfCan(f)@Union(PDF,"failed"))$EDF case PDF

    isConstant?(f:EDF):Boolean ==
      -- tests whether the function can be retracted to a constant (DoubleFloat)
      (retractIfCan(f)@Union(DF,"failed"))$EDF case DF

    denominatorIsPolynomial?(args:NIA):Boolean ==
      -- tests if the denominator can be retracted to polynomial
      a:= copy args
      a.fn:=denominator(args.fn)
      (functionIsPolynomial?(a))@Boolean

    denIsPolynomial?(f:EDF):Boolean ==
      -- tests if the denominator can be retracted to polynomial
      (isPolynomial?(denominator f))@Boolean

    listInRange(l:LDF,range:SOCDF):LDF ==
      -- returns a list with only those elements internal to the range range
      [t for t in l | in?(t,range)]

    loseUntil(l:SDF,a:DF):SDF ==
      empty?(l)$SDF => l
      f := first(l)$SDF
      (abs(f) <= abs(a)) => loseUntil(rest(l)$SDF,a)
      l

    retainUntil(l:SDF,a:DF,b:DF,flag:Boolean):SDF ==
      empty?(l)$SDF => l
      f := first(l)$SDF
```

```
    (in?(f)$ExpertSystemContinuityPackage1(a,b)) =>
      concat(f,retainUntil(rest(l),a,b,false))
    flag => empty()$SDF
    retainUntil(rest(l),a,b,true)

  streamInRange(l:SDF,range:SOCDF):SDF ==
    -- returns a stream with only those elements internal to the range range
    a := getlo(range := dfRange(range))
    b := gethi(range)
    explicitlyFinite?(l) =>
      select(in?$ExpertSystemContinuityPackage1(a,b),l)$SDF
    negative?(a*b) => retainUntil(l,a,b,false)
    negative?(a) =>
      l := loseUntil(l,b)
      retainUntil(l,a,b,false)
    l := loseUntil(l,a)
    retainUntil(l,a,b,false)

  getStream(n:Symbol,s:String):SDF ==
    import RS
    entry?(n,bfKeys()$BasicFunctions)$(List(Symbol)) =>
      c := bfEntry(n)$BasicFunctions
      (s = "zeros")@Boolean => c.zeros
      (s = "singularities")@Boolean => c.singularities
      (s = "ones")@Boolean => c.ones
    empty()$SDF

  polynomialZeros(fn:PFI,var:Symbol,range:SOCDF):LDF ==
    up := unmakeSUP(univariate(fn)$PFI)$UP(var,FI)
    range := dfRange(range)
    r:Record(left:FI,right:FI) := [df2fi(getlo(range)), df2fi(gethi(range))]
    ans:List(Record(left:FI,right:FI)) :=
        realZeros(up,r,1/1000000000000000000)$RealZeroPackageQ(UP(var,FI))
    listInRange(dflist(ans),range)

  functionIsFracPolynomial?(args:NIA):Boolean ==
    -- tests whether the function can be retracted to a fraction
    -- where both numerator and denominator are polynomial
    (retractIfCan(args.fn)@Union(FPDF,"failed"))$EDF case FPDF

  problemPoints(f:EDF,var:Symbol,range:SOCDF):LDF ==
    (denIsPolynomial?(f))@Boolean =>
      c := retract(edf2efi(denominator(f)))@PFI
      polynomialZeros(c,var,range)
    empty()$LDF
```

```
    zerosOf(e:EDF,vars:List Symbol,range:SOCDF):SDF ==
      (u := isQuotient(e)) case EDF =>
        singularitiesOf(u,vars,range)
      k := kernels(e)$EDF
      ((nk := # k) = 0)@Boolean => empty()$SDF -- constant found.
      (nk = 1)@Boolean =>                      -- single expression found.
        ker := first(k)$LKEDF
        n := name(operator(ker)$KEDF)$BO
        entry?(n,vars) =>                   -- polynomial found.
          c := retract(edf2efi(e))@PFI
          coerce(polynomialZeros(c,n,range))$SDF
        a := first(argument(ker)$KEDF)$LEDF
        (not (n = log :: Symbol)@Boolean) and ((w := isPlus a) case LEDF) =>
          var:Symbol := first(variables(a))
          c:EDF := w.2
          c1:EDF := w.1
--          entry?(c1,[b::EDF for b in vars]) and (one?(# vars)) =>
          entry?(c1,[b::EDF for b in vars]) and ((# vars) = 1) =>
            c2:DF := edf2df c
            c3 := c2 :: OCDF
            varEdf := var :: EDF
            varEqn := equation(varEdf,c1-c)$EEDF
            range2 := (lo(range)+c3)..(hi(range)+c3)
            s := zerosOf(subst(e,varEqn)$EDF,vars,range2)
            st := map(t1 +-> t1-c2,s)$StreamFunctions2(DF,DF)
            streamInRange(st,range)
          zerosOf(a,vars,range)
        (t := isPlus(e)$EDF) case LEDF =>    -- constant + expression
          # t > 2 => empty()$SDF
          entry?(a,[b::EDF for b in vars]) =>   -- finds entries like sqrt(x)
            st := getStream(n,"ones")
            o := edf2df(second(t)$LEDF)
--            one?(o) or one?(-o) =>           -- is it like (f(x) -/+ 1)
            (o = 1) or (-o = 1) =>           -- is it like (f(x) -/+ 1)
              st := map(t2 +-> -t2/o,st)$StreamFunctions2(DF,DF)
              streamInRange(st,range)
            empty()$SDF
          empty()$SDF
        entry?(a,[b::EDF for b in vars]) =>     -- finds entries like sqrt(x)
          st := getStream(n,"zeros")
          streamInRange(st,range)
        (n = tan :: Symbol)@Boolean =>
          concat([zerosOf(a,vars,range),singularitiesOf(a,vars,range)])
        (n = sin :: Symbol)@Boolean =>
          concat([zerosOf(a,vars,range),singularitiesOf(a,vars,range)])
        empty()$SDF
```

```
        (t := isPlus(e)$EDF) case LEDF => empty()$SDF  -- INCOMPLETE!!!
        (v := isTimes(e)$EDF) case LEDF =>
          concat([zerosOf(u,vars,range) for u in v])
        empty()$SDF

      singularitiesOf(e:EDF,vars:List Symbol,range:SOCDF):SDF ==
        (u := isQuotient(e)) case EDF =>
          zerosOf(u,vars,range)
        (t := isPlus e) case LEDF =>
          concat([singularitiesOf(u,vars,range) for u in t])
        (v := isTimes e) case LEDF =>
          concat([singularitiesOf(u,vars,range) for u in v])
        (k := mainKernel e) case KEDF =>
          n := name(operator k)
          entry?(n,vars) => coerce(problemPoints(e,n,range))$SDF
          a:EDF := (argument k).1
          (not (n = log :: Symbol)@Boolean) and ((w := isPlus a) case LEDF) =>
            var:Symbol := first(variables(a))
            c:EDF := w.2
            c1:EDF := w.1
--            entry?(c1,[b::EDF for b in vars]) and (one?(# vars)) =>
            entry?(c1,[b::EDF for b in vars]) and ((# vars) = 1) =>
              c2:DF := edf2df c
              c3 := c2 :: OCDF
              varEdf := var :: EDF
              varEqn := equation(varEdf,c1-c)$EEDF
              range2 := (lo(range)+c3)..(hi(range)+c3)
              s := singularitiesOf(subst(e,varEqn)$EDF,vars,range2)
              st := map(t3 +-> t3-c2,s)$StreamFunctions2(DF,DF)
              streamInRange(st,range)
            singularitiesOf(a,vars,range)
          entry?(a,[b::EDF for b in vars]) =>
            st := getStream(n,"singularities")
            streamInRange(st,range)
          (n = log :: Symbol)@Boolean =>
            concat([zerosOf(a,vars,range),singularitiesOf(a,vars,range)])
          singularitiesOf(a,vars,range)
        empty()$SDF

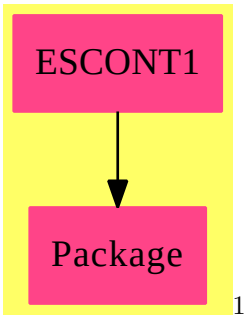
      singularitiesOf(v:VEDF,vars:List Symbol,range:SOCDF):SDF ==
        ls := [singularitiesOf(u,vars,range) for u in entries(v)$VEDF]
        concat(ls)$SDF
```

⟨*ESCONT.dotabb*⟩≡

```
"ESCONT" [color="#FF4488",href="bookvol10.4.pdf#nameddest=ESCONT"]
"ACFS" [color="#4488FF",href="bookvol10.2.pdf#nameddest=ACFS"]
"ESCONT" -> "ACFS"
```

6.37 package ESCONT1 ExpertSystemContinuityPackage1

6.38 ExpertSystemContinuityPackage1



1.00

Exports:
in?

⟨*package ESCONT1 ExpertSystemContinuityPackage1*⟩≡

```
)abbrev package ESCONT1 ExpertSystemContinuityPackage1
++ Author: Brian Dupee
++ Date Created: May 1994
++ Date Last Updated: June 1995
++ Basic Operations: problemPoints, singularitiesOf, zerosOf
++ Related Constructors:
++ Description:
++ ExpertSystemContinuityPackage1 exports a function to check range inclusion

ExpertSystemContinuityPackage1(A:DF,B:DF): E == I where
  EF2   ==> ExpressionFunctions2
  FI    ==> Fraction Integer
  EFI   ==> Expression Fraction Integer
  PFI   ==> Polynomial Fraction Integer
  DF    ==> DoubleFloat
  LDF   ==> List DoubleFloat
  EDF   ==> Expression DoubleFloat
  VEDF  ==> Vector Expression DoubleFloat
  SDF   ==> Stream DoubleFloat
  SS    ==> Stream String
  EEDF  ==> Equation Expression DoubleFloat
  LEDF  ==> List Expression DoubleFloat
  KEDF  ==> Kernel Expression DoubleFloat
  LKEDF ==> List Kernel Expression DoubleFloat
  PDF   ==> Polynomial DoubleFloat
  FPDF  ==> Fraction Polynomial DoubleFloat
```

```
OCDF  ==> OrderedCompletion DoubleFloat
SOCDF ==> Segment OrderedCompletion DoubleFloat
NIA   ==> Record(var:Symbol,fn:EDF,range:SOCDF,abserr:DF,relerr:DF)
UP    ==> UnivariatePolynomial
BO    ==> BasicOperator
RS    ==> Record(zeros: SDF,ones: SDF,singularities: SDF)

E ==> with

  in?:DF -> Boolean
    ++ in?(p) tests whether point p is internal to the range [\spad{A..B}]

I ==> add

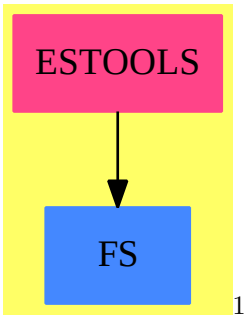
  in?(p:DF):Boolean ==
    a:Boolean := (p < B)$DF
    b:Boolean := (A < p)$DF
    (a and b)@Boolean
```

⟨*ESCONT1.dotabb*⟩≡

```
"ESCONT1" [color="#FF4488",href="bookvol10.4.pdf#nameddest=ESCONT1"]
"Package" [color="#FF4488"]
"ESCONT1" -> "Package"
```

6.39 package ESTOOLS ExpertSystemToolsPackage

6.40 ExpertSystemToolsPackage



1.00

Exports:

att2Result	concat	convert	dflist	dfRange
df2ef	df2fi	df2mf	df2st	edf2df
edf2ef	edf2efi	edf2fi	ef2edf	expenseOfEvaluation
fi2df	f2df	f2st	gethi	getlo
iflist2Result	in?	isQuotient	ldf2lst	ldf2vmf
mat	measure2Result	measure2Result	numberOfOperations	ocf2ocdf
outputMeasure	pdf2df	pdf2ef	sdf2lst	socf2socdf
vedf2vef				

⟨*package ESTOOLS ExpertSystemToolsPackage*⟩≡

```
)abbrev package ESTOOLS ExpertSystemToolsPackage
++ Author: Brian Dupee
++ Date Created: May 1994
++ Date Last Updated: July 1996
++ Basic Operations:
++ Description:
++ \axiom{ExpertSystemToolsPackage} contains some useful functions for use
++ by the computational agents of numerical solvers.
ExpertSystemToolsPackage():E == I where
  LEDF  ==> List Expression DoubleFloat
  KEDF  ==> Kernel Expression DoubleFloat
  LKEDF ==> List Kernel Expression DoubleFloat
  VEDF  ==> Vector Expression DoubleFloat
  VEF   ==> Vector Expression Float
  VMF   ==> Vector MachineFloat
  EF2   ==> ExpressionFunctions2
  EFI   ==> Expression Fraction Integer
  MDF   ==> Matrix DoubleFloat
  LDF   ==> List DoubleFloat
```

```
PDF   ==> Polynomial DoubleFloat
EDF   ==> Expression DoubleFloat
EF    ==> Expression Float
SDF   ==> Stream DoubleFloat
DF    ==> DoubleFloat
F     ==> Float
MF    ==> MachineFloat
INT   ==> Integer
NNI   ==> NonNegativeInteger
LS    ==> List Symbol
ST    ==> String
LST   ==> List String
SS    ==> Stream String
FI    ==> Fraction Integer
R     ==> Ring
OR    ==> OrderedRing
ON    ==> Record(additions:INT,multiplications:INT,exponentiations:INT,function
RVE   ==> Record(val:EDF,exponent:INT)
BO    ==> BasicOperator
OCF   ==> OrderedCompletion Float
OCDF  ==> OrderedCompletion DoubleFloat
SOCF  ==> Segment OrderedCompletion Float
SOCDF ==> Segment OrderedCompletion DoubleFloat
Measure       ==> Record(measure:F, name:String, explanations:List String)
Measure2      ==> Record(measure:F, name:String, explanations:List String, extr
CTYPE ==> Union(continuous: "Continuous at the end points",
             lowerSingular: "There is a singularity at the lower end point",
              upperSingular: "There is a singularity at the upper end point",
               bothSingular: "There are singularities at both end points",
                notEvaluated: "End point continuity not yet evaluated")
RTYPE ==> Union(finite: "The range is finite",
              lowerInfinite: "The bottom of range is infinite",
                upperInfinite: "The top of range is infinite",
                  bothInfinite: "Both top and bottom points are infinite",
                    notEvaluated: "Range not yet evaluated")
STYPE ==> Union(str:SDF,
                   notEvaluated:"Internal singularities not yet evaluated")
ATT   ==> Record(endPointContinuity:CTYPE,singularitiesStream:STYPE,range:RTYPE
IFV   ==> Record(stiffness:F,stability:F,expense:F,accuracy:F,intermediateResul

E ==> with

  f2df:F -> DF
    ++ f2df(f) is a function to convert a \axiomType{Float} to a
    ++ \axiomType{DoubleFloat}
  ef2edf:EF -> EDF
```

```
  ++ ef2edf(f) is a function to convert an \axiomType{Expression Float}
  ++ to an \axiomType{Expression DoubleFloat}
ocf2ocdf: OCF -> OCDF
  ++ ocf2ocdf(a) is a function to convert an \axiomType{OrderedCompletion
  ++ Float} to an \axiomType{OrderedCompletion DoubleFloat}
socf2socdf: SOCF -> SOCDF
  ++ socf2socdf(a) is a function to convert a \axiomType{Segment OrderedCompletion Float
  ++ to a \axiomType{Segment OrderedCompletion DoubleFloat}
convert: List SOCF -> List SOCDF
  ++ convert(l) is a function to convert a \axiomType{Segment OrderedCompletion Float}
  ++ to a \axiomType{Segment OrderedCompletion DoubleFloat}
df2fi :DF -> FI
  ++ df2fi(n) is a function to convert a \axiomType{DoubleFloat} to a
  ++ \axiomType{Fraction Integer}
edf2fi :EDF -> FI
  ++ edf2fi(n) maps \axiomType{Expression DoubleFloat} to
  ++ \axiomType{Fraction Integer}
  ++ It is an error if n is not coercible to Fraction Integer
edf2df :EDF -> DF
  ++ edf2df(n) maps \axiomType{Expression DoubleFloat} to
  ++ \axiomType{DoubleFloat}
  ++ It is an error if \spad{n} is not coercible to DoubleFloat
isQuotient:EDF -> Union(EDF,"failed")
  ++ isQuotient(expr) returns the quotient part of the input
  ++ expression or \spad{"failed"} if the expression is not of that form.
expenseOfEvaluation:VEDF -> F
  ++ expenseOfEvaluation(o) gives an approximation of the cost of
  ++ evaluating a list of expressions in terms of the number of basic
  ++ operations.
  ++ < 0.3 inexpensive ; 0.5 neutral ; > 0.7 very expensive
  ++ 400 'operation units' -> 0.75
  ++ 200 'operation units' -> 0.5
  ++ 83 'operation units' -> 0.25
  ++ ** = 4 units , function calls = 10 units.
numberOfOperations:VEDF -> ON
  ++ numberOfOperations(ode) counts additions, multiplications,
  ++ exponentiations and function calls in the input set of expressions.
edf2efi:EDF -> EFI
  ++ edf2efi(e) coerces \axiomType{Expression DoubleFloat} into
  ++ \axiomType{Expression Fraction Integer}
dfRange:SOCDF -> SOCDF
  ++ dfRange(r) converts a range including
  ++ \inputbitmap{\htbmdir{}/plusminus.bitmap} \infty
  ++ to \axiomType{DoubleFloat} equavalents.
dflist:List(Record(left:FI,right:FI)) -> LDF
  ++ dflist(l) returns a list of \axiomType{DoubleFloat} equivalents of list l
```

```
df2mf:DF -> MF
  ++ df2mf(n) coerces a \axiomType{DoubleFloat} to \axiomType{MachineFloat}
ldf2vmf:LDF -> VMF
  ++ ldf2vmf(l) coerces a \axiomType{List DoubleFloat} to
  ++ \axiomType{List MachineFloat}
edf2ef:EDF -> EF
  ++ edf2ef(e) maps \axiomType{Expression DoubleFloat} to
  ++ \axiomType{Expression Float}
vedf2vef:VEDF -> VEF
  ++ vedf2vef(v) maps \axiomType{Vector Expression DoubleFloat} to
  ++ \axiomType{Vector Expression Float}
in?:(DF,SOCDF) -> Boolean
  ++ in?(p,range) tests whether point p is internal to the
  ++ range range
df2st:DF -> ST
  ++ df2st(n) coerces a \axiomType{DoubleFloat} to \axiomType{String}
f2st:F -> ST
  ++ f2st(n) coerces a \axiomType{Float} to \axiomType{String}
ldf2lst:LDF -> LST
  ++ ldf2lst(ln) coerces a \axiomType{List DoubleFloat} to \axiomType{List St
sdf2lst:SDF -> LST
  ++ sdf2lst(ln) coerces a \axiomType{Stream DoubleFloat} to \axiomType{Strin
getlo : SOCDF -> DF
  ++ getlo(u) gets the \axiomType{DoubleFloat} equivalent of
  ++ the first endpoint of the range \spad{u}
gethi : SOCDF -> DF
  ++ gethi(u) gets the \axiomType{DoubleFloat} equivalent of
  ++ the second endpoint of the range \spad{u}
concat:(Result,Result) -> Result
  ++ concat(a,b) adds two aggregates of type \axiomType{Result}.
concat:(List Result) -> Result
  ++ concat(l) concatenates a list of aggregates of type \axiomType{Result}
outputMeasure:F -> ST
  ++ outputMeasure(n) rounds \spad{n} to 3 decimal places and outputs
  ++ it as a string
measure2Result:Measure -> Result
  ++ measure2Result(m) converts a measure record into a \axiomType{Result}
measure2Result:Measure2 -> Result
  ++ measure2Result(m) converts a measure record into a \axiomType{Result}
att2Result:ATT -> Result
  ++ att2Result(m) converts a attributes record into a \axiomType{Result}
iflist2Result:IFV -> Result
  ++ iflist2Result(m) converts a attributes record into a \axiomType{Result}
pdf2ef:PDF -> EF
  ++ pdf2ef(p) coerces a \axiomType{Polynomial DoubleFloat} to
  ++ \axiomType{Expression Float}
```

```
    pdf2df:PDF -> DF
      ++ pdf2df(p) coerces a \axiomType{Polynomial DoubleFloat} to
      ++ \axiomType{DoubleFloat}. It is an error if \axiom{p} is not
      ++ retractable to DoubleFloat.
    df2ef:DF -> EF
      ++ df2ef(a) coerces a \axiomType{DoubleFloat} to \axiomType{Expression Float}
    fi2df:FI -> DF
      ++ fi2df(f) coerces a \axiomType{Fraction Integer} to \axiomType{DoubleFloat}
    mat:(LDF,NNI) -> MDF
      ++ mat(a,n) constructs a one-dimensional matrix of a.

  I ==> add

    mat(a:LDF,n:NNI):MDF ==
      empty?(a)$LDF => zero(1,n)$MDF
      matrix(list([i for i in a for j in 1..n])$(List LDF))$MDF

    f2df(f:F):DF == (convert(f)@DF)$F

    ef2edf(f:EF):EDF == map(f2df,f)$EF2(F,DF)

    fi2df(f:FI):DF == coerce(f)$DF

    ocf2ocdf(a:OCF):OCDF ==
      finite? a => (f2df(retract(a)@F))::OCDF
      a pretend OCDF

    socf2socdf(a:SOCF):SOCDF ==
      segment(ocf2ocdf(lo a),ocf2ocdf(hi a))

    convert(l:List SOCF):List SOCDF == [socf2socdf a for a in l]

    pdf2df(p:PDF):DF == retract(p)@DF

    df2ef(a:DF):EF ==
      b := convert(a)@Float
      coerce(b)$EF

    pdf2ef(p:PDF):EF == df2ef(pdf2df(p))

    edf2fi(m:EDF):FI == retract(retract(m)@DF)@FI

    edf2df(m:EDF):DF == retract(m)@DF

    df2fi(r:DF):FI == (retract(r)@FI)$DF
```

```
dfRange(r:SOCDF):SOCDF ==
  if infinite?(lo(r))$OCDF then r := -(max()$DF :: OCDF)..hi(r)$SOCDF
  if infinite?(hi(r))$OCDF then r := lo(r)$SOCDF..(max()$DF :: OCDF)
  r

dflist(l:List(Record(left:FI,right:FI))):LDF == [u.left :: DF for u in l]

edf2efi(f:EDF):EFI == map(df2fi,f)$EF2(DF,FI)

df2st(n:DF):String == (convert((convert(n)@Float)$DF)@ST)$Float

f2st(n:F):String == (convert(n)@ST)$Float

ldf2lst(ln:LDF):LST == [df2st f for f in ln]

sdf2lst(ln:SDF):LST ==
  explicitlyFinite? ln =>
    m := map(df2st,ln)$StreamFunctions2(DF,ST)
    if index?(20,m)$SS then
      split!(m,20)
      m := concat(m,".......")
    m := complete(m)$SS
    entries(m)$SS
  empty()$LST

df2mf(n:DF):MF == (df2fi(n))::MF

ldf2vmf(l:LDF):VMF ==
  m := [df2mf(n) for n in l]
  vector(m)$VMF

edf2ef(e:EDF):EF == map(convert$DF,e)$EF2(DF,Float)

vedf2vef(vedf:VEDF):VEF == vector([edf2ef e for e in members(vedf)])

getlo(u:SOCDF):DF == retract(lo(u))@DF

gethi(u:SOCDF):DF == retract(hi(u))@DF

in?(p:DF,range:SOCDF):Boolean ==
  top := gethi(range)
  bottom := getlo(range)
  a:Boolean := (p < top)$DF
  b:Boolean := (p > bottom)$DF
  (a and b)@Boolean
```

```
    isQuotient(expr:EDF):Union(EDF,"failed") ==
      (k := mainKernel expr) case KEDF =>
        (expr = inv(f := k :: KEDF :: EDF)$EDF)$EDF => f
--        one?(numerator expr) => denominator expr
        (numerator expr) = 1 => denominator expr
        "failed"
      "failed"

    numberOfOperations1(fn:EDF,numbersSoFar:ON):ON ==
      (u := isQuotient(fn)) case EDF =>
        numbersSoFar := numberOfOperations1(u,numbersSoFar)
      (p := isPlus(fn)) case LEDF =>
        p := coerce(p)@LEDF
        np := #p
        numbersSoFar.additions := (numbersSoFar.additions)+np-1
        for i in 1..np repeat
          numbersSoFar := numberOfOperations1(p.i,numbersSoFar)
        numbersSoFar
      (t:=isTimes(fn)) case LEDF =>
        t := coerce(t)@LEDF
        nt := #t
        numbersSoFar.multiplications := (numbersSoFar.multiplications)+nt-1
        for i in 1..nt repeat
          numbersSoFar := numberOfOperations1(t.i,numbersSoFar)
        numbersSoFar
      if (e:=isPower(fn)) case RVE then
        e := coerce(e)@RVE
        e.exponent>1 =>
          numbersSoFar.exponentiations := inc(numbersSoFar.exponentiations)
          numbersSoFar := numberOfOperations1(e.val,numbersSoFar)
      lk := kernels(fn)
      #lk = 1 =>        -- #lk = 0 => constant found (no further action)
        k := first(lk)$LKEDF
        n := name(operator(k)$KEDF)$BO
        entry?(n,variables(fn)$EDF)$LS => numbersSoFar  -- solo variable found
        a := first(argument(k)$KEDF)$LEDF
        numbersSoFar.functionCalls := inc(numbersSoFar.functionCalls)$INT
        numbersSoFar := numberOfOperations1(a,numbersSoFar)
      numbersSoFar

    numberOfOperations(ode:VEDF):ON ==
      n:ON := [0,0,0,0]
      for i in 1..#ode repeat
        n:ON := numberOfOperations1(ode.i,n)
      n
```

```
expenseOfEvaluation(o:VEDF):F ==
  ln:ON := numberOfOperations(o)
  a := ln.additions
  m := ln.multiplications
  e := ln.exponentiations
  f := 10*ln.functionCalls
  n := (a + m + 4*e + 10*e)
  (1.0-exp((-n::F/288.0))$F)

concat(a:Result,b:Result):Result ==
  membersOfa := (members(a)@List(Record(key:Symbol,entry:Any)))
  membersOfb := (members(b)@List(Record(key:Symbol,entry:Any)))
  allMembers:=
    concat(membersOfa,membersOfb)$List(Record(key:Symbol,entry:Any))
  construct(allMembers)

concat(l:List Result):Result ==
  import List Result
  empty? l => empty()$Result
  f := first l
  if empty?(r := rest l) then
    f
  else
    concat(f,concat r)

outputMeasure(m:F):ST ==
  fl:Float := round(m*(f:= 1000.0))/f
  convert(fl)@ST

measure2Result(m:Measure):Result ==
  mm := coerce(m.measure)$AnyFunctions1(Float)
  mmr:Record(key:Symbol,entry:Any) := [bestMeasure@Symbol,mm]
  mn := coerce(m.name)$AnyFunctions1(ST)
  mnr:Record(key:Symbol,entry:Any) := [nameOfRoutine@Symbol,mn]
  me := coerce(m.explanations)$AnyFunctions1(List String)
  mer:Record(key:Symbol,entry:Any) := [allMeasures@Symbol,me]
  mr := construct([mmr,mnr,mer])$Result
  met := coerce(mr)$AnyFunctions1(Result)
  meth:Record(key:Symbol,entry:Any):=[method@Symbol,met]
  construct([meth])$Result

measure2Result(m:Measure2):Result ==
  mm := coerce(m.measure)$AnyFunctions1(Float)
  mmr:Record(key:Symbol,entry:Any) := [bestMeasure@Symbol,mm]
  mn := coerce(m.name)$AnyFunctions1(ST)
  mnr:Record(key:Symbol,entry:Any) := [nameOfRoutine@Symbol,mn]
```

```
    me := coerce(m.explanations)$AnyFunctions1(List String)
    mer:Record(key:Symbol,entry:Any) := [allMeasures@Symbol,me]
    mx := coerce(m.extra)$AnyFunctions1(Result)
    mxr:Record(key:Symbol,entry:Any) := [other@Symbol,mx]
    mr := construct([mmr,mnr,mer,mxr])$Result
    met := coerce(mr)$AnyFunctions1(Result)
    meth:Record(key:Symbol,entry:Any):=[method@Symbol,met]
    construct([meth])$Result

  att2Result(att:ATT):Result ==
    aepc := coerce(att.endPointContinuity)$AnyFunctions1(CTYPE)
    ar := coerce(att.range)$AnyFunctions1(RTYPE)
    as := coerce(att.singularitiesStream)$AnyFunctions1(STYPE)
    aa:List Any := [aepc,ar,as]
    aaa := coerce(aa)$AnyFunctions1(List Any)
    aar:Record(key:Symbol,entry:Any) := [attributes@Symbol,aaa]
    construct([aar])$Result

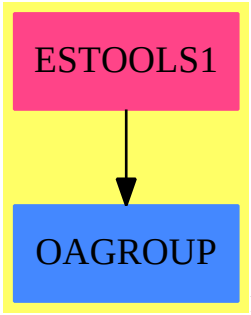
  iflist2Result(ifv:IFV):Result ==
    ifvs:List String :=
      [concat(["stiffness: ",outputMeasure(ifv.stiffness)]),
        concat(["stability: ",outputMeasure(ifv.stability)]),
         concat(["expense: ",outputMeasure(ifv.expense)]),
          concat(["accuracy: ",outputMeasure(ifv.accuracy)]),
           concat(["intermediateResults: ",outputMeasure(ifv.intermediateResults)])]
    ifa:= coerce(ifvs)$AnyFunctions1(List String)
    ifr:Record(key:Symbol,entry:Any) := [intensityFunctions@Symbol,ifa]
    construct([ifr])$Result
```

⟨*ESTOOLS.dotabb*⟩≡

```
"ESTOOLS" [color="#FF4488",href="bookvol10.4.pdf#nameddest=ESTOOLS"]
"FS" [color="#4488FF",href="bookvol10.2.pdf#nameddest=FS"]
"ESTOOLS" -> "FS"
```

6.41 package ESTOOLS1 ExpertSystemToolsPackage1

6.42 ExpertSystemToolsPackage1



1.00

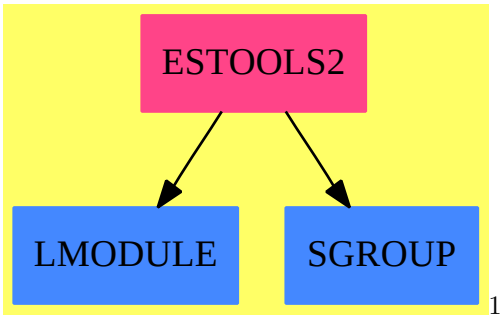
Exports:
neglist

⟨*package ESTOOLS1 ExpertSystemToolsPackage1*⟩≡
```
)abbrev package ESTOOLS1 ExpertSystemToolsPackage1
++ Author: Brian Dupee
++ Date Created: February 1995
++ Date Last Updated: February 1995
++ Basic Operations: neglist
++ Description:
++ \axiom{ExpertSystemToolsPackage1} contains some useful functions for use
++ by the computational agents of Ordinary Differential Equation solvers.
ExpertSystemToolsPackage1(R1:OR): E == I where
  OR    ==> OrderedRing
  E ==>  with
    neglist:List R1 -> List R1
      ++ neglist(l) returns only the negative elements of the list \spad{l}
  I ==> add
    neglist(l:List R1):List R1 == [u for u in l | negative?(u)$R1]
```

⟨*ESTOOLS1.dotabb*⟩≡
```
"ESTOOLS1" [color="#FF4488",href="bookvol10.4.pdf#nameddest=ESTOOLS1"]
"OAGROUP" [color="#4488FF",href="bookvol10.2.pdf#nameddest=OAGROUP"]
"ESTOOLS1" -> "OAGROUP"
```

6.43 package ESTOOLS2 ExpertSystemToolsPackage2

6.44 ExpertSystemToolsPackage2



1.00

Exports:
map

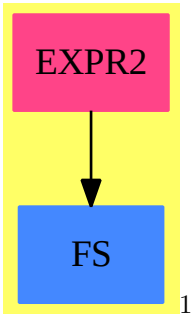
⟨*package ESTOOLS2 ExpertSystemToolsPackage2*⟩≡
```
)abbrev package ESTOOLS2 ExpertSystemToolsPackage2
++ Author: Brian Dupee
++ Date Created: February 1995
++ Date Last Updated: July 1996
++ Basic Operations: map
++ Related Constructors: Matrix
++ Description:
++ \axiom{ExpertSystemToolsPackage2} contains some useful functions for use
++ by the computational agents of Ordinary Differential Equation solvers.
ExpertSystemToolsPackage2(R1:R,R2:R): E == I where
  R     ==> Ring
  E ==>  with
    map:(R1->R2,Matrix R1) -> Matrix R2
      ++ map(f,m) applies a mapping f:R1 -> R2 onto a matrix
      ++ \spad{m} in R1 returning a matrix in R2
  I ==> add
    map(f:R1->R2,m:Matrix R1):Matrix R2 ==
      matrix([[f u for u in v] for v in listOfLists(m)$(Matrix R1)])$(Matrix R2)
```

⟨*ESTOOLS2.dotabb*⟩≡

```
"ESTOOLS2" [color="#FF4488",href="bookvol10.4.pdf#nameddest=ESTOOLS2"]
"LMODULE" [color="#4488FF",href="bookvol10.2.pdf#nameddest=LMODULE"]
"SGROUP" [color="#4488FF",href="bookvol10.2.pdf#nameddest=SGROUP"]
"ESTOOLS2" -> "LMODULE"
"ESTOOLS2" -> "SGROUP"
```

6.45 package EXPR2 ExpressionFunctions2

6.46 ExpressionFunctions2



1.00

Exports:
map

⟨*package EXPR2 ExpressionFunctions2*⟩≡

```
)abbrev package EXPR2 ExpressionFunctions2
++ Lifting of maps to Expressions
++ Author: Manuel Bronstein
++ Description: Lifting of maps to Expressions.
++ Date Created: 16 Jan 1989
++ Date Last Updated: 22 Jan 1990
ExpressionFunctions2(R:OrderedSet, S:OrderedSet):
 Exports == Implementation where
  K   ==> Kernel R
  F2  ==> FunctionSpaceFunctions2(R, Expression R, S, Expression S)
  E2  ==> ExpressionSpaceFunctions2(Expression R, Expression S)

  Exports ==> with
    map: (R -> S, Expression R) -> Expression S
      ++ map(f, e) applies f to all the constants appearing in e.

  Implementation == add
    if S has Ring and R has Ring then
      map(f, r) == map(f, r)$F2
    else
      map(f, r) == map(x1 +-> map(f, x1), retract r)$E2
```

⟨*EXPR2.dotabb*⟩≡

```
"EXPR2" [color="#FF4488",href="bookvol10.4.pdf#nameddest=EXPR2"]
"FS" [color="#4488FF",href="bookvol10.2.pdf#nameddest=FS"]
"EXPR2" -> "FS"
```

6.47 package EXPRSOL ExpressionSolve

6.47.1 Bugs

```
seriesSolve(sin f x / cos x, f, x, [1])$EXPRSOL(INT, EXPR INT, UFPS EXPR INT, UFPS SUPEXPR EXPR INT)
```

returns

```
(((0 . 1) 0 . 1) NonNullStream #<compiled-function |STREAM;generate;M$;62!0|> . UNPRINTABLE)
```

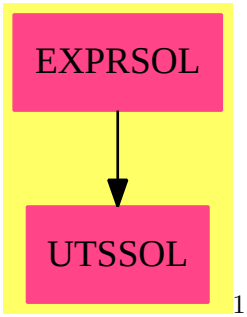
but

```
U ==> UFPS SUPEXPR EXPR INT
seriesSolve(s +-> sin s *((cos monomial(1,1)$U)**-1)$U, f, x, [0])$EXPRSOL(INT, EXPR INT, UFPS EXPR INT,
```

works. This is probably due to missing "/" in UFPS.

```
  I'd really like to be able to specify a function that works for all domains
  in a category. For example, {\tt{}x\ +->\ y(x){\char94}2\ +\ sin\ x\ +\ x} should \lq work\rq\
  for {\tt{}EXPR\ INT} as well as for {\tt{}UTS\ INT}, both being domains having
  {\tt{}TranscendentalFunctionCategory}.
```

6.48 ExpressionSolve



1.00

Exports:
replaceDiffs seriesSolve

⟨*package EXPRSOL ExpressionSolve*⟩≡

```
)abbrev package EXPRSOL ExpressionSolve
ExpressionSolve(R, F, UTSF, UTSSUPF): Exports == Implementation where
    R: Join(OrderedSet, IntegralDomain, ConvertibleTo InputForm)
    F: FunctionSpace R
    UTSF: UnivariateTaylorSeriesCategory F
    SUP  ==> SparseUnivariatePolynomialExpressions
    UTSSUPF: UnivariateTaylorSeriesCategory SUP F
```

```
    OP   ==> BasicOperator
    SY   ==> Symbol
    NNI  ==> NonNegativeInteger
    MKF ==> MakeBinaryCompiledFunction(F, UTSSUPF, UTSSUPF, UTSSUPF)

    Exports == with

        seriesSolve: (F, OP, SY, List F) -> UTSF
        replaceDiffs: (F, OP, Symbol) -> F

    Implementation == add
```

⟨*implementation: EXPRSOL ExpressionSolve*⟩

The general method is to transform the given expression into a form which can then be compiled. There is currently no other way in Axiom to transform an expression into a function.

We need to replace the differentiation operator by the corresponding function in the power series category, and make composition explicit. Furthermore, we need to replace the variable by the corresponding variable in the power series. It turns out that the compiler doesn't find the right definition of monomial(1,1). Thus we introduce it as a second argument. In fact, maybe that's even cleaner. Also, we need to tell the compiler that kernels that are independent of the main variable should be coerced to elements of the coefficient ring, since it will complain otherwise.

```
I cannot find an example for this behaviour right now. However, if I do use
the coerce, the following fails:

   seriesSolve(h x -1-x*h x *h(q*x), h, x, [1])
```

⟨*implementation: EXPRSOL ExpressionSolve*⟩≡

```
          opelt := operator("elt"::Symbol)$OP
          opdiff := operator("D"::Symbol)$OP
          opcoerce := operator("coerce"::Symbol)$OP

  --        replaceDiffs: (F, OP, Symbol) -> F
          replaceDiffs (expr, op, sy) ==
              lk := kernels expr
              for k in lk repeat
  --                if freeOf?(coerce k, sy) then
  --                    expr := subst(expr, [k], [opcoerce [coerce k]])

                  if is?(k, op) then
                      arg := first argument k
                      if arg = sy::F
                      then expr := subst(expr, [k], [(name op)::F])
                      else expr := subst(expr, [k], [opelt [(name op)::F,
                                                            replaceDiffs(arg, op,
                                                            sy)]])
  --                    => "iterate"

                  if is?(k, %diff) then
                      args := argument k
                      differentiand :=
                       replaceDiffs(subst(args.1, args.2 = args.3), op, sy)
                      expr := subst(expr, [k], [opdiff differentiand])
  --                    => "iterate"
              expr
```

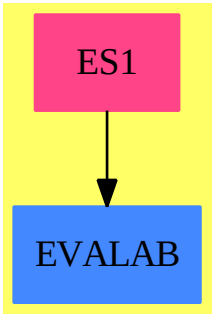
```
        seriesSolve(expr, op, sy, l) ==
            ex := replaceDiffs(expr, op, sy)
            f := compiledFunction(ex, name op, sy)$MKF
            seriesSolve(x+->f(x, monomial(1,1)$UTSSUPF), l)_
              $TaylorSolve(F, UTSF, UTSSUPF)
```

⟨*EXPRSOL.dotabb*⟩≡

```
"EXPRSOL" [color="#FF4488",href="bookvol10.4.pdf#nameddest=EXPRSOL"]
"UTSSOL" [color="#FF4488",href="bookvol10.4.pdf#nameddest=UTSSOL"]
"EXPRSOL" -> "UTSSOL"
```

6.49 package ES1 ExpressionSpaceFunctions1

6.50 ExpressionSpaceFunctions1



1.00

Exports:
map

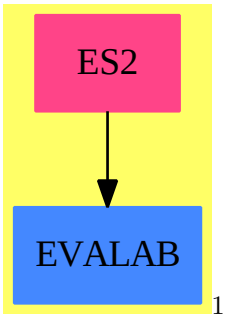
⟨*package ES1 ExpressionSpaceFunctions1*⟩≡
```
)abbrev package ES1 ExpressionSpaceFunctions1
++ Lifting of maps from expression spaces to kernels over them
++ Author: Manuel Bronstein
++ Date Created: 23 March 1988
++ Date Last Updated: 19 April 1991
++ Description:
++   This package allows a map from any expression space into any object
++   to be lifted to a kernel over the expression set, using a given
++   property of the operator of the kernel.
-- should not be exposed
ExpressionSpaceFunctions1(F:ExpressionSpace, S:Type): with
    map: (F -> S, String, Kernel F) -> S
      ++ map(f, p, k) uses the property p of the operator
      ++ of k, in order to lift f and apply it to k.

  == add
    --  prop  contains an evaluation function List S -> S
    map(F2S, prop, k) ==
      args := [F2S x for x in argument k]$List(S)
      (p := property(operator k, prop)) case None =>
                                  ((p::None) pretend (List S -> S)) args
      error "Operator does not have required property"
```

⟨*ES1.dotabb*⟩≡
```
"ES1" [color="#FF4488",href="bookvol10.4.pdf#nameddest=ES1"]
"EVALAB" [color="#4488FF",href="bookvol10.2.pdf#nameddest=EVALAB"]
"ES1" -> "EVALAB"
```

6.51 package ES2 ExpressionSpaceFunctions2

6.52 ExpressionSpaceFunctions2



1.00

Exports:
map

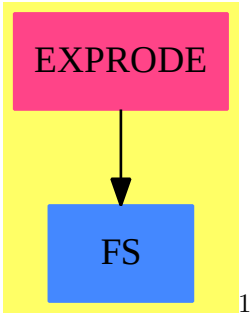
⟨*package ES2 ExpressionSpaceFunctions2*⟩≡
```
)abbrev package ES2 ExpressionSpaceFunctions2
++ Lifting of maps from expression spaces to kernels over them
++ Author: Manuel Bronstein
++ Date Created: 23 March 1988
++ Date Last Updated: 19 April 1991
++ Description:
++ This package allows a mapping E -> F to be lifted to a kernel over E;
++ This lifting can fail if the operator of the kernel cannot be applied
++ in F; Do not use this package with E = F, since this may
++ drop some properties of the operators.
ExpressionSpaceFunctions2(E:ExpressionSpace, F:ExpressionSpace): with
    map: (E -> F, Kernel E) -> F
      ++ map(f, k) returns \spad{g = op(f(a1),...,f(an))} where
      ++ \spad{k = op(a1,...,an)}.
  == add
    map(f, k) ==
      (operator(operator k)$F) [f x for x in argument k]$List(F)
```

⟨*ES2.dotabb*⟩≡

```
"ES2" [color="#FF4488",href="bookvol10.4.pdf#nameddest=ES2"]
"EVALAB" [color="#4488FF",href="bookvol10.2.pdf#nameddest=EVALAB"]
"ES2" -> "EVALAB"
```

6.53 package EXPRODE ExpressionSpaceODE-Solver

6.54 ExpressionSpaceODESolver



1.00

Exports:
seriesSolve

⟨*package EXPRODE ExpressionSpaceODESolver*⟩≡

```
)abbrev package EXPRODE ExpressionSpaceODESolver
++ Taylor series solutions of ODE's
++ Author: Manuel Bronstein
++ Date Created: 5 Mar 1990
++ Date Last Updated: 30 September 1993
++ Description: Taylor series solutions of explicit ODE's;
++ Keywords: differential equation, ODE, Taylor series
ExpressionSpaceODESolver(R, F): Exports == Implementation where
  R: Join(OrderedSet, IntegralDomain, ConvertibleTo InputForm)
  F: FunctionSpace R

  K   ==> Kernel F
  P   ==> SparseMultivariatePolynomial(R, K)
  OP  ==> BasicOperator
  SY  ==> Symbol
  UTS ==> UnivariateTaylorSeries(F, x, center)
  MKF ==> MakeUnaryCompiledFunction(F, UTS, UTS)
  MKL ==> MakeUnaryCompiledFunction(F, List UTS, UTS)
  A1  ==> AnyFunctions1(UTS)
  AL1 ==> AnyFunctions1(List UTS)
  EQ  ==> Equation F
  ODE ==> UnivariateTaylorSeriesODESolver(F, UTS)

  Exports ==> with
    seriesSolve: (EQ, OP, EQ, EQ) -> Any
      ++ seriesSolve(eq,y,x=a, y a = b) returns a Taylor series solution
```

```
    ++ of eq around x = a with initial condition \spad{y(a) = b}.
    ++ Note: eq must be of the form
    ++ \spad{f(x, y x) y'(x) + g(x, y x) = h(x, y x)}.
  seriesSolve: (EQ, OP, EQ, List F) -> Any
    ++ seriesSolve(eq,y,x=a,[b0,...,b(n-1)]) returns a Taylor series
    ++ solution of eq around \spad{x = a} with initial conditions
    ++ \spad{y(a) = b0}, \spad{y'(a) = b1},
    ++ \spad{y''(a) = b2}, ...,\spad{y(n-1)(a) = b(n-1)}
    ++ eq must be of the form
    ++ \spad{f(x, y x, y'(x),..., y(n-1)(x)) y(n)(x) +
    ++ g(x,y x,y'(x),...,y(n-1)(x)) = h(x,y x, y'(x),..., y(n-1)(x))}.
  seriesSolve: (List EQ, List OP, EQ, List EQ) -> Any
    ++ seriesSolve([eq1,...,eqn],[y1,...,yn],x = a,[y1 a = b1,...,yn a = bn])
    ++ returns a taylor series solution of \spad{[eq1,...,eqn]} around
    ++ \spad{x = a} with initial conditions \spad{yi(a) = bi}.
    ++ Note: eqi must be of the form
    ++ \spad{fi(x, y1 x, y2 x,..., yn x) y1'(x) +
    ++ gi(x, y1 x, y2 x,..., yn x) = h(x, y1 x, y2 x,..., yn x)}.
  seriesSolve: (List EQ, List OP, EQ, List F) -> Any
    ++ seriesSolve([eq1,...,eqn], [y1,...,yn], x=a, [b1,...,bn])
    ++ is equivalent to
    ++ \spad{seriesSolve([eq1,...,eqn], [y1,...,yn], x = a,
    ++ [y1 a = b1,..., yn a = bn])}.
  seriesSolve: (List F, List OP, EQ, List F) -> Any
    ++ seriesSolve([eq1,...,eqn], [y1,...,yn], x=a, [b1,...,bn])
    ++ is equivalent to
    ++ \spad{seriesSolve([eq1=0,...,eqn=0], [y1,...,yn], x=a, [b1,...,bn])}.
  seriesSolve: (List F, List OP, EQ, List EQ) -> Any
    ++ seriesSolve([eq1,...,eqn], [y1,...,yn],
    ++ x = a,[y1 a = b1,..., yn a = bn])
    ++ is equivalent to
    ++ \spad{seriesSolve([eq1=0,...,eqn=0], [y1,...,yn], x = a,
    ++ [y1 a = b1,..., yn a = bn])}.
  seriesSolve: (EQ, OP, EQ, F) -> Any
    ++ seriesSolve(eq,y, x=a, b) is equivalent to
    ++ \spad{seriesSolve(eq, y, x=a, y a = b)}.
  seriesSolve: (F, OP, EQ, F) -> Any
    ++ seriesSolve(eq, y, x = a, b) is equivalent to
    ++ \spad{seriesSolve(eq = 0, y, x = a, y a = b)}.
  seriesSolve: (F, OP, EQ, EQ) -> Any
    ++ seriesSolve(eq, y, x = a, y a = b) is equivalent to
    ++ \spad{seriesSolve(eq=0, y, x=a, y a = b)}.
  seriesSolve: (F, OP, EQ, List F) -> Any
    ++ seriesSolve(eq, y, x = a, [b0,...,bn]) is equivalent to
    ++ \spad{seriesSolve(eq = 0, y, x = a, [b0,...,b(n-1)])}.
```

```
  Implementation ==> add
    checkCompat: (OP, EQ, EQ) -> F
    checkOrder1: (F, OP, K, SY, F) -> F
    checkOrderN: (F, OP, K, SY, F, NonNegativeInteger) -> F
    checkSystem: (F, List K, List F) -> F
    div2exquo  : F -> F
    smp2exquo  : P -> F
    k2exquo    : K -> F
    diffRhs    : (F, F) -> F
    diffRhsK   : (K, F) -> F
    findCompat : (F, List EQ) -> F
    findEq     : (K, SY, List F) -> F
    localInteger: F -> F

    opelt := operator("elt"::Symbol)$OP
    --opex  := operator("exquo"::Symbol)$OP
    opex  := operator("fixedPointExquo"::Symbol)$OP
    opint := operator("integer"::Symbol)$OP

    Rint? := R has IntegerNumberSystem

    localInteger n == (Rint? => n; opint n)
    diffRhs(f, g) == diffRhsK(retract(f)@K, g)

    k2exquo k ==
      is?(op := operator k, "%diff"::Symbol) =>
        error "Improper differential equation"
      kernel(op, [div2exquo f for f in argument k]$List(F))

    smp2exquo p ==
      map(k2exquo,x+->x::F,p)_
       $PolynomialCategoryLifting(IndexedExponents K,K, R, P, F)

    div2exquo f ==
--      one?(d := denom f) => f
      ((d := denom f) = 1) => f
      opex(smp2exquo numer f, smp2exquo d)

-- if g is of the form a * k + b, then return -b/a
    diffRhsK(k, g) ==
      h := univariate(g, k)
      (degree(numer h) <= 1) and ground? denom h =>
        - coefficient(numer h, 0) / coefficient(numer h, 1)
      error "Improper differential equation"

    checkCompat(y, eqx, eqy) ==
```

```
    lhs(eqy) =$F y(rhs eqx) => rhs eqy
    error "Improper initial value"

  findCompat(yx, l) ==
    for eq in l repeat
      yx =$F lhs eq => return rhs eq
    error "Improper initial value"

  findEq(k, x, sys) ==
    k := retract(differentiate(k::F, x))@K
    for eq in sys repeat
      member?(k, kernels eq) => return eq
    error "Improper differential equation"

  checkOrder1(diffeq, y, yx, x, sy) ==
    div2exquo subst(diffRhs(differentiate(yx::F,x),diffeq),[yx],[sy])

  checkOrderN(diffeq, y, yx, x, sy, n) ==
    zero? n => error "No initial value(s) given"
    m     := (minIndex(l := [retract(f := yx::F)@K]$List(K)))::F
    lv    := [opelt(sy, localInteger m)]$List(F)
    for i in 2..n repeat
      l  := concat(retract(f := differentiate(f, x))@K, l)
      lv := concat(opelt(sy, localInteger(m := m + 1)), lv)
    div2exquo subst(diffRhs(differentiate(f, x), diffeq), l, lv)

  checkSystem(diffeq, yx, lv) ==
    for k in kernels diffeq repeat
      is?(k, "%diff"::SY) =>
        return div2exquo subst(diffRhsK(k, diffeq), yx, lv)
    0

  seriesSolve(l:List EQ, y:List OP, eqx:EQ, eqy:List EQ) ==
    seriesSolve([lhs deq - rhs deq for deq in l]$List(F), y, eqx, eqy)

  seriesSolve(l:List EQ, y:List OP, eqx:EQ, y0:List F) ==
    seriesSolve([lhs deq - rhs deq for deq in l]$List(F), y, eqx, y0)

  seriesSolve(l:List F, ly:List OP, eqx:EQ, eqy:List EQ) ==
    seriesSolve(l, ly, eqx,
                [findCompat(y rhs eqx, eqy) for y in ly]$List(F))

  seriesSolve(diffeq:EQ, y:OP, eqx:EQ, eqy:EQ) ==
    seriesSolve(lhs diffeq - rhs diffeq, y, eqx, eqy)

  seriesSolve(diffeq:EQ, y:OP, eqx:EQ, y0:F) ==
```

```
    seriesSolve(lhs diffeq - rhs diffeq, y, eqx, y0)

  seriesSolve(diffeq:EQ, y:OP, eqx:EQ, y0:List F) ==
    seriesSolve(lhs diffeq - rhs diffeq, y, eqx, y0)

  seriesSolve(diffeq:F, y:OP, eqx:EQ, eqy:EQ) ==
    seriesSolve(diffeq, y, eqx, checkCompat(y, eqx, eqy))

  seriesSolve(diffeq:F, y:OP, eqx:EQ, y0:F) ==
    x      := symbolIfCan(retract(lhs eqx)@K)::SY
    sy     := name y
    yx     := retract(y lhs eqx)@K
    f      := checkOrder1(diffeq, y, yx, x, sy::F)
    center := rhs eqx
    coerce(ode1(compiledFunction(f, sy)$MKF, y0)$ODE)$A1

  seriesSolve(diffeq:F, y:OP, eqx:EQ, y0:List F) ==
    x      := symbolIfCan(retract(lhs eqx)@K)::SY
    sy     := new()$SY
    yx     := retract(y lhs eqx)@K
    f      := checkOrderN(diffeq, y, yx, x, sy::F, #y0)
    center := rhs eqx
    coerce(ode(compiledFunction(f, sy)$MKL, y0)$ODE)$A1

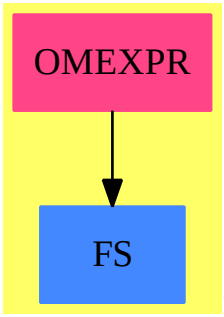
  seriesSolve(sys:List F, ly:List OP, eqx:EQ, l0:List F) ==
    x      := symbolIfCan(kx := retract(lhs eqx)@K)::SY
    fsy    := (sy := new()$SY)::F
    m      := (minIndex(l0) - 1)::F
    yx     := concat(kx, [retract(y lhs eqx)@K for y in ly]$List(K))
    lelt   := [opelt(fsy, localInteger(m := m+1)) for k in yx]$List(F)
    sys    := [findEq(k, x, sys) for k in rest yx]
    l      := [checkSystem(eq, yx, lelt) for eq in sys]$List(F)
    center := rhs eqx
    coerce(mpsode(l0,[compiledFunction(f,sy)$MKL for f in l])$ODE)$AL1
```

⟨*EXPRODE.dotabb*⟩≡

```
"EXPRODE" [color="#FF4488",href="bookvol10.4.pdf#nameddest=EXPRODE"]
"FS" [color="#4488FF",href="bookvol10.2.pdf#nameddest=FS"]
"EXPRODE" -> "FS"
```

6.55 package OMEXPR ExpressionToOpenMath

6.56 ExpressionToOpenMath



1.00

Exports:
OMwrite

⟨*package OMEXPR ExpressionToOpenMath*⟩≡
```
)abbrev package OMEXPR ExpressionToOpenMath
++ Author: Mike Dewar & Vilya Harvey
++ Date Created:
++ Date Last Updated:
++ Basic Functions:
++ Related Constructors:
++ Also See:
++ AMS Classifications:
++ Keywords:
++ References:
++ Description: \spadtype{ExpressionToOpenMath} provides support for
++ converting objects of type \spadtype{Expression} into OpenMath.
ExpressionToOpenMath(R: Join(OpenMath, OrderedSet, Ring)): with
  OMwrite  : Expression R -> String
  OMwrite  : (Expression R, Boolean) -> String
  OMwrite  : (OpenMathDevice, Expression R) -> Void
  OMwrite  : (OpenMathDevice, Expression R, Boolean) -> Void
 == add
  import Expression R
  SymInfo ==> Record(cd:String, name:String)
  import SymInfo
  import Record(key: Symbol, entry: SymInfo)
  import AssociationList(Symbol, SymInfo)
  import OMENC

  ----------------------------
  -- Local translation tables.
```

```
----------------------------

nullaryFunctionAList : AssociationList(Symbol, SymInfo) := construct [_
  [pi, ["nums1", "pi"]] ]

unaryFunctionAList : AssociationList(Symbol, SymInfo) := construct [_
  [exp,  ["transc1", "exp"]],_
  [log,  ["transc1", "ln"]],_
  [sin,  ["transc1", "sin"]],_
  [cos,  ["transc1", "cos"]],_
  [tan,  ["transc1", "tan"]],_
  [cot,  ["transc1", "cot"]],_
  [sec,  ["transc1", "sec"]],_
  [csc,  ["transc1", "csc"]],_
  [asin, ["transc1", "arcsin"]],_
  [acos, ["transc1", "arccos"]],_
  [atan, ["transc1", "arctan"]],_
  [acot, ["transc1", "arccot"]],_
  [asec, ["transc1", "arcsec"]],_
  [acsc, ["transc1", "arccsc"]],_
  [sinh, ["transc1", "sinh"]],_
  [cosh, ["transc1", "cosh"]],_
  [tanh, ["transc1", "tanh"]],_
  [coth, ["transc1", "coth"]],_
  [sech, ["transc1", "sech"]],_
  [csch, ["transc1", "csch"]],_
  [asinh, ["transc1", "arcsinh"]],_
  [acosh, ["transc1", "arccosh"]],_
  [atanh, ["transc1", "arctanh"]],_
  [acoth, ["transc1", "arccoth"]],_
  [asech, ["transc1", "arcsech"]],_
  [acsch, ["transc1", "arccsch"]],_
  [factorial, ["integer1", "factorial"]],_
  [abs, ["arith1", "abs"]] ]

  -- Still need the following unary functions:
  --  digamma
  --  Gamma
  --  airyAi
  --  airyBi
  --  erf
  --  Ei
  --  Si
  --  Ci
  --  li
  --  dilog
```

```
  -- Still need the following binary functions:
  --      Gamma(a, x)
  --      Beta(x,y)
  --      polygamma(k,x)
  --      besselJ(v,x)
  --      besselY(v,x)
  --      besselI(v,x)
  --      besselK(v,x)
  --      permutation(n, m)
  --      summation(x:%, n:Symbol) : as opposed to "definite" sum
  --      product(x:%, n:Symbol)   : ditto

------------------------
-- Forward declarations.
------------------------

outputOMExpr  : (OpenMathDevice, Expression R) -> Void

-------------------------
-- Local helper functions
-------------------------

outputOMArith1(dev: OpenMathDevice, sym: String, args: List Expression R): Void ==
  OMputApp(dev)
  OMputSymbol(dev, "arith1", sym)
  for arg in args repeat
    OMwrite(dev, arg, false)
  OMputEndApp(dev)

outputOMLambda(dev: OpenMathDevice, ex: Expression R, var: Expression R): Void ==
  OMputBind(dev)
  OMputSymbol(dev, "fns1", "lambda")
  OMputBVar(dev)
  OMwrite(dev, var, false)
  OMputEndBVar(dev)
  OMwrite(dev, ex, false)
  OMputEndBind(dev)

outputOMInterval(dev: OpenMathDevice, lo: Expression R, hi: Expression R): Void ==
  OMputApp(dev)
  OMputSymbol(dev, "interval1", "interval")
  OMwrite(dev, lo, false)
  OMwrite(dev, hi, false)
  OMputEndApp(dev)
```

```
outputOMIntInterval(dev: OpenMathDevice, lo: Expression R, hi: Expression R): V
  OMputApp(dev)
  OMputSymbol(dev, "interval1", "integer__interval")
  OMwrite(dev, lo, false)
  OMwrite(dev, hi, false)
  OMputEndApp(dev)

outputOMBinomial(dev: OpenMathDevice, args: List Expression R): Void ==
  not #args=2 => error "Wrong number of arguments to binomial"
  OMputApp(dev)
  OMputSymbol(dev, "combinat1", "binomial")
  for arg in args repeat
    OMwrite(dev, arg, false)
  OMputEndApp(dev)

outputOMPower(dev: OpenMathDevice, args: List Expression R): Void ==
  not #args=2 => error "Wrong number of arguments to power"
  outputOMArith1(dev, "power", args)

outputOMDefsum(dev: OpenMathDevice, args: List Expression R): Void ==
  #args ^= 5 => error "Unexpected number of arguments to a defsum"
  OMputApp(dev)
  OMputSymbol(dev, "arith1", "sum")
  outputOMIntInterval(dev, args.4, args.5)
  outputOMLambda(dev, eval(args.1, args.2, args.3), args.3)
  OMputEndApp(dev)

outputOMDefprod(dev: OpenMathDevice, args: List Expression R): Void ==
  #args ^= 5 => error "Unexpected number of arguments to a defprod"
  OMputApp(dev)
  OMputSymbol(dev, "arith1", "product")
  outputOMIntInterval(dev, args.4, args.5)
  outputOMLambda(dev, eval(args.1, args.2, args.3), args.3)
  OMputEndApp(dev)

outputOMDefint(dev: OpenMathDevice, args: List Expression R): Void ==
  #args ^= 5 => error "Unexpected number of arguments to a defint"
  OMputApp(dev)
  OMputSymbol(dev, "calculus1", "defint")
  outputOMInterval(dev, args.4, args.5)
  outputOMLambda(dev, eval(args.1, args.2, args.3), args.3)
  OMputEndApp(dev)

outputOMInt(dev: OpenMathDevice, args: List Expression R): Void ==
  #args ^= 3 => error "Unexpected number of arguments to a defint"
  OMputApp(dev)
```

```
      OMputSymbol(dev, "calculus1", "int")
      outputOMLambda(dev, eval(args.1, args.2, args.3), args.3)
      OMputEndApp(dev)

    outputOMFunction(dev: OpenMathDevice, op: Symbol, args: List Expression R): Void ==
      nargs := #args
      zero? nargs =>
        omOp: Union(SymInfo, "failed") := search(op, nullaryFunctionAList)
        omOp case "failed" =>
          error concat ["No OpenMath definition for nullary function ", coerce op]
        OMputSymbol(dev, omOp.cd, omOp.name)
--      one? nargs =>
      (nargs = 1) =>
        omOp: Union(SymInfo, "failed") := search(op, unaryFunctionAList)
        omOp case "failed" =>
          error concat ["No OpenMath definition for unary function ", coerce op]
        OMputApp(dev)
        OMputSymbol(dev, omOp.cd, omOp.name)
        for arg in args repeat
          OMwrite(dev, arg, false)
        OMputEndApp(dev)
      -- Most of the binary operators cannot be handled trivialy like the
      -- unary ones since they have bound variables of one kind or another.
      -- The special functions should be straightforward, but we don't have
      -- a CD for them yet :-)
      op = %defint  => outputOMDefint(dev, args)
      op = integral => outputOMInt(dev, args)
      op = %defsum  => outputOMDefsum(dev, args)
      op = %defprod => outputOMDefprod(dev, args)
      op = %power   => outputOMPower(dev, args)
      op = binomial => outputOMBinomial(dev, args)
      error concat ["No OpenMath definition for function ", string op]

    outputOMExpr(dev: OpenMathDevice, ex: Expression R): Void ==
      ground? ex => OMwrite(dev, ground ex, false)
      not((v := retractIfCan(ex)@Union(Symbol,"failed")) case "failed") =>
        OMputVariable(dev, v)
      not((w := isPlus ex) case "failed") => outputOMArith1(dev, "plus", w)
      not((w := isTimes ex) case "failed") => outputOMArith1(dev, "times", w)
      --not((y := isMult ex) case "failed") =>
      --  outputOMArith("times", [OMwrite(y.coef)$Integer,
      --          OMwrite(coerce y.var)])
      -- At the time of writing we don't need both isExpt and isPower
      -- here but they may be relevent when we integrate this stuff into
      -- the main Expression code.  Note that if we don't check that
      -- the exponent is non-trivial we get thrown into an infinite recursion.
```

```
--    not (((x := isExpt ex) case "failed") or one? x.exponent) =>
    not (((x := isExpt ex) case "failed") or (x.exponent = 1)) =>
      not((s := symbolIfCan(x.var)@Union(Symbol,"failed")) case "failed") =>
        --outputOMPower(dev, [s::Expression(R), (x.exponent)::Expression(R)])
        OMputApp(dev)
        OMputSymbol(dev, "arith1", "power")
        OMputVariable(dev, s)
        OMputInteger(dev, x.exponent)
        OMputEndApp(dev)
      -- TODO: add error handling code here...
--    not (((z := isPower ex) case "failed") or one? z.exponent) =>
    not (((z := isPower ex) case "failed") or (z.exponent = 1)) =>
      outputOMPower(dev, [ z.val, z.exponent::Expression R ])
      --OMputApp(dev)
      --OMputSymbol(dev, "arith1", "power")
      --outputOMExpr(dev, z.val)
      --OMputInteger(dev, z.exponent)
      --OMputEndApp(dev)
    -- Must only be one top-level Kernel by this point
    k : Kernel Expression R := first kernels ex
    outputOMFunction(dev, name operator k, argument k)


  ----------
  -- Exports
  ----------

  OMwrite(ex: Expression R): String ==
    s: String := ""
    sp := OM_-STRINGTOSTRINGPTR(s)$Lisp
    dev: OpenMathDevice := OMopenString(sp pretend String, OMencodingXML())
    OMputObject(dev)
    outputOMExpr(dev, ex)
    OMputEndObject(dev)
    OMclose(dev)
    s := OM_-STRINGPTRTOSTRING(sp)$Lisp pretend String
    s

  OMwrite(ex: Expression R, wholeObj: Boolean): String ==
    s: String := ""
    sp := OM_-STRINGTOSTRINGPTR(s)$Lisp
    dev: OpenMathDevice := OMopenString(sp pretend String, OMencodingXML())
    if wholeObj then
      OMputObject(dev)
    outputOMExpr(dev, ex)
    if wholeObj then
```

```
      OMputEndObject(dev)
    OMclose(dev)
    s := OM_-STRINGPTRTOSTRING(sp)$Lisp pretend String
    s

  OMwrite(dev: OpenMathDevice, ex: Expression R): Void ==
    OMputObject(dev)
    outputOMExpr(dev, ex)
    OMputEndObject(dev)

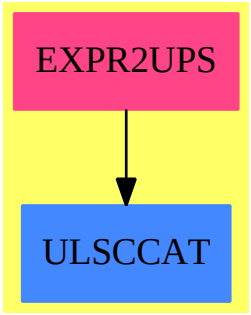
  OMwrite(dev: OpenMathDevice, ex: Expression R, wholeObj: Boolean): Void ==
    if wholeObj then
      OMputObject(dev)
    outputOMExpr(dev, ex)
    if wholeObj then
      OMputEndObject(dev)
```

⟨*OMEXPR.dotabb*⟩≡

```
"OMEXPR" [color="#FF4488",href="bookvol10.4.pdf#nameddest=OMEXPR"]
"FS" [color="#4488FF",href="bookvol10.2.pdf#nameddest=FS"]
"OMEXPR" -> "FS"
```

6.57 package EXPR2UPS ExpressionToUnivariatePowerSeries

6.58 ExpressionToUnivariatePowerSeries



1.00

Exports:
laurent puiseux series taylor

⟨*package EXPR2UPS ExpressionToUnivariatePowerSeries*⟩≡

```
)abbrev package EXPR2UPS ExpressionToUnivariatePowerSeries
++ Author: Clifton J. Williamson
++ Date Created: 9 May 1989
++ Date Last Updated: 20 September 1993
++ Basic Operations: taylor, laurent, puiseux, series
++ Related Domains: UnivariateTaylorSeries, UnivariateLaurentSeries,
++   UnivariatePuiseuxSeries, Expression
++ Also See: FunctionSpaceToUnivariatePowerSeries
++ AMS Classifications:
++ Keywords: Taylor series, Laurent series, Puiseux series
++ Examples:
++ References:
++ Description:
++   This package provides functions to convert functional expressions
++   to power series.
ExpressionToUnivariatePowerSeries(R,FE): Exports == Implementation where
  R  : Join(GcdDomain,OrderedSet,RetractableTo Integer,_
            LinearlyExplicitRingOver Integer)
  FE : Join(AlgebraicallyClosedField,TranscendentalFunctionCategory,_
            FunctionSpace R)

  EQ     ==> Equation
  I      ==> Integer
  NNI    ==> NonNegativeInteger
  RN     ==> Fraction Integer
  SY     ==> Symbol
```

```
UTS    ==> UnivariateTaylorSeries
ULS    ==> UnivariateLaurentSeries
UPXS   ==> UnivariatePuiseuxSeries
GSER   ==> GeneralUnivariatePowerSeries
EFULS  ==> ElementaryFunctionsUnivariateLaurentSeries
EFUPXS ==> ElementaryFunctionsUnivariatePuiseuxSeries
FS2UPS ==> FunctionSpaceToUnivariatePowerSeries
Prob   ==> Record(func:String,prob:String)
ANY1   ==> AnyFunctions1

Exports ==> with
  taylor: SY -> Any
    ++ \spad{taylor(x)} returns x viewed as a Taylor series.
  taylor: FE -> Any
    ++ \spad{taylor(f)} returns a Taylor expansion of the expression f.
    ++ Note: f should have only one variable; the series will be
    ++ expanded in powers of that variable.
  taylor: (FE,NNI) -> Any
    ++ \spad{taylor(f,n)} returns a Taylor expansion of the expression f.
    ++ Note: f should have only one variable; the series will be
    ++ expanded in powers of that variable and terms will be computed
    ++ up to order at least n.
  taylor: (FE,EQ FE) -> Any
    ++ \spad{taylor(f,x = a)} expands the expression f as a Taylor series
    ++ in powers of \spad{(x - a)}.
  taylor: (FE,EQ FE,NNI) -> Any
    ++ \spad{taylor(f,x = a)} expands the expression f as a Taylor series
    ++ in powers of \spad{(x - a)}; terms will be computed up to order
    ++ at least n.

  laurent: SY -> Any
    ++ \spad{laurent(x)} returns x viewed as a Laurent series.
  laurent: FE -> Any
    ++ \spad{laurent(f)} returns a Laurent expansion of the expression f.
    ++ Note: f should have only one variable; the series will be
    ++ expanded in powers of that variable.
  laurent: (FE,I) -> Any
    ++ \spad{laurent(f,n)} returns a Laurent expansion of the expression f.
    ++ Note: f should have only one variable; the series will be
    ++ expanded in powers of that variable and terms will be computed
    ++ up to order at least n.
  laurent: (FE,EQ FE) -> Any
    ++ \spad{laurent(f,x = a)} expands the expression f as a Laurent series
    ++ in powers of \spad{(x - a)}.
  laurent: (FE,EQ FE,I) -> Any
    ++ \spad{laurent(f,x = a,n)} expands the expression f as a Laurent
```

```
    ++ series in powers of \spad{(x - a)}; terms will be computed up to order
    ++ at least n.
  puiseux: SY -> Any
    ++ \spad{puiseux(x)} returns x viewed as a Puiseux series.
  puiseux: FE -> Any
    ++ \spad{puiseux(f)} returns a Puiseux expansion of the expression f.
    ++ Note: f should have only one variable; the series will be
    ++ expanded in powers of that variable.
  puiseux: (FE,RN) -> Any
    ++ \spad{puiseux(f,n)} returns a Puiseux expansion of the expression f.
    ++ Note: f should have only one variable; the series will be
    ++ expanded in powers of that variable and terms will be computed
    ++ up to order at least n.
  puiseux: (FE,EQ FE) -> Any
    ++ \spad{puiseux(f,x = a)} expands the expression f as a Puiseux series
    ++ in powers of \spad{(x - a)}.
  puiseux: (FE,EQ FE,RN) -> Any
    ++ \spad{puiseux(f,x = a,n)} expands the expression f as a Puiseux
    ++ series in powers of \spad{(x - a)}; terms will be computed up to order
    ++ at least n.

  series: SY -> Any
    ++ \spad{series(x)} returns x viewed as a series.
  series: FE -> Any
    ++ \spad{series(f)} returns a series expansion of the expression f.
    ++ Note: f should have only one variable; the series will be
    ++ expanded in powers of that variable.
  series: (FE,RN) -> Any
    ++ \spad{series(f,n)} returns a series expansion of the expression f.
    ++ Note: f should have only one variable; the series will be
    ++ expanded in powers of that variable and terms will be computed
    ++ up to order at least n.
  series: (FE,EQ FE) -> Any
    ++ \spad{series(f,x = a)} expands the expression f as a series
    ++ in powers of (x - a).
  series: (FE,EQ FE,RN) -> Any
    ++ \spad{series(f,x = a,n)} expands the expression f as a series
    ++ in powers of (x - a); terms will be computed up to order
    ++ at least n.

Implementation ==> add
  performSubstitution: (FE,SY,FE) -> FE
  performSubstitution(fcn,x,a) ==
    zero? a => fcn
    xFE := x :: FE
    eval(fcn,xFE = xFE + a)
```

```
iTaylor: (FE,SY,FE) -> Any
iTaylor(fcn,x,a) ==
  pack := FS2UPS(R,FE,I,ULS(FE,x,a),_
                 EFULS(FE,UTS(FE,x,a),ULS(FE,x,a)),x)
  ans := exprToUPS(fcn,false,"just do it")$pack
  ans case %problem =>
    ans.%problem.prob = "essential singularity" =>
      error "No Taylor expansion: essential singularity"
    ans.%problem.func = "log" =>
      error "No Taylor expansion: logarithmic singularity"
    ans.%problem.func = "nth root" =>
      error "No Taylor expansion: fractional powers in expansion"
    error "No Taylor expansion"
  uls := ans.%series
  (uts := taylorIfCan uls) case "failed" =>
    error "No Taylor expansion: pole"
  any1 := ANY1(UTS(FE,x,a))
  coerce(uts :: UTS(FE,x,a))$any1

taylor(x:SY) ==
  uts := UTS(FE,x,0$FE); any1 := ANY1(uts)
  coerce(monomial(1,1)$uts)$any1

taylor(fcn:FE) ==
  null(vars := variables fcn) =>
    error "taylor: expression has no variables"
  not null rest vars =>
    error "taylor: expression has more than one variable"
  taylor(fcn,(first(vars) :: FE) = 0)

taylor(fcn:FE,n:NNI) ==
  null(vars := variables fcn) =>
    error "taylor: expression has no variables"
  not null rest vars =>
    error "taylor: expression has more than one variable"
  x := first vars
  uts := UTS(FE,x,0$FE); any1 := ANY1(uts)
  series := retract(taylor(fcn,(x :: FE) = 0))$any1
  coerce(extend(series,n))$any1

taylor(fcn:FE,eq:EQ FE) ==
  (xx := retractIfCan(lhs eq)@Union(SY,"failed")) case "failed" =>
    error "taylor: left hand side must be a variable"
  x := xx :: SY; a := rhs eq
  iTaylor(performSubstitution(fcn,x,a),x,a)
```

```
taylor(fcn,eq,n) ==
  (xx := retractIfCan(lhs eq)@Union(SY,"failed")) case "failed" =>
    error "taylor: left hand side must be a variable"
  x := xx :: SY; a := rhs eq
  any1 := ANY1(UTS(FE,x,a))
  series := retract(iTaylor(performSubstitution(fcn,x,a),x,a))$any1
  coerce(extend(series,n))$any1

iLaurent: (FE,SY,FE) -> Any
iLaurent(fcn,x,a) ==
  pack := FS2UPS(R,FE,I,ULS(FE,x,a),_
                 EFULS(FE,UTS(FE,x,a),ULS(FE,x,a)),x)
  ans := exprToUPS(fcn,false,"just do it")$pack
  ans case %problem =>
    ans.%problem.prob = "essential singularity" =>
      error "No Laurent expansion: essential singularity"
    ans.%problem.func = "log" =>
      error "No Laurent expansion: logarithmic singularity"
    ans.%problem.func = "nth root" =>
      error "No Laurent expansion: fractional powers in expansion"
    error "No Laurent expansion"
  any1 := ANY1(ULS(FE,x,a))
  coerce(ans.%series)$any1

laurent(x:SY) ==
  uls := ULS(FE,x,0$FE); any1 := ANY1(uls)
  coerce(monomial(1,1)$uls)$any1

laurent(fcn:FE) ==
  null(vars := variables fcn) =>
    error "laurent: expression has no variables"
  not null rest vars =>
    error "laurent: expression has more than one variable"
  laurent(fcn,(first(vars) :: FE) = 0)

laurent(fcn:FE,n:I) ==
  null(vars := variables fcn) =>
    error "laurent: expression has no variables"
  not null rest vars =>
    error "laurent: expression has more than one variable"
  x := first vars
  uls := ULS(FE,x,0$FE); any1 := ANY1(uls)
  series := retract(laurent(fcn,(x :: FE) = 0))$any1
  coerce(extend(series,n))$any1
```

```
laurent(fcn:FE,eq:EQ FE) ==
  (xx := retractIfCan(lhs eq)@Union(SY,"failed")) case "failed" =>
    error "taylor: left hand side must be a variable"
  x := xx :: SY; a := rhs eq
  iLaurent(performSubstitution(fcn,x,a),x,a)

laurent(fcn,eq,n) ==
  (xx := retractIfCan(lhs eq)@Union(SY,"failed")) case "failed" =>
    error "taylor: left hand side must be a variable"
  x := xx :: SY; a := rhs eq
  any1 := ANY1(ULS(FE,x,a))
  series := retract(iLaurent(performSubstitution(fcn,x,a),x,a))$any1
  coerce(extend(series,n))$any1

iPuiseux: (FE,SY,FE) -> Any
iPuiseux(fcn,x,a) ==
  pack := FS2UPS(R,FE,RN,UPXS(FE,x,a),_
                 EFUPXS(FE,ULS(FE,x,a),UPXS(FE,x,a),_
                 EFULS(FE,UTS(FE,x,a),ULS(FE,x,a))),x)
  ans := exprToUPS(fcn,false,"just do it")$pack
  ans case %problem =>
    ans.%problem.prob = "essential singularity" =>
      error "No Puiseux expansion: essential singularity"
    ans.%problem.func = "log" =>
      error "No Puiseux expansion: logarithmic singularity"
    error "No Puiseux expansion"
  any1 := ANY1(UPXS(FE,x,a))
  coerce(ans.%series)$any1

puiseux(x:SY) ==
  upxs := UPXS(FE,x,0$FE); any1 := ANY1(upxs)
  coerce(monomial(1,1)$upxs)$any1

puiseux(fcn:FE) ==
  null(vars := variables fcn) =>
    error "puiseux: expression has no variables"
  not null rest vars =>
    error "puiseux: expression has more than one variable"
  puiseux(fcn,(first(vars) :: FE) = 0)

puiseux(fcn:FE,n:RN) ==
  null(vars := variables fcn) =>
    error "puiseux: expression has no variables"
  not null rest vars =>
    error "puiseux: expression has more than one variable"
  x := first vars
```

```
  upxs := UPXS(FE,x,0$FE); any1 := ANY1(upxs)
  series := retract(puiseux(fcn,(x :: FE) = 0))$any1
  coerce(extend(series,n))$any1

puiseux(fcn:FE,eq:EQ FE) ==
  (xx := retractIfCan(lhs eq)@Union(SY,"failed")) case "failed" =>
    error "taylor: left hand side must be a variable"
  x := xx :: SY; a := rhs eq
  iPuiseux(performSubstitution(fcn,x,a),x,a)

puiseux(fcn,eq,n) ==
  (xx := retractIfCan(lhs eq)@Union(SY,"failed")) case "failed" =>
    error "taylor: left hand side must be a variable"
  x := xx :: SY; a := rhs eq
  any1 := ANY1(UPXS(FE,x,a))
  series := retract(iPuiseux(performSubstitution(fcn,x,a),x,a))$any1
  coerce(extend(series,n))$any1

iSeries: (FE,SY,FE) -> Any
iSeries(fcn,x,a) ==
  pack := FS2UPS(R,FE,RN,UPXS(FE,x,a), _
                 EFUPXS(FE,ULS(FE,x,a),UPXS(FE,x,a), _
                 EFULS(FE,UTS(FE,x,a),ULS(FE,x,a))),x)
  ans := exprToUPS(fcn,false,"just do it")$pack
  ans case %problem =>
    ansG := exprToGenUPS(fcn,false,"just do it")$pack
    ansG case %problem =>
      ansG.%problem.prob = "essential singularity" =>
        error "No series expansion: essential singularity"
      error "No series expansion"
    anyone := ANY1(GSER(FE,x,a))
    coerce((ansG.%series) :: GSER(FE,x,a))$anyone
  any1 := ANY1(UPXS(FE,x,a))
  coerce(ans.%series)$any1

series(x:SY) ==
  upxs := UPXS(FE,x,0$FE); any1 := ANY1(upxs)
  coerce(monomial(1,1)$upxs)$any1

series(fcn:FE) ==
  null(vars := variables fcn) =>
    error "series: expression has no variables"
  not null rest vars =>
    error "series: expression has more than one variable"
  series(fcn,(first(vars) :: FE) = 0)
```

```
series(fcn:FE,n:RN) ==
  null(vars := variables fcn) =>
    error "series: expression has no variables"
  not null rest vars =>
    error "series: expression has more than one variable"
  x := first vars
  upxs := UPXS(FE,x,0$FE); any1 := ANY1(upxs)
  series := retract(series(fcn,(x :: FE) = 0))$any1
  coerce(extend(series,n))$any1

series(fcn:FE,eq:EQ FE) ==
  (xx := retractIfCan(lhs eq)@Union(SY,"failed")) case "failed" =>
    error "taylor: left hand side must be a variable"
  x := xx :: SY; a := rhs eq
  iSeries(performSubstitution(fcn,x,a),x,a)

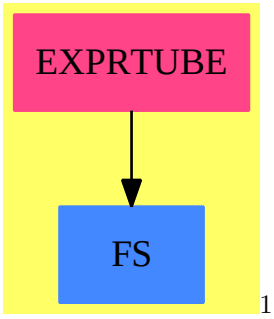
series(fcn,eq,n) ==
  (xx := retractIfCan(lhs eq)@Union(SY,"failed")) case "failed" =>
    error "taylor: left hand side must be a variable"
  x := xx :: SY; a := rhs eq
  any1 := ANY1(UPXS(FE,x,a))
  series := retract(iSeries(performSubstitution(fcn,x,a),x,a))$any1
  coerce(extend(series,n))$any1
```

⟨*EXPR2UPS.dotabb*⟩≡

```
"EXPR2UPS" [color="#FF4488",href="bookvol10.4.pdf#nameddest=EXPR2UPS"]
"ULSCCAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=ULSCCAT"]
"EXPR2UPS" -> "ULSCCAT"
```

6.59 package EXPRTUBE ExpressionTubePlot

6.60 ExpressionTubePlot



1.00

Exports:
constantToUnaryFunction tubePlot

⟨*package EXPRTUBE ExpressionTubePlot*⟩≡
```
)abbrev package EXPRTUBE ExpressionTubePlot
++ Author: Clifton J. Williamson
++ Date Created: Bastille Day 1989
++ Date Last Updated: 5 June 1990
++ Keywords:
++ Examples:
++ Package for constructing tubes around 3-dimensional parametric curves.
ExpressionTubePlot(): Exports == Implementation where
  B   ==> Boolean
  I   ==> Integer
  FE  ==> Expression Integer
  SY  ==> Symbol
  SF  ==> DoubleFloat
  L   ==> List
  S   ==> String
  SEG ==> Segment
  F2F ==> MakeFloatCompiledFunction(FE)
  Pt  ==> Point SF
  PLOT3 ==> Plot3D
  TUBE  ==> TubePlot Plot3D

  Exports ==> with
    constantToUnaryFunction: SF -> (SF -> SF)
      ++ constantToUnaryFunction(s) is a local function which takes the
      ++ value of s, which may be a function of a constant, and returns
      ++ a function which always returns the value \spadtype{DoubleFloat} s.
    tubePlot: (FE,FE,FE,SF -> SF,SEG SF,SF -> SF,I) -> TUBE
```

```
      ++ tubePlot(f,g,h,colorFcn,a..b,r,n) puts a tube of radius r(t) with
      ++ n points on each circle about the curve \spad{x = f(t)},
      ++ \spad{y = g(t)}, \spad{z = h(t)} for t in \spad{[a,b]}.
      ++ The tube is considered to be open.
    tubePlot: (FE,FE,FE,SF -> SF,SEG SF,SF -> SF,I,S) -> TUBE
      ++ tubePlot(f,g,h,colorFcn,a..b,r,n,s) puts a tube of radius \spad{r(t)}
      ++ with n points on each circle about the curve \spad{x = f(t)},
      ++ \spad{y = g(t)},
      ++ \spad{z = h(t)} for t in \spad{[a,b]}. If s = "closed", the tube is
      ++ considered to be closed; if s = "open", the tube is considered
      ++ to be open.
    tubePlot: (FE,FE,FE,SF -> SF,SEG SF,SF,I) -> TUBE
      ++ tubePlot(f,g,h,colorFcn,a..b,r,n) puts a tube of radius r with
      ++ n points on each circle about the curve \spad{x = f(t)},
      ++ \spad{y = g(t)}, \spad{z = h(t)} for t in \spad{[a,b]}.
      ++ The tube is considered to be open.
    tubePlot: (FE,FE,FE,SF -> SF,SEG SF,SF,I,S) -> TUBE
      ++ tubePlot(f,g,h,colorFcn,a..b,r,n,s) puts a tube of radius r with
      ++ n points on each circle about the curve \spad{x = f(t)},
      ++ \spad{y = g(t)}, \spad{z = h(t)} for t in \spad{[a,b]}.
      ++ If s = "closed", the tube is
      ++ considered to be closed; if s = "open", the tube is considered
      ++ to be open.

  Implementation ==> add
    import Plot3D
    import F2F
    import TubePlotTools

--% variables

    getVariable: (FE,FE,FE) -> SY
    getVariable(x,y,z) ==
      varList1 := variables x
      varList2 := variables y
      varList3 := variables z
      (not (# varList1 <= 1)) or (not (# varList2 <= 1)) or _
       (not (# varList3 <= 1)) =>
        error "tubePlot: only one variable may be used"
      null varList1 =>
        null varList2 =>
          null varList3 =>
            error "tubePlot: a variable must appear in functions"
          first varList3
        t2 := first varList2
        null varList3 => t2
```

```
        not (first varList3 = t2) =>
          error "tubePlot: only one variable may be used"
      t1 := first varList1
      null varList2 =>
        null varList3 => t1
        not (first varList3 = t1) =>
          error "tubePlot: only one variable may be used"
        t1
      t2 := first varList2
      null varList3 =>
        not (t1 = t2) =>
          error "tubePlot: only one variable may be used"
        t1
      not (first varList3 = t1) or not (t2 = t1) =>
        error "tubePlot: only one variable may be used"
      t1

--% tubes: variable radius

    tubePlot(x:FE,y:FE,z:FE,colorFcn:SF -> SF,_
             tRange:SEG SF,radFcn:SF -> SF,n:I,string:S) ==
      -- check value of n
      n < 3 => error "tubePlot: n should be at least 3"
      -- check string
      flag : B :=
        string = "closed" => true
        string = "open" => false
        error "tubePlot: last argument should be open or closed"
      -- check variables
      t := getVariable(x,y,z)
      -- coordinate functions
      xFunc := makeFloatFunction(x,t)
      yFunc := makeFloatFunction(y,t)
      zFunc := makeFloatFunction(z,t)
      -- derivatives of coordinate functions
      xp := differentiate(x,t)
      yp := differentiate(y,t)
      zp := differentiate(z,t)
      -- derivative of arc length
      sp := sqrt(xp ** 2 + yp ** 2 + zp ** 2)
      -- coordinates of unit tangent vector
      Tx := xp/sp; Ty := yp/sp; Tz := zp/sp
      -- derivatives of coordinates of unit tangent vector
      Txp := differentiate(Tx,t)
      Typ := differentiate(Ty,t)
      Tzp := differentiate(Tz,t)
```

```
      -- K = curvature = length of curvature vector
      K := sqrt(Txp ** 2 + Typ ** 2 + Tzp ** 2)
      -- coordinates of principal normal vector
      Nx := Txp / K; Ny := Typ / K; Nz := Tzp / K
      -- functions SF->SF giving coordinates of principal normal vector
      NxFunc := makeFloatFunction(Nx,t);
      NyFunc := makeFloatFunction(Ny,t);
      NzFunc := makeFloatFunction(Nz,t);
      -- coordinates of binormal vector
      Bx := Ty * Nz - Tz * Ny
      By := Tz * Nx - Tx * Nz
      Bz := Tx * Ny - Ty * Nx
      -- functions SF -> SF giving coordinates of binormal vector
      BxFunc := makeFloatFunction(Bx,t);
      ByFunc := makeFloatFunction(By,t);
      BzFunc := makeFloatFunction(Bz,t);
      -- create Plot3D
      parPlot := plot(xFunc,yFunc,zFunc,colorFcn,tRange)
      tvals := first tValues parPlot
      curvePts := first listBranches parPlot
      cosSin := cosSinInfo n
      loopList : L L Pt := nil()
      while not null tvals repeat
        -- note: tvals and curvePts have the same number of elements
        tval := first tvals; tvals := rest tvals
        ctr := first curvePts; curvePts := rest curvePts
        pNormList : L SF :=
          [NxFunc tval,NyFunc tval,NzFunc tval,colorFcn tval]
        pNorm : Pt := point pNormList
        bNormList : L SF :=
          [BxFunc tval,ByFunc tval,BzFunc tval,colorFcn tval]
        bNorm : Pt := point bNormList
        lps := loopPoints(ctr,pNorm,bNorm,radFcn tval,cosSin)
        loopList := cons(lps,loopList)
      tube(parPlot,reverse_! loopList,flag)

    tubePlot(x:FE,y:FE,z:FE,colorFcn:SF -> SF,_
             tRange:SEG SF,radFcn:SF -> SF,n:I) ==
      tubePlot(x,y,z,colorFcn,tRange,radFcn,n,"open")

--% tubes: constant radius

    project: (SF,SF) -> SF
    project(x,y) == x

    constantToUnaryFunction x == s +-> project(x,s)
```

```
tubePlot(x:FE,y:FE,z:FE,colorFcn:SF -> SF,_
         tRange:SEG SF,rad:SF,n:I,s:S) ==
  tubePlot(x,y,z,colorFcn,tRange,constantToUnaryFunction rad,n,s)

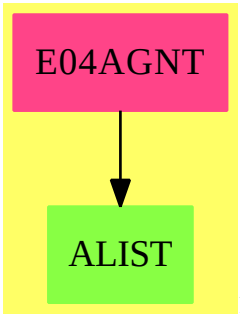
tubePlot(x:FE,y:FE,z:FE,colorFcn:SF -> SF,_
         tRange:SEG SF,rad:SF,n:I) ==
  tubePlot(x,y,z,colorFcn,tRange,rad,n,"open")
```

⟨*EXPRTUBE.dotabb*⟩≡

```
"EXPRTUBE" [color="#FF4488",href="bookvol10.4.pdf#nameddest=EXPRTUBE"]
"FS" [color="#4488FF",href="bookvol10.2.pdf#nameddest=FS"]
"EXPRTUBE" -> "FS"
```

6.61 package E04AGNT e04AgentsPackage

6.62 e04AgentsPackage



1.00

Exports:

changeNameToObjf	expenseOfEvaluation	finiteBound	linear?	linearMatrix
linearPart	nonLinearPart	optAttributes	quadratic?	simpleBounds?
sortConstraints	splitLinear	sumOfSquares	varList	variables

⟨*package E04AGNT e04AgentsPackage*⟩≡

```
)abbrev package E04AGNT e04AgentsPackage
++ Author: Brian Dupee
++ Date Created: February 1996
++ Date Last Updated: June 1996
++ Basic Operations: simple? linear?, quadratic?, nonLinear?
++ Description:
++ \axiomType{e04AgentsPackage} is a package of numerical agents to be used
++ to investigate attributes of an input function so as to decide the
++ \axiomFun{measure} of an appropriate numerical optimization routine.
MDF     ==> Matrix DoubleFloat
VEDF    ==> Vector Expression DoubleFloat
EDF     ==> Expression DoubleFloat
EFI     ==> Expression Fraction Integer
PFI     ==> Polynomial Fraction Integer
FI      ==> Fraction Integer
F       ==> Float
DF      ==> DoubleFloat
OCDF    ==> OrderedCompletion DoubleFloat
LOCDF   ==> List OrderedCompletion DoubleFloat
LEDF    ==> List Expression DoubleFloat
PDF     ==> Polynomial DoubleFloat
LDF     ==> List DoubleFloat
INT     ==> Integer
NNI     ==> NonNegativeInteger
LS      ==> List Symbol
```

```
  EF2     ==> ExpressionFunctions2
  NOA     ==> Record(fn:EDF, init:LDF, lb:LOCDF, cf:LEDF, ub:LOCDF)
  LSA     ==> Record(lfn:LEDF, init:LDF)

  e04AgentsPackage(): E == I where
    E ==> with
      finiteBound:(LOCDF,DF) -> LDF
        ++ finiteBound(l,b) repaces all instances of an infinite entry in
        ++ \axiom{l} by a finite entry \axiom{b} or \axiom{-b}.
      sortConstraints:NOA -> NOA
        ++ sortConstraints(args) uses a simple bubblesort on the list of
        ++ constraints using the degree of the expression on which to sort.
        ++ Of course, it must match the bounds to the constraints.
      sumOfSquares:EDF -> Union(EDF,"failed")
        ++ sumOfSquares(f) returns either an expression for which the square is
        ++ the original function of "failed".
      splitLinear:EDF -> EDF
        ++ splitLinear(f) splits the linear part from an expression which it
        ++ returns.
      simpleBounds?:LEDF -> Boolean
        ++ simpleBounds?(l) returns true if the list of expressions l are
        ++ simple.
      linear?:LEDF -> Boolean
        ++ linear?(l) returns true if all the bounds l are either linear or
        ++ simple.
      linear?:EDF -> Boolean
        ++ linear?(e) tests if \axiom{e} is a linear function.
      linearMatrix:(LEDF, NNI) -> MDF
        ++ linearMatrix(l,n) returns a matrix of coefficients of the linear
        ++ functions in \axiom{l}.  If l is empty, the matrix has at least one
        ++ row.
      linearPart:LEDF -> LEDF
        ++ linearPart(l) returns the list of linear functions of \axiom{l}.
      nonLinearPart:LEDF -> LEDF
        ++ nonLinearPart(l) returns the list of non-linear functions of \axiom{l}.
      quadratic?:EDF -> Boolean
        ++ quadratic?(e) tests if \axiom{e} is a quadratic function.
      variables:LSA -> LS
        ++ variables(args) returns the list of variables in \axiom{args.lfn}
      varList:(EDF,NNI) -> LS
        ++ varList(e,n) returns a list of \axiom{n} indexed variables with name
        ++ as in \axiom{e}.
      changeNameToObjf:(Symbol,Result) -> Result
        ++ changeNameToObjf(s,r) changes the name of item \axiom{s} in \axiom{r}
        ++ to objf.
      expenseOfEvaluation:LSA -> F
```

```
    ++ expenseOfEvaluation(o) returns the intensity value of the
    ++ cost of evaluating the input set of functions.  This is in terms
    ++ of the number of ``operational units''.  It returns a value
    ++ in the range [0,1].
  optAttributes:Union(noa:NOA,lsa:LSA) -> List String
    ++ optAttributes(o) is a function for supplying a list of attributes
    ++ of an optimization problem.

I ==> add

  import ExpertSystemToolsPackage, ExpertSystemContinuityPackage

  sumOfSquares2:EFI -> Union(EFI,"failed")
  nonLinear?:EDF -> Boolean
  finiteBound2:(OCDF,DF) -> DF
  functionType:EDF -> String

  finiteBound2(a:OCDF,b:DF):DF ==
    not finite?(a) =>
      positive?(a) => b
      -b
    retract(a)@DF

  finiteBound(l:LOCDF,b:DF):LDF == [finiteBound2(i,b) for i in l]

  sortConstraints(args:NOA):NOA ==
    Args := copy args
    c:LEDF := Args.cf
    l:LOCDF := Args.lb
    u:LOCDF := Args.ub
    m:INT := (# c) - 1
    n:INT := (# l) - m
    for j in m..1 by -1 repeat
      for i in 1..j repeat
        s:EDF := c.i
        t:EDF := c.(i+1)
        if linear?(t) and (nonLinear?(s) or quadratic?(s)) then
          swap!(c,i,i+1)$LEDF
          swap!(l,n+i-1,n+i)$LOCDF
          swap!(u,n+i-1,n+i)$LOCDF
    Args

  changeNameToObjf(s:Symbol,r:Result):Result ==
    a := remove!(s,r)$Result
    a case Any =>
      insert!([objf@Symbol,a],r)$Result
```

```
    r
  r

sum(a:EDF,b:EDF):EDF == a+b

variables(args:LSA): LS == variables(reduce(sum,(args.lfn)))

sumOfSquares(f:EDF):Union(EDF,"failed") ==
  e := edf2efi(f)
  s:Union(EFI,"failed") := sumOfSquares2(e)
  s case EFI =>
    map(fi2df,s)$EF2(FI,DF)
  "failed"

sumOfSquares2(f:EFI):Union(EFI,"failed") ==
  p := retractIfCan(f)@Union(PFI,"failed")
  p case PFI =>
    r := squareFreePart(p)$PFI
    (p=r)@Boolean => "failed"
    tp := totalDegree(p)$PFI
    tr := totalDegree(r)$PFI
    t := tp quo tr
    found := false
    q := r
    for i in 2..t by 2 repeat
      s := q**2
      (s=p)@Boolean =>
        found := true
        leave
      q := r**i
    if found then
      q :: EFI
    else
      "failed"
  "failed"

splitLinear(f:EDF):EDF ==
  out := 0$EDF
  (l := isPlus(f)$EDF) case LEDF =>
    for i in l repeat
      if not quadratic? i then
        out := out + i
    out
  out

edf2pdf(f:EDF):PDF == (retract(f)@PDF)$EDF
```

```
    varList(e:EDF,n:NNI):LS ==
      s := name(first(variables(edf2pdf(e))$PDF)$LS)$Symbol
      [subscript(s,[t::OutputForm]) for t in expand([1..n])$Segment(Integer)]

    functionType(f:EDF):String ==
      n := #(variables(f))$EDF
      p := (retractIfCan(f)@Union(PDF,"failed"))$EDF
      p case PDF =>
        d := totalDegree(p)$PDF
--        one?(n*d) => "simple"
        (n*d) = 1 => "simple"
--        one?(d) => "linear"
        (d = 1) => "linear"
        (d=2)@Boolean => "quadratic"
        "non-linear"
      "non-linear"

    simpleBounds?(l: LEDF):Boolean ==
      a := true
      for e in l repeat
        not (functionType(e) = "simple")@Boolean =>
          a := false
          leave
      a

    simple?(e:EDF):Boolean == (functionType(e) = "simple")@Boolean

    linear?(e:EDF):Boolean == (functionType(e) = "linear")@Boolean

    quadratic?(e:EDF):Boolean == (functionType(e) = "quadratic")@Boolean

    nonLinear?(e:EDF):Boolean == (functionType(e) = "non-linear")@Boolean

    linear?(l: LEDF):Boolean ==
      a := true
      for e in l repeat
        s := functionType(e)
        (s = "quadratic")@Boolean or (s = "non-linear")@Boolean =>
          a := false
          leave
      a

    simplePart(l:LEDF):LEDF == [i for i in l | simple?(i)]

    linearPart(l:LEDF):LEDF == [i for i in l | linear?(i)]
```

```
nonLinearPart(l:LEDF):LEDF ==
  [i for i in l | not linear?(i) and not simple?(i)]

linearMatrix(l:LEDF, n:NNI):MDF ==
  empty?(l) => mat([],n)
  L := linearPart l
  M := zero(max(1,# L)$NNI,n)$MDF
  vars := varList(first(l)$LEDF,n)
  row:INT := 1
  for a in L repeat
    for j in monomials(edf2pdf(a))$PDF repeat
      col:INT := 1
      for c in vars repeat
        if ((first(variables(j)$PDF)$LS)=c)@Boolean then
          M(row,col):= first(coefficients(j)$PDF)$LDF
        col := col+1
    row := row + 1
  M

expenseOfEvaluation(o:LSA):F ==
  expenseOfEvaluation(vector(copy o.lfn)$VEDF)

optAttributes(o:Union(noa:NOA,lsa:LSA)):List String ==
  o case noa =>
    n := o.noa
    s1:String := "The object function is " functionType(n.fn)
    if empty?(n.lb) then
      s2:String := "There are no bounds on the variables"
    else
      s2:String := "There are simple bounds on the variables"
    c := n.cf
    if empty?(c) then
      s3:String := "There are no constraint functions"
    else
      t := #(c)
      lin := #(linearPart(c))
      nonlin := #(nonLinearPart(c))
      s3:String := "There are " string(lin)$String " linear and "_
                     string(nonlin)$String " non-linear constraints"
    [s1,s2,s3]
  l := o.lsa
  s:String := "non-linear"
  if linear?(l.lfn) then
    s := "linear"
  ["The object functions are " s]
```

⟨*E04AGNT.dotabb*⟩≡

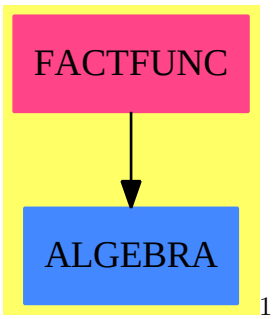
```
"E04AGNT" [color="#FF4488",href="bookvol10.4.pdf#nameddest=E04AGNT"]
"ALIST" [color="#88FF44",href="bookvol10.3.pdf#nameddest=ALIST"]
"E04AGNT" -> "ALIST"
```

Chapter 7

Chapter F

7.1 package FACTFUNC FactoredFunctions

7.2 FactoredFunctions



1.00

Exports:
log nthRoot

⟨*package FACTFUNC FactoredFunctions*⟩≡
```
)abbrev package FACTFUNC FactoredFunctions
++ Author: Manuel Bronstein
++ Date Created: 2 Feb 1988
++ Date Last Updated: 25 Jun 1990
++ Description: computes various functions on factored arguments.
-- not visible to the user
FactoredFunctions(M:IntegralDomain): Exports == Implementation where
  N ==> NonNegativeInteger

  Exports ==> with
    nthRoot: (Factored M,N) -> Record(exponent:N,coef:M,radicand:List M)
```

```
      ++ nthRoot(f, n) returns \spad{(p, r, [r1,...,rm])} such that
      ++ the nth-root of f is equal to \spad{r * pth-root(r1 * ... * rm)},
      ++ where r1,...,rm are distinct factors of f,
      ++ each of which has an exponent smaller than p in f.
    log : Factored M -> List Record(coef:N, logand:M)
      ++ log(f) returns \spad{[(a1,b1),...,(am,bm)]} such that
      ++ the logarithm of f is equal to \spad{a1*log(b1) + ... + am*log(bm)}.

  Implementation ==> add
    nthRoot(ff, n) ==
      coeff:M       := 1
--      radi:List(M)  := (one? unit ff => empty(); [unit ff])
      radi:List(M)  := (((unit ff) = 1) => empty(); [unit ff])
      lf            := factors ff
      d:N :=
        empty? radi => gcd(concat(n, [t.exponent::N for t in lf]))::N
        1
      n             := n quo d
      for term in lf repeat
        qr    := divide(term.exponent::N quo d, n)
        coeff := coeff * term.factor ** qr.quotient
        not zero?(qr.remainder) =>
          radi := concat_!(radi, term.factor ** qr.remainder)
      [n, coeff, radi]

    log ff ==
      ans := unit ff
      concat([1, unit ff],
             [[term.exponent::N, term.factor] for term in factors ff])
```

⟨*FACTFUNC.dotabb*⟩≡

```
"FACTFUNC" [color="#FF4488",href="bookvol10.4.pdf#nameddest=FACTFUNC"]
"ALGEBRA" [color="#4488FF",href="bookvol10.2.pdf#nameddest=ALGEBRA"]
"FACTFUNC" -> "ALGEBRA"
```

7.3 package FR2 FactoredFunctions2

⟨*FactoredFunctions2.input*⟩≡

```
)set break resume
)sys rm -f FactoredFunctions2.output
)spool FactoredFunctions2.output
)set message test on
)set message auto off
)clear all
--S 1 of 6
double(x) == x + x
--R
--R                                                                    Type: Void
--E 1

--S 2 of 6
f := factor(720)
--R
--R
--R         4 2
--R   (2)  2 3 5
--R                                                  Type: Factored Integer
--E 2

--S 3 of 6
map(double,f)
--R
--R   Compiling function double with type Integer -> Integer
--R
--R           4 2
--R   (3)  2 4 6 10
--R                                                  Type: Factored Integer
--E 3

--S 4 of 6
makePoly(b) == x + b
--R
--R                                                                    Type: Void
--E 4

--S 5 of 6
g := map(makePoly,f)
--R
--R   Compiling function makePoly with type Integer -> Polynomial Integer
--R
--R                       4       2
```

```
--R   (5)  (x + 1)(x + 2) (x + 3) (x + 5)
--R                                                Type: Factored Polynomial Integer
--E 5

--S 6 of 6
nthFlag(g,1)
--R
--R
--R   (6)  "nil"
--R                                                          Type: Union("nil",...)
--E 6
)spool
)lisp (bye)
```

⟨*FactoredFunctions2.help*⟩≡

```
==========================================================================
FactoredFunctions2 examples
==========================================================================

The FactoredFunctions2 package implements one operation, map, for
applying an operation to every base in a factored object and to the unit.

  double(x) == x + x
                          Type: Void

  f := factor(720)
     4 2
    2 3 5
                          Type: Factored Integer

Actually, the map operation used in this example comes from Factored
itself, since double takes an integer argument and returns an integer
result.

  map(double,f)
       4 2
    2 4 6 10
                          Type: Factored Integer

If we want to use an operation that returns an object that has a type
different from the operation's argument, the map in Factored cannot be
used and we use the one in FactoredFunctions2.

  makePoly(b) == x + b
                    4       2
    (x + 1)(x + 2) (x + 3) (x + 5)
                          Type: Factored Polynomial Integer

In fact, the "2" in the name of the package means that we might
be using factored objects of two different types.

  g := map(makePoly,f)

It is important to note that both versions of map destroy any information
known about the bases (the fact that they are prime, for instance).

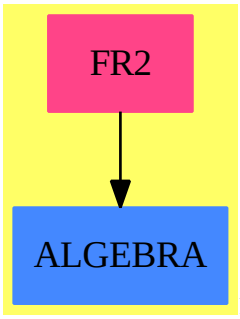
The flags for each base are set to "nil" in the object returned by map.

  nthFlag(g,1)
    "nil"
```

```
                          Type: Union("nil",...)

  See Also:
  o )help Factored
  o )show FactoredFunctions2
```

7.4 FactoredFunctions2



1.00

Exports:
map

⟨*package FR2 FactoredFunctions2*⟩≡
```
)abbrev package FR2 FactoredFunctions2
++ Author: Robert S. Sutor
++ Date Created: 1987
++ Change History:
++ Basic Operations: map
++ Related Constructors: Factored
++ Also See:
++ AMS Classifications: 11A51, 11Y05
++ Keywords: map, factor
++ References:
++ Description:
++   \spadtype{FactoredFunctions2} contains functions that involve
++   factored objects whose underlying domains may not be the same.
++   For example, \spadfun{map} might be used to coerce an object of
++   type \spadtype{Factored(Integer)} to
++   \spadtype{Factored(Complex(Integer))}.
FactoredFunctions2(R, S): Exports == Implementation where
  R: IntegralDomain
  S: IntegralDomain

  Exports ==> with
    map: (R -> S, Factored R) -> Factored S
      ++ map(fn,u) is used to apply the function \userfun{fn} to every
      ++ factor of \spadvar{u}. The new factored object will have all its
      ++ information flags set to "nil". This function is used, for
      ++ example, to coerce every factor base to another type.

  Implementation ==> add
    map(func, f) ==
      func(unit f) *
```

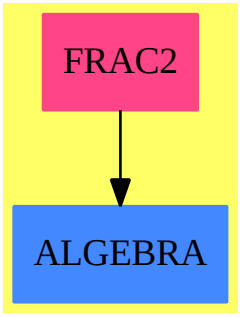
```
            _*/[nilFactor(func(g.factor), g.exponent) for g in factors f]
```

⟨*FR2.dotabb*⟩≡

```
"FR2" [color="#FF4488",href="bookvol10.4.pdf#nameddest=FR2"]
"ALGEBRA" [color="#4488FF",href="bookvol10.2.pdf#nameddest=ALGEBRA"]
"FR2" -> "ALGEBRA"
```

7.5 package FRUTIL FactoredFunctionUtilities

7.6 FactoredFunctionUtilities



1.00

Exports:
mergeFactors refine

⟨*package FRUTIL FactoredFunctionUtilities*⟩≡

```
)abbrev package FRUTIL FactoredFunctionUtilities
++ Author:
++ Date Created:
++ Change History:
++ Basic Operations: refine, mergeFactors
++ Related Constructors: Factored
++ Also See:
++ AMS Classifications: 11A51, 11Y05
++ Keywords: factor
++ References:
++ Description:
++   \spadtype{FactoredFunctionUtilities} implements some utility
++   functions for manipulating factored objects.
FactoredFunctionUtilities(R): Exports == Implementation where
  R: IntegralDomain
  FR ==> Factored R

  Exports ==> with
    refine: (FR, R-> FR) -> FR
      ++ refine(u,fn) is used to apply the function \userfun{fn} to
      ++ each factor of \spadvar{u} and then build a new factored
      ++ object from the results.  For example, if \spadvar{u} were
      ++ created by calling \spad{nilFactor(10,2)} then
      ++ \spad{refine(u,factor)} would create a factored object equal
      ++ to that created by \spad{factor(100)} or
      ++ \spad{primeFactor(2,2) * primeFactor(5,2)}.
```

```
    mergeFactors: (FR,FR) -> FR
      ++ mergeFactors(u,v) is used when the factorizations of \spadvar{u}
      ++ and \spadvar{v} are known to be disjoint, e.g. resulting from a
      ++ content/primitive part split. Essentially, it creates a new
      ++ factored object by multiplying the units together and appending
      ++ the lists of factors.

  Implementation ==> add
    fg: FR
    func: R -> FR
    fUnion ==> Union("nil", "sqfr", "irred", "prime")
    FF     ==> Record(flg: fUnion, fctr: R, xpnt: Integer)

    mergeFactors(f,g) ==
      makeFR(unit(f)*unit(g),append(factorList f,factorList g))

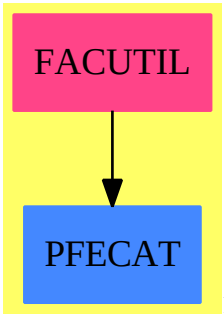
    refine(f, func) ==
       u := unit(f)
       l: List FF := empty()
       for item in factorList f repeat
         fitem := func item.fctr
         u := u*unit(fitem) ** (item.xpnt :: NonNegativeInteger)
         if item.xpnt = 1 then
            l := concat(factorList fitem,l)
         else l := concat([[v.flg,v.fctr,v.xpnt*item.xpnt]
                           for v in factorList fitem],l)
       makeFR(u,l)
```

⟨*FRUTIL.dotabb*⟩≡

```
"FRUTIL" [color="#FF4488",href="bookvol10.4.pdf#nameddest=FRUTIL"]
"ALGEBRA" [color="#4488FF",href="bookvol10.2.pdf#nameddest=ALGEBRA"]
"FRUTIL" -> "ALGEBRA"
```

7.7 package FACUTIL FactoringUtilities

7.8 FactoringUtilities



1.00

Exports:

completeEval	degree	lowerPolynomial	normalDeriv	raisePolynomial
ran	variables			

⟨*package FACUTIL FactoringUtilities*⟩≡

```
)abbrev package FACUTIL FactoringUtilities
++ Author: Barry Trager
++ Date Created: March 12, 1992
++ Date Last Updated:
++ Basic Functions:
++ Related Constructors:
++ Also See:
++ AMS Classifications:
++ Keywords:
++ References:
++ Description:
++ This package provides utilities used by the factorizers
++ which operate on polynomials represented as univariate polynomials
++ with multivariate coefficients.

FactoringUtilities(E,OV,R,P) : C == T where
   E : OrderedAbelianMonoidSup
   OV : OrderedSet
   R : Ring
   P : PolynomialCategory(R,E,OV)

   SUP ==> SparseUnivariatePolynomial
   NNI ==> NonNegativeInteger
   Z   ==> Integer

   C == with
```

```
    completeEval   :      (SUP P,List OV,List R)        -> SUP R
      ++ completeEval(upoly, lvar, lval) evaluates the polynomial upoly
      ++ with each variable in lvar replaced by the corresponding value
      ++ in lval. Substitutions are done for all variables in upoly
      ++ producing a univariate polynomial over R.
    degree         :       (SUP P,List OV)              -> List NNI
      ++ degree(upoly, lvar) returns a list containing the maximum
      ++ degree for each variable in lvar.
    variables      :           SUP P                    -> List OV
      ++ variables(upoly) returns the list of variables for the coefficients
      ++ of upoly.
    lowerPolynomial:           SUP P                    -> SUP R
      ++ lowerPolynomial(upoly) converts upoly to be a univariate polynomial
      ++ over R. An error if the coefficients contain variables.
    raisePolynomial:           SUP R                    -> SUP P
      ++ raisePolynomial(rpoly) converts rpoly from a univariate polynomial
      ++ over r to be a univariate polynomial with polynomial coefficients.
    normalDeriv    :        (SUP P,Z)                   -> SUP P
      ++ normalDeriv(poly,i) computes the ith derivative of poly divided
      ++ by i!.
    ran        :                Z                      -> R
      ++ ran(k) computes a random integer between -k and k as a member of R.

T == add

  lowerPolynomial(f:SUP P) : SUP R ==
    zero? f => 0$SUP(R)
    monomial(ground leadingCoefficient f, degree f)$SUP(R) +
        lowerPolynomial(reductum f)

  raisePolynomial(u:SUP R) : SUP P ==
    zero? u => 0$SUP(P)
    monomial(leadingCoefficient(u)::P, degree u)$SUP(P) +
        raisePolynomial(reductum u)

  completeEval(f:SUP P,lvar:List OV,lval:List R) : SUP R ==
    zero? f => 0$SUP(R)
    monomial(ground eval(leadingCoefficient f,lvar,lval),degree f)$SUP(R) +
           completeEval(reductum f,lvar,lval)

  degree(f:SUP P,lvar:List OV) : List NNI ==
    coefs := coefficients f
    ldeg:= ["max"/[degree(fc,xx) for fc in coefs] for xx in lvar]

  variables(f:SUP P) : List OV ==
    "setUnion"/[variables cf for cf in coefficients f]
```

```
      if R has FiniteFieldCategory then
         ran(k:Z):R == random()$R
      else
         ran(k:Z):R == (random(2*k+1)$Z -k)::R

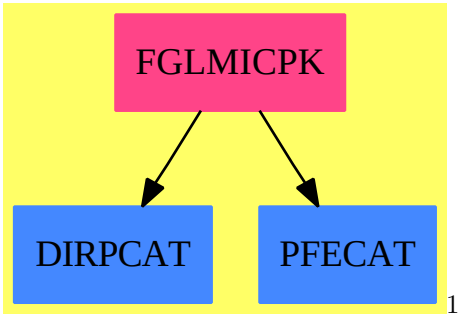
   -- Compute the normalized m derivative
      normalDeriv(f:SUP P,m:Z) : SUP P==
        (n1:Z:=degree f) < m => 0$SUP(P)
        n1=m => (leadingCoefficient f)::SUP(P)
        k:=binomial(n1,m)
        ris:SUP:=0$SUP(P)
        n:Z:=n1
        while n>= m repeat
          while n1>n repeat
            k:=(k*(n1-m)) quo n1
            n1:=n1-1
          ris:=ris+monomial(k*leadingCoefficient f,(n-m)::NNI)
          f:=reductum f
          n:=degree f
        ris
```

⟨*FACUTIL.dotabb*⟩≡

```
"FACUTIL" [color="#FF4488",href="bookvol10.4.pdf#nameddest=FACUTIL"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"FACUTIL" -> "PFECAT"
```

7.9 package FGLMICPK FGLMIfCanPackage

7.10 FGLMIfCanPackage



1.00

Exports:
fglmIfCan groebner zeroDimensional?

⟨*package FGLMICPK FGLMIfCanPackage*⟩≡
```
)abbrev package FGLMICPK FGLMIfCanPackage
++ Author: Marc Moreno Maza
++ Date Created: 08/02/1999
++ Date Last Updated: 08/02/1999
++ Description:
++ This is just an interface between several packages and domains.
++ The goal is to compute lexicographical Groebner bases
++ of sets of polynomial with type \spadtype{Polynomial R}
++ by the {\em FGLM} algorithm if this is possible (i.e.
++ if the input system generates a zero-dimensional ideal).
++ Version: 1.
FGLMIfCanPackage(R,ls): Exports == Implementation where
  R: GcdDomain
  ls: List Symbol
  V ==> OrderedVariableList ls
  N ==> NonNegativeInteger
  Z ==> Integer
  B ==> Boolean
  Q1 ==> Polynomial R
  Q2 ==> HomogeneousDistributedMultivariatePolynomial(ls,R)
  Q3 ==> DistributedMultivariatePolynomial(ls,R)
  E2 ==> HomogeneousDirectProduct(#ls,NonNegativeInteger)
  E3 ==>  DirectProduct(#ls,NonNegativeInteger)
  poltopol ==> PolToPol(ls, R)
  lingrobpack ==> LinGroebnerPackage(ls,R)
  groebnerpack2 ==> GroebnerPackage(R,E2,V,Q2)
  groebnerpack3 ==> GroebnerPackage(R,E3,V,Q3)
```

```
Exports ==  with

   zeroDimensional?: List(Q1) -> B
       ++ \axiom{zeroDimensional?(lq1)} returns true iff
       ++ \axiom{lq1} generates a zero-dimensional ideal
       ++ w.r.t. the variables of \axiom{ls}.
   fglmIfCan: List(Q1) -> Union(List(Q1), "failed")
       ++ \axiom{fglmIfCan(lq1)} returns the lexicographical Groebner
       ++ basis of \axiom{lq1} by using the {\em FGLM} strategy,
       ++ if \axiom{zeroDimensional?(lq1)} holds.
   groebner: List(Q1) -> List(Q1)
       ++ \axiom{groebner(lq1)} returns the lexicographical Groebner
       ++ basis of \axiom{lq1}. If \axiom{lq1} generates a zero-dimensional
       ++ ideal then the {\em FGLM} strategy is used, otherwise
       ++ the {\em Sugar} strategy is used.

Implementation == add

   zeroDim?(lq2: List Q2): Boolean ==
     lq2 := groebner(lq2)$groebnerpack2
     empty? lq2 => false
     #lq2 < #ls => false
     lv: List(V) := [(variable(s)$V)::V for s in ls]
     for q2 in lq2 while not empty?(lv) repeat
        m := leadingMonomial(q2)
        x := mainVariable(m)::V
        if ground?(leadingCoefficient(univariate(m,x))) then
             lv := remove(x, lv)
     empty? lv

   zeroDimensional?(lq1: List(Q1)): Boolean ==
     lq2: List(Q2) := [pToHdmp(q1)$poltopol for q1 in lq1]
     zeroDim?(lq2)

   fglmIfCan(lq1:List(Q1)): Union(List(Q1),"failed") ==
     lq2: List(Q2) := [pToHdmp(q1)$poltopol for q1 in lq1]
     lq2 := groebner(lq2)$groebnerpack2
     not zeroDim?(lq2) => "failed"::Union(List(Q1),"failed")
     lq3: List(Q3) := totolex(lq2)$lingrobpack
     lq1 := [dmpToP(q3)$poltopol for q3 in lq3]
     lq1::Union(List(Q1),"failed")

   groebner(lq1:List(Q1)): List(Q1) ==
     lq2: List(Q2) := [pToHdmp(q1)$poltopol for q1 in lq1]
     lq2 := groebner(lq2)$groebnerpack2
     not zeroDim?(lq2) =>
```

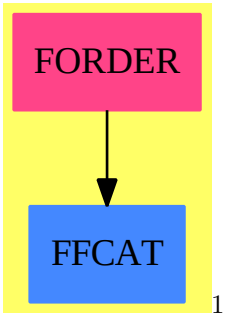
```
    lq3: List(Q3) := [pToDmp(q1)$poltopol for q1 in lq1]
    lq3 := groebner(lq3)$groebnerpack3
    [dmpToP(q3)$poltopol for q3 in lq3]
  lq3: List(Q3) := totolex(lq2)$lingrobpack
  [dmpToP(q3)$poltopol for q3 in lq3]
```

⟨*FGLMICPK.dotabb*⟩≡

```
"FGLMICPK" [color="#FF4488",href="bookvol10.4.pdf#nameddest=FGLMICPK"]
"DIRPCAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=DIRPCAT"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"FGLMICPK" -> "DIRPCAT"
"FGLMICPK" -> "PFECAT"
```

7.11 package FORDER FindOrderFinite

7.12 FindOrderFinite



1.00

Exports:
order

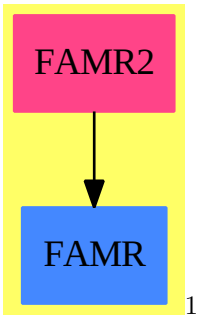
⟨*package FORDER FindOrderFinite*⟩≡
```
)abbrev package FORDER FindOrderFinite
++ Finds the order of a divisor over a finite field
++ Author: Manuel Bronstein
++ Date Created: 1988
++ Date Last Updated: 11 Jul 1990
FindOrderFinite(F, UP, UPUP, R): Exports == Implementation where
  F   : Join(Finite, Field)
  UP  : UnivariatePolynomialCategory F
  UPUP: UnivariatePolynomialCategory Fraction UP
  R   : FunctionFieldCategory(F, UP, UPUP)

  Exports ==> with
    order: FiniteDivisor(F, UP, UPUP, R) -> NonNegativeInteger
        ++ order(x) \undocumented
  Implementation ==> add
    order d ==
      dd := d := reduce d
      for i in 1.. repeat
        principal? dd => return(i::NonNegativeInteger)
        dd := reduce(d + dd)
```

⟨*FORDER.dotabb*⟩≡
```
"FORDER" [color="#FF4488",href="bookvol10.4.pdf#nameddest=FORDER"]
"FFCAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=FFCAT"]
"FORDER" -> "FFCAT"
```

7.13 package FAMR2 FiniteAbelianMonoidRing-Functions2

7.14 FiniteAbelianMonoidRingFunctions2



1.00

Exports:
map

⟨*package FAMR2 FiniteAbelianMonoidRingFunctions2*⟩≡

```
)abbrev package FAMR2 FiniteAbelianMonoidRingFunctions2
++ Author: Martin Rubey
++ Description:
++ This package provides a mapping function for
++ \spadtype{FiniteAbelianMonoidRing}
++ The packages defined in this file provide fast fraction free rational
++ interpolation algorithms. (see FAMR2, FFFG, FFFGF, NEWTON)
FiniteAbelianMonoidRingFunctions2(E: OrderedAbelianMonoid,
                                  R1: Ring,
                                  A1: FiniteAbelianMonoidRing(R1, E),
                                  R2: Ring,
                                  A2: FiniteAbelianMonoidRing(R2, E)) _
                                 : Exports == Implementation where
  Exports == with

    map: (R1 -> R2, A1) -> A2
      ++ \spad{map}(f, a) applies the map f to each coefficient in a. It is
      ++ assumed that f maps 0 to 0

  Implementation == add

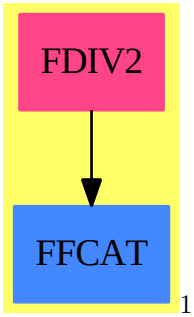
    map(f: R1 -> R2, a: A1): A2 ==
      if zero? a then 0$A2
      else
        monomial(f leadingCoefficient a, degree a)$A2 + map(f, reductum a)
```

⟨*FAMR2.dotabb*⟩≡

```
"FAMR2" [color="#FF4488",href="bookvol10.4.pdf#nameddest=FAMR2"]
"FAMR" [color="#4488FF",href="bookvol10.2.pdf#nameddest=FAMR"]
"FAMR2" -> "FAMR"
```

7.15 package FDIV2 FiniteDivisorFunctions2

7.16 FiniteDivisorFunctions2



1.00

Exports:
map

⟨*package FDIV2 FiniteDivisorFunctions2*⟩≡

```
)abbrev package FDIV2 FiniteDivisorFunctions2
++ Lift a map to finite divisors.
++ Author: Manuel Bronstein
++ Date Created: 1988
++ Date Last Updated: 19 May 1993
FiniteDivisorFunctions2(R1, UP1, UPUP1, F1, R2, UP2, UPUP2, F2):
 Exports == Implementation where
  R1   : Field
  UP1  : UnivariatePolynomialCategory R1
  UPUP1: UnivariatePolynomialCategory Fraction UP1
  F1   : FunctionFieldCategory(R1, UP1, UPUP1)
  R2   : Field
  UP2  : UnivariatePolynomialCategory R2
  UPUP2: UnivariatePolynomialCategory Fraction UP2
  F2   : FunctionFieldCategory(R2, UP2, UPUP2)

  Exports ==> with
    map: (R1 -> R2, FiniteDivisor(R1, UP1, UPUP1, F1)) ->
                                        FiniteDivisor(R2, UP2, UPUP2, F2)
        ++ map(f,d) \undocumented{}

  Implementation ==> add
    import UnivariatePolynomialCategoryFunctions2(R1,UP1,R2,UP2)
    import FunctionFieldCategoryFunctions2(R1,UP1,UPUP1,F1,R2,UP2,UPUP2,F2)
    import FractionalIdealFunctions2(UP1, Fraction UP1, UPUP1, F1,
                                     UP2, Fraction UP2, UPUP2, F2)
```

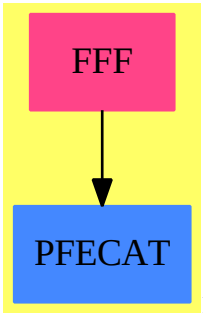
```
      map(f, d) ==
        rec := decompose d
        divisor map(f, rec.principalPart) +
          divisor map((s:UP1):UP2 +-> map(f,s), rec.id)
```

⟨*FDIV2.dotabb*⟩≡

```
  "FDIV2" [color="#FF4488",href="bookvol10.4.pdf#nameddest=FDIV2"]
  "FFCAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=FFCAT"]
  "FDIV2" -> "FFCAT"
```

7.17 package FFF FiniteFieldFunctions

7.18 FiniteFieldFunctions



1.00

Exports:

createLowComplexityNormalBasis	createLowComplexityTable	createMultiplicationMatrix
createMultiplicationTable	createZechTable	sizeMultiplication

⟨*package FFF FiniteFieldFunctions*⟩≡

```
)abbrev package FFF FiniteFieldFunctions
++ Author: J. Grabmeier, A. Scheerhorn
++ Date Created: 21 March 1991
++ Date Last Updated: 31 March 1991
++ Basic Operations:
++ Related Constructors:
++ Also See:
++ AMS Classifications:
++ References:
++  Lidl, R. & Niederreiter, H., "Finite Fields",
++   Encycl. of Math. 20, Addison-Wesley, 1983
++  J. Grabmeier, A. Scheerhorn: Finite Fields in AXIOM.
++   AXIOM Technical Report Series, ATR/5 NP2522.
++ Description:
++  FiniteFieldFunctions(GF) is a package with functions
++  concerning finite extension fields of the finite ground field {\em GF},
++  e.g. Zech logarithms.
++ Keywords: finite field, Zech logarithm, Jacobi logarithm, normal basis

FiniteFieldFunctions(GF): Exports == Implementation where
  GF  : FiniteFieldCategory  -- the ground field

  PI   ==> PositiveInteger
  NNI  ==> NonNegativeInteger
  I    ==> Integer
  SI   ==> SingleInteger
```

```
SUP  ==> SparseUnivariatePolynomial GF
V    ==> Vector
M    ==> Matrix
L    ==> List
OUT  ==> OutputForm
SAE  ==> SimpleAlgebraicExtension
ARR  ==> PrimitiveArray(SI)
TERM ==> Record(value:GF,index:SI)
MM   ==> ModMonic(GF,SUP)
PF   ==> PrimeField

Exports ==> with

    createZechTable: SUP -> ARR
      ++ createZechTable(f) generates a Zech logarithm table for the cyclic
      ++ group representation of a extension of the ground field by the
      ++ primitive polynomial {\em f(x)}, i.e. \spad{Z(i)},
      ++ defined by {\em x**Z(i) = 1+x**i} is stored at index i.
      ++ This is needed in particular
      ++ to perform addition of field elements in finite fields represented
      ++ in this way. See \spadtype{FFCGP}, \spadtype{FFCGX}.
    createMultiplicationTable: SUP -> V L TERM
      ++ createMultiplicationTable(f) generates a multiplication
      ++ table for the normal basis of the field extension determined
      ++ by {\em f}. This is needed to perform multiplications
      ++ between elements represented as coordinate vectors to this basis.
      ++ See \spadtype{FFNBP}, \spadtype{FFNBX}.
    createMultiplicationMatrix: V L TERM -> M GF
      ++ createMultiplicationMatrix(m) forms the multiplication table
      ++ {\em m} into a matrix over the ground field.
      -- only useful for the user to visualise the multiplication table
      -- in a nice form
    sizeMultiplication: V L TERM -> NNI
      ++ sizeMultiplication(m) returns the number of entries
      ++ of the multiplication table {\em m}.
      -- the time of the multiplication of field elements depends
      -- on this size
    createLowComplexityTable: PI -> Union(Vector List TERM,"failed")
      ++ createLowComplexityTable(n) tries to find
      ++ a low complexity normal basis of degree {\em n} over {\em GF}
      ++ and returns its multiplication matrix
      ++ Fails, if it does not find a low complexity basis
    createLowComplexityNormalBasis: PI -> Union(SUP, V L TERM)
      ++ createLowComplexityNormalBasis(n) tries to find a
      ++ a low complexity normal basis of degree {\em n} over {\em GF}
      ++ and returns its multiplication matrix
```

```
        ++ If no low complexity basis is found it calls
        ++ \axiomFunFrom{createNormalPoly}{FiniteFieldPolynomialPackage}(n)
        ++ to produce a normal
        ++ polynomial of degree {\em n} over {\em GF}

    Implementation ==> add


      createLowComplexityNormalBasis(n) ==
        (u:=createLowComplexityTable(n)) case "failed" =>
          createNormalPoly(n)$FiniteFieldPolynomialPackage(GF)
        u::(V L TERM)

  -- try to find a low complexity normal basis multiplication table
  -- of the field of extension degree n
  -- the algorithm is from:
  -- Wassermann A., Konstruktion von Normalbasen,
  -- Bayreuther Mathematische Schriften 31 (1989),1-9.

      createLowComplexityTable(n) ==
        q:=size()$GF
        -- this algorithm works only for prime fields
        p:=characteristic()$GF
        -- search of a suitable parameter k
        k:NNI:=0
        for i in 1..n-1  while (k=0) repeat
          if prime?(i*n+1) and not(p = (i*n+1)) then
            primitive?(q::PF(i*n+1))$PF(i*n+1) =>
                a:NNI:=1
                k:=i
                t1:PF(k*n+1):=(q::PF(k*n+1))**n
            gcd(n,a:=discreteLog(q::PF(n*i+1))$PF(n*i+1))$I = 1 =>
                k:=i
                t1:=primitiveElement()$PF(k*n+1)**n
        k = 0 => "failed"
        -- initialize some start values
        multmat:M PF(p):=zero(n,n)
        p1:=(k*n+1)
        pkn:=q::PF(p1)
        t:=t1 pretend PF(p1)
        if odd?(k) then
            jt:I:=(n quo 2)+1
            vt:I:=positiveRemainder((k-a) quo 2,k)+1
          else
            jt:I:=1
            vt:I:=(k quo 2)+1
```

```
    -- compute matrix
    vec:Vector I:=zero(p1 pretend NNI)
    for x in 1..k repeat
      for l in 1..n repeat
        vec.((t**(x-1) * pkn**(l-1)) pretend Integer+1):=_
                                          positiveRemainder(l,p1)
    lvj:M I:=zero(k::NNI,n)
    for v in 1..k repeat
      for j in 1..n repeat
        if (j^=jt) or (v^=vt) then
          help:PF(p1):=t**(v-1)*pkn**(j-1)+1@PF(p1)
          setelt(lvj,v,j,vec.(help pretend I +1))
    for j in 1..n repeat
      if j^=jt then
        for v in 1..k repeat
          lvjh:=elt(lvj,v,j)
          setelt(multmat,j,lvjh,elt(multmat,j,lvjh)+1)
    for i in 1..n repeat
      setelt(multmat,jt,i,positiveRemainder(-k,p)::PF(p))
    for v in 1..k repeat
      if v^=vt then
        lvjh:=elt(lvj,v,jt)
        setelt(multmat,jt,lvjh,elt(multmat,jt,lvjh)+1)
    -- multmat
    m:=nrows(multmat)$(M PF(p))
    multtable:V L TERM:=new(m,nil()$(L TERM))$(V L TERM)
    for i in 1..m repeat
      l:L TERM:=nil()$(L TERM)
      v:V PF(p):=row(multmat,i)
      for j in (1::I)..(m::I) repeat
        if (v.j ^= 0) then
          -- take -v.j to get trace 1 instead of -1
          term:TERM:=[(convert(-v.j)@I)::GF,(j-2) pretend SI]$TERM
          l:=cons(term,l)$(L TERM)
      qsetelt_!(multtable,i,copy l)$(V L TERM)
    multtable

  sizeMultiplication(m) ==
    s:NNI:=0
    for i in 1..#m repeat
      s := s + #(m.i)
    s

  createMultiplicationTable(f:SUP) ==
    sizeGF:NNI:=size()$GF -- the size of the ground field
    m:PI:=degree(f)$SUP pretend PI
```

```
    m=1 =>
      [[[-coefficient(f,0)$SUP,(-1)::SI]$TERM]$(L TERM)]::(V L TERM)
    m1:I:=m-1
    -- initialize basis change matrices
    setPoly(f)$MM
    e:=reduce(monomial(1,1)$SUP)$MM ** sizeGF
    w:=1$MM
    qpow:PrimitiveArray(MM):=new(m,0)
    qpow.0:=1$MM
    for i in 1..m1 repeat
      qpow.i:=(w:=w*e)
    -- qpow.i = x**(i*q)
    qexp:PrimitiveArray(MM):=new(m,0)
    qexp.0:=reduce(monomial(1,1)$SUP)$MM
    mat:M GF:=zero(m,m)$(M GF)
    qsetelt_!(mat,2,1,1$GF)$(M GF)
    h:=qpow.1
    qexp.1:=h
    setColumn_!(mat,2,Vectorise(h)$MM)$(M GF)
    for i in 2..m1 repeat
      g:=0$MM
      while h ^= 0 repeat
        g:=g + leadingCoefficient(h) * qpow.degree(h)$MM
        h:=reductum(h)$MM
      qexp.i:=g
      setColumn_!(mat,i+1,Vectorise(h:=g)$MM)$(M GF)
    -- loop invariant: qexp.i = x**(q**i)
    mat1:=inverse(mat)$(M GF)
    mat1 = "failed" =>
      error "createMultiplicationTable: polynomial must be normal"
    mat:=mat1 :: (M GF)
    -- initialize multiplication table
    multtable:V L TERM:=new(m,nil()$(L TERM))$(V L TERM)
    for i in 1..m repeat
      l:L TERM:=nil()$(L TERM)
      v:V GF:=mat *$(M GF) Vectorise(qexp.(i-1) *$MM qexp.0)$MM
      for j in (1::SI)..(m::SI) repeat
        if (v.j ^= 0$GF) then
          term:TERM:=[(v.j),j-(2::SI)]$TERM
          l:=cons(term,l)$(L TERM)
      qsetelt_!(multtable,i,copy l)$(V L TERM)
    multtable


  createZechTable(f:SUP) ==
    sizeGF:NNI:=size()$GF -- the size of the ground field
```

```
      m:=degree(f)$SUP::PI
      qm1:SI:=(sizeGF ** m -1) pretend SI
      zechlog:ARR:=new(((sizeGF ** m + 1) quo 2)::NNI,-1::SI)$ARR
      helparr:ARR:=new(sizeGF ** m::NNI,0$SI)$ARR
      primElement:=reduce(monomial(1,1)$SUP)$SAE(GF,SUP,f)
      a:=primElement
      for i in 1..qm1-1 repeat
        helparr.(lookup(a -$SAE(GF,SUP,f) 1$SAE(GF,SUP,f)_
           )$SAE(GF,SUP,f)):=i::SI
        a:=a * primElement
      characteristic() = 2 =>
        a:=primElement
        for i in 1..(qm1 quo 2) repeat
          zechlog.i:=helparr.lookup(a)$SAE(GF,SUP,f)
          a:=a * primElement
        zechlog
      a:=1$SAE(GF,SUP,f)
      for i in 0..((qm1-2) quo 2) repeat
        zechlog.i:=helparr.lookup(a)$SAE(GF,SUP,f)
        a:=a * primElement
      zechlog

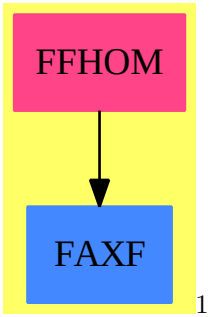
    createMultiplicationMatrix(m) ==
      n:NNI:=#m
      mat:M GF:=zero(n,n)$(M GF)
      for i in 1..n repeat
        for t in m.i repeat
          qsetelt_!(mat,i,t.index+2,t.value)
      mat
```

⟨*FFF.dotabb*⟩≡

```
"FFF" [color="#FF4488",href="bookvol10.4.pdf#nameddest=FFF"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"FFF" -> "PFECAT"
```

7.19 package FFHOM FiniteFieldHomomorphisms

7.20 FiniteFieldHomomorphisms



1.00

Exports:
coerce

⟨*package FFHOM FiniteFieldHomomorphisms*⟩≡
```
)abbrev package FFHOM FiniteFieldHomomorphisms
++ Authors: J.Grabmeier, A.Scheerhorn
++ Date Created: 26.03.1991
++ Date Last Updated:
++ Basic Operations:
++ Related Constructors: FiniteFieldCategory, FiniteAlgebraicExtensionField
++ Also See:
++ AMS Classifications:
++ Keywords: finite field, homomorphism, isomorphism
++ References:
++  R.Lidl, H.Niederreiter: Finite Field, Encycoldia of Mathematics and
++  Its Applications, Vol. 20, Cambridge Univ. Press, 1983, ISBN 0 521 30240 4
++  J. Grabmeier, A. Scheerhorn: Finite Fields in AXIOM.
++  AXIOM Technical Report Series, ATR/5 NP2522.
++ Description:
++  FiniteFieldHomomorphisms(F1,GF,F2) exports coercion functions of
++  elements between the fields {\em F1} and {\em F2}, which both must be
++  finite simple algebraic extensions of the finite ground field {\em GF}.
FiniteFieldHomomorphisms(F1,GF,F2): Exports == Implementation where
  F1: FiniteAlgebraicExtensionField(GF)
  GF: FiniteFieldCategory
  F2: FiniteAlgebraicExtensionField(GF)
 -- the homorphism can only convert elements w.r.t. the last extension .
  -- Adding a function 'groundField()' which returns the groundfield of GF
  -- as a variable of type FiniteFieldCategory in the new compiler, one
  -- could build up 'convert' recursively to get an homomorphism w.r.t
  -- the whole extension.
```

```
   I   ==> Integer
   NNI ==> NonNegativeInteger
   SI  ==> SingleInteger
   PI  ==> PositiveInteger
   SUP ==> SparseUnivariatePolynomial
   M   ==> Matrix GF
   FFP ==> FiniteFieldExtensionByPolynomial
   FFPOL2 ==> FiniteFieldPolynomialPackage2
   FFPOLY ==> FiniteFieldPolynomialPackage
   OUT ==> OutputForm

   Exports ==> with

     coerce: F1  ->  F2
       ++ coerce(x) is the homomorphic image of x from
       ++ {\em F1} in {\em F2}. Thus {\em coerce} is a
       ++ field homomorphism between the fields extensions
       ++ {\em F1} and {\em F2} both over ground field {\em GF}
       ++ (the second argument to the package).
       ++ Error: if the extension degree of {\em F1} doesn't divide
       ++ the extension degree of {\em F2}.
       ++ Note that the other coercion function in the
       ++ \spadtype{FiniteFieldHomomorphisms} is a left inverse.

     coerce: F2  ->  F1
       ++ coerce(x) is the homomorphic image of x from
       ++ {\em F2} in {\em F1}, where {\em coerce} is a
       ++ field homomorphism between the fields extensions
       ++ {\em F2} and {\em F1} both over ground field {\em GF}
       ++ (the second argument to the package).
       ++ Error: if the extension degree of {\em F2} doesn't divide
       ++ the extension degree of {\em F1}.
       ++ Note that the other coercion function in the
       ++ \spadtype{FiniteFieldHomomorphisms} is a left inverse.
     -- coerce(coerce(x:F1)@F2)@F1 = x and coerce(coerce(y:F2)@F1)@F2 = y

   Implementation ==> add

 -- global variables ================================================

     degree1:NNI:= extensionDegree()$F1
     degree2:NNI:= extensionDegree()$F2
     -- the degrees of the last extension

     -- a necessary condition for the one field being an subfield of
```

```
    -- the other one is, that the respective extension degrees are
    -- multiples
    if max(degree1,degree2) rem min(degree1,degree2) ^= 0 then
      error "FFHOM: one extension degree must divide the other one"

    conMat1to2:M:= zero(degree2,degree1)$M
    -- conversion Matix for the conversion direction F1 -> F2
    conMat2to1:M:= zero(degree1,degree2)$M
    -- conversion Matix for the conversion direction F2 -> F1

    repType1:=representationType()$F1
    repType2:=representationType()$F2
    -- the representation types of the fields

    init?:Boolean:=true
    -- gets false after initialization

    defPol1:=definingPolynomial()$F1
    defPol2:=definingPolynomial()$F2
    -- the defining polynomials of the fields


  -- functions ============================================================


    compare: (SUP GF,SUP GF) -> Boolean
    -- compares two polynomials

    convertWRTsameDefPol12: F1  ->  F2
    convertWRTsameDefPol21: F2  ->  F1
    -- homomorphism if the last extension of F1 and F2 was build up
    -- using the same defining polynomials

    convertWRTdifferentDefPol12: F1  ->  F2
    convertWRTdifferentDefPol21: F2  ->  F1
    -- homomorphism if the last extension of F1 and F2 was build up
    -- with different defining polynomials

    initialize: () -> Void
    -- computes the conversion matrices

    compare(g:(SUP GF),f:(SUP GF)) ==
      degree(f)$(SUP GF)  >$NNI degree(g)$(SUP GF) => true
      degree(f)$(SUP GF) <$NNI degree(g)$(SUP GF) => false
      equal:Integer:=0
      for i in degree(f)$(SUP GF)..0 by -1 while equal=0 repeat
```

```
      not zero?(coefficient(f,i)$(SUP GF))$GF and _
           zero?(coefficient(g,i)$(SUP GF))$GF => equal:=1
      not zero?(coefficient(g,i)$(SUP GF))$GF and _
           zero?(coefficient(f,i)$(SUP GF))$GF => equal:=(-1)
      (f1:=lookup(coefficient(f,i)$(SUP GF))$GF) >$PositiveInteger _
       (g1:=lookup(coefficient(g,i)$(SUP GF))$GF) =>  equal:=1
      f1 <$PositiveInteger g1 => equal:=(-1)
    equal=1 => true
    false

  initialize() ==
    -- 1) in the case of equal def. polynomials initialize is called only
    --  if one of the rep. types is "normal" and the other one is "polynomial"
    --  we have to compute the basis change matrix 'mat', which i-th
    --  column are the coordinates of a**(q**i), the i-th component of
    --  the normal basis ('a' the root of the def. polynomial and q the
    --  size of the groundfield)
    defPol1 =$(SUP GF) defPol2 =>
      -- new code using reducedQPowers
      mat:=zero(degree1,degree1)$M
      arr:=reducedQPowers(defPol1)$FFPOLY(GF)
      for i in 1..degree1 repeat
        setColumn_!(mat,i,vectorise(arr.(i-1),degree1)$SUP(GF))$M
        -- old code
        -- here one of the representation types must be "normal"
        --a:=basis()$FFP(GF,defPol1).2  -- the root of the def. polynomial
        --setColumn_!(mat,1,coordinates(a)$FFP(GF,defPol1))$M
        --for i in 2..degree1 repeat
        --  a:= a **$FFP(GF,defPol1) size()$GF
        --  setColumn_!(mat,i,coordinates(a)$FFP(GF,defPol1))$M
        --for the direction "normal" -> "polynomial" we have to multiply the
        -- coordinate vector of an element of the normal basis field with
        -- the matrix 'mat'. In this case 'mat' is the correct conversion
        -- matrix for the conversion of F1 to F2, its inverse the correct
        -- inversion matrix for the conversion of F2 to F1
      repType1 = "normal" =>  -- repType2 = "polynomial"
        conMat1to2:=copy(mat)
        conMat2to1:=copy(inverse(mat)$M :: M)
        --we finish the function for one case, hence reset initialization flag
        init? := false
        void()$Void
        -- print("'normal' <=> 'polynomial' matrices initialized"::OUT)
      -- in the other case we have to change the matrices
      -- repType2 = "normal" and repType1 = "polynomial"
      conMat2to1:=copy(mat)
      conMat1to2:=copy(inverse(mat)$M :: M)
```

```
  -- print("'normal' <=> 'polynomial' matrices initialized"::OUT)
  --we finish the function for one case, hence reset initialization flag
  init? := false
  void()$Void
-- 2) in the case of different def. polynomials we have to order the
--    fields to get the same isomorphism, if the package is called with
--    the fields F1 and F2 swapped.
dPbig:= defPol2
rTbig:= repType2
dPsmall:= defPol1
rTsmall:= repType1
degbig:=degree2
degsmall:=degree1
if compare(defPol2,defPol1) then
  degsmall:=degree2
  degbig:=degree1
  dPbig:= defPol1
  rTbig:= repType1
  dPsmall:= defPol2
  rTsmall:= repType2
-- 3) in every case we need a conversion between the polynomial
--  represented fields. Therefore we compute 'root' as a root of the
--  'smaller' def. polynomial in the 'bigger' field.
--  We compute the matrix 'matsb', which i-th column are the coordinates
--  of the (i-1)-th power of root, i=1..degsmall. Multiplying a
--  coordinate vector of an element of the 'smaller' field by this
--  matrix, we got the coordinates of the corresponding element in the
--  'bigger' field.
-- compute the root of dPsmall in the 'big' field
root:=rootOfIrreduciblePoly(dPsmall)$FFPOL2(FFP(GF,dPbig),GF)
-- set up matrix for polynomial conversion
matsb:=zero(degbig,degsmall)$M
qsetelt_!(matsb,1,1,1$GF)$M
a:=root
for i in 2..degsmall repeat
  setColumn_!(matsb,i,coordinates(a)$FFP(GF,dPbig))$M
  a := a *$FFP(GF,dPbig) root
--  the conversion from 'big' to 'small': we can't invert matsb
--  directly, because it has degbig rows and degsmall columns and
--  may be no square matrix. Therfore we construct a square matrix
--  mat from degsmall linear independent rows of matsb and invert it.
--  Now we get the conversion matrix 'matbs' for the conversion from
--  'big' to 'small' by putting the columns of mat at the indices
--  of the linear independent rows of matsb to columns of matbs.
ra:I:=1   -- the rank
mat:M:=transpose(row(matsb,1))$M -- has already rank 1
```

```
rowind:I:=2
iVec:Vector I:=new(degsmall,1$I)$(Vector I)
while ra < degsmall repeat
  if rank(vertConcat(mat,transpose(row(matsb,rowind))$M)$M)$M > ra then
    mat:=vertConcat(mat,transpose(row(matsb,rowind))$M)$M
    ra:=ra+1
    iVec.ra := rowind
  rowind:=rowind + 1
mat:=inverse(mat)$M :: M
matbs:=zero(degsmall,degbig)$M
for i in 1..degsmall repeat
  setColumn_!(matbs,iVec.i,column(mat,i)$M)$M
-- print(matsb::OUT)
-- print(matbs::OUT)
-- 4) if the 'bigger' field is "normal" we have to compose the
--  polynomial conversion with a conversion from polynomial to normal
--  between the FFP(GF,dPbig) and FFNBP(GF,dPbig) the 'bigger'
--  field. Therefore we compute a conversion matrix 'mat' as in 1)
--  Multiplying with the inverse of 'mat' yields the desired
--  conversion from polynomial to normal. Multiplying this matrix by
--  the above computed 'matsb' we got the matrix for converting form
--  'small polynomial' to 'big normal'.
-- set up matrix 'mat' for polynomial to normal
if rTbig = "normal" then
  arr:=reducedQPowers(dPbig)$FFPOLY(GF)
  mat:=zero(degbig,degbig)$M
  for i in 1..degbig repeat
    setColumn_!(mat,i,vectorise(arr.(i-1),degbig)$SUP(GF))$M
  -- old code
  --a:=basis()$FFP(GF,dPbig).2  -- the root of the def.Polynomial
  --setColumn_!(mat,1,coordinates(a)$FFP(GF,dPbig))$M
  --for i in 2..degbig repeat
  --  a:= a **$FFP(GF,dPbig) size()$GF
  --  setColumn_!(mat,i,coordinates(a)$FFP(GF,dPbig))$M
  -- print(inverse(mat)$M::OUT)
  matsb:= (inverse(mat)$M :: M) * matsb
  -- print("inv *.."::OUT)
  matbs:=matbs * mat
  -- 5) if the 'smaller' field is "normal" we have first to convert
  --    from 'small normal' to 'small polynomial', that is from
  --    FFNBP(GF,dPsmall) to FFP(GF,dPsmall). Therefore we compute a
  --    conversion matrix 'mat' as in 1). Multiplying with  'mat'
  --    yields the desired conversion from normal to polynomial.
  --    Multiplying the above computed 'matsb' with 'mat' we got the
  --    matrix for converting form 'small normal' to 'big normal'.
-- set up matrix 'mat' for normal to polynomial
```

```
    if rTsmall = "normal" then
      arr:=reducedQPowers(dPsmall)$FFPOLY(GF)
      mat:=zero(degsmall,degsmall)$M
      for i in 1..degsmall repeat
        setColumn_!(mat,i,vectorise(arr.(i-1),degsmall)$SUP(GF))$M
    -- old code
    --b:FFP(GF,dPsmall):=basis()$FFP(GF,dPsmall).2
    --setColumn_!(mat,1,coordinates(b)$FFP(GF,dPsmall))$M
    --for i in 2..degsmall repeat
    --  b:= b **$FFP(GF,dPsmall) size()$GF
    --  setColumn_!(mat,i,coordinates(b)$FFP(GF,dPsmall))$M
      -- print(mat::OUT)
      matsb:= matsb * mat
      matbs:= (inverse(mat) :: M) * matbs
    -- now 'matsb' is the corret conversion matrix for 'small' to 'big'
    -- and 'matbs' the corret one for 'big' to 'small'.
    -- depending on the above ordering the conversion matrices are
    -- initialized
    dPbig =$(SUP GF) defPol2 =>
      conMat1to2 :=matsb
      conMat2to1 :=matbs
      -- print(conMat1to2::OUT)
      -- print(conMat2to1::OUT)
      -- print("conversion matrices initialized"::OUT)
      --we finish the function for one case, hence reset initialization flag
      init? := false
      void()$Void
    conMat1to2 :=matbs
    conMat2to1 :=matsb
    -- print(conMat1to2::OUT)
    -- print(conMat2to1::OUT)
    -- print("conversion matrices initialized"::OUT)
    --we finish the function for one case, hence reset initialization flag
    init? := false
    void()$Void


  coerce(x:F1) ==
    inGroundField?(x)$F1 => retract(x)$F1 :: F2
    -- if x is already in GF then we can use a simple coercion
    defPol1 =$(SUP GF) defPol2 => convertWRTsameDefPol12(x)
    convertWRTdifferentDefPol12(x)

  convertWRTsameDefPol12(x:F1)  ==
    repType1 = repType2 => x pretend F2
    -- same groundfields, same defining polynomials, same
```

```
      -- representation types --> F1 = F2, x is already in F2
      repType1 = "cyclic" =>
        x = 0$F1 => 0$F2
      -- the SI corresponding to the cyclic representation is the exponent of
      -- the primitiveElement, therefore we exponentiate the primitiveElement
      -- of F2 by it.
        primitiveElement()$F2 **$F2 (x pretend SI)
      repType2 = "cyclic" =>
        x = 0$F1 => 0$F2
      -- to get the exponent, we have to take the discrete logarithm of the
      -- element in the given field.
        (discreteLog(x)$F1 pretend SI) pretend F2
      -- here one of the representation types is "normal"
      if init? then initialize()
      -- here a conversion matrix is necessary, (see initialize())
      represents(conMat1to2 *$(Matrix GF) coordinates(x)$F1)$F2

    convertWRTdifferentDefPol12(x:F1) ==
      if init? then initialize()
      -- if we want to convert into a 'smaller' field, we have to test,
      -- whether the element is in the subfield of the 'bigger' field, which
      -- corresponds to the 'smaller' field
      if degree1 > degree2 then
        if positiveRemainder(degree2,degree(x)$F1)^= 0 then
          error "coerce: element doesn't belong to smaller field"
      represents(conMat1to2 *$(Matrix GF) coordinates(x)$F1)$F2

  -- the three functions below equal the three functions above up to
  -- '1' exchanged by '2' in all domain and variable names


    coerce(x:F2) ==
      inGroundField?(x)$F2 => retract(x)$F2 :: F1
      -- if x is already in GF then we can use a simple coercion
      defPol1 =$(SUP GF) defPol2 => convertWRTsameDefPol21(x)
      convertWRTdifferentDefPol21(x)

    convertWRTsameDefPol21(x:F2)  ==
      repType1 = repType2 => x pretend F1
      -- same groundfields, same defining polynomials,
      -- same representation types --> F1 = F2, that is:
      -- x is already in F1
      repType2 = "cyclic" =>
        x = 0$F2 => 0$F1
        primitiveElement()$F1 **$F1 (x pretend SI)
      repType1 = "cyclic" =>
```

```
      x = 0$F2 => 0$F1
      (discreteLog(x)$F2 pretend SI) pretend F1
    -- here one of the representation types is "normal"
    if init? then initialize()
    represents(conMat2to1 *$(Matrix GF) coordinates(x)$F2)$F1

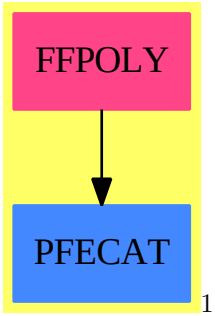
  convertWRTdifferentDefPol21(x:F2) ==
    if init? then initialize()
    if degree2 > degree1 then
      if positiveRemainder(degree1,degree(x)$F2)^= 0 then
        error "coerce: element doesn't belong to smaller field"
    represents(conMat2to1 *$(Matrix GF) coordinates(x)$F2)$F1
```

⟨*FFHOM.dotabb*⟩≡

```
"FFHOM" [color="#FF4488",href="bookvol10.4.pdf#nameddest=FFHOM"]
"FAXF" [color="#4488FF",href="bookvol10.2.pdf#nameddest=FAXF"]
"FFHOM" -> "FAXF"
```

7.21 package FFPOLY FiniteFieldPolynomialPackage

7.22 FiniteFieldPolynomialPackage



1.00

Exports:

createIrreduciblePoly	createNormalPoly	createNormalPrimitivePoly
createPrimitiveNormalPoly	createPrimitivePoly	leastAffineMultiple
nextIrreduciblePoly	nextNormalPoly	nextNormalPrimitivePoly
nextPrimitiveNormalPoly	nextPrimitivePoly	normal?
numberOfIrreduciblePoly	numberOfNormalPoly	numberOfPrimitivePoly
primitive?	random	reducedQPowers

⟨*package FFPOLY FiniteFieldPolynomialPackage*⟩≡

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)abbrev package FFPOLY FiniteFieldPolynomialPackage
++ Author: A. Bouyer, J. Grabmeier, A. Scheerhorn, R. Sutor, B. Trager
++ Date Created: January 1991
++ Date Last Updated: 1 June 1994
++ Basic Operations:
++ Related Constructors:
++ Also See:
++ AMS Classifications:
++ Keywords: finite field, polynomial, irreducible polynomial, normal
++   polynomial, primitive polynomial, random polynomials
++ References:
++   [LS] Lenstra, H. W. & Schoof, R. J., "Primitivive Normal Bases
++        for Finite Fields", Math. Comp. 48, 1987, pp. 217-231
++   [LN] Lidl, R. & Niederreiter, H., "Finite Fields",
++        Encycl. of Math. 20, Addison-Wesley, 1983
++  J. Grabmeier, A. Scheerhorn: Finite Fields in Axiom.
++   Axiom Technical Report Series, to appear.
++ Description:
++   This package provides a number of functions for generating, counting
++   and testing irreducible, normal, primitive, random polynomials
++   over finite fields.
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FiniteFieldPolynomialPackage GF : Exports == Implementation where

  GF : FiniteFieldCategory

  I    ==> Integer
  L    ==> List
  NNI  ==> NonNegativeInteger
  PI   ==> PositiveInteger
  Rec  ==> Record(expnt:NNI, coeff:GF)
  Repr ==> L Rec
  SUP  ==> SparseUnivariatePolynomial GF

  Exports ==> with
 --    qEulerPhiCyclotomic : PI -> PI
--      ++ qEulerPhiCyclotomic(n)$FFPOLY(GF) yields the q-Euler's function
--      ++ of the n-th cyclotomic polynomial over the field {\em GF} of
--      ++ order q (cf. [LN] p.122);
--      ++ error if n is a multiple of the field characteristic.
    primitive? : SUP -> Boolean
      ++ primitive?(f) tests whether the polynomial f over a finite
      ++ field is primitive, i.e. all its roots are primitive.
    normal? : SUP -> Boolean
      ++ normal?(f) tests whether the polynomial f over a finite field is
      ++ normal, i.e. its roots are linearly independent over the field.
    numberOfIrreduciblePoly : PI -> PI
      ++ numberOfIrreduciblePoly(n)$FFPOLY(GF) yields the number of
      ++ monic irreducible univariate polynomials of degree n
      ++ over the finite field {\em GF}.
    numberOfPrimitivePoly : PI -> PI
      ++ numberOfPrimitivePoly(n)$FFPOLY(GF) yields the number of
      ++ primitive polynomials of degree n over the finite field {\em GF}.
    numberOfNormalPoly : PI -> PI
      ++ numberOfNormalPoly(n)$FFPOLY(GF) yields the number of
      ++ normal polynomials of degree n over the finite field {\em GF}.
    createIrreduciblePoly : PI -> SUP
      ++ createIrreduciblePoly(n)$FFPOLY(GF) generates a monic irreducible
      ++ univariate polynomial of degree n over the finite field {\em GF}.
    createPrimitivePoly : PI -> SUP
      ++ createPrimitivePoly(n)$FFPOLY(GF) generates a primitive polynomial
      ++ of degree n over the finite field {\em GF}.
    createNormalPoly : PI -> SUP
      ++ createNormalPoly(n)$FFPOLY(GF) generates a normal polynomial
      ++ of degree n over the finite field {\em GF}.
    createNormalPrimitivePoly : PI -> SUP
      ++ createNormalPrimitivePoly(n)$FFPOLY(GF) generates a normal and
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    ++ primitive polynomial of degree n over the field {\em GF}.
    ++ Note: this function is equivalent to createPrimitiveNormalPoly(n)
  createPrimitiveNormalPoly : PI -> SUP
    ++ createPrimitiveNormalPoly(n)$FFPOLY(GF) generates a normal and
    ++ primitive polynomial of degree n over the field {\em GF}.
    ++ polynomial of degree n over the field {\em GF}.
  nextIrreduciblePoly : SUP -> Union(SUP, "failed")
    ++ nextIrreduciblePoly(f) yields the next monic irreducible polynomial
    ++ over a finite field {\em GF} of the same degree as f in the following
    ++ order, or "failed" if there are no greater ones.
    ++ Error: if f has degree 0.
    ++ Note: the input polynomial f is made monic.
    ++ Also, \spad{f < g} if
    ++ the number of monomials of f is less
    ++ than this number for g.
    ++ If f and g have the same number of monomials,
    ++ the lists of exponents are compared lexicographically.
    ++ If these lists are also equal, the lists of coefficients
    ++ are compared according to the lexicographic ordering induced by
    ++ the ordering of the elements of {\em GF} given by {\em lookup}.
  nextPrimitivePoly : SUP -> Union(SUP, "failed")
    ++ nextPrimitivePoly(f) yields the next primitive polynomial over
    ++ a finite field {\em GF} of the same degree as f in the following
    ++ order, or "failed" if there are no greater ones.
    ++ Error: if f has degree 0.
    ++ Note: the input polynomial f is made monic.
    ++ Also, \spad{f < g} if the {\em lookup} of the constant term
    ++ of f is less than
    ++ this number for g.
    ++ If these values are equal, then \spad{f < g} if
    ++ if the number of monomials of f is less than that for g or if
    ++ the lists of exponents of f are lexicographically less than the
    ++ corresponding list for g.
    ++ If these lists are also equal, the lists of coefficients are
    ++ compared according to the lexicographic ordering induced by
    ++ the ordering of the elements of {\em GF} given by {\em lookup}.
  nextNormalPoly : SUP -> Union(SUP, "failed")
    ++ nextNormalPoly(f) yields the next normal polynomial over
    ++ a finite field {\em GF} of the same degree as f in the following
    ++ order, or "failed" if there are no greater ones.
    ++ Error: if f has degree 0.
    ++ Note: the input polynomial f is made monic.
    ++ Also, \spad{f < g} if the {\em lookup} of the coefficient
    ++ of the term of degree
    ++ {\em n-1} of f is less than that for g.
    ++ In case these numbers are equal, \spad{f < g} if
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      ++ if the number of monomials of f is less that for g or if
      ++ the list of exponents of f are lexicographically less than the
      ++ corresponding list for g.
      ++ If these lists are also equal, the lists of coefficients are
      ++ compared according to the lexicographic ordering induced by
      ++ the ordering of the elements of {\em GF} given by {\em lookup}.
    nextNormalPrimitivePoly : SUP -> Union(SUP, "failed")
      ++ nextNormalPrimitivePoly(f) yields the next normal primitive polynomial
      ++ over a finite field {\em GF} of the same degree as f in the following
      ++ order, or "failed" if there are no greater ones.
      ++ Error: if f has degree 0.
      ++ Note: the input polynomial f is made monic.
      ++ Also, \spad{f < g} if the {\em lookup} of the constant
      ++ term of f is less than
      ++ this number for g or if
      ++ {\em lookup} of the coefficient of the term of degree {\em n-1}
      ++ of f is less than this number for g.
      ++ Otherwise, \spad{f < g}
      ++ if the number of monomials of f is less than
      ++ that for g or if the lists of exponents for f are
      ++ lexicographically less than those for g.
      ++ If these lists are also equal, the lists of coefficients are
      ++ compared according to the lexicographic ordering induced by
      ++ the ordering of the elements of {\em GF} given by {\em lookup}.
      ++ This operation is equivalent to nextPrimitiveNormalPoly(f).
    nextPrimitiveNormalPoly : SUP -> Union(SUP, "failed")
      ++ nextPrimitiveNormalPoly(f) yields the next primitive normal polynomial
      ++ over a finite field {\em GF} of the same degree as f in the following
      ++ order, or "failed" if there are no greater ones.
      ++ Error: if f has degree 0.
      ++ Note: the input polynomial f is made monic.
      ++ Also, \spad{f < g} if the {\em lookup} of the
      ++ constant term of f is less than
      ++ this number for g or, in case these numbers are equal, if the
      ++ {\em lookup} of the coefficient of the term of degree {\em n-1}
      ++ of f is less than this number for g.
      ++ If these numbers are equals, \spad{f < g}
      ++ if the number of monomials of f is less than
      ++ that for g, or if the lists of exponents for f are lexicographically
      ++ less than those for g.
      ++ If these lists are also equal, the lists of coefficients are
      ++ coefficients according to the lexicographic ordering induced by
      ++ the ordering of the elements of {\em GF} given by {\em lookup}.
      ++ This operation is equivalent to nextNormalPrimitivePoly(f).
--    random : () -> SUP
--      ++ random()$FFPOLY(GF) generates a random monic polynomial
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--      ++ of random degree over the field {\em GF}
    random : PI -> SUP
      ++ random(n)$FFPOLY(GF) generates a random monic polynomial
      ++ of degree n over the finite field {\em GF}.
    random : (PI, PI) -> SUP
      ++ random(m,n)$FFPOLY(GF) generates a random monic polynomial
      ++ of degree d over the finite field {\em GF}, d between m and n.
    leastAffineMultiple: SUP  -> SUP
      ++ leastAffineMultiple(f) computes the least affine polynomial which
      ++ is divisible by the polynomial f over the finite field {\em GF},
      ++ i.e. a polynomial whose exponents are 0 or a power of q, the
      ++ size of {\em GF}.
    reducedQPowers: SUP  -> PrimitiveArray SUP
      ++ reducedQPowers(f)
      ++ generates \spad{[x,x**q,x**(q**2),...,x**(q**(n-1))]}
      ++ reduced modulo f where \spad{q = size()$GF} and \spad{n = degree f}.
    --
    -- we intend to implement also the functions
    -- cyclotomicPoly: PI -> SUP, order: SUP -> PI,
    -- and maybe a new version of irreducible?


  Implementation ==> add

    import IntegerNumberTheoryFunctions
    import DistinctDegreeFactorize(GF, SUP)


    MM := ModMonic(GF, SUP)

    sizeGF : PI := size()$GF :: PI

    revListToSUP(l:Repr):SUP ==
        newl:Repr := empty()
        -- cannot use map since copy for Record is an XLAM
        for t in l repeat newl := cons(copy t, newl)
        newl pretend SUP

    listToSUP(l:Repr):SUP ==
        newl:Repr := [copy t for t in l]
        newl pretend SUP

    nextSubset : (L NNI, NNI) -> Union(L NNI, "failed")
      -- for a list s of length m with 1 <= s.1 < ... < s.m <= bound,
      -- nextSubset(s, bound) yields the immediate successor of s
      -- (resp. "failed" if s = [1,...,bound])
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  -- where s < t if and only if:
  -- (i)  #s < #t; or
  -- (ii) #s = #t and s < t in the lexicographical order;
  -- (we have chosen to fix the signature with NNI instead of PI
  --  to avoid coercions in the main functions)

reducedQPowers(f) ==
  m:PI:=degree(f)$SUP pretend PI
  m1:I:=m-1
  setPoly(f)$MM
  e:=reduce(monomial(1,1)$SUP)$MM ** sizeGF
  w:=1$MM
  qpow:PrimitiveArray SUP:=new(m,0)
  qpow.0:=1$SUP
  for i in 1..m1 repeat  qpow.i:=lift(w:=w*e)$MM
  qexp:PrimitiveArray SUP:=new(m,0)
  m = 1 =>
    qexp.(0$I):= (-coefficient(f,0$NNI)$SUP)::SUP
    qexp
  qexp.0$I:=monomial(1,1)$SUP
  h:=qpow.1
  qexp.1:=h
  for i in 2..m1 repeat
    g:=0$SUP
    while h ^= 0 repeat
      g:=g + leadingCoefficient(h) * qpow.degree(h)
      h:=reductum(h)
    qexp.i:=(h:=g)
  qexp

leastAffineMultiple(f) ==
-- [LS] p.112
  qexp:=reducedQPowers(f)
  n:=degree(f)$SUP
  b:Matrix GF:= transpose matrix [entries vectorise
       (qexp.i,n) for i in 0..n-1]
  col1:Matrix GF:= new(n,1,0)
  col1(1,1)  := 1
  ns : List Vector GF := nullSpace (horizConcat(col1,b) )
  ----------------------------------------------------------------
  -- perhaps one should use that the first vector in ns is already
  -- the right one
  ----------------------------------------------------------------
  dim:=n+2
  coeffVector : Vector GF
  until empty? ns repeat
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        newCoeffVector := ns.1
        i : PI :=(n+1) pretend PI
        while newCoeffVector(i) = 0 repeat
          i := (i - 1) pretend PI
        if i < dim then
          dim := i
          coeffVector := newCoeffVector
        ns := rest ns
      (coeffVector(1)::SUP) +(+/[monomial(coeffVector.k, _
               sizeGF**((k-2)::NNI))$SUP for k in 2..dim])

--    qEulerPhiCyclotomic n ==
--      n = 1 => (sizeGF - 1) pretend PI
--      p : PI := characteristic()$GF :: PI
--      (n rem p) = 0 => error
--        "cyclotomic polynomial not defined for this argument value"
--      q  : PI := sizeGF
--      -- determine the multiplicative order of q modulo n
--      e  : PI := 1
--      qe : PI := q
--      while (qe rem n) ^= 1 repeat
--        e  := e + 1
--        qe := qe * q
--      ((qe - 1) ** ((eulerPhi(n) quo e) pretend PI) ) pretend PI

    numberOfIrreduciblePoly n ==
      -- we compute the number Nq(n) of monic irreducible polynomials
      -- of degree n over the field GF of order q by the formula
      -- Nq(n) = (1/n)* sum(moebiusMu(n/d)*q**d) where the sum extends
      -- over all divisors d of n (cf. [LN] p.93, Th. 3.25)
      n = 1 => sizeGF
      -- the contribution of d = 1 :
      lastd : PI  := 1
      qd    : PI  := sizeGF
      sum   :  I  := moebiusMu(n) * qd
      -- the divisors d > 1 of n :
      divisorsOfn : L PI := rest(divisors n) pretend L PI
      for d in divisorsOfn repeat
        qd := qd * (sizeGF) ** ((d - lastd) pretend PI)
        sum := sum + moebiusMu(n quo d) * qd
        lastd := d
      (sum quo n) :: PI

    numberOfPrimitivePoly n == (eulerPhi((sizeGF ** n) - 1) quo n) :: PI
      -- [each root of a primitive polynomial of degree n over a field
      --  with q elements is a generator of the multiplicative group
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    --  of a field of order q**n (definition), and the number of such
    --  generators is precisely eulerPhi(q**n - 1)]

  numberOfNormalPoly n ==
    -- we compute the number Nq(n) of normal polynomials of degree n
    -- in GF[X], with GF of order q, by the formula
    -- Nq(n) = (1/n) * qPhi(X**n - 1) (cf. [LN] p.124) where,
    -- for any polynomial f in GF[X] of positive degree n,
    -- qPhi(f) = q**n * (1 - q**(-n1)) *...* (1 - q**(-nr)) =
    -- q**n * ((q**(n1)-1) / q**(n1)) *...* ((q**(nr)-1) / q**(n_r)),
    -- the ni being the degrees of the distinct irreducible factors
    -- of f in its canonical factorization over GF
    -- ([LN] p.122, Lemma 3.69).
    -- hence, if n = m * p**r where p is the characteristic of GF
    -- and gcd(m,p) = 1, we get
    -- Nq(n) = (1/n)* q**(n-m) * qPhi(X**m - 1)
    -- now X**m - 1 is the product of the (pairwise relatively prime)
    -- cyclotomic polynomials Qd(X) for which d divides m
    -- ([LN] p.64, Th. 2.45), and each Qd(X) factors into
    -- eulerPhi(d)/e (distinct) monic irreducible polynomials in GF[X]
    -- of the same degree e, where e is the least positive integer k
    -- such that d divides q**k - 1 ([LN] p.65, Th. 2.47)
    n = 1 => (sizeGF - 1) :: NNI :: PI
    m : PI := n
    p : PI := characteristic()$GF :: PI
    q : PI := sizeGF
    while (m rem p) = 0 repeat   -- find m such that
      m := (m quo p) :: PI       -- n = m * p**r and gcd(m,p) = 1
    m = 1 =>
       -- know that n is a power of p
      (((q ** ((n-1)::NNI) )  * (q - 1) ) quo n) :: PI
    prod : I := q - 1
    divisorsOfm : L PI := rest(divisors m) pretend L PI
    for d in divisorsOfm repeat
      -- determine the multiplicative order of q modulo d
      e  : PI := 1
      qe : PI := q
      while (qe rem d) ^= 1 repeat
        e  := e + 1
        qe := qe * q
      prod := prod * _
        ((qe - 1) ** ((eulerPhi(d) quo e) pretend PI) ) pretend PI
    (q**((n-m) pretend PI) * prod quo n) pretend PI

  primitive? f ==
    -- let GF be a field of order q; a monic polynomial f in GF[X]
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    -- of degree n is primitive over GF if and only if its constant
    -- term is non-zero, f divides X**(q**n - 1) - 1 and,
    -- for each prime divisor d of q**n - 1,
    -- f does not divide X**((q**n - 1) / d) - 1
    -- (cf. [LN] p.89, Th. 3.16, and p.87, following Th. 3.11)
    n : NNI := degree f
    n = 0 => false
    leadingCoefficient f ^= 1 => false
    coefficient(f, 0) = 0 => false
    q  : PI := sizeGF
    qn1: PI := (q**n - 1) :: NNI :: PI
    setPoly f
    x := reduce(monomial(1,1)$SUP)$MM -- X rem f represented in MM
    --
    -- may be improved by tabulating the residues x**(i*q)
    -- for i = 0,...,n-1 :
    --
    lift(x ** qn1)$MM ^= 1 => false -- X**(q**n - 1) rem f in GF[X]
    lrec  : L Record(factor:I, exponent:I) := factors(factor qn1)
    lfact : L PI := []              -- collect the prime factors
    for rec in lrec repeat          -- of q**n - 1
      lfact := cons((rec.factor) :: PI, lfact)
    for d in lfact repeat
      if (expt := (qn1 quo d)) >= n then
        lift(x ** expt)$MM = 1 => return false
    true

  normal? f ==
    -- let GF be a field with q elements; a monic irreducible
    -- polynomial f in GF[X] of degree n is normal if its roots
    -- x, x**q, ... , x**(q**(n-1)) are linearly independent over GF
    n : NNI := degree f
    n = 0 => false
    leadingCoefficient f ^= 1 => false
    coefficient(f, 0) = 0 => false
    n = 1 => true
    not irreducible? f => false
    g:=reducedQPowers(f)
    l:=[entries vectorise(g.i,n)$SUP for i in 0..(n-1)::NNI]
    rank(matrix(l)$Matrix(GF)) = n => true
    false

  nextSubset(s, bound) ==
    m : NNI := #(s)
    m = 0 => [1]
    -- find the first element s(i) of s such that s(i) + 1 < s(i+1) :
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    noGap : Boolean := true
    i : NNI := 0
    restOfs : L NNI
    while noGap and not empty?(restOfs := rest s) repeat
    -- after i steps (0 <= i <= m-1) we have s = [s(i), ... , s(m)]
    -- and restOfs = [s(i+1), ... , s(m)]
      secondOfs := first restOfs    -- s(i+1)
      firstOfsPlus1 := first s + 1  -- s(i) + 1
      secondOfs = firstOfsPlus1 =>
        s := restOfs
        i := i + 1
      setfirst_!(s, firstOfsPlus1)  -- s := [s(i)+1, s(i+1),..., s(m)]
      noGap := false
    if noGap then                   -- here s = [s(m)]
      firstOfs := first s
      firstOfs < bound => setfirst_!(s, firstOfs + 1) -- s := [s(m)+1]
      m < bound =>
          setfirst_!(s, m + 1)      -- s := [m+1]
          i := m
      return "failed"               -- (here m = s(m) = bound)
    for j in i..1 by -1 repeat  -- reconstruct the destroyed
      s := cons(j, s)           -- initial part of s
    s

  nextIrreduciblePoly f ==
    n : NNI := degree f
    n = 0 => error "polynomial must have positive degree"
    -- make f monic
    if (lcf := leadingCoefficient f) ^= 1 then f := (inv lcf) * f
    -- if f = fn*X**n + ... + f{i0}*X**{i0} with the fi non-zero
    -- then fRepr := [[n,fn], ... , [i0,f{i0}]]
    fRepr : Repr := f pretend Repr
    fcopy : Repr := []
    -- we can not simply write fcopy := copy fRepr because
    -- the input(!) f would be modified by assigning
    -- a new value to one of its records
    for term in fRepr repeat
      fcopy := cons(copy term, fcopy)
    if term.expnt ^= 0 then
      fcopy := cons([0,0]$Rec, fcopy)
    tailpol : Repr := []
    headpol : Repr := fcopy  -- [[0,f0], ... , [n,fn]] where
                             -- fi is non-zero for i > 0
    fcopy   := reverse fcopy
    weight  : NNI := (#(fcopy) - 1) :: NNI -- #s(f) as explained above
    taillookuplist : L NNI := []
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-- the zeroes in the headlookuplist stand for the fi
-- whose lookup's were not yet computed :
headlookuplist : L NNI := new(weight, 0)
s  : L NNI := [] -- we will compute s(f) only if necessary
n1 : NNI := (n - 1) :: NNI
repeat
  -- (run through the possible weights)
  while not empty? headlookuplist repeat
    -- find next polynomial in the above order with fixed weight;
    -- assume at this point we have
    -- headpol = [[i1,f{i1}], [i2,f{i2}], ... , [n,1]]
    -- and tailpol = [[k,fk], ... , [0,f0]] (with k < i1)
    term := first headpol
    j := first headlookuplist
    if j = 0 then j := lookup(term.coeff)$GF
    j := j + 1 -- lookup(f{i1})$GF + 1
    j rem sizeGF = 0 =>
      -- in this case one has to increase f{i2}
      tailpol := cons(term, tailpol) -- [[i1,f{i1}],...,[0,f0]]
      headpol := rest headpol        -- [[i2,f{i2}],...,[n,1]]
      taillookuplist := cons(j, taillookuplist)
      headlookuplist := rest headlookuplist
    -- otherwise set f{i1} := index(j)$GF
    setelt(first headpol, coeff, index(j :: PI)$GF)
    setfirst_!(headlookuplist, j)
    if empty? taillookuplist then
      pol := revListToSUP(headpol)
      --
      -- may be improved by excluding reciprocal polynomials
      --
      irreducible? pol => return pol
    else
      -- go back to fk
      headpol := cons(first tailpol, headpol) -- [[k,fk],...,[n,1]]
      tailpol := rest tailpol
      headlookuplist := cons(first taillookuplist, headlookuplist)
      taillookuplist := rest taillookuplist
  -- must search for polynomial with greater weight
  if empty? s then -- compute s(f)
    restfcopy := rest fcopy
    for entry in restfcopy repeat s := cons(entry.expnt, s)
  weight = n => return "failed"
  s1 := nextSubset(rest s, n1) :: L NNI
  s := cons(0, s1)
  weight := #s
  taillookuplist := []
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    headlookuplist := cons(sizeGF, new((weight-1) :: NNI, 1))
    tailpol := []
    headpol := [] -- [[0,0], [s.2,1], ... , [s.weight,1], [n,1]] :
    s1 := cons(n, reverse s1)
    while not empty? s1 repeat
      headpol := cons([first s1, 1]$Rec, headpol)
      s1 := rest s1
    headpol := cons([0, 0]$Rec, headpol)

nextPrimitivePoly f ==
  n : NNI := degree f
  n = 0 => error "polynomial must have positive degree"
  -- make f monic
  if (lcf := leadingCoefficient f) ^= 1 then f := (inv lcf) * f
  -- if f = fn*X**n + ... + f{i0}*X**{i0} with the fi non-zero
  -- then fRepr := [[n,fn], ... , [i0,f{i0}]]
  fRepr : Repr := f pretend Repr
  fcopy : Repr := []
  -- we can not simply write fcopy := copy fRepr because
  -- the input(!) f would be modified by assigning
  -- a new value to one of its records
  for term in fRepr repeat
    fcopy := cons(copy term, fcopy)
  if term.expnt ^= 0 then
    term  := [0,0]$Rec
    fcopy := cons(term, fcopy)
  fcopy   := reverse fcopy
  xn : Rec := first fcopy
  c0 : GF  := term.coeff
  l  : NNI := lookup(c0)$GF rem sizeGF
  n = 1 =>
    -- the polynomial X + c is primitive if and only if -c
    -- is a primitive element of GF
    q1 : NNI  := (sizeGF - 1) :: NNI
    while l < q1 repeat -- find next c such that -c is primitive
      l := l + 1
      c := index(l :: PI)$GF
      primitive?(-c)$GF =>
        return [xn, [0,c]$Rec] pretend SUP
    "failed"
  weight : NNI := (#(fcopy) - 1) :: NNI -- #s(f)+1 as explained above
  s  : L NNI := [] -- we will compute s(f) only if necessary
  n1 : NNI := (n - 1) :: NNI
  -- a necessary condition for a monic polynomial f of degree n
  -- over GF to be primitive is that (-1)**n * f(0) be a
  -- primitive element of GF (cf. [LN] p.90, Th. 3.18)
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```
c  : GF  := c0
while l < sizeGF repeat
  -- (run through the possible values of the constant term)
  noGenerator : Boolean := true
  while noGenerator and l < sizeGF repeat
    -- find least c >= c0 such that (-1)^n c0 is primitive
    primitive?((-1)**n * c)$GF => noGenerator := false
    l := l + 1
    c := index(l :: PI)$GF
  noGenerator => return "failed"
  constterm : Rec := [0, c]$Rec
  if c = c0 and weight > 1 then
    headpol : Repr := rest reverse fcopy -- [[i0,f{i0}],...,[n,1]]
                                         -- fi is non-zero for i>0
    -- the zeroes in the headlookuplist stand for the fi
    -- whose lookup's were not yet computed :
    headlookuplist : L NNI := new(weight, 0)
  else
    -- X**n + c can not be primitive for n > 1 (cf. [LN] p.90,
    -- Th. 3.18); next possible polynomial is X**n + X + c
    headpol : Repr := [[1,0]$Rec, xn] -- 0*X + X**n
    headlookuplist : L NNI := [sizeGF]
    s := [0,1]
    weight := 2
  tailpol : Repr := []
  taillookuplist : L NNI := []
  notReady : Boolean := true
  while notReady repeat
    -- (run through the possible weights)
    while not empty? headlookuplist repeat
      -- find next polynomial in the above order with fixed
      -- constant term and weight; assume at this point we have
      -- headpol = [[i1,f{i1}], [i2,f{i2}], ... , [n,1]] and
      -- tailpol = [[k,fk],...,[k0,fk0]] (k0<...<k<i1<i2<...<n)
      term := first headpol
      j := first headlookuplist
      if j = 0 then j := lookup(term.coeff)$GF
      j := j + 1 -- lookup(f{i1})$GF + 1
      j rem sizeGF = 0 =>
        -- in this case one has to increase f{i2}
        tailpol := cons(term, tailpol) -- [[i1,f{i1}],...,[k0,f{k0}]]
        headpol := rest headpol        -- [[i2,f{i2}],...,[n,1]]
        taillookuplist := cons(j, taillookuplist)
        headlookuplist := rest headlookuplist
      -- otherwise set f{i1} := index(j)$GF
      setelt(first headpol, coeff, index(j :: PI)$GF)
```

```
            setfirst_!(headlookuplist, j)
            if empty? taillookuplist then
              pol := revListToSUP cons(constterm, headpol)
              --
              -- may be improved by excluding reciprocal polynomials
              --
              primitive? pol => return pol
            else
              -- go back to fk
              headpol := cons(first tailpol, headpol) -- [[k,fk],...,[n,1]]
              tailpol := rest tailpol
              headlookuplist := cons(first taillookuplist,
                                     headlookuplist)
              taillookuplist := rest taillookuplist
          if weight = n then notReady := false
          else
            -- must search for polynomial with greater weight
            if empty? s then -- compute s(f)
              restfcopy := rest fcopy
              for entry in restfcopy repeat s := cons(entry.expnt, s)
            s1 := nextSubset(rest s, n1) :: L NNI
            s  := cons(0, s1)
            weight := #s
            taillookuplist := []
            headlookuplist := cons(sizeGF, new((weight-2) :: NNI, 1))
            tailpol := []
            -- headpol = [[s.2,0], [s.3,1], ... , [s.weight,1], [n,1]] :
            headpol := [[first s1, 0]$Rec]
            while not empty? (s1 := rest s1) repeat
              headpol := cons([first s1, 1]$Rec, headpol)
            headpol := reverse cons([n, 1]$Rec, headpol)
        -- next polynomial must have greater constant term
        l := l + 1
        c := index(l :: PI)$GF
      "failed"

    nextNormalPoly f ==
      n : NNI := degree f
      n = 0 => error "polynomial must have positive degree"
      -- make f monic
      if (lcf := leadingCoefficient f) ^= 1 then f := (inv lcf) * f
      -- if f = fn*X**n + ... + f{i0}*X**{i0} with the fi non-zero
      -- then fRepr := [[n,fn], ... , [i0,f{i0}]]
      fRepr : Repr := f pretend Repr
      fcopy : Repr := []
      -- we can not simply write fcopy := copy fRepr because
```

```
-- the input(!) f would be modified by assigning
-- a new value to one of its records
for term in fRepr repeat
  fcopy := cons(copy term, fcopy)
if term.expnt ^= 0 then
  term  := [0,0]$Rec
  fcopy := cons(term, fcopy)
fcopy     := reverse fcopy -- [[n,1], [r,fr], ... , [0,f0]]
xn : Rec  := first fcopy
middlepol : Repr := rest fcopy -- [[r,fr], ... , [0,f0]]
a0 : GF  := (first middlepol).coeff -- fr
l  : NNI := lookup(a0)$GF rem sizeGF
n = 1 =>
  -- the polynomial X + a is normal if and only if a is not zero
  l = sizeGF - 1 => "failed"
  [xn, [0, index((l+1) :: PI)$GF]$Rec] pretend SUP
n1 : NNI := (n  - 1) :: NNI
n2 : NNI := (n1 - 1) :: NNI
-- if the polynomial X**n + a * X**(n-1) + ... is normal then
-- a = -(x + x**q +...+ x**(q**n)) can not be zero (where q = #GF)
a  : GF  := a0
-- if a = 0 then set a := 1
if l = 0 then
  l := 1
  a := 1$GF
while l < sizeGF repeat
  -- (run through the possible values of a)
  if a = a0 then
    -- middlepol = [[0,f0], ... , [m,fm]] with m < n-1
    middlepol := reverse rest middlepol
    weight : NNI := #middlepol -- #s(f) as explained above
    -- the zeroes in the middlelookuplist stand for the fi
    -- whose lookup's were not yet computed :
    middlelookuplist : L NNI := new(weight, 0)
    s : L NNI := [] -- we will compute s(f) only if necessary
  else
    middlepol := [[0,0]$Rec]
    middlelookuplist : L NNI := [sizeGF]
    s : L NNI := [0]
    weight : NNI := 1
  headpol : Repr := [xn, [n1, a]$Rec] -- X**n + a * X**(n-1)
  tailpol : Repr := []
  taillookuplist : L NNI := []
  notReady : Boolean := true
  while notReady repeat
    -- (run through the possible weights)
```

```
while not empty? middlelookuplist repeat
  -- find next polynomial in the above order with fixed
  -- a and weight; assume at this point we have
  -- middlepol = [[i1,f{i1}], [i2,f{i2}], ... , [m,fm]] and
  -- tailpol = [[k,fk],...,[0,f0]] ( with k<i1<i2<...<m)
  term := first middlepol
  j := first middlelookuplist
  if j = 0 then j := lookup(term.coeff)$GF
  j := j + 1 -- lookup(f{i1})$GF + 1
  j rem sizeGF = 0 =>
    -- in this case one has to increase f{i2}
    -- tailpol = [[i1,f{i1}],...,[0,f0]]
    tailpol   := cons(term, tailpol)
    middlepol := rest middlepol -- [[i2,f{i2}],...,[m,fm]]
    taillookuplist   := cons(j, taillookuplist)
    middlelookuplist := rest middlelookuplist
  -- otherwise set f{i1} := index(j)$GF
  setelt(first middlepol, coeff, index(j :: PI)$GF)
  setfirst_!(middlelookuplist, j)
  if empty? taillookuplist then
    pol := listToSUP append(headpol, reverse middlepol)
    --
    -- may be improved by excluding reciprocal polynomials
    --
    normal? pol => return pol
  else
    -- go back to fk
    -- middlepol = [[k,fk],...,[m,fm]]
    middlepol := cons(first tailpol, middlepol)
    tailpol := rest tailpol
    middlelookuplist := cons(first taillookuplist,
                                      middlelookuplist)
    taillookuplist := rest taillookuplist
if weight = n1 then notReady := false
else
  -- must search for polynomial with greater weight
  if empty? s then -- compute s(f)
    restfcopy := rest rest fcopy
    for entry in restfcopy repeat s := cons(entry.expnt, s)
  s1 := nextSubset(rest s, n2) :: L NNI
  s  := cons(0, s1)
  weight := #s
  taillookuplist := []
  middlelookuplist := cons(sizeGF, new((weight-1) :: NNI, 1))
  tailpol   := []
  -- middlepol = [[0,0], [s.2,1], ... , [s.weight,1]] :
```

```
        middlepol := []
        s1 := reverse s1
        while not empty? s1 repeat
          middlepol := cons([first s1, 1]$Rec, middlepol)
          s1 := rest s1
        middlepol := cons([0,0]$Rec, middlepol)
      -- next polynomial must have greater a
      l := l + 1
      a := index(l :: PI)$GF
    "failed"

  nextNormalPrimitivePoly f ==
    n : NNI := degree f
    n = 0 => error "polynomial must have positive degree"
    -- make f monic
    if (lcf := leadingCoefficient f) ^= 1 then f := (inv lcf) * f
    -- if f = fn*X**n + ... + f{i0}*X**{i0} with the fi non-zero
    -- then fRepr := [[n,fn], ... , [i0,f{i0}]]
    fRepr : Repr := f pretend Repr
    fcopy : Repr := []
    -- we can not simply write fcopy := copy fRepr because
    -- the input(!) f would be modified by assigning
    -- a new value to one of its records
    for term in fRepr repeat
      fcopy := cons(copy term, fcopy)
    if term.expnt ^= 0 then
      term  := [0,0]$Rec
      fcopy := cons(term, fcopy)
    fcopy   := reverse fcopy -- [[n,1], [r,fr], ... , [0,f0]]
    xn : Rec := first fcopy
    c0 : GF  := term.coeff
    lc : NNI := lookup(c0)$GF rem sizeGF
    n = 1 =>
      -- the polynomial X + c is primitive if and only if -c
      -- is a primitive element of GF
      q1 : NNI  := (sizeGF - 1) :: NNI
      while lc < q1 repeat -- find next c such that -c is primitive
        lc := lc + 1
        c  := index(lc :: PI)$GF
        primitive?(-c)$GF =>
          return [xn, [0,c]$Rec] pretend SUP
      "failed"
    n1 : NNI := (n  - 1) :: NNI
    n2 : NNI := (n1 - 1) :: NNI
    middlepol : Repr := rest fcopy -- [[r,fr],...,[i0,f{i0}],[0,f0]]
    a0 : GF  := (first middlepol).coeff
```

```
la : NNI := lookup(a0)$GF rem sizeGF
-- if the polynomial X**n + a * X**(n-1) +...+ c is primitive and
-- normal over GF then (-1)**n * c is a primitive element of GF
-- (cf. [LN] p.90, Th. 3.18), and a = -(x + x**q +...+ x**(q**n))
-- is not zero (where q = #GF)
c : GF  := c0
a : GF  := a0
-- if a = 0 then set a := 1
if la = 0 then
  la := 1
  a  := 1$GF
while lc < sizeGF repeat
  -- (run through the possible values of the constant term)
  noGenerator : Boolean := true
  while noGenerator and lc < sizeGF repeat
    -- find least c >= c0 such that (-1)**n * c0 is primitive
    primitive?((-1)**n * c)$GF => noGenerator := false
    lc := lc + 1
    c  := index(lc :: PI)$GF
  noGenerator => return "failed"
  constterm : Rec := [0, c]$Rec
  while la < sizeGF repeat
  -- (run through the possible values of a)
    headpol : Repr := [xn, [n1, a]$Rec] -- X**n + a X**(n-1)
    if c = c0 and a = a0 then
      -- middlepol = [[i0,f{i0}], ... , [m,fm]] with m < n-1
      middlepol := rest reverse rest middlepol
      weight : NNI := #middlepol + 1 -- #s(f)+1 as explained above
      -- the zeroes in the middlelookuplist stand for the fi
      -- whose lookup's were not yet computed :
      middlelookuplist : L NNI := new((weight-1) :: NNI, 0)
      s : L NNI := [] -- we will compute s(f) only if necessary
    else
      pol := listToSUP append(headpol, [constterm])
      normal? pol and primitive? pol => return pol
      middlepol := [[1,0]$Rec]
      middlelookuplist : L NNI := [sizeGF]
      s : L NNI := [0,1]
      weight : NNI := 2
    tailpol : Repr := []
    taillookuplist : L NNI := []
    notReady : Boolean := true
    while notReady repeat
    -- (run through the possible weights)
      while not empty? middlelookuplist repeat
        -- find next polynomial in the above order with fixed
```

```
    -- c, a and weight; assume at this point we have
    -- middlepol = [[i1,f{i1}], [i2,f{i2}], ... , [m,fm]]
    -- tailpol = [[k,fk],...,[k0,fk0]] (k0<...<k<i1<...<m)
    term := first middlepol
    j := first middlelookuplist
    if j = 0 then j := lookup(term.coeff)$GF
    j := j + 1 -- lookup(f{i1})$GF + 1
    j rem sizeGF = 0 =>
      -- in this case one has to increase f{i2}
      -- tailpol = [[i1,f{i1}],...,[k0,f{k0}]]
      tailpol   := cons(term, tailpol)
      middlepol := rest middlepol -- [[i2,f{i2}],...,[m,fm]]
      taillookuplist   := cons(j, taillookuplist)
      middlelookuplist := rest middlelookuplist
    -- otherwise set f{i1} := index(j)$GF
    setelt(first middlepol, coeff, index(j :: PI)$GF)
    setfirst_!(middlelookuplist, j)
    if empty? taillookuplist then
      pol := listToSUP append(headpol, reverse
                      cons(constterm, middlepol))
      --
      -- may be improved by excluding reciprocal polynomials
      --
      normal? pol and primitive? pol => return pol
    else
      -- go back to fk
      -- middlepol = [[k,fk],...,[m,fm]]
      middlepol := cons(first tailpol, middlepol)
      tailpol := rest tailpol
      middlelookuplist := cons(first taillookuplist,
                                       middlelookuplist)
      taillookuplist := rest taillookuplist
  if weight = n1 then notReady := false
  else
    -- must search for polynomial with greater weight
    if empty? s then -- compute s(f)
      restfcopy := rest rest fcopy
      for entry in restfcopy repeat s := cons(entry.expnt, s)
    s1 := nextSubset(rest s, n2) :: L NNI
    s  := cons(0, s1)
    weight := #s
    taillookuplist := []
    middlelookuplist := cons(sizeGF, new((weight-2)::NNI, 1))
    tailpol   := []
    -- middlepol = [[s.2,0], [s.3,1], ... , [s.weight,1] :
    middlepol := [[first s1, 0]$Rec]
```

```
            while not empty? (s1 := rest s1) repeat
              middlepol := cons([first s1, 1]$Rec, middlepol)
            middlepol := reverse middlepol
        -- next polynomial must have greater a
        la := la + 1
        a  := index(la :: PI)$GF
      -- next polynomial must have greater constant term
      lc := lc + 1
      c  := index(lc :: PI)$GF
      la := 1
      a  := 1$GF
    "failed"

  nextPrimitiveNormalPoly f == nextNormalPrimitivePoly f

  createIrreduciblePoly n ==
    x := monomial(1,1)$SUP
    n = 1 => x
    xn := monomial(1,n)$SUP
    n >= sizeGF => nextIrreduciblePoly(xn + x) :: SUP
    -- (since in this case there is most no irreducible binomial X+a)
    odd? n => nextIrreduciblePoly(xn + 1) :: SUP
    nextIrreduciblePoly(xn) :: SUP

  createPrimitivePoly n ==
  -- (see also the comments in the code of nextPrimitivePoly)
    xn := monomial(1,n)$SUP
    n = 1 => xn + monomial(-primitiveElement()$GF, 0)$SUP
    c0 : GF := (-1)**n * primitiveElement()$GF
    constterm : Rec := [0, c0]$Rec
    -- try first (probably faster) the polynomials
    -- f = X**n + f{n-1}*X**(n-1) +...+ f1*X + c0 for which
    -- fi is 0 or 1 for i=1,...,n-1,
    -- and this in the order used to define nextPrimitivePoly
    s  : L NNI := [0,1]
    weight : NNI := 2
    s1 : L NNI := [1]
    n1 : NNI := (n - 1) :: NNI
    notReady : Boolean := true
    while notReady repeat
      polRepr : Repr := [constterm]
      while not empty? s1 repeat
        polRepr := cons([first s1, 1]$Rec, polRepr)
        s1 := rest s1
      polRepr := cons([n, 1]$Rec, polRepr)
      --
```

```
      -- may be improved by excluding reciprocal polynomials
      --
      primitive? (pol := listToSUP polRepr) => return pol
      if weight = n then notReady := false
      else
        s1 := nextSubset(rest s, n1) :: L NNI
        s  := cons(0, s1)
        weight := #s
    -- if there is no primitive f of the above form
    -- search now from the beginning, allowing arbitrary
    -- coefficients f_i, i = 1,...,n-1
    nextPrimitivePoly(xn + monomial(c0, 0)$SUP) :: SUP

  createNormalPoly n  ==
    n = 1 => monomial(1,1)$SUP + monomial(-1,0)$SUP
    -- get a normal polynomial f = X**n + a * X**(n-1) + ...
    -- with a = -1
    -- [recall that if f is normal over the field GF of order q
    -- then a = -(x + x**q +...+ x**(q**n)) can not be zero;
    -- hence the existence of such an f follows from the
    -- normal basis theorem ([LN] p.60, Th. 2.35) and the
    -- surjectivity of the trace ([LN] p.55, Th. 2.23 (iii))]
    nextNormalPoly(monomial(1,n)$SUP
                    + monomial(-1, (n-1) :: NNI)$SUP) :: SUP

  createNormalPrimitivePoly n ==
    xn := monomial(1,n)$SUP
    n = 1 => xn + monomial(-primitiveElement()$GF, 0)$SUP
    n1  : NNI := (n - 1) :: NNI
    c0  : GF  := (-1)**n * primitiveElement()$GF
    constterm  := monomial(c0, 0)$SUP
    -- try first the polynomials f = X**n + a *  X**(n-1) + ...
    -- with a = -1
    pol := xn + monomial(-1, n1)$SUP + constterm
    normal? pol and primitive? pol => pol
    res := nextNormalPrimitivePoly(pol)
    res case SUP => res
    -- if there is no normal primitive f with a = -1
    -- get now one with arbitrary (non-zero) a
    -- (the existence is proved in [LS])
    pol := xn + monomial(1, n1)$SUP + constterm
    normal? pol and primitive? pol => pol
    nextNormalPrimitivePoly(pol) :: SUP

  createPrimitiveNormalPoly n == createNormalPrimitivePoly n
```

```
--    qAdicExpansion m ==
--      ragits : List I := wholeRagits(m :: (RadixExpansion sizeGF))
--      pol  : SUP := 0
--      expt : NNI := #ragits
--      for i in ragits repeat
--        expt := (expt - 1) :: NNI
--        if i ^= 0 then pol := pol + monomial(index(i::PI)$GF, expt)
--      pol

--    random == qAdicExpansion(random()$I)

--    random n ==
--      pol := monomial(1,n)$SUP
--      n1 : NNI := (n - 1) :: NNI
--      for i in 0..n1 repeat
--        if (c := random()$GF) ^= 0 then
--          pol := pol + monomial(c, i)$SUP
--      pol

    random n ==
      polRepr : Repr := []
      n1 : NNI := (n - 1) :: NNI
      for i in 0..n1 repeat
        if (c := random()$GF) ^= 0 then
          polRepr := cons([i, c]$Rec, polRepr)
      cons([n, 1$GF]$Rec, polRepr) pretend SUP

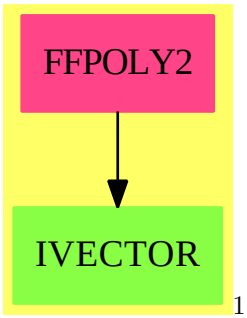
    random(m,n) ==
      if m > n then (m,n) := (n,m)
      d : NNI := (n - m) :: NNI
      if d > 1 then n := ((random()$I rem (d::PI)) + m) :: PI
      random(n)
```

⟨*FFPOLY.dotabb*⟩≡

```
"FFPOLY" [color="#FF4488",href="bookvol10.4.pdf#nameddest=FFPOLY"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"FFPOLY" -> "PFECAT"
```

7.23 package FFPOLY2 FiniteFieldPolynomialPackage2

7.24 FiniteFieldPolynomialPackage2



1.00

Exports:
rootOfIrreduciblePoly

⟨*package FFPOLY2 FiniteFieldPolynomialPackage2*⟩≡

```
)abbrev package FFPOLY2 FiniteFieldPolynomialPackage2
++ Authors: J.Grabmeier, A.Scheerhorn
++ Date Created: 26.03.1991
++ Date Last Updated:
++ Basic Operations: rootOfIrreduciblePoly
++ Related Constructors: FiniteFieldCategory
++ Also See:
++ AMS Classifications:
++ Keywords: finite field, zeros of polynomials, Berlekamp's trace algorithm
++ References:
++  R.Lidl, H.Niederreiter: Finite Field, Encycoldia of Mathematics and
++  Its Applications, Vol. 20, Cambridge Univ. Press, 1983, ISBN 0 521 30240 4
++  AXIOM Technical Report Series, to appear.
++ Description:
++  FiniteFieldPolynomialPackage2(F,GF) exports some functions concerning
++  finite fields, which depend on a finite field {\em GF} and an
++  algebraic extension F of {\em GF}, e.g. a zero of a polynomial
++  over {\em GF} in F.
FiniteFieldPolynomialPackage2(F,GF):Exports == Implementation where
  F:FieldOfPrimeCharacteristic with
      coerce: GF -> F
        ++ coerce(x) \undocumented{}
      lookup: F -> PositiveInteger
        ++ lookup(x) \undocumented{}
      basis: PositiveInteger -> Vector F
        ++ basis(n) \undocumented{}
```

```
      Frobenius: F -> F
        ++ Frobenius(x) \undocumented{}
  -- F should be a algebraic extension of the finite field GF, either an
  -- algebraic closure of GF or a simple algebraic extension field of GF
  GF:FiniteFieldCategory

  I   ==> Integer
  NNI ==> NonNegativeInteger
  PI  ==> PositiveInteger
  SUP ==> SparseUnivariatePolynomial
  MM  ==> ModMonic(GF,SUP GF)
  OUT ==> OutputForm
  M   ==> Matrix
  V   ==> Vector
  L   ==> List
  FFPOLY ==> FiniteFieldPolynomialPackage(GF)
  SUPF2 ==> SparseUnivariatePolynomialFunctions2(GF,F)

  Exports ==> with

    rootOfIrreduciblePoly:SUP GF -> F
      ++ rootOfIrreduciblePoly(f) computes one root of the monic,
      ++ irreducible polynomial f,
      ++ which degree must divide the extension degree
      ++ of {\em F} over {\em GF},
      ++ i.e. f splits into linear factors over {\em F}.


  Implementation ==> add

-- we use berlekamps trace algorithm
-- it is not checked whether the polynomial is irreducible over GF]]
    rootOfIrreduciblePoly(pf) ==
--    not irreducible(pf)$FFPOLY =>
--      error("polynomial has to be irreducible")
      sizeGF:=size()$GF
      -- if the polynomial is of degree one, we're ready
      deg:=degree(pf)$(SUP GF)::PI
      deg = 0 => error("no roots")
      deg = 1 => -coefficient(pf,0)$(SUP GF)::F
      p : SUP F := map(coerce,pf)$SUPF2
      -- compute qexp, qexp(i) = x **(size()GF ** i) mod p
      -- with this list it's easier to compute the gcd(p(x),trace(x))
      qexp:=reducedQPowers(pf)$FFPOLY
      stillToFactor:=p
      -- take linear independent elements, the basis of F over GF
```

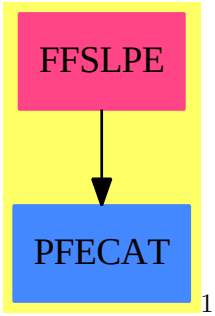
```
basis:Vector F:=basis(deg)$F
basispointer:I:=1
-- as p is irreducible over GF, 0 can't be a root of p
-- therefore we can use the predicate zero?(root) for indicating
-- whether a root is found
root:=0$F
while zero?(root)$F repeat
  beta:F:=basis.basispointer
  -- gcd(trace(x)+gf,p(x)) has degree 0,that's why we skip beta=1
  if beta = 1$F then
    basispointer:=basispointer + 1
    beta:= basis.basispointer
  basispointer:=basispointer+1
  -- compute the polynomial trace(beta * x) mod p(x) using explist
  trModp:SUP F:= map(coerce,qexp.0)$SUPF2 * beta
  for i in 1..deg-1 repeat
    beta:=Frobenius(beta)
    trModp:=trModp +$(SUP F) beta *$(SUP F) map(coerce,qexp.i)$SUPF2
  -- if it is of degree 0, it doesn't help us finding a root
  if degree(trModp)$(SUP F) > 0 then
    -- for all elements gf of GF do
    for j in 1..sizeGF repeat
      -- compute gcd(trace(beta * x) + gf,stillToFactor)
      h:=gcd(stillToFactor,trModp +$(SUP F) _
       (index(j pretend PI)$GF::F::(SUP F)))$(SUP F)
      -- make the gcd polynomial monic
      if leadingCoefficient(h)$(SUP F) ^= 1$F then
        h:= (inv leadingCoefficient(h)) * h
      degh:=degree(h)$(SUP F)
      degSTF:=degree(stillToFactor)$(SUP F)
      -- if the gcd has degree one we are ready
      degh = 1 => root:=-coefficient(h,0)$(SUP F)
      -- if the quotient of stillToFactor and the gcd has
      -- degree one, we're also ready
      degSTF - degh = 1 =>
        root:= -coefficient(stillToFactor quo h,0)$(SUP F)
      -- otherwise the gcd helps us finding a root, only if its
      -- degree is between 2 and degree(stillToFactor)-2
      if degh > 1 and degh < degSTF then
        2*degh > degSTF => stillToFactor := stillToFactor quo h
        stillToFactor := h
root
```

⟨*FFPOLY2.dotabb*⟩≡

```
"FFPOLY2" [color="#FF4488",href="bookvol10.4.pdf#nameddest=FFPOLY2"]
"IVECTOR" [color="#88FF44",href="bookvol10.3.pdf#nameddest=IVECTOR"]
"FFPOLY2" -> "IVECTOR"
```

7.25 package FFSLPE FiniteFieldSolveLinearPolynomialEquation

7.26 FiniteFieldSolveLinearPolynomialEquation



1.00

Exports:
solveLinearPolynomialEquation

⟨*package FFSLPE FiniteFieldSolveLinearPolynomialEquation*⟩≡

```
)abbrev package FFSLPE FiniteFieldSolveLinearPolynomialEquation
++ Author: Davenport
++ Date Created: 1991
++ Date Last Updated:
++ Basic Functions:
++ Related Constructors:
++ Also See:
++ AMS Classifications:
++ Keywords:
++ References:
++ Description:
++ This package solves linear diophantine equations for Bivariate polynomials
++ over finite fields

FiniteFieldSolveLinearPolynomialEquation(F:FiniteFieldCategory,
                                         FP:UnivariatePolynomialCategory F,
                                         FPP:UnivariatePolynomialCategory FP): with
   solveLinearPolynomialEquation: (List FPP, FPP) -> Union(List FPP,"failed")
              ++ solveLinearPolynomialEquation([f1, ..., fn], g)
              ++ (where the fi are relatively prime to each other)
              ++ returns a list of ai such that
              ++ \spad{g/prod fi = sum ai/fi}
              ++ or returns "failed" if no such list of ai's exists.
  == add
     oldlp:List FPP := []
     slpePrime: FP := monomial(1,1)
```

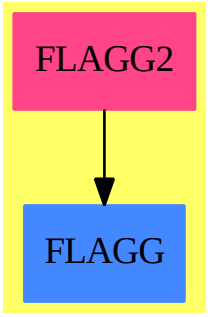
```
oldtable:Vector List FPP := []
lp: List FPP
p: FPP
import DistinctDegreeFactorize(F,FP)
solveLinearPolynomialEquation(lp,p) ==
  if (oldlp ^= lp) then
     -- we have to generate a new table
     deg:= +/[degree u for u in lp]
     ans:Union(Vector List FPP,"failed"):="failed"
     slpePrime:=monomial(1,1)+monomial(1,0)   -- x+1: our starting guess
     while (ans case "failed") repeat
       ans:=tablePow(deg,slpePrime,lp)$GenExEuclid(FP,FPP)
       if (ans case "failed") then
          slpePrime:= nextItem(slpePrime)::FP
          while (degree slpePrime > 1) and
                not irreducible? slpePrime repeat
            slpePrime := nextItem(slpePrime)::FP
     oldtable:=(ans:: Vector List FPP)
  answer:=solveid(p,slpePrime,oldtable)
  answer
```

⟨*FFSLPE.dotabb*⟩≡

```
"FFSLPE" [color="#FF4488",href="bookvol10.4.pdf#nameddest=FFSLPE"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"FFSLPE" -> "PFECAT"
```

7.27 package FLAGG2 FiniteLinearAggregate-Functions2

7.28 FiniteLinearAggregateFunctions2



1.00

Exports:
map reduce scan

⟨*package FLAGG2 FiniteLinearAggregateFunctions2*⟩≡

```
)abbrev package FLAGG2 FiniteLinearAggregateFunctions2
--% FiniteLinearAggregateFunctions2

++ Author: ???
++ Date Created: ???
++ Date Last Updated: ???
++ Description:
++ FiniteLinearAggregateFunctions2 provides functions involving two
++ FiniteLinearAggregates where the underlying domains might be
++ different. An example of this might be creating a list of rational
++ numbers by mapping a function across a list of integers where the
++ function divides each integer by 1000.

FiniteLinearAggregateFunctions2(S, A, R, B):
 Exports == Implementation where
  S, R: Type
  A   : FiniteLinearAggregate S
  B   : FiniteLinearAggregate R

  Exports ==> with
    map    : (S -> R, A) -> B
     ++ map(f,a) applies function f to each member of aggregate
     ++ \spad{a} resulting in a new aggregate over a
     ++ possibly different underlying domain.
    reduce : ((S, R) -> R, A, R) -> R
     ++ reduce(f,a,r) applies function f to each
```

```
    ++ successive element of the
    ++ aggregate \spad{a} and an accumulant initialized to r.
    ++ For example,
    ++ \spad{reduce(_+$Integer,[1,2,3],0)}
    ++ does \spad{3+(2+(1+0))}. Note: third argument r
    ++ may be regarded as the
    ++ identity element for the function f.
  scan   : ((S, R) -> R, A, R) -> B
    ++ scan(f,a,r) successively applies
    ++ \spad{reduce(f,x,r)} to more and more leading sub-aggregates
    ++ x of aggregrate \spad{a}.
    ++ More precisely, if \spad{a} is \spad{[a1,a2,...]}, then
    ++ \spad{scan(f,a,r)} returns
    ++ \spad{[reduce(f,[a1],r),reduce(f,[a1,a2],r),...]}.
Implementation ==> add
  if A has ListAggregate(S) then         -- A is a list-oid
    reduce(fn, l, ident) ==
      empty? l => ident
      reduce(fn, rest l, fn(first l, ident))

    if B has ListAggregate(R) or not(B has shallowlyMutable) then
      -- A is a list-oid, and B is either list-oids or not mutable
      map(f, l) == construct [f s for s in entries l]

      scan(fn, l, ident) ==
        empty? l => empty()
        val := fn(first l, ident)
        concat(val, scan(fn, rest l, val))

    else                       -- A is a list-oid, B a mutable array-oid
      map(f, l) ==
        i := minIndex(w := new(#l,NIL$Lisp)$B)
        for a in entries l repeat (qsetelt_!(w, i, f a); i := inc i)
        w

      scan(fn, l, ident) ==
        i := minIndex(w := new(#l,NIL$Lisp)$B)
        vl := ident
        for a in entries l repeat
          vl := qsetelt_!(w, i, fn(a, vl))
          i := inc i
        w

  else                               -- A is an array-oid
    reduce(fn, v, ident) ==
      val := ident
```

```
    for i in minIndex v .. maxIndex v repeat
      val := fn(qelt(v, i), val)
    val

  if B has ListAggregate(R) then   -- A is an array-oid, B a list-oid
    map(f, v) ==
      construct [f qelt(v, i) for i in minIndex v .. maxIndex v]

    scan(fn, v, ident) ==
      w := empty()$B
      for i in minIndex v .. maxIndex v repeat
        ident := fn(qelt(v, i), ident)
        w := concat(ident, w)
      reverse_! w

  else                              -- A and B are array-oid's
    if B has shallowlyMutable then -- B is also mutable
      map(f, v) ==
        w := new(#v,NIL$Lisp)$B
        for i in minIndex w .. maxIndex w repeat
          qsetelt_!(w, i, f qelt(v, i))
        w

      scan(fn, v, ident) ==
        w   := new(#v,NIL$Lisp)$B
        vl := ident
        for i in minIndex v .. maxIndex v repeat
          vl := qsetelt_!(w, i, fn(qelt(v, i), vl))
        w

    else                                    -- B non mutable array-oid
      map(f, v) ==
        construct [f qelt(v, i) for i in minIndex v .. maxIndex v]

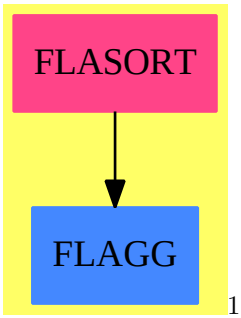
      scan(fn, v, ident) ==
        w := empty()$B
        for i in minIndex v .. maxIndex v repeat
          ident := fn(qelt(v, i), ident)
          w := concat(w, ident)
        w
```

⟨*FLAGG2.dotabb*⟩≡

```
"FLAGG2" [color="#FF4488",href="bookvol10.4.pdf#nameddest=FLAGG2"]
"FLAGG" [color="#4488FF",href="bookvol10.2.pdf#nameddest=FLAGG"]
"FLAGG2" -> "FLAGG"
```

7.29 package FLASORT FiniteLinearAggregate-Sort

7.30 FiniteLinearAggregateSort



1.00

Exports:
heapSort quickSort shellSort

⟨*package FLASORT FiniteLinearAggregateSort*⟩≡

```
)abbrev package FLASORT FiniteLinearAggregateSort
++ FiniteLinearAggregateSort
++ Sort package (in-place) for shallowlyMutable Finite Linear Aggregates
++ Author: Michael Monagan Sep/88
++ RelatedOperations: sort
++ Description:
++  This package exports 3 sorting algorithms which work over
++  FiniteLinearAggregates.
-- the following package is only instantiated over %
-- thus shouldn't be cached. We prevent it
-- from being cached by declaring it to be mutableDomains

)bo PUSH('FiniteLinearAggregateSort, $mutableDomains)

FiniteLinearAggregateSort(S, V): Exports == Implementation where
  S: Type
  V: FiniteLinearAggregate(S) with shallowlyMutable

  B ==> Boolean
  I ==> Integer

  Exports ==> with
    quickSort: ((S, S) -> B, V) -> V
      ++ quickSort(f, agg) sorts the aggregate agg with the ordering function
      ++ f using the quicksort algorithm.
    heapSort : ((S, S) -> B, V) -> V
```

```
    ++ heapSort(f, agg) sorts the aggregate agg with the ordering function
    ++ f using the heapsort algorithm.
  shellSort: ((S, S) -> B, V) -> V
    ++ shellSort(f, agg) sorts the aggregate agg with the ordering function
    ++ f using the shellSort algorithm.

Implementation ==> add
  siftUp   : ((S, S) -> B, V, I, I) -> Void
  partition: ((S, S) -> B, V, I, I, I) -> I
  QuickSort: ((S, S) -> B, V, I, I) -> V

  quickSort(l, r) == QuickSort(l, r, minIndex r, maxIndex r)

  siftUp(l, r, i, n) ==
    t := qelt(r, i)
    while (j := 2*i+1) < n repeat
      if (k := j+1) < n and l(qelt(r, j), qelt(r, k)) then j := k
      if l(t,qelt(r,j)) then
         qsetelt_!(r, i, qelt(r, j))
         qsetelt_!(r, j, t)
         i := j
      else leave

  heapSort(l, r) ==
    not zero? minIndex r => error "not implemented"
    n := (#r)::I
    for k in shift(n,-1) - 1 .. 0 by -1 repeat siftUp(l, r, k, n)
    for k in n-1 .. 1 by -1 repeat
       swap_!(r, 0, k)
       siftUp(l, r, 0, k)
    r

  partition(l, r, i, j, k) ==
    -- partition r[i..j] such that r.s <= r.k <= r.t
    x := qelt(r, k)
    t := qelt(r, i)
    qsetelt_!(r, k, qelt(r, j))
    while i < j repeat
       if l(x,t) then
         qsetelt_!(r, j, t)
         j := j-1
         t := qsetelt_!(r, i, qelt(r, j))
       else (i := i+1; t := qelt(r, i))
    qsetelt_!(r, j, x)
    j
```

```
      QuickSort(l, r, i, j) ==
        n := j - i
  --      if one? n and l(qelt(r, j), qelt(r, i)) then swap_!(r, i, j)
        if (n = 1) and l(qelt(r, j), qelt(r, i)) then swap_!(r, i, j)
        n < 2 => return r
        -- for the moment split at the middle item
        k := partition(l, r, i, j, i + shift(n,-1))
        QuickSort(l, r, i, k - 1)
        QuickSort(l, r, k + 1, j)

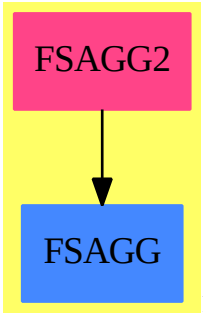
      shellSort(l, r) ==
        m := minIndex r
        n := maxIndex r
        -- use Knuths gap sequence: 1,4,13,40,121,...
        g := 1
        while g <= (n-m) repeat g := 3*g+1
        g := g quo 3
        while g > 0 repeat
           for i in m+g..n repeat
              j := i-g
              while j >= m and l(qelt(r, j+g), qelt(r, j)) repeat
                 swap_!(r,j,j+g)
                 j := j-g
           g := g quo 3
        r
```

⟨*FLASORT.dotabb*⟩≡

```
"FLASORT" [color="#FF4488",href="bookvol10.4.pdf#nameddest=FLASORT"]
"FLAGG" [color="#4488FF",href="bookvol10.2.pdf#nameddest=FLAGG"]
"FLASORT" -> "FLAGG"
```

7.31 package FSAGG2 FiniteSetAggregateFunctions2

7.32 FiniteSetAggregateFunctions2



1.00

Exports:
map reduce scan

⟨*package FSAGG2 FiniteSetAggregateFunctions2*⟩≡

```
)abbrev package FSAGG2 FiniteSetAggregateFunctions2

--% FiniteSetAggregateFunctions2

++ Author: Robert S. Sutor
++ Date Created: 15 May 1990
++ Date Last Updated: 14 Oct 1993
++ Description:
++ FiniteSetAggregateFunctions2 provides functions involving two
++ finite set aggregates where the underlying domains might be
++ different. An example of this is to create a set of rational
++ numbers by mapping a function across a set of integers, where the
++ function divides each integer by 1000.

FiniteSetAggregateFunctions2(S, A, R, B): Exports == Implementation where
   S, R: SetCategory
   A   : FiniteSetAggregate S
   B   : FiniteSetAggregate R

   Exports ==> with
     map    : (S -> R, A) -> B
      ++ map(f,a) applies function f to each member of
      ++ aggregate \spad{a}, creating a new aggregate with
      ++ a possibly different underlying domain.
     reduce : ((S, R) -> R, A, R) -> R
      ++ reduce(f,a,r) applies function f to each
```

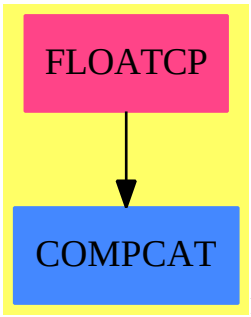
```
      ++ successive element of the aggregate \spad{a} and an
      ++ accumulant initialised to r.
      ++ For example,
      ++ \spad{reduce(_+$Integer,[1,2,3],0)}
      ++ does a \spad{3+(2+(1+0))}.
      ++ Note: third argument r may be regarded
      ++ as an identity element for the function.
    scan   : ((S, R) -> R, A, R) -> B
      ++ scan(f,a,r) successively applies \spad{reduce(f,x,r)}
      ++ to more and more leading sub-aggregates x of
      ++ aggregate \spad{a}.
      ++ More precisely, if \spad{a} is \spad{[a1,a2,...]}, then
      ++ \spad{scan(f,a,r)} returns
      ++ \spad {[reduce(f,[a1],r),reduce(f,[a1,a2],r),...]}.
  Implementation ==> add
    map(fn, a) ==
      set(map(fn, parts a)$ListFunctions2(S, R))$B
    reduce(fn, a, ident) ==
      reduce(fn, parts a, ident)$ListFunctions2(S, R)
    scan(fn, a, ident) ==
      set(scan(fn, parts a, ident)$ListFunctions2(S, R))$B
```

⟨*FSAGG2.dotabb*⟩≡

```
"FSAGG2" [color="#FF4488",href="bookvol10.4.pdf#nameddest=FSAGG2"]
"FSAGG" [color="#4488FF",href="bookvol10.2.pdf#nameddest=FSAGG"]
"FSAGG2" -> "FSAGG"
```

7.33 package FLOATCP FloatingComplexPackage

7.34 FloatingComplexPackage



1.00

Exports:
complexRoots complexSolve

⟨*package FLOATCP FloatingComplexPackage*⟩≡
```
)abbrev package FLOATCP FloatingComplexPackage
++ Author: P. Gianni
++ Date Created: January 1990
++ Date Last Updated:
++ Basic Functions:
++ Related Constructors: SystemSolvePackage, RadicalSolvePackage,
++ FloatingRealPackage
++ Also See:
++ AMS Classifications:
++ Keywords:
++ References:
++ Description:
++    This is a package for the approximation of complex solutions for
++ systems of equations of rational functions with complex rational
++ coefficients. The results are expressed as either complex rational
++ numbers or complex floats depending on the type of the precision
++ parameter which can be either a rational number or a floating point number.
FloatingComplexPackage(Par): Cat == Cap where
    Par : Join(Field, OrderedRing)
    K   ==> GI
    FPK ==> Fraction P K
    C   ==> Complex
    I       ==> Integer
    NNI     ==> NonNegativeInteger
    P       ==> Polynomial
    EQ      ==> Equation
```

```
L        ==> List
SUP      ==> SparseUnivariatePolynomial
RN       ==> Fraction Integer
NF       ==> Float
CF       ==> Complex Float
GI       ==> Complex Integer
GRN      ==> Complex RN
SE       ==> Symbol
RFI      ==> Fraction P I
INFSP ==> InnerNumericFloatSolvePackage


Cat == with

   complexSolve:    (L FPK,Par) -> L L EQ P C Par
     ++ complexSolve(lp,eps) finds all the complex solutions to
     ++ precision eps of the system lp of rational functions
     ++ over the complex rationals with respect to all the
     ++ variables appearing in lp.

   complexSolve:    (L EQ FPK,Par) -> L L EQ P C Par
     ++ complexSolve(leq,eps) finds all the complex solutions
     ++ to precision eps of the system leq of equations
     ++ of rational functions over complex rationals
     ++ with respect to all the variables appearing in lp.

   complexSolve:    (FPK,Par) -> L EQ P C Par
     ++ complexSolve(p,eps) find all the complex solutions of the
     ++ rational function p with complex rational coefficients
     ++ with respect to all the variables appearing in p,
     ++ with precision eps.

   complexSolve:    (EQ FPK,Par) ->  L EQ P C Par
     ++ complexSolve(eq,eps) finds all the complex solutions of the
     ++ equation eq of rational functions with rational rational coefficients
     ++ with respect to all the variables appearing in eq,
     ++ with precision eps.

   complexRoots : (FPK,Par) -> L C Par
     ++ complexRoots(rf, eps) finds all the complex solutions of a
     ++ univariate rational function with rational number coefficients.
     ++ The solutions are computed to precision eps.

   complexRoots : (L FPK,L SE,Par) -> L L C Par
     ++ complexRoots(lrf, lv, eps) finds all the complex solutions of a
     ++ list of rational functions with rational number coefficients
```

```
      ++ with respect the the variables appearing in lv.
      ++ Each solution is computed to precision eps and returned as
      ++ list corresponding to the order of variables in lv.

  Cap == add

    -- find the complex zeros of an univariate polynomial --
    complexRoots(q:FPK,eps:Par) : L C Par ==
      p:=numer q
      complexZeros(univariate p,eps)$ComplexRootPackage(SUP GI, Par)

    -- find the complex zeros of an univariate polynomial --
    complexRoots(lp:L FPK,lv:L SE,eps:Par) : L L C Par ==
      lnum:=[numer p for p in lp]
      lden:=[dp for p in lp |(dp:=denom p)^=1]
      innerSolve(lnum,lden,lv,eps)$INFSP(K,C Par,Par)

    complexSolve(lp:L FPK,eps : Par) : L L EQ  P C Par ==
      lnum:=[numer p for p in lp]
      lden:=[dp for p in lp |(dp:=denom p)^=1]
      lv:="setUnion"/[variables np for np in lnum]
      if lden^=[] then
       lv:=setUnion(lv,"setUnion"/[variables dp for dp in lden])
      [[equation(x::(P C Par),r::(P C Par)) for x in lv for r in nres]
        for nres in innerSolve(lnum,lden,lv,eps)$INFSP(K,C Par,Par)]

    complexSolve(le:L EQ FPK,eps : Par) : L L EQ  P C Par ==
      lp:=[lhs ep - rhs ep for ep in le]
      lnum:=[numer p for p in lp]
      lden:=[dp for p in lp |(dp:=denom p)^=1]
      lv:="setUnion"/[variables np for np in lnum]
      if lden^=[] then
       lv:=setUnion(lv,"setUnion"/[variables dp for dp in lden])
      [[equation(x::(P C Par),r::(P C Par)) for x in lv for r in nres]
        for nres in innerSolve(lnum,lden,lv,eps)$INFSP(K,C Par,Par)]

    complexSolve(p : FPK,eps : Par) : L EQ  P C Par ==
      (mvar := mainVariable numer p ) case "failed" =>
              error "no variable found"
      x:P C Par:=mvar::SE::(P C Par)
      [equation(x,val::(P C Par)) for val in complexRoots(p,eps)]

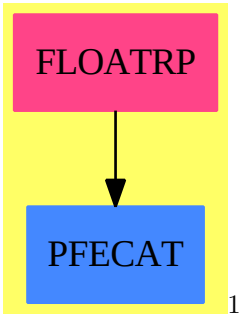
    complexSolve(eq : EQ FPK,eps : Par) : L EQ  P C Par ==
      complexSolve(lhs eq - rhs eq,eps)
```

⟨*FLOATCP.dotabb*⟩≡

```
"FLOATCP" [color="#FF4488",href="bookvol10.4.pdf#nameddest=FLOATCP"]
"COMPCAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=COMPCAT"]
"FLOATCP" -> "COMPCAT"
```

7.35 package FLOATRP FloatingRealPackage

7.36 FloatingRealPackage



1.00

Exports:
realRoots solve

⟨*package FLOATRP FloatingRealPackage*⟩≡
```
)abbrev package FLOATRP FloatingRealPackage
++ Author: P. Gianni
++ Date Created: January 1990
++ Date Last Updated:
++ Basic Functions:
++ Related Constructors: SystemSolvePackage, RadicalSolvePackage,
++ FloatingComplexPackage
++ Also See:
++ AMS Classifications:
++ Keywords:
++ References:
++ Description:
++    This is a package for the approximation of real solutions for
++ systems of polynomial equations over the rational numbers.
++ The results are expressed as either rational numbers or floats
++ depending on the type of the precision parameter which can be
++ either a rational number or a floating point number.
FloatingRealPackage(Par): Cat == Cap where
    I        ==> Integer
    NNI      ==> NonNegativeInteger
    P        ==> Polynomial
    EQ       ==> Equation
    L        ==> List
    SUP      ==> SparseUnivariatePolynomial
    RN       ==> Fraction Integer
    NF       ==> Float
    CF       ==> Complex Float
```

```
GI       ==> Complex Integer
GRN      ==> Complex RN
SE       ==> Symbol
RFI      ==> Fraction P I
INFSP ==> InnerNumericFloatSolvePackage

Par : Join(OrderedRing, Field)  -- RN or NewFloat

Cat == with

   solve:    (L RFI,Par) -> L L EQ P Par
     ++ solve(lp,eps) finds all of the real solutions of the
     ++ system lp of rational functions over the rational numbers
     ++ with respect to all the variables appearing in lp,
     ++ with precision eps.

   solve:    (L EQ RFI,Par) -> L L EQ P Par
     ++ solve(leq,eps) finds all of the real solutions of the
     ++ system leq of equationas of rational functions
     ++ with respect to all the variables appearing in lp,
     ++ with precision eps.

   solve:    (RFI,Par) ->  L EQ P Par
     ++ solve(p,eps) finds all of the real solutions of the
     ++ univariate rational function p with rational coefficients
     ++ with respect to the unique variable appearing in p,
     ++ with precision eps.

   solve:    (EQ RFI,Par) ->  L EQ P Par
     ++ solve(eq,eps) finds all of the real solutions of the
     ++ univariate equation eq of rational functions
     ++ with respect to the unique variables appearing in eq,
     ++ with precision eps.

   realRoots:    (L RFI,L SE,Par) -> L L Par
     ++ realRoots(lp,lv,eps) computes the list of the real
     ++ solutions of the list lp of rational functions with rational
     ++ coefficients with respect to the variables in lv,
     ++ with precision eps. Each solution is expressed as a list
     ++ of numbers in order corresponding to the variables in lv.

   realRoots : (RFI,Par) -> L Par
     ++ realRoots(rf, eps) finds the real zeros of a univariate
     ++ rational function with precision given by eps.

Cap == add
```

```
makeEq(nres:L Par,lv:L SE) : L EQ P Par ==
    [equation(x::(P Par),r::(P Par)) for x in lv for r in nres]

-- find the real zeros of an univariate rational polynomial --
realRoots(p:RFI,eps:Par) : L Par ==
  innerSolve1(numer p,eps)$INFSP(I,Par,Par)

-- real zeros of the system of polynomial lp --
realRoots(lp:L RFI,lv:L SE,eps: Par) : L L Par ==
  lnum:=[numer p for p in lp]
  lden:=[dp for p in lp |(dp:=denom p)^=1]
  innerSolve(lnum,lden,lv,eps)$INFSP(I,Par,Par)

solve(lp:L RFI,eps : Par) : L L EQ  P Par ==
  lnum:=[numer p for p in lp]
  lden:=[dp for p in lp |(dp:=denom p)^=1]
  lv:="setUnion"/[variables np for np in lnum]
  if lden^=[] then
   lv:=setUnion(lv,"setUnion"/[variables dp for dp in lden])
  [makeEq(numres,lv) for numres
     in innerSolve(lnum,lden,lv,eps)$INFSP(I,Par,Par)]

solve(le:L EQ RFI,eps : Par) : L L EQ  P Par ==
  lp:=[lhs ep - rhs ep for ep in le]
  lnum:=[numer p for p in lp]
  lden:=[dp for p in lp |(dp:=denom p)^=1]
  lv:="setUnion"/[variables np for np in lnum]
  if lden^=[] then
   lv:=setUnion(lv,"setUnion"/[variables dp for dp in lden])
  [makeEq(numres,lv) for numres
    in innerSolve(lnum,lden,lv,eps)$INFSP(I,Par,Par)]

solve(p : RFI,eps : Par) :  L EQ  P Par ==
  (mvar := mainVariable numer p ) case "failed" =>
       error "no variable found"
  x:P Par:=mvar::SE::(P Par)
  [equation(x,val::(P Par)) for val in realRoots(p,eps)]

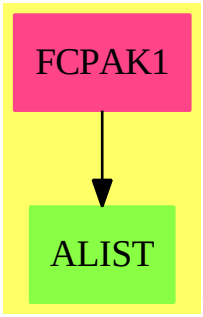
solve(eq : EQ RFI,eps : Par) :  L EQ  P Par ==
  solve(lhs eq - rhs eq,eps)
```

⟨*FLOATRP.dotabb*⟩≡

```
"FLOATRP" [color="#FF4488",href="bookvol10.4.pdf#nameddest=FLOATRP"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"FLOATRP" -> "PFECAT"
```

7.37 package FCPAK1 FortranCodePackage1

7.38 FortranCodePackage1



1.00

Exports:
identitySquareMatrix zeroMatrix zeroSquareMatrix zeroVector

⟨*package FCPAK1 FortranCodePackage1*⟩≡
```
)abbrev package FCPAK1 FortranCodePackage1
++ Author: Grant Keady and Godfrey Nolan
++ Date Created: April 1993
++ Date Last Updated:
++ Basic Operations:
++ Related Constructors:
++ Also See:
++ AMS Classifications:
++ Keywords:
++ References:
++ Description:
++  \spadtype{FortranCodePackage1} provides some utilities for
++  producing useful objects in FortranCode domain.
++  The Package may be used with the FortranCode domain and its
++  \spad{printCode} or possibly via an outputAsFortran.
++  (The package provides items of use in connection with ASPs
++  in the AXIOM-NAG link and, where appropriate, naming accords
++  with that in IRENA.)
++  The easy-to-use functions use Fortran loop variables I1, I2,
++  and it is users' responsibility to check that this is sensible.
++  The advanced functions use SegmentBinding to allow users control
++  over Fortran loop variable names.
-- Later might add functions to build
-- diagonalMatrix from List, i.e. the FC version of the corresponding
-- AXIOM function from MatrixCategory;
-- bandedMatrix, i.e. the full-matrix-FC version of the corresponding
-- AXIOM function in BandedMatrix Domain
```

```
-- bandedSymmetricMatrix, i.e. the full-matrix-FC version of the corresponding
-- AXIOM function in BandedSymmetricMatrix Domain

FortranCodePackage1: Exports  == Implementation where

  NNI    ==> NonNegativeInteger
  PI     ==> PositiveInteger
  PIN    ==> Polynomial(Integer)
  SBINT  ==> SegmentBinding(Integer)
  SEGINT ==> Segment(Integer)
  LSBINT ==> List(SegmentBinding(Integer))
  SBPIN  ==> SegmentBinding(Polynomial(Integer))
  SEGPIN ==> Segment(Polynomial(Integer))
  LSBPIN ==> List(SegmentBinding(Polynomial(Integer)))
  FC     ==> FortranCode
  EXPRESSION  ==> Union(Expression Integer,Expression Float,Expression Complex Integer,Expre

  Exports == with

    zeroVector: (Symbol,PIN) -> FC
      ++ zeroVector(s,p) \undocumented{}

    zeroMatrix: (Symbol,PIN,PIN) -> FC
      ++ zeroMatrix(s,p,q) uses loop variables in the Fortran, I1 and I2

    zeroMatrix: (Symbol,SBPIN,SBPIN) -> FC
      ++ zeroMatrix(s,b,d) in this version gives the user control
      ++ over names of Fortran variables used in loops.

    zeroSquareMatrix: (Symbol,PIN) -> FC
      ++ zeroSquareMatrix(s,p) \undocumented{}

    identitySquareMatrix: (Symbol,PIN) -> FC
      ++ identitySquareMatrix(s,p) \undocumented{}

  Implementation ==> add
    import FC

    zeroVector(fname:Symbol,n:PIN):FC ==
      ue:Expression(Integer) := 0
      i1:Symbol := "I1"::Symbol
      lp1:PIN := 1::PIN
      hp1:PIN := n
      segp1:SEGPIN:= segment(lp1,hp1)$SEGPIN
      segbp1:SBPIN := equation(i1,segp1)$SBPIN
      ip1:PIN := i1::PIN
```

```
  indices:List(PIN) := [ip1]
  fa:FC := forLoop(segbp1,assign(fname,indices,ue)$FC)$FC
  fa

zeroMatrix(fname:Symbol,m:PIN,n:PIN):FC ==
  ue:Expression(Integer) := 0
  i1:Symbol := "I1"::Symbol
  lp1:PIN := 1::PIN
  hp1:PIN := m
  segp1:SEGPIN:= segment(lp1,hp1)$SEGPIN
  segbp1:SBPIN := equation(i1,segp1)$SBPIN
  i2:Symbol := "I2"::Symbol
  hp2:PIN := n
  segp2:SEGPIN:= segment(lp1,hp2)$SEGPIN
  segbp2:SBPIN := equation(i2,segp2)$SBPIN
  ip1:PIN := i1::PIN
  ip2:PIN := i2::PIN
  indices:List(PIN) := [ip1,ip2]
  fa:FC :=forLoop(segbp1,forLoop(segbp2,assign(fname,indices,ue)$FC)$FC)$FC
  fa

zeroMatrix(fname:Symbol,segbp1:SBPIN,segbp2:SBPIN):FC ==
  ue:Expression(Integer) := 0
  i1:Symbol := variable(segbp1)$SBPIN
  i2:Symbol := variable(segbp2)$SBPIN
  ip1:PIN := i1::PIN
  ip2:PIN := i2::PIN
  indices:List(PIN) := [ip1,ip2]
  fa:FC :=forLoop(segbp1,forLoop(segbp2,assign(fname,indices,ue)$FC)$FC)$FC
  fa

zeroSquareMatrix(fname:Symbol,n:PIN):FC ==
  ue:Expression(Integer) := 0
  i1:Symbol := "I1"::Symbol
  lp1:PIN := 1::PIN
  hp1:PIN := n
  segp1:SEGPIN:= segment(lp1,hp1)$SEGPIN
  segbp1:SBPIN := equation(i1,segp1)$SBPIN
  i2:Symbol := "I2"::Symbol
  segbp2:SBPIN := equation(i2,segp1)$SBPIN
  ip1:PIN := i1::PIN
  ip2:PIN := i2::PIN
  indices:List(PIN) := [ip1,ip2]
  fa:FC :=forLoop(segbp1,forLoop(segbp2,assign(fname,indices,ue)$FC)$FC)$FC
  fa
```

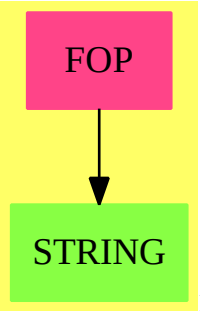
```
identitySquareMatrix(fname:Symbol,n:PIN):FC ==
  ue:Expression(Integer) := 0
  u1:Expression(Integer) := 1
  i1:Symbol := "I1"::Symbol
  lp1:PIN := 1::PIN
  hp1:PIN := n
  segp1:SEGPIN:= segment(lp1,hp1)$SEGPIN
  segbp1:SBPIN := equation(i1,segp1)$SBPIN
  i2:Symbol := "I2"::Symbol
  segbp2:SBPIN := equation(i2,segp1)$SBPIN
  ip1:PIN := i1::PIN
  ip2:PIN := i2::PIN
  indice1:List(PIN) := [ip1,ip1]
  indices:List(PIN) := [ip1,ip2]
  fc:FC := forLoop(segbp2,assign(fname,indices,ue)$FC)$FC
  f1:FC := assign(fname,indice1,u1)$FC
  fl:List(FC) := [fc,f1]
  fa:FC := forLoop(segbp1,block(fl)$FC)$FC
  fa
```

⟨*FCPAK1.dotabb*⟩≡

```
"FCPAK1" [color="#FF4488",href="bookvol10.4.pdf#nameddest=FCPAK1"]
"ALIST" [color="#88FF44",href="bookvol10.3.pdf#nameddest=ALIST"]
"FCPAK1" -> "ALIST"
```

7.39 package FOP FortranOutputStackPackage

7.40 FortranOutputStackPackage



1.00

Exports:

popFortranOutputStack	clearFortranOutputStack	pushFortranOutputStack
pushFortranOutputStack	showFortranOutputStack	topFortranOutputStack

⟨*package FOP FortranOutputStackPackage*⟩≡

```
)abbrev package FOP FortranOutputStackPackage
-- Because of a bug in the compiler:
)bo $noSubsumption:=false

++ Author: Mike Dewar
++ Date Created:  October 1992
++ Date Last Updated:
++ Basic Operations:
++ Related Domains:
++ Also See:
++ AMS Classifications:
++ Keywords:
++ Examples:
++ References:
++ Description: Code to manipulate Fortran Output Stack
FortranOutputStackPackage() : specification == implementation where

  specification == with

    clearFortranOutputStack : () -> Stack String
      ++ clearFortranOutputStack() clears the Fortran output stack
    showFortranOutputStack : () -> Stack String
      ++ showFortranOutputStack() returns the Fortran output stack
    popFortranOutputStack : () -> Void
      ++ popFortranOutputStack() pops the Fortran output stack
    pushFortranOutputStack : FileName -> Void
```

```
      ++ pushFortranOutputStack(f) pushes f onto the Fortran output stack
    pushFortranOutputStack : String -> Void
      ++ pushFortranOutputStack(f) pushes f onto the Fortran output stack
    topFortranOutputStack : () -> String
      ++ topFortranOutputStack() returns the top element of the Fortran
      ++ output stack

  implementation == add

    import MoreSystemCommands

    -- A stack of filenames for Fortran output.  We are sharing this with
    -- the standard Fortran output code, so want to be a bit careful about
    -- how we interact with what the user does independently.  We get round
    -- potential problems by always examining the top element of the stack
    -- before we push.  If the user has redirected output then we alter our
    -- top value accordingly.
    fortranOutputStack : Stack String := empty()@(Stack String)

    topFortranOutputStack():String == string(_$fortranOutputFile$Lisp)

    pushFortranOutputStack(fn:FileName):Void ==
      if empty? fortranOutputStack then
        push!(string(_$fortranOutputFile$Lisp),fortranOutputStack)
      else if not(top(fortranOutputStack)=string(_$fortranOutputFile$Lisp)) then
        pop! fortranOutputStack
        push!(string(_$fortranOutputFile$Lisp),fortranOutputStack)
      push!( fn::String,fortranOutputStack)
      systemCommand concat(["set output fortran quiet ", fn::String])$String
      void()

    pushFortranOutputStack(fn:String):Void ==
      if empty? fortranOutputStack then
        push!(string(_$fortranOutputFile$Lisp),fortranOutputStack)
      else if not(top(fortranOutputStack)=string(_$fortranOutputFile$Lisp)) then
        pop! fortranOutputStack
        push!(string(_$fortranOutputFile$Lisp),fortranOutputStack)
      push!( fn,fortranOutputStack)
      systemCommand concat(["set output fortran quiet ", fn])$String
      void()

    popFortranOutputStack():Void ==
      if not empty? fortranOutputStack then pop! fortranOutputStack
      if empty? fortranOutputStack then push!("CONSOLE",fortranOutputStack)
      systemCommand concat(["set output fortran quiet append ",_
                            top fortranOutputStack])$String
```

```
        void()

      clearFortranOutputStack():Stack String ==
        fortranOutputStack := empty()@(Stack String)

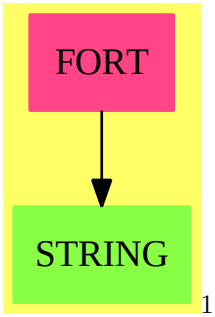
      showFortranOutputStack():Stack String ==
        fortranOutputStack
```

⟨*FOP.dotabb*⟩≡

```
"FOP" [color="#FF4488",href="bookvol10.4.pdf#nameddest=FOP"]
"STRING" [color="#88FF44",href="bookvol10.3.pdf#nameddest=STRING"]
"FOP" -> "STRING"
```

7.41 package FORT FortranPackage

7.42 FortranPackage



1.00

Exports:
outputAsFortran linkToFortran setLegalFortranSourceExtensions

⟨*package FORT FortranPackage*⟩≡
```
)abbrev package FORT FortranPackage
-- Because of a bug in the compiler:
)bo $noSubsumption:=true

++ Author: Mike Dewar
++ Date Created: October 6 1991
++ Date Last Updated: 13 July 1994
++ Basic Operations: linkToFortran
++ Related Constructors:
++ Also See:
++ AMS Classifications:
++ Keywords:
++ References:
++ Description: provides an interface to the boot code for calling Fortran
FortranPackage(): Exports == Implementation where
 FST ==> FortranScalarType
 SEX ==> SExpression
 L   ==> List
 S   ==> Symbol
 FOP ==> FortranOutputStackPackage
 U   ==> Union(array:L S,scalar:S)

 Exports ==> with
  linkToFortran: (S, L U, L L U, L S) -> SEX
        ++ linkToFortran(s,l,ll,lv) \undocumented{}
  linkToFortran: (S, L U, L L U, L S, S) -> SEX
        ++ linkToFortran(s,l,ll,lv,t) \undocumented{}
```

```
    linkToFortran: (S,L S,TheSymbolTable,L S) -> SEX
          ++ linkToFortran(s,l,t,lv) \undocumented{}
    outputAsFortran: FileName -> Void
          ++ outputAsFortran(fn) \undocumented{}
    setLegalFortranSourceExtensions: List String -> List String
          ++ setLegalFortranSourceExtensions(l) \undocumented{}

   Implementation ==> add

    legalFortranSourceExtensions : List String := ["f"]

    setLegalFortranSourceExtensions(l:List String):List String ==
      legalFortranSourceExtensions := l

    checkExtension(fn : FileName) : String ==
      -- Does it end in a legal extension ?
      stringFn := fn::String
      not member?(extension fn,legalFortranSourceExtensions) =>
        error [stringFn,"is not a legal Fortran Source File."]
      stringFn

    outputAsFortran(fn:FileName):Void ==
--    source : String := checkExtension fn
      source : String := fn::String
      not readable? fn =>
        popFortranOutputStack()$FOP
        error([source,"is not readable"]@List(String))
      target : String := topFortranOutputStack()$FOP
      command : String :=
        concat(["sys rm -f ",target," ; cp ",source," ",target])$String
      systemCommand(command)$MoreSystemCommands
      void()$Void

    linkToFortran(name:S,args:L U, decls:L L U, res:L(S)):SEX ==
      makeFort(name,args,decls,res,NIL$Lisp,NIL$Lisp)$Lisp

    linkToFortran(name:S,args:L U, decls:L L U, res:L(S),returnType:S):SEX ==
      makeFort(name,args,decls,res,returnType,NIL$Lisp)$Lisp

    dimensions(type:FortranType):SEX ==
      convert([convert(convert(u)@InputForm)@SEX _
        for u in dimensionsOf(type)])@SEX

    ftype(name:S,type:FortranType):SEX ==
      [name,scalarTypeOf(type),dimensions(type),external? type]$Lisp
```

```
makeAspList(asp:S,syms:TheSymbolTable):SExpression==
  symtab : SymbolTable := symbolTableOf(asp,syms)
  [asp,returnTypeOf(asp,syms),argumentListOf(asp,syms), _
   [ftype(u,fortranTypeOf(u,symtab)) for u in parametersOf symtab]]$Lisp

linkToFortran(name:S,aArgs:L S,syms:TheSymbolTable,res:L S):SEX ==
  arguments : L S := argumentListOf(name,syms)$TheSymbolTable
  dummies : L S := setDifference(arguments,aArgs)
  symbolTable:SymbolTable := symbolTableOf(name,syms)
  symbolList := newTypeLists(symbolTable)
  rt:Union(fst: FST,void: "void") := returnTypeOf(name,syms)$TheSymbolTable

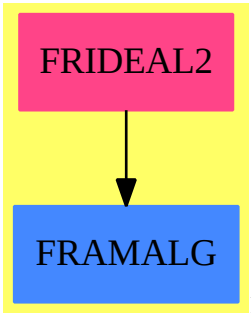
  -- Look for arguments which are subprograms
  asps :=[makeAspList(u,syms) for u in externalList(symbolTable)$SymbolTable]
  rt case fst =>
    makeFort1(name,arguments,aArgs,dummies,symbolList,res,(rt.fst)::S,asps)$Lisp
  makeFort1(name,arguments,aArgs,dummies,symbolList,res,NIL$Lisp,asps)$Lisp
```

⟨*FORT.dotabb*⟩≡

```
"FORT" [color="#FF4488",href="bookvol10.4.pdf#nameddest=FORT"]
"STRING" [color="#88FF44",href="bookvol10.3.pdf#nameddest=STRING"]
"FORT" -> "STRING"
```

7.43 package FRIDEAL2 FractionalIdealFunctions2

7.44 FractionalIdealFunctions2



1.00

Exports:
map

⟨*package FRIDEAL2 FractionalIdealFunctions2*⟩≡

```
)abbrev package FRIDEAL2 FractionalIdealFunctions2
++ Lifting of morphisms to fractional ideals.
++ Author: Manuel Bronstein
++ Date Created: 1 Feb 1989
++ Date Last Updated: 27 Feb 1990
++ Keywords: ideal, algebra, module.
FractionalIdealFunctions2(R1, F1, U1, A1, R2, F2, U2, A2):
 Exports == Implementation where
  R1, R2: EuclideanDomain
  F1: QuotientFieldCategory R1
  U1: UnivariatePolynomialCategory F1
  A1: Join(FramedAlgebra(F1, U1), RetractableTo F1)
  F2: QuotientFieldCategory R2
  U2: UnivariatePolynomialCategory F2
  A2: Join(FramedAlgebra(F2, U2), RetractableTo F2)

  Exports ==> with
    map: (R1 -> R2, FractionalIdeal(R1, F1, U1, A1)) ->
                                          FractionalIdeal(R2, F2, U2, A2)
        ++ map(f,i) \undocumented{}

  Implementation ==> add
    fmap: (F1 -> F2, A1) -> A2

    fmap(f, a) ==
      v := coordinates a
      represents
```

```
          [f qelt(v, i) for i in minIndex v .. maxIndex v]$Vector(F2)

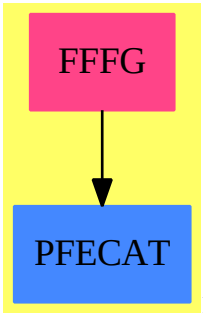
      map(f, i) ==
        b := basis i
        ideal [fmap(s +-> f(numer s) / f(denom s), qelt(b, j))
               for j in minIndex b .. maxIndex b]$Vector(A2)
```

⟨*FRIDEAL2.dotabb*⟩≡

```
"FRIDEAL2" [color="#FF4488",href="bookvol10.4.pdf#nameddest=FRIDEAL2"]
"FRAMALG" [color="#4488FF",href="bookvol10.2.pdf#nameddest=FRAMALG"]
"FRIDEAL2" -> "FRAMALG"
```

7.45 package FFFG FractionFreeFastGaussian

7.46 FractionFreeFastGaussian



1.00

Exports:

DiffAction	DiffC	ShiftAction	ShiftC
fffg	generalCoefficient	generalInterpolation	interpolate
qShiftAction	qShiftC		

⟨*package FFFG FractionFreeFastGaussian*⟩≡

```
)abbrev package FFFG FractionFreeFastGaussian
++ Author: Martin Rubey
++ Description:
++ This package implements the interpolation algorithm proposed in Beckermann,
++ Bernhard and Labahn, George, Fraction-free computation of matrix rational
++ interpolants and matrix GCDs, SIAM Journal on Matrix Analysis and
++ Applications 22.
++ The packages defined in this file provide fast fraction free rational
++ interpolation algorithms. (see FAMR2, FFFG, FFFGF, NEWTON)
FractionFreeFastGaussian(D, V): Exports == Implementation where
  D: Join(IntegralDomain, GcdDomain)
  V: AbelianMonoidRing(D, NonNegativeInteger) -- for example, SUP D

  SUP  ==> SparseUnivariatePolynomial

  cFunction ==> (NonNegativeInteger, Vector SUP D) -> D

  CoeffAction ==> (NonNegativeInteger, NonNegativeInteger, V) -> D

  Exports == with

    fffg: (List D, cFunction, List NonNegativeInteger) -> Matrix SUP D
      ++ \spad{fffg} is the general algorithm as proposed by Beckermann and
      ++ Labahn.
      ++
```

```
    ++ The first argument is the list of c_{i,i}.  These are the only values
    ++ of C explicitely needed in \spad{fffg}.
    ++
    ++ The second argument c, computes c_k(M), i.e., c_k(.) is the dual basis
    ++ of the vector space V, but also knows about the special multiplication
    ++ rule as descibed in Equation (2).  Note that the information about f
    ++ is therefore encoded in c.
    ++
    ++ The third argument is the vector of degree bounds n, as introduced in
    ++ Definition 2.1.  In particular, the sum of the entries is the order of
    ++ the Mahler system computed.

  interpolate: (List D, List D, NonNegativeInteger) -> Fraction SUP D
    ++ \spad{interpolate(xlist, ylist, deg} returns the rational function with
    ++ numerator degree at most \spad{deg} and denominator degree at most
    ++ \spad{#xlist-deg-1}  that interpolates the given points using
    ++ fraction free arithmetic. Note that rational interpolation does not
    ++ guarantee that all given points are interpolated correctly:
    ++ unattainable points may make this impossible.
```

```
  The following function could be moved to {\tt{}FFFGF}, parallel to
  {\tt{}generalInterpolation}.  However, the reason for moving
  {\tt{}generalInterpolation} for fractions to a separate package was the need of
  a generic signature, hence the extra argument {\tt{}VF} to {\tt{}FFFGF}.  In the
  special case of rational interpolation, this extra argument is not necessary,
  since we are always returning a fraction of {\tt{}SUP}s, and ignore {\tt{}V}.  In
  fact, {\tt{}V} is not needed for {\tt{}fffg} itself, only if we want to specify a
  {\tt{}CoeffAction}.

  Thus, maybe it would be better to move {\tt{}fffg} to a separate package?
```

⟨*package FFFG FractionFreeFastGaussian*⟩+≡

```
      interpolate: (List Fraction D, List Fraction D, NonNegativeInteger)
                  -> Fraction SUP D
        ++ \spad{interpolate(xlist, ylist, deg} returns the rational function with
        ++ numerator degree \spad{deg} that interpolates the given points using
        ++ fraction free arithmetic.

      generalInterpolation: (List D, CoeffAction,
                             Vector V, List NonNegativeInteger) -> Matrix SUP D
        ++ \spad{generalInterpolation(C, CA, f, eta)} performs Hermite-Pade
        ++ approximation using the given action CA of polynomials on the elements
        ++ of f. The result is guaranteed to be correct up to order
        ++ |eta|-1. Given that eta is a "normal" point, the degrees on the
        ++ diagonal are given by eta. The degrees of column i are in this case
        ++ eta + e.i - [1,1,...,1], where the degree of zero is -1.
        ++
        ++ The first argument C is the list of coefficients c_{k,k} in the
        ++ expansion <x^k> z g(x) = sum_{i=0}^k c_{k,i} <x^i> g(x).
        ++
        ++ The second argument, CA(k, l, f), should return the coefficient of x^k
        ++ in z^l f(x).

      generalInterpolation: (List D, CoeffAction,
                             Vector V, NonNegativeInteger, NonNegativeInteger)
                           -> Stream Matrix SUP D
        ++ \spad{generalInterpolation(C, CA, f, sumEta, maxEta)} applies
        ++ \spad{generalInterpolation(C, CA, f, eta)} for all possible \spad{eta}
        ++ with maximal entry \spad{maxEta} and sum of entries at most
        ++ \spad{sumEta}.
        ++
        ++ The first argument C is the list of coefficients c_{k,k} in the
        ++ expansion <x^k> z g(x) = sum_{i=0}^k c_{k,i} <x^i> g(x).
        ++
        ++ The second argument, CA(k, l, f), should return the coefficient of x^k
        ++ in z^l f(x).
```

```
    generalCoefficient: (CoeffAction, Vector V,
                         NonNegativeInteger, Vector SUP D) -> D
      ++ \spad{generalCoefficient(action, f, k, p)} gives the coefficient of
      ++ x^k in p(z)\dot f(x), where the action of z^l on a polynomial in x is
      ++ given by action, i.e., action(k, l, f) should return the coefficient
      ++ of x^k in z^l f(x).

    ShiftAction: (NonNegativeInteger, NonNegativeInteger, V) -> D
      ++ \spad{ShiftAction(k, l, g)} gives the coefficient of x^k in z^l g(x),
      ++ where \spad{z*(a+b*x+c*x^2+d*x^3+...) = (b*x+2*c*x^2+3*d*x^3+...)}. In
      ++ terms of sequences, z*u(n)=n*u(n).

    ShiftC: NonNegativeInteger -> List D
      ++ \spad{ShiftC} gives the coefficients c_{k,k} in the expansion <x^k> z
      ++ g(x) = sum_{i=0}^k c_{k,i} <x^i> g(x), where z acts on g(x) by
      ++ shifting. In fact, the result is [0,1,2,...]

    DiffAction: (NonNegativeInteger, NonNegativeInteger, V) -> D
      ++ \spad{DiffAction(k, l, g)} gives the coefficient of x^k in z^l g(x),
      ++ where z*(a+b*x+c*x^2+d*x^3+...) = (a*x+b*x^2+c*x^3+...), i.e.,
      ++ multiplication with x.

    DiffC: NonNegativeInteger -> List D
      ++ \spad{DiffC} gives the coefficients c_{k,k} in the expansion <x^k> z
      ++ g(x) = sum_{i=0}^k c_{k,i} <x^i> g(x), where z acts on g(x) by
      ++ shifting. In fact, the result is [0,0,0,...]

    qShiftAction: (D, NonNegativeInteger, NonNegativeInteger, V) -> D
      ++ \spad{qShiftAction(q, k, l, g)} gives the coefficient of x^k in z^l
      ++ g(x), where z*(a+b*x+c*x^2+d*x^3+...) =
      ++ (a+q*b*x+q^2*c*x^2+q^3*d*x^3+...). In terms of sequences,
      ++ z*u(n)=q^n*u(n).

    qShiftC: (D, NonNegativeInteger) -> List D
      ++ \spad{qShiftC} gives the coefficients c_{k,k} in the expansion <x^k> z
      ++ g(x) = sum_{i=0}^k c_{k,i} <x^i> g(x), where z acts on g(x) by
      ++ shifting. In fact, the result is [1,q,q^2,...]

  Implementation ==> add

--------------------------------------------------------------------------------
-- Shift Operator
--------------------------------------------------------------------------------

-- ShiftAction(k, l, f) is the CoeffAction appropriate for the shift operator.
```

```
    ShiftAction(k: NonNegativeInteger, l: NonNegativeInteger, f: V): D ==
      k**l*coefficient(f, k)


    ShiftC(total: NonNegativeInteger): List D ==
      [i::D for i in 0..total-1]

--------------------------------------------------------------------------------
-- q-Shift Operator
--------------------------------------------------------------------------------

-- q-ShiftAction(k, l, f) is the CoeffAction appropriate for the q-shift operator

    qShiftAction(q: D, k: NonNegativeInteger, l: NonNegativeInteger, f: V): D ==
      q**(k*l)*coefficient(f, k)


    qShiftC(q: D, total: NonNegativeInteger): List D ==
      [q**i for i in 0..total-1]

--------------------------------------------------------------------------------
-- Differentiation Operator
--------------------------------------------------------------------------------

-- DiffAction(k, l, f) is the CoeffAction appropriate for the differentiation
-- operator.

    DiffAction(k: NonNegativeInteger, l: NonNegativeInteger, f: V): D ==
      coefficient(f, (k-l)::NonNegativeInteger)


    DiffC(total: NonNegativeInteger): List D ==
      [0 for i in 1..total]

--------------------------------------------------------------------------------
-- general - suitable for functions f
--------------------------------------------------------------------------------

-- get the coefficient of z^k in the scalar product of p and f, the action
-- being defined by coeffAction

    generalCoefficient(coeffAction: CoeffAction, f: Vector V,
                       k: NonNegativeInteger, p: Vector SUP D): D ==
      res: D := 0
      for i in 1..#f repeat
```

```
-- Defining a and b and summing only over those coefficients that might be
-- nonzero makes a huge difference in speed
        a := f.i
        b := p.i
        for l in minimumDegree b..degree b repeat
            if not zero? coefficient(b, l)
            then res := res + coefficient(b, l) * coeffAction(k, l, a)
      res


    generalInterpolation(C: List D, coeffAction: CoeffAction,
                         f: Vector V,
                         eta: List NonNegativeInteger): Matrix SUP D ==

      c: cFunction := (x,y) +-> generalCoefficient(coeffAction, f,
                                        (x-1)::NonNegativeInteger, y)
      fffg(C, c, eta)



-------------------------------------------------------------------------------
-- general - suitable for functions f - trying all possible degree combinations
-------------------------------------------------------------------------------
```

The following function returns the lexicographically next vector with non-negative components smaller than p with the same sum as v.

⟨*package FFFG FractionFreeFastGaussian*⟩+≡

```
nextVector!(p: NonNegativeInteger, v: List NonNegativeInteger)
           : Union("failed", List NonNegativeInteger) ==
  n := #v
  pos := position(x +-> x < p, v)
  zero? pos => return "failed"
  if pos = 1 then
    sum: Integer := v.1
    for i in 2..n repeat
      if v.i < p and sum > 0 then
        v.i := v.i + 1
        sum := sum - 1
        for j in 1..i-1 repeat
          if sum > p then
            v.j := p
            sum := sum - p
          else
            v.j := sum::NonNegativeInteger
            sum := 0
        return v
      else sum := sum + v.i
    return "failed"
  else
    v.pos     := v.pos + 1
    v.(pos-1) := (v.(pos-1) - 1)::NonNegativeInteger

  v
```

The following function returns the stream of all possible degree vectors, beginning with v, where the degree vectors are sorted in reverse lexicographic order. Furthermore, the entries are all less or equal to p and their sum equals the sum of the entries of v. We assume that the entries of v are also all less or equal to p.

⟨*package FFFG FractionFreeFastGaussian*⟩+≡

```
vectorStream(p: NonNegativeInteger, v: List NonNegativeInteger)
            : Stream List NonNegativeInteger == delay
  next := nextVector!(p, copy v)
  (next case "failed") => empty()$Stream(List NonNegativeInteger)
  cons(next, vectorStream(p, next))
```

`vectorStream2` skips every second entry of `vectorStream`.

⟨*package FFFG FractionFreeFastGaussian*⟩+≡

```
vectorStream2(p: NonNegativeInteger, v: List NonNegativeInteger)
             : Stream List NonNegativeInteger == delay
  next := nextVector!(p, copy v)
  (next case "failed") => empty()$Stream(List NonNegativeInteger)
  next2 := nextVector!(p, copy next)
  (next2 case "failed") => cons(next, empty())
  cons(next2, vectorStream2(p, next2))
```

This version of `generalInterpolation` returns a stream of solutions, one for each possible degree vector. Thus, it only needs to apply the previously defined `generalInterpolation` to each degree vector. These are generated by `vectorStream` and `vectorStream2` respectively.

If `f` consists of two elements only, we can skip every second degree vector: note that `fffg`, and thus also `generalInterpolation`, returns a matrix with `#f` columns, each corresponding to a solution of the interpolation problem. More precisely, the i^{th} column is a solution with degrees `eta`$-(1,1,\ldots,1)+e_i$. Thus, in the case of 2×2 matrices, `vectorStream` would produce solutions corresponding to $(d,0),(d-1,1);(d-1,1),(d-2,2);(d-2,2),(d-3,3)\ldots$, i.e., every second matrix is redundant.

Although some redundancy exists also for higher dimensional `f`, the scheme becomes much more complicated, thus we did not implement it.

⟨*package FFFG FractionFreeFastGaussian*⟩+≡

```
generalInterpolation(C: List D, coeffAction: CoeffAction,
                     f: Vector V,
                     sumEta: NonNegativeInteger,
                     maxEta: NonNegativeInteger)
                    : Stream Matrix SUP D ==

  ⟨generate an initial degree vector⟩

  if #f = 2 then
    map(x +-> generalInterpolation(C, coeffAction, f, x),
        cons(eta, vectorStream2(maxEta, eta)))
       $StreamFunctions2(List NonNegativeInteger,
                         Matrix SUP D)
  else
    map(x +-> generalInterpolation(C, coeffAction, f, x),
        cons(eta, vectorStream(maxEta, eta)))
       $StreamFunctions2(List NonNegativeInteger,
                         Matrix SUP D)
```

We need to generate an initial degree vector, being the minimal element in reverse lexicographic order, i.e., $m, m, \ldots, m, k, 0, 0, \ldots$, where m is `maxEta` and k is the remainder of `sumEta` divided by `maxEta`. This is done by the following code:

⟨*generate an initial degree vector*⟩≡
```
sum: Integer := sumEta
entry: Integer
eta: List NonNegativeInteger
    := [(if sum < maxEta _
         then (entry := sum; sum := 0) _
         else (entry := maxEta; sum := sum - maxEta); _
         entry::NonNegativeInteger) for i in 1..#f]
```

We want to generate all vectors with sum of entries being at most `sumEta`. Therefore the following is incorrect.

⟨*BUG generate an initial degree vector*⟩≡
```
-- (sum > 0) => empty()$Stream(Matrix SUP D)
```

⟨*package FFFG FractionFreeFastGaussian*⟩+≡

```
--------------------------------------------------------------------------------
-- rational interpolation
--------------------------------------------------------------------------------

    interpolate(x: List Fraction D, y: List Fraction D, d: NonNegativeInteger)
              : Fraction SUP D ==
      gx := splitDenominator(x)$InnerCommonDenominator(D, Fraction D, _
                                                       List D, _
                                                       List Fraction D)
      gy := splitDenominator(y)$InnerCommonDenominator(D, Fraction D, _
                                                       List D, _
                                                       List Fraction D)
      r := interpolate(gx.num, gy.num, d)
      elt(numer r, monomial(gx.den,1))/(gy.den*elt(denom r, monomial(gx.den,1)))


    interpolate(x: List D, y: List D, d: NonNegativeInteger): Fraction SUP D ==
-- berechne Interpolante mit Graden d und N-d-1
      if (N := #x) ~= #y then
        error "interpolate: number of points and values must match"
      if N <= d then
        error "interpolate: numerator degree must be smaller than number of data points"
      c: cFunction := (s,u) +-> y.s * elt(u.2, x.s) - elt(u.1, x.s)
      eta: List NonNegativeInteger := [d, (N-d)::NonNegativeInteger]
      M := fffg(x, c, eta)

      if zero?(M.(2,1)) then M.(1,2)/M.(2,2)
                        else M.(1,1)/M.(2,1)
```

Because of Lemma 5.3, `M.1.(2,1)` and `M.1.(2,2)` cannot both vanish, since `eta_sigma` is always σ-normal by Theorem 7.2 and therefore also para-normal, see Definition 4.2.

Because of Lemma 5.1 we have that `M.1.(*,2)` is a solution of the interpolation problem, if `M.1.(2,1)` vanishes.

⟨*package FFFG FractionFreeFastGaussian*⟩+≡

```
--------------------------------------------------------------------------------
-- fffg
--------------------------------------------------------------------------------
```

`recurrence` computes the new matrix M, according to the following formulas (cf. Table 2 in Beckermann and Labahn):

$$\begin{gathered}\text{Increase order}\\ \text{for } \ell = 1 \ldots m,\ \ell \neq \pi\\ \mathbf{M}_{\sigma+1}^{(.,\ell)} := \left(\mathbf{M}_{\sigma}^{(.,\ell)} r^{(\pi)} - \mathbf{M}_{\sigma}^{(.,\pi)} r^{(\ell)}\right)/d_\sigma\\ \text{Increase order in column } \pi\\ \mathbf{M}_{\sigma+1}^{(.,\pi)} := (z - c_{\sigma,\sigma})\,\mathbf{M}_{\sigma}^{(.,\pi)}\\ \text{Adjust degree constraints:}\\ \mathbf{M}_{\sigma+1}^{(.,\pi)} := \left(\mathbf{M}_{\sigma+1}^{(.,\pi)} r^{(\pi)} - \textstyle\sum_{\ell\neq\pi} \mathbf{M}_{\sigma+1}^{(.,\ell)} p^{(\ell)}\right)/d_\sigma\end{gathered}$$

Since we do not need the matrix $\mathbf{M}_\sigma$ anymore, we drop the index and update the matrix destructively. In the following, we write `Ck` for $c_{\sigma,\sigma}$.

⟨*package FFFG FractionFreeFastGaussian*⟩+≡

```
  -- a major part of the time is spent here
      recurrence(M: Matrix SUP D, pi: NonNegativeInteger, m: NonNegativeInteger,
                 r: Vector D, d: D, z: SUP D, Ck: D, p: Vector D): Matrix SUP D ==

          rPi: D := qelt(r, pi)
          polyf: SUP D := rPi * (z - Ck::SUP D)

          for i in 1..m repeat
              MiPi: SUP D    := qelt(M, i, pi)
              newMiPi: SUP D := polyf * MiPi

  -- update columns ~= pi and calculate their sum
              for l in 1..m | l ~= pi repeat
                  rl: D  := qelt(r, l)
  -- I need the coercion to SUP D, since exquo returns an element of
  -- Union("failed", SUP D)...
                  Mil: SUP D := ((qelt(M, i, l) * rPi - MiPi * rl) exquo d)::SUP D
                  qsetelt!(M, i, l, Mil)

                  pl: D  := qelt(p, l)
                  newMiPi := newMiPi - pl * Mil

  -- update column pi
              qsetelt!(M, i, pi, (newMiPi exquo d)::SUP D)

          M


      fffg(C: List D, c: cFunction, eta: List NonNegativeInteger): Matrix SUP D ==
  -- eta is the vector of degrees. We compute M with degrees eta+e_i-1, i=1..m
          z: SUP D := monomial(1, 1)
```

```
        m: NonNegativeInteger := #eta
        M: Matrix SUP D := scalarMatrix(m, 1)
        d: D := 1
        K: NonNegativeInteger := reduce(_+, eta)
        etak: Vector NonNegativeInteger := zero(m)
        r: Vector D := zero(m)
        p: Vector D := zero(m)
        Lambda: List Integer
        lambdaMax: Integer
        lambda: NonNegativeInteger

        for k in 1..K repeat
-- k = sigma+1

            for l in 1..m repeat r.l := c(k, column(M, l))

            Lambda := [eta.l-etak.l for l in 1..m | r.l ~= 0]

-- if Lambda is empty, then M, d and etak remain unchanged. Otherwise, we look
-- for the next closest para-normal point.

            (empty? Lambda) => "iterate"

            lambdaMax := reduce(max, Lambda)
            lambda := 1
            while eta.lambda-etak.lambda < lambdaMax or r.lambda = 0 repeat
                lambda := lambda + 1

-- Calculate leading coefficients

            for l in 1..m | l ~= lambda repeat
                if etak.l > 0 then
                    p.l := coefficient(M.(l, lambda),
                                       (etak.l-1)::NonNegativeInteger)
                else
                    p.l := 0

-- increase order and adjust degree constraints

            M := recurrence(M, lambda, m, r, d, z, C.k, p)

            d := r.lambda
            etak.lambda := etak.lambda + 1

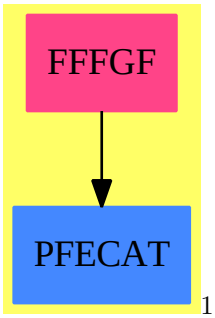
        M
```

⟨*FFFG.dotabb*⟩≡

```
"FFFG" [color="#FF4488",href="bookvol10.4.pdf#nameddest=FFFG"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"FFFG" -> "PFECAT"
```

7.47 package FFFGF FractionFreeFastGaussian-Fractions

7.48 FractionFreeFastGaussianFractions



1.00

Exports:
generalInterpolation

⟨*package FFFGF FractionFreeFastGaussianFractions*⟩≡

```
)abbrev package FFFGF FractionFreeFastGaussianFractions
++ Author: Martin Rubey
++ Description:
++ This package lifts the interpolation functions from
++ \spadtype{FractionFreeFastGaussian} to fractions.
++ The packages defined in this file provide fast fraction free rational
++ interpolation algorithms. (see FAMR2, FFFG, FFFGF, NEWTON)
FractionFreeFastGaussianFractions(D, V, VF): Exports == Implementation where
  D: Join(IntegralDomain, GcdDomain)
  V: FiniteAbelianMonoidRing(D, NonNegativeInteger)
  VF: FiniteAbelianMonoidRing(Fraction D, NonNegativeInteger)

  F     ==> Fraction D
  NNI   ==> NonNegativeInteger
  SUP   ==> SparseUnivariatePolynomial
  FFFG  ==> FractionFreeFastGaussian
  FAMR2 ==> FiniteAbelianMonoidRingFunctions2

  cFunction ==> (NNI, Vector SUP D) -> D

  CoeffAction ==> (NNI, NNI, V) -> D
-- coeffAction(k, l, f) is the coefficient of x^k in z^l f(x)

  Exports == with

    generalInterpolation: (List D, CoeffAction, Vector VF, List NNI)
```

```
                          -> Matrix SUP D
      ++ \spad{generalInterpolation(l, CA, f, eta)} performs Hermite-Pade
      ++ approximation using the given action CA of polynomials on the elements
      ++ of f. The result is guaranteed to be correct up to order
      ++ |eta|-1. Given that eta is a "normal" point, the degrees on the
      ++ diagonal are given by eta. The degrees of column i are in this case
      ++ eta + e.i - [1,1,...,1], where the degree of zero is -1.

    generalInterpolation: (List D, CoeffAction, Vector VF, NNI, NNI)
                          -> Stream Matrix SUP D
      ++ \spad{generalInterpolation(l, CA, f, sumEta, maxEta)} applies
      ++ generalInterpolation(l, CA, f, eta) for all possible eta with maximal
      ++ entry maxEta and sum of entries sumEta

  Implementation == add

    multiplyRows!(v: Vector D, M: Matrix SUP D): Matrix SUP D ==
      n := #v
      for i in 1..n repeat
        for j in 1..n repeat
          M.(i,j) := v.i*M.(i,j)

      M

    generalInterpolation(C: List D, coeffAction: CoeffAction,
                         f: Vector VF, eta: List NNI): Matrix SUP D ==
      n := #f
      g: Vector V   := new(n, 0)
      den: Vector D := new(n, 0)

      for i in 1..n repeat
        c := coefficients(f.i)
        den.i := commonDenominator(c)$CommonDenominator(D, F, List F)
        g.i :=
          map(x +-> retract(x*den.i)@D, f.i)$FAMR2(NNI, Fraction D, VF, D, V)

      M := generalInterpolation(C, coeffAction, g, eta)$FFFG(D, V)

-- The following is necessary since I'm multiplying each row with a factor, not
-- each column. Possibly I could factor out gcd den, but I'm not sure whether
-- this is efficient.

      multiplyRows!(den, M)

    generalInterpolation(C: List D, coeffAction: CoeffAction,
                         f: Vector VF, sumEta: NNI, maxEta: NNI)
```

```
                          : Stream Matrix SUP D ==

        n := #f
        g: Vector V   := new(n, 0)
        den: Vector D := new(n, 0)

        for i in 1..n repeat
          c := coefficients(f.i)
          den.i := commonDenominator(c)$CommonDenominator(D, F, List F)
          g.i :=
            map(x +-> retract(x*den.i)@D, f.i)$FAMR2(NNI, Fraction D, VF, D, V)

        c: cFunction :=
         (x,y) +-> generalCoefficient(coeffAction, g, (x-1)::NNI, y)$FFFG(D, V)


        MS: Stream Matrix SUP D
           := generalInterpolation(C, coeffAction, g, sumEta, maxEta)$FFFG(D, V)

  -- The following is necessary since I'm multiplying each row with a factor, not
  -- each column. Possibly I could factor out gcd den, but I'm not sure whether
  -- this is efficient.

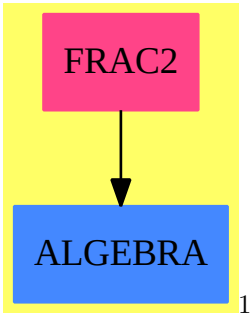
        map(x +-> multiplyRows!(den, x), MS)$Stream(Matrix SUP D)
```

⟨*FFFGF.dotabb*⟩≡

```
  "FFFGF" [color="#FF4488",href="bookvol10.4.pdf#nameddest=FFFGF"]
  "PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
  "FFFGF" -> "PFECAT"
```

7.49 package FRAC2 FractionFunctions2

7.50 FractionFunctions2



1.00

Exports:
map

⟨*package FRAC2 FractionFunctions2*⟩≡
```
)abbrev package FRAC2 FractionFunctions2
++ Author:
++ Date Created:
++ Date Last Updated:
++ Basic Functions:
++ Related Constructors:
++ Also See:
++ AMS Classifications:
++ Keywords:
++ References:
++ Description: This package extends a map between integral domains to
++ a map between Fractions over those domains by applying the map to the
++ numerators and denominators.
FractionFunctions2(A, B): Exports == Impl where
  A, B: IntegralDomain

  R ==> Fraction A
  S ==> Fraction B

  Exports ==> with
    map: (A -> B, R) -> S
      ++ map(func,frac) applies the function func to the numerator
      ++ and denominator of the fraction frac.

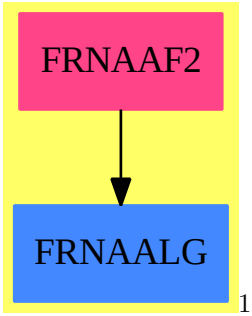
  Impl ==> add
    map(f, r) == map(f, r)$QuotientFieldCategoryFunctions2(A, B, R, S)
```

⟨*FRAC2.dotabb*⟩≡

```
"FRAC2" [color="#FF4488",href="bookvol10.4.pdf#nameddest=FRAC2"]
"ALGEBRA" [color="#4488FF",href="bookvol10.2.pdf#nameddest=ALGEBRA"]
"FRAC2" -> "ALGEBRA"
```

7.51 package FRNAAF2 FramedNonAssociativeAlgebraFunctions2

7.52 FramedNonAssociativeAlgebraFunctions2



1.00

Exports:
map

⟨*package FRNAAF2 FramedNonAssociativeAlgebraFunctions2*⟩≡

```
)abbrev package FRNAAF2 FramedNonAssociativeAlgebraFunctions2
++ Author: Johannes Grabmeier
++ Date Created: 28 February 1992
++ Date Last Updated: 28 February 1992
++ Basic Operations: map
++ Related Constructors:
++ Also See:
++ AMS Classifications:
++ Keywords: non-associative algebra
++ References:
++ Description:
++  FramedNonAssociativeAlgebraFunctions2 implements functions between
++  two framed non associative algebra domains defined over different rings.
++  The function map is used to coerce between algebras over different
++  domains having the same structural constants.

FramedNonAssociativeAlgebraFunctions2(AR,R,AS,S) : Exports ==
  Implementation where
    R  : CommutativeRing
    S  : CommutativeRing
    AR : FramedNonAssociativeAlgebra R
    AS : FramedNonAssociativeAlgebra S
    V ==> Vector
    Exports ==> with
      map:     (R -> S, AR) -> AS
        ++ map(f,u) maps f onto the coordinates of u to get an element
```

```
      ++ in \spad{AS} via identification of the basis of \spad{AR}
      ++ as beginning part of the basis of \spad{AS}.
    Implementation ==> add
     map(fn : R -> S, u : AR): AS ==
      rank()$AR > rank()$AS => error("map: ranks of algebras do not fit")
      vr : V R := coordinates u
      vs : V S := map(fn,vr)$VectorFunctions2(R,S)
```

This line used to read:

```
rank()$AR = rank()$AR => represents(vs)$AS
```

but the test is clearly always true and cannot be what was intended. Gregory Vanuxem supplied the fix below.

⟨*package FRNAAF2 FramedNonAssociativeAlgebraFunctions2*⟩+≡

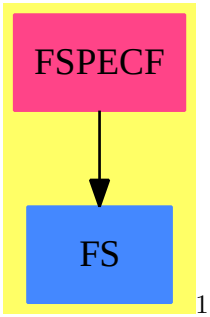
```
      rank()$AR = rank()$AS => represents(vs)$AS
      ba := basis()$AS
      represents(vs,[ba.i for i in 1..rank()$AR])
```

⟨*FRNAAF2.dotabb*⟩≡

```
"FRNAAF2" [color="#FF4488",href="bookvol10.4.pdf#nameddest=FRNAAF2"]
"FRNAALG" [color="#4488FF",href="bookvol10.2.pdf#nameddest=FRNAALG"]
"FRNAAF2" -> "FRNAALG"
```

7.53 package FSPECF FunctionalSpecialFunction

7.54 FunctionalSpecialFunction



1.00

Exports:

Beta	Gamma	abs	airyAi	airyBi
belong?	besselI	besselJ	besselK	besselY
digamma	iiAiryAi	iiAiryBi	iiBesselI	iiBesselJ
iiBesselK	iiBesselY	iiBeta	iiGamma	iiabs
iidigamma	iipolygamma	operator	polygamma	

⟨*package FSPECF FunctionalSpecialFunction*⟩≡

```
)abbrev package FSPECF FunctionalSpecialFunction
++ Provides the special functions
++ Author: Manuel Bronstein
++ Date Created: 18 Apr 1989
++ Date Last Updated: 4 October 1993
++ Description: Provides some special functions over an integral domain.
++ Keywords: special, function.
FunctionalSpecialFunction(R, F): Exports == Implementation where
  R: Join(OrderedSet, IntegralDomain)
  F: FunctionSpace R

  OP  ==> BasicOperator
  K   ==> Kernel F
  SE  ==> Symbol
  SPECIALDIFF  ==> "%specialDiff"

  Exports ==> with
    belong? : OP -> Boolean
      ++ belong?(op) is true if op is a special function operator;
    operator: OP -> OP
      ++ operator(op) returns a copy of op with the domain-dependent
      ++ properties appropriate for F;
      ++ error if op is not a special function operator
```

```
abs     : F -> F
  ++ abs(f) returns the absolute value operator applied to f
Gamma   : F -> F
  ++ Gamma(f) returns the formal Gamma function applied to f
Gamma   : (F,F) -> F
  ++ Gamma(a,x) returns the incomplete Gamma function applied to a and x
Beta:      (F,F) -> F
  ++ Beta(x,y) returns the beta function applied to x and y
digamma:   F->F
  ++ digamma(x) returns the digamma function applied to x
polygamma: (F,F) ->F
  ++ polygamma(x,y) returns the polygamma function applied to x and y
besselJ:   (F,F) -> F
  ++ besselJ(x,y) returns the besselj function applied to x and y
besselY:   (F,F) -> F
  ++ besselY(x,y) returns the bessely function applied to x and y
besselI:   (F,F) -> F
  ++ besselI(x,y) returns the besseli function applied to x and y
besselK:   (F,F) -> F
  ++ besselK(x,y) returns the besselk function applied to x and y
airyAi:    F -> F
  ++ airyAi(x) returns the airyai function applied to x
airyBi:    F -> F
  ++ airyBi(x) returns the airybi function applied to x
```

In case we want to have more special function operators here, do not forget to add them to the list `specop` in `CommonOperators`. Otherwise they will not have the 'special' attribute and will not be recognised here. One effect could be that

```
myNewSpecOp(1::Expression Integer)::Expression DoubleFloat
```

might not re-evaluate the operator.

⟨*package FSPECF FunctionalSpecialFunction*⟩+≡

```
      iiGamma : F -> F
        ++ iiGamma(x) should be local but conditional;
      iiabs     : F -> F
        ++ iiabs(x) should be local but conditional;
      iiBeta     : List F -> F
        ++ iiGamma(x) should be local but conditional;
      iidigamma  : F -> F
        ++ iidigamma(x) should be local but conditional;
      iipolygamma: List F -> F
        ++ iipolygamma(x) should be local but conditional;
      iiBesselJ  : List F -> F
        ++ iiBesselJ(x) should be local but conditional;
      iiBesselY  : List F -> F
        ++ iiBesselY(x) should be local but conditional;
      iiBesselI  : List F -> F
        ++ iiBesselI(x) should be local but conditional;
      iiBesselK  : List F -> F
        ++ iiBesselK(x) should be local but conditional;
      iiAiryAi   : F -> F
        ++ iiAiryAi(x) should be local but conditional;
      iiAiryBi   : F -> F
        ++ iiAiryBi(x) should be local but conditional;

    Implementation ==> add
      iabs      : F -> F
      iGamma    : F -> F
      iBeta     : (F, F) -> F
      idigamma  : F -> F
      iiipolygamma: (F, F) -> F
      iiiBesselJ  : (F, F) -> F
      iiiBesselY  : (F, F) -> F
      iiiBesselI  : (F, F) -> F
      iiiBesselK  : (F, F) -> F
      iAiryAi   : F -> F
      iAiryBi   : F -> F

      opabs       := operator("abs"::Symbol)$CommonOperators
      opGamma     := operator("Gamma"::Symbol)$CommonOperators
```

```
    opGamma2    := operator("Gamma2"::Symbol)$CommonOperators
    opBeta      := operator("Beta"::Symbol)$CommonOperators
    opdigamma   := operator("digamma"::Symbol)$CommonOperators
    oppolygamma := operator("polygamma"::Symbol)$CommonOperators
    opBesselJ   := operator("besselJ"::Symbol)$CommonOperators
    opBesselY   := operator("besselY"::Symbol)$CommonOperators
    opBesselI   := operator("besselI"::Symbol)$CommonOperators
    opBesselK   := operator("besselK"::Symbol)$CommonOperators
    opAiryAi    := operator("airyAi"::Symbol)$CommonOperators
    opAiryBi    := operator("airyBi"::Symbol)$CommonOperators

    abs x          == opabs x
    Gamma(x)       == opGamma(x)
    Gamma(a,x)     == opGamma2(a,x)
    Beta(x,y)      == opBeta(x,y)
    digamma x      == opdigamma(x)
    polygamma(k,x)== oppolygamma(k,x)
    besselJ(a,x)  == opBesselJ(a,x)
    besselY(a,x)  == opBesselY(a,x)
    besselI(a,x)  == opBesselI(a,x)
    besselK(a,x)  == opBesselK(a,x)
    airyAi(x)      == opAiryAi(x)
    airyBi(x)      == opAiryBi(x)

    belong? op == has?(op, "special")

    operator op ==
      is?(op, "abs"::Symbol)      => opabs
      is?(op, "Gamma"::Symbol)    => opGamma
      is?(op, "Gamma2"::Symbol)   => opGamma2
      is?(op, "Beta"::Symbol)     => opBeta
      is?(op, "digamma"::Symbol)  => opdigamma
      is?(op, "polygamma"::Symbol)=> oppolygamma
      is?(op, "besselJ"::Symbol)  => opBesselJ
      is?(op, "besselY"::Symbol)  => opBesselY
      is?(op, "besselI"::Symbol)  => opBesselI
      is?(op, "besselK"::Symbol)  => opBesselK
      is?(op, "airyAi"::Symbol)   => opAiryAi
      is?(op, "airyBi"::Symbol)   => opAiryBi

      error "Not a special operator"

    -- Could put more unconditional special rules for other functions here
    iGamma x ==
--      one? x => x
      (x = 1) => x
```

```
  kernel(opGamma, x)

iabs x ==
  zero? x => 0
  is?(x, opabs) => x
  x < 0 => kernel(opabs, -x)
  kernel(opabs, x)

iBeta(x, y) == kernel(opBeta, [x, y])
idigamma x == kernel(opdigamma, x)
iiipolygamma(n, x) == kernel(oppolygamma, [n, x])
iiiBesselJ(x, y) == kernel(opBesselJ, [x, y])
iiiBesselY(x, y) == kernel(opBesselY, [x, y])
iiiBesselI(x, y) == kernel(opBesselI, [x, y])
iiiBesselK(x, y) == kernel(opBesselK, [x, y])
iAiryAi x == kernel(opAiryAi, x)
iAiryBi x == kernel(opAiryBi, x)


-- Could put more conditional special rules for other functions here

if R has abs : R -> R then
  iiabs x ==
    (r := retractIfCan(x)@Union(Fraction Polynomial R, "failed"))
      case "failed" => iabs x
    f := r::Fraction Polynomial R
    (a := retractIfCan(numer f)@Union(R, "failed")) case "failed" or
      (b := retractIfCan(denom f)@Union(R,"failed")) case "failed" => iabs x
    abs(a::R)::F / abs(b::R)::F

else iiabs x == iabs x

if R has SpecialFunctionCategory then
  iiGamma x ==
    (r:=retractIfCan(x)@Union(R,"failed")) case "failed" => iGamma x
    Gamma(r::R)::F

  iiBeta l ==
    (r:=retractIfCan(first l)@Union(R,"failed")) case "failed" or _
    (s:=retractIfCan(second l)@Union(R,"failed")) case "failed" _
        => iBeta(first l, second l)
    Beta(r::R, s::R)::F

  iidigamma x ==
    (r:=retractIfCan(x)@Union(R,"failed")) case "failed" => idigamma x
    digamma(r::R)::F
```

```
    iipolygamma l ==
      (s:=retractIfCan(first l)@Union(R,"failed")) case "failed" or _
      (r:=retractIfCan(second l)@Union(R,"failed")) case "failed" _
          => iiipolygamma(first l, second l)
      polygamma(s::R, r::R)::F

    iiBesselJ l ==
      (r:=retractIfCan(first l)@Union(R,"failed")) case "failed" or _
      (s:=retractIfCan(second l)@Union(R,"failed")) case "failed" _
          => iiiBesselJ(first l, second l)
      besselJ(r::R, s::R)::F

    iiBesselY l ==
      (r:=retractIfCan(first l)@Union(R,"failed")) case "failed" or _
      (s:=retractIfCan(second l)@Union(R,"failed")) case "failed" _
          => iiiBesselY(first l, second l)
      besselY(r::R, s::R)::F

    iiBesselI l ==
      (r:=retractIfCan(first l)@Union(R,"failed")) case "failed" or _
      (s:=retractIfCan(second l)@Union(R,"failed")) case "failed" _
          => iiiBesselI(first l, second l)
      besselI(r::R, s::R)::F

    iiBesselK l ==
      (r:=retractIfCan(first l)@Union(R,"failed")) case "failed" or _
      (s:=retractIfCan(second l)@Union(R,"failed")) case "failed" _
          => iiiBesselK(first l, second l)
      besselK(r::R, s::R)::F

    iiAiryAi x ==
      (r:=retractIfCan(x)@Union(R,"failed")) case "failed" => iAiryAi x
      airyAi(r::R)::F

    iiAiryBi x ==
      (r:=retractIfCan(x)@Union(R,"failed")) case "failed" => iAiryBi x
      airyBi(r::R)::F

  else
    if R has RetractableTo Integer then
      iiGamma x ==
        (r := retractIfCan(x)@Union(Integer, "failed")) case Integer
          and (r::Integer >= 1) => factorial(r::Integer - 1)::F
        iGamma x
    else
```

```
      iiGamma x == iGamma x

    iiBeta l == iBeta(first l, second l)
    iidigamma x == idigamma x
    iipolygamma l == iiipolygamma(first l, second l)
    iiBesselJ l == iiiBesselJ(first l, second l)
    iiBesselY l == iiiBesselY(first l, second l)
    iiBesselI l == iiiBesselI(first l, second l)
    iiBesselK l == iiiBesselK(first l, second l)
    iiAiryAi x == iAiryAi x
    iiAiryBi x == iAiryBi x

  -- Default behaviour is to build a kernel
  evaluate(opGamma, iiGamma)$BasicOperatorFunctions1(F)
  evaluate(opabs, iiabs)$BasicOperatorFunctions1(F)
--  evaluate(opGamma2    ,iiGamma2   )$BasicOperatorFunctions1(F)
  evaluate(opBeta      ,iiBeta     )$BasicOperatorFunctions1(F)
  evaluate(opdigamma   ,iidigamma  )$BasicOperatorFunctions1(F)
  evaluate(oppolygamma ,iipolygamma)$BasicOperatorFunctions1(F)
  evaluate(opBesselJ   ,iiBesselJ  )$BasicOperatorFunctions1(F)
  evaluate(opBesselY   ,iiBesselY  )$BasicOperatorFunctions1(F)
  evaluate(opBesselI   ,iiBesselI  )$BasicOperatorFunctions1(F)
  evaluate(opBesselK   ,iiBesselK  )$BasicOperatorFunctions1(F)
  evaluate(opAiryAi    ,iiAiryAi   )$BasicOperatorFunctions1(F)
  evaluate(opAiryBi    ,iiAiryBi   )$BasicOperatorFunctions1(F)
```

7.54.1 differentiation of special functions

In the following we define the symbolic derivatives of the special functions we provide. The formulas we use for the Bessel functions can be found in Milton Abramowitz and Irene A. Stegun, eds. (1965). Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables. New York: Dover. ISBN 0-486-61272-4, Equations 9.1.27 and 9.6.26. Up to `patch--50` the formula for K missed the minus sign. (Issue #355)

We do not attempt to provide formulas for the derivative with respect to the first argument currently. Instead, we leave such derivatives unevaluated.

⟨*package FSPECF FunctionalSpecialFunction*⟩+≡

```
  import Fraction Integer
  ahalf:  F    := recip(2::F)::F
  athird: F    := recip(2::F)::F
  twothirds: F := 2*recip(3::F)::F
```

We need to get hold of the differentiation operator as modified by `FunctionSpace`. Otherwise, for example, display will be ugly. We accomplish that by differentiating an operator, which will certainly result in a single kernel only.

⟨*package FSPECF FunctionalSpecialFunction*⟩+≡

```
dummyArg: SE := new()$SE
opdiff := operator first kernels D((operator(new()$SE)$BasicOperator)
                                        (dummyArg::F), dummyArg)
```

The differentiation operator `opdiff` takes three arguments corresponding to

$$F_{,i}(a_1, a_2, \ldots, a_n) :$$

1. $F(a_1, ..., dm, ...a_n)$, where the i^{th} argument is a dummy variable,
2. dm, the dummy variable, and
3. a_i, the point at which the differential is evaluated.

In the following, it seems to be safe to use the same dummy variable troughout. At least, this is done also in `FunctionSpace`, and did not cause problems.

The operation `symbolicGrad` returns the first component of the gradient of `op l`.

⟨*package FSPECF FunctionalSpecialFunction*⟩+≡

```
dm := new()$SE :: F

iBesselJ(l: List F, t: SE): F ==
    n := first l; x := second l
    differentiate(n, t)*kernel(opdiff, [opBesselJ [dm, x], dm, n])
      + differentiate(x, t) * ahalf * (besselJ (n-1,x) - besselJ (n+1,x))

iBesselY(l: List F, t: SE): F ==
    n := first l; x := second l
    differentiate(n, t)*kernel(opdiff, [opBesselY [dm, x], dm, n])
      + differentiate(x, t) * ahalf * (besselY (n-1,x) - besselY (n+1,x))

iBesselI(l: List F, t: SE): F ==
    n := first l; x := second l
    differentiate(n, t)*kernel(opdiff, [opBesselI [dm, x], dm, n])
      + differentiate(x, t)* ahalf * (besselI (n-1,x) + besselI (n+1,x))

iBesselK(l: List F, t: SE): F ==
    n := first l; x := second l
    differentiate(n, t)*kernel(opdiff, [opBesselK [dm, x], dm, n])
      - differentiate(x, t)* ahalf * (besselK (n-1,x) + besselK (n+1,x))
```

For the moment we throw an error if we try to differentiate `polygamma` with respect to the first argument.

⟨*package FSPECF FunctionalSpecialFunction*⟩+≡

```
ipolygamma(l: List F, x: SE): F ==
    member?(x, variables first l) =>
        error "cannot differentiate polygamma with respect to the first argum
    n := first l; y := second l
    differentiate(y, x)*polygamma(n+1, y)
iBetaGrad1(l: List F): F ==
    x := first l; y := second l
    Beta(x,y)*(digamma x - digamma(x+y))
iBetaGrad2(l: List F): F ==
    x := first l; y := second l
    Beta(x,y)*(digamma y - digamma(x+y))

if F has ElementaryFunctionCategory then
  iGamma2(l: List F, t: SE): F ==
    a := first l; x := second l
    differentiate(a, t)*kernel(opdiff, [opGamma2 [dm, x], dm, a])
      - differentiate(x, t)* x ** (a - 1) * exp(-x)
  setProperty(opGamma2, SPECIALDIFF, iGamma2@((List F, SE)->F)
                                          pretend None)
```

Finally, we tell Axiom to use these functions for differentiation. Note that up to `patch--50`, the properties for the Bessel functions were set using `derivative(oppolygamma, [lzero, ipolygammaGrad])`, where `lzero` returned zero always. Trying to replace `lzero` by a function that returns the first component of the gradient failed, it resulted in an infinite loop for `integrate(D(besselJ(a,x),a),a)`.

⟨*package FSPECF FunctionalSpecialFunction*⟩+≡

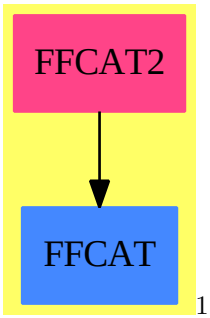
```
derivative(opabs,       (x:F):F +-> abs(x) * inv(x))
derivative(opGamma,     (x:F):F +-> digamma x * Gamma x)
derivative(opBeta,      [iBetaGrad1, iBetaGrad2])
derivative(opdigamma,   (x:F):F +-> polygamma(1, x))
setProperty(oppolygamma, SPECIALDIFF, ipolygamma@((List F, SE)->F)
                                            pretend None)
setProperty(opBesselJ, SPECIALDIFF, iBesselJ@((List F, SE)->F)
                                          pretend None)
setProperty(opBesselY, SPECIALDIFF, iBesselY@((List F, SE)->F)
                                          pretend None)
setProperty(opBesselI, SPECIALDIFF, iBesselI@((List F, SE)->F)
                                          pretend None)
setProperty(opBesselK, SPECIALDIFF, iBesselK@((List F, SE)->F)
                                          pretend None)
```

⟨*FSPECF.dotabb*⟩≡

```
"FSPECF" [color="#FF4488",href="bookvol10.4.pdf#nameddest=FSPECF"]
"FS" [color="#4488FF",href="bookvol10.2.pdf#nameddest=FS"]
"FSPECF" -> "FS"
```

7.55 package FFCAT2 FunctionFieldCategoryFunctions2

7.56 FunctionFieldCategoryFunctions2



1.00

Exports:
map

⟨*package FFCAT2 FunctionFieldCategoryFunctions2*⟩≡

```
)abbrev package FFCAT2 FunctionFieldCategoryFunctions2
++ Lifts a map from rings to function fields over them
++ Author: Manuel Bronstein
++ Date Created: May 1988
++ Date Last Updated: 26 Jul 1988
++ Description: Lifts a map from rings to function fields over them.
FunctionFieldCategoryFunctions2(R1, UP1, UPUP1, F1, R2, UP2, UPUP2, F2):
 Exports == Implementation where
  R1   : UniqueFactorizationDomain
  UP1  : UnivariatePolynomialCategory R1
  UPUP1: UnivariatePolynomialCategory Fraction UP1
  F1   : FunctionFieldCategory(R1, UP1, UPUP1)
  R2   : UniqueFactorizationDomain
  UP2  : UnivariatePolynomialCategory R2
  UPUP2: UnivariatePolynomialCategory Fraction UP2
  F2   : FunctionFieldCategory(R2, UP2, UPUP2)

  Exports ==> with
    map: (R1 -> R2, F1) -> F2
      ++ map(f, p) lifts f to F1 and applies it to p.

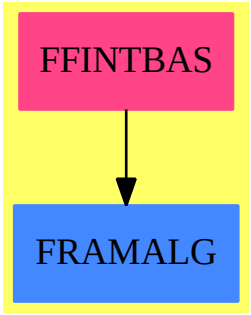
  Implementation ==> add
    map(f, f1) ==
      reduce(map(f, lift f1)$MultipleMap(R1, UP1, UPUP1, R2, UP2, UPUP2))
```

⟨*FFCAT2.dotabb*⟩≡

```
"FFCAT2" [color="#FF4488",href="bookvol10.4.pdf#nameddest=FFCAT2"]
"FFCAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=FFCAT"]
"FFCAT2" -> "FFCAT"
```

7.57 package FFINTBAS FunctionFieldIntegralBasis

7.58 FunctionFieldIntegralBasis



1.00

Exports:
integralBasis localIntegralBasis

⟨*package FFINTBAS FunctionFieldIntegralBasis*⟩≡

```
)abbrev package FFINTBAS FunctionFieldIntegralBasis
++ Integral bases for function fields of dimension one
++ Author: Victor Miller
++ Date Created: 9 April 1990
++ Date Last Updated: 20 September 1994
++ Keywords:
++ Examples:
++ References:
++ Description:
++ In this package R is a Euclidean domain and F is a framed algebra
++ over R.  The package provides functions to compute the integral
++ closure of R in the quotient field of F.  It is assumed that
++ \spad{char(R/P) = char(R)} for any prime P of R.  A typical instance of
++ this is when \spad{R = K[x]} and F is a function field over R.


FunctionFieldIntegralBasis(R,UP,F): Exports == Implementation where
  R  : EuclideanDomain with
       squareFree: $ -> Factored $
               ++ squareFree(x) returns a square-free factorisation of x
  UP : UnivariatePolynomialCategory R
  F  : FramedAlgebra(R,UP)

  I   ==> Integer
  Mat ==> Matrix R
  NNI ==> NonNegativeInteger
```

```
Exports ==> with
  integralBasis : () -> Record(basis: Mat, basisDen: R, basisInv:Mat)
    ++ \spad{integralBasis()} returns a record
    ++ \spad{[basis,basisDen,basisInv]} containing information regarding
    ++ the integral closure of R in the quotient field of F, where
    ++ F is a framed algebra with R-module basis \spad{w1,w2,...,wn}.
    ++ If \spad{basis} is the matrix \spad{(aij, i = 1..n, j = 1..n)}, then
    ++ the \spad{i}th element of the integral basis is
    ++ \spad{vi = (1/basisDen) * sum(aij * wj, j = 1..n)}, i.e. the
    ++ \spad{i}th row of \spad{basis} contains the coordinates of the
    ++ \spad{i}th basis vector.  Similarly, the \spad{i}th row of the
    ++ matrix \spad{basisInv} contains the coordinates of \spad{wi} with
    ++ respect to the basis \spad{v1,...,vn}: if \spad{basisInv} is the
    ++ matrix \spad{(bij, i = 1..n, j = 1..n)}, then
    ++ \spad{wi = sum(bij * vj, j = 1..n)}.
  localIntegralBasis : R -> Record(basis: Mat, basisDen: R, basisInv:Mat)
    ++ \spad{integralBasis(p)} returns a record
    ++ \spad{[basis,basisDen,basisInv]} containing information regarding
    ++ the local integral closure of R at the prime \spad{p} in the quotient
    ++ field of F, where F is a framed algebra with R-module basis
    ++ \spad{w1,w2,...,wn}.
    ++ If \spad{basis} is the matrix \spad{(aij, i = 1..n, j = 1..n)}, then
    ++ the \spad{i}th element of the local integral basis is
    ++ \spad{vi = (1/basisDen) * sum(aij * wj, j = 1..n)}, i.e. the
    ++ \spad{i}th row of \spad{basis} contains the coordinates of the
    ++ \spad{i}th basis vector.  Similarly, the \spad{i}th row of the
    ++ matrix \spad{basisInv} contains the coordinates of \spad{wi} with
    ++ respect to the basis \spad{v1,...,vn}: if \spad{basisInv} is the
    ++ matrix \spad{(bij, i = 1..n, j = 1..n)}, then
    ++ \spad{wi = sum(bij * vj, j = 1..n)}.

Implementation ==> add
  import IntegralBasisTools(R, UP, F)
  import ModularHermitianRowReduction(R)
  import TriangularMatrixOperations(R, Vector R, Vector R, Matrix R)

  squaredFactors: R -> R
  squaredFactors px ==
         */[(if ffe.exponent > 1 then ffe.factor else 1$R)
              for ffe in factors squareFree px]

  iIntegralBasis: (Mat,R,R) -> Record(basis: Mat, basisDen: R, basisInv:Mat)
  iIntegralBasis(tfm,disc,sing) ==
    -- tfm = trace matrix of current order
    n := rank()$F; tfm0 := copy tfm; disc0 := disc
```

```
    rb := scalarMatrix(n, 1); rbinv := scalarMatrix(n, 1)
    -- rb    = basis matrix of current order
    -- rbinv = inverse basis matrix of current order
    -- these are wrt the original basis for F
    rbden : R := 1; index : R := 1; oldIndex : R := 1
    -- rbden = denominator for current basis matrix
    -- index = index of original order in current order
    not sizeLess?(1, sing) => [rb, rbden, rbinv]
    repeat
      -- compute the p-radical
      idinv := transpose squareTop rowEchelon(tfm, sing)
      -- [u1,..,un] are the coordinates of an element of the p-radical
      -- iff [u1,..,un] * idinv is in sing * R^n
      id := rowEchelon LowTriBddDenomInv(idinv, sing)
      -- id = basis matrix of the p-radical
      idinv := UpTriBddDenomInv(id, sing)
      -- id * idinv = sing * identity
      -- no need to check for inseparability in this case
      rbinv := idealiser(id * rb, rbinv * idinv, sing * rbden)
      index := diagonalProduct rbinv
      rb := rowEchelon LowTriBddDenomInv(rbinv, rbden * sing)
      g := matrixGcd(rb,sing,n)
      if sizeLess?(1,g) then rb := (rb exquo g) :: Mat
      rbden := rbden * (sing quo g)
      rbinv := UpTriBddDenomInv(rb, rbden)
      disc := disc0 quo (index * index)
      indexChange := index quo oldIndex; oldIndex := index
      sing := gcd(indexChange, squaredFactors disc)
      not sizeLess?(1, sing) => return [rb, rbden, rbinv]
      tfm := ((rb * tfm0 * transpose rb) exquo (rbden * rbden)) :: Mat

  integralBasis() ==
    n := rank()$F; p := characteristic()$F
    (not zero? p) and (n >= p) =>
      error "integralBasis: possible wild ramification"
    tfm := traceMatrix()$F; disc := determinant tfm
    sing := squaredFactors disc    -- singularities of relative Spec
    iIntegralBasis(tfm,disc,sing)

  localIntegralBasis prime ==
    n := rank()$F; p := characteristic()$F
    (not zero? p) and (n >= p) =>
      error "integralBasis: possible wild ramification"
    tfm := traceMatrix()$F; disc := determinant tfm
    (disc exquo (prime * prime)) case "failed" =>
      [scalarMatrix(n,1),1,scalarMatrix(n,1)]
```

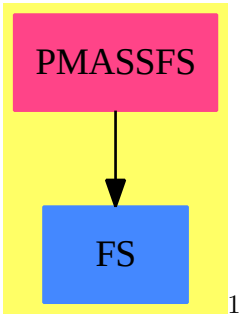
```
iIntegralBasis(tfm,disc,prime)
```

⟨*FFINTBAS.dotabb*⟩≡

```
"FFINTBAS" [color="#FF4488",href="bookvol10.4.pdf#nameddest=FFINTBAS"]
"FRAMALG" [color="#4488FF",href="bookvol10.2.pdf#nameddest=FRAMALG"]
"FFINTBAS" -> "FRAMALG"
```

7.59 package PMASSFS FunctionSpaceAssertions

7.60 FunctionSpaceAssertions



1.00

Exports:
assert constant multiple optional

⟨*package PMASSFS FunctionSpaceAssertions*⟩≡
```
)abbrev package PMASSFS FunctionSpaceAssertions
++ Assertions for pattern-matching
++ Author: Manuel Bronstein
++ Description: Attaching assertions to symbols for pattern matching;
++ Date Created: 21 Mar 1989
++ Date Last Updated: 23 May 1990
++ Keywords: pattern, matching.
FunctionSpaceAssertions(R, F): Exports == Implementation where
  R: OrderedSet
  F: FunctionSpace R

  K  ==> Kernel F
  PMOPT   ==> "%pmoptional"
  PMMULT  ==> "%pmmultiple"
  PMCONST ==> "%pmconstant"

  Exports ==> with
    assert  : (F, String) -> F
      ++ assert(x, s) makes the assertion s about x.
      ++ Error: if x is not a symbol.
    constant: F -> F
      ++ constant(x) tells the pattern matcher that x should
      ++ match only the symbol 'x and no other quantity.
      ++ Error: if x is not a symbol.
    optional: F -> F
      ++ optional(x) tells the pattern matcher that x can match
      ++ an identity (0 in a sum, 1 in a product or exponentiation).
```

```
      ++ Error: if x is not a symbol.
    multiple: F -> F
      ++ multiple(x) tells the pattern matcher that x should
      ++ preferably match a multi-term quantity in a sum or product.
      ++ For matching on lists, multiple(x) tells the pattern matcher
      ++ that x should match a list instead of an element of a list.
      ++ Error: if x is not a symbol.

  Implementation ==> add
    ass  : (K, String) -> F
    asst : (K, String) -> F
    mkk  : BasicOperator -> F

    mkk op == kernel(op, empty()$List(F))

    ass(k, s) ==
      has?(op := operator k, s) => k::F
      mkk assert(copy op, s)

    asst(k, s) ==
      has?(op := operator k, s) => k::F
      mkk assert(op, s)

    assert(x, s) ==
      retractIfCan(x)@Union(Symbol, "failed") case Symbol =>
        asst(retract(x)@K, s)
      error "assert must be applied to symbols only"

    constant x ==
      retractIfCan(x)@Union(Symbol, "failed") case Symbol =>
        ass(retract(x)@K, PMCONST)
      error "constant must be applied to symbols only"

    optional x ==
      retractIfCan(x)@Union(Symbol, "failed") case Symbol =>
        ass(retract(x)@K, PMOPT)
      error "optional must be applied to symbols only"

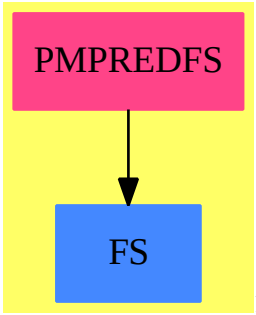
    multiple x ==
      retractIfCan(x)@Union(Symbol, "failed") case Symbol =>
        ass(retract(x)@K, PMMULT)
      error "multiple must be applied to symbols only"
```

⟨*PMASSFS.dotabb*⟩≡

```
"PMASSFS" [color="#FF4488",href="bookvol10.4.pdf#nameddest=PMASSFS"]
"FS" [color="#4488FF",href="bookvol10.2.pdf#nameddest=FS"]
"PMASSFS" -> "FS"
```

7.61 package PMPREDFS FunctionSpaceAttachPredicates

7.62 FunctionSpaceAttachPredicates



1.00

Exports:
suchThat

⟨*package PMPREDFS FunctionSpaceAttachPredicates*⟩≡
```
)abbrev package PMPREDFS FunctionSpaceAttachPredicates
++ Predicates for pattern-matching.
++ Author: Manuel Bronstein
++ Description: Attaching predicates to symbols for pattern matching.
++ Date Created: 21 Mar 1989
++ Date Last Updated: 23 May 1990
++ Keywords: pattern, matching.
FunctionSpaceAttachPredicates(R, F, D): Exports == Implementation where
  R: OrderedSet
  F: FunctionSpace R
  D: Type

  K  ==> Kernel F
  PMPRED  ==> "%pmpredicate"

  Exports ==> with
    suchThat: (F, D -> Boolean) -> F
      ++ suchThat(x, foo) attaches the predicate foo to x;
      ++ error if x is not a symbol.
    suchThat: (F, List(D -> Boolean)) -> F
      ++ suchThat(x, [f1, f2, ..., fn]) attaches the predicate
      ++ f1 and f2 and ... and fn to x.
      ++ Error: if x is not a symbol.

  Implementation ==> add
    import AnyFunctions1(D -> Boolean)
```

```
st   : (K, List Any) -> F
preds: K -> List Any
mkk  : BasicOperator -> F

suchThat(p:F, f:D -> Boolean) == suchThat(p, [f])
mkk op                        == kernel(op, empty()$List(F))

preds k ==
  (u := property(operator k, PMPRED)) case "failed" => empty()
  (u::None) pretend List(Any)

st(k, l) ==
  mkk assert(setProperty(copy operator k, PMPRED,
             concat(preds k, l) pretend None), string(new()$Symbol))

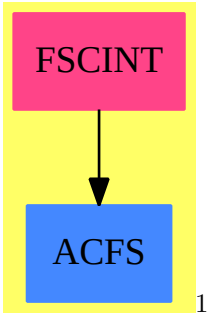
suchThat(p:F, l:List(D -> Boolean)) ==
  retractIfCan(p)@Union(Symbol, "failed") case Symbol =>
    st(retract(p)@K, [f::Any for f in l])
  error "suchThat must be applied to symbols only"
```

⟨*PMPREDFS.dotabb*⟩≡

```
"PMPREDFS" [color="#FF4488",href="bookvol10.4.pdf#nameddest=PMPREDFS"]
"FS" [color="#4488FF",href="bookvol10.2.pdf#nameddest=FS"]
"PMPREDFS" -> "FS"
```

7.63 package FSCINT FunctionSpaceComplex-Integration

7.64 FunctionSpaceComplexIntegration



1.00

Exports:
complexIntegrate internalIntegrate internalIntegrate0

⟨*package FSCINT FunctionSpaceComplexIntegration*⟩≡

```
)abbrev package FSCINT FunctionSpaceComplexIntegration
++ Top-level complex function integration
++ Author: Manuel Bronstein
++ Date Created: 4 February 1988
++ Date Last Updated: 11 June 1993
++ Description:
++   \spadtype{FunctionSpaceComplexIntegration} provides functions for the
++   indefinite integration of complex-valued functions.
++ Keywords: function, integration.
FunctionSpaceComplexIntegration(R, F): Exports == Implementation where
  R : Join(EuclideanDomain, OrderedSet, CharacteristicZero,
           RetractableTo Integer, LinearlyExplicitRingOver Integer)
  F : Join(TranscendentalFunctionCategory,
           AlgebraicallyClosedFunctionSpace R)

  SE  ==> Symbol
  G   ==> Complex R
  FG  ==> Expression G
  IR  ==> IntegrationResult F

  Exports ==> with
    internalIntegrate : (F, SE) -> IR
        ++ internalIntegrate(f, x) returns the integral of \spad{f(x)dx}
        ++ where x is viewed as a complex variable.
    internalIntegrate0: (F, SE) -> IR
        ++ internalIntegrate0 should be a local function, but is conditional.
```

```
    complexIntegrate  : (F, SE) -> F
        ++ complexIntegrate(f, x) returns the integral of \spad{f(x)dx}
        ++ where x is viewed as a complex variable.

  Implementation ==> add
    import IntegrationTools(R, F)
    import ElementaryIntegration(R, F)
    import ElementaryIntegration(G, FG)
    import AlgebraicManipulations(R, F)
    import AlgebraicManipulations(G, FG)
    import TrigonometricManipulations(R, F)
    import IntegrationResultToFunction(R, F)
    import IntegrationResultFunctions2(FG, F)
    import ElementaryFunctionStructurePackage(R, F)
    import ElementaryFunctionStructurePackage(G, FG)
    import InnerTrigonometricManipulations(R, F, FG)

    K2KG: Kernel F -> Kernel FG

    K2KG k                 == retract(tan F2FG first argument k)@Kernel(FG)

    complexIntegrate(f, x) ==
      removeConstantTerm(complexExpand internalIntegrate(f, x), x)

    if R has Join(ConvertibleTo Pattern Integer, PatternMatchable Integer)
      and F has Join(LiouvillianFunctionCategory, RetractableTo SE) then
        import PatternMatchIntegration(R, F)
        internalIntegrate0(f, x) ==
          intPatternMatch(f, x, lfintegrate, pmComplexintegrate)

    else internalIntegrate0(f, x) == lfintegrate(f, x)

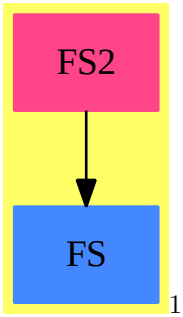
    internalIntegrate(f, x) ==
      f := distribute(f, x::F)
      any?(x1+->has?(operator x1, "rtrig"),
       [k for k in tower(g := realElementary(f, x))
        | member?(x, variables(k::F))]$List(Kernel F))$List(Kernel F) =>
          h := trigs2explogs(F2FG g, [K2KG k for k in tower f
                        | is?(k, "tan"::SE) or is?(k, "cot"::SE)], [x])
          real?(g := FG2F h) =>
            internalIntegrate0(rootSimp(rischNormalize(g, x).func), x)
          real?(g := FG2F(h := rootSimp(rischNormalize(h, x).func))) =>
                                                  internalIntegrate0(g, x)
          map(FG2F, lfintegrate(h, x))
      internalIntegrate0(rootSimp(rischNormalize(g, x).func), x)
```

⟨*FSCINT.dotabb*⟩≡

```
"FSCINT" [color="#FF4488",href="bookvol10.4.pdf#nameddest=FSCINT"]
"ACFS" [color="#4488FF",href="bookvol10.2.pdf#nameddest=ACFS"]
"FSCINT" -> "ACFS"
```

7.65 package FS2 FunctionSpaceFunctions2

7.66 FunctionSpaceFunctions2



1.00

Exports:
map

⟨*package FS2 FunctionSpaceFunctions2*⟩≡

```
)abbrev package FS2 FunctionSpaceFunctions2
++ Lifting of maps to function spaces
++ Author: Manuel Bronstein
++ Date Created: 22 March 1988
++ Date Last Updated: 3 May 1994
++ Description:
++   This package allows a mapping R -> S to be lifted to a mapping
++   from a function space over R to a function space over S;
FunctionSpaceFunctions2(R, A, S, B): Exports == Implementation where
  R, S: Join(Ring, OrderedSet)
  A   : FunctionSpace R
  B   : FunctionSpace S

  K  ==> Kernel A
  P  ==> SparseMultivariatePolynomial(R, K)

  Exports ==> with
    map: (R -> S, A) -> B
      ++ map(f, a) applies f to all the constants in R appearing in \spad{a}.

  Implementation ==> add
    smpmap: (R -> S, P) -> B

    smpmap(fn, p) ==
      map(x+->map(z+->map(fn, z),x)$ExpressionSpaceFunctions2(A,B),
          y+->fn(y)::B,p)_
           $PolynomialCategoryLifting(IndexedExponents K, K, R, P, B)
```

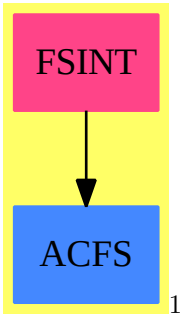
```
      if R has IntegralDomain then
        if S has IntegralDomain then
          map(f, x) == smpmap(f, numer x) / smpmap(f, denom x)
        else
          map(f, x) == smpmap(f, numer x) * (recip(smpmap(f, denom x))::B)
      else
        map(f, x) == smpmap(f, numer x)
```

⟨*FS2.dotabb*⟩≡

```
  "FS2" [color="#FF4488",href="bookvol10.4.pdf#nameddest=FS2"]
  "FS" [color="#4488FF",href="bookvol10.2.pdf#nameddest=FS"]
  "FS2" -> "FS"
```

7.67 package FSINT FunctionSpaceIntegration

7.68 FunctionSpaceIntegration



1.00

Exports:
integrate

⟨*package FSINT FunctionSpaceIntegration*⟩≡

```
)abbrev package FSINT FunctionSpaceIntegration
++ Top-level real function integration
++ Author: Manuel Bronstein
++ Date Created: 4 February 1988
++ Date Last Updated: 11 June 1993
++ Keywords: function, integration.
++ Description:
++   \spadtype{FunctionSpaceIntegration} provides functions for the
++   indefinite integration of real-valued functions.
++ Examples: )r INTEF INPUT
FunctionSpaceIntegration(R, F): Exports == Implementation where
  R : Join(EuclideanDomain, OrderedSet, CharacteristicZero,
           RetractableTo Integer, LinearlyExplicitRingOver Integer)
  F : Join(TranscendentalFunctionCategory, PrimitiveFunctionCategory,
           AlgebraicallyClosedFunctionSpace R)

  B   ==> Boolean
  G   ==> Complex R
  K   ==> Kernel F
  P   ==> SparseMultivariatePolynomial(R, K)
  SE  ==> Symbol
  IR  ==> IntegrationResult F
  FG  ==> Expression G
  ALGOP ==> "%alg"
  TANTEMP ==> "%temptan"::SE

  Exports ==> with
```

```
    integrate: (F, SE) -> Union(F, List F)
        ++ integrate(f, x) returns the integral of \spad{f(x)dx}
        ++ where x is viewed as a real variable.

  Implementation ==> add
    import IntegrationTools(R, F)
    import ElementaryIntegration(R, F)
    import ElementaryIntegration(G, FG)
    import AlgebraicManipulations(R, F)
    import TrigonometricManipulations(R, F)
    import IntegrationResultToFunction(R, F)
    import TranscendentalManipulations(R, F)
    import IntegrationResultFunctions2(FG, F)
    import FunctionSpaceComplexIntegration(R, F)
    import ElementaryFunctionStructurePackage(R, F)
    import InnerTrigonometricManipulations(R, F, FG)
    import PolynomialCategoryQuotientFunctions(IndexedExponents K,
                    K, R, SparseMultivariatePolynomial(R, K), F)

    K2KG      : K -> Kernel FG
    postSubst : (F, List F, List K, B, List K, SE) -> F
    rinteg    : (IR, F, SE, B, B) -> Union(F, List F)
    mkPrimh   : (F, SE, B, B) -> F
    trans?    : F -> B
    goComplex?: (B, List K, List K) -> B
    halfangle : F -> F
    Khalf     : K -> F
    tan2temp  : K -> K

    optemp:BasicOperator := operator(TANTEMP, 1)

    K2KG k     == retract(tan F2FG first argument k)@Kernel(FG)
    tan2temp k == kernel(optemp, argument k, height k)$K

    trans? f ==
      any?(x1+->is?(x1,"log"::SE) or is?(x1,"exp"::SE) or is?(x1,"atan"::SE),
           operators f)$List(BasicOperator)

    mkPrimh(f, x, h, comp) ==
      f := real f
      if comp then f := removeSinSq f
      g := mkPrim(f, x)
      h and trans? g => htrigs g
      g

    rinteg(i, f, x, h, comp) ==
```

```
      not elem? i => integral(f, x)$F
      empty? rest(l := [mkPrimh(f, x, h, comp) for f in expand i]) => first l
      l

  -- replace tan(a/2)**2 by (1-cos a)/(1+cos a) if tan(a/2) is in ltan
    halfangle a ==
      a := 2 * a
      (1 - cos a) / (1 + cos a)

    Khalf k ==
      a := 2 * first argument k
      sin(a) / (1 + cos a)

  -- ltan = list of tangents in the integrand after real normalization
    postSubst(f, lv, lk, comp, ltan, x) ==
      for v in lv for k in lk repeat
        if ((u := retractIfCan(v)@Union(K, "failed")) case K) then
           if has?(operator(kk := u::K), ALGOP) then
             f := univariate(f, kk, minPoly kk) (kk::F)
           f := eval(f, [u::K], [k::F])
      if not(comp or empty? ltan) then
        ltemp := [tan2temp k for k in ltan]
        f := eval(f, ltan, [k::F for k in ltemp])
        f := eval(f, TANTEMP, 2, halfangle)
        f := eval(f, ltemp, [Khalf k for k in ltemp])
      removeConstantTerm(f, x)

  -- can handle a single unnested tangent directly, otherwise go complex for now
  -- l is the list of all the kernels containing x
  -- ltan is the list of all the tangents in l
    goComplex?(rt, l, ltan) ==
      empty? ltan => rt
      not empty? rest rest l

    integrate(f, x) ==
      not real? f => complexIntegrate(f, x)
      f   := distribute(f, x::F)
      tf  := [k for k in tower f | member?(x,variables(k::F)@List(SE))]$List(K)
      ltf := select(x1+->is?(operator x1, "tan"::SE), tf)
      ht  := any?(x1+->has?(operator x1, "htrig"), tf)
      rec := rischNormalize(realElementary(f, x), x)
      g   := rootSimp(rec.func)
      tg  := [k for k in tower g | member?(x, variables(k::F))]$List(K)
      ltg := select(x1+->is?(operator x1, "tan"::SE), tg)
      rtg := any?(x1+->has?(operator x1, "rtrig"), tg)
      el  := any?(x1+->has?(operator x1, "elem"), tg)
```

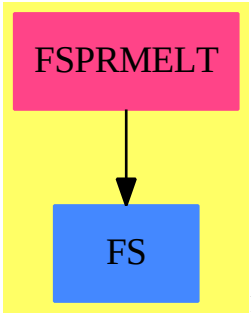
```
        i:IR
        if (comp := goComplex?(rtg, tg, ltg)) then
          i := map(FG2F, lfintegrate(trigs2explogs(F2FG g,
                         [K2KG k for k in tf | is?(k, "tan"::SE) or
                              is?(k, "cot"::SE)], [x]), x))
        else i := lfintegrate(g, x)
        ltg := setDifference(ltg, ltf)   -- tan's added by normalization
        (u := rinteg(i, f, x, el and ht, comp)) case F =>
          postSubst(u::F, rec.vals, rec.kers, comp, ltg, x)
        [postSubst(h, rec.vals, rec.kers, comp, ltg, x) for h in u::List(F)]
```

⟨*FSINT.dotabb*⟩≡

```
"FSINT" [color="#FF4488",href="bookvol10.4.pdf#nameddest=FSINT"]
"ACFS" [color="#4488FF",href="bookvol10.2.pdf#nameddest=ACFS"]
"FSINT" -> "ACFS"
```

7.69 package FSPRMELT FunctionSpacePrimitiveElement

7.70 FunctionSpacePrimitiveElement



1.00

Exports:
primitiveElement

⟨*package FSPRMELT FunctionSpacePrimitiveElement*⟩≡

```
)abbrev package FSPRMELT FunctionSpacePrimitiveElement
++ Computation of primitive elements.
++ Author: Manuel Bronstein
++ Date Created: 6 Jun 1990
++ Date Last Updated: 25 April 1991
++ Description:
++   FunctionsSpacePrimitiveElement provides functions to compute
++   primitive elements in functions spaces;
++ Keywords: algebraic, extension, primitive.
FunctionSpacePrimitiveElement(R, F): Exports == Implementation where
  R: Join(IntegralDomain, OrderedSet, CharacteristicZero)
  F: FunctionSpace R

  SY  ==> Symbol
  P   ==> Polynomial F
  K   ==> Kernel F
  UP  ==> SparseUnivariatePolynomial F
  REC ==> Record(primelt:F, poly:List UP, prim:UP)

  Exports ==> with
    primitiveElement: List F -> Record(primelt:F, poly:List UP, prim:UP)
      ++ primitiveElement([a1,...,an]) returns \spad{[a, [q1,...,qn], q]}
      ++ such that then \spad{k(a1,...,an) = k(a)},
      ++ \spad{ai = qi(a)}, and \spad{q(a) = 0}.
      ++ This operation uses the technique of
      ++ \spadglossSee{groebner bases}{Groebner basis}.
```

```
    if F has AlgebraicallyClosedField then
      primitiveElement: (F,F)->Record(primelt:F,pol1:UP,pol2:UP,prim:UP)
        ++ primitiveElement(a1, a2) returns \spad{[a, q1, q2, q]}
        ++ such that \spad{k(a1, a2) = k(a)},
        ++ \spad{ai = qi(a)}, and \spad{q(a) = 0}.
        ++ The minimal polynomial for a2 may involve a1, but the
        ++ minimal polynomial for a1 may not involve a2;
        ++ This operations uses \spadfun{resultant}.

  Implementation ==> add
    import PrimitiveElement(F)
    import AlgebraicManipulations(R, F)
    import PolynomialCategoryLifting(IndexedExponents K,
                          K, R, SparseMultivariatePolynomial(R, K), P)

    F2P: (F, List SY) -> P
    K2P: (K, List SY) -> P

    F2P(f, l) ==
     inv(denom(f)::F)*map((k1:K):P+->K2P(k1,l),(r1:R):P+->r1::F::P, numer f)

    K2P(k, l) ==
      ((v := symbolIfCan k) case SY) and member?(v::SY, l) => v::SY::P
      k::F::P

    primitiveElement l ==
      u    := string(uu := new()$SY)
      vars := [concat(u, string i)::SY for i in 1..#l]
      vv   := [kernel(v)$K :: F for v in vars]
      kers := [retract(a)@K for a in l]
      pols := [F2P(subst(ratDenom((minPoly k) v, kers), kers, vv), vars)
                                              for k in kers for v in vv]
      rec := primitiveElement(pols, vars, uu)
      [+/[c * a for c in rec.coef for a in l], rec.poly, rec.prim]

    if F has AlgebraicallyClosedField then
      import PolynomialCategoryQuotientFunctions(IndexedExponents K,
                          K, R, SparseMultivariatePolynomial(R, K), F)

      F2UP: (UP, K, UP) -> UP
      getpoly: (UP, F) -> UP

      F2UP(p, k, q) ==
        ans:UP := 0
        while not zero? p repeat
          f   := univariate(leadingCoefficient p, k)
```

```
          ans := ans + ((numer f) q)
                       * monomial(inv(retract(denom f)@F), degree p)
          p   := reductum p
        ans

      primitiveElement(a1, a2) ==
        a   := (aa := new()$SY)::F
        b   := (bb := new()$SY)::F
        l   := [aa, bb]$List(SY)
        p1  := minPoly(k1 := retract(a1)@K)
        p2  := map((z1:F):F+->subst(ratDenom(z1, [k1]), [k1], [a]),
                                              minPoly(retract(a2)@K))
        rec := primitiveElement(F2P(p1 a, l), aa, F2P(p2 b, l), bb)
        w   := rec.coef1 * a1 + rec.coef2 * a2
        g   := rootOf(rec.prim)
        zero?(rec.coef1) =>
          c2g := inv(rec.coef2 :: F) * g
          r := gcd(p1, univariate(p2 c2g, retract(a)@K, p1))
          q := getpoly(r, g)
          [w, q, rec.coef2 * monomial(1, 1)$UP, rec.prim]
        ic1 := inv(rec.coef1 :: F)
        gg  := (ic1 * g)::UP - monomial(rec.coef2 * ic1, 1)$UP
        kg  := retract(g)@K
        r   := gcd(p1 gg, F2UP(p2, retract(a)@K, gg))
        q   := getpoly(r, g)
        [w, monomial(ic1, 1)$UP - rec.coef2 * ic1 * q, q, rec.prim]

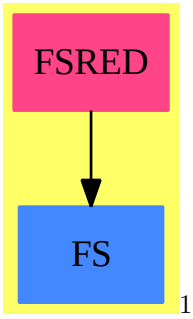
      getpoly(r, g) ==
--      one? degree r =>
        (degree r = 1) =>
          k := retract(g)@K
          univariate(-coefficient(r,0)/leadingCoefficient r,k,minPoly k)
        error "GCD not of degree 1"
```

⟨*FSPRMELT.dotabb*⟩≡

```
"FSPRMELT" [color="#FF4488",href="bookvol10.4.pdf#nameddest=FSPRMELT"]
"FS" [color="#4488FF",href="bookvol10.2.pdf#nameddest=FS"]
"FSPRMELT" -> "FS"
```

7.71 package FSRED FunctionSpaceReduce

7.72 FunctionSpaceReduce



1.00

Exports:
bringDown newReduc

⟨*package FSRED FunctionSpaceReduce*⟩≡
```
)abbrev package FSRED FunctionSpaceReduce
++ Reduction from a function space to the rational numbers
++ Author: Manuel Bronstein
++ Date Created: 1988
++ Date Last Updated: 11 Jul 1990
++ Description:
++ This package provides function which replaces transcendental kernels
++ in a function space by random integers. The correspondence between
++ the kernels and the integers is fixed between calls to new().
++ Keywords: function, space, redcution.
FunctionSpaceReduce(R, F): Exports == Implementation where
  R: Join(OrderedSet, IntegralDomain, RetractableTo Integer)
  F: FunctionSpace R

  Z   ==> Integer
  Q   ==> Fraction Integer
  UP  ==> SparseUnivariatePolynomial Q
  K   ==> Kernel F
  ALGOP  ==> "%alg"

  Exports ==> with
    bringDown: F -> Q
        ++ bringDown(f) \undocumented
    bringDown: (F, K) -> UP
        ++ bringDown(f,k) \undocumented
    newReduc : () -> Void
        ++ newReduc() \undocumented
```

```
Implementation ==> add
  import SparseUnivariatePolynomialFunctions2(F, Q)
  import PolynomialCategoryQuotientFunctions(IndexedExponents K,
                       K, R, SparseMultivariatePolynomial(R, K), F)

  K2Z : K -> F

  redmap := table()$AssociationList(K, Z)

  newReduc() ==
    for k in keys redmap repeat remove_!(k, redmap)
    void

  bringDown(f, k) ==
    ff := univariate(f, k)
    (bc := extendedEuclidean(map(bringDown, denom ff),
              m := map(bringDown, minPoly k), 1)) case "failed" =>
                   error "denominator is 0"
    (map(bringDown, numer ff) * bc.coef1) rem m

  bringDown f ==
    retract(eval(f, lk := kernels f, [K2Z k for k in lk]))@Q

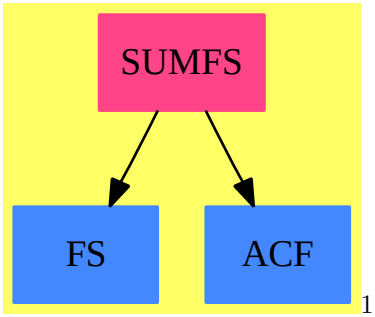
  K2Z k ==
    has?(operator k, ALGOP) => error "Cannot reduce constant field"
    (u := search(k, redmap)) case "failed" =>
                                    setelt(redmap, k, random()$Z)::F
    u::Z::F
```

⟨*FSRED.dotabb*⟩≡

```
"FSRED" [color="#FF4488",href="bookvol10.4.pdf#nameddest=FSRED"]
"FS" [color="#4488FF",href="bookvol10.2.pdf#nameddest=FS"]
"FSRED" -> "FS"
```

7.73 package SUMFS FunctionSpaceSum

7.74 FunctionSpaceSum



1.00

Exports:
sum

⟨*package SUMFS FunctionSpaceSum*⟩≡
```
)abbrev package SUMFS FunctionSpaceSum
++ Top-level sum function
++ Author: Manuel Bronstein
++ Date Created: ???
++ Date Last Updated: 19 April 1991
++ Description: computes sums of top-level expressions;
FunctionSpaceSum(R, F): Exports == Implementation where
  R: Join(IntegralDomain, OrderedSet,
          RetractableTo Integer, LinearlyExplicitRingOver Integer)
  F: Join(FunctionSpace R, CombinatorialOpsCategory,
          AlgebraicallyClosedField, TranscendentalFunctionCategory)

  SE  ==> Symbol
  K   ==> Kernel F

  Exports ==> with
    sum: (F, SE) -> F
      ++ sum(a(n), n) returns A(n) such that A(n+1) - A(n) = a(n);
    sum: (F, SegmentBinding F) -> F
      ++ sum(f(n), n = a..b) returns f(a) + f(a+1) + ... + f(b);

  Implementation ==> add
    import ElementaryFunctionStructurePackage(R, F)
    import GosperSummationMethod(IndexedExponents K, K, R,
                                 SparseMultivariatePolynomial(R, K), F)

    innersum: (F, K) -> Union(F, "failed")
```

```
notRF?  : (F, K) -> Boolean
newk    : () -> K

newk() == kernel(new()$SE)

sum(x:F, s:SegmentBinding F) ==
  k := kernel(variable s)@K
  (u := innersum(x, k)) case "failed" => summation(x, s)
  eval(u::F, k, 1 + hi segment s) - eval(u::F, k, lo segment s)

sum(x:F, v:SE) ==
  (u := innersum(x, kernel(v)@K)) case "failed" => summation(x,v)
  u::F

notRF?(f, k) ==
  for kk in tower f repeat
    member?(k, tower(kk::F)) and (symbolIfCan(kk) case "failed") =>
      return true
  false

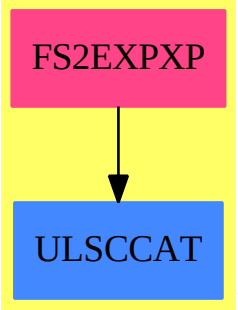
innersum(x, k) ==
  zero? x => 0
  notRF?(f := normalize(x / (x1 := eval(x, k, k::F - 1))), k) =>
    "failed"
  (u := GospersMethod(f, k, newk)) case "failed" => "failed"
  x1 * eval(u::F, k, k::F - 1)
```

⟨*SUMFS.dotabb*⟩≡

```
"SUMFS" [color="#FF4488",href="bookvol10.4.pdf#nameddest=SUMFS"]
"FS" [color="#4488FF",href="bookvol10.2.pdf#nameddest=FS"]
"ACF" [color="#4488FF",href="bookvol10.2.pdf#nameddest=ACF"]
"SUMFS" -> "FS"
"SUMFS" -> "ACF"
```

7.75 package FS2EXPXP FunctionSpaceToExponentialExpansion

7.76 FunctionSpaceToExponentialExpansion



1.00

Exports:
localAbs exprToXXP

⟨*package FS2EXPXP FunctionSpaceToExponentialExpansion*⟩≡
```
)abbrev package FS2EXPXP FunctionSpaceToExponentialExpansion
++ Author: Clifton J. Williamson
++ Date Created: 17 August 1992
++ Date Last Updated: 2 December 1994
++ Basic Operations:
++ Related Domains: ExponentialExpansion, UnivariatePuiseuxSeries(FE,x,cen)
++ Also See: FunctionSpaceToUnivariatePowerSeries
++ AMS Classifications:
++ Keywords: elementary function, power series
++ Examples:
++ References:
++ Description:
++   This package converts expressions in some function space to exponential
++   expansions.
FunctionSpaceToExponentialExpansion(R,FE,x,cen):_
     Exports == Implementation where
  R     : Join(GcdDomain,OrderedSet,RetractableTo Integer,_
               LinearlyExplicitRingOver Integer)
  FE    : Join(AlgebraicallyClosedField,TranscendentalFunctionCategory,_
               FunctionSpace R)
  x     : Symbol
  cen   : FE
  B       ==> Boolean
  BOP     ==> BasicOperator
  Expon   ==> Fraction Integer
  I       ==> Integer
```

```
NNI      ==> NonNegativeInteger
K        ==> Kernel FE
L        ==> List
RN       ==> Fraction Integer
S        ==> String
SY       ==> Symbol
PCL      ==> PolynomialCategoryLifting(IndexedExponents K,K,R,SMP,FE)
POL      ==> Polynomial R
SMP      ==> SparseMultivariatePolynomial(R,K)
SUP      ==> SparseUnivariatePolynomial Polynomial R
UTS      ==> UnivariateTaylorSeries(FE,x,cen)
ULS      ==> UnivariateLaurentSeries(FE,x,cen)
UPXS     ==> UnivariatePuiseuxSeries(FE,x,cen)
EFULS    ==> ElementaryFunctionsUnivariateLaurentSeries(FE,UTS,ULS)
EFUPXS   ==> ElementaryFunctionsUnivariatePuiseuxSeries(FE,ULS,UPXS,EFULS)
FS2UPS   ==> FunctionSpaceToUnivariatePowerSeries(R,FE,RN,UPXS,EFUPXS,x)
EXPUPXS  ==> ExponentialOfUnivariatePuiseuxSeries(FE,x,cen)
UPXSSING ==> UnivariatePuiseuxSeriesWithExponentialSingularity(R,FE,x,cen)
XXP      ==> ExponentialExpansion(R,FE,x,cen)
Problem  ==> Record(func:String,prob:String)
Result   ==> Union(%series:UPXS,%problem:Problem)
XResult  ==> Union(%expansion:XXP,%problem:Problem)
SIGNEF   ==> ElementaryFunctionSign(R,FE)

Exports ==> with
  exprToXXP : (FE,B) -> XResult
    ++ exprToXXP(fcn,posCheck?) converts the expression \spad{fcn} to
    ++ an exponential expansion.  If \spad{posCheck?} is true,
    ++ log's of negative numbers are not allowed nor are nth roots of
    ++ negative numbers with n even.  If \spad{posCheck?} is false,
    ++ these are allowed.
  localAbs: FE -> FE
    ++ localAbs(fcn) = \spad{abs(fcn)} or \spad{sqrt(fcn**2)} depending
    ++ on whether or not FE has a function \spad{abs}.  This should be
    ++ a local function, but the compiler won't allow it.

Implementation ==> add

  import FS2UPS  -- conversion of functional expressions to Puiseux series
  import EFUPXS  -- partial transcendental funtions on UPXS

  ratIfCan            : FE -> Union(RN,"failed")
  stateSeriesProblem  : (S,S) -> Result
  stateProblem        : (S,S) -> XResult
  newElem             : FE -> FE
  smpElem             : SMP -> FE
```

```
    k2Elem              : K -> FE
    iExprToXXP          : (FE,B) -> XResult
    listToXXP           : (L FE,B,XXP,(XXP,XXP) -> XXP) -> XResult
    isNonTrivPower      : FE -> Union(Record(val:FE,exponent:I),"failed")
    negativePowerOK?    : UPXS -> Boolean
    powerToXXP          : (FE,I,B) -> XResult
    carefulNthRootIfCan : (UPXS,NNI,B) -> Result
    nthRootXXPIfCan     : (XXP,NNI,B) -> XResult
    nthRootToXXP        : (FE,NNI,B) -> XResult
    genPowerToXXP       : (L FE,B) -> XResult
    kernelToXXP         : (K,B) -> XResult
    genExp              : (UPXS,B) -> Result
    exponential         : (UPXS,B) -> XResult
    expToXXP            : (FE,B) -> XResult
    genLog              : (UPXS,B) -> Result
    logToXXP            : (FE,B) -> XResult
    applyIfCan          : (UPXS -> Union(UPXS,"failed"),FE,S,B) -> XResult
    applyBddIfCan       : (FE,UPXS -> Union(UPXS,"failed"),FE,S,B) -> XResult
    tranToXXP           : (K,FE,B) -> XResult
    contOnReals?        : S -> B
    bddOnReals?         : S -> B
    opsInvolvingX       : FE -> L BOP
    opInOpList?         : (SY,L BOP) -> B
    exponential?        : FE -> B
    productOfNonZeroes? : FE -> B
    atancotToXXP        : (FE,FE,B,I) -> XResult

    ZEROCOUNT : RN := 1000/1
    -- number of zeroes to be removed when taking logs or nth roots

--% retractions

    ratIfCan fcn == retractIfCan(fcn)@Union(RN,"failed")

--% 'problems' with conversion

    stateSeriesProblem(function,problem) ==
      -- records the problem which occured in converting an expression
      -- to a power series
      [[function,problem]]

    stateProblem(function,problem) ==
      -- records the problem which occured in converting an expression
      -- to an exponential expansion
      [[function,problem]]
```

```
  --% normalizations

      newElem f ==
        -- rewrites a functional expression; all trig functions are
        -- expressed in terms of sin and cos; all hyperbolic trig
        -- functions are expressed in terms of exp; all inverse
        -- hyperbolic trig functions are expressed in terms of exp
        -- and log
        smpElem(numer f) / smpElem(denom f)

      smpElem p == map(k2Elem,(x1:R):FE+->x1::FE,p)$PCL

      k2Elem k ==
      -- rewrites a kernel; all trig functions are
      -- expressed in terms of sin and cos; all hyperbolic trig
      -- functions are expressed in terms of exp
        null(args := [newElem a for a in argument k]) => k :: FE
        iez  := inv(ez  := exp(z := first args))
        sinz := sin z; cosz := cos z
        is?(k,"tan" :: SY)  => sinz / cosz
        is?(k,"cot" :: SY)  => cosz / sinz
        is?(k,"sec" :: SY)  => inv cosz
        is?(k,"csc" :: SY)  => inv sinz
        is?(k,"sinh" :: SY) => (ez - iez) / (2 :: FE)
        is?(k,"cosh" :: SY) => (ez + iez) / (2 :: FE)
        is?(k,"tanh" :: SY) => (ez - iez) / (ez + iez)
        is?(k,"coth" :: SY) => (ez + iez) / (ez - iez)
        is?(k,"sech" :: SY) => 2 * inv(ez + iez)
        is?(k,"csch" :: SY) => 2 * inv(ez - iez)
        is?(k,"acosh" :: SY) => log(sqrt(z**2 - 1) + z)
        is?(k,"atanh" :: SY) => log((z + 1) / (1 - z)) / (2 :: FE)
        is?(k,"acoth" :: SY) => log((z + 1) / (z - 1)) / (2 :: FE)
        is?(k,"asech" :: SY) => log((inv z) + sqrt(inv(z**2) - 1))
        is?(k,"acsch" :: SY) => log((inv z) + sqrt(1 + inv(z**2)))
        (operator k) args

  --% general conversion function

      exprToXXP(fcn,posCheck?) == iExprToXXP(newElem fcn,posCheck?)

      iExprToXXP(fcn,posCheck?) ==
        -- converts a functional expression to an exponential expansion
        --!! The following line is commented out so that expressions of
        --!! the form a**b will be normalized to exp(b * log(a)) even if
        --!! 'a' and 'b' do not involve the limiting variable 'x'.
        --!!                          - cjw 1 Dec 94
```

```
      --not member?(x,variables fcn) => [monomial(fcn,0)$UPXS :: XXP]
      (poly := retractIfCan(fcn)@Union(POL,"failed")) case POL =>
        [exprToUPS(fcn,false,"real:two sides").%series :: XXP]
      (sum := isPlus fcn) case L(FE) =>
        listToXXP(sum::L(FE),posCheck?,0,(y1:XXP,y2:XXP):XXP +-> y1+y2)
      (prod := isTimes fcn) case L(FE) =>
        listToXXP(prod :: L(FE),posCheck?,1,(y1:XXP,y2:XXP):XXP +-> y1*y2)
      (expt := isNonTrivPower fcn) case Record(val:FE,exponent:I) =>
        power := expt :: Record(val:FE,exponent:I)
        powerToXXP(power.val,power.exponent,posCheck?)
      (ker := retractIfCan(fcn)@Union(K,"failed")) case K =>
        kernelToXXP(ker :: K,posCheck?)
      error "exprToXXP: neither a sum, product, power, nor kernel"

  --% sums and products

    listToXXP(list,posCheck?,ans,op) ==
      -- converts each element of a list of expressions to an exponential
      -- expansion and returns the sum of these expansions, when 'op' is +
      -- and 'ans' is 0, or the product of these expansions, when 'op' is *
      -- and 'ans' is 1
      while not null list repeat
        (term := iExprToXXP(first list,posCheck?)) case %problem =>
          return term
        ans := op(ans,term.%expansion)
        list := rest list
      [ans]

  --% nth roots and integral powers

    isNonTrivPower fcn ==
      -- is the function a power with exponent other than 0 or 1?
      (expt := isPower fcn) case "failed" => "failed"
      power := expt :: Record(val:FE,exponent:I)
  --    one? power.exponent => "failed"
      (power.exponent = 1) => "failed"
      power

    negativePowerOK? upxs ==
      -- checks the lower order coefficient of a Puiseux series;
      -- the coefficient may be inverted only if
      -- (a) the only function involving x is 'log', or
      -- (b) the lowest order coefficient is a product of exponentials
      --     and functions not involving x
      deg := degree upxs
      if (coef := coefficient(upxs,deg)) = 0 then
```

```
      deg := order(upxs,deg + ZEROCOUNT :: Expon)
      (coef := coefficient(upxs,deg)) = 0 =>
        error "inverse of series with many leading zero coefficients"
    xOpList := opsInvolvingX coef
    -- only function involving x is 'log'
    (null xOpList) => true
    (null rest xOpList and is?(first xOpList,"log" :: SY)) => true
    -- lowest order coefficient is a product of exponentials and
    -- functions not involving x
    productOfNonZeroes? coef => true
    false

  powerToXXP(fcn,n,posCheck?) ==
    -- converts an integral power to an exponential expansion
    (b := iExprToXXP(fcn,posCheck?)) case %problem => b
    xxp := b.%expansion
    n > 0 => [xxp ** n]
    -- a Puiseux series will be reciprocated only if n < 0 and
    -- numerator of 'xxp' has exactly one monomial
    numberOfMonomials(num := numer xxp) > 1 => [xxp ** n]
    negativePowerOK? leadingCoefficient num =>
      (rec := recip num) case "failed" => error "FS2EXPXP: can't happen"
      nn := (-n) :: NNI
      [(((denom xxp) ** nn) * ((rec :: UPXSSING) ** nn)) :: XXP]
    --!! we may want to create a fraction instead of trying to
    --!! reciprocate the numerator
    stateProblem("inv","lowest order coefficient involves x")

  carefulNthRootIfCan(ups,n,posCheck?) ==
    -- similar to 'nthRootIfCan', but it is fussy about the series
    -- it takes as an argument.  If 'n' is EVEN and 'posCheck?'
    -- is truem then the leading coefficient of the series must
    -- be POSITIVE.  In this case, if 'rightOnly?' is false, the
    -- order of the series must be zero.  The idea is that the
    -- series represents a real function of a real variable, and
    -- we want a unique real nth root defined on a neighborhood
    -- of zero.
    n < 1 => error "nthRoot: n must be positive"
    deg := degree ups
    if (coef := coefficient(ups,deg)) = 0 then
      deg := order(ups,deg + ZEROCOUNT :: Expon)
      (coef := coefficient(ups,deg)) = 0 =>
        error "log of series with many leading zero coefficients"
    -- if 'posCheck?' is true, we do not allow nth roots of negative
    -- numbers when n in even
    if even?(n :: I) then
```

```
        if posCheck? and ((signum := sign(coef)$SIGNEF) case I) then
          (signum :: I) = -1 =>
            return stateSeriesProblem("nth root","root of negative number")
      (ans := nthRootIfCan(ups,n)) case "failed" =>
        stateSeriesProblem("nth root","no nth root")
      [ans :: UPXS]

    nthRootXXPIfCan(xxp,n,posCheck?) ==
      num := numer xxp; den := denom xxp
      not zero?(reductum num) or not zero?(reductum den) =>
       stateProblem("nth root","several monomials in numerator or denominator")
      nInv : RN := 1/n
      newNum :=
        coef : UPXS :=
          root := carefulNthRootIfCan(leadingCoefficient num,n,posCheck?)
          root case %problem => return [root.%problem]
          root.%series
        deg := (nInv :: FE) * (degree num)
        monomial(coef,deg)
      newDen :=
        coef : UPXS :=
          root := carefulNthRootIfCan(leadingCoefficient den,n,posCheck?)
          root case %problem => return [root.%problem]
          root.%series
        deg := (nInv :: FE) * (degree den)
        monomial(coef,deg)
      [newNum/newDen]

    nthRootToXXP(arg,n,posCheck?) ==
      -- converts an nth root to a power series
      -- this is not used in the limit package, so the series may
      -- have non-zero order, in which case nth roots may not be unique
      (result := iExprToXXP(arg,posCheck?)) case %problem => [result.%problem]
      ans := nthRootXXPIfCan(result.%expansion,n,posCheck?)
      ans case %problem => [ans.%problem]
      [ans.%expansion]

  --% general powers f(x) ** g(x)

    genPowerToXXP(args,posCheck?) ==
      -- converts a power f(x) ** g(x) to an exponential expansion
      (logBase := logToXXP(first args,posCheck?)) case %problem =>
        logBase
      (expon := iExprToXXP(second args,posCheck?)) case %problem =>
        expon
      xxp := (expon.%expansion) * (logBase.%expansion)
```

```
      (f := retractIfCan(xxp)@Union(UPXS,"failed")) case "failed" =>
        stateProblem("exp","multiply nested exponential")
      exponential(f,posCheck?)

  --% kernels

    kernelToXXP(ker,posCheck?) ==
      -- converts a kernel to a power series
      (sym := symbolIfCan(ker)) case Symbol =>
        (sym :: Symbol) = x => [monomial(1,1)$UPXS :: XXP]
        [monomial(ker :: FE,0)$UPXS :: XXP]
      empty?(args := argument ker) => [monomial(ker :: FE,0)$UPXS :: XXP]
      empty? rest args =>
        arg := first args
        is?(ker,"%paren" :: Symbol) => iExprToXXP(arg,posCheck?)
        is?(ker,"log" :: Symbol) => logToXXP(arg,posCheck?)
        is?(ker,"exp" :: Symbol) => expToXXP(arg,posCheck?)
        tranToXXP(ker,arg,posCheck?)
      is?(ker,"%power" :: Symbol) => genPowerToXXP(args,posCheck?)
      is?(ker,"nthRoot" :: Symbol) =>
        n := retract(second args)@I
        nthRootToXXP(first args,n :: NNI,posCheck?)
      stateProblem(string name ker,"unknown kernel")

  --% exponentials and logarithms

    genExp(ups,posCheck?) ==
      -- If the series has order zero and the constant term a0 of the
      -- series involves x, the function tries to expand exp(a0) as
      -- a power series.
      (deg := order(ups,1)) < 0 =>
        -- this "can't happen"
        error "exp of function with sigularity"
      deg > 0 => [exp(ups)]
      lc := coefficient(ups,0); varOpList := opsInvolvingX lc
      not opInOpList?("log" :: Symbol,varOpList) => [exp(ups)]
      -- try to fix exp(lc) if necessary
      expCoef := normalize(exp lc,x)$ElementaryFunctionStructurePackage(R,FE)
      result := exprToGenUPS(expCoef,posCheck?,"real:right side")$FS2UPS
      --!! will deal with problems in limitPlus in EXPEXPAN
      --result case %problem => result
      result case %problem => [exp(ups)]
      [(result.%series) * exp(ups - monomial(lc,0))]

    exponential(f,posCheck?) ==
      singPart := truncate(f,0) - (coefficient(f,0) :: UPXS)
```

```
    taylorPart := f - singPart
    expon := exponential(singPart)$EXPUPXS
    (coef := genExp(taylorPart,posCheck?)) case %problem => [coef.%problem]
    [monomial(coef.%series,expon)$UPXSSING :: XXP]

  expToXXP(arg,posCheck?) ==
    (result := iExprToXXP(arg,posCheck?)) case %problem => result
    xxp := result.%expansion
    (f := retractIfCan(xxp)@Union(UPXS,"failed")) case "failed" =>
      stateProblem("exp","multiply nested exponential")
    exponential(f,posCheck?)

  genLog(ups,posCheck?) ==
    deg := degree ups
    if (coef := coefficient(ups,deg)) = 0 then
      deg := order(ups,deg + ZEROCOUNT)
      (coef := coefficient(ups,deg)) = 0 =>
        error "log of series with many leading zero coefficients"
    -- if 'posCheck?' is true, we do not allow logs of negative numbers
    if posCheck? then
      if ((signum := sign(coef)$SIGNEF) case I) then
        (signum :: I) = -1 =>
          return stateSeriesProblem("log","negative leading coefficient")
    lt := monomial(coef,deg)$UPXS
    -- check to see if lowest order coefficient is a negative rational
    negRat? : Boolean :=
      ((rat := ratIfCan coef) case RN) =>
        (rat :: RN) < 0 => true
        false
      false
    logTerm : FE :=
      mon : FE := (x :: FE) - (cen :: FE)
      pow : FE := mon ** (deg :: FE)
      negRat? => log(coef * pow)
      term1 : FE := (deg :: FE) * log(mon)
      log(coef) + term1
    [monomial(logTerm,0)$UPXS + log(ups/lt)]

  logToXXP(arg,posCheck?) ==
    (result := iExprToXXP(arg,posCheck?)) case %problem => result
    xxp := result.%expansion
    num := numer xxp; den := denom xxp
    not zero?(reductum num) or not zero?(reductum den) =>
      stateProblem("log","several monomials in numerator or denominator")
    numCoefLog : UPXS :=
      (res := genLog(leadingCoefficient num,posCheck?)) case %problem =>
```

```
          return [res.%problem]
        res.%series
      denCoefLog : UPXS :=
        (res := genLog(leadingCoefficient den,posCheck?)) case %problem =>
          return [res.%problem]
        res.%series
      numLog := (exponent degree num) + numCoefLog
      denLog := (exponent degree den) + denCoefLog  --?? num?
      [(numLog - denLog) :: XXP]

  --% other transcendental functions

    applyIfCan(fcn,arg,fcnName,posCheck?) ==
      -- converts fcn(arg) to an exponential expansion
      (xxpArg := iExprToXXP(arg,posCheck?)) case %problem => xxpArg
      xxp := xxpArg.%expansion
      (f := retractIfCan(xxp)@Union(UPXS,"failed")) case "failed" =>
        stateProblem(fcnName,"multiply nested exponential")
      upxs := f :: UPXS
      (deg := order(upxs,1)) < 0 =>
        stateProblem(fcnName,"essential singularity")
      deg > 0 => [fcn(upxs) :: UPXS :: XXP]
      lc := coefficient(upxs,0); xOpList := opsInvolvingX lc
      null xOpList => [fcn(upxs) :: UPXS :: XXP]
      opInOpList?("log" :: SY,xOpList) =>
        stateProblem(fcnName,"logs in constant coefficient")
      contOnReals? fcnName => [fcn(upxs) :: UPXS :: XXP]
      stateProblem(fcnName,"x in constant coefficient")

    applyBddIfCan(fe,fcn,arg,fcnName,posCheck?) ==
      -- converts fcn(arg) to a generalized power series, where the
      -- function fcn is bounded for real values
      -- if fcn(arg) has an essential singularity as a complex
      -- function, we return fcn(arg) as a monomial of degree 0
      (xxpArg := iExprToXXP(arg,posCheck?)) case %problem =>
        trouble := xxpArg.%problem
        trouble.prob = "essential singularity" => [monomial(fe,0)$UPXS :: XXP]
        xxpArg
      xxp := xxpArg.%expansion
      (f := retractIfCan(xxp)@Union(UPXS,"failed")) case "failed" =>
        stateProblem("exp","multiply nested exponential")
      (ans := fcn(f :: UPXS)) case "failed" => [monomial(fe,0)$UPXS :: XXP]
      [ans :: UPXS :: XXP]

    CONTFCNS : L S := ["sin","cos","atan","acot","exp","asinh"]
    -- functions which are defined and continuous at all real numbers
```

```
BDDFCNS : L S := ["sin","cos","atan","acot"]
-- functions which are bounded on the reals

contOnReals? fcn == member?(fcn,CONTFCNS)
bddOnReals? fcn  == member?(fcn,BDDFCNS)

opsInvolvingX fcn ==
  opList := [op for k in tower fcn | unary?(op := operator k) _
             and member?(x,variables first argument k)]
  removeDuplicates opList

opInOpList?(name,opList) ==
  for op in opList repeat
    is?(op,name) => return true
  false

exponential? fcn ==
  -- is 'fcn' of the form exp(f)?
  (ker := retractIfCan(fcn)@Union(K,"failed")) case K =>
    is?(ker :: K,"exp" :: Symbol)
  false

productOfNonZeroes? fcn ==
  -- is 'fcn' a product of non-zero terms, where 'non-zero'
  -- means an exponential or a function not involving x
  exponential? fcn => true
  (prod := isTimes fcn) case "failed" => false
  for term in (prod :: L(FE)) repeat
    (not exponential? term) and member?(x,variables term) =>
      return false
  true

tranToXXP(ker,arg,posCheck?) ==
  -- converts op(arg) to a power series for certain functions
  -- op in trig or hyperbolic trig categories
  -- N.B. when this function is called, 'k2elem' will have been
  -- applied, so the following functions cannot appear:
  -- tan, cot, sec, csc, sinh, cosh, tanh, coth, sech, csch
  -- acosh, atanh, acoth, asech, acsch
  is?(ker,"sin" :: SY) =>
    applyBddIfCan(ker :: FE,sinIfCan,arg,"sin",posCheck?)
  is?(ker,"cos" :: SY) =>
    applyBddIfCan(ker :: FE,cosIfCan,arg,"cos",posCheck?)
  is?(ker,"asin" :: SY) =>
    applyIfCan(asinIfCan,arg,"asin",posCheck?)
```

```
    is?(ker,"acos" :: SY) =>
      applyIfCan(acosIfCan,arg,"acos",posCheck?)
    is?(ker,"atan" :: SY) =>
      atancotToXXP(ker :: FE,arg,posCheck?,1)
    is?(ker,"acot" :: SY) =>
      atancotToXXP(ker :: FE,arg,posCheck?,-1)
    is?(ker,"asec" :: SY) =>
      applyIfCan(asecIfCan,arg,"asec",posCheck?)
    is?(ker,"acsc" :: SY) =>
      applyIfCan(acscIfCan,arg,"acsc",posCheck?)
    is?(ker,"asinh" :: SY) =>
      applyIfCan(asinhIfCan,arg,"asinh",posCheck?)
    stateProblem(string name ker,"unknown kernel")

  if FE has abs: FE -> FE then
    localAbs fcn == abs fcn
  else
    localAbs fcn == sqrt(fcn * fcn)

  signOfExpression: FE -> FE
  signOfExpression arg == localAbs(arg)/arg

  atancotToXXP(fe,arg,posCheck?,plusMinus) ==
    -- converts atan(f(x)) to a generalized power series
    atanFlag : String := "real: right side"; posCheck? : Boolean := true
    (result := exprToGenUPS(arg,posCheck?,atanFlag)$FS2UPS) case %problem =>
      trouble := result.%problem
      trouble.prob = "essential singularity" => [monomial(fe,0)$UPXS :: XXP]
      [result.%problem]
    ups := result.%series; coef := coefficient(ups,0)
    -- series involves complex numbers
    (ord := order(ups,0)) = 0 and coef * coef = -1 =>
      y := differentiate(ups)/(1 + ups*ups)
      yCoef := coefficient(y,-1)
      [(monomial(log yCoef,0)+integrate(y - monomial(yCoef,-1)$UPXS)) :: XXP]
    cc : FE :=
      ord < 0 =>
        (rn := ratIfCan(ord :: FE)) case "failed" =>
          -- this condition usually won't occur because exponents will
          -- be integers or rational numbers
          return stateProblem("atan","branch problem")
        lc := coefficient(ups,ord)
        (signum := sign(lc)$SIGNEF) case "failed" =>
          -- can't determine sign
          posNegPi2 := signOfExpression(lc) * pi()/(2 :: FE)
          plusMinus = 1 => posNegPi2
```

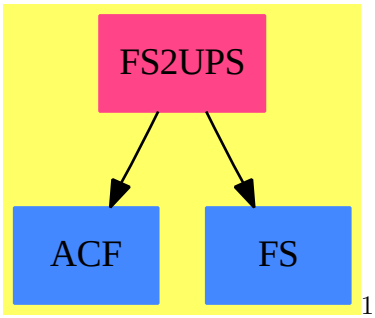
```
            pi()/(2 :: FE) - posNegPi2
          (n := signum :: Integer) = -1 =>
            plusMinus = 1 => -pi()/(2 :: FE)
            pi()
          plusMinus = 1 => pi()/(2 :: FE)
          0
        atan coef
      [((cc :: UPXS) + integrate(differentiate(ups)/(1 + ups*ups))) :: XXP]
```

⟨*FS2EXPXP.dotabb*⟩≡

```
"FS2EXPXP" [color="#FF4488",href="bookvol10.4.pdf#nameddest=FS2EXPXP"]
"ULSCCAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=ULSCCAT"]
"FS2EXPXP" -> "ULSCCAT"
```

7.77 package FS2UPS FunctionSpaceToUnivariatePowerSeries

7.78 FunctionSpaceToUnivariatePowerSeries



1.00

Exports:
localAbs exprToGenUPS exprToUPS

⟨*package FS2UPS FunctionSpaceToUnivariatePowerSeries*⟩≡

```
)abbrev package FS2UPS FunctionSpaceToUnivariatePowerSeries
++ Author: Clifton J. Williamson
++ Date Created: 21 March 1989
++ Date Last Updated: 2 December 1994
++ Basic Operations:
++ Related Domains:
++ Also See:
++ AMS Classifications:
++ Keywords: elementary function, power series
++ Examples:
++ References:
++ Description:
++   This package converts expressions in some function space to power
++   series in a variable x with coefficients in that function space.
++   The function \spadfun{exprToUPS} converts expressions to power series
++   whose coefficients do not contain the variable x. The function
++   \spadfun{exprToGenUPS} converts functional expressions to power series
++   whose coefficients may involve functions of \spad{log(x)}.
FunctionSpaceToUnivariatePowerSeries(R,FE,Expon,UPS,TRAN,x):_
 Exports == Implementation where
  R     : Join(GcdDomain,OrderedSet,RetractableTo Integer,_
              LinearlyExplicitRingOver Integer)
  FE    : Join(AlgebraicallyClosedField,TranscendentalFunctionCategory,_
              FunctionSpace R)
            with
              coerce: Expon -> %
```

```
              ++ coerce(e) converts an 'exponent' e to an 'expression'
  Expon : OrderedRing
  UPS   : Join(UnivariatePowerSeriesCategory(FE,Expon),Field,_
               TranscendentalFunctionCategory)
            with
              differentiate: % -> %
                ++ differentiate(x) returns the derivative of x since we
                ++ need to be able to differentiate a power series
              integrate: % -> %
                ++ integrate(x) returns the integral of x since
                ++ we need to be able to integrate a power series
  TRAN  : PartialTranscendentalFunctions UPS
  x     : Symbol
  B       ==> Boolean
  BOP     ==> BasicOperator
  I       ==> Integer
  NNI     ==> NonNegativeInteger
  K       ==> Kernel FE
  L       ==> List
  RN      ==> Fraction Integer
  S       ==> String
  SY      ==> Symbol
  PCL     ==> PolynomialCategoryLifting(IndexedExponents K,K,R,SMP,FE)
  POL     ==> Polynomial R
  SMP     ==> SparseMultivariatePolynomial(R,K)
  SUP     ==> SparseUnivariatePolynomial Polynomial R
  Problem ==> Record(func:String,prob:String)
  Result  ==> Union(%series:UPS,%problem:Problem)
  SIGNEF  ==> ElementaryFunctionSign(R,FE)

  Exports ==> with
    exprToUPS : (FE,B,S) -> Result
      ++ exprToUPS(fcn,posCheck?,atanFlag) converts the expression
      ++ \spad{fcn} to a power series.  If \spad{posCheck?} is true,
      ++ log's of negative numbers are not allowed nor are nth roots of
      ++ negative numbers with n even.  If \spad{posCheck?} is false,
      ++ these are allowed.  \spad{atanFlag} determines how the case
      ++ \spad{atan(f(x))}, where \spad{f(x)} has a pole, will be treated.
      ++ The possible values of \spad{atanFlag} are \spad{"complex"},
      ++ \spad{"real: two sides"}, \spad{"real: left side"},
      ++ \spad{"real: right side"}, and \spad{"just do it"}.
      ++ If \spad{atanFlag} is \spad{"complex"}, then no series expansion
      ++ will be computed because, viewed as a function of a complex
      ++ variable, \spad{atan(f(x))} has an essential singularity.
      ++ Otherwise, the sign of the leading coefficient of the series
      ++ expansion of \spad{f(x)} determines the constant coefficient
```

```
    ++ in the series expansion of \spad{atan(f(x))}.  If this sign cannot
    ++ be determined, a series expansion is computed only when
    ++ \spad{atanFlag} is \spad{"just do it"}.  When the leading term
    ++ in the series expansion of \spad{f(x)} is of odd degree (or is a
    ++ rational degree with odd numerator), then the constant coefficient
    ++ in the series expansion of \spad{atan(f(x))} for values to the
    ++ left differs from that for values to the right.  If \spad{atanFlag}
    ++ is \spad{"real: two sides"}, no series expansion will be computed.
    ++ If \spad{atanFlag} is \spad{"real: left side"} the constant
    ++ coefficient for values to the left will be used and if \spad{atanFlag}
    ++ \spad{"real: right side"} the constant coefficient for values to the
    ++ right will be used.
    ++ If there is a problem in converting the function to a power series,
    ++ a record containing the name of the function that caused the problem
    ++ and a brief description of the problem is returned.
    ++ When expanding the expression into a series it is assumed that
    ++ the series is centered at 0.  For a series centered at a, the
    ++ user should perform the substitution \spad{x -> x + a} before calling
    ++ this function.

  exprToGenUPS : (FE,B,S) -> Result
    ++ exprToGenUPS(fcn,posCheck?,atanFlag) converts the expression
    ++ \spad{fcn} to a generalized power series.  If \spad{posCheck?}
    ++ is true, log's of negative numbers are not allowed nor are nth roots
    ++ of negative numbers with n even. If \spad{posCheck?} is false,
    ++ these are allowed.  \spad{atanFlag} determines how the case
    ++ \spad{atan(f(x))}, where \spad{f(x)} has a pole, will be treated.
    ++ The possible values of \spad{atanFlag} are \spad{"complex"},
    ++ \spad{"real: two sides"}, \spad{"real: left side"},
    ++ \spad{"real: right side"}, and \spad{"just do it"}.
    ++ If \spad{atanFlag} is \spad{"complex"}, then no series expansion
    ++ will be computed because, viewed as a function of a complex
    ++ variable, \spad{atan(f(x))} has an essential singularity.
    ++ Otherwise, the sign of the leading coefficient of the series
    ++ expansion of \spad{f(x)} determines the constant coefficient
    ++ in the series expansion of \spad{atan(f(x))}.  If this sign cannot
    ++ be determined, a series expansion is computed only when
    ++ \spad{atanFlag} is \spad{"just do it"}.  When the leading term
    ++ in the series expansion of \spad{f(x)} is of odd degree (or is a
    ++ rational degree with odd numerator), then the constant coefficient
    ++ in the series expansion of \spad{atan(f(x))} for values to the
    ++ left differs from that for values to the right.  If \spad{atanFlag}
    ++ is \spad{"real: two sides"}, no series expansion will be computed.
    ++ If \spad{atanFlag} is \spad{"real: left side"} the constant
    ++ coefficient for values to the left will be used and if \spad{atanFlag}
    ++ \spad{"real: right side"} the constant coefficient for values to the
```

```
      ++ right will be used.
      ++ If there is a problem in converting the function to a power
      ++ series, we return a record containing the name of the function
      ++ that caused the problem and a brief description of the problem.
      ++ When expanding the expression into a series it is assumed that
      ++ the series is centered at 0.  For a series centered at a, the
      ++ user should perform the substitution \spad{x -> x + a} before calling
      ++ this function.
    localAbs: FE -> FE
      ++ localAbs(fcn) = \spad{abs(fcn)} or \spad{sqrt(fcn**2)} depending
      ++ on whether or not FE has a function \spad{abs}.  This should be
      ++ a local function, but the compiler won't allow it.

  Implementation ==> add

    ratIfCan            : FE -> Union(RN,"failed")
    carefulNthRootIfCan : (UPS,NNI,B,B) -> Result
    stateProblem        : (S,S) -> Result
    polyToUPS           : SUP -> UPS
    listToUPS           : (L FE,(FE,B,S) -> Result,B,S,UPS,(UPS,UPS) -> UPS)_
                                        -> Result
    isNonTrivPower      : FE -> Union(Record(val:FE,exponent:I),"failed")
    powerToUPS          : (FE,I,B,S) -> Result
    kernelToUPS         : (K,B,S) -> Result
    nthRootToUPS        : (FE,NNI,B,S) -> Result
    logToUPS            : (FE,B,S) -> Result
    atancotToUPS        : (FE,B,S,I) -> Result
    applyIfCan          : (UPS -> Union(UPS,"failed"),FE,S,B,S) -> Result
    tranToUPS           : (K,FE,B,S) -> Result
    powToUPS            : (L FE,B,S) -> Result
    newElem             : FE -> FE
    smpElem             : SMP -> FE
    k2Elem              : K -> FE
    contOnReals?        : S -> B
    bddOnReals?         : S -> B
    iExprToGenUPS       : (FE,B,S) -> Result
    opsInvolvingX       : FE -> L BOP
    opInOpList?         : (SY,L BOP) -> B
    exponential?        : FE -> B
    productOfNonZeroes? : FE -> B
    powerToGenUPS       : (FE,I,B,S) -> Result
    kernelToGenUPS      : (K,B,S) -> Result
    nthRootToGenUPS     : (FE,NNI,B,S) -> Result
    logToGenUPS         : (FE,B,S) -> Result
    expToGenUPS         : (FE,B,S) -> Result
    expGenUPS           : (UPS,B,S) -> Result
```

```
atancotToGenUPS     : (FE,FE,B,S,I) -> Result
genUPSApplyIfCan    : (UPS -> Union(UPS,"failed"),FE,S,B,S) -> Result
applyBddIfCan       : (FE,UPS -> Union(UPS,"failed"),FE,S,B,S) -> Result
tranToGenUPS        : (K,FE,B,S) -> Result
powToGenUPS         : (L FE,B,S) -> Result

ZEROCOUNT : I := 1000
-- number of zeroes to be removed when taking logs or nth roots

ratIfCan fcn == retractIfCan(fcn)@Union(RN,"failed")

carefulNthRootIfCan(ups,n,posCheck?,rightOnly?) ==
  -- similar to 'nthRootIfCan', but it is fussy about the series
  -- it takes as an argument.  If 'n' is EVEN and 'posCheck?'
  -- is truem then the leading coefficient of the series must
  -- be POSITIVE.  In this case, if 'rightOnly?' is false, the
  -- order of the series must be zero.  The idea is that the
  -- series represents a real function of a real variable, and
  -- we want a unique real nth root defined on a neighborhood
  -- of zero.
  n < 1 => error "nthRoot: n must be positive"
  deg := degree ups
  if (coef := coefficient(ups,deg)) = 0 then
    deg := order(ups,deg + ZEROCOUNT :: Expon)
    (coef := coefficient(ups,deg)) = 0 =>
      error "log of series with many leading zero coefficients"
  -- if 'posCheck?' is true, we do not allow nth roots of negative
  -- numbers when n in even
  if even?(n :: I) then
    if posCheck? and ((signum := sign(coef)$SIGNEF) case I) then
      (signum :: I) = -1 =>
        return stateProblem("nth root","negative leading coefficient")
      not rightOnly? and not zero? deg => -- nth root not unique
        return stateProblem("nth root","series of non-zero order")
  (ans := nthRootIfCan(ups,n)) case "failed" =>
    stateProblem("nth root","no nth root")
  [ans :: UPS]

stateProblem(function,problem) ==
  -- records the problem which occured in converting an expression
  -- to a power series
  [[function,problem]]

exprToUPS(fcn,posCheck?,atanFlag) ==
  -- converts a functional expression to a power series
  --!! The following line is commented out so that expressions of
```

```
    --!! the form a**b will be normalized to exp(b * log(a)) even if
    --!! 'a' and 'b' do not involve the limiting variable 'x'.
    --!!                          - cjw 1 Dec 94
    --not member?(x,variables fcn) => [monomial(fcn,0)]
    (poly := retractIfCan(fcn)@Union(POL,"failed")) case POL =>
      [polyToUPS univariate(poly :: POL,x)]
    (sum := isPlus fcn) case L(FE) =>
      listToUPS(sum :: L(FE),exprToUPS,posCheck?,atanFlag,0,
         (y1,y2) +-> y1 + y2)
    (prod := isTimes fcn) case L(FE) =>
      listToUPS(prod :: L(FE),exprToUPS,posCheck?,atanFlag,1,
         (y1,y2) +-> y1 * y2)
    (expt := isNonTrivPower fcn) case Record(val:FE,exponent:I) =>
      power := expt :: Record(val:FE,exponent:I)
      powerToUPS(power.val,power.exponent,posCheck?,atanFlag)
    (ker := retractIfCan(fcn)@Union(K,"failed")) case K =>
      kernelToUPS(ker :: K,posCheck?,atanFlag)
    error "exprToUPS: neither a sum, product, power, nor kernel"

  polyToUPS poly ==
    -- converts a polynomial to a power series
    zero? poly => 0
    -- we don't start with 'ans := 0' as this may lead to an
    -- enormous number of leading zeroes in the power series
    deg  := degree poly
    coef := leadingCoefficient(poly) :: FE
    ans  := monomial(coef,deg :: Expon)$UPS
    poly := reductum poly
    while not zero? poly repeat
      deg  := degree poly
      coef := leadingCoefficient(poly) :: FE
      ans  := ans + monomial(coef,deg :: Expon)$UPS
      poly := reductum poly
    ans

  listToUPS(list,feToUPS,posCheck?,atanFlag,ans,op) ==
    -- converts each element of a list of expressions to a power
    -- series and returns the sum of these series, when 'op' is +
    -- and 'ans' is 0, or the product of these series, when 'op' is *
    -- and 'ans' is 1
    while not null list repeat
      (term := feToUPS(first list,posCheck?,atanFlag)) case %problem =>
        return term
      ans := op(ans,term.%series)
      list := rest list
    [ans]
```

```
    isNonTrivPower fcn ==
      -- is the function a power with exponent other than 0 or 1?
      (expt := isPower fcn) case "failed" => "failed"
      power := expt :: Record(val:FE,exponent:I)
--      one? power.exponent => "failed"
      (power.exponent = 1) => "failed"
      power

    powerToUPS(fcn,n,posCheck?,atanFlag) ==
      -- converts an integral power to a power series
      (b := exprToUPS(fcn,posCheck?,atanFlag)) case %problem => b
      n > 0 => [(b.%series) ** n]
      -- check lowest order coefficient when n < 0
      ups := b.%series; deg := degree ups
      if (coef := coefficient(ups,deg)) = 0 then
        deg := order(ups,deg + ZEROCOUNT :: Expon)
        (coef := coefficient(ups,deg)) = 0 =>
          error "inverse of series with many leading zero coefficients"
      [ups ** n]

    kernelToUPS(ker,posCheck?,atanFlag) ==
      -- converts a kernel to a power series
      (sym := symbolIfCan(ker)) case Symbol =>
        (sym :: Symbol) = x => [monomial(1,1)]
        [monomial(ker :: FE,0)]
      empty?(args := argument ker) => [monomial(ker :: FE,0)]
      not member?(x, variables(ker :: FE)) => [monomial(ker :: FE,0)]
      empty? rest args =>
        arg := first args
        is?(ker,"abs" :: Symbol) =>
          nthRootToUPS(arg*arg,2,posCheck?,atanFlag)
        is?(ker,"%paren" :: Symbol) => exprToUPS(arg,posCheck?,atanFlag)
        is?(ker,"log" :: Symbol) => logToUPS(arg,posCheck?,atanFlag)
        is?(ker,"exp" :: Symbol) =>
          applyIfCan(expIfCan,arg,"exp",posCheck?,atanFlag)
        tranToUPS(ker,arg,posCheck?,atanFlag)
      is?(ker,"%power" :: Symbol) => powToUPS(args,posCheck?,atanFlag)
      is?(ker,"nthRoot" :: Symbol) =>
        n := retract(second args)@I
        nthRootToUPS(first args,n :: NNI,posCheck?,atanFlag)
      stateProblem(string name ker,"unknown kernel")

    nthRootToUPS(arg,n,posCheck?,atanFlag) ==
      -- converts an nth root to a power series
      -- this is not used in the limit package, so the series may
```

```
    -- have non-zero order, in which case nth roots may not be unique
    (result := exprToUPS(arg,posCheck?,atanFlag)) case %problem => result
    ans := carefulNthRootIfCan(result.%series,n,posCheck?,false)
    ans case %problem => ans
    [ans.%series]

  logToUPS(arg,posCheck?,atanFlag) ==
    -- converts a logarithm log(f(x)) to a power series
    -- f(x) must have order 0 and if 'posCheck?' is true,
    -- then f(x) must have a non-negative leading coefficient
    (result := exprToUPS(arg,posCheck?,atanFlag)) case %problem => result
    ups := result.%series
    not zero? order(ups,1) =>
      stateProblem("log","series of non-zero order")
    coef := coefficient(ups,0)
    -- if 'posCheck?' is true, we do not allow logs of negative numbers
    if posCheck? then
      if ((signum := sign(coef)$SIGNEF) case I) then
        (signum :: I) = -1 =>
          return stateProblem("log","negative leading coefficient")
    [logIfCan(ups) :: UPS]

  if FE has abs: FE -> FE then
    localAbs fcn == abs fcn
  else
    localAbs fcn == sqrt(fcn * fcn)

  signOfExpression: FE -> FE
  signOfExpression arg == localAbs(arg)/arg

  atancotToUPS(arg,posCheck?,atanFlag,plusMinus) ==
    -- converts atan(f(x)) to a power series
    (result := exprToUPS(arg,posCheck?,atanFlag)) case %problem => result
    ups := result.%series; coef := coefficient(ups,0)
    (ord := order(ups,0)) = 0 and coef * coef = -1 =>
      -- series involves complex numbers
      return stateProblem("atan","logarithmic singularity")
    cc : FE :=
      ord < 0 =>
        atanFlag = "complex" =>
          return stateProblem("atan","essential singularity")
        (rn := ratIfCan(ord :: FE)) case "failed" =>
          -- this condition usually won't occur because exponents will
          -- be integers or rational numbers
          return stateProblem("atan","branch problem")
        if (atanFlag = "real: two sides") and (odd? numer(rn :: RN)) then
```

```
          -- expansions to the left and right of zero have different
          -- constant coefficients
          return stateProblem("atan","branch problem")
        lc := coefficient(ups,ord)
        (signum := sign(lc)$SIGNEF) case "failed" =>
          -- can't determine sign
          atanFlag = "just do it" =>
            plusMinus = 1 => pi()/(2 :: FE)
            0
          posNegPi2 := signOfExpression(lc) * pi()/(2 :: FE)
          plusMinus = 1 => posNegPi2
          pi()/(2 :: FE) - posNegPi2
          --return stateProblem("atan","branch problem")
        left? : B := atanFlag = "real: left side"; n := signum :: Integer
        (left? and n = 1) or (not left? and n = -1) =>
          plusMinus = 1 => -pi()/(2 :: FE)
          pi()
        plusMinus = 1 => pi()/(2 :: FE)
        0
      atan coef
    [(cc :: UPS) + integrate(plusMinus * differentiate(ups)/(1 + ups*ups))]

  applyIfCan(fcn,arg,fcnName,posCheck?,atanFlag) ==
    -- converts fcn(arg) to a power series
    (ups := exprToUPS(arg,posCheck?,atanFlag)) case %problem => ups
    ans := fcn(ups.%series)
    ans case "failed" => stateProblem(fcnName,"essential singularity")
    [ans :: UPS]

  tranToUPS(ker,arg,posCheck?,atanFlag) ==
    -- converts ker to a power series for certain functions
    -- in trig or hyperbolic trig categories
    is?(ker,"sin" :: SY) =>
      applyIfCan(sinIfCan,arg,"sin",posCheck?,atanFlag)
    is?(ker,"cos" :: SY) =>
      applyIfCan(cosIfCan,arg,"cos",posCheck?,atanFlag)
    is?(ker,"tan" :: SY) =>
      applyIfCan(tanIfCan,arg,"tan",posCheck?,atanFlag)
    is?(ker,"cot" :: SY) =>
      applyIfCan(cotIfCan,arg,"cot",posCheck?,atanFlag)
    is?(ker,"sec" :: SY) =>
      applyIfCan(secIfCan,arg,"sec",posCheck?,atanFlag)
    is?(ker,"csc" :: SY) =>
      applyIfCan(cscIfCan,arg,"csc",posCheck?,atanFlag)
    is?(ker,"asin" :: SY) =>
      applyIfCan(asinIfCan,arg,"asin",posCheck?,atanFlag)
```

```
      is?(ker,"acos" :: SY) =>
        applyIfCan(acosIfCan,arg,"acos",posCheck?,atanFlag)
      is?(ker,"atan" :: SY) => atancotToUPS(arg,posCheck?,atanFlag,1)
      is?(ker,"acot" :: SY) => atancotToUPS(arg,posCheck?,atanFlag,-1)
      is?(ker,"asec" :: SY) =>
        applyIfCan(asecIfCan,arg,"asec",posCheck?,atanFlag)
      is?(ker,"acsc" :: SY) =>
        applyIfCan(acscIfCan,arg,"acsc",posCheck?,atanFlag)
      is?(ker,"sinh" :: SY) =>
        applyIfCan(sinhIfCan,arg,"sinh",posCheck?,atanFlag)
      is?(ker,"cosh" :: SY) =>
        applyIfCan(coshIfCan,arg,"cosh",posCheck?,atanFlag)
      is?(ker,"tanh" :: SY) =>
        applyIfCan(tanhIfCan,arg,"tanh",posCheck?,atanFlag)
      is?(ker,"coth" :: SY) =>
        applyIfCan(cothIfCan,arg,"coth",posCheck?,atanFlag)
      is?(ker,"sech" :: SY) =>
        applyIfCan(sechIfCan,arg,"sech",posCheck?,atanFlag)
      is?(ker,"csch" :: SY) =>
        applyIfCan(cschIfCan,arg,"csch",posCheck?,atanFlag)
      is?(ker,"asinh" :: SY) =>
        applyIfCan(asinhIfCan,arg,"asinh",posCheck?,atanFlag)
      is?(ker,"acosh" :: SY) =>
        applyIfCan(acoshIfCan,arg,"acosh",posCheck?,atanFlag)
      is?(ker,"atanh" :: SY) =>
        applyIfCan(atanhIfCan,arg,"atanh",posCheck?,atanFlag)
      is?(ker,"acoth" :: SY) =>
        applyIfCan(acothIfCan,arg,"acoth",posCheck?,atanFlag)
      is?(ker,"asech" :: SY) =>
        applyIfCan(asechIfCan,arg,"asech",posCheck?,atanFlag)
      is?(ker,"acsch" :: SY) =>
        applyIfCan(acschIfCan,arg,"acsch",posCheck?,atanFlag)
      stateProblem(string name ker,"unknown kernel")

    powToUPS(args,posCheck?,atanFlag) ==
      -- converts a power f(x) ** g(x) to a power series
      (logBase := logToUPS(first args,posCheck?,atanFlag)) case %problem =>
        logBase
      (expon := exprToUPS(second args,posCheck?,atanFlag)) case %problem =>
        expon
      ans := expIfCan((expon.%series) * (logBase.%series))
      ans case "failed" => stateProblem("exp","essential singularity")
      [ans :: UPS]

-- Generalized power series: power series in x, where log(x) and
-- bounded functions of x are allowed to appear in the coefficients
```

```
  -- of the series.  Used for evaluating REAL limits at x = 0.

      newElem f ==
      -- rewrites a functional expression; all trig functions are
      -- expressed in terms of sin and cos; all hyperbolic trig
      -- functions are expressed in terms of exp
        smpElem(numer f) / smpElem(denom f)

      smpElem p == map(k2Elem,(x1:R):FE +-> x1::FE,p)$PCL

      k2Elem k ==
      -- rewrites a kernel; all trig functions are
      -- expressed in terms of sin and cos; all hyperbolic trig
      -- functions are expressed in terms of exp
        null(args := [newElem a for a in argument k]) => k::FE
        iez  := inv(ez  := exp(z := first args))
        sinz := sin z; cosz := cos z
        is?(k,"tan" :: Symbol)  => sinz / cosz
        is?(k,"cot" :: Symbol)  => cosz / sinz
        is?(k,"sec" :: Symbol)  => inv cosz
        is?(k,"csc" :: Symbol)  => inv sinz
        is?(k,"sinh" :: Symbol) => (ez - iez) / (2 :: FE)
        is?(k,"cosh" :: Symbol) => (ez + iez) / (2 :: FE)
        is?(k,"tanh" :: Symbol) => (ez - iez) / (ez + iez)
        is?(k,"coth" :: Symbol) => (ez + iez) / (ez - iez)
        is?(k,"sech" :: Symbol) => 2 * inv(ez + iez)
        is?(k,"csch" :: Symbol) => 2 * inv(ez - iez)
        (operator k) args

      CONTFCNS : L S := ["sin","cos","atan","acot","exp","asinh"]
      -- functions which are defined and continuous at all real numbers

      BDDFCNS : L S := ["sin","cos","atan","acot"]
      -- functions which are bounded on the reals

      contOnReals? fcn == member?(fcn,CONTFCNS)
      bddOnReals? fcn  == member?(fcn,BDDFCNS)

      exprToGenUPS(fcn,posCheck?,atanFlag) ==
        -- converts a functional expression to a generalized power
        -- series; "generalized" means that log(x) and bounded functions
        -- of x are allowed to appear in the coefficients of the series
        iExprToGenUPS(newElem fcn,posCheck?,atanFlag)

      iExprToGenUPS(fcn,posCheck?,atanFlag) ==
        -- converts a functional expression to a generalized power
```

```
    -- series without first normalizing the expression
    --!! The following line is commented out so that expressions of
    --!! the form a**b will be normalized to exp(b * log(a)) even if
    --!! 'a' and 'b' do not involve the limiting variable 'x'.
    --!!                          - cjw 1 Dec 94
    --not member?(x,variables fcn) => [monomial(fcn,0)]
    (poly := retractIfCan(fcn)@Union(POL,"failed")) case POL =>
      [polyToUPS univariate(poly :: POL,x)]
    (sum := isPlus fcn) case L(FE) =>
      listToUPS(sum :: L(FE),iExprToGenUPS,posCheck?,atanFlag,0,
         (y1,y2) +-> y1 + y2)
    (prod := isTimes fcn) case L(FE) =>
      listToUPS(prod :: L(FE),iExprToGenUPS,posCheck?,atanFlag,1,
         (y1,y2) +-> y1 * y2)
    (expt := isNonTrivPower fcn) case Record(val:FE,exponent:I) =>
      power := expt :: Record(val:FE,exponent:I)
      powerToGenUPS(power.val,power.exponent,posCheck?,atanFlag)
    (ker := retractIfCan(fcn)@Union(K,"failed")) case K =>
      kernelToGenUPS(ker :: K,posCheck?,atanFlag)
    error "exprToGenUPS: neither a sum, product, power, nor kernel"

  opsInvolvingX fcn ==
    opList := [op for k in tower fcn | unary?(op := operator k) _
                and member?(x,variables first argument k)]
    removeDuplicates opList

  opInOpList?(name,opList) ==
    for op in opList repeat
      is?(op,name) => return true
    false

  exponential? fcn ==
    -- is 'fcn' of the form exp(f)?
    (ker := retractIfCan(fcn)@Union(K,"failed")) case K =>
      is?(ker :: K,"exp" :: Symbol)
    false

  productOfNonZeroes? fcn ==
    -- is 'fcn' a product of non-zero terms, where 'non-zero'
    -- means an exponential or a function not involving x
    exponential? fcn => true
    (prod := isTimes fcn) case "failed" => false
    for term in (prod :: L(FE)) repeat
      (not exponential? term) and member?(x,variables term) =>
        return false
    true
```

```
powerToGenUPS(fcn,n,posCheck?,atanFlag) ==
  -- converts an integral power to a generalized power series
  -- if n < 0 and the lowest order coefficient of the series
  -- involves x, we are careful about inverting this coefficient
  -- the coefficient is inverted only if
  -- (a) the only function involving x is 'log', or
  -- (b) the lowest order coefficient is a product of exponentials
  --     and functions not involving x
  (b := exprToGenUPS(fcn,posCheck?,atanFlag)) case %problem => b
  n > 0 => [(b.%series) ** n]
  -- check lowest order coefficient when n < 0
  ups := b.%series; deg := degree ups
  if (coef := coefficient(ups,deg)) = 0 then
    deg := order(ups,deg + ZEROCOUNT :: Expon)
    (coef := coefficient(ups,deg)) = 0 =>
      error "inverse of series with many leading zero coefficients"
  xOpList := opsInvolvingX coef
  -- only function involving x is 'log'
  (null xOpList) => [ups ** n]
  (null rest xOpList and is?(first xOpList,"log" :: SY)) =>
    [ups ** n]
  -- lowest order coefficient is a product of exponentials and
  -- functions not involving x
  productOfNonZeroes? coef => [ups ** n]
  stateProblem("inv","lowest order coefficient involves x")

kernelToGenUPS(ker,posCheck?,atanFlag) ==
  -- converts a kernel to a generalized power series
  (sym := symbolIfCan(ker)) case Symbol =>
    (sym :: Symbol) = x => [monomial(1,1)]
    [monomial(ker :: FE,0)]
  empty?(args := argument ker) => [monomial(ker :: FE,0)]
  empty? rest args =>
    arg := first args
    is?(ker,"abs" :: Symbol) =>
      nthRootToGenUPS(arg*arg,2,posCheck?,atanFlag)
    is?(ker,"%paren" :: Symbol) => iExprToGenUPS(arg,posCheck?,atanFlag)
    is?(ker,"log" :: Symbol) => logToGenUPS(arg,posCheck?,atanFlag)
    is?(ker,"exp" :: Symbol) => expToGenUPS(arg,posCheck?,atanFlag)
    tranToGenUPS(ker,arg,posCheck?,atanFlag)
  is?(ker,"%power" :: Symbol) => powToGenUPS(args,posCheck?,atanFlag)
  is?(ker,"nthRoot" :: Symbol) =>
    n := retract(second args)@I
    nthRootToGenUPS(first args,n :: NNI,posCheck?,atanFlag)
  stateProblem(string name ker,"unknown kernel")
```

```
nthRootToGenUPS(arg,n,posCheck?,atanFlag) ==
  -- convert an nth root to a power series
  -- used for computing right hand limits, so the series may have
  -- non-zero order, but may not have a negative leading coefficient
  -- when n is even
  (result := iExprToGenUPS(arg,posCheck?,atanFlag)) case %problem =>
    result
  ans := carefulNthRootIfCan(result.%series,n,posCheck?,true)
  ans case %problem => ans
  [ans.%series]

logToGenUPS(arg,posCheck?,atanFlag) ==
  -- converts a logarithm log(f(x)) to a generalized power series
  (result := iExprToGenUPS(arg,posCheck?,atanFlag)) case %problem =>
    result
  ups := result.%series; deg := degree ups
  if (coef := coefficient(ups,deg)) = 0 then
    deg := order(ups,deg + ZEROCOUNT :: Expon)
    (coef := coefficient(ups,deg)) = 0 =>
      error "log of series with many leading zero coefficients"
  -- if 'posCheck?' is true, we do not allow logs of negative numbers
  if posCheck? then
    if ((signum := sign(coef)$SIGNEF) case I) then
      (signum :: I) = -1 =>
        return stateProblem("log","negative leading coefficient")
  -- create logarithmic term, avoiding log's of negative rationals
  lt := monomial(coef,deg)$UPS; cen := center lt
  -- check to see if lowest order coefficient is a negative rational
  negRat? : Boolean :=
    ((rat := ratIfCan coef) case RN) =>
      (rat :: RN) < 0 => true
      false
    false
  logTerm : FE :=
    mon : FE := (x :: FE) - (cen :: FE)
    pow : FE := mon ** (deg :: FE)
    negRat? => log(coef * pow)
    term1 : FE := (deg :: FE) * log(mon)
    log(coef) + term1
  [monomial(logTerm,0) + log(ups/lt)]

expToGenUPS(arg,posCheck?,atanFlag) ==
  -- converts an exponential exp(f(x)) to a generalized
  -- power series
  (ups := iExprToGenUPS(arg,posCheck?,atanFlag)) case %problem => ups
```

```
    expGenUPS(ups.%series,posCheck?,atanFlag)

  expGenUPS(ups,posCheck?,atanFlag) ==
    -- computes the exponential of a generalized power series.
    -- If the series has order zero and the constant term a0 of the
    -- series involves x, the function tries to expand exp(a0) as
    -- a power series.
    (deg := order(ups,1)) < 0 =>
      stateProblem("exp","essential singularity")
    deg > 0 => [exp ups]
    lc := coefficient(ups,0); xOpList := opsInvolvingX lc
    not opInOpList?("log" :: SY,xOpList) => [exp ups]
    -- try to fix exp(lc) if necessary
    expCoef :=
      normalize(exp lc,x)$ElementaryFunctionStructurePackage(R,FE)
    opInOpList?("log" :: SY,opsInvolvingX expCoef) =>
      stateProblem("exp","logs in constant coefficient")
    result := exprToGenUPS(expCoef,posCheck?,atanFlag)
    result case %problem => result
    [(result.%series) * exp(ups - monomial(lc,0))]

  atancotToGenUPS(fe,arg,posCheck?,atanFlag,plusMinus) ==
    -- converts atan(f(x)) to a generalized power series
    (result := exprToGenUPS(arg,posCheck?,atanFlag)) case %problem =>
      trouble := result.%problem
      trouble.prob = "essential singularity" => [monomial(fe,0)$UPS]
      result
    ups := result.%series; coef := coefficient(ups,0)
    -- series involves complex numbers
    (ord := order(ups,0)) = 0 and coef * coef = -1 =>
      y := differentiate(ups)/(1 + ups*ups)
      yCoef := coefficient(y,-1)
      [monomial(log yCoef,0) + integrate(y - monomial(yCoef,-1)$UPS)]
    cc : FE :=
      ord < 0 =>
        atanFlag = "complex" =>
          return stateProblem("atan","essential singularity")
        (rn := ratIfCan(ord :: FE)) case "failed" =>
          -- this condition usually won't occur because exponents will
          -- be integers or rational numbers
          return stateProblem("atan","branch problem")
        if (atanFlag = "real: two sides") and (odd? numer(rn :: RN)) then
          -- expansions to the left and right of zero have different
          -- constant coefficients
          return stateProblem("atan","branch problem")
        lc := coefficient(ups,ord)
```

```
      (signum := sign(lc)$SIGNEF) case "failed" =>
        -- can't determine sign
        atanFlag = "just do it" =>
          plusMinus = 1 => pi()/(2 :: FE)
          0
        posNegPi2 := signOfExpression(lc) * pi()/(2 :: FE)
        plusMinus = 1 => posNegPi2
        pi()/(2 :: FE) - posNegPi2
        --return stateProblem("atan","branch problem")
      left? : B := atanFlag = "real: left side"; n := signum :: Integer
      (left? and n = 1) or (not left? and n = -1) =>
        plusMinus = 1 => -pi()/(2 :: FE)
        pi()
      plusMinus = 1 => pi()/(2 :: FE)
      0
    atan coef
  [(cc :: UPS) + integrate(differentiate(ups)/(1 + ups*ups))]

genUPSApplyIfCan(fcn,arg,fcnName,posCheck?,atanFlag) ==
  -- converts fcn(arg) to a generalized power series
  (series := iExprToGenUPS(arg,posCheck?,atanFlag)) case %problem =>
    series
  ups := series.%series
  (deg := order(ups,1)) < 0 =>
    stateProblem(fcnName,"essential singularity")
  deg > 0 => [fcn(ups) :: UPS]
  lc := coefficient(ups,0); xOpList := opsInvolvingX lc
  null xOpList => [fcn(ups) :: UPS]
  opInOpList?("log" :: SY,xOpList) =>
    stateProblem(fcnName,"logs in constant coefficient")
  contOnReals? fcnName => [fcn(ups) :: UPS]
  stateProblem(fcnName,"x in constant coefficient")

applyBddIfCan(fe,fcn,arg,fcnName,posCheck?,atanFlag) ==
  -- converts fcn(arg) to a generalized power series, where the
  -- function fcn is bounded for real values
  -- if fcn(arg) has an essential singularity as a complex
  -- function, we return fcn(arg) as a monomial of degree 0
  (ups := iExprToGenUPS(arg,posCheck?,atanFlag)) case %problem =>
    trouble := ups.%problem
    trouble.prob = "essential singularity" => [monomial(fe,0)$UPS]
    ups
  (ans := fcn(ups.%series)) case "failed" => [monomial(fe,0)$UPS]
  [ans :: UPS]

tranToGenUPS(ker,arg,posCheck?,atanFlag) ==
```

```
    -- converts op(arg) to a power series for certain functions
    -- op in trig or hyperbolic trig categories
    -- N.B. when this function is called, 'k2elem' will have been
    -- applied, so the following functions cannot appear:
    -- tan, cot, sec, csc, sinh, cosh, tanh, coth, sech, csch
    is?(ker,"sin" :: SY) =>
      applyBddIfCan(ker :: FE,sinIfCan,arg,"sin",posCheck?,atanFlag)
    is?(ker,"cos" :: SY) =>
      applyBddIfCan(ker :: FE,cosIfCan,arg,"cos",posCheck?,atanFlag)
    is?(ker,"asin" :: SY) =>
      genUPSApplyIfCan(asinIfCan,arg,"asin",posCheck?,atanFlag)
    is?(ker,"acos" :: SY) =>
      genUPSApplyIfCan(acosIfCan,arg,"acos",posCheck?,atanFlag)
    is?(ker,"atan" :: SY) =>
      atancotToGenUPS(ker :: FE,arg,posCheck?,atanFlag,1)
    is?(ker,"acot" :: SY) =>
      atancotToGenUPS(ker :: FE,arg,posCheck?,atanFlag,-1)
    is?(ker,"asec" :: SY) =>
      genUPSApplyIfCan(asecIfCan,arg,"asec",posCheck?,atanFlag)
    is?(ker,"acsc" :: SY) =>
      genUPSApplyIfCan(acscIfCan,arg,"acsc",posCheck?,atanFlag)
    is?(ker,"asinh" :: SY) =>
      genUPSApplyIfCan(asinhIfCan,arg,"asinh",posCheck?,atanFlag)
    is?(ker,"acosh" :: SY) =>
      genUPSApplyIfCan(acoshIfCan,arg,"acosh",posCheck?,atanFlag)
    is?(ker,"atanh" :: SY) =>
      genUPSApplyIfCan(atanhIfCan,arg,"atanh",posCheck?,atanFlag)
    is?(ker,"acoth" :: SY) =>
      genUPSApplyIfCan(acothIfCan,arg,"acoth",posCheck?,atanFlag)
    is?(ker,"asech" :: SY) =>
      genUPSApplyIfCan(asechIfCan,arg,"asech",posCheck?,atanFlag)
    is?(ker,"acsch" :: SY) =>
      genUPSApplyIfCan(acschIfCan,arg,"acsch",posCheck?,atanFlag)
    stateProblem(string name ker,"unknown kernel")

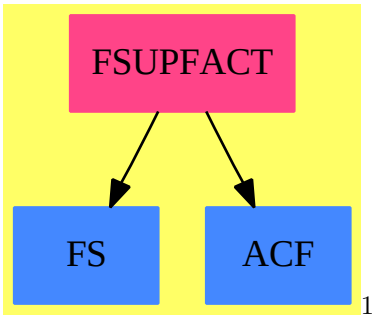
  powToGenUPS(args,posCheck?,atanFlag) ==
    -- converts a power f(x) ** g(x) to a generalized power series
    (logBase := logToGenUPS(first args,posCheck?,atanFlag)) case %problem =>
      logBase
    expon := iExprToGenUPS(second args,posCheck?,atanFlag)
    expon case %problem => expon
    expGenUPS((expon.%series) * (logBase.%series),posCheck?,atanFlag)
```

⟨*FS2UPS.dotabb*⟩≡

```
"FS2UPS" [color="#FF4488",href="bookvol10.4.pdf#nameddest=FS2UPS"]
"ACF" [color="#4488FF",href="bookvol10.2.pdf#nameddest=ACF"]
"FS" [color="#4488FF",href="bookvol10.2.pdf#nameddest=FS"]
"FS2UPS" -> "ACF"
"FS2UPS" -> "FS"
```

7.79 package FSUPFACT FunctionSpaceUnivariatePolynomialFactor

7.80 FunctionSpaceUnivariatePolynomialFactor



1.00

Exports:
anfactor ffactor qfactor UP2ifCan

⟨*package FSUPFACT FunctionSpaceUnivariatePolynomialFactor*⟩≡

```
)abbrev package FSUPFACT FunctionSpaceUnivariatePolynomialFactor
++ Used internally by IR2F
++ Author: Manuel Bronstein
++ Date Created: 12 May 1988
++ Date Last Updated: 22 September 1993
++ Keywords: function, space, polynomial, factoring
FunctionSpaceUnivariatePolynomialFactor(R, F, UP):
 Exports == Implementation where
  R : Join(IntegralDomain, OrderedSet, RetractableTo Integer)
  F : FunctionSpace R
  UP: UnivariatePolynomialCategory F

  Q   ==> Fraction Integer
  K   ==> Kernel F
  AN  ==> AlgebraicNumber
  PQ  ==> SparseMultivariatePolynomial(Q, K)
  PR  ==> SparseMultivariatePolynomial(R, K)
  UPQ ==> SparseUnivariatePolynomial Q
  UPA ==> SparseUnivariatePolynomial AN
  FR  ==> Factored UP
  FRQ ==> Factored UPQ
  FRA ==> Factored UPA

  Exports ==> with
    ffactor: UP -> FR
      ++ ffactor(p) tries to factor a univariate polynomial p over F
```

```
    qfactor: UP -> Union(FRQ, "failed")
      ++ qfactor(p) tries to factor p over fractions of integers,
      ++ returning "failed" if it cannot
    UP2ifCan: UP  -> Union(overq: UPQ, overan: UPA, failed: Boolean)
      ++ UP2ifCan(x) should be local but conditional.
    if F has RetractableTo AN then
      anfactor: UP -> Union(FRA, "failed")
        ++ anfactor(p) tries to factor p over algebraic numbers,
        ++ returning "failed" if it cannot

  Implementation ==> add
    import AlgFactor(UPA)
    import RationalFactorize(UPQ)

    P2QifCan : PR  -> Union(PQ, "failed")
    UPQ2UP   : (SparseUnivariatePolynomial PQ, F) -> UP
    PQ2F     : (PQ, F) -> F
    ffactor0 : UP -> FR

    dummy := kernel(new()$Symbol)$K

    if F has RetractableTo AN then
      UPAN2F: UPA -> UP
      UPQ2AN: UPQ -> UPA

      UPAN2F p ==
        map(x+->x::F, p)$UnivariatePolynomialCategoryFunctions2(AN,UPA,F,UP)

      UPQ2AN p ==
        map(x+->x::AN, p)$UnivariatePolynomialCategoryFunctions2(Q,UPQ,AN,UPA)

      ffactor p ==
        (pq := anfactor p) case FRA =>
                          map(UPAN2F, pq::FRA)$FactoredFunctions2(UPA, UP)
        ffactor0 p

      anfactor p ==
        (q := UP2ifCan p) case overq =>
                     map(UPQ2AN, factor(q.overq))$FactoredFunctions2(UPQ, UPA)
        q case overan => factor(q.overan)
        "failed"

      UP2ifCan p ==
        ansq := 0$UPQ ; ansa := 0$UPA
        goforq? := true
        while p ^= 0 repeat
```

```
        if goforq? then
          rq := retractIfCan(leadingCoefficient p)@Union(Q, "failed")
          if rq case Q then
            ansq := ansq + monomial(rq::Q, degree p)
            ansa := ansa + monomial(rq::Q::AN, degree p)
          else
            goforq? := false
            ra := retractIfCan(leadingCoefficient p)@Union(AN, "failed")
            if ra case AN then ansa := ansa + monomial(ra::AN, degree p)
                          else return [true]
        else
          ra := retractIfCan(leadingCoefficient p)@Union(AN, "failed")
          if ra case AN then ansa := ansa + monomial(ra::AN, degree p)
                        else return [true]
          p := reductum p
        goforq? => [ansq]
        [ansa]

    else
      UPQ2F: UPQ -> UP

      UPQ2F p ==
        map(x+->x::F, p)$UnivariatePolynomialCategoryFunctions2(Q,UPQ,F,UP)

      ffactor p ==
        (pq := qfactor p) case FRQ =>
                          map(UPQ2F, pq::FRQ)$FactoredFunctions2(UPQ, UP)
        ffactor0 p

      UP2ifCan p ==
        ansq := 0$UPQ
        while p ^= 0 repeat
          rq := retractIfCan(leadingCoefficient p)@Union(Q, "failed")
          if rq case Q then ansq := ansq + monomial(rq::Q, degree p)
                       else return [true]
          p := reductum p
        [ansq]

    ffactor0 p ==
      smp := numer(ep := p(dummy::F))
      (q := P2QifCan smp) case "failed" => p::FR
      map(x+->UPQ2UP(univariate(x, dummy), denom(ep)::F), factor(q::PQ
            )$MRationalFactorize(IndexedExponents K, K, Integer,
                 PQ))$FactoredFunctions2(PQ, UP)

    UPQ2UP(p, d) ==
```

```
      map(x+->PQ2F(x, d), p)$UnivariatePolynomialCategoryFunctions2(PQ,
                                  SparseUnivariatePolynomial PQ, F, UP)

    PQ2F(p, d) ==
      map((x:K):F+->x::F, (y:Q):F+->y::F, p)_
        $PolynomialCategoryLifting(IndexedExponents K, K, Q, PQ, F) / d

    qfactor p ==
      (q := UP2ifCan p) case overq => factor(q.overq)
      "failed"

    P2QifCan p ==
      and/[retractIfCan(c::F)@Union(Q, "failed") case Q
           for c in coefficients p] =>
            map(x+->x::PQ, y+->retract(y::F)@Q :: PQ, p)_
              $PolynomialCategoryLifting(IndexedExponents K,K,R,PR,PQ)
      "failed"
```

⟨*FSUPFACT.dotabb*⟩≡

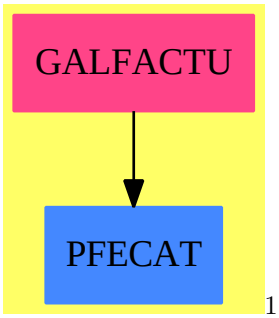
```
"FSUPFACT" [color="#FF4488",href="bookvol10.4.pdf#nameddest=FSUPFACT"]
"FS" [color="#4488FF",href="bookvol10.2.pdf#nameddest=FS"]
"ACF" [color="#4488FF",href="bookvol10.2.pdf#nameddest=ACF"]
"FSUPFACT" -> "FS"
"FSUPFACT" -> "ACF"
```

Chapter 8

Chapter G

8.1 package GALFACTU GaloisGroupFactorizationUtilities

8.2 GaloisGroupFactorizationUtilities



1.00

Exports:

beauzamyBound	bombieriNorm	height	infinityNorm	length
norm	quadraticNorm	rootBound	singleFactorBound	

⟨*package GALFACTU GaloisGroupFactorizationUtilities*⟩≡

```
)abbrev package GALFACTU GaloisGroupFactorizationUtilities
++ Author: Frederic Lehobey
++ Date Created: 30 June 1994
++ Date Last Updated: 19 October 1995
++ Basic Functions:
++ Related Constructors:
++ Also See:
++ AMS Classifications:
```

```
++ Keywords:
++ References:
++ [1] Bernard Beauzamy, Products of polynomials and a priori estimates for
++ coefficients in polynomial decompositions: a sharp result,
++ J. Symbolic Computation (1992) 13, 463-472
++ [2] David W. Boyd, Bounds for the Height of a Factor of a Polynomial in
++ Terms of Bombieri's Norms: I. The Largest Factor,
++ J. Symbolic Computation (1993) 16, 115-130
++ [3] David W. Boyd, Bounds for the Height of a Factor of a Polynomial in
++ Terms of Bombieri's Norms: II. The Smallest Factor,
++ J. Symbolic Computation (1993) 16, 131-145
++ [4] Maurice Mignotte, Some Useful Bounds,
++ Computing, Suppl. 4, 259-263 (1982), Springer-Verlag
++ [5] Donald E. Knuth, The Art of Computer Programming, Vol. 2, (Seminumerical
++ Algorithms) 1st edition, 2nd printing, Addison-Wesley 1971, p. 397-398
++ [6] Bernard Beauzamy, Vilmar Trevisan and Paul S. Wang, Polynomial
++ Factorization: Sharp Bounds, Efficient Algorithms,
++ J. Symbolic Computation (1993) 15, 393-413
++ [7] Augustin-Lux Cauchy, Exercices de Math\'ematiques Quatri\`eme Ann\'ee.
++ De Bure Fr\`eres, Paris 1829 (reprinted Oeuvres, II S\'erie, Tome IX,
++ Gauthier-Villars, Paris, 1891).
++ Description:
++ \spadtype{GaloisGroupFactorizationUtilities} provides functions
++ that will be used by the factorizer.

GaloisGroupFactorizationUtilities(R,UP,F): Exports == Implementation where
  R : Ring
  UP : UnivariatePolynomialCategory R
  F : Join(FloatingPointSystem,RetractableTo(R),Field,
   TranscendentalFunctionCategory,ElementaryFunctionCategory)
  N ==> NonNegativeInteger
  P ==> PositiveInteger
  Z ==> Integer

  Exports ==> with
    beauzamyBound: UP -> Z -- See [1]
      ++ beauzamyBound(p) returns a bound on the larger coefficient of any
      ++ factor of p.
    bombieriNorm: UP -> F -- See [1]
      ++ bombieriNorm(p) returns quadratic Bombieri's norm of p.
    bombieriNorm: (UP,P) -> F -- See [2] and [3]
      ++ bombieriNorm(p,n) returns the nth Bombieri's norm of p.
    rootBound: UP -> Z -- See [4] and [5]
      ++ rootBound(p) returns a bound on the largest norm of the complex roots
      ++ of p.
    singleFactorBound: (UP,N) -> Z -- See [6]
```

```
    ++ singleFactorBound(p,r) returns a bound on the infinite norm of
    ++ the factor of p with smallest Bombieri's norm. r is a lower bound
    ++ for the number of factors of p. p shall be of degree higher or equal
    ++ to 2.
  singleFactorBound: UP -> Z -- See [6]
    ++ singleFactorBound(p,r) returns a bound on the infinite norm of
    ++ the factor of p with smallest Bombieri's norm. p shall be of degree
    ++ higher or equal to 2.
  norm: (UP,P) -> F
    ++ norm(f,p) returns the lp norm of the polynomial f.
  quadraticNorm: UP -> F
    ++ quadraticNorm(f) returns the l2 norm of the polynomial f.
  infinityNorm: UP -> F
    ++ infinityNorm(f) returns the maximal absolute value of the coefficients
    ++ of the polynomial f.
  height: UP -> F
    ++ height(p) returns the maximal absolute value of the coefficients of
    ++ the polynomial p.
  length: UP -> F
    ++ length(p) returns the sum of the absolute values of the coefficients
    ++ of the polynomial p.

Implementation ==> add

  import GaloisGroupUtilities(F)

  height(p:UP):F == infinityNorm(p)

  length(p:UP):F == norm(p,1)

  norm(f:UP,p:P):F ==
    n : F := 0
    for c in coefficients f repeat
      n := n+abs(c::F)**p
    nthRoot(n,p::N)

  quadraticNorm(f:UP):F == norm(f,2)

  infinityNorm(f:UP):F ==
    n : F := 0
    for c in coefficients f repeat
      n := max(n,c::F)
    n

  singleFactorBound(p:UP,r:N):Z == -- See [6]
    n : N := degree p
```

```
  r := max(2,r)
  n < r => error "singleFactorBound: Bad arguments."
  nf : F := n :: F
  num : F := nthRoot(bombieriNorm(p),r)
  if F has Gamma: F -> F then
    num := num*nthRoot(Gamma(nf+1$F),2*r)
    den : F := Gamma(nf/((2*r)::F)+1$F)
  else
    num := num*(2::F)**(5/8+n/2)*exp(1$F/(4*nf))
    den : F := (pi()$F*nf)**(3/8)
  safeFloor( num/den )

singleFactorBound(p:UP):Z == singleFactorBound(p,2) -- See [6]

rootBound(p:UP):Z == -- See [4] and [5]
  n := degree p
  zero? n => 0
  lc := abs(leadingCoefficient(p)::F)
  b1 : F := 0 -- Mignotte
  b2 : F := 0 -- Knuth
  b3 : F := 0 -- Zassenhaus in [5]
  b4 : F := 0 -- Cauchy in [7]
  c : F := 0
  cl : F := 0
  for i in 1..n repeat
    c := abs(coefficient(p,(n-i)::N)::F)
    b1 := max(b1,c)
    cl := c/lc
    b2 := max(b2,nthRoot(cl,i))
    b3 := max(b3,nthRoot(cl/pascalTriangle(n,i),i))
    b4 := max(b4,nthRoot(n*cl,i))
  min(1+safeCeiling(b1/lc),min(safeCeiling(2*b2),min(safeCeiling(b3/
   (nthRoot(2::F,n)-1)),safeCeiling(b4))))

beauzamyBound(f:UP):Z == -- See [1]
  d := degree f
  zero? d => safeFloor bombieriNorm f
  safeFloor( (bombieriNorm(f)*(3::F)**(3/4+d/2))/
   (2*sqrt(pi()$F*(d::F))) )

bombieriNorm(f:UP,p:P):F == -- See [2] and [3]
  d := degree f
  b := abs(coefficient(f,0)::F)
  if zero? d then return b
   else b := b**p
  b := b+abs(leadingCoefficient(f)::F)**p
```

```
        dd := (d-1) quo 2
        for i in 1..dd repeat
          b := b+(abs(coefficient(f,i)::F)**p+abs(coefficient(f,(d-i)::N)::F)**p)
           /pascalTriangle(d,i)
        if even? d then
          dd := dd+1
          b := b+abs(coefficient(f, dd::N)::F)**p/pascalTriangle(d,dd)
        nthRoot(b,p::N)

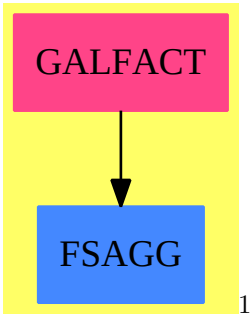
      bombieriNorm(f:UP):F == bombieriNorm(f,2) -- See [1]
```

⟨*GALFACTU.dotabb*⟩≡

```
"GALFACTU" [color="#FF4488",href="bookvol10.4.pdf#nameddest=GALFACTU"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"GALFACTU" -> "PFECAT"
```

8.3 package GALFACT GaloisGroupFactorizer

8.4 GaloisGroupFactorizer



1.00

Exports:

btwFact	degreePartition	eisensteinIrreducible?	factor
factorSquareFree	henselFact	makeFR	modularFactor
numberOfFactors	stopMusserTrials	tryFunctionalDecomposition	tryFunctionalDe
useEisensteinCriterion?	useSingleFactorBound	useSingleFactorBound?	

⟨*package GALFACT GaloisGroupFactorizer*⟩≡

```
)abbrev package GALFACT GaloisGroupFactorizer
++ Author: Frederic Lehobey
++ Date Created: 28 June 1994
++ Date Last Updated: 11 July 1997
++ Basic Operations: factor
++ Related Domains:
++ Also See:
++ AMS Classifications:
++ Keywords: factorization
++ Examples:
++ References:
++ [1] Bernard Beauzamy, Vilmar Trevisan and Paul S. Wang, Polynomial
++ Factorization: Sharp Bounds, Efficient Algorithms,
++ J. Symbolic Computation (1993) 15, 393-413
++ [2] John Brillhart, Note on Irreducibility Testing,
++ Mathematics of Computation, vol. 35, num. 35, Oct. 1980, 1379-1381
++ [3] David R. Musser, On the Efficiency of a Polynomial Irreducibility Test,
++ Journal of the ACM, Vol. 25, No. 2, April 1978, pp. 271-282
++ Description: \spadtype{GaloisGroupFactorizer} provides functions
++ to factor resolvents.
-- improvements to do :
--   + reformulate the lifting problem in completeFactor -- See [1] (hard)
--   + implement algorithm RC -- See [1] (easy)
--   + use Dedekind's criterion to prove sometimes irreducibility (easy)
```

```
--     or even to improve early detection of true factors (hard)
--   + replace Sets by Bits
GaloisGroupFactorizer(UP): Exports == Implementation where
  Z ==> Integer
  UP: UnivariatePolynomialCategory Z
  N ==> NonNegativeInteger
  P ==> PositiveInteger
  CYC ==> CyclotomicPolynomialPackage()
  SUPZ ==> SparseUnivariatePolynomial Z

  ParFact ==> Record(irr: UP, pow: Z)
  FinalFact ==> Record(contp: Z, factors: List ParFact)
  DDRecord ==> Record(factor: UP, degree: Z) -- a Distinct-Degree factor
  DDList ==> List DDRecord
  MFact ==> Record(prime: Z,factors: List UP) -- Modular Factors
  LR ==> Record(left: UP, right: UP) -- Functional decomposition

  Exports ==> with
    makeFR: FinalFact -> Factored UP
      ++ makeFR(flist) turns the final factorization of henselFact into a
      ++ \spadtype{Factored} object.
    degreePartition: DDList -> Multiset N
      ++ degreePartition(ddfactorization) returns the degree partition of
      ++ the polynomial f modulo p where ddfactorization is the distinct
      ++ degree factorization of f computed by
      ++ \spadfunFrom{ddFact}{ModularDistinctDegreeFactorizer}
      ++ for some prime p.
    musserTrials: () -> P
      ++ musserTrials() returns the number of primes that are tried in
      ++ \spadfun{modularFactor}.
    musserTrials: P -> P
      ++ musserTrials(n) sets to n the number of primes to be tried in
      ++ \spadfun{modularFactor} and returns the previous value.
    stopMusserTrials: () -> P
      ++ stopMusserTrials() returns the bound on the number of factors for
      ++ which \spadfun{modularFactor} stops to look for an other prime. You
      ++ will have to remember that the step of recombining the extraneous
      ++ factors may take up to \spad{2**stopMusserTrials()} trials.
    stopMusserTrials: P -> P
      ++ stopMusserTrials(n) sets to n the bound on the number of factors for
      ++ which \spadfun{modularFactor} stops to look for an other prime. You
      ++ will have to remember that the step of recombining the extraneous
      ++ factors may take up to \spad{2**n} trials. Returns the previous
      ++ value.
    numberOfFactors: DDList -> N
      ++ numberOfFactors(ddfactorization) returns the number of factors of
```

```
  ++ the polynomial f modulo p where ddfactorization is the distinct
  ++ degree factorization of f computed by
  ++ \spadfunFrom{ddFact}{ModularDistinctDegreeFactorizer}
  ++ for some prime p.
modularFactor: UP -> MFact
  ++ modularFactor(f) chooses a "good" prime and returns the factorization
  ++ of f modulo this prime in a form that may be used by
  ++ \spadfunFrom{completeHensel}{GeneralHenselPackage}. If prime is zero
  ++ it means that f has been proved to be irreducible over the integers
  ++ or that f is a unit (i.e. 1 or -1).
  ++ f shall be primitive (i.e. content(p)=1) and square free (i.e.
  ++ without repeated factors).
useSingleFactorBound?: () -> Boolean
  ++ useSingleFactorBound?() returns \spad{true} if algorithm with single
  ++ factor bound is used for factorization, \spad{false} for algorithm
  ++ with overall bound.
useSingleFactorBound: Boolean -> Boolean
  ++ useSingleFactorBound(b) chooses the algorithm to be used by the
  ++ factorizers: \spad{true} for algorithm with single
  ++ factor bound, \spad{false} for algorithm with overall bound.
  ++ Returns the previous value.
useEisensteinCriterion?: () -> Boolean
  ++ useEisensteinCriterion?() returns \spad{true} if factorizers
  ++ check Eisenstein's criterion before factoring.
useEisensteinCriterion: Boolean -> Boolean
  ++ useEisensteinCriterion(b) chooses whether factorizers check
  ++ Eisenstein's criterion before factoring: \spad{true} for
  ++ using it, \spad{false} else. Returns the previous value.
eisensteinIrreducible?: UP -> Boolean
  ++ eisensteinIrreducible?(p) returns \spad{true} if p can be
  ++ shown to be irreducible by Eisenstein's criterion,
  ++ \spad{false} is inconclusive.
tryFunctionalDecomposition?: () -> Boolean
  ++ tryFunctionalDecomposition?() returns \spad{true} if
  ++ factorizers try functional decomposition of polynomials before
  ++ factoring them.
tryFunctionalDecomposition: Boolean -> Boolean
  ++ tryFunctionalDecomposition(b) chooses whether factorizers have
  ++ to look for functional decomposition of polynomials
  ++ (\spad{true}) or not (\spad{false}). Returns the previous value.
factor: UP -> Factored UP
  ++ factor(p) returns the factorization of p over the integers.
factor: (UP,N) -> Factored UP
  ++ factor(p,r) factorizes the polynomial p using the single factor bound
  ++ algorithm and knowing that p has at least r factors.
factor: (UP,List N) -> Factored UP
```

```
    ++ factor(p,listOfDegrees) factorizes the polynomial p using the single
    ++ factor bound algorithm and knowing that p has for possible
    ++ splitting of its degree listOfDegrees.
  factor: (UP,List N,N) -> Factored UP
    ++ factor(p,listOfDegrees,r) factorizes the polynomial p using the single
    ++ factor bound algorithm, knowing that p has for possible
    ++ splitting of its degree listOfDegrees and that p has at least r
    ++ factors.
  factor: (UP,N,N) -> Factored UP
    ++ factor(p,d,r) factorizes the polynomial p using the single
    ++ factor bound algorithm, knowing that d divides the degree of all
    ++ factors of p and that p has at least r factors.
  factorSquareFree: UP -> Factored UP
    ++ factorSquareFree(p) returns the factorization of p which is supposed
    ++ not having any repeated factor (this is not checked).
  factorSquareFree: (UP,N) -> Factored UP
    ++ factorSquareFree(p,r) factorizes the polynomial p using the single
    ++ factor bound algorithm and knowing that p has at least r factors.
    ++ f is supposed not having any repeated factor (this is not checked).
  factorSquareFree: (UP,List N) -> Factored UP
    ++ factorSquareFree(p,listOfDegrees) factorizes the polynomial p using
    ++ the single factor bound algorithm and knowing that p has for possible
    ++ splitting of its degree listOfDegrees.
    ++ f is supposed not having any repeated factor (this is not checked).
  factorSquareFree: (UP,List N,N) -> Factored UP
    ++ factorSquareFree(p,listOfDegrees,r) factorizes the polynomial p using
    ++ the single factor bound algorithm, knowing that p has for possible
    ++ splitting of its degree listOfDegrees and that p has at least r
    ++ factors.
    ++ f is supposed not having any repeated factor (this is not checked).
  factorSquareFree: (UP,N,N) -> Factored UP
    ++ factorSquareFree(p,d,r) factorizes the polynomial p using the single
    ++ factor bound algorithm, knowing that d divides the degree of all
    ++ factors of p and that p has at least r factors.
    ++ f is supposed not having any repeated factor (this is not checked).
  factorOfDegree: (P,UP) -> Union(UP,"failed")
    ++ factorOfDegree(d,p) returns a factor of p of degree d.
  factorOfDegree: (P,UP,N) -> Union(UP,"failed")
    ++ factorOfDegree(d,p,r) returns a factor of p of degree
    ++ d knowing that p has at least r factors.
  factorOfDegree: (P,UP,List N) -> Union(UP,"failed")
    ++ factorOfDegree(d,p,listOfDegrees) returns a factor
    ++ of p of degree d knowing that p has for possible splitting of its
    ++ degree listOfDegrees.
  factorOfDegree: (P,UP,List N,N) -> Union(UP,"failed")
    ++ factorOfDegree(d,p,listOfDegrees,r) returns a factor
```

```
      ++ of p of degree d knowing that p has for possible splitting of its
      ++ degree listOfDegrees, and that p has at least r factors.
    factorOfDegree: (P,UP,List N,N,Boolean) -> Union(UP,"failed")
      ++ factorOfDegree(d,p,listOfDegrees,r,sqf) returns a
      ++ factor of p of degree d knowing that p has for possible splitting of
      ++ its degree listOfDegrees, and that p has at least r factors.
      ++ If \spad{sqf=true} the polynomial is assumed to be square free (i.e.
      ++ without repeated factors).
    henselFact: (UP,Boolean) -> FinalFact
      ++ henselFact(p,sqf) returns the factorization of p, the result
      ++ is a Record such that \spad{contp=}content p,
      ++ \spad{factors=}List of irreducible factors of p with exponent.
      ++ If \spad{sqf=true} the polynomial is assumed to be square free (i.e.
      ++ without repeated factors).
    btwFact: (UP,Boolean,Set N,N) -> FinalFact
      ++ btwFact(p,sqf,pd,r) returns the factorization of p, the result
      ++ is a Record such that \spad{contp=}content p,
      ++ \spad{factors=}List of irreducible factors of p with exponent.
      ++ If \spad{sqf=true} the polynomial is assumed to be square free (i.e.
      ++ without repeated factors).
      ++ pd is the \spadtype{Set} of possible degrees. r is a lower bound for
      ++ the number of factors of p. Please do not use this function in your
      ++ code because its design may change.

  Implementation ==> add

    fUnion ==> Union("nil", "sqfr", "irred", "prime")
    FFE ==> Record(flg:fUnion, fctr:UP, xpnt:Z) -- Flag-Factor-Exponent
    DDFact ==> Record(prime:Z, ddfactors:DDList) -- Distinct Degree Factors
    HLR ==> Record(plist:List UP, modulo:Z) -- HenselLift Record

    mussertrials: P := 5
    stopmussertrials: P := 8
    usesinglefactorbound: Boolean := true
    tryfunctionaldecomposition: Boolean := true
    useeisensteincriterion: Boolean := true

    useEisensteinCriterion?():Boolean == useeisensteincriterion

    useEisensteinCriterion(b:Boolean):Boolean ==
      (useeisensteincriterion,b) := (b,useeisensteincriterion)
      b

    tryFunctionalDecomposition?():Boolean == tryfunctionaldecomposition

    tryFunctionalDecomposition(b:Boolean):Boolean ==
```

```
      (tryfunctionaldecomposition,b) := (b,tryfunctionaldecomposition)
      b

    useSingleFactorBound?():Boolean == usesinglefactorbound

    useSingleFactorBound(b:Boolean):Boolean ==
      (usesinglefactorbound,b) := (b,usesinglefactorbound)
      b

    stopMusserTrials():P == stopmussertrials

    stopMusserTrials(n:P):P ==
      (stopmussertrials,n) := (n,stopmussertrials)
      n

    musserTrials():P == mussertrials

    musserTrials(n:P):P ==
      (mussertrials,n) := (n,mussertrials)
      n

    import GaloisGroupFactorizationUtilities(Z,UP,Float)

    import GaloisGroupPolynomialUtilities(Z,UP)

    import IntegerPrimesPackage(Z)
    import IntegerFactorizationPackage(Z)

    import ModularDistinctDegreeFactorizer(UP)

    eisensteinIrreducible?(f:UP):Boolean ==
      rf := reductum f
      c: Z := content rf
      zero? c => false
      unit? c => false
      lc := leadingCoefficient f
      tc := lc
      while not zero? rf repeat
        tc := leadingCoefficient rf
        rf := reductum rf
      for p in factors(factor c)$Factored(Z) repeat
--        if (one? p.exponent) and (not zero? (lc rem p.factor)) and
        if (p.exponent = 1) and (not zero? (lc rem p.factor)) and
         (not zero? (tc rem ((p.factor)**2))) then return true
      false
```

```
    numberOfFactors(ddlist:DDList):N ==
      n: N := 0
      d: Z := 0
      for dd in ddlist repeat
        n := n +
          zero? (d := degree(dd.factor)::Z) => 1
          (d quo dd.degree)::N
      n

    -- local function, returns the a Set of shifted elements
    shiftSet(s:Set N,shift:N):Set N == set [ e+shift for e in parts s ]

    -- local function, returns the "reductum" of an Integer (as chain of bits)
    reductum(n:Z):Z == n-shift(1,length(n)-1)

    -- local function, returns an integer with level lowest bits set to 1
    seed(level:Z):Z == shift(1,level)-1

    -- local function, returns the next number (as a chain of bit) for
    -- factor reconciliation of a given level (which is the number of
    -- extraneaous factors involved) or "End of level" if not any
    nextRecNum(levels:N,level:Z,n:Z):Union("End of level",Z) ==
      if (l := length n)<levels then return(n+shift(1,l-1))
      (n=shift(seed(level),levels-level)) => "End of level"
      b: Z := 1
      while ((l-b) = (lr := length(n := reductum n)))@Boolean repeat b := b+1
      reductum(n)+shift(seed(b+1),lr)

    -- local function, return the set of N, 0..n
    fullSet(n:N):Set N == set [ i for i in 0..n ]

    modularFactor(p:UP):MFact ==
--      not one? abs(content(p)) =>
      not (abs(content(p)) = 1) =>
       error "modularFactor: the polynomial is not primitive."
      zero? (n := degree p) => [0,[p]]

      -- declarations --
      cprime: Z := 2
      trials: List DDFact := empty()
      d: Set N := fullSet(n)
      dirred: Set N := set [0,n]
      s: Set N := empty()
      ddlist: DDList := empty()
      degfact: N := 0
      nf: N := stopmussertrials+1
```

```
    i: Z

    -- Musser, see [3] --
    diffp := differentiate p
    for i in 1..mussertrials | nf>stopmussertrials repeat
      -- test 1: cprime divides leading coefficient
      -- test 2: "bad" primes: (in future: use Dedekind's Criterion)
      while (zero? ((leadingCoefficient p) rem cprime)) or
       (not zero? degree gcd(p,diffp,cprime)) repeat
        cprime := nextPrime(cprime)
      ddlist := ddFact(p,cprime)
      -- degree compatibility: See [3] --
      s := set [0]
      for f in ddlist repeat
        degfact := f.degree::N
        if not zero? degfact then
          for j in 1..(degree(f.factor) quo degfact) repeat
            s := union(s, shiftSet(s,degfact))
      trials := cons([cprime,ddlist]$DDFact,trials)
      d := intersect(d, s)
      d = dirred => return [0,[p]] -- p is irreducible
      cprime := nextPrime(cprime)
      nf := numberOfFactors ddlist

    -- choose the one with the smallest number of factors
    choice := first trials
    nfc := numberOfFactors(choice.ddfactors)
    for t in rest trials repeat
      nf := numberOfFactors(t.ddfactors)
      if nf<nfc or ((nf=nfc) and (t.prime>choice.prime)) then
        nfc := nf
        choice := t
    cprime := choice.prime
    -- HenselLift$GHENSEL expects the degree 0 factor first
    [cprime,separateFactors(choice.ddfactors,cprime)]

  degreePartition(ddlist:DDList):Multiset N ==
    dp: Multiset N := empty()
    d: N := 0
    dd: N := 0
    for f in ddlist repeat
      zero? (d := degree(f.factor)) => dp := insert!(0,dp)
      dd := f.degree::N
      dp := insert!(dd,dp,d quo dd)
    dp
```

```
import GeneralHenselPackage(Z,UP)
import UnivariatePolynomialDecompositionPackage(Z,UP)
import BrillhartTests(UP) -- See [2]

-- local function, finds the factors of f primitive, square-free, with
-- positive leading coefficient and non zero trailing coefficient,
-- using the overall bound technique. If pdecomp is true then look
-- for a functional decomposition of f.
henselfact(f:UP,pdecomp:Boolean):List UP ==
  if brillhartIrreducible? f or
   (useeisensteincriterion => eisensteinIrreducible? f ; false)
  then return [f]
  cf: Union(LR,"failed")
  if pdecomp and tryfunctionaldecomposition then
    cf := monicDecomposeIfCan f
  else
    cf := "failed"
  cf case "failed" =>
    m := modularFactor f
    zero? (cprime := m.prime) => m.factors
    b: P := (2*leadingCoefficient(f)*beauzamyBound(f)) :: P
    completeHensel(f,m.factors,cprime,b)
  lrf := cf::LR
  "append"/[ henselfact(g(lrf.right),false) for g in
   henselfact(lrf.left,true) ]

-- local function, returns the complete factorization of its arguments,
-- using the single-factor bound technique
completeFactor(f:UP,lf:List UP,cprime:Z,pk:P,r:N,d:Set N):List UP ==
  lc := leadingCoefficient f
  f0 := coefficient(f,0)
  ltrue: List UP := empty()
  found? := true
  degf: N := 0
  degg: N := 0
  g0: Z := 0
  g: UP := 0
  rg: N := 0
  nb: Z := 0
  lg: List UP := empty()
  b: P := 1
  dg: Set N := empty()
  llg: HLR := [empty(),0]
  levels: N := #lf
  level: Z := 1
  ic: Union(Z,"End of level") := 0
```

```
i: Z := 0
while level<levels repeat
  -- try all possible factors with degree in d
  ic := seed(level)
  while ((not found?) and (ic case Z)) repeat
    i := ic::Z
    degg := 0
    g0 := 1 -- LC algorithm
    for j in 1..levels repeat
      if bit?(i,j-1) then
        degg := degg+degree lf.j
        g0 := g0*coefficient(lf.j,0) -- LC algorithm
    g0 := symmetricRemainder(lc*g0,pk) -- LC algorithm
    if member?(degg,d) and (((lc*f0) exquo g0) case Z) then
      --                        LC algorithm
      g := lc::UP -- build the possible factor -- LC algorithm
      for j in 1..levels repeat if bit?(i,j-1) then g := g*lf.j
      g := primitivePart reduction(g,pk)
      f1 := f exquo g
      if f1 case UP then -- g is a true factor
        found? := true
        -- remove the factors of g from lf
        nb := 1
        for j in 1..levels repeat
          if bit?(i,j-1) then
            swap!(lf,j,nb)
            nb := nb+1
        lg := lf
        lf := rest(lf,level::N)
        setrest!(rest(lg,(level-1)::N),empty()$List(UP))
        f := f1::UP
        lc := leadingCoefficient f
        f0 := coefficient(f,0)
        -- is g irreducible?
        dg := select(x+->x <= degg,d)
        if not(dg=set [0,degg]) then -- implies degg >= 2
          rg := max(2,r+level-levels)::N
          b := (2*leadingCoefficient(g)*singleFactorBound(g,rg)) :: P
          if b>pk and (not brillhartIrreducible?(g)) and
            (useeisensteincriterion => not eisensteinIrreducible?(g) ;
            true)
          then
            -- g may be reducible
            llg := HenselLift(g,lg,cprime,b)
            gpk: P := (llg.modulo)::P
            -- In case exact factorisation has been reached by
```

```
                    -- HenselLift before coefficient bound.
                    if gpk<b then
                      lg := llg.plist
                    else
                      lg := completeFactor(g,llg.plist,cprime,gpk,rg,dg)
                  else lg := [ g ] -- g irreducible
                else lg := [ g ] -- g irreducible
                ltrue := append(ltrue,lg)
                r := max(2,(r-#lg))::N
                degf := degree f
                d := select(x+->x <= degf,d)
                if degf<=1 then -- lf exhausted
--                  if one? degf then
                  if (degf = 1) then
                    ltrue := cons(f,ltrue)
                  return ltrue -- 1st exit, all factors found
                else -- can we go on with the same pk?
                  b := (2*lc*singleFactorBound(f,r)) :: P
                  if b>pk then -- unlucky: no we can't
                    llg := HenselLift(f,lf,cprime,b) -- I should reformulate
                     -- the lifting probleme, but hadn't time for that.
                     -- In any case, such case should be quite exceptional.
                    lf := llg.plist
                    pk := (llg.modulo)::P
                    -- In case exact factorisation has been reached by
                    -- HenselLift before coefficient bound.
                    if pk<b then return append(lf,ltrue) -- 2nd exit
                    level := 1
              ic := nextRecNum(levels,level,i)
            if found? then
              levels := #lf
              found? := false
            if not (ic case Z) then level := level+1
          cons(f,ltrue) -- 3rd exit, the last factor was irreducible but not "true"

        -- local function, returns the set of elements "divided" by an integer
        divideSet(s:Set N, n:N):Set N ==
          l: List N := [ 0 ]
          for e in parts s repeat
            if (ee := (e exquo n)$N) case N then l := cons(ee::N,l)
          set(l)

        -- Beauzamy-Trevisan-Wang FACTOR, see [1] with some refinements
        -- and some differences. f is assumed to be primitive, square-free
        -- and with positive leading coefficient. If pdecomp is true then
        -- look for a functional decomposition of f.
```

```
btwFactor(f:UP,d:Set N,r:N,pdecomp:Boolean):List UP ==
  df := degree f
  not (max(d) = df) => error "btwFact: Bad arguments"
  reverse?: Boolean := false
  negativelc?: Boolean := false

  (d = set [0,df]) => [ f ]
  if abs(coefficient(f,0))<abs(leadingCoefficient(f)) then
    f := reverse f
    reverse? := true
  brillhartIrreducible? f or
   (useeisensteincriterion => eisensteinIrreducible?(f) ; false) =>
    if reverse? then [ reverse f ] else [ f ]
  if leadingCoefficient(f)<0 then
    f := -f
    negativelc? := true
  cf: Union(LR,"failed")
  if pdecomp and tryfunctionaldecomposition then
    cf := monicDecomposeIfCan f
  else
    cf := "failed"
  if cf case "failed" then
    m := modularFactor f
    zero? (cprime := m.prime) =>
      if reverse? then
        if negativelc? then return [ -reverse f ]
        else return [ reverse f ]
      else if negativelc? then return [ -f ]
           else return [ f ]
    if noLinearFactor? f then d := remove(1,d)
    lc := leadingCoefficient f
    f0 := coefficient(f,0)
    b: P := (2*lc*singleFactorBound(f,r)) :: P -- LC algorithm
    lm := HenselLift(f,m.factors,cprime,b)
    lf := lm.plist
    pk: P := (lm.modulo)::P
    if ground? first lf then lf := rest lf
    -- in case exact factorisation has been reached by HenselLift
    -- before coefficient bound
    if not pk < b then lf := completeFactor(f,lf,cprime,pk,r,d)
  else
    lrf := cf::LR
    dh := degree lrf.right
    lg := btwFactor(lrf.left,divideSet(d,dh),2,true)
    lf: List UP := empty()
    for i in 1..#lg repeat
```

```
          g := lg.i
          dgh := (degree g)*dh
          df := subtractIfCan(df,dgh)::N
          lfg := btwFactor(g(lrf.right),
           select(x+->x <= dgh,d),max(2,r-df)::N,false)
          lf := append(lf,lfg)
          r := max(2,r-#lfg)::N
        if reverse? then lf := [ reverse(fact) for fact in lf ]
        for i in 1..#lf repeat
          if leadingCoefficient(lf.i)<0 then lf.i := -lf.i
          -- because we assume f with positive leading coefficient
        lf

      makeFR(flist:FinalFact):Factored UP ==
        ctp := factor flist.contp
        fflist: List FFE := empty()
        for ff in flist.factors repeat
          fflist := cons(["prime", ff.irr, ff.pow]$FFE, fflist)
        for fc in factorList ctp repeat
          fflist := cons([fc.flg, fc.fctr::UP, fc.xpnt]$FFE, fflist)
        makeFR(unit(ctp)::UP, fflist)

      import IntegerRoots(Z)

      -- local function, factorizes a quadratic polynomial
      quadratic(p:UP):List UP ==
        a := leadingCoefficient p
        b := coefficient(p,1)
        d := b**2-4*a*coefficient(p,0)
        r := perfectSqrt(d)
        r case "failed" => [p]
        b := b+(r::Z)
        a := 2*a
        d := gcd(a,b)
--      if not one? d then
        if not (d = 1) then
          a := a quo d
          b := b quo d
        f: UP := monomial(a,1)+monomial(b,0)
        cons(f,[(p exquo f)::UP])

      isPowerOf2(n:Z): Boolean ==
         n = 1 => true
         qr: Record(quotient: Z, remainder: Z) := divide(n,2)
         qr.remainder = 1 => false
         isPowerOf2 qr.quotient
```

```
      subMinusX(supPol: SUPZ): UP ==
         minusX: SUPZ := monomial(-1,1)$SUPZ
         unmakeSUP(elt(supPol,minusX)$SUPZ)

      henselFact(f:UP, sqf:Boolean):FinalFact ==
        factorlist: List(ParFact) := empty()

        -- make m primitive
        c: Z := content f
        f := (f exquo c)::UP

        -- make the leading coefficient positive
        if leadingCoefficient f < 0 then
          c := -c
          f := -f

        -- is x**d factor of f
        if (d := minimumDegree f) > 0 then
          f := monicDivide(f,monomial(1,d)).quotient
          factorlist := [[monomial(1,1),d]$ParFact]

        d := degree f

        -- is f constant?
        zero? d => [c,factorlist]$FinalFact

        -- is f linear?
--      one? d => [c,cons([f,1]$ParFact,factorlist)]$FinalFact
        (d = 1) => [c,cons([f,1]$ParFact,factorlist)]$FinalFact

        lcPol: UP := leadingCoefficient(f) :: UP

        -- is f cyclotomic (x**n - 1)?
        -lcPol = reductum(f) =>    -- if true, both will = 1
          for fac in map(z+->unmakeSUP(z)$UP,
           cyclotomicDecomposition(d)$CYC)$ListFunctions2(SUPZ,UP) repeat
            factorlist := cons([fac,1]$ParFact,factorlist)
          [c,factorlist]$FinalFact

        -- is f odd cyclotomic (x**(2*n+1) + 1)?
        odd?(d) and (lcPol = reductum(f)) =>
          for sfac in cyclotomicDecomposition(d)$CYC repeat
             fac := subMinusX sfac
             if leadingCoefficient fac < 0 then fac := -fac
             factorlist := cons([fac,1]$ParFact,factorlist)
```

```
        [c,factorlist]$FinalFact

      -- is the poly of the form x**n + 1 with n a power of 2?
      -- if so, then irreducible
      isPowerOf2(d) and (lcPol = reductum(f)) =>
        factorlist := cons([f,1]$ParFact,factorlist)
        [c,factorlist]$FinalFact

      -- other special cases to implement...

      -- f is square-free :
      sqf => [c, append([[pf,1]$ParFact for pf in henselfact(f,true)],
       factorlist)]$FinalFact

      -- f is not square-free :
      sqfflist := factors squareFree f
      for sqfr in sqfflist repeat
        mult := sqfr.exponent
        sqff := sqfr.factor
        d := degree sqff
--        one? d => factorlist := cons([sqff,mult]$ParFact,factorlist)
        (d = 1) => factorlist := cons([sqff,mult]$ParFact,factorlist)
        d=2 =>
          factorlist := append([[pf,mult]$ParFact for pf in quadratic(sqff)],
           factorlist)
        factorlist := append([[pf,mult]$ParFact for pf in
         henselfact(sqff,true)],factorlist)
      [c,factorlist]$FinalFact

    btwFact(f:UP, sqf:Boolean, fd:Set N, r:N):FinalFact ==
      d := degree f
      not(max(fd)=d) => error "btwFact: Bad arguments"
      factorlist: List(ParFact) := empty()

      -- make m primitive
      c: Z := content f
      f := (f exquo c)::UP

      -- make the leading coefficient positive
      if leadingCoefficient f < 0 then
        c := -c
        f := -f

      -- is x**d factor of f
      if (maxd := minimumDegree f) > 0 then
        f := monicDivide(f,monomial(1,maxd)).quotient
```

```
        factorlist := [[monomial(1,1),maxd]$ParFact]
        r := max(2,r-maxd)::N
        d := subtractIfCan(d,maxd)::N
        fd := select(x+->x <= d,fd)

      -- is f constant?
      zero? d => [c,factorlist]$FinalFact

      -- is f linear?
--      one? d => [c,cons([f,1]$ParFact,factorlist)]$FinalFact
      (d = 1) => [c,cons([f,1]$ParFact,factorlist)]$FinalFact

      lcPol: UP := leadingCoefficient(f) :: UP

      -- is f cyclotomic (x**n - 1)?
      -lcPol = reductum(f) =>    -- if true, both will = 1
        for fac in map(z+->unmakeSUP(z)$UP,
         cyclotomicDecomposition(d)$CYC)$ListFunctions2(SUPZ,UP) repeat
          factorlist := cons([fac,1]$ParFact,factorlist)
        [c,factorlist]$FinalFact

      -- is f odd cyclotomic (x**(2*n+1) + 1)?
      odd?(d) and (lcPol = reductum(f)) =>
        for sfac in cyclotomicDecomposition(d)$CYC repeat
           fac := subMinusX sfac
           if leadingCoefficient fac < 0 then fac := -fac
           factorlist := cons([fac,1]$ParFact,factorlist)
        [c,factorlist]$FinalFact

      -- is the poly of the form x**n + 1 with n a power of 2?
      -- if so, then irreducible
      isPowerOf2(d) and (lcPol = reductum(f)) =>
        factorlist := cons([f,1]$ParFact,factorlist)
        [c,factorlist]$FinalFact

      -- other special cases to implement...

      -- f is square-free :
      sqf => [c, append([[pf,1]$ParFact for pf in btwFactor(f,fd,r,true)],
       factorlist)]$FinalFact

      -- f is not square-free :
      sqfflist := factors squareFree(f)
--      if one?(#(sqfflist)) then -- indeed f was a power of a square-free
      if ((#(sqfflist)) = 1) then -- indeed f was a power of a square-free
        r := max(r quo ((first sqfflist).exponent),2)::N
```

```
      else
        r := 2
      for sqfr in sqfflist repeat
        mult := sqfr.exponent
        sqff := sqfr.factor
        d := degree sqff
--        one? d =>
        (d = 1) =>
          factorlist := cons([sqff,mult]$ParFact,factorlist)
          maxd := (max(fd)-mult)::N
          fd := select(x+->x <= maxd,fd)
        d=2 =>
          factorlist := append([[pf,mult]$ParFact for pf in quadratic(sqff)],
           factorlist)
          maxd := (max(fd)-2*mult)::N
          fd := select(x+->x <= maxd,fd)
        factorlist := append([[pf,mult]$ParFact for pf in
         btwFactor(sqff,select(x+->x <= d,fd),r,true)],factorlist)
        maxd := (max(fd)-d*mult)::N
        fd := select(x+->x <= maxd,fd)
      [c,factorlist]$FinalFact

    factor(f:UP):Factored UP ==
      makeFR
        usesinglefactorbound => btwFact(f,false,fullSet(degree f),2)
        henselFact(f,false)

    -- local function, returns true if the sum of the elements of the list
    -- is not the degree.
    errorsum?(d:N,ld:List N):Boolean == not (d = +/ld)

    -- local function, turns list of degrees into a Set
    makeSet(ld:List N):Set N ==
      s := set [0]
      for d in ld repeat s := union(s,shiftSet(s,d))
      s

    factor(f:UP,ld:List N,r:N):Factored UP ==
      errorsum?(degree f,ld) => error "factor: Bad arguments"
      makeFR btwFact(f,false,makeSet(ld),r)

    factor(f:UP,r:N):Factored UP == makeFR btwFact(f,false,fullSet(degree f),r)

    factor(f:UP,ld:List N):Factored UP == factor(f,ld,2)

    factor(f:UP,d:N,r:N):Factored UP ==
```

```
      n := (degree f) exquo d
      n case "failed" => error "factor: Bad arguments"
      factor(f,new(n::N,d)$List(N),r)

    factorSquareFree(f:UP):Factored UP ==
      makeFR
        usesinglefactorbound => btwFact(f,true,fullSet(degree f),2)
        henselFact(f,true)

    factorSquareFree(f:UP,ld:List(N),r:N):Factored UP ==
      errorsum?(degree f,ld) => error "factorSquareFree: Bad arguments"
      makeFR btwFact(f,true,makeSet(ld),r)

    factorSquareFree(f:UP,r:N):Factored UP ==
      makeFR btwFact(f,true,fullSet(degree f),r)

    factorSquareFree(f:UP,ld:List N):Factored UP == factorSquareFree(f,ld,2)

    factorSquareFree(f:UP,d:N,r:N):Factored UP ==
      n := (degree f) exquo d
      n case "failed" => error "factorSquareFree: Bad arguments"
      factorSquareFree(f,new(n::N,d)$List(N),r)

    factorOfDegree(d:P,p:UP,ld:List N,r:N,sqf:Boolean):Union(UP,"failed") ==
      dp := degree p
      errorsum?(dp,ld) => error "factorOfDegree: Bad arguments"
--      (one? (d::N)) and noLinearFactor?(p) => "failed"
      ((d::N) = 1) and noLinearFactor?(p) => "failed"
      lf := btwFact(p,sqf,makeSet(ld),r).factors
      for f in lf repeat
        degree(f.irr)=d => return f.irr
      "failed"

    factorOfDegree(d:P,p:UP,ld:List N,r:N):Union(UP,"failed") ==
      factorOfDegree(d,p,ld,r,false)

    factorOfDegree(d:P,p:UP,r:N):Union(UP,"failed") ==
      factorOfDegree(d,p,new(degree p,1)$List(N),r,false)

    factorOfDegree(d:P,p:UP,ld:List N):Union(UP,"failed") ==
      factorOfDegree(d,p,ld,2,false)

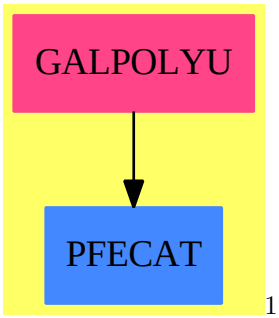
    factorOfDegree(d:P,p:UP):Union(UP,"failed") ==
      factorOfDegree(d,p,new(degree p,1)$List(N),2,false)
```

⟨*GALFACT.dotabb*⟩≡

```
"GALFACT" [color="#FF4488",href="bookvol10.4.pdf#nameddest=GALFACT"]
"FSAGG" [color="#4488FF",href="bookvol10.2.pdf#nameddest=FSAGG"]
"GALFACT" -> "FSAGG"
```

8.5 package GALPOLYU GaloisGroupPolynomialUtilities

8.6 GaloisGroupPolynomialUtilities



1.00

Exports:

degreePartition	factorOfDegree	factorsOfDegree	monic?	reverse
scaleRoots	shiftRoots	unvectorise		

⟨*package GALPOLYU GaloisGroupPolynomialUtilities*⟩≡

```
)abbrev package GALPOLYU GaloisGroupPolynomialUtilities
++ Author: Frederic Lehobey
++ Date Created: 30 June 1994
++ Date Last Updated: 15 July 1994
++ Basic Operations:
++ Related Domains:
++ Also See:
++ AMS Classifications:
++ Keywords:
++ Examples:
++ References:
++ Description: \spadtype{GaloisGroupPolynomialUtilities} provides useful
++ functions for univariate polynomials which should be added to
++ \spadtype{UnivariatePolynomialCategory} or to \spadtype{Factored}
++ (July 1994).

GaloisGroupPolynomialUtilities(R,UP): Exports == Implementation where
  R : Ring
  UP : UnivariatePolynomialCategory R
  N ==> NonNegativeInteger
  P ==> PositiveInteger

  Exports ==> with
    monic?: UP -> Boolean
      ++ monic?(p) tests if p is monic (i.e. leading coefficient equal to 1).
```

```
    unvectorise: Vector R -> UP
      ++ unvectorise(v) returns the polynomial which has for coefficients the
      ++ entries of v in the increasing order.
    reverse: UP -> UP
      ++ reverse(p) returns the reverse polynomial of p.
    scaleRoots: (UP,R) -> UP
      ++ scaleRoots(p,c) returns the polynomial which has c times the roots
      ++ of p.
    shiftRoots: (UP,R) -> UP
      ++ shiftRoots(p,c) returns the polynomial which has for roots c added
      ++ to the roots of p.
    degreePartition: Factored UP -> Multiset N
      ++ degreePartition(f) returns the degree partition (i.e. the multiset
      ++ of the degrees of the irreducible factors) of
      ++ the polynomial f.
    factorOfDegree: (P, Factored UP) -> UP
      ++ factorOfDegree(d,f) returns a factor of degree d of the factored
      ++ polynomial f. Such a factor shall exist.
    factorsOfDegree: (P, Factored UP) -> List UP
      ++ factorsOfDegree(d,f) returns the factors of degree d of the factored
      ++ polynomial f.

  Implementation ==> add

    import Factored UP

    factorsOfDegree(d:P,r:Factored UP):List UP ==
      lfact : List UP := empty()
      for fr in factors r | degree(fr.factor)=(d::N) repeat
        for i in 1..fr.exponent repeat
          lfact := cons(fr.factor,lfact)
      lfact

    factorOfDegree(d:P,r:Factored UP):UP ==
      factor : UP := 0
      for i in 1..numberOfFactors r repeat
        factor := nthFactor(r,i)
        if degree(factor)=(d::N) then return factor
      error "factorOfDegree: Bad arguments"

    degreePartition(r:Factored UP):Multiset N ==
      multiset([ degree(nthFactor(r,i)) for i in 1..numberOfFactors r ])

--    monic?(p:UP):Boolean == one? leadingCoefficient p
    monic?(p:UP):Boolean == (leadingCoefficient p) = 1
```

```
    unvectorise(v:Vector R):UP ==
      p : UP := 0
      for i in 1..#v repeat p := p + monomial(v(i),(i-1)::N)
      p

    reverse(p:UP):UP ==
      r : UP := 0
      n := degree(p)
      for i in 0..n repeat r := r + monomial(coefficient(p,(n-i)::N),i)
      r

    scaleRoots(p:UP,c:R):UP ==
--      one? c => p
      (c = 1) => p
      n := degree p
      zero? c => monomial(leadingCoefficient p,n)
      r : UP := 0
      mc : R := 1
      for i in n..0 by -1 repeat
        r := r + monomial(mc*coefficient(p,i),i)
        mc := mc*c
      r

    import UnivariatePolynomialCategoryFunctions2(R,UP,UP,
     SparseUnivariatePolynomial UP)

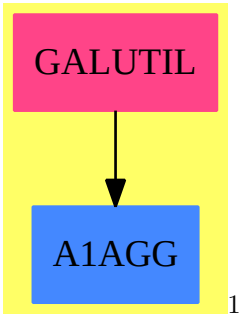
    shiftRoots(p:UP,c:R):UP == elt(map(coerce,p),monomial(1,1)$UP-c::UP)::UP
```

⟨*GALPOLYU.dotabb*⟩≡

```
"GALPOLYU" [color="#FF4488",href="bookvol10.4.pdf#nameddest=GALPOLYU"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"GALPOLYU" -> "PFECAT"
```

8.7 package GALUTIL GaloisGroupUtilities

8.8 GaloisGroupUtilities



1.00

Exports:
fillPascalTriangle pascalTriangle rangePascalTriangle safeCeiling safeFloor safetyMargin sizePascalTriangle

⟨*package GALUTIL GaloisGroupUtilities*⟩≡
```
)abbrev package GALUTIL GaloisGroupUtilities
++ Author: Frederic Lehobey
++ Date Created: 29 June 1994
++ Date Last Updated: 30 June 1994
++ Basic Functions:
++ Related Constructors:
++ Also See:
++ AMS Classifications:
++ Keywords:
++ References:
++ Description:
++ \spadtype{GaloisGroupUtilities} provides several useful functions.

GaloisGroupUtilities(R): Exports == Implementation where
  N ==> NonNegativeInteger
  Z ==> Integer
  R : Ring

  Exports ==> with
    pascalTriangle: (N,Z) -> R
      ++ pascalTriangle(n,r) returns the binomial coefficient
      ++ \spad{C(n,r)=n!/(r! (n-r)!)}
      ++ and stores it in a table to prevent recomputation.
    rangePascalTriangle: N -> N
      ++ rangePascalTriangle(n) sets the maximal number of lines which
      ++ are stored and returns the previous value.
```

```
    rangePascalTriangle: () -> N
      ++ rangePascalTriangle() returns the maximal number of lines stored.
    sizePascalTriangle: () -> N
      ++ sizePascalTriangle() returns the number of entries currently stored
      ++ in the table.
    fillPascalTriangle: () -> Void
      ++ fillPascalTriangle() fills the stored table.

    if R has FloatingPointSystem then
      safeCeiling: R -> Z
        ++ safeCeiling(x) returns the integer which is greater than any integer
        ++ with the same floating point number representation.
      safeFloor: R -> Z
        ++ safeFloor(x) returns the integer which is lower or equal to the
        ++ largest integer which has the same floating point number
        ++ representation.
      safetyMargin: N -> N
        ++ safetyMargin(n) sets to n the number of low weight digits we do not
        ++ trust in the floating point representation and returns the previous
        ++ value (for use by \spadfun{safeCeiling}).
      safetyMargin: () -> N
        ++ safetyMargin() returns the number of low weight digits we do not
        ++ trust in the floating point representation (used by
        ++ \spadfun{safeCeiling}).

  Implementation ==> add

    if R has FloatingPointSystem then
      safetymargin : N := 6

      safeFloor(x:R):Z ==
        if (shift := order(x)-precision()$R+safetymargin) >= 0 then
          x := x+float(1,shift)
        retract(floor(x))@Z

      safeCeiling(x:R):Z ==
        if (shift := order(x)-precision()$R+safetymargin) >= 0 then
          x := x+float(1,shift)
        retract(ceiling(x))@Z

      safetyMargin(n:N):N ==
        (safetymargin,n) := (n,safetymargin)
        n

      safetyMargin():N == safetymargin
```

```
    pascaltriangle : FlexibleArray(R) := empty()
    ncomputed : N := 3
    rangepascaltriangle : N := 216

    pascalTriangle(n:N, r:Z):R ==
      negative? r => 0
      (d := n-r) < r => pascalTriangle(n,d)
      zero? r => 1$R
--      one? r => n :: R
      (r = 1) => n :: R
      n > rangepascaltriangle =>
       binomial(n,r)$IntegerCombinatoricFunctions(Z) :: R
      n <= ncomputed =>
        m := divide(n-4,2)
        mq := m.quotient
        pascaltriangle((mq+1)*(mq+m.remainder)+r-1)
      -- compute the missing lines
      for i in (ncomputed+1)..n repeat
        for j in 2..(i quo 2) repeat
          pascaltriangle := concat!(pascaltriangle,pascalTriangle((i-1)
           :: N, j-1)+pascalTriangle((i-1) :: N,j))
        ncomputed := i
      pascalTriangle(n,r)

    rangePascalTriangle(n:N):N ==
      if n<ncomputed then
        if n<3 then
          pascaltriangle := delete!(pascaltriangle,1..#pascaltriangle)
          ncomputed := 3
        else
          d := divide(n-3,2)
          dq := d.quotient
          pascaltriangle := delete!(pascaltriangle,((dq+1)*(dq+d.remainder)
           +1)..#pascaltriangle)
          ncomputed := n
      (rangepascaltriangle,n) := (n,rangepascaltriangle)
      n

    rangePascalTriangle():N == rangepascaltriangle

    sizePascalTriangle():N == #pascaltriangle

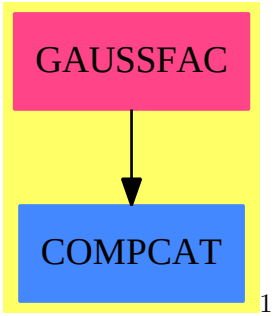
    fillPascalTriangle():Void == pascalTriangle(rangepascaltriangle,2)
```

⟨*GALUTIL.dotabb*⟩≡

```
"GALUTIL" [color="#FF4488",href="bookvol10.4.pdf#nameddest=GALUTIL"]
"A1AGG" [color="#4488FF",href="bookvol10.2.pdf#nameddest=A1AGG"]
"GALUTIL" -> "A1AGG"
```

8.9 package GAUSSFAC GaussianFactorization-Package

8.10 GaussianFactorizationPackage



1.00

Exports:
factor prime? sumSquares

⟨*package GAUSSFAC GaussianFactorizationPackage*⟩≡

```
)abbrev package GAUSSFAC GaussianFactorizationPackage
++ Author: Patrizia Gianni
++ Date Created: Summer 1986
++ Date Last Updated:
++ Basic Functions:
++ Related Constructors:
++ Also See:
++ AMS Classifications:
++ Keywords:
++ References:
++ Description: Package for the factorization of complex or gaussian
++ integers.
GaussianFactorizationPackage() : C == T
 where
  NNI  ==  NonNegativeInteger
  Z      ==> Integer
  ZI     ==> Complex Z
  FRZ    ==> Factored ZI
  fUnion ==> Union("nil", "sqfr", "irred", "prime")
  FFE    ==> Record(flg:fUnion, fctr:ZI, xpnt:Integer)

  C  == with
     factor      :     ZI     ->     FRZ
       ++ factor(zi) produces the complete factorization of the complex
       ++ integer zi.
     sumSquares  :     Z      ->    List Z
```

```
    ++ sumSquares(p) construct \spad{a} and b such that \spad{a**2+b**2}
    ++ is equal to
    ++ the integer prime p, and otherwise returns an error.
    ++ It will succeed if the prime number p is 2 or congruent to 1
    ++ mod 4.
  prime?      :     ZI     ->    Boolean
    ++ prime?(zi) tests if the complex integer zi is prime.

T == add
  import IntegerFactorizationPackage Z

  reduction(u:Z,p:Z):Z ==
    p=0 => u
    positiveRemainder(u,p)

  merge(p:Z,q:Z):Union(Z,"failed") ==
    p = q => p
    p = 0 => q
    q = 0 => p
    "failed"

  exactquo(u:Z,v:Z,p:Z):Union(Z,"failed") ==
     p=0 => u exquo v
     v rem p = 0 => "failed"
     positiveRemainder((extendedEuclidean(v,p,u)::Record(coef1:Z,coef2:Z)).coef1,p)

  FMod := ModularRing(Z,Z,reduction,merge,exactquo)

  fact2:ZI:= complex(1,1)

          ----  find the solution of x**2+1 mod q  ----
  findelt(q:Z) : Z ==
    q1:=q-1
    r:=q1
    r1:=r exquo 4
    while ^(r1 case "failed") repeat
      r:=r1::Z
      r1:=r exquo 2
    s : FMod := reduce(1,q)
    qq1:FMod :=reduce(q1,q)
    for i in 2.. while (s=1 or s=qq1) repeat
      s:=reduce(i,q)**(r::NNI)
    t:=s
    while t^=qq1 repeat
      s:=t
      t:=t**2
```

```
  s::Z


---- write p, congruent to 1 mod 4, as a sum of two squares ----
sumsq1(p:Z) : List Z ==
  s:= findelt(p)
  u:=p
  while u**2>p repeat
    w:=u rem s
    u:=s
    s:=w
  [u,s]

       ---- factorization of an integer  ----
intfactor(n:Z) : Factored ZI ==
  lfn:= factor n
  r : List FFE :=[]
  unity:ZI:=complex(unit lfn,0)
  for term in (factorList lfn) repeat
    n:=term.fctr
    exp:=term.xpnt
    n=2 =>
      r :=concat(["prime",fact2,2*exp]$FFE,r)
      unity:=unity*complex(0,-1)**(exp rem 4)::NNI

    (n rem 4) = 3 => r:=concat(["prime",complex(n,0),exp]$FFE,r)

    sz:=sumsq1(n)
    z:=complex(sz.1,sz.2)
    r:=concat(["prime",z,exp]$FFE,
             concat(["prime",conjugate(z),exp]$FFE,r))
  makeFR(unity,r)

      ---- factorization of a gaussian number  ----
factor(m:ZI) : FRZ ==
  m=0 => primeFactor(0,1)
  a:= real m

  (b:= imag m)=0 => intfactor(a) :: FRZ

  a=0 =>
    ris:=intfactor(b)
    unity:= unit(ris)*complex(0,1)
    makeFR(unity,factorList ris)

  d:=gcd(a,b)
```

```
    result : List FFE :=[]
    unity:ZI:=1$ZI

    if d^=1 then
      a:=(a exquo d)::Z
      b:=(b exquo d)::Z
      r:= intfactor(d)
      result:=factorList r
      unity:=unit r
      m:=complex(a,b)

    n:Z:=a**2+b**2
    factn:= factorList(factor n)
    part:FFE:=["prime",0$ZI,0]
    for term in factn repeat
      n:=term.fctr
      exp:=term.xpnt
      n=2 =>
        part:= ["prime",fact2,exp]$FFE
        m:=m quo (fact2**exp:NNI)
        result:=concat(part,result)

      (n rem 4) = 3 =>
        g0:=complex(n,0)
        part:= ["prime",g0,exp quo 2]$FFE
        m:=m quo g0
        result:=concat(part,result)

      z:=gcd(m,complex(n,0))
      part:= ["prime",z,exp]$FFE
      z:=z**(exp:NNI)
      m:=m quo z
      result:=concat(part,result)

    if m^=1 then unity:=unity * m
    makeFR(unity,result)

        ----  write p prime like sum of two squares  ----
  sumSquares(p:Z) : List Z ==
    p=2 => [1,1]
    p rem 4 ^= 1 => error "no solutions"
    sumsq1(p)


  prime?(a:ZI) : Boolean ==
     n : Z := norm a
```

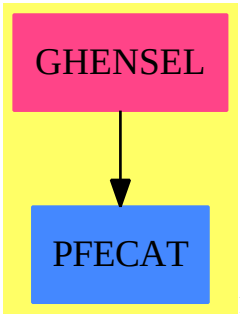
```
n=0 => false            -- zero
n=1 => false            -- units
prime?(n)$IntegerPrimesPackage(Z)  => true
re : Z := real a
im : Z := imag a
re^=0 and im^=0 => false
p : Z := abs(re+im)     -- a is of the form p, -p, %i*p or -%i*p
p rem 4 ^= 3 => false
-- return-value true, if p is a rational prime,
-- and false, otherwise
prime?(p)$IntegerPrimesPackage(Z)
```

⟨*GAUSSFAC.dotabb*⟩≡

```
"GAUSSFAC" [color="#FF4488",href="bookvol10.4.pdf#nameddest=GAUSSFAC"]
"COMPCAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=COMPCAT"]
"GAUSSFAC" -> "COMPCAT"
```

8.11 package GHENSEL GeneralHenselPackage

8.12 GeneralHenselPackage



1.00

Exports:
completeHensel HenselLift reduction

⟨*package GHENSEL GeneralHenselPackage*⟩≡
```
)abbrev package GHENSEL GeneralHenselPackage
++ Author : P.Gianni
++ General Hensel Lifting
++ Used for Factorization of bivariate polynomials over a finite field.
GeneralHenselPackage(RP,TP):C == T where
   RP :   EuclideanDomain
   TP :   UnivariatePolynomialCategory RP

   PI ==> PositiveInteger

   C == with
      HenselLift: (TP,List(TP),RP,PI) -> Record(plist:List(TP), modulo:RP)
        ++ HenselLift(pol,lfacts,prime,bound) lifts lfacts,
        ++ that are the factors of pol mod prime,
        ++ to factors of pol mod prime**k > bound. No recombining is done .

      completeHensel: (TP,List(TP),RP,PI) -> List TP
        ++ completeHensel(pol,lfact,prime,bound) lifts lfact,
        ++ the factorization mod prime of pol,
        ++ to the factorization mod prime**k>bound.
        ++ Factors are recombined on the way.

      reduction     :  (TP,RP)  ->  TP
        ++ reduction(u,pol) computes the symmetric reduction of u mod pol

   T == add
     GenExEuclid: (List(FP),List(FP),FP) -> List(FP)
```

```
HenselLift1: (TP,List(TP),List(FP),List(FP),RP,RP,F) -> List(TP)
mQuo: (TP,RP) -> TP

reduceCoef(c:RP,p:RP):RP ==
   zero? p => c
   RP is Integer => symmetricRemainder(c,p)
   c rem p

reduction(u:TP,p:RP):TP ==
   zero? p => u
   RP is Integer => map(x+->symmetricRemainder(x,p),u)
   map(x+->x rem p,u)

merge(p:RP,q:RP):Union(RP,"failed") ==
    p = q => p
    p = 0 => q
    q = 0 => p
    "failed"

modInverse(c:RP,p:RP):RP ==
   (extendedEuclidean(c,p,1)::Record(coef1:RP,coef2:RP)).coef1

exactquo(u:TP,v:TP,p:RP):Union(TP,"failed") ==
   invlcv:=modInverse(leadingCoefficient v,p)
   r:=monicDivide(u,reduction(invlcv*v,p))
   reduction(r.remainder,p) ^=0 => "failed"
   reduction(invlcv*r.quotient,p)

FP:=EuclideanModularRing(RP,TP,RP,reduction,merge,exactquo)

mQuo(poly:TP,n:RP) : TP == map(x+->x quo n,poly)

GenExEuclid(fl:List FP,cl:List FP,rhs:FP) :List FP ==
   [clp*rhs rem flp for clp in cl for flp in fl]

-- generate the possible factors
genFact(fln:List TP,factlist:List List TP) : List List TP ==
  factlist=[] => [[pol] for pol in fln]
  maxd := +/[degree f for f in fln] quo 2
  auxfl:List List TP := []
  for poly in fln while factlist^=[] repeat
    factlist := [term for term in factlist | ^member?(poly,term)]
    dp := degree poly
    for term in factlist repeat
      (+/[degree f for f in term]) + dp > maxd => "next term"
      auxfl := cons(cons(poly,term),auxfl)
```

```
  auxfl

HenselLift1(poly:TP,fln:List TP,fl1:List FP,cl1:List FP,
            prime:RP,Modulus:RP,cinv:RP):List TP ==
   lcp := leadingCoefficient poly
   rhs := reduce(mQuo(poly - lcp * */fln,Modulus),prime)
   zero? rhs => fln
   lcinv:=reduce(cinv::TP,prime)
   vl := GenExEuclid(fl1,cl1,lcinv*rhs)
   [flp + Modulus*(vlp::TP) for flp in fln for vlp in vl]

HenselLift(poly:TP,tl1:List TP,prime:RP,bound:PI) ==
   -- convert tl1
   constp:TP:=0
   if degree first tl1 = 0 then
      constp:=tl1.first
      tl1 := rest tl1
   fl1:=[reduce(ttl,prime) for ttl in tl1]
   cl1 := multiEuclidean(fl1,1)::List FP
   Modulus:=prime
   fln :List TP := [ffl1::TP for ffl1 in fl1]
   lcinv:RP:=retract((inv
             (reduce((leadingCoefficient poly)::TP,prime)))::TP)
   while euclideanSize(Modulus)<bound repeat
      nfln:=HenselLift1(poly,fln,fl1,cl1,prime,Modulus,lcinv)
      fln = nfln and zero?(err:=poly-*/fln) => leave "finished"
      fln := nfln
      Modulus := prime*Modulus
   if constp^=0 then fln:=cons(constp,fln)
   [fln,Modulus]

completeHensel(m:TP,tl1:List TP,prime:RP,bound:PI) ==
 hlift:=HenselLift(m,tl1,prime,bound)
 Modulus:RP:=hlift.modulo
 fln:List TP:=hlift.plist
 nm := degree m
 u:Union(TP,"failed")
 aux,auxl,finallist:List TP
 auxfl,factlist:List List TP
 factlist := []
 dfn :NonNegativeInteger := nm
 lcm1 := leadingCoefficient m
 mm := lcm1*m
 while dfn>0 and (factlist := genFact(fln,factlist))^=[] repeat
   auxfl := []
   while factlist^=[] repeat
```

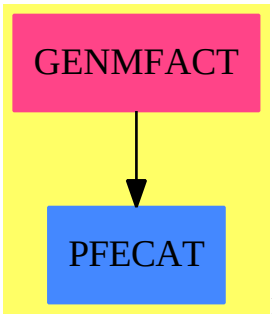
```
            auxl := factlist.first
            factlist := factlist.rest
            tc := reduceCoef((lcm1 * */[coefficient(poly,0)
                           for poly in auxl]), Modulus)
            coefficient(mm,0) exquo tc case "failed" =>
              auxfl := cons(auxl,auxfl)
            pol := */[poly for poly in auxl]
            poly :=reduction(lcm1*pol,Modulus)
            u := mm exquo poly
            u case "failed"  => auxfl := cons(auxl,auxfl)
            poly1: TP := primitivePart poly
            m := mQuo((u::TP),leadingCoefficient poly1)
            lcm1 := leadingCoefficient(m)
            mm := lcm1*m
            finallist := cons(poly1,finallist)
            dfn := degree m
            aux := []
            for poly in fln repeat
              ^member?(poly,auxl) => aux := cons(poly,aux)
              auxfl := [term for term in auxfl | ^member?(poly,term)]
              factlist := [term for term in factlist |^member?(poly,term)]
            fln := aux
          factlist := auxfl
        if dfn > 0 then finallist := cons(m,finallist)
        finallist
```

⟨*GHENSEL.dotabb*⟩≡

```
"GHENSEL" [color="#FF4488",href="bookvol10.4.pdf#nameddest=GHENSEL"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"GHENSEL" -> "PFECAT"
```

8.13 package GENMFACT GeneralizedMultivariateFactorize

8.14 GeneralizedMultivariateFactorize



1.00

Exports:
factor

⟨*package GENMFACT GeneralizedMultivariateFactorize*⟩≡
```
)abbrev package GENMFACT GeneralizedMultivariateFactorize
++ Author: P. Gianni
++ Date Created: 1983
++ Date Last Updated: Sept. 1990
++ Basic Functions:
++ Related Constructors: MultFiniteFactorize, AlgebraicMultFact, MultivariateFactorize
++ Also See:
++ AMS Classifications:
++ Keywords:
++ References:
++ Description:
++   This is the top level package for doing multivariate factorization
++ over basic domains like \spadtype{Integer} or \spadtype{Fraction Integer}.

GeneralizedMultivariateFactorize(OV,E,S,R,P) : C == T
 where
  R          :   IntegralDomain
                   -- with factor on R[x]
  S          :   IntegralDomain
  OV    :   OrderedSet  with
                convert : % -> Symbol
                  ++ convert(x) converts x to a symbol
                variable: Symbol -> Union(%, "failed")
                  ++ variable(s) makes an element from symbol s or fails.
  E          :   OrderedAbelianMonoidSup
  P          :   PolynomialCategory(R,E,OV)
```

```
C == with
  factor      :      P  ->  Factored P
    ++ factor(p) factors the multivariate polynomial p over its coefficient
    ++ domain

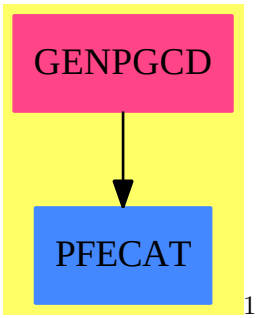
T == add
  factor(p:P) : Factored P ==
    R has FiniteFieldCategory => factor(p)$MultFiniteFactorize(OV,E,R,P)
    R is Polynomial(S) and S has EuclideanDomain =>
       factor(p)$MPolyCatPolyFactorizer(E,OV,S,P)
    R is Fraction(S) and S has CharacteristicZero and
      S has EuclideanDomain =>
          factor(p)$MRationalFactorize(E,OV,S,P)
    R is Fraction Polynomial S =>
       factor(p)$MPolyCatRationalFunctionFactorizer(E,OV,S,P)
    R has CharacteristicZero and R has EuclideanDomain =>
             factor(p)$MultivariateFactorize(OV,E,R,P)
    squareFree p
```

⟨*GENMFACT.dotabb*⟩≡

```
"GENMFACT" [color="#FF4488",href="bookvol10.4.pdf#nameddest=GENMFACT"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"GENMFACT" -> "PFECAT"
```

8.15 package GENPGCD GeneralPolynomialGcdPackage

8.16 GeneralPolynomialGcdPackage



1.00

Exports:
gcdPolynomial randomR

⟨*package GENPGCD GeneralPolynomialGcdPackage*⟩≡
```
)abbrev package GENPGCD GeneralPolynomialGcdPackage
++ Description:
++ This package provides operations for GCD computations
++ on polynomials
GeneralPolynomialGcdPackage(E,OV,R,P):C == T where
    R     :  PolynomialFactorizationExplicit
    P     :  PolynomialCategory(R,E,OV)
    OV    :  OrderedSet
    E     :  OrderedAbelianMonoidSup

    SUPP      ==> SparseUnivariatePolynomial P
--JHD    ContPrim  ==> Record(cont:P,prim:P)

    C == with
           gcdPolynomial     :   (SUPP,SUPP) -> SUPP
                ++ gcdPolynomial(p,q) returns the GCD of p and q
           randomR        : ()                ->R
                ++ randomR() should be local but conditional
--JHD      gcd               :   (P,P)    -> P
--JHD      gcd               :   List P   -> P
--JHD      gcdprim           :   (P,P)    -> P
--JHD      gcdprim           :   List P   -> P

--JHD      gcdcofact         :   List P   -> List P
--JHD      gcdcofactprim     :   List P   -> List P
```

```
--JHD      primitate          :   (P,OV)   -> P
--JHD      primitate          :    SUPP    -> SUPP

--JHD      content            :     P      -> P
--JHD      content            :   List P   -> List P
--JHD      contprim           :   List P   -> List ContPrim

--JHD      monomContent       :   (P,OV)   -> P
--JHD      monomContent       :    SUPP    -> SUPP


    T == add

      SUPR     ==> SparseUnivariatePolynomial R
--JHD      SUPLGcd  ==> Record(locgcd:SUPP,goodint:List R)
--JHD      LGcd     ==> Record(locgcd:P,goodint:List R)
--JHD      UTerm    ==> Record(lpol:List SUPR,lint:List R,mpol:P)
--JHD--JHD      pmod:R   :=  (prevPrime(2**26)$IntegerPrimesPackage(Integer))::R

--JHD      import MultivariateLifting(E,OV,R,P,pmod)
      import UnivariatePolynomialCategoryFunctions2(R,SUPR,P,SUPP)
      import UnivariatePolynomialCategoryFunctions2(P,SUPP,R,SUPR)
                --------  Local  Functions  --------

--JHD      abs              :      P       -> P
      better          :    (P,P)     -> Boolean
--JHD      failtest         :   (P,P,P)    -> Boolean
--JHD      gcdMonom         :  (P,P,OV)    -> P
--JHD      gcdTermList      :    (P,P)     -> P
--JHD      gcdPrim          :  (P,P,OV)    -> P
--JHD      gcdSameMainvar   :  (P,P,OV)    -> P
--JHD      internal         :  (P,P,OV)    -> P
--JHD      good             :  (P,List OV) -> Record(upol:SUPR,inval:List R)
--JHD      gcdPrs           : (P,P,NNI,OV) -> Union(P,"failed")
--JHD
--JHD      chooseVal        :     (P,P,List OV)          -> UTerm
--JHD      localgcd         :     (P,P,List OV)          -> LGcd
--JHD      notCoprime       :  (P,P, List NNI,List OV)   -> P
--JHD      imposelc         : (List SUPR,List OV,List R,List P)  -> List SUPR

--JHD      lift? :(P,P,UTerm,List NNI,List OV)  -> Union("failed",P)
--      lift  :(P,SUPR,SUPR,P,List OV,List NNI,List R) -> P
      lift  :  (SUPR,SUPP,SUPR,List OV,List R) -> Union(SUPP,"failed")
         -- lifts first and third arguments as factors of the second
         -- fourth is number of variables.
--JHD      monomContent       :   (P,OV)   -> P
```

```
      monomContentSup   :    SUPP    -> SUPP
 --
 --JHD  gcdcofact         :   List P   -> List P

      gcdTrivial      :  (SUPP,SUPP)   -> SUPP
      gcdSameVariables:  (SUPP,SUPP,List OV)    -> SUPP
      recursivelyGCDCoefficients: (SUPP,List OV,SUPP,List OV) -> SUPP
      flatten         : (SUPP,List OV) -> SUPP
                        -- evaluates out all variables in the second
                        -- argument, leaving a polynomial of the same
                        -- degree
 --    eval            : (SUPP,List OV,List R) -> SUPP
      variables       :  SUPP          -> List OV
                     ---- JHD's exported functions ---
      gcdPolynomial(p1:SUPP,p2:SUPP) ==
        zero? p1 => p2
        zero? p2 => p1
        0=degree p1  => gcdTrivial(p1,p2)
        0=degree p2  => gcdTrivial(p2,p1)
        if degree p1 < degree p2 then (p1,p2):=(p2,p1)
        p1 exquo p2 case SUPP  => (unitNormal p2).canonical
        c1:= monomContentSup(p1)
        c2:= monomContentSup(p2)
        p1:= (p1 exquo c1)::SUPP
        p2:= (p2 exquo c2)::SUPP
        (p1 exquo p2) case SUPP => (unitNormal p2).canonical * gcd(c1,c2)
        vp1:=variables p1
        vp2:=variables p2
        v1:=setDifference(vp1,vp2)
        v2:=setDifference(vp2,vp1)
        #v1 = 0 and #v2 = 0 => gcdSameVariables(p1,p2,vp1)*gcd(c1,c2)
                  -- all variables are in common
        v:=setDifference(vp1,v1)
        pp1:=flatten(p1,v1)
        pp2:=flatten(p2,v2)
        g:=gcdSameVariables(pp1,pp2,v)
 --       one? g => gcd(c1,c2)::SUPP
        (g = 1) => gcd(c1,c2)::SUPP
        (#v1 = 0 or not (p1 exquo g) case "failed") and
                     -- if #vi = 0 then pp1 = p1, so we know g divides
              (#v2 = 0 or not (p2 exquo g) case "failed")
            => g*gcd(c1,c2)  -- divdes them both, so is the gcd
        -- OK, so it's not the gcd: try again
        v:=variables g -- there can be at most these variables in answer
        v1:=setDifference(vp1,v)
        v2:=setDifference(vp2,v)
```

```
        if (#v1 = 0) then g:= gcdSameVariables(g,flatten(p2,v2),v)
        else if (#v2=0) then g:=gcdSameVariables(g,flatten(p1,v1),v)
        else g:=gcdSameVariables(g,flatten(p1,v1)-flatten(p2,v2),v)
--        one? g => gcd(c1,c2)::SUPP
        (g = 1) => gcd(c1,c2)::SUPP
        (#v1 = 0 or not (p1 exquo g) case "failed") and
              (#v2 = 0 or not (p2 exquo g) case "failed")
            => g*gcd(c1,c2)::SUPP  -- divdes them both, so is the gcd
        v:=variables g -- there can be at most these variables in answer
        v1:=setDifference(vp1,v)
        if #v1 ^= 0 then
           g:=recursivelyGCDCoefficients(g,v,p1,v1)
--           one? g => return gcd(c1,c2)::SUPP
           (g = 1) => return gcd(c1,c2)::SUPP
           v:=variables g -- there can be at most these variables in answer
        v2:=setDifference(vp2,v)
        recursivelyGCDCoefficients(g,v,p2,v2)*gcd(c1,c2)
      if R has StepThrough then
         randomCount:R := init()
         randomR() ==
            (v:=nextItem(randomCount)) case R =>
                randomCount:=v
                v
            SAY("Taking next stepthrough range in GeneralPolynomialGcdPackage")$L
            randomCount:=init()
            randomCount
      else
            randomR() == (random$Integer() rem 100)::R
                     ---- JHD's local functions ---
      gcdSameVariables(p1:SUPP,p2:SUPP,lv:List OV) ==
            -- two non-trivial primitive (or, at least, we don't care
            -- about content)
            -- polynomials with precisely the same degree
          #lv = 0 => map((x:R):P+->x::P,gcdPolynomial(map(ground,p1),
                                              map(ground,p2)))
          degree p2 = 1 =>
            p1 exquo p2 case SUPP => p2
            1
          gcdLC:=gcd(leadingCoefficient p1,leadingCoefficient p2)
          lr:=[randomR() for vv in lv]
          count:NonNegativeInteger:=0
          while count<10 repeat
            while zero? eval(gcdLC,lv,lr) and count<10 repeat
                  lr:=[randomR() for vv in lv]
                  count:=count+1
            count = 10 => error "too many evaluations in GCD code"
```

```
        up1:SUPR:=map(y+->ground eval(y,lv,lr),p1)
        up2:SUPR:=map(z+->ground eval(z,lv,lr),p2)
        u:=gcdPolynomial(up1,up2)
        degree u = 0 => return 1
        -- let's pick a second one, just to check
        lrr:=[randomR() for vv in lv]
        while zero? eval(gcdLC,lv,lrr) and count<10 repeat
              lrr:=[randomR() for vv in lv]
              count:=count+1
        count = 10 => error "too many evaluations in GCD code"
        vp1:SUPR:=map(x1+->ground eval(x1,lv,lrr),p1)
        vp2:SUPR:=map(y1+->ground eval(y1,lv,lrr),p2)
        v:=gcdPolynomial(vp1,vp2)
        degree v = 0 => return 1
        if degree v < degree u then
           u:=v
           up1:=vp1
           up2:=vp2
           lr:=lrr
        up1:=(up1 exquo u)::SUPR
        degree gcd(u,up1) = 0 =>
            ans:=lift(u,p1,up1,lv,lr)
            ans case SUPP => return ans
            "next"
        up2:=(up2 exquo u)::SUPR
        degree gcd(u,up2) = 0 =>
            ans:=lift(u,p2,up2,lv,lr)
            ans case SUPP => return ans
            "next"
        -- so neither cofactor is relatively prime
        count:=0
        while count < 10 repeat
           r:=randomR()
           uu:=up1+r*up2
           degree gcd(u,uu)=0 =>
             ans:= lift(u,p1+r::P *p2,uu,lv,lr)
             ans case SUPP => return ans
             "next"
        error "too many evaluations in GCD code"
      count >= 10 => error "too many evaluations in GCD code"
    lift(gR:SUPR,p:SUPP,cfR:SUPR,lv:List OV,lr:List R) ==
      -- lift the coprime factorisation gR*cfR = (univariate of p)
      -- where the variables lv have been evaluated at lr
      lcp:=leadingCoefficient p
      g:=monomial(lcp,degree gR)+map(x+->x::P,reductum gR)
      cf:=monomial(lcp,degree cfR)+map(y+->y::P,reductum cfR)
```

```
      p:=lcp*p       -- impose leaidng coefficient of p on each factor
      while lv ^= [] repeat
         v:=first lv
         r:=first lr
         lv:=rest lv
         lr:=rest lr
         thisp:=map(x1+->eval(x1,lv,lr),p)
         d:="max"/[degree(c,v) for c in coefficients p]
         prime:=v::P - r::P
         pn:=prime
         origFactors:=[g,cf]::List SUPP
         for n in 1..d repeat
            Ecart:=(thisp- g*cf) exquo pn
            Ecart case "failed" =>
               error "failed lifting in hensel in Complex Polynomial GCD"
            zero? Ecart => leave
            step:=solveLinearPolynomialEquation(origFactors,
                     map(x2+->eval(x2,v,r),Ecart::SUPP))
            step case "failed" => return "failed"
            g:=g+pn*first step
            cf:=cf+pn*second step
            pn:=pn*prime
         thisp ^= g*cf => return "failed"
      g
    recursivelyGCDCoefficients(g:SUPP,v:List OV,p:SUPP,pv:List OV) ==
       mv:=first pv   -- take each coefficient w.r.t. mv
       pv:=rest pv    -- and recurse on pv as necessary
       d:="max"/[degree(u,mv) for u in coefficients p]
       for i in 0..d repeat
           p1:=map(x+->coefficient(x,mv,i),p)
           oldg:=g
           if pv = [] then g:=gcdSameVariables(g,p1,v)
           else g:=recursivelyGCDCoefficients(p,v,p1,pv)
--           one? g => return 1
           (g = 1) => return 1
           g^=oldg =>
              oldv:=v
              v:=variables g
              pv:=setUnion(pv,setDifference(v,oldv))
       g
    flatten(p1:SUPP,lv:List OV) ==
       #lv = 0 => p1
       lr:=[ randomR() for vv in lv]
       dg:=degree p1
       while dg ^= degree (ans:= map(x+->eval(x,lv,lr),p1)) repeat
         lr:=[ randomR() for vv in lv]
```

```
         ans
--      eval(p1:SUPP,lv:List OV,lr:List R) == map(eval(#1,lv,lr),p1)
      variables(p1:SUPP) ==
        removeDuplicates ("concat"/[variables u for u in coefficients p1])
      gcdTrivial(p1:SUPP,p2:SUPP) ==
        -- p1 is non-zero, but has degree zero
        -- p2 is non-zero
        cp1:=leadingCoefficient p1
--        one? cp1 => 1
        (cp1 = 1) => 1
        degree p2 = 0 => gcd(cp1,leadingCoefficient p2)::SUPP
        un?:=unit? cp1
        while not zero? p2 and not un? repeat
           cp1:=gcd(leadingCoefficient p2,cp1)
           un?:=unit? cp1
           p2:=reductum p2
        un? => 1
        cp1::SUPP

                    ---- Local  functions ----
--JHD    -- test if something wrong happened in the gcd
--JHD      failtest(f:P,p1:P,p2:P) : Boolean ==
--JHD        (p1 exquo f) case "failed" or (p2 exquo f) case "failed"
--JHD
--JHD    -- Choose the integers
--JHD      chooseVal(p1:P,p2:P,lvar:List OV):UTerm ==
--JHD        x:OV:=lvar.first
--JHD        lvr:=lvar.rest
--JHD        d1:=degree(p1,x)
--JHD        d2:=degree(p2,x)
--JHD        dd:NNI:=0$NNI
--JHD        nvr:NNI:=#lvr
--JHD        lval:List R :=[]
--JHD        range:I:=8
--JHD        for i in 1..  repeat
--JHD          range:=2*range
--JHD          lval:=[(random()$I rem (2*range) - range)::R  for i in 1..nvr]
--JHD          uf1:SUPR:=univariate eval(p1,lvr,lval)
--JHD          degree uf1 ^= d1 => "new point"
--JHD          uf2:SUPR:=univariate eval(p2,lvr,lval)
--JHD          degree uf2 ^= d2 => "new point"
--JHD          u:=gcd(uf1,uf2)
--JHD          du:=degree u
--JHD         --the univariate gcd is 1
--JHD          if du=0 then return [[1$SUPR],lval,0$P]$UTerm
--JHD
```

```
--JHD          ugcd:List SUPR:=[u,(uf1 exquo u)::SUPR,(uf2 exquo u)::SUPR]
--JHD          uterm:=[ugcd,lval,0$P]$UTerm
--JHD          dd=0 => dd:=du
--JHD
--JHD        --the degree is not changed
--JHD          du=dd =>
--JHD
--JHD           --test if one of the polynomials is the gcd
--JHD            dd=d1 =>
--JHD              if ^((f:=p2 exquo p1) case "failed") then
--JHD                return [[u],lval,p1]$UTerm
--JHD              if dd^=d2 then dd:=(dd-1)::NNI
--JHD
--JHD            dd=d2 =>
--JHD              if ^((f:=p1 exquo p2) case "failed") then
--JHD                return [[u],lval,p2]$UTerm
--JHD              dd:=(dd-1)::NNI
--JHD            return uterm
--JHD
--JHD         --the new gcd has degree less
--JHD          du<dd => dd:=du
--JHD
--JHD      good(f:P,lvr:List OV):Record(upol:SUPR,inval:List R) ==
--JHD        nvr:NNI:=#lvr
--JHD        range:I:=1
--JHD        ltry:List List R:=[]
--JHD        while true repeat
--JHD          range:=2*range
--JHD          lval:=[(random()$I rem (2*range) -range)::R  for i in 1..nvr]
--JHD          member?(lval,ltry) => "new point"
--JHD          ltry:=cons(lval,ltry)
--JHD          uf:=univariate eval(f,lvr,lval)
--JHD          if degree gcd(uf,differentiate uf)=0 then return [uf,lval]
--JHD
--JHD      -- impose the right lc
--JHD      imposelc(lipol:List SUPR,
--JHD               lvar:List OV,lval:List R,leadc:List P):List SUPR ==
--JHD        result:List SUPR :=[]
--JHD        lvar:=lvar.rest
--JHD        for pol in lipol for leadpol in leadc repeat
--JHD           p1:= univariate eval(leadpol,lvar,lval) * pol
--JHD           result:= cons((p1 exquo leadingCoefficient pol)::SUPR,result)
--JHD        reverse result
--JHD
--JHD     --Compute the gcd between not coprime polynomials
--JHD      notCoprime(g:P,p2:P,ldeg:List NNI,lvar:List OV) : P ==
```

```
--JHD        x:OV:=lvar.first
--JHD        lvar1:List OV:=lvar.rest
--JHD        lg1:=gcdcofact([g,differentiate(g,x)])
--JHD        g1:=lg1.1
--JHD        lg:LGcd:=localgcd(g1,p2,lvar)
--JHD        (l,lval):=(lg.locgcd,lg.goodint)
--JHD        p2:=(p2 exquo l)::P
--JHD        (gd1,gd2):=(l,l)
--JHD        ul:=univariate(eval(l,lvar1,lval))
--JHD        dl:=degree ul
--JHD        if degree gcd(ul,differentiate ul) ^=0 then
--JHD          newchoice:=good(l,lvar.rest)
--JHD          ul:=newchoice.upol
--JHD          lval:=newchoice.inval
--JHD        ug1:=univariate(eval(g1,lvar1,lval))
--JHD        ulist:=[ug1,univariate eval(p2,lvar1,lval)]
--JHD        lcpol:=[leadingCoefficient univariate(g1,x),
--JHD                leadingCoefficient univariate(p2,x)]
--JHD        while true repeat
--JHD          d:SUPR:=gcd(cons(ul,ulist))
--JHD          if degree d =0 then return gd1
--JHD          lquo:=(ul exquo d)::SUPR
--JHD          if degree lquo ^=0 then
--JHD            lgcd:=gcd(cons(leadingCoefficient univariate(l,x),lcpol))
--JHD            gd2:=lift(l,d,lquo,lgcd,lvar,ldeg,lval)
--JHD            l:=gd2
--JHD            ul:=univariate(eval(l,lvar1,lval))
--JHD            dl:=degree ul
--JHD          gd1:=gd1*gd2
--JHD          ulist:=[(uf exquo d)::SUPR for uf in ulist]
--JHD
--JHD -- we suppose that the poly have the same mainvar, deg p1<deg p2 and the
--JHD -- polys primitive
--JHD      internal(p1:P,p2:P,x:OV) : P ==
--JHD        lvar:List OV:=sort(#1>#2,setUnion(variables p1,variables p2))
--JHD        d1:=degree(p1,x)
--JHD        d2:=degree(p2,x)
--JHD        result: P:=localgcd(p1,p2,lvar).locgcd
--JHD        -- special cases
--JHD        result=1 => 1$P
--JHD        (dr:=degree(result,x))=d1 or dr=d2  => result
--JHD        while failtest(result,p1,p2) repeat
--JHD          SAY$Lisp  "retrying gcd"
--JHD          result:=localgcd(p1,p2,lvar).locgcd
--JHD        result
--JHD
```

```
--JHD    --local function for the gcd : it returns the evaluation point too
--JHD      localgcd(p1:P,p2:P,lvar:List(OV)) : LGcd ==
--JHD        x:OV:=lvar.first
--JHD        uterm:=chooseVal(p1,p2,lvar)
--JHD        listpol:= uterm.lpol
--JHD        ud:=listpol.first
--JHD        dd:= degree ud
--JHD
--JHD        --the univariate gcd is 1
--JHD        dd=0 => [1$P,uterm.lint]$LGcd
--JHD
--JHD        --one of the polynomials is the gcd
--JHD        dd=degree(p1,x) or dd=degree(p2,x) =>
--JHD                                          [uterm.mpol,uterm.lint]$LGcd
--JHD        ldeg:List NNI:=map(min,degree(p1,lvar),degree(p2,lvar))
--JHD
--JHD       -- if there is a polynomial g s.t. g/gcd and gcd are coprime ...
--JHD        -- I can lift
--JHD        (h:=lift?(p1,p2,uterm,ldeg,lvar)) case "failed" =>
--JHD          [notCoprime(p1,p2,ldeg,lvar),uterm.lint]$LGcd
--JHD        [h::P,uterm.lint]$LGcd
--JHD
--JHD
--JHD  -- content, internal functions return the poly if it is a monomial
--JHD      monomContent(p:P,var:OV):P ==
--JHD        ground? p => 1$P
--JHD        md:= minimumDegree(p,var)
--JHD        ((var::P)**md)*(gcd sort(better,coefficients univariate(p,var)))

      monomContentSup(u:SUPP):SUPP ==
        degree(u) = 0$NonNegativeInteger => 1$SUPP
        md:= minimumDegree u
        gcd(sort(better,coefficients u)) * monomial(1$P,md)$SUPP

--JHD  -- change the polynomials to have positive lc
--JHD      abs(p:P): P == unitNormal(p).canonical

  -- Ordering for gcd purposes
      better(p1:P,p2:P):Boolean ==
        ground? p1 => true
        ground? p2 => false
        degree(p1,mainVariable(p1)::OV) < degree(p2,mainVariable(p2)::OV)

   -- PRS algorithm
   -- gcdPrs(p1:P,p2:P,d:NNI,var:OV):Union(P,"failed") ==
   --   u1:= univariate(p1,var)
```

```
    --   u2:= univariate(p2,var)
    --   finished:Boolean:= false
    --   until finished repeat
    --     dd:NNI:=(degree u1 - degree u2)::NNI
    --     lc1:SUPP:=leadingCoefficient u2 * reductum u1
    --     lc2:SUPP:=leadingCoefficient u1 * reductum u2
    --     u3:SUPP:= primitate((lc1-lc2)*monomial(1$P,dd))$%
    --     (d3:=degree(u3)) <= d => finished:= true
    --     u1:= u2
    --     u2:= u3
    --     if d3 > degree(u1) then (u1,u2):= (u2,u1)
    --   g:= (u2 exquo u3)
    --   g case SUPP => abs multivariate(u3,var)
    --   "failed"

  -- Gcd between polynomial p1 and p2 with
  -- mainVariable p1 < x=mainVariable p2
--JHD      gcdTermList(p1:P,p2:P) : P ==
--JHD        termList:=sort(better,
--JHD           cons(p1,coefficients univariate(p2,(mainVariable p2)::OV)))
--JHD        q:P:=termList.first
--JHD        for term in termList.rest until q = 1$P repeat q:= gcd(q,term)
--JHD        q
--JHD
--JHD  -- Gcd between polynomials with the same mainVariable
--JHD      gcdSameMainvar(p1:P,p2:P,mvar:OV): P ==
--JHD        if degree(p1,mvar) < degree(p2,mvar) then (p1,p2):= (p2,p1)
--JHD        (p1 exquo p2) case P => abs p2
--JHD        c1:= monomContent(p1,mvar)$%
--JHD        c1 = p1 => gcdMonom(p1,p2,mvar)
--JHD        c2:= monomContent(p2,mvar)$%
--JHD        c2 = p2 => gcdMonom(p2,p1,mvar)
--JHD        p1:= (p1 exquo c1)::P
--JHD        p2:= (p2 exquo c2)::P
--JHD        if degree(p1,mvar) < degree(p2,mvar) then (p1,p2):= (p2,p1)
--JHD        (p1 exquo p2) case P => abs(p2) * gcd(c1,c2)
--JHD        abs(gcdPrim(p1,p2,mvar)) * gcd(c1,c2)
--JHD
--JHD    --  make the polynomial primitive with respect to var
--JHD      primitate(p:P,var:OV):P == (p exquo monomContent(p,var))::P
--JHD
--JHD      primitate(u:SUPP):SUPP == (u exquo monomContentSup u)::SUPP
--JHD
--JHD    -- gcd between primitive polynomials with the same mainVariable
--JHD      gcdPrim(p1:P,p2:P,mvar:OV):P ==
--JHD        vars:= removeDuplicates append(variables p1,variables p2)
```

```
--JHD        #vars=1 => multivariate(gcd(univariate p1,univariate p2),mvar)
--JHD        vars:=delete(vars,position(mvar,vars))
--JHD        --d:= degModGcd(p1,p2,mvar,vars)
--JHD        --d case "failed" => internal(p2,p1,mvar)
--JHD        --deg:= d:NNI
--JHD        --deg = 0$NNI => 1$P
--JHD        --deg = degree(p1,mvar) =>
--JHD        --  (p2 exquo p1) case P => abs(p1)  -- already know that
--JHD                                            -- ^(p1 exquo p2)
--JHD        --  internal(p2,p1,mvar)
--JHD        --cheapPrs?(p1,p2,deg,mvar) =>
--JHD        --  g:= gcdPrs(p1,p2,deg,mvar)
--JHD        --  g case P => g::P
--JHD        --  internal(p2,p1,mvar)
--JHD        internal(p2,p1,mvar)
--JHD
--JHD    -- gcd between a monomial and a polynomial
--JHD       gcdMonom(m:P,p:P,var:OV):P ==
--JHD         ((var::P) ** min(minimumDegree(m,var),minimumDegree(p,var))) *
--JHD           gcdTermList(leadingCoefficient(univariate(m,var)),p)
--JHD
--JHD     --If there is a pol s.t. pol/gcd and gcd are coprime I can lift
--JHD       lift?(p1:P,p2:P,uterm:UTerm,ldeg:List NNI,
--JHD                       lvar:List OV) : Union("failed",P) ==
--JHD         x:OV:=lvar.first
--JHD         leadpol:Boolean:=false
--JHD         (listpol,lval):=(uterm.lpol,uterm.lint)
--JHD         d:=listpol.first
--JHD         listpol:=listpol.rest
--JHD         nolift:Boolean:=true
--JHD         for uf in listpol repeat
--JHD               --note uf and d not necessarily primitive
--JHD           degree gcd(uf,d) =0 => nolift:=false
--JHD         nolift => "failed"
--JHD         f:P:=([p1,p2]$List(P)).(position(uf,listpol))
--JHD         lgcd:=gcd(leadingCoefficient univariate(p1,x),
--JHD                   leadingCoefficient univariate(p2,x))
--JHD         lift(f,d,uf,lgcd,lvar,ldeg,lval)
--JHD
--JHD    -- interface with the general "lifting" function
--JHD       lift(f:P,d:SUPR,uf:SUPR,lgcd:P,lvar:List OV,
--JHD                         ldeg:List NNI,lval:List R):P ==
--JHD         x:OV:=lvar.first
--JHD         leadpol:Boolean:=false
--JHD         lcf:P
--JHD         lcf:=leadingCoefficient univariate(f,x)
```

```
--JHD        df:=degree(f,x)
--JHD        leadlist:List(P):=[]
--JHD
--JHD        if lgcd^=1$P then
--JHD          leadpol:=true
--JHD          f:=lgcd*f
--JHD          ldeg:=[n0+n1 for n0 in ldeg for n1 in degree(lgcd,lvar)]
--JHD          lcd:R:=leadingCoefficient d
--JHD          if ground? lgcd then d:=((retract lgcd) *d exquo lcd)::SUPR
--JHD          else d:=(retract(eval(lgcd,lvar.rest,lval)) * d exquo lcd)::SUPR
--JHD          uf:=lcd*uf
--JHD        leadlist:=[lgcd,lcf]
--JHD        lg:=imposelc([d,uf],lvar,lval,leadlist)
--JHD        plist:=lifting(univariate(f,x),lvar,lg,lval,leadlist,ldeg)::List P
--JHD        (p0:P,p1:P):=(plist.first,plist.2)
--JHD        if univariate eval(p0,rest lvar,lval) ^= lg.first then
--JHD           (p0,p1):=(p1,p0)
--JHD        ^leadpol => p0
--JHD        cprim:=contprim([p0])
--JHD        cprim.first.prim
--JHD
--JHD  -- Gcd for two multivariate polynomials
--JHD      gcd(p1:P,p2:P) : P ==
--JHD        (p1:= abs(p1)) = (p2:= abs(p2)) => p1
--JHD        ground? p1 =>
--JHD          p1 = 1$P => p1
--JHD          p1 = 0$P => p2
--JHD          ground? p2 => gcd((retract p1)@R,(retract p2)@R)::P
--JHD          gcdTermList(p1,p2)
--JHD        ground? p2 =>
--JHD          p2 = 1$P => p2
--JHD          p2 = 0$P => p1
--JHD          gcdTermList(p2,p1)
--JHD        mv1:= mainVariable(p1)::OV
--JHD        mv2:= mainVariable(p2)::OV
--JHD        mv1 = mv2 => gcdSameMainvar(p1,p2,mv1)
--JHD        mv1 < mv2 => gcdTermList(p1,p2)
--JHD        gcdTermList(p2,p1)
--JHD
--JHD  -- Gcd for a list of multivariate polynomials
--JHD      gcd(listp:List P) : P ==
--JHD        lf:=sort(better,listp)
--JHD        f:=lf.first
--JHD        for g in lf.rest repeat
--JHD          f:=gcd(f,g)
--JHD          if f=1$P then return f
```

```
--JHD        f
--JHD    -- Gcd and cofactors for a list of polynomials
--JHD       gcdcofact(listp : List P) : List P ==
--JHD          h:=gcd listp
--JHD          cons(h,[(f exquo h) :: P  for f in listp])
--JHD
--JHD    -- Gcd for primitive polynomials
--JHD       gcdprim(p1:P,p2:P):P ==
--JHD         (p1:= abs(p1)) = (p2:= abs(p2)) => p1
--JHD         ground? p1 =>
--JHD           ground? p2 => gcd((retract p1)@R,(retract p2)@R)::P
--JHD           p1 = 0$P => p2
--JHD           1$P
--JHD         ground? p2 =>
--JHD           p2 = 0$P => p1
--JHD           1$P
--JHD         mv1:= mainVariable(p1)::OV
--JHD         mv2:= mainVariable(p2)::OV
--JHD         mv1 = mv2 =>
--JHD           md:=min(minimumDegree(p1,mv1),minimumDegree(p2,mv1))
--JHD           mp:=1$P
--JHD           if md>1 then
--JHD             mp:=(mv1::P)**md
--JHD             p1:=(p1 exquo mp)::P
--JHD             p2:=(p2 exquo mp)::P
--JHD           mp*gcdPrim(p1,p2,mv1)
--JHD         1$P
--JHD
--JHD    -- Gcd for a list of primitive multivariate polynomials
--JHD       gcdprim(listp:List P) : P ==
--JHD         lf:=sort(better,listp)
--JHD         f:=lf.first
--JHD         for g in lf.rest repeat
--JHD           f:=gcdprim(f,g)
--JHD           if f=1$P then return f
--JHD         f
--JHD    -- Gcd and cofactors for a list of primitive polynomials
--JHD       gcdcofactprim(listp : List P) : List P ==
--JHD          h:=gcdprim listp
--JHD          cons(h,[(f exquo h) :: P  for f in listp])
--JHD
--JHD    -- content of a polynomial (with respect to its main var)
--JHD       content(f:P):P ==
--JHD         ground? f => f
--JHD         x:OV:=(mainVariable f)::OV
--JHD         gcd sort(better,coefficients univariate(f,x))
```

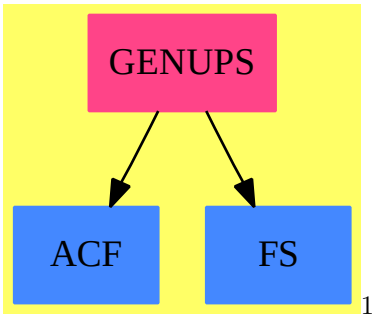
```
--JHD
--JHD   -- contents of a list of polynomials
--JHD      content(listf:List P) : List P == [content f for f in listf]
--JHD
--JHD   -- contents and primitive parts of a list  of polynomials
--JHD      contprim(listf:List P) : List ContPrim ==
--JHD        prelim :List P := content listf
--JHD        [[q,(f exquo q)::P]$ContPrim for q in prelim for f in listf]
--JHD
```

⟨*GENPGCD.dotabb*⟩≡

```
"GENPGCD" [color="#FF4488",href="bookvol10.4.pdf#nameddest=GENPGCD"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"GENPGCD" -> "PFECAT"
```

8.17 package GENUPS GenerateUnivariatePowerSeries

8.18 GenerateUnivariatePowerSeries



1.00

Exports:
laurent puiseux series taylor

⟨*package GENUPS GenerateUnivariatePowerSeries*⟩≡
```
)abbrev package GENUPS GenerateUnivariatePowerSeries
++ Author: Clifton J. Williamson
++ Date Created: 29 April 1990
++ Date Last Updated: 31 May 1990
++ Basic Operations:
++ Related Domains:
++ Also See:
++ AMS Classifications:
++ Keywords: series, Taylor, Laurent, Puiseux
++ Examples:
++ References:
++ Description:
++   \spadtype{GenerateUnivariatePowerSeries} provides functions that create
++   power series from explicit formulas for their \spad{n}th coefficient.
GenerateUnivariatePowerSeries(R,FE): Exports == Implementation where
  R  : Join(IntegralDomain,OrderedSet,RetractableTo Integer,_
            LinearlyExplicitRingOver Integer)
  FE : Join(AlgebraicallyClosedField,TranscendentalFunctionCategory,_
            FunctionSpace R)
  ANY1 ==> AnyFunctions1
  EQ   ==> Equation
  I    ==> Integer
  NNI  ==> NonNegativeInteger
  RN   ==> Fraction Integer
  SEG  ==> UniversalSegment
  ST   ==> Stream
```

```
SY   ==> Symbol
UTS  ==> UnivariateTaylorSeries
ULS  ==> UnivariateLaurentSeries
UPXS ==> UnivariatePuiseuxSeries

Exports ==> with
  taylor: (I -> FE,EQ FE) -> Any
    ++ \spad{taylor(n +-> a(n),x = a)} returns
    ++ \spad{sum(n = 0..,a(n)*(x-a)**n)}.
  taylor: (FE,SY,EQ FE) -> Any
    ++ \spad{taylor(a(n),n,x = a)} returns \spad{sum(n = 0..,a(n)*(x-a)**n)}.
  taylor: (I -> FE,EQ FE,SEG NNI) -> Any
    ++ \spad{taylor(n +-> a(n),x = a,n0..)} returns
    ++ \spad{sum(n=n0..,a(n)*(x-a)**n)};
    ++ \spad{taylor(n +-> a(n),x = a,n0..n1)} returns
    ++ \spad{sum(n = n0..,a(n)*(x-a)**n)}.
  taylor: (FE,SY,EQ FE,SEG NNI) -> Any
    ++ \spad{taylor(a(n),n,x = a,n0..)} returns
    ++ \spad{sum(n = n0..,a(n)*(x-a)**n)};
    ++ \spad{taylor(a(n),n,x = a,n0..n1)} returns
    ++ \spad{sum(n = n0..,a(n)*(x-a)**n)}.

  laurent: (I -> FE,EQ FE,SEG I) -> Any
    ++ \spad{laurent(n +-> a(n),x = a,n0..)} returns
    ++ \spad{sum(n = n0..,a(n) * (x - a)**n)};
    ++ \spad{laurent(n +-> a(n),x = a,n0..n1)} returns
    ++ \spad{sum(n = n0..n1,a(n) * (x - a)**n)}.
  laurent: (FE,SY,EQ FE,SEG I) -> Any
    ++ \spad{laurent(a(n),n,x=a,n0..)} returns
    ++ \spad{sum(n = n0..,a(n) * (x - a)**n)};
    ++ \spad{laurent(a(n),n,x=a,n0..n1)} returns
    ++ \spad{sum(n = n0..n1,a(n) * (x - a)**n)}.

  puiseux: (RN -> FE,EQ FE,SEG RN,RN) -> Any
    ++ \spad{puiseux(n +-> a(n),x = a,r0..,r)} returns
    ++ \spad{sum(n = r0,r0 + r,r0 + 2*r..., a(n) * (x - a)**n)};
    ++ \spad{puiseux(n +-> a(n),x = a,r0..r1,r)} returns
    ++ \spad{sum(n = r0 + k*r while n <= r1, a(n) * (x - a)**n)}.
  puiseux: (FE,SY,EQ FE,SEG RN,RN) -> Any
    ++ \spad{puiseux(a(n),n,x = a,r0..,r)} returns
    ++ \spad{sum(n = r0,r0 + r,r0 + 2*r..., a(n) * (x - a)**n)};
    ++ \spad{puiseux(a(n),n,x = a,r0..r1,r)} returns
    ++ \spad{sum(n = r0 + k*r while n <= r1, a(n) * (x - a)**n)}.

  series: (I -> FE,EQ FE) -> Any
    ++ \spad{series(n +-> a(n),x = a)} returns
```

```
    ++ \spad{sum(n = 0..,a(n)*(x-a)**n)}.
  series: (FE,SY,EQ FE) -> Any
    ++ \spad{series(a(n),n,x = a)} returns
    ++ \spad{sum(n = 0..,a(n)*(x-a)**n)}.
  series: (I -> FE,EQ FE,SEG I) -> Any
    ++ \spad{series(n +-> a(n),x = a,n0..)} returns
    ++ \spad{sum(n = n0..,a(n) * (x - a)**n)};
    ++ \spad{series(n +-> a(n),x = a,n0..n1)} returns
    ++ \spad{sum(n = n0..n1,a(n) * (x - a)**n)}.
  series: (FE,SY,EQ FE,SEG I) -> Any
    ++ \spad{series(a(n),n,x=a,n0..)} returns
    ++ \spad{sum(n = n0..,a(n) * (x - a)**n)};
    ++ \spad{series(a(n),n,x=a,n0..n1)} returns
    ++ \spad{sum(n = n0..n1,a(n) * (x - a)**n)}.
  series: (RN -> FE,EQ FE,SEG RN,RN) -> Any
    ++ \spad{series(n +-> a(n),x = a,r0..,r)} returns
    ++ \spad{sum(n = r0,r0 + r,r0 + 2*r..., a(n) * (x - a)**n)};
    ++ \spad{series(n +-> a(n),x = a,r0..r1,r)} returns
    ++ \spad{sum(n = r0 + k*r while n <= r1, a(n) * (x - a)**n)}.
  series: (FE,SY,EQ FE,SEG RN,RN) -> Any
    ++ \spad{series(a(n),n,x = a,r0..,r)} returns
    ++ \spad{sum(n = r0,r0 + r,r0 + 2*r..., a(n) * (x - a)**n)};
    ++ \spad{series(a(n),n,x = a,r0..r1,r)} returns
    ++ \spad{sum(n = r0 + k*r while n <= r1, a(n) * (x - a)**n)}.

Implementation ==> add

  genStream: (I -> FE,I) -> ST FE
  genStream(f,n) == delay concat(f(n),genStream(f,n + 1))

  genFiniteStream: (I -> FE,I,I) -> ST FE
  genFiniteStream(f,n,m) == delay
    n > m => empty()
    concat(f(n),genFiniteStream(f,n + 1,m))

  taylor(f,eq) ==
    (xx := retractIfCan(lhs eq)@Union(SY,"failed")) case "failed" =>
      error "taylor: left hand side must be a variable"
    x := xx :: SY; a := rhs eq
    coerce(series(genStream(f,0))$UTS(FE,x,a))$ANY1(UTS(FE,x,a))

  taylor(an:FE,n:SY,eq:EQ FE) ==
    taylor((i:I):FE +-> eval(an,(n::FE) = (i::FE)),eq)

  taylor(f:I -> FE,eq:EQ FE,seg:SEG NNI) ==
    (xx := retractIfCan(lhs eq)@Union(SY,"failed")) case "failed" =>
```

```
      error "taylor: left hand side must be a variable"
    x := xx :: SY; a := rhs eq
    hasHi seg =>
      n0 := lo seg; n1 := hi seg
      if n1 < n0 then (n0,n1) := (n1,n0)
      uts := series(genFiniteStream(f,n0,n1))$UTS(FE,x,a)
      uts := uts * monomial(1,n0)$UTS(FE,x,a)
      coerce(uts)$ANY1(UTS(FE,x,a))
    n0 := lo seg
    uts := series(genStream(f,n0))$UTS(FE,x,a)
    uts := uts * monomial(1,n0)$UTS(FE,x,a)
    coerce(uts)$ANY1(UTS(FE,x,a))

  taylor(an,n,eq,seg) ==
    taylor((i:I):FE +-> eval(an,(n::FE) = (i::FE)),eq,seg)

  laurent(f,eq,seg) ==
    (xx := retractIfCan(lhs eq)@Union(SY,"failed")) case "failed" =>
      error "taylor: left hand side must be a variable"
    x := xx :: SY; a := rhs eq
    hasHi seg =>
      n0 := lo seg; n1 := hi seg
      if n1 < n0 then (n0,n1) := (n1,n0)
      uts := series(genFiniteStream(f,n0,n1))$UTS(FE,x,a)
      coerce(laurent(n0,uts)$ULS(FE,x,a))$ANY1(ULS(FE,x,a))
    n0 := lo seg
    uts := series(genStream(f,n0))$UTS(FE,x,a)
    coerce(laurent(n0,uts)$ULS(FE,x,a))$ANY1(ULS(FE,x,a))

  laurent(an,n,eq,seg) ==
    laurent((i:I):FE +-> eval(an,(n::FE) = (i::FE)),eq,seg)

  modifyFcn:(RN -> FE,I,I,I,I) -> FE
  modifyFcn(f,n0,nn,q,m) == (zero?((m - n0) rem nn) => f(m/q); 0)

  puiseux(f,eq,seg,r) ==
    (xx := retractIfCan(lhs eq)@Union(SY,"failed")) case "failed" =>
      error "puiseux: left hand side must be a variable"
    x := xx :: SY; a := rhs eq
    not positive? r => error "puiseux: last argument must be positive"
    hasHi seg =>
      r0 := lo seg; r1 := hi seg
      if r1 < r0 then (r0,r1) := (r1,r0)
      p0 := numer r0; q0 := denom r0
      p1 := numer r1; q1 := denom r1
      p2 := numer r; q2 := denom r
```

```
      q := lcm(lcm(q0,q1),q2)
      n0 := p0 * (q quo q0); n1 := p1 * (q quo q1)
      nn := p2 * (q quo q2)
      ulsUnion :=
        laurent((i:I):FE+->modifyFcn(f,n0,nn,q,i),eq,segment(n0,n1))
      uls := retract(ulsUnion)$ANY1(ULS(FE,x,a))
      coerce(puiseux(1/q,uls)$UPXS(FE,x,a))$ANY1(UPXS(FE,x,a))
    p0 := numer(r0 := lo seg); q0 := denom r0
    p2 := numer r; q2 := denom r
    q := lcm(q0,q2)
    n0 := p0 * (q quo q0); nn := p2 * (q quo q2)
    ulsUnion :=
      laurent((i:I):FE+->modifyFcn(f,n0,nn,q,i),eq,segment n0)
    uls := retract(ulsUnion)$ANY1(ULS(FE,x,a))
    coerce(puiseux(1/q,uls)$UPXS(FE,x,a))$ANY1(UPXS(FE,x,a))

  puiseux(an,n,eq,r0,m) ==
    puiseux((r:RN):FE+->eval(an,(n::FE) = (r::FE)),eq,r0,m)

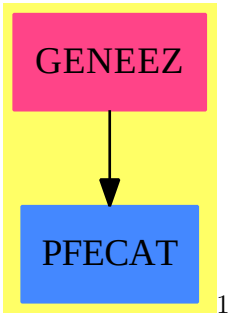
  series(f:I -> FE,eq:EQ FE) == puiseux(r+->f(numer r),eq,segment 0,1)
  series(an:FE,n:SY,eq:EQ FE) == puiseux(an,n,eq,segment 0,1)
  series(f:I -> FE,eq:EQ FE,seg:SEG I) ==
    ratSeg : SEG RN := map(x+->x::RN,seg)$UniversalSegmentFunctions2(I,RN)
    puiseux((r:RN):FE+->f(numer r),eq,ratSeg,1)
  series(an:FE,n:SY,eq:EQ FE,seg:SEG I) ==
    ratSeg : SEG RN := map(i+->i::RN,seg)$UniversalSegmentFunctions2(I,RN)
    puiseux(an,n,eq,ratSeg,1)
  series(f:RN -> FE,eq:EQ FE,seg:SEG RN,r:RN) == puiseux(f,eq,seg,r)
  series(an:FE,n:SY,eq:EQ FE,seg:SEG RN,r:RN) == puiseux(an,n,eq,seg,r)
```

⟨*GENUPS.dotabb*⟩≡

```
"GENUPS" [color="#FF4488",href="bookvol10.4.pdf#nameddest=GENUPS"]
"ACF" [color="#4488FF",href="bookvol10.2.pdf#nameddest=ACF"]
"FS" [color="#4488FF",href="bookvol10.2.pdf#nameddest=FS"]
"GENUPS" -> "ACF"
"GENUPS" -> "FS"
```

8.19 package GENEEZ GenExEuclid

8.20 GenExEuclid



1.00

Exports:
compBound reduction solveid tablePow testModulus

⟨*package GENEEZ GenExEuclid*⟩≡

```
)abbrev package GENEEZ GenExEuclid
++ Author : P.Gianni.
++ January 1990
++ The equation \spad{Af+Bg=h} and its generalization to n polynomials
++ is solved for solutions over the R, euclidean domain.
++ A table containing the solutions of \spad{Af+Bg=x**k} is used.
++ The operations are performed modulus a prime
++ which are in principle big enough,
++ but the solutions are tested and, in case of failure, a hensel
++ lifting process is used to get to the right solutions.
++ It will be used in the factorization of multivariate polynomials
++ over finite field, with \spad{R=F[x]}.

GenExEuclid(R,BP) : C == T
 where
  R   :   EuclideanDomain
  PI  ==> PositiveInteger
  NNI ==> NonNegativeInteger
  BP  :   UnivariatePolynomialCategory R
  L   ==> List

  C == with
       reduction: (BP,R) -> BP
         ++ reduction(p,prime) reduces the polynomial p modulo prime of R.
         ++ Note: this function is exported only because it's conditional.
       compBound: (BP,L BP) -> NNI
         ++ compBound(p,lp)
```

```
      ++ computes a bound for the coefficients of the solution
      ++ polynomials.
      ++ Given a polynomial right hand side p,
      ++ and a list lp of left hand side polynomials.
      ++ Exported because it depends on the valuation.
    tablePow :    (NNI,R,L BP)     -> Union(Vector(L BP),"failed")
      ++ tablePow(maxdeg,prime,lpol) constructs the table with the
      ++ coefficients of the Extended Euclidean Algorithm for lpol.
      ++ Here the right side is \spad{x**k}, for k less or equal to maxdeg.
      ++ The operation returns "failed" when the elements
      ++ are not coprime modulo prime.
    solveid  : (BP,R,Vector L BP) -> Union(L BP,"failed")
      ++ solveid(h,table) computes the coefficients of the
      ++ extended euclidean algorithm for a list of polynomials
      ++ whose tablePow is table and with right side h.

    testModulus : (R, L BP)    -> Boolean
      ++ testModulus(p,lp) returns true if the the prime p
      ++ is valid for the list of polynomials lp, i.e. preserves
      ++ the degree and they remain relatively prime.

T == add
  if R has multiplicativeValuation then
    compBound(m:BP,listpolys:L BP) : NNI ==
      ldeg:=[degree f for f in listpolys]
      n:NNI:= (+/[df for df in ldeg])
      normlist:=[ +/[euclideanSize(u)**2 for u in coefficients f]
                      for f in listpolys]
      nm:= +/[euclideanSize(u)**2 for u in coefficients m]
      normprod := */[g**((n-df)::NNI) for g in normlist for df in ldeg]
      2*(approxSqrt(normprod * nm)$IntegerRoots(Integer))::NNI
  else if R has additiveValuation then
    -- a fairly crude Hadamard-style bound for the solution
    -- based on regarding the problem as a system of linear equations.
    compBound(m:BP,listpolys:L BP) : NNI ==
      "max"/[euclideanSize u for u in coefficients m] +
        +/["max"/[euclideanSize u for u in coefficients p]
           for p in listpolys]
  else
    compBound(m:BP,listpolys:L BP) : NNI ==
      error "attempt to use compBound without a well-understood valuation"
  if R has IntegerNumberSystem then
    reduction(u:BP,p:R):BP ==
      p = 0 => u
      map(x +-> symmetricRemainder(x,p),u)
  else reduction(u:BP,p:R):BP ==
```

```
    p = 0 => u
    map(x +-> x rem p,u)

merge(p:R,q:R):Union(R,"failed") ==
     p = q => p
     p = 0 => q
     q = 0 => p
     "failed"

modInverse(c:R,p:R):R ==
    (extendedEuclidean(c,p,1)::Record(coef1:R,coef2:R)).coef1

exactquo(u:BP,v:BP,p:R):Union(BP,"failed") ==
    invlcv:=modInverse(leadingCoefficient v,p)
    r:=monicDivide(u,reduction(invlcv*v,p))
    reduction(r.remainder,p) ^=0 => "failed"
    reduction(invlcv*r.quotient,p)

FP:=EuclideanModularRing(R,BP,R,reduction,merge,exactquo)

--make table global variable!
table:Vector L BP
import GeneralHenselPackage(R,BP)

                  --local  functions
makeProducts :    L BP   -> L BP
liftSol: (L BP,BP,R,R,Vector L BP,BP,NNI) -> Union(L BP,"failed")

reduceList(lp:L BP,lmod:R): L FP ==[reduce(ff,lmod) for ff in lp]

coerceLFP(lf:L FP):L BP == [fm::BP for fm in lf]

liftSol(oldsol:L BP,err:BP,lmod:R,lmodk:R,
       table:Vector L BP,m:BP,bound:NNI):Union(L BP,"failed") ==
  euclideanSize(lmodk) > bound => "failed"
  d:=degree err
  ftab:Vector L FP :=
    map(x +-> reduceList(x,lmod),table)$VectorFunctions2(List BP,List FP)
  sln:L FP:=[0$FP for xx in ftab.1 ]
  for i in 0 .. d |(cc:=coefficient(err,i)) ^=0 repeat
    sln:=[slp+reduce(cc::BP,lmod)*pp
          for pp in ftab.(i+1) for slp in sln]
  nsol:=[f-lmodk*reduction(g::BP,lmod) for f in oldsol for g in sln]
  lmodk1:=lmod*lmodk
  nsol:=[reduction(slp,lmodk1) for slp in nsol]
  lpolys:L BP:=table.(#table)
```

```
  (fs:=+/[f*g for f in lpolys for g in nsol]) = m => nsol
  a:BP:=((fs-m) exquo lmodk1)::BP
  liftSol(nsol,a,lmod,lmodk1,table,m,bound)

makeProducts(listPol:L BP):L BP ==
  #listPol < 2 => listPol
  #listPol = 2 => reverse listPol
  f:= first listPol
  ll := rest listPol
  [*/ll,:[f*g for g in makeProducts ll]]

testModulus(pmod, listPol) ==
     redListPol := reduceList(listPol, pmod)
     for pol in listPol for rpol in redListPol repeat
          degree(pol) ^= degree(rpol::BP) => return false
     while not empty? redListPol repeat
         rpol := first redListPol
         redListPol := rest redListPol
         for rpol2 in redListPol repeat
            gcd(rpol, rpol2) ^= 1 => return false
     true

if R has Field then
  tablePow(mdeg:NNI,pmod:R,listPol:L BP) ==
    multiE:=multiEuclidean(listPol,1$BP)
    multiE case "failed" => "failed"
    ptable:Vector L BP :=new(mdeg+1,[])
    ptable.1:=multiE
    x:BP:=monomial(1,1)
    for i in 2..mdeg repeat ptable.i:=
        [tpol*x rem fpol for tpol in ptable.(i-1) for fpol in listPol]
    ptable.(mdeg+1):=makeProducts listPol
    ptable

  solveid(m:BP,pmod:R,table:Vector L BP) : Union(L BP,"failed") ==
       -- Actually, there's no possibility of failure
    d:=degree m
    sln:L BP:=[0$BP for xx in table.1]
    for i in 0 .. d | coefficient(m,i)^=0 repeat
      sln:=[slp+coefficient(m,i)*pp
            for pp in table.(i+1) for slp in sln]
    sln

else

  tablePow(mdeg:NNI,pmod:R,listPol:L BP) ==
```

```
    listP:L FP:= [reduce(pol,pmod) for pol in listPol]
    multiE:=multiEuclidean(listP,1$FP)
    multiE case "failed" => "failed"
    ftable:Vector L FP :=new(mdeg+1,[])
    fl:L FP:= [ff::FP for ff in multiE]
    ftable.1:=[fpol for fpol in fl]
    x:FP:=reduce(monomial(1,1),pmod)
    for i in 2..mdeg repeat ftable.i:=
        [tpol*x rem fpol for tpol in ftable.(i-1) for fpol in listP]
    ptable:= map(coerceLFP,ftable)$VectorFunctions2(List FP,List BP)
    ptable.(mdeg+1):=makeProducts listPol
    ptable

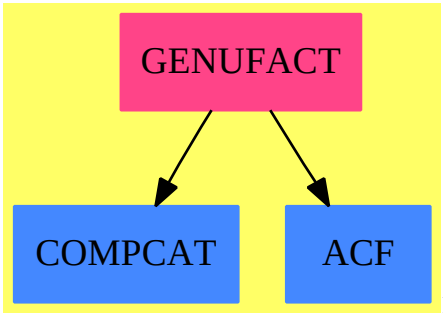
  solveid(m:BP,pmod:R,table:Vector L BP) : Union(L BP,"failed") ==
    d:=degree m
    ftab:Vector L FP:=
      map(x+->reduceList(x,pmod),table)$VectorFunctions2(List BP,List FP)
    lpolys:L BP:=table.(#table)
    sln:L FP:=[0$FP for xx in ftab.1]
    for i in 0 .. d | coefficient(m,i)^=0 repeat
      sln:=[slp+reduce(coefficient(m,i)::BP,pmod)*pp
            for pp in ftab.(i+1) for slp in sln]
    soln:=[slp::BP for slp in sln]
    (fs:=+/[f*g for f in lpolys for g in soln]) = m=> soln
    -- Compute bound
    bound:=compBound(m,lpolys)
    a:BP:=((fs-m) exquo pmod)::BP
    liftSol(soln,a,pmod,pmod,table,m,bound)
```

⟨*GENEEZ.dotabb*⟩≡

```
"GENEEZ" [color="#FF4488",href="bookvol10.4.pdf#nameddest=GENEEZ"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"GENEEZ" -> "PFECAT"
```

8.21 package GENUFACT GenUFactorize

8.22 GenUFactorize



1.00

Exports:
factor

⟨*package GENUFACT GenUFactorize*⟩≡
```
)abbrev package GENUFACT GenUFactorize
++ Description
++ This package provides operations for the factorization
++ of univariate polynomials with integer
++ coefficients. The factorization is done by "lifting" the
++ finite "berlekamp's" factorization
GenUFactorize(R) : public == private where
  R    :    EuclideanDomain
  PR   ==>  SparseUnivariatePolynomial R  -- with factor
            -- should be UnivariatePolynomialCategory
  NNI  ==>  NonNegativeInteger
  SUP  ==>  SparseUnivariatePolynomial


  public == with
     factor      :           PR  -> Factored PR
        ++ factor(p) returns the factorisation of p

  private == add

    -- Factorisation currently fails when algebraic extensions have multiple
    -- generators.
    factorWarning(f:OutputForm):Void ==
      import AnyFunctions1(String)
      import AnyFunctions1(OutputForm)
      outputList(["WARNING (genufact): No known algorithm to factor "::Any, _
              f::Any, _
```

```
                ", trying square-free."::Any])$OutputPackage

      factor(f:PR) : Factored PR ==
        R is Integer => (factor f)$GaloisGroupFactorizer(PR)

        R is Fraction Integer  =>
                                  (factor f)$RationalFactorize(PR)

  --      R has Field and R has Finite =>
        R has FiniteFieldCategory =>
                                  (factor f)$DistinctDegreeFactorize(R,PR)

        R is (Complex Integer) => (factor f)$ComplexFactorization(Integer,PR)

        R is (Complex Fraction Integer) =>
                              (factor f)$ComplexFactorization(Fraction Integer,PR)

        R is AlgebraicNumber =>   (factor f)$AlgFactor(PR)

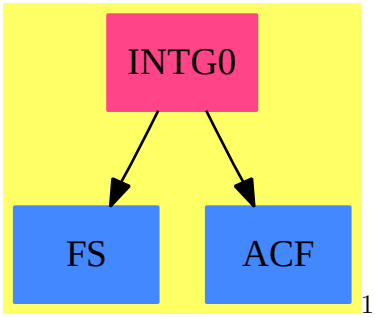
     -- following is to handle SAE
        R has generator : () -> R =>
          var := symbol(convert(generator()::OutputForm)@InputForm)
          up:=UnivariatePolynomial(var,Fraction Integer)
          R has MonogenicAlgebra(Fraction Integer, up) =>
             factor(f)$SimpleAlgebraicExtensionAlgFactor(up, R, PR)
          upp:=UnivariatePolynomial(var,Fraction Polynomial Integer)
          R has MonogenicAlgebra(Fraction Polynomial Integer, upp) =>
             factor(f)$SAERationalFunctionAlgFactor(upp, R, PR)
          factorWarning(f::OutputForm)
          squareFree f
        factorWarning(f::OutputForm)
        squareFree f
```

⟨*GENUFACT.dotabb*⟩≡

```
"GENUFACT" [color="#FF4488",href="bookvol10.4.pdf#nameddest=GENUFACT"]
"COMPCAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=COMPCAT"]
"ACF" [color="#4488FF",href="bookvol10.2.pdf#nameddest=ACF"]
"GENUFACT" -> "COMPCAT"
"GENUFACT" -> "ACF"
```

8.23 package INTG0 GenusZeroIntegration

8.24 GenusZeroIntegration



1.00

Exports:

lift	multivariate	palgLODE0	palgRDE0
palgextint0	palgint0	palglimint0	univariate

⟨*package INTG0 GenusZeroIntegration*⟩≡

```
)abbrev package INTG0 GenusZeroIntegration
++ Rationalization of several types of genus 0 integrands;
++ Author: Manuel Bronstein
++ Date Created: 11 October 1988
++ Date Last Updated: 24 June 1994
++ Description:
++ This internal package rationalises integrands on curves of the form:
++   \spad{y\^2 = a x\^2 + b x + c}
++   \spad{y\^2 = (a x + b) / (c x + d)}
++   \spad{f(x, y) = 0} where f has degree 1 in x
++ The rationalization is done for integration, limited integration,
++ extended integration and the risch differential equation;
GenusZeroIntegration(R, F, L): Exports == Implementation where
  R: Join(GcdDomain, RetractableTo Integer, OrderedSet, CharacteristicZero,
          LinearlyExplicitRingOver Integer)
  F: Join(FunctionSpace R, AlgebraicallyClosedField,
          TranscendentalFunctionCategory)
  L: SetCategory

  SY  ==> Symbol
  Q   ==> Fraction Integer
  K   ==> Kernel F
  P   ==> SparseMultivariatePolynomial(R, K)
  UP  ==> SparseUnivariatePolynomial F
  RF  ==> Fraction UP
  UPUP ==> SparseUnivariatePolynomial RF
```

```
IR  ==> IntegrationResult F
LOG ==> Record(coeff:F, logand:F)
U1  ==> Union(F, "failed")
U2  ==> Union(Record(ratpart:F, coeff:F),"failed")
U3  ==> Union(Record(mainpart:F, limitedlogs:List LOG), "failed")
REC ==> Record(coeff:F, var:List K, val:List F)
ODE ==> Record(particular: Union(F, "failed"), basis: List F)
LODO==> LinearOrdinaryDifferentialOperator1 RF

Exports ==> with
  palgint0   : (F, K, K, F, UP)    -> IR
    ++ palgint0(f, x, y, d, p) returns the integral of \spad{f(x,y)dx}
    ++ where y is an algebraic function of x satisfying
    ++ \spad{d(x)\^2 y(x)\^2 = P(x)}.
  palgint0   : (F, K, K, K, F, RF) -> IR
    ++ palgint0(f, x, y, z, t, c) returns the integral of \spad{f(x,y)dx}
    ++ where y is an algebraic function of x satisfying
    ++ \spad{f(x,y)dx = c f(t,y) dy}; c and t are rational functions of y.
    ++ Argument z is a dummy variable not appearing in \spad{f(x,y)}.
  palgextint0: (F, K, K, F, F, UP) -> U2
    ++ palgextint0(f, x, y, g, d, p) returns functions \spad{[h, c]} such
    ++ that \spad{dh/dx = f(x,y) - c g}, where y is an algebraic function
    ++ of x satisfying \spad{d(x)\^2 y(x)\^2 = P(x)},
    ++ or "failed" if no such functions exist.
  palgextint0: (F, K, K, F, K, F, RF) -> U2
    ++ palgextint0(f, x, y, g, z, t, c) returns functions \spad{[h, d]} such
    ++ that \spad{dh/dx = f(x,y) - d g}, where y is an algebraic function
    ++ of x satisfying \spad{f(x,y)dx = c f(t,y) dy}, and c and t are
    ++ rational functions of y.
    ++ Argument z is a dummy variable not appearing in \spad{f(x,y)}.
    ++ The operation returns "failed" if no such functions exist.
  palglimint0: (F, K, K, List F, F, UP) -> U3
    ++ palglimint0(f, x, y, [u1,...,un], d, p) returns functions
    ++ \spad{[h,[[ci, ui]]]} such that the ui's are among \spad{[u1,...,un]}
    ++ and \spad{d(h + sum(ci log(ui)))/dx = f(x,y)} if such functions exist,
    ++ and "failed" otherwise.
    ++ Argument y is an algebraic function of x satisfying
    ++ \spad{d(x)\^2y(x)\^2 = P(x)}.
  palglimint0: (F, K, K, List F, K, F, RF) -> U3
    ++ palglimint0(f, x, y, [u1,...,un], z, t, c) returns functions
    ++ \spad{[h,[[ci, ui]]]} such that the ui's are among \spad{[u1,...,un]}
    ++ and \spad{d(h + sum(ci log(ui)))/dx = f(x,y)} if such functions exist,
    ++ and "failed" otherwise.
    ++ Argument y is an algebraic function of x satisfying
    ++ \spad{f(x,y)dx = c f(t,y) dy}; c and t are rational functions of y.
  palgRDE0   : (F, F, K, K, (F, F, SY) -> U1, F, UP) -> U1
```

```
    ++ palgRDE0(f, g, x, y, foo, d, p) returns a function \spad{z(x,y)}
    ++ such that \spad{dz/dx + n * df/dx z(x,y) = g(x,y)} if such a z exists,
    ++ and "failed" otherwise.
    ++ Argument y is an algebraic function of x satisfying
    ++ \spad{d(x)\^2y(x)\^2 = P(x)}.
    ++ Argument foo, called by \spad{foo(a, b, x)}, is a function that solves
    ++ \spad{du/dx + n * da/dx u(x) = u(x)}
    ++ for an unknown \spad{u(x)} not involving y.
  palgRDE0   : (F, F, K, K, (F, F, SY) -> U1, K, F, RF) -> U1
    ++ palgRDE0(f, g, x, y, foo, t, c) returns a function \spad{z(x,y)}
    ++ such that \spad{dz/dx + n * df/dx z(x,y) = g(x,y)} if such a z exists,
    ++ and "failed" otherwise.
    ++ Argument y is an algebraic function of x satisfying
    ++ \spad{f(x,y)dx = c f(t,y) dy}; c and t are rational functions of y.
    ++ Argument \spad{foo}, called by \spad{foo(a, b, x)}, is a function that
    ++ solves \spad{du/dx + n * da/dx u(x) = u(x)}
    ++ for an unknown \spad{u(x)} not involving y.
  univariate: (F, K, K, UP) -> UPUP
      ++ univariate(f,k,k,p) \undocumented
  multivariate: (UPUP, K, F) -> F
      ++ multivariate(u,k,f) \undocumented
  lift: (UP, K) -> UPUP
      ++ lift(u,k) \undocumented
  if L has LinearOrdinaryDifferentialOperatorCategory F then
    palgLODE0  : (L, F, K, K, F, UP) -> ODE
      ++ palgLODE0(op, g, x, y, d, p) returns the solution of \spad{op f = g}.
      ++ Argument y is an algebraic function of x satisfying
      ++ \spad{d(x)\^2y(x)\^2 = P(x)}.
    palgLODE0  : (L, F, K, K, K, F, RF) -> ODE
      ++ palgLODE0(op,g,x,y,z,t,c) returns the solution of \spad{op f = g}
      ++ Argument y is an algebraic function of x satisfying
      ++ \spad{f(x,y)dx = c f(t,y) dy}; c and t are rational functions of y.

Implementation ==> add
  import RationalIntegration(F, UP)
  import AlgebraicManipulations(R, F)
  import IntegrationResultFunctions2(RF, F)
  import ElementaryFunctionStructurePackage(R, F)
  import SparseUnivariatePolynomialFunctions2(F, RF)
  import PolynomialCategoryQuotientFunctions(IndexedExponents K,
                                                K, R, P, F)

  mkRat    : (F, REC, List K) -> RF
  mkRatlx  : (F, K, K, F, K, RF) -> RF
  quadsubst: (K, K, F, UP) -> Record(diff:F, subs:REC, newk:List K)
  kerdiff  : (F, F) -> List K
```

```
    checkroot: (F, List K) -> F
    univ     : (F, List K, K) -> RF

    dummy := kernel(new()$SY)@K

    kerdiff(sa, a)         == setDifference(kernels sa, kernels a)
    checkroot(f, l)        == (empty? l => f; rootNormalize(f, first l))
    univ(c, l, x)          == univariate(checkroot(c, l), x)
    univariate(f, x, y, p) == lift(univariate(f, y, p), x)
    lift(p, k)             == map(x1+->univariate(x1, k), p)

    palgint0(f, x, y, den, radi) ==
      -- y is a square root so write f as f1 y + f0 and integrate separately
      ff := univariate(f, x, y, minPoly y)
      f0 := reductum ff
      pr := quadsubst(x, y, den, radi)
      map(f1+->f1(x::F), integrate(retract(f0)@RF)) +
        map(f1+->f1(pr.diff),
            integrate
              mkRat(multivariate(leadingMonomial ff,x,y::F), pr.subs, pr.newk))

-- the algebraic relation is (den * y)**2 = p  where p is a * x**2 + b * x + c
-- if p is squarefree, then parametrize in the following form:
--     u  = y - x \sqrt{a}
--     x  = (u^2 - c) / (b - 2 u \sqrt{a}) = h(u)
--     dx = h'(u) du
--     y  = (u + a h(u)) / den = g(u)
-- if a is a perfect square,
--     u  = (y - \sqrt{c}) / x
--     x  = (b - 2 u \sqrt{c}) / (u^2 - a) = h(u)
--     dx = h'(u) du
--     y  = (u h(u) + \sqrt{c}) / den = g(u)
-- otherwise.
-- if p is a square p = a t^2, then we choose only one branch for now:
--     u  = x
--     x  = u = h(u)
--     dx = du
--     y  = t \sqrt{a} / den = g(u)
-- returns [u(x,y), [h'(u), [x,y], [h(u), g(u)], l] in both cases,
-- where l is empty if no new square root was needed,
-- l := [k] if k is the new square root kernel that was created.
    quadsubst(x, y, den, p) ==
      u   := dummy::F
      b   := coefficient(p, 1)
      c   := coefficient(p, 0)
      sa  := rootSimp sqrt(a := coefficient(p, 2))
```

```
      zero?(b * b - 4 * a * c) =>    -- case where p = a (x + b/(2a))^2
        [x::F, [1, [x, y], [u, sa * (u + b / (2*a)) / eval(den,x,u)]], empty()]
      empty? kerdiff(sa, a) =>
        bm2u := b - 2 * u * sa
        q    := eval(den, x, xx := (u**2 - c) / bm2u)
        yy   := (ua := u + xx * sa) / q
        [y::F - x::F * sa, [2 * ua / bm2u, [x, y], [xx, yy]], empty()]
      u2ma:= u**2 - a
      sc  := rootSimp sqrt c
      q   := eval(den, x, xx := (b - 2 * u * sc) / u2ma)
      yy  := (ux := xx * u + sc) / q
      [(y::F - sc) / x::F, [- 2 * ux / u2ma, [x ,y], [xx, yy]], kerdiff(sc, c)]

    mkRatlx(f,x,y,t,z,dx) ==
      rat := univariate(eval(f, [x, y], [t, z::F]), z) * dx
      numer(rat) / denom(rat)

    mkRat(f, rec, l) ==
      rat:=univariate(checkroot(rec.coeff * eval(f,rec.var,rec.val), l), dummy)
      numer(rat) / denom(rat)

    palgint0(f, x, y, z, xx, dx) ==
      map(x1+->multivariate(x1, y), integrate mkRatlx(f, x, y, xx, z, dx))

    palgextint0(f, x, y, g, z, xx, dx) ==
      map(x1+->multivariate(x1, y),
            extendedint(mkRatlx(f,x,y,xx,z,dx), mkRatlx(g,x,y,xx,z,dx)))

    palglimint0(f, x, y, lu, z, xx, dx) ==
      map(x1+->multivariate(x1, y), limitedint(mkRatlx(f, x, y, xx, z, dx),
                              [mkRatlx(u, x, y, xx, z, dx) for u in lu]))

    palgRDE0(f, g, x, y, rischde, z, xx, dx) ==
      (u := rischde(eval(f, [x, y], [xx, z::F]),
                     multivariate(dx, z) * eval(g, [x, y], [xx, z::F]),
                         symbolIfCan(z)::SY)) case "failed" => "failed"
      eval(u::F, z, y::F)

  -- given p = sum_i a_i(X) Y^i, returns  sum_i a_i(x) y^i
    multivariate(p, x, y) ==
      (map((x1:RF):F+->multivariate(x1, x),
           p)$SparseUnivariatePolynomialFunctions2(RF, F))
              (y)

    palgextint0(f, x, y, g, den, radi) ==
      pr := quadsubst(x, y, den, radi)
```

```
    map(f1+->f1(pr.diff),
        extendedint(mkRat(f, pr.subs, pr.newk), mkRat(g, pr.subs, pr.newk)))

  palglimint0(f, x, y, lu, den, radi) ==
    pr := quadsubst(x, y, den, radi)
    map(f1+->f1(pr.diff),
       limitedint(mkRat(f, pr.subs, pr.newk),
                  [mkRat(u, pr.subs, pr.newk) for u in lu]))

  palgRDE0(f, g, x, y, rischde, den, radi) ==
    pr := quadsubst(x, y, den, radi)
    (u := rischde(checkroot(eval(f, pr.subs.var, pr.subs.val), pr.newk),
                checkroot(pr.subs.coeff * eval(g, pr.subs.var, pr.subs.val),
                          pr.newk), symbolIfCan(dummy)::SY)) case "failed"
                                 => "failed"
    eval(u::F, dummy, pr.diff)

  if L has LinearOrdinaryDifferentialOperatorCategory F then
    import RationalLODE(F, UP)

    palgLODE0(eq, g, x, y, den, radi) ==
      pr := quadsubst(x, y, den, radi)
      d := monomial(univ(inv(pr.subs.coeff), pr.newk, dummy), 1)$LODO
      di:LODO := 1                  -- will accumulate the powers of d
      op:LODO := 0                  -- will accumulate the new LODO
      for i in 0..degree eq repeat
        op := op + univ(eval(coefficient(eq, i), pr.subs.var, pr.subs.val),
                      pr.newk, dummy) * di
        di := d * di
      rec := ratDsolve(op,univ(eval(g,pr.subs.var,pr.subs.val),pr.newk,dummy))
      bas:List(F) := [b(pr.diff) for b in rec.basis]
      rec.particular case "failed" => ["failed", bas]
      [((rec.particular)::RF) (pr.diff), bas]

    palgLODE0(eq, g, x, y, kz, xx, dx) ==
      d := monomial(univariate(inv multivariate(dx, kz), kz), 1)$LODO
      di:LODO := 1                  -- will accumulate the powers of d
      op:LODO := 0                  -- will accumulate the new LODO
      lk:List(K) := [x, y]
      lv:List(F) := [xx, kz::F]
      for i in 0..degree eq repeat
        op := op + univariate(eval(coefficient(eq, i), lk, lv), kz) * di
        di := d * di
      rec := ratDsolve(op, univariate(eval(g, lk, lv), kz))
      bas:List(F) := [multivariate(b, y) for b in rec.basis]
      rec.particular case "failed" => ["failed", bas]
```

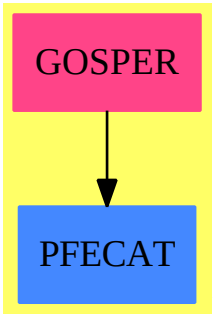
```
[multivariate((rec.particular)::RF, y), bas]
```

⟨*INTG0.dotabb*⟩≡

```
"INTG0" [color="#FF4488",href="bookvol10.4.pdf#nameddest=INTG0"]
"FS" [color="#4488FF",href="bookvol10.2.pdf#nameddest=FS"]
"ACF" [color="#4488FF",href="bookvol10.2.pdf#nameddest=ACF"]
"INTG0" -> "FS"
"INTG0" -> "ACF"
```

8.25 package GOSPER GosperSummationMethod

8.26 GosperSummationMethod



1.00

Exports:
GospersMethod

⟨*package GOSPER GosperSummationMethod*⟩≡

```
)abbrev package GOSPER GosperSummationMethod
++ Gosper's summation algorithm
++ Author: SMW
++ Date Created: ???
++ Date Last Updated: 19 August 1991
++ Description: Gosper's summation algorithm.
GosperSummationMethod(E, V, R, P, Q): Exports == Impl where
    E: OrderedAbelianMonoidSup
    V: OrderedSet
    R: IntegralDomain
    P: PolynomialCategory(R, E, V)
    Q: Join(RetractableTo Fraction Integer, Field with
              (coerce: P -> %; numer : % -> P; denom : % -> P))

    I   ==> Integer
    RN  ==> Fraction I
    PQ  ==> SparseMultivariatePolynomial(RN, V)
    RQ  ==> Fraction PQ

    Exports ==> with
        GospersMethod: (Q, V, () -> V) -> Union(Q, "failed")
            ++ GospersMethod(b, n, new) returns a rational function
            ++ \spad{rf(n)} such that \spad{a(n) * rf(n)} is the indefinite
            ++ sum of \spad{a(n)}
            ++ with respect to upward difference on \spad{n}, i.e.
            ++ \spad{a(n+1) * rf(n+1) - a(n) * rf(n) = a(n)},
            ++ where \spad{b(n) = a(n)/a(n-1)} is a rational function.
```

```
        ++ Returns "failed" if no such rational function \spad{rf(n)}
        ++ exists.
        ++ Note: \spad{new} is a nullary function returning a new
        ++ V every time.
        ++ The condition on \spad{a(n)} is that \spad{a(n)/a(n-1)}
        ++ is a rational function of \spad{n}.
        --++  \spad{sum(a(n), n) = rf(n) * a(n)}.

    Impl ==> add
      import PolynomialCategoryQuotientFunctions(E, V, R, P, Q)
      import LinearSystemMatrixPackage(RQ,Vector RQ,Vector RQ,Matrix RQ)

      InnerGospersMethod: (RQ, V, () -> V) -> Union(RQ, "failed")
      GosperPQR:   (PQ, PQ, V, () -> V)       -> List PQ
      GosperDegBd: (PQ, PQ, PQ, V, () -> V)    -> I
      GosperF:     (I, PQ, PQ, PQ, V, () -> V) -> Union(RQ, "failed")
      linearAndNNIntRoot: (PQ, V) -> Union(I, "failed")
      deg0:    (PQ, V) -> I       -- degree with deg 0 = -1.
      pCoef:   (PQ, PQ) -> PQ  -- pCoef(p, a*b**2)
      RF2QIfCan: Q -> Union(RQ, "failed")
      UP2QIfCan: P -> Union(PQ,"failed")
      RFQ2R    : RQ -> Q
      PQ2R     : PQ -> Q
      rat?     : R  -> Boolean

      deg0(p, v) == (zero? p => -1; degree(p, v))
      rat? x     == retractIfCan(x::P::Q)@Union(RN, "failed") case RN
      RFQ2R f    == PQ2R(numer f) / PQ2R(denom f)

      PQ2R p ==
        map(x+->x::P::Q, y+->y::Q, p)$PolynomialCategoryLifting(
                                      IndexedExponents V, V, RN, PQ, Q)

      GospersMethod(aquo, n, newV) ==
        ((q := RF2QIfCan aquo) case "failed") or
          ((u := InnerGospersMethod(q::RQ, n, newV)) case "failed") =>
             "failed"
        RFQ2R(u::RQ)

      RF2QIfCan f ==
        (n := UP2QIfCan numer f) case "failed" => "failed"
        (d := UP2QIfCan denom f) case "failed" => "failed"
        n::PQ / d::PQ

      UP2QIfCan p ==
        every?(rat?, coefficients p) =>
```

```
      map(x +-> x::PQ,
          y +-> (retractIfCan(y::P::Q)@Union(RN, "failed"))::RN::PQ,p)_
           $PolynomialCategoryLifting(E, V, R, P, PQ)
    "failed"

  InnerGospersMethod(aquo, n, newV) ==
        -- 1. Define coprime polys an,anm1 such that
        --      an/anm1=a(n)/a(n-1)
        an   := numer aquo
        anm1 := denom aquo

        -- 2. Define p,q,r such that
        --      a(n)/a(n-1) = (p(n)/p(n-1)) * (q(n)/r(n))
        --    and
        --      gcd(q(n), r(n+j)) = 1, for all j: NNI.
        pqr:= GosperPQR(an, anm1, n, newV)
        pn := first pqr; qn := second pqr; rn := third pqr

        -- 3. If the sum is a rational fn, there is a poly f with
        --      sum(a(n), n) = q(n+1)/p(n) * a(n) * f(n).

        -- 4. Bound the degree of f(n).
        (k := GosperDegBd(pn, qn, rn, n, newV)) < 0 => "failed"

        -- 5. Find a polynomial f of degree at most k, satisfying
        --      p(n) = q(n+1)*f(n) - r(n)*f(n-1)
        (ufn := GosperF(k, pn, qn, rn, n, newV)) case "failed" =>
          "failed"
        fn  := ufn::RQ

        -- 6. The sum is q(n-1)/p(n)*f(n) * a(n). We leave out a(n).
        --qnm1 := eval(qn,n,n::PQ - 1)
        --qnm1/pn * fn
        qn1 := eval(qn,n,n::PQ + 1)
        qn1/pn * fn

  GosperF(k, pn, qn, rn, n, newV) ==
        mv := newV(); mp := mv::PQ; np := n::PQ
        fn:       PQ := +/[mp**(i+1) * np**i for i in 0..k]
        fnminus1: PQ := eval(fn, n, np-1)
        qnplus1        := eval(qn, n, np+1)
        zro  := qnplus1 * fn - rn * fnminus1 - pn
        zron := univariate(zro, n)
        dz   := degree zron
        mat: Matrix RQ := zero(dz+1, (k+1)::NonNegativeInteger)
        vec: Vector RQ := new(dz+1, 0)
```

```
        while zron ^= 0 repeat
            cz := leadingCoefficient zron
            dz := degree zron
            zron := reductum zron
            mz := univariate(cz, mv)
            while mz ^= 0 repeat
                cmz := leadingCoefficient(mz)::RQ
                dmz := degree mz
                mz := reductum mz
                dmz = 0 => vec(dz + minIndex vec) := -cmz
                qsetelt_!(mat, dz + minRowIndex mat,
                              dmz + minColIndex(mat) - 1, cmz)
        (soln := particularSolution(mat, vec)) case "failed" => "failed"
        vec := soln::Vector RQ
        (+/[np**i * vec(i + minIndex vec) for i in 0..k])@RQ

    GosperPQR(an, anm1, n, newV) ==
        np := n::PQ   -- polynomial version of n
        -- Initial guess.
        pn: PQ := 1
        qn: PQ := an
        rn: PQ := anm1
        -- Find all j: NNI giving common factors to q(n) and r(n+j).
        j     := newV()
        rnj   := eval(rn, n, np + j::PQ)
        res   := resultant(qn, rnj, n)
        fres  := factor(res)$MRationalFactorize(IndexedExponents V,
                                                V, I, PQ)
        js    := [rt::I for fe in factors fres
                    | (rt := linearAndNNIntRoot(fe.factor,j)) case I]
        -- For each such j, change variables to remove the gcd.
        for rt in js repeat
            rtp:= rt::PQ  -- polynomial version of rt
            gn := gcd(qn, eval(rn,n,np+rtp))
            qn := (qn exquo gn)::PQ
            rn := (rn exquo eval(gn, n, np-rtp))::PQ
            pn := pn * */[eval(gn, n, np-i::PQ) for i in 0..rt-1]
        [pn, qn, rn]

      -- Find a degree bound for the polynomial f(n) which satisfies
      --   p(n) = q(n+1)*f(n) - r(n)*f(n-1).
    GosperDegBd(pn, qn, rn, n, newV) ==
        np := n::PQ
        qnplus1  := eval(qn, n, np+1)
        lplus  := deg0(qnplus1 + rn,  n)
        lminus := deg0(qnplus1 - rn, n)
```

```
        degp   := deg0(pn, n)
        k := degp - max(lplus-1, lminus)
        lplus <= lminus => k
        -- Find L(k), such that
        --   p(n) = L(k)*c[k]*n**(k + lplus - 1) + ...
        -- To do this, write f(n) and f(n-1) symbolically.
        --   f(n)  = c[k]*n**k + c[k-1]*n**(k-1) +O(n**(k-2))
        --   f(n-1)=c[k]*n**k + (c[k-1]-k*c[k])*n**(k-1)+O(n**(k-2))
        kk := newV()::PQ
        ck := newV()::PQ
        ckm1 := newV()::PQ
        nkm1:= newV()::PQ
        nk := np*nkm1
        headfn   := ck*nk +         ckm1*nkm1
        headfnm1 := ck*nk + (ckm1-kk*ck)*nkm1
        -- Then p(n) = q(n+1)*f(n) - r(n)*f(n-1) gives L(k).
        pk   := qnplus1 * headfn - rn * headfnm1
        lcpk := pCoef(pk, ck*np*nkm1)
        -- The degree bd is now given by k, and the root of L.
        k0 := linearAndNNIntRoot(lcpk, mainVariable(kk)::V)
        k0 case "failed" => k
        max(k0::I, k)

    pCoef(p, nom) ==
        not monomial? nom =>
          error "pCoef requires a monomial 2nd arg"
        vlist := variables nom
        for v in vlist while p ^= 0 repeat
            unom:= univariate(nom,v)
            pow:=degree unom
            nom:=leadingCoefficient unom
            up  := univariate(p, v)
            p   := coefficient(up, pow)
        p

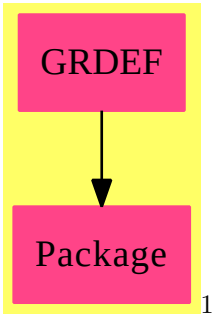
    linearAndNNIntRoot(mp, v) ==
        p := univariate(mp, v)
        degree p ^= 1 => "failed"
        (p1 := retractIfCan(coefficient(p, 1))@Union(RN,"failed"))
         case "failed" or
          (p0 := retractIfCan(coefficient(p, 0))@Union(RN,"failed"))
           case "failed" => "failed"
        rt := -(p0::RN)/(p1::RN)
        rt < 0 or denom rt ^= 1 => "failed"
        numer rt
```

⟨*GOSPER.dotabb*⟩≡

```
"GOSPER" [color="#FF4488",href="bookvol10.4.pdf#nameddest=GOSPER"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"GOSPER" -> "PFECAT"
```

8.27 package GRDEF GraphicsDefaults

8.28 GraphicsDefaults



1.00

Exports:
adaptive clipPointsDefault drawToScale maxPoints minPoints screenResolution

⟨*package GRDEF GraphicsDefaults*⟩≡

```
)abbrev package GRDEF GraphicsDefaults
++ Author: Clifton J. Williamson
++ Date Created: 8 January 1990
++ Date Last Updated: 8 January 1990
++ Basic Operations: clipPointsDefault, drawToScale, adaptive, maxPoints,
++ minPoints, screenResolution
++ Related Constructors:
++ Also See:
++ AMS Classifications:
++ Keywords:
++ References:
++ Description: TwoDimensionalPlotSettings sets global flags and constants
++ for 2-dimensional plotting.

GraphicsDefaults(): Exports == Implementation where
  B  ==> Boolean
  I  ==> Integer
  SF ==> DoubleFloat
  maxWidth  ==> 1000
  maxHeight ==> 1000

  Exports ==> with
    clipPointsDefault: () -> B
      ++ clipPointsDefault() determines whether or not automatic clipping is
      ++ to be done.
    drawToScale: () -> B
      ++ drawToScale() determines whether or not plots are to be drawn to scale.
```

```
    clipPointsDefault: B -> B
      ++ clipPointsDefault(true) turns on automatic clipping;
      ++ \spad{clipPointsDefault(false)} turns off automatic clipping.
      ++ The default setting is true.
    drawToScale: B -> B
      ++ drawToScale(true) causes plots to be drawn to scale.
      ++ \spad{drawToScale(false)} causes plots to be drawn so that they
      ++ fill up the viewport window.
      ++ The default setting is false.

  --% settings from the two-dimensional plot package

    adaptive: () -> B
      ++ adaptive() determines whether plotting will be done adaptively.
    maxPoints: () -> I
      ++ maxPoints() returns the maximum number of points in a plot.
    minPoints: () -> I
      ++ minPoints() returns the minimum number of points in a plot.
    screenResolution: () -> I
      ++ screenResolution() returns the screen resolution n.

    adaptive: B -> B
      ++ adaptive(true) turns adaptive plotting on;
      ++ \spad{adaptive(false)} turns adaptive plotting off.
    maxPoints: I -> I
      ++ maxPoints() sets the maximum number of points in a plot.
    minPoints: I -> I
      ++ minPoints() sets the minimum number of points in a plot.
    screenResolution: I -> I
      ++ screenResolution(n) sets the screen resolution to n.

  Implementation ==> add

  --% global flags and constants

    CLIPPOINTSDEFAULT : B := true
    TOSCALE  : B := false

  --% functions

    clipPointsDefault()     == CLIPPOINTSDEFAULT
    drawToScale()  == TOSCALE

    clipPointsDefault b    == CLIPPOINTSDEFAULT := b
    drawToScale b == TOSCALE := b
```

```
--% settings from the two-dimensional plot package

    adaptive() == adaptive?()$Plot
    minPoints() == minPoints()$Plot
    maxPoints() == maxPoints()$Plot
    screenResolution() == screenResolution()$Plot

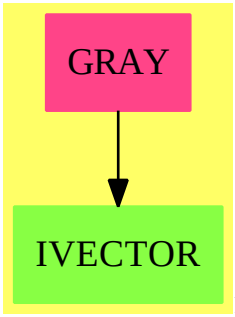
    adaptive b == setAdaptive(b)$Plot
    minPoints n == setMinPoints(n)$Plot
    maxPoints n == setMaxPoints(n)$Plot
    screenResolution n == setScreenResolution(n)$Plot
```

⟨*GRDEF.dotabb*⟩≡

```
"GRDEF" [color="#FF4488",href="bookvol10.4.pdf#nameddest=GRDEF"]
"Package" [color="#FF4488"]
"GRDEF" -> "Package"
```

8.29 package GRAY GrayCode

8.30 GrayCode



1.00

Exports:
firstSubsetGray nextSubsetGray

⟨*package GRAY GrayCode*⟩≡
```
)abbrev package GRAY GrayCode
++ Authors: Johannes Grabmeier, Oswald Gschnitzer
++ Date Created: 7 August 1989
++ Date Last Updated: 23 August 1990
++ Basic Operations: nextSubsetGray
++ Related Constructors: Permanent
++ Also See: SymmetricGroupCombinatoric Functions
++ AMS Classifications:
++ Keywords: gray code, subsets of finite sets
++ References:
++  Henryk Minc: Evaluation of Permanents,
++    Proc. of the Edinburgh Math. Soc.(1979), 22/1 pp 27-32.
++  Nijenhuis and Wilf : Combinatorical Algorithms, Academic
++    Press, New York 1978.
++   S.G.Williamson, Combinatorics for Computer Science,
++    Computer Science Press, 1985.
++ Description:
++  GrayCode provides a function for efficiently running
++  through all subsets of a finite set, only changing one element
++  by another one.
GrayCode: public == private where

  PI ==> PositiveInteger
  I  ==> Integer
  V  ==> Vector

  public ==> with
```

```
    nextSubsetGray: (V V I,PI) -> V V I
      ++ nextSubsetGray(ww,n) returns a vector {\em vv} whose components
      ++ have the following meanings:\begin{items}
      ++ \item {\em vv.1}: a vector of length n whose entries are 0 or 1. This
      ++    can be interpreted as a code for a subset of the set 1,...,n;
      ++    {\em vv.1} differs from {\em ww.1} by exactly one entry;
      ++ \item {\em vv.2.1} is the number of the entry of {\em vv.1} which
      ++    will be changed next time;
      ++ \item {\em vv.2.1 = n+1} means that {\em vv.1} is the last subset;
      ++    trying to compute nextSubsetGray(vv) if {\em vv.2.1 = n+1}
      ++    will produce an error!
      ++ \end{items}
      ++ The other components of {\em vv.2} are needed to compute
      ++ nextSubsetGray efficiently.
      ++ Note: this is an implementation of [Williamson, Topic II, 3.54,
      ++ p. 112] for the special case {\em r1 = r2 = ... = rn = 2};
      ++ Note: nextSubsetGray produces a side-effect, i.e.
      ++ {\em nextSubsetGray(vv)} and {\em vv := nextSubsetGray(vv)}
      ++ will have the same effect.

    firstSubsetGray: PI -> V V I
      ++ firstSubsetGray(n) creates the first vector {\em ww} to start a
      ++ loop using {\em nextSubsetGray(ww,n)}

  private ==> add

    firstSubsetGray(n : PI) ==
      vv : V V I := new(2,[])
      vv.1 := new(n,0) : V I
      vv.2 := new(n+1,1) : V I
      for i in 1..(n+1) repeat
        vv.2.i := i
      vv

    nextSubsetGray(vv : V V I,n : PI) ==
      subs : V I := vv.1    -- subset
      lab : V I := vv.2     -- labels
      c : I := lab(1)    -- element which is to be changed next
      lab(1):= 1
      if subs.c = 0 then subs.c := 1
      else subs.c := 0
      lab.c := lab(c+1)
      lab(c+1) := c+1
      vv
```

⟨*GRAY.dotabb*⟩≡

```
"GRAY" [color="#FF4488",href="bookvol10.4.pdf#nameddest=GRAY"]
"IVECTOR" [color="#88FF44",href="bookvol10.3.pdf#nameddest=IVECTOR"]
"GRAY" -> "IVECTOR"
```

8.31 package GBF GroebnerFactorizationPackage

⟨*GroebnerFactorizationPackage.input*⟩≡

```
)set break resume
)sys rm -f GroebnerFactorizationPackage.output
)spool GroebnerFactorizationPackage.output
)set message test on
)set message auto off
)clear all
--S 1 of 3
mfzn : SQMATRIX(6,DMP([x,y,z],Fraction INT)) := [ [0,1,1,1,1,1], [1,0,1,8/3,x,8/3], [1,1,0,1
--R
--R
--R        +0  1  1  1  1  1+
--R        |                 |
--R        |         8     8|
--R        |1  0  1  -  x  -|
--R        |         3     3|
--R        |                 |
--R        |            8    |
--R        |1  1  0  1  -  y|
--R        |            3    |
--R        |                 |
--R   (1)  |   8           8|
--R        |1  -  1  0  1  -|
--R        |   3           3|
--R        |                 |
--R        |      8          |
--R        |1  x  -  1  0  1|
--R        |      3          |
--R        |                 |
--R        |   8     8       |
--R        |1  -  y  -  1  0|
--R        +   3     3      +
--RType: SquareMatrix(6,DistributedMultivariatePolynomial([x,y,z],Fraction Integer))
--E 1

--S 2 of 3
eq := determinant mfzn
--R
--R
--R   (2)
--R      2 2   22  2    25  2   22    2   388       250     25  2   250     14575
--R   - x y  + -- x y - -- x  + -- x y  - --- x y - --- x - -- y  - --- y + -----
```

```
--R             3        9       3         9         27      9       27       81
--R           Type: DistributedMultivariatePolynomial([x,y,z],Fraction Integer)
--E 2

--S 3 of 3
groebnerFactorize [eq,eval(eq, [x,y,z],[y,z,x]), eval(eq,[x,y,z],[z,x,y])]
--R
--R
--R   (3)
--R   [
--R                   22           22     22     121
--R     [x y + x z - -- x + y z - -- y - -- z + ---,
--R                   3            3      3      3
--R         2   22       25        2   22       25     22  2   388     250
--R      x z  - -- x z + -- x + y z  - -- y z + -- y - -- z  + --- z + ---,
--R             3        9             3        9      3       9       27
--R       2 2   22  2    25  2   22    2   388       250     25  2   250     1457
--R      y z  - -- y z + -- y  - -- y z  + --- y z + --- y + -- z  + --- z - ----
--R             3        9       3         9         27      9       27       81
--R     ,
--R             21994  2   21994     4427     463
--R    [x + y - -----,y  - ----- y + ----,z - ---],
--R              5625       5625      675      87
--R      2   1       11     5     265        2   38     265
--R    [x  - - x z - -- x - - z + ---,y - z,z  - -- z + ---],
--R          2        2     6      18            3      9
--R         25     11     11        11     11     11        5     5     5
--R    [x - --,y - --,z - --], [x - --,y - --,z - --], [x + -,y + -,z + -],
--R          9      3      3         3      3      3        3     3     3
--R         19     5     5
--R    [x - --,y + -,z + -]]
--R          3     3     3
--R  Type: List List DistributedMultivariatePolynomial([x,y,z],Fraction Integer)
--E 3
)spool
)lisp (bye)
```

⟨*GroebnerFactorizationPackage.help*⟩≡

```
==========================================================================
GroebnerFactorizationPackage examples
==========================================================================

Solving systems of polynomial equations with the Groebner basis
algorithm can often be very time consuming because, in general, the
algorithm has exponential run-time.  These systems, which often come
from concrete applications, frequently have symmetries which are not
taken advantage of by the algorithm.  However, it often happens in
this case that the polynomials which occur during the Groebner
calculations are reducible.  Since Axiom has an excellent polynomial
factorization algorithm, it is very natural to combine the Groebner
and factorization algorithms.

GroebnerFactorizationPackage exports the groebnerFactorize operation
which implements a modified Groebner basis algorithm.  In this
algorithm, each polynomial that is to be put into the partial list of
the basis is first factored.  The remaining calculation is split into
as many parts as there are irreducible factors.  Call these factors
p1,...,pN. In the branches corresponding to p2,...,pN, the factor p1
can be divided out, and so on.  This package also contains operations
that allow you to specify the polynomials that are not zero on the
common roots of the final Groebner basis.

Here is an example from chemistry.  In a theoretical model of the
cyclohexan C6H12, the six carbon atoms each sit in the center of
gravity of a tetrahedron that has two hydrogen atoms and two carbon
atoms at its corners.  We first normalize and set the length of each
edge to 1.  Hence, the distances of one fixed carbon atom to each of
its immediate neighbours is 1.  We will denote the distances to the
other three carbon atoms by x, y and z.

A. Dress developed a theory to decide whether a set of points
and distances between them can be realized in an n-dimensional space.
Here, of course, we have n = 3.

  mfzn : SQMATRIX(6,DMP([x,y,z],Fraction INT)) := _
   [ [0,1,1,1,1,1], [1,0,1,8/3,x,8/3], [1,1,0,1,8/3,y], _
     [1,8/3,1,0,1,8/3], [1,x,8/3,1,0,1], [1,8/3,y,8/3,1,0] ]
        +0  1  1  1  1  1+
        |                 |
        |         8     8 |
        |1  0  1  -  x  - |
        |         3     3 |
        |                 |
```

```
        |           8   |
        |1  1  0  1  -  y|
        |           3   |
        |               |
        |   8           8|
        |1  -  1  0  1  -|
        |   3           3|
        |                |
        |      8         |
        |1  x  -  1  0  1|
        |      3         |
        |                |
        |   8     8      |
        |1  -  y  -  1  0|
        +   3     3      +
Type: SquareMatrix(6,DistributedMultivariatePolynomial([x,y,z],
                     Fraction Integer))
```

For the cyclohexan, the distances have to satisfy this equation.

```
eq := determinant mfzn
    2 2   22  2    25  2   22    2   388       250     25  2   250     14575
 - x y  + -- x y - -- x  + -- x y  - --- x y - --- x - -- y  - --- y + -----
          3        9       3         9         27      9       27       81
         Type: DistributedMultivariatePolynomial([x,y,z],Fraction Integer)
```

They also must satisfy the equations given by cyclic shifts of the indeterminates.

```
groebnerFactorize [eq,eval(eq, [x,y,z],[y,z,x]), eval(eq,[x,y,z],[z,x,y])]
 [
              22           22     22     121
  [x y + x z - -- x + y z - -- y - -- z + ---,
               3            3      3      3
     2   22       25        2   22       25     22  2   388     250
   x z  - -- x z + -- x + y z  - -- y z + -- y - -- z  + --- z + ---,
          3        9             3        9      3       9       27
    2 2   22  2    25  2   22    2   388       250     25  2   250     14575
   y z  - -- y z + -- y  - -- y z  + --- y z + --- y + -- z  + --- z - -----]
          3        9       3         9         27      9       27       81
 ,
         21994  2   21994     4427     463
 [x + y - -----,y  - ----- y + ----,z - ---],
          5625       5625      675      87
   2   1       11     5     265        2   38     265
 [x  - - x z - -- x - - z + ---,y - z,z  - -- z + ---],
```

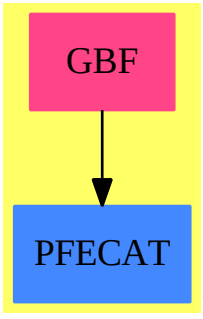
```
      2        2     6      18             3      9
     25     11     11       11     11     11       5     5     5
[x - --,y - --,z - --], [x - --,y - --,z - --], [x + -,y + -,z + -],
      9      3      3        3      3      3       3     3     3
     19     5     5
[x - --,y + -,z + -]]
      3     3     3
  Type: List List DistributedMultivariatePolynomial([x,y,z],Fraction Integer)
```

The union of the solutions of this list is the solution of our original problem. If we impose positivity conditions, we get two relevant ideals. One ideal is zero-dimensional, namely x = y = z =11/3, and this determines the "boat" form of the cyclohexan. The other ideal is one-dimensional, which means that we have a solution space given by one parameter. This gives the "chair" form of the cyclohexan. The parameter describes the angle of the "back of the chair."

groebnerFactorize has an optional Boolean-valued second argument. When it is true partial results are displayed, since it may happen that the calculation does not terminate in a reasonable time. See the source code for GroebnerFactorizationPackage in groebf.spad.pamphlet for more details about the algorithms used.

See Also:
o)display operations groebnerFactorize
o)show GroebnerFactorizationPackage
o) show GroebnerPackage
o)show EuclideanGroebnerBasisPackage

8.32 GroebnerFactorizationPackage



1.00

Exports:
factorGroebnerBasis groebnerFactorize

⟨*package GBF GroebnerFactorizationPackage*⟩≡
```
)abbrev package GBF GroebnerFactorizationPackage
++ Author: H. Michael Moeller, Johannes Grabmeier
++ Date Created: 24 August 1989
++ Date Last Updated: 01 January 1992
++ Basic Operations: groebnerFactorize factorGroebnerBasis
++ Related Constructors:
++ Also See: GroebnerPackage, Ideal, IdealDecompositionPackage
++ AMS Classifications:
++ Keywords: groebner basis, groebner factorization, ideal decomposition
++ References:
++ Description:
++   \spadtype{GroebnerFactorizationPackage} provides the function
++   groebnerFactor" which uses the factorization routines of \Language{} to
++   factor each polynomial under consideration while doing the groebner basis
++   algorithm. Then it writes the ideal as an intersection of ideals
++   determined by the irreducible factors. Note that the whole ring may
++   occur as well as other redundancies. We also use the fact, that from the
++   second factor on we can assume that the preceding factors are
++   not equal to 0 and we divide all polynomials under considerations
++   by the elements of this list of "nonZeroRestrictions".
++   The result is a list of groebner bases, whose union of solutions
++   of the corresponding systems of equations is the solution of
++   the system of equation corresponding to the input list.
++   The term ordering is determined by the polynomial type used.
++   Suggested types include
++   \spadtype{DistributedMultivariatePolynomial},
++   \spadtype{HomogeneousDistributedMultivariatePolynomial},
++   \spadtype{GeneralDistributedMultivariatePolynomial}.

GroebnerFactorizationPackage(Dom, Expon, VarSet, Dpol): T == C where
```

```
Dom :    Join(EuclideanDomain,CharacteristicZero)
Expon :  OrderedAbelianMonoidSup
VarSet : OrderedSet
Dpol: PolynomialCategory(Dom, Expon, VarSet)
MF       ==>   MultivariateFactorize(VarSet,Expon,Dom,Dpol)
sugarPol ==>   Record(totdeg: NonNegativeInteger, pol : Dpol)
critPair ==>   Record(lcmfij: Expon,totdeg: NonNegativeInteger, poli: Dpol, polj: Dpol )
L        ==>   List
B        ==>   Boolean
NNI      ==>   NonNegativeInteger
OUT      ==>   OutputForm

T ==> with

  factorGroebnerBasis : L Dpol -> L L Dpol
    ++ factorGroebnerBasis(basis) checks whether the basis contains
    ++ reducible polynomials and uses these to split the basis.
  factorGroebnerBasis : (L Dpol, Boolean) -> L L Dpol
    ++ factorGroebnerBasis(basis,info) checks whether the basis contains
    ++ reducible polynomials and uses these to split the basis.
    ++ If argument {\em info} is true, information is printed about
    ++ partial results.
  groebnerFactorize : (L Dpol, L Dpol) -> L L Dpol
    ++ groebnerFactorize(listOfPolys, nonZeroRestrictions) returns
    ++ a list of groebner basis. The union of their solutions
    ++ is the solution of the system of equations given by {\em listOfPolys}
    ++ under the restriction that the polynomials of {\em nonZeroRestrictions}
    ++ don't vanish.
    ++ At each stage the polynomial p under consideration (either from
    ++ the given basis or obtained from a reduction of the next S-polynomial)
    ++ is factorized. For each irreducible factors of p, a
    ++ new {\em createGroebnerBasis} is started
    ++ doing the usual updates with the factor
    ++ in place of p.
  groebnerFactorize : (L Dpol, L Dpol, Boolean) -> L L Dpol
    ++ groebnerFactorize(listOfPolys, nonZeroRestrictions, info) returns
    ++ a list of groebner basis. The union of their solutions
    ++ is the solution of the system of equations given by {\em listOfPolys}
    ++ under the restriction that the polynomials of {\em nonZeroRestrictions}
    ++ don't vanish.
    ++ At each stage the polynomial p under consideration (either from
    ++ the given basis or obtained from a reduction of the next S-polynomial)
    ++ is factorized. For each irreducible factors of p a
    ++ new {\em createGroebnerBasis} is started
    ++ doing the usual updates with the factor in place of p.
```

```
    ++ If argument {\em info} is true, information is printed about
    ++ partial results.
  groebnerFactorize : L Dpol  -> L L Dpol
    ++ groebnerFactorize(listOfPolys) returns
    ++ a list of groebner bases. The union of their solutions
    ++ is the solution of the system of equations given by {\em listOfPolys}.
    ++ At each stage the polynomial p under consideration (either from
    ++ the given basis or obtained from a reduction of the next S-polynomial)
    ++ is factorized. For each irreducible factors of p, a
    ++ new {\em createGroebnerBasis} is started
    ++ doing the usual updates with the factor
    ++ in place of p.
    ++
    ++X mfzn : SQMATRIX(6,DMP([x,y,z],Fraction INT)) := _
    ++X   [ [0,1,1,1,1,1], [1,0,1,8/3,x,8/3], [1,1,0,1,8/3,y], _
    ++X   [1,8/3,1,0,1,8/3], [1,x,8/3,1,0,1], [1,8/3,y,8/3,1,0] ]
    ++X eq := determinant mfzn
    ++X groebnerFactorize _
    ++X   [eq,eval(eq, [x,y,z],[y,z,x]), eval(eq,[x,y,z],[z,x,y])]
  groebnerFactorize : (L Dpol, Boolean)  -> L L Dpol
    ++ groebnerFactorize(listOfPolys, info) returns
    ++ a list of groebner bases. The union of their solutions
    ++ is the solution of the system of equations given by {\em listOfPolys}.
    ++ At each stage the polynomial p under consideration (either from
    ++ the given basis or obtained from a reduction of the next S-polynomial)
    ++ is factorized. For each irreducible factors of p, a
    ++ new {\em createGroebnerBasis} is started
    ++ doing the usual updates with the factor
    ++ in place of p.
    ++ If {\em info} is true, information is printed about partial results.

C ==> add

  import GroebnerInternalPackage(Dom,Expon,VarSet,Dpol)
  -- next to help compiler to choose correct signatures:
  info: Boolean
  -- signatures of local functions

  newPairs : (L sugarPol, Dpol) -> L critPair
    -- newPairs(lp, p) constructs list of critical pairs from the list of
    -- {\em lp} of input polynomials and a given further one p.
    -- It uses criteria M and T to reduce the list.
  updateCritPairs : (L critPair, L critPair, Dpol) -> L critPair
    -- updateCritPairs(lcP1,lcP2,p) applies criterion B to {\em lcP1} using
    -- p. Then this list is merged with {\em lcP2}.
  updateBasis : (L sugarPol, Dpol, NNI) -> L sugarPol
```

```
  -- updateBasis(li,p,deg) every polynomial in {\em li} is dropped if
  -- its leading term is a multiple of the leading term of p.
  -- The result is this list enlarged by p.
createGroebnerBases : (L sugarPol, L Dpol, L Dpol, L Dpol, L critPair,_
                       L L Dpol, Boolean) -> L L Dpol
  -- createGroebnerBases(basis, redPols, nonZeroRestrictions, inputPolys,
  --   lcP,listOfBases): This function is used to be called from
  -- groebnerFactorize.
  -- basis: part of a Groebner basis, computed so far
  -- redPols: Polynomials from the ideal to be used for reducing,
  --   we don't throw away polynomials
  -- nonZeroRestrictions: polynomials not zero in the common zeros
  --   of the polynomials in the final (Groebner) basis
  -- inputPolys: assumed to be in descending order
  -- lcP: list of critical pairs built from polynomials of the
  --   actual basis
  -- listOfBases: Collects the (Groebner) bases constructed by this
  --   recursive algorithm at different stages.
  --   we print info messages if info is true
createAllFactors: Dpol -> L Dpol
  -- factor reduced critpair polynomial

-- implementation of local functions


createGroebnerBases(basis, redPols, nonZeroRestrictions, inputPolys,_
    lcP, listOfBases, info) ==
  doSplitting? : B := false
  terminateWithBasis : B := false
  allReducedFactors : L Dpol := []
  nP : Dpol  -- actual polynomial under consideration
  p :  Dpol  -- next polynomial from input list
  h :  Dpol  -- next polynomial from critical pairs
  stopDividing : Boolean
  --    STEP 1   do the next polynomials until a splitting is possible
  -- In the first step we take the first polynomial of "inputPolys"
  -- if empty, from list of critical pairs "lcP" and do the following:
  -- Divide it, if possible, by the polynomials from "nonZeroRestrictions".
  -- We factorize it and reduce each irreducible  factor with respect to
  -- "basis". If 0$Dpol occurs in the list we update the list and continue
  -- with next polynomial.
  -- If there are at least two (irreducible) factors
  -- in the list of factors we finish STEP 1 and set a boolean variable
  -- to continue with STEP 2, the splitting step.
  -- If there is just one of it, we do the following:
  -- If it is 1$Dpol we stop the whole calculation and put
```

```
-- [1$Dpol] into the listOfBases
-- Otherwise we update the "basis" and the other lists and continue
-- with next polynomial.

while (not doSplitting?) and (not terminateWithBasis) repeat
  terminateWithBasis := (null inputPolys and null lcP)
  not terminateWithBasis =>  -- still polynomials left
    -- determine next polynomial "nP"
    nP :=
      not null inputPolys =>
        p := first inputPolys
        inputPolys := rest inputPolys
        -- we know that p is not equal to 0 or 1, but, although,
        -- the inputPolys and the basis are ordered, we cannot assume
        -- that p is reduced w.r.t. basis, as the ordering is only quasi
        -- and we could have equal leading terms, and due to factorization
        -- polynomials of smaller leading terms, hence reduce p first:
        hMonic redPol(p,redPols)
      -- now we have inputPolys empty and hence lcP is not empty:
      -- create S-Polynomial from first critical pair:
      h := sPol first lcP
      lcP := rest lcP
      hMonic redPol(h,redPols)

    nP = 1$Dpol =>
      basis := [[0,1$Dpol]$sugarPol]
      terminateWithBasis := true

    -- if "nP" ^= 0, then  we continue, otherwise we determine next "nP"
    nP ^= 0$Dpol =>
      -- now we divide "nP", if possible, by the polynomials
      -- from "nonZeroRestrictions"
      for q in nonZeroRestrictions repeat
        stopDividing := false
        until stopDividing repeat
          nPq := nP exquo q
          stopDividing := (nPq case "failed")
          if not stopDividing then nP := autoCoerce nPq
          stopDividing := stopDividing or zero? degree nP

      zero? degree nP =>
        basis := [[0,1$Dpol]$sugarPol]
        terminateWithBasis := true  -- doSplitting? is still false

      -- a careful analysis has to be done, when and whether the
      -- following reduction and case nP=1 is necessary
```

```
        nP := hMonic redPol(nP,redPols)
        zero? degree nP =>
          basis := [[0,1$Dpol]$sugarPol]
          terminateWithBasis := true  -- doSplitting? is still false

        -- if "nP" ^= 0, then  we continue, otherwise we determine next "nP"
        nP ^= 0$Dpol =>
          -- now we factorize "nP", which is not constant
          irreducibleFactors : L Dpol := createAllFactors(nP)
          -- if there are more than 1 factors we reduce them and split
          (doSplitting? := not null rest irreducibleFactors) =>
            -- and reduce and normalize the factors
            for fnP in irreducibleFactors repeat
              fnP := hMonic redPol(fnP,redPols)
              -- no factor reduces to 0, as then "fP" would have been
              -- reduced to zero,
              -- but 1 may occur, which we will drop in a later version.
              allReducedFactors := cons(fnP, allReducedFactors)
            -- end of "for fnP in irreducibleFactors repeat"

            -- we want that the smaller factors are dealt with first
            allReducedFactors := reverse allReducedFactors
          -- now the case of exactly 1 factor, but certainly not
          -- further reducible with respect to "redPols"
          nP := first irreducibleFactors
          -- put "nP" into "basis" and update "lcP" and "redPols":
          lcP : L critPair := updateCritPairs(lcP,newPairs(basis,nP),nP)
          basis := updateBasis(basis,nP,virtualDegree nP)
          redPols := concat(redPols,nP)
      -- end of "while not doSplitting? and not terminateWithBasis repeat"

      --    STEP 2  splitting step

      doSplitting? =>
        for fnP in allReducedFactors repeat
          if fnP ^= 1$Dpol
            then
              newInputPolys : L Dpol  := _
                sort((x,y) +-> degree x > degree y ,cons(fnP,inputPolys))
              listOfBases := createGroebnerBases(basis, redPols, _
                nonZeroRestrictions,newInputPolys,lcP,listOfBases,info)
              -- update "nonZeroRestrictions"
              nonZeroRestrictions := cons(fnP,nonZeroRestrictions)
            else
              if info then
```

```
              messagePrint("we terminated with [1]")$OUT
            listOfBases := cons([1$Dpol],listOfBases)

      -- we finished with all the branches on one level and hence
      -- finished this call of createGroebnerBasis. Therefore
      -- we terminate with the actual "listOfBasis" as
      -- everything is done in the recursions
      listOfBases
    -- end of "doSplitting? =>"

    --    STEP 3 termination step

    --  we found a groebner basis and put it into the list "listOfBases"
    --  (auto)reduce each basis element modulo the others
    newBasis :=
      minGbasis(sort((x,y)+->degree x > degree y,[p.pol for p in basis]))
    -- now check whether the normalized basis again has reducible
    -- polynomials, in this case continue splitting!
    if info then
      messagePrint("we found a groebner basis and check whether it ")$OUT
      messagePrint("contains reducible polynomials")$OUT
      print(newBasis::OUT)$OUT
      -- here we should create an output form which is reusable by the system
      -- print(convert(newBasis::OUT)$InputForm :: OUT)$OUT
    removeDuplicates append(factorGroebnerBasis(newBasis, info), listOfBases)

  createAllFactors(p: Dpol) ==
    loF : L Dpol := [el.fctr for el in factorList factor(p)$MF]
    sort((x,y) +-> degree x < degree y, loF)
  newPairs(lp : L sugarPol,p : Dpol) ==
    totdegreeOfp : NNI := virtualDegree p
    -- next list lcP contains all critPair constructed from
    -- p and and the polynomials q in lp
    lcP: L critPair := _
      --[[sup(degree q, degreeOfp), q, p]$critPair for q in lp]
      [makeCrit(q, p, totdegreeOfp) for q in lp]
    -- application of the criteria to reduce the list lcP
    critMTonD1 sort(critpOrder,lcP)
  updateCritPairs(oldListOfcritPairs, newListOfcritPairs, p)==
    updatD (newListOfcritPairs, critBonD(p,oldListOfcritPairs))
  updateBasis(lp, p, deg) == updatF(p,deg,lp)

  -- exported functions

  factorGroebnerBasis basis == factorGroebnerBasis(basis, false)
```

```
factorGroebnerBasis (basis, info) ==
  foundAReducible : Boolean := false
  for p in basis while not foundAReducible repeat
    -- we use fact that polynomials have content 1
    foundAReducible := 1 < #[el.fctr for el in factorList factor(p)$MF]
  not foundAReducible =>
    if info then  messagePrint("factorGroebnerBasis: no reducible polynomials in this bas
    [basis]
  -- improve! Use the fact that the irreducible ones already
  -- build part of the basis, use the done factorizations, etc.
  if info then  messagePrint("factorGroebnerBasis:_
     we found reducible polynomials and continue splitting")$OUT
  createGroebnerBases([],[],[],basis,[],[],info)

groebnerFactorize(basis, nonZeroRestrictions) ==
  groebnerFactorize(basis, nonZeroRestrictions, false)

groebnerFactorize(basis, nonZeroRestrictions, info) ==
  basis = [] => [basis]
  basis := remove((x:Dpol):Boolean +->(x = 0$Dpol),basis)
  basis = [] => [[0$Dpol]]
  -- normalize all input polynomial
  basis := [hMonic p for p in basis]
  member?(1$Dpol,basis) => [[1$Dpol]]
  basis :=  sort((x,y) +-> degree x > degree y, basis)
  createGroebnerBases([],[],nonZeroRestrictions,basis,[],[],info)

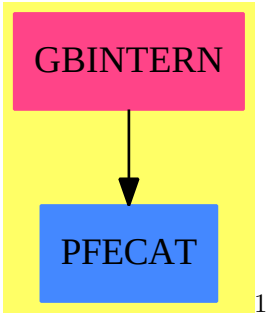
groebnerFactorize(basis) == groebnerFactorize(basis, [], false)
groebnerFactorize(basis,info) == groebnerFactorize(basis, [], info)
```

⟨*GBF.dotabb*⟩≡

```
"GBF" [color="#FF4488",href="bookvol10.4.pdf#nameddest=GBF"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"GBF" -> "PFECAT"
```

8.33 package GBINTERN GroebnerInternalPackage

8.34 GroebnerInternalPackage



1.00

Exports:

credPol	critB	critBonD	critM	critMTonD1
critMonD1	critpOrder	critT	fprindINFO	gbasis
hMonic	lepol	makeCrit	minGbasis	prinb
prindINFO	prinpolINFO	prinshINFO	redPo	redPol
sPol	updatD	updatF	virtualDegree	

⟨*package GBINTERN GroebnerInternalPackage*⟩≡

```
)abbrev package GBINTERN GroebnerInternalPackage
++ Author:
++ Date Created:
++ Date Last Updated:
++ Keywords:
++ Description
++ This package provides low level tools for Groebner basis computations
GroebnerInternalPackage(Dom, Expon, VarSet, Dpol): T == C where
 Dom:   GcdDomain
 Expon: OrderedAbelianMonoidSup
 VarSet: OrderedSet
 Dpol:  PolynomialCategory(Dom, Expon, VarSet)
 NNI    ==> NonNegativeInteger
   ------  Definition of Record critPair and Prinp

 critPair ==> Record( lcmfij: Expon, totdeg: NonNegativeInteger,
                      poli: Dpol, polj: Dpol )
 sugarPol ==> Record( totdeg: NonNegativeInteger, pol : Dpol)
 Prinp    ==> Record( ci:Dpol,tci:Integer,cj:Dpol,tcj:Integer,c:Dpol,
                tc:Integer,rc:Dpol,trc:Integer,tF:Integer,tD:Integer)
 Prinpp   ==> Record( ci:Dpol,tci:Integer,cj:Dpol,tcj:Integer,c:Dpol,
                tc:Integer,rc:Dpol,trc:Integer,tF:Integer,tDD:Integer,
```

```
                tDF:Integer)
  T== with

      credPol:  (Dpol, List(Dpol))  -> Dpol
         ++ credPol \undocumented
      redPol:   (Dpol, List(Dpol))  -> Dpol
         ++ redPol \undocumented
      gbasis:  (List(Dpol), Integer, Integer) -> List(Dpol)
         ++ gbasis \undocumented
      critT:  critPair   -> Boolean
         ++ critT \undocumented
      critM:  (Expon, Expon) -> Boolean
         ++ critM \undocumented
      critB:  (Expon, Expon, Expon, Expon) -> Boolean
         ++ critB \undocumented
      critBonD:  (Dpol, List(critPair)) -> List(critPair)
         ++ critBonD \undocumented
      critMTonD1: (List(critPair)) -> List(critPair)
         ++ critMTonD1 \undocumented
      critMonD1:  (Expon, List(critPair)) -> List(critPair)
         ++ critMonD1 \undocumented
      redPo: (Dpol, List(Dpol) )  ->  Record(poly:Dpol, mult:Dom)
         ++ redPo \undocumented
      hMonic:  Dpol  -> Dpol
         ++ hMonic \undocumented
      updatF: (Dpol, NNI, List(sugarPol) ) -> List(sugarPol)
         ++ updatF \undocumented
      sPol:  critPair  -> Dpol
         ++ sPol \undocumented
      updatD: (List(critPair), List(critPair)) -> List(critPair)
         ++ updatD \undocumented
      minGbasis: List(Dpol) -> List(Dpol)
         ++ minGbasis \undocumented
      lepol: Dpol -> Integer
         ++ lepol \undocumented
      prinshINFO : Dpol -> Void
         ++ prinshINFO \undocumented
      prindINFO: (critPair, Dpol, Dpol,Integer,Integer,Integer) -> Integer
         ++ prindINFO \undocumented
      fprindINFO: (critPair, Dpol, Dpol, Integer,Integer,Integer
                  ,Integer) ->  Integer
         ++ fprindINFO \undocumented
      prinpolINFO: List(Dpol) -> Void
         ++ prinpolINFO \undocumented
      prinb: Integer-> Void
         ++ prinb \undocumented
```

```
      critpOrder: (critPair, critPair) -> Boolean
         ++ critpOrder \undocumented
      makeCrit: (sugarPol, Dpol, NonNegativeInteger) -> critPair
         ++ makeCrit \undocumented
      virtualDegree : Dpol -> NonNegativeInteger
         ++ virtualDegree \undocumented

  C== add
    Ex ==> OutputForm
    import OutputForm

    ------  Definition of intermediate functions
    if Dpol has totalDegree: Dpol -> NonNegativeInteger then
      virtualDegree p == totalDegree p
    else
      virtualDegree p == 0

    ------  ordering of critpairs

    critpOrder(cp1,cp2) ==
      cp1.totdeg < cp2.totdeg => true
      cp2.totdeg < cp1.totdeg => false
      cp1.lcmfij < cp2.lcmfij

    ------    creating a critical pair

    makeCrit(sp1, p2, totdeg2) ==
      p1 := sp1.pol
      deg := sup(degree(p1), degree(p2))
      e1 := subtractIfCan(deg, degree(p1))::Expon
      e2 := subtractIfCan(deg, degree(p2))::Expon
      tdeg := max(sp1.totdeg + virtualDegree(monomial(1,e1)),
                  totdeg2 + virtualDegree(monomial(1,e2)))
      [deg, tdeg, p1, p2]$critPair

    ------    calculate basis

    gbasis(Pol: List(Dpol), xx1: Integer, xx2: Integer ) ==
      D, D1: List(critPair)
      ---------   create D and Pol

      Pol1:= sort((z1,z2) +-> degree z1 > degree z2, Pol)
      basPols:= updatF(hMonic(first Pol1),virtualDegree(first Pol1),[])
      Pol1:= rest(Pol1)
      D:= nil
      while _^ null Pol1 repeat
```

```
   h:= hMonic(first(Pol1))
   Pol1:= rest(Pol1)
   toth := virtualDegree h
   D1:= [makeCrit(x,h,toth) for x in basPols]
   D:= updatD(critMTonD1(sort(critpOrder, D1)),
              critBonD(h,D))
   basPols:= updatF(h,toth,basPols)
D:= sort(critpOrder, D)
xx:= xx2
--------  loop

redPols := [x.pol for x in basPols]
while _^ null D repeat
    D0:= first D
    s:= hMonic(sPol(D0))
    D:= rest(D)
    h:= hMonic(redPol(s,redPols))
    if xx1 = 1  then
         prinshINFO(h)
    h = 0  =>
     if xx2 = 1 then
      prindINFO(D0,s,h,# basPols, # D,xx)
      xx:= 2
     " go to top of while "
    degree(h) = 0 =>
      D:= nil
      if xx2 = 1 then
       prindINFO(D0,s,h,# basPols, # D,xx)
       xx:= 2
      basPols:= updatF(h,0,[])
      leave "out of while"
    D1:= [makeCrit(x,h,D0.totdeg) for x in basPols]
    D:= updatD(critMTonD1(sort(critpOrder, D1)),
              critBonD(h,D))
    basPols:= updatF(h,D0.totdeg,basPols)
    redPols := concat(redPols,h)
    if xx2 = 1 then
       prindINFO(D0,s,h,# basPols, # D,xx)
       xx:= 2
Pol := [x.pol for x in basPols]
if xx2 = 1 then
  prinpolINFO(Pol)
  messagePrint("    THE GROEBNER BASIS POLYNOMIALS")
if xx1 = 1 and xx2 ^= 1 then
  messagePrint("    THE GROEBNER BASIS POLYNOMIALS")
Pol
```

```
          ------------------------------------

          --- erase multiple of e in D2 using crit M

critMonD1(e: Expon, D2: List(critPair))==
   null D2 => nil
   x:= first(D2)
   critM(e, x.lcmfij) => critMonD1(e, rest(D2))
   cons(x, critMonD1(e, rest(D2)))

          ----------------------------

          --- reduce D1 using crit T and crit M

critMTonD1(D1: List(critPair))==
        null D1 => nil
        f1:= first(D1)
        s1:= #(D1)
        cT1:= critT(f1)
        s1= 1 and cT1 => nil
        s1= 1 => D1
        e1:= f1.lcmfij
        r1:= rest(D1)
        e1 = (first r1).lcmfij  =>
           cT1 =>   critMTonD1(cons(f1, rest(r1)))
           critMTonD1(r1)
        D1 := critMonD1(e1, r1)
        cT1 => critMTonD1(D1)
        cons(f1, critMTonD1(D1))

          ----------------------------

          --- erase elements in D fullfilling crit B

critBonD(h:Dpol, D: List(critPair))==
      null D => nil
      x:= first(D)
      critB(degree(h), x.lcmfij, degree(x.poli), degree(x.polj)) =>
        critBonD(h, rest(D))
      cons(x, critBonD(h, rest(D)))

          ----------------------------

          --- concat F and h and erase multiples of h in F
```

```
updatF(h: Dpol, deg:NNI, F: List(sugarPol)) ==
    null F => [[deg,h]]
    f1:= first(F)
    critM(degree(h), degree(f1.pol))  => updatF(h, deg, rest(F))
    cons(f1, updatF(h, deg, rest(F)))

          ----------------------------

          --- concat ordered critical pair lists D1 and D2

updatD(D1: List(critPair), D2: List(critPair)) ==
   null D1 => D2
   null D2 => D1
   dl1:= first(D1)
   dl2:= first(D2)
   critpOrder(dl1,dl2) => cons(dl1, updatD(D1.rest, D2))
   cons(dl2, updatD(D1, D2.rest))

         ----------------------------

         --- remove gcd from pair of coefficients

gcdCo(c1:Dom, c2:Dom):Record(co1:Dom,co2:Dom) ==
   d:=gcd(c1,c2)
   [(c1 exquo d)::Dom, (c2 exquo d)::Dom]

         --- calculate S-polynomial of a critical pair

sPol(p:critPair)==
   Tij := p.lcmfij
   fi := p.poli
   fj := p.polj
   cc := gcdCo(leadingCoefficient fi, leadingCoefficient fj)
   reductum(fi)*monomial(cc.co2,subtractIfCan(Tij, degree fi)::Expon) -
     reductum(fj)*monomial(cc.co1,subtractIfCan(Tij, degree fj)::Expon)

         ---------------------------

         --- reduce critpair polynomial mod F
         --- iterative version

redPo(s: Dpol, F: List(Dpol)) ==
   m:Dom := 1
   Fh := F
   while _^ ( s = 0 or null F ) repeat
     f1:= first(F)
```

```
      s1:= degree(s)
      e: Union(Expon, "failed")
      (e:= subtractIfCan(s1, degree(f1))) case Expon  =>
         cc:=gcdCo(leadingCoefficient f1, leadingCoefficient s)
         s:=cc.co1*reductum(s) - monomial(cc.co2,e)*reductum(f1)
         m := m*cc.co1
         F:= Fh
      F:= rest F
    [s,m]

redPol(s: Dpol, F: List(Dpol)) ==  credPol(redPo(s,F).poly,F)

          ----------------------------

          --- crit T  true, if e1 and e2 are disjoint

critT(p: critPair) == p.lcmfij =  (degree(p.poli) + degree(p.polj))

          ----------------------------

          --- crit M - true, if lcm#2 multiple of lcm#1

critM(e1: Expon, e2: Expon) ==
       en: Union(Expon, "failed")
       (en:=subtractIfCan(e2, e1)) case Expon

          ----------------------------

          --- crit B - true, if eik is a multiple of eh and eik ^equal
          ---          lcm(eh,ei) and eik ^equal lcm(eh,ek)

critB(eh:Expon, eik:Expon, ei:Expon, ek:Expon) ==
    critM(eh, eik) and (eik ^= sup(eh, ei)) and (eik ^= sup(eh, ek))

          ----------------------------

          ---  make polynomial monic case Domain a Field

hMonic(p: Dpol) ==
      p= 0 => p
      -- inv(leadingCoefficient(p))*p
      primitivePart p

          ----------------------------

          ---  reduce all terms of h mod F  (iterative version )
```

```
credPol(h: Dpol, F: List(Dpol) ) ==
     null F => h
     h0:Dpol:= monomial(leadingCoefficient h, degree h)
     while (h:=reductum h) ^= 0 repeat
        hred:= redPo(h, F)
        h := hred.poly
        h0:=(hred.mult)*h0 + monomial(leadingCoefficient(h),degree h)
     h0

         -------------------------------

         ----  calculate minimal basis for ordered F

minGbasis(F: List(Dpol)) ==
     null F => nil
     newbas := minGbasis rest F
     cons(hMonic credPol( first(F), newbas),newbas)

         -------------------------------

         ----  calculate number of terms of polynomial

lepol(p1:Dpol)==
   n: Integer
   n:= 0
   while p1 ^= 0 repeat
      n:= n + 1
      p1:= reductum(p1)
   n

         ----  print blanc lines

prinb(n: Integer)==
   for x in 1..n repeat
      messagePrint("    ")

         ----  print reduced critpair polynom

prinshINFO(h: Dpol)==
        prinb(2)
        messagePrint(" reduced Critpair - Polynom :")
        prinb(2)
        print(h::Ex)
        prinb(2)
```

```
          --------------------------------

          ----  print info string

    prindINFO(cp: critPair, ps: Dpol, ph: Dpol, i1:Integer,
              i2:Integer, n:Integer) ==
        ll: List Prinp
        a: Dom
        cpi:= cp.poli
        cpj:= cp.polj
        if n = 1 then
         prinb(1)
         messagePrint("you choose option  -info-  ")
         messagePrint("abbrev. for the following information strings are")
         messagePrint("  ci  =>  Leading monomial  for critpair calculation")
         messagePrint("  tci =>  Number of terms of polynomial i")
         messagePrint("  cj  =>  Leading monomial  for critpair calculation")
         messagePrint("  tcj =>  Number of terms of polynomial j")
         messagePrint("  c   =>  Leading monomial of critpair polynomial")
         messagePrint("  tc  =>  Number of terms of critpair polynomial")
         messagePrint("  rc  =>  Leading monomial of redcritpair polynomial")
         messagePrint("  trc =>  Number of terms of redcritpair polynomial")
         messagePrint("  tF  =>  Number of polynomials in reduction list F")
         messagePrint("  tD  =>  Number of critpairs still to do")
         prinb(4)
         n:= 2
        prinb(1)
        a:= 1
        ph = 0  =>
           ps = 0 =>
             ll:= [[monomial(a,degree(cpi)),lepol(cpi),
                   monomial(a,degree(cpj)),
                    lepol(cpj),ps,0,ph,0,i1,i2]$Prinp]
             print(ll::Ex)
             prinb(1)
             n
           ll:= [[monomial(a,degree(cpi)),lepol(cpi),
              monomial(a,degree(cpj)),lepol(cpj),monomial(a,degree(ps)),
               lepol(ps), ph,0,i1,i2]$Prinp]
           print(ll::Ex)
           prinb(1)
           n
        ll:= [[monomial(a,degree(cpi)),lepol(cpi),
             monomial(a,degree(cpj)),lepol(cpj),monomial(a,degree(ps)),
              lepol(ps),monomial(a,degree(ph)),lepol(ph),i1,i2]$Prinp]
        print(ll::Ex)
```

```
        prinb(1)
        n

             ------------------------------

             ----  print the groebner basis polynomials

    prinpolINFO(pl: List(Dpol))==
        n:Integer
        n:= # pl
        prinb(1)
        n = 1 =>
          messagePrint("  There is 1  Groebner Basis Polynomial ")
          prinb(2)
        messagePrint("  There are ")
        prinb(1)
        print(n::Ex)
        prinb(1)
        messagePrint("  Groebner Basis Polynomials. ")
        prinb(2)

    fprindINFO(cp: critPair, ps: Dpol, ph: Dpol, i1:Integer,
              i2:Integer, i3:Integer, n: Integer) ==
        ll: List Prinpp
        a: Dom
        cpi:= cp.poli
        cpj:= cp.polj
        if n = 1 then
         prinb(1)
         messagePrint("you choose option  -info-  ")
         messagePrint("abbrev. for the following information strings are")
         messagePrint("  ci  =>  Leading monomial  for critpair calculation")
         messagePrint("  tci =>  Number of terms of polynomial i")
         messagePrint("  cj  =>  Leading monomial  for critpair calculation")
         messagePrint("  tcj =>  Number of terms of polynomial j")
         messagePrint("  c   =>  Leading monomial of critpair polynomial")
         messagePrint("  tc  =>  Number of terms of critpair polynomial")
         messagePrint("  rc  =>  Leading monomial of redcritpair polynomial")
         messagePrint("  trc =>  Number of terms of redcritpair polynomial")
         messagePrint("  tF  =>  Number of polynomials in reduction list F")
         messagePrint("  tD  =>  Number of critpairs still to do")
         messagePrint("  tDF =>  Number of subproblems still to do")
         prinb(4)
         n:= 2
        prinb(1)
        a:= 1
```

```
        ph = 0  =>
           ps = 0 =>
             ll:= [[monomial(a,degree(cpi)),lepol(cpi),
               monomial(a,degree(cpj)),
                lepol(cpj),ps,0,ph,0,i1,i2,i3]$Prinpp]
             print(ll::Ex)
             prinb(1)
             n
           ll:= [[monomial(a,degree(cpi)),lepol(cpi),
             monomial(a,degree(cpj)),lepol(cpj),monomial(a,degree(ps)),
              lepol(ps), ph,0,i1,i2,i3]$Prinpp]
           print(ll::Ex)
           prinb(1)
           n
        ll:= [[monomial(a,degree(cpi)),lepol(cpi),
             monomial(a,degree(cpj)),lepol(cpj),monomial(a,degree(ps)),
              lepol(ps),monomial(a,degree(ph)),lepol(ph),i1,i2,i3]$Prinpp]
        print(ll::Ex)
        prinb(1)
        n
```

⟨*GBINTERN.dotabb*⟩≡

```
"GBINTERN" [color="#FF4488",href="bookvol10.4.pdf#nameddest=GBINTERN"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"GBINTERN" -> "PFECAT"
```

8.35 package GB GroebnerPackage

⟨*GroebnerPackage.input*⟩≡

```
)set break resume
)sys rm -f GroebnerPackage.output
)spool GroebnerPackage.output
)set message test on
)set message auto off
)clear all
--S 1 of 24
s1:DMP([w,p,z,t,s,b],FRAC(INT)):= 45*p + 35*s - 165*b - 36
--R
--R   (1)  45p + 35s - 165b - 36
--R      Type: DistributedMultivariatePolynomial([w,p,z,t,s,b],Fraction Integer)
--E 1

--S 2 of 24
s2:DMP([w,p,z,t,s,b],FRAC(INT)):= 35*p + 40*z + 25*t - 27*s
--R
--R   (2)  35p + 40z + 25t - 27s
--R      Type: DistributedMultivariatePolynomial([w,p,z,t,s,b],Fraction Integer)
--E 2

--S 3 of 24
s3:DMP([w,p,z,t,s,b],FRAC(INT)):= 15*w + 25*p*s + 30*z - 18*t - 165*b**2
--R
--R                                      2
--R   (3)  15w + 25p s + 30z - 18t - 165b
--R      Type: DistributedMultivariatePolynomial([w,p,z,t,s,b],Fraction Integer)
--E 3

--S 4 of 24
s4:DMP([w,p,z,t,s,b],FRAC(INT)):= -9*w + 15*p*t + 20*z*s
--R
--R   (4)  - 9w + 15p t + 20z s
--R      Type: DistributedMultivariatePolynomial([w,p,z,t,s,b],Fraction Integer)
--E 4

--S 5 of 24
s5:DMP([w,p,z,t,s,b],FRAC(INT)):= w*p + 2*z*t - 11*b**3
--R
--R                         3
--R   (5)  w p + 2z t - 11b
--R      Type: DistributedMultivariatePolynomial([w,p,z,t,s,b],Fraction Integer)
--E 5
```

```
--S 6 of 24
s6:DMP([w,p,z,t,s,b],FRAC(INT)):= 99*w - 11*b*s + 3*b**2
--R
--R                          2
--R   (6)  99w - 11s b + 3b
--R      Type: DistributedMultivariatePolynomial([w,p,z,t,s,b],Fraction Integer)
--E 6

--S 7 of 24
s7:DMP([w,p,z,t,s,b],FRAC(INT)):= b**2 + 33/50*b + 2673/10000
--R
--R         2   33      2673
--R   (7)  b  + -- b + -----
--R             50     10000
--R      Type: DistributedMultivariatePolynomial([w,p,z,t,s,b],Fraction Integer)
--E 7

--S 8 of 24
sn7:=[s1,s2,s3,s4,s5,s6,s7]
--R
--R   (8)
--R   [45p + 35s - 165b - 36, 35p + 40z + 25t - 27s,
--R                                    2                                    3
--R    15w + 25p s + 30z - 18t - 165b , - 9w + 15p t + 20z s, w p + 2z t - 11b ,
--R                     2   2   33      2673
--R    99w - 11s b + 3b , b  + -- b + -----]
--R                            50     10000
--R Type: List DistributedMultivariatePolynomial([w,p,z,t,s,b],Fraction Integer)
--E 8

--S 9 of 24
groebner(sn7)
--R
--R   (9)
--R         19      1323      31     153      49     1143      37      27
--R   [w + --- b + -----, p - -- b - ---, z + -- b + ----, t - -- b + ---,
--R        120     20000      18     200      36     2000      15     250
--R        5      9    2   33      2673
--R    s - - b - ---, b  + -- b + -----]
--R        2     200       50     10000
--R Type: List DistributedMultivariatePolynomial([w,p,z,t,s,b],Fraction Integer)
--E 9

--S 10 of 24
groebner(sn7,"redcrit")
--R
```

```
--R
--R    reduced Critpair - Polynom :
--R
--R
--R       5     61     77      7
--R   z + - t - -- s + -- b + --
--R       8     45     24     10
--R
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R         66        603     278  2   11       672     2277     415881
--R   t s - -- t b + ---- t - --- s  + -- s b - --- s - ---- b - ------
--R         29       1450     435      29       725     7250     725000
--R
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R       100  2   160       104      37      79
--R   t + --- s  - --- s b - --- s - --- b - ---
--R       189       63        63     105     125
--R
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R    3   1026  2    5424  2   2529       1326807     12717      660717
--R   s  - ---- s b - ---- s  - ---- s b - ------- s + ----- b + -------
--R         145       3625       725        362500      6250     3625000
--R
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R      2     91248294  2   6550614        7087292937     20020838931
--R     s b + --------- s  - ------- s b + ----------- s - ----------- b
```

```
--R           128176525      5127061       12817652500     12817652500
--R   +
--R       37595502243
--R     - -----------
--R       51270610000
--R
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R      2   4746183626079988       1015195815329760     30723564870033201
--R     s  - ---------------- s b - ---------------- s - ---------------- b
--R           987357073521193        987357073521193     24683926838029825
--R   +
--R       3696123458901625353
--R     - -------------------
--R       2468392683802982500
--R
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R   0
--R
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R           16827373608076633182513471     12627931635816456985349 64
--R     s b + -------------------------- s - ------------------------- b
--R           23063714246644859914108300     57659285616612149785270 75
--R   +
--R      9159434520598111965243603 3
--R     ---------------------------
--R     1441482140415303744631768 75
--R
--R
--R
--R
--R    reduced Critpair - Polynom :
```

```
--R
--R
--R       5      9
--R   s - - b - ---
--R       2     200
--R
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R   0
--R
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R   0
--R
--R
--R       THE GROEBNER BASIS POLYNOMIALS
--R
--R   (10)
--R         19      1323      31     153      49     1143      37      27
--R   [w + --- b + -----, p - -- b - ---, z + -- b + ----, t - -- b + ---,
--R        120     20000      18     200      36     2000      15     250
--R        5      9    2   33      2673
--R    s - - b - ---, b  + -- b + -----]
--R        2     200       50     10000
--R Type: List DistributedMultivariatePolynomial([w,p,z,t,s,b],Fraction Integer)
--E 10

--S 11 of 24
groebner(sn7,"info")
--R
--R   you choose option  -info-
--R   abbrev. for the following information strings are
--R     ci  =>  Leading monomial  for critpair calculation
--R     tci =>  Number of terms of polynomial i
--R     cj  =>  Leading monomial  for critpair calculation
--R     tcj =>  Number of terms of polynomial j
--R     c   =>  Leading monomial of critpair polynomial
--R     tc  =>  Number of terms of critpair polynomial
```

```
--R     rc  =>  Leading monomial of redcritpair polynomial
--R     trc =>  Number of terms of redcritpair polynomial
--R     tF  =>  Number of polynomials in reduction list F
--R     tD  =>  Number of critpairs still to do
--R
--R
--R
--R
--R
--R   [[ci= p,tci= 4,cj= p,tcj= 4,c= z,tc= 5,rc= z,trc= 5,tF= 4,tD= 3]]
--R
--R
--R   [[ci= w,tci= 3,cj= w,tcj= 5,c= p t,tc= 6,rc= t s,trc= 8,tF= 5,tD= 2]]
--R
--R
--R   [[ci= w,tci= 3,cj= w,tcj= 3,c= p t,tc= 4,rc= t,trc= 6,tF= 5,tD= 2]]
--R
--R
--R                                                      3
--R   [[ci= t s,tci= 8,cj= t,tcj= 6,c= t b,tc= 9,rc= s ,trc= 7,tF= 6,tD= 1]]
--R
--R
--R                                                        2
--R   [[ci= w p,tci= 3,cj= w,tcj= 3,c= p s b,tc= 4,rc= s b,trc= 6,tF= 7,tD= 2]]
--R
--R
--R          2             2             2             2
--R   [[ci= b ,tci= 3,cj= s b,tcj= 6,c= s b,tc= 6,rc= s ,trc= 5,tF= 6,tD= 2]]
--R
--R
--R          2              2            2
--R   [[ci= s b,tci= 6,cj= s ,tcj= 5,c= s ,tc= 7,rc= 0,trc= 0,tF= 6,tD= 1]]
--R
--R
--R          3             2            2
--R   [[ci= s ,tci= 7,cj= s ,tcj= 5,c= s b,tc= 6,rc= s b,trc= 4,tF= 7,tD= 2]]
--R
--R
--R          2
--R   [[ci= b ,tci= 3,cj= s b,tcj= 4,c= s b,tc= 4,rc= s,trc= 3,tF= 6,tD= 2]]
--R
--R
--R   [[ci= s b,tci= 4,cj= s,tcj= 3,c= s,tc= 4,rc= 0,trc= 0,tF= 6,tD= 1]]
--R
--R
--R          2
```

```
--R   [[ci= s ,tci= 5,cj= s,tcj= 3,c= s b,tc= 4,rc= 0,trc= 0,tF= 6,tD= 0]]
--R
--R
--R     There are
--R
--R   6
--R
--R     Groebner Basis Polynomials.
--R
--R
--R       THE GROEBNER BASIS POLYNOMIALS
--R
--R   (11)
--R         19      1323      31     153      49     1143      37      27
--R   [w + --- b + -----, p - -- b - ---, z + -- b + ----, t - -- b + ---,
--R        120     20000      18     200      36     2000      15     250
--R        5      9    2   33      2673
--R    s - - b - ---, b  + -- b + -----]
--R        2     200       50     10000
--R Type: List DistributedMultivariatePolynomial([w,p,z,t,s,b],Fraction Integer)
--E 11

--S 12 of 24
groebner(sn7,"redcrit","info")
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R       5     61     77      7
--R   z + - t - -- s + -- b + --
--R       8     45     24     10
--R
--R
--R
--R   you choose option  -info-
--R   abbrev. for the following information strings are
--R     ci  =>  Leading monomial  for critpair calculation
--R     tci =>  Number of terms of polynomial i
--R     cj  =>  Leading monomial  for critpair calculation
--R     tcj =>  Number of terms of polynomial j
--R     c   =>  Leading monomial of critpair polynomial
--R     tc  =>  Number of terms of critpair polynomial
--R     rc  =>  Leading monomial of redcritpair polynomial
--R     trc =>  Number of terms of redcritpair polynomial
--R     tF  =>  Number of polynomials in reduction list F
```

```
--R     tD  =>  Number of critpairs still to do
--R
--R
--R
--R
--R
--R   [[ci= p,tci= 4,cj= p,tcj= 4,c= z,tc= 5,rc= z,trc= 5,tF= 4,tD= 3]]
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R         66        603     278  2   11       672     2277     415881
--R   t s - -- t b + ---- t - --- s  + -- s b - --- s - ---- b - ------
--R         29       1450     435      29       725     7250     725000
--R
--R
--R
--R   [[ci= w,tci= 3,cj= w,tcj= 5,c= p t,tc= 6,rc= t s,trc= 8,tF= 5,tD= 2]]
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R       100  2   160       104      37      79
--R   t + --- s  - --- s b - --- s - --- b - ---
--R       189       63        63     105     125
--R
--R
--R
--R   [[ci= w,tci= 3,cj= w,tcj= 3,c= p t,tc= 4,rc= t,trc= 6,tF= 5,tD= 2]]
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R    3   1026  2    5424  2   2529       1326807     12717      660717
--R   s  - ---- s b - ---- s  - ---- s b - ------- s + ----- b + -------
--R         145       3625       725        362500      6250     3625000
--R
--R
--R
--R                                                        3
```

```
--R   [[ci= t s,tci= 8,cj= t,tcj= 6,c= t b,tc= 9,rc= s ,trc= 7,tF= 6,tD= 1]]
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R      2     91248294  2   6550614        7087292937     20020838931
--R     s b + --------- s  - ------- s b + ----------- s - ----------- b
--R           128176525      5127061       12817652500     12817652500
--R   +
--R       37595502243
--R     - -----------
--R       51270610000
--R
--R
--R
--R                                                      2
--R   [[ci= w p,tci= 3,cj= w,tcj= 3,c= p s b,tc= 4,rc= s b,trc= 6,tF= 7,tD= 2]]
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R      2   4746183626079988       1015195815329760     30723564870033201
--R     s  - ---------------- s b - ---------------- s - ----------------- b
--R           987357073521193        987357073521193     24683926838029825
--R   +
--R       3696123458901625353
--R     - -------------------
--R       2468392683802982500
--R
--R
--R
--R          2             2             2             2
--R   [[ci= b ,tci= 3,cj= s b,tcj= 6,c= s b,tc= 6,rc= s ,trc= 5,tF= 6,tD= 2]]
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R   0
--R
--R
```

```
--R
--R          2              2            2
--R   [[ci= s b,tci= 6,cj= s ,tcj= 5,c= s ,tc= 7,rc= 0,trc= 0,tF= 6,tD= 1]]
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R           168273736080766331825134 71     126279316358164569853496 4
--R     s b + ------------------------- s - ------------------------- b
--R           230637142466448599141083 00     576592856166121497852707 5
--R   +
--R      915943452059811196524360 33
--R     ---------------------------
--R     144148214041530374463176875
--R
--R
--R
--R          3              2            2
--R   [[ci= s ,tci= 7,cj= s ,tcj= 5,c= s b,tc= 6,rc= s b,trc= 4,tF= 7,tD= 2]]
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R       5      9
--R   s - - b - ---
--R       2     200
--R
--R
--R
--R          2
--R   [[ci= b ,tci= 3,cj= s b,tcj= 4,c= s b,tc= 4,rc= s,trc= 3,tF= 6,tD= 2]]
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R   0
--R
--R
--R
--R   [[ci= s b,tci= 4,cj= s,tcj= 3,c= s,tc= 4,rc= 0,trc= 0,tF= 6,tD= 1]]
```

```
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R   0
--R
--R
--R
--R          2
--R   [[ci= s ,tci= 5,cj= s,tcj= 3,c= s b,tc= 4,rc= 0,trc= 0,tF= 6,tD= 0]]
--R
--R
--R     There are
--R
--R   6
--R
--R     Groebner Basis Polynomials.
--R
--R
--R       THE GROEBNER BASIS POLYNOMIALS
--R
--R   (12)
--R         19      1323      31     153      49     1143      37      27
--R   [w + --- b + -----, p - -- b - ---, z + -- b + ----, t - -- b + ---,
--R        120     20000      18     200      36     2000      15     250
--R        5      9    2   33      2673
--R    s - - b - ---, b  + -- b + -----]
--R        2     200       50     10000
--R Type: List DistributedMultivariatePolynomial([w,p,z,t,s,b],Fraction Integer)
--E 12

--S 13 of 24
hs1:HDMP([w,p,z,t,s,b],FRAC(INT)):= 45*p + 35*s - 165*b - 36
--R
--R   (13)  45p + 35s - 165b - 36
--RType: HomogeneousDistributedMultivariatePolynomial([w,p,z,t,s,b],Fraction Integer)
--E 13

--S 14 of 24
hs2:HDMP([w,p,z,t,s,b],FRAC(INT)):= 35*p + 40*z + 25*t - 27*s
--R
--R   (14)  35p + 40z + 25t - 27s
--RType: HomogeneousDistributedMultivariatePolynomial([w,p,z,t,s,b],Fraction Integer)
--E 14
```

```
--S 15 of 24
hs3:HDMP([w,p,z,t,s,b],FRAC(INT)):= 15*w + 25*p*s + 30*z - 18*t - 165*b**2
--R                      2
--R   (15)  25p s - 165b  + 15w + 30z - 18t
--RType: HomogeneousDistributedMultivariatePolynomial([w,p,z,t,s,b],Fraction Inte
--E 15

--S 16 of 24
hs4:HDMP([w,p,z,t,s,b],FRAC(INT)):= -9*w + 15*p*t + 20*z*s
--R
--R   (16)  15p t + 20z s - 9w
--RType: HomogeneousDistributedMultivariatePolynomial([w,p,z,t,s,b],Fraction Inte
--E 16

--S 17 of 24
hs5:HDMP([w,p,z,t,s,b],FRAC(INT)):= w*p + 2*z*t - 11*b**3
--R
--R              3
--R   (17)  - 11b  + w p + 2z t
--RType: HomogeneousDistributedMultivariatePolynomial([w,p,z,t,s,b],Fraction Inte
--E 17

--S 18 of 24
hs6:HDMP([w,p,z,t,s,b],FRAC(INT)):= 99*w - 11*b*s + 3*b**2
--R
--R                      2
--R   (18)  - 11s b + 3b  + 99w
--RType: HomogeneousDistributedMultivariatePolynomial([w,p,z,t,s,b],Fraction Inte
--E 18

--S 19 of 24
hs7:HDMP([w,p,z,t,s,b],FRAC(INT)):= b**2 + 33/50*b + 2673/10000
--R
--R          2   33      2673
--R   (19)  b  + -- b + -----
--R              50     10000
--RType: HomogeneousDistributedMultivariatePolynomial([w,p,z,t,s,b],Fraction Inte
--E 19

--S 20 of 24
hsn7:=[hs1,hs2,hs3,hs4,hs5,hs6,hs7]
--R
--R   (20)
--R   [45p + 35s - 165b - 36, 35p + 40z + 25t - 27s,
--R                2                                          3
```

```
--R    25p s - 165b  + 15w + 30z - 18t, 15p t + 20z s - 9w, - 11b  + w p + 2z t,
--R                2         2   33      2673
--R    - 11s b + 3b  + 99w, b  + -- b + -----]
--R                              50     10000
--RType: List HomogeneousDistributedMultivariatePolynomial([w,p,z,t,s,b],Fraction Integer)
--E 20

--S 21 of 24
groebner(hsn7)
--R
--R   (21)
--R     2   33      2673       19      1323      31     153      49     1143
--R   [b  + -- b + -----, w + --- b + -----, p - -- b - ---, z + -- b + ----,
--R         50     10000      120     20000      18     200      36     2000
--R        37      27      5      9
--R    t - -- b + ---, s - - b - ---]
--R        15     250      2     200
--RType: List HomogeneousDistributedMultivariatePolynomial([w,p,z,t,s,b],Fraction Integer)
--E 21

--S 22 of 24
groebner(hsn7,"redcrit")
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R       5     61     77      7
--R   z + - t - -- s + -- b + --
--R       8     45     24     10
--R
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R    2   216     189     78      99     10557
--R   s  - --- w + --- t - -- s + --- b - -----
--R         5      100     25     500     12500
--R
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
```

```
--R
--R         66       17541     5886     10588      9273     8272413
--R   t s - -- t b - ----- w + ---- t - ----- s - ----- b - -------
--R         29        725      3625      3625     36250     7250000
--R
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R      2   28       44       143       962712     420652     5166944
--R     t  + -- w s - -- w b + --- t b - ------ w + ------ t - ------- s
--R          45       15       725        18125      90625      815625
--R   +
--R     5036339     83580953
--R     ------- b - --------
--R     5437500     90625000
--R
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R         33      297        81
--R   w b + -- w + ----- s - ----- b
--R         50     10000     10000
--R
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R          21        33      6723      2031      104247
--R   w s + --- t b - --- w + ----- s - ----- b + -------
--R         100       250     50000     25000     5000000
--R
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R         2373       41563      17253      578853      258751      11330361
```

```
--R   w t + ---- t b - ----- w + ------ t + ------- s - ------- b + ---------
--R         7250       36250     290000     7250000     3625000     362500000
--R
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R   0
--R
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R         51061712      91248294     1516761889      481096937      5789482077
--R   t b - -------- w + --------- t - ---------- s + ---------- b + -----------
--R          5127061     128176525     1922647875     1281765250     51270610000
--R
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R         2962071220563579     1229379913128787     4524811449715289
--R     w + ---------------- t - ---------------- s + ---------------- b
--R          98138188260880       36801820597830       490690941304400
--R   +
--R     59240140318722273
--R     -----------------
--R     12267273532610000
--R
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R   0
--R
--R
--R
--R
```

```
--R    reduced Critpair - Polynom :
--R
--R
--R       172832706542351932      47302810289036749      2736061156820726
--R   t - ------------------ s + ------------------ b + ----------------
--R       155991468675747195     155991468675747195     17332385408416355
--R
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R       5      9
--R   s - - b - ---
--R       2     200
--R
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R   0
--R
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R   0
--R
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R   0
--R
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
```

```
--R
--R   0
--R
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R   0
--R
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R   0
--R
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R   0
--R
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R   0
--R
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R   0
--R
--R
--R       THE GROEBNER BASIS POLYNOMIALS
--R
```

```
--R   (22)
--R     2   33      2673       19      1323      31     153      49     1143
--R   [b  + -- b + -----, w + --- b + -----, p - -- b - ---, z + -- b + ----,
--R         50     10000      120     20000      18     200      36     2000
--R        37      27      5      9
--R    t - -- b + ---, s - - b - ---]
--R        15     250      2     200
--RType: List HomogeneousDistributedMultivariatePolynomial([w,p,z,t,s,b],Fraction
--E 22

--S 23 of 24
groebner(hsn7,"info")
--R
--R   you choose option  -info-
--R   abbrev. for the following information strings are
--R     ci  =>  Leading monomial  for critpair calculation
--R     tci =>  Number of terms of polynomial i
--R     cj  =>  Leading monomial  for critpair calculation
--R     tcj =>  Number of terms of polynomial j
--R     c   =>  Leading monomial of critpair polynomial
--R     tc  =>  Number of terms of critpair polynomial
--R     rc  =>  Leading monomial of redcritpair polynomial
--R     trc =>  Number of terms of redcritpair polynomial
--R     tF  =>  Number of polynomials in reduction list F
--R     tD  =>  Number of critpairs still to do
--R
--R
--R
--R
--R
--R   [[ci= p,tci= 4,cj= p,tcj= 4,c= z,tc= 5,rc= z,trc= 5,tF= 4,tD= 5]]
--R
--R
--R                                                    2
--R   [[ci= p s,tci= 5,cj= p,tcj= 4,c= z s,tc= 7,rc= s ,trc= 6,tF= 5,tD= 5]]
--R
--R
--R   [[ci= p t,tci= 3,cj= p,tcj= 4,c= z t,tc= 5,rc= t s,trc= 7,tF= 6,tD= 6]]
--R
--R
--R          3             2                          2
--R   [[ci= b ,tci= 3,cj= b ,tcj= 3,c= w p,tc= 4,rc= t ,trc= 9,tF= 7,tD= 6]]
--R
--R
--R                         2           3
--R   [[ci= s b,tci= 3,cj= b ,tcj= 3,c= b ,tc= 4,rc= w b,trc= 4,tF= 8,tD= 7]]
```

```
--R
--R
--R                          2             2
--R   [[ci= s b,tci= 3,cj= s ,tcj= 6,c= s b ,tc= 7,rc= w s,trc= 6,tF= 9,tD= 9]]
--R
--R
--R                                        2
--R   [[ci= s b,tci= 3,cj= t s,tcj= 7,c= t b ,tc= 7,rc= w t,trc= 7,tF= 10,tD= 11]]
--R
--R
--R                                        2
--R   [[ci= p s,tci= 5,cj= s b,tcj= 3,c= p b ,tc= 6,rc= 0,trc= 0,tF= 10,tD= 10]]
--R
--R
--R          2
--R   [[ci= s ,tci= 6,cj= t s,tcj= 7,c= t s b,tc= 10,rc= t b,trc= 6,tF= 11,tD= 13]]
--R
--R
--R          2
--R   [[ci= b ,tci= 3,cj= t b,tcj= 6,c= w b,tc= 6,rc= w,trc= 5,tF= 9,tD= 14]]
--R
--R
--R          2
--R   [[ci= b ,tci= 3,cj= w b,tcj= 4,c= s b,tc= 3,rc= 0,trc= 0,tF= 9,tD= 13]]
--R
--R
--R                                        2
--R   [[ci= s b,tci= 3,cj= t b,tcj= 6,c= t b ,tc= 7,rc= t,trc= 4,tF= 7,tD= 11]]
--R
--R
--R                                        2
--R   [[ci= s b,tci= 3,cj= w b,tcj= 4,c= w b ,tc= 5,rc= s,trc= 3,tF= 6,tD= 9]]
--R
--R
--R                                      2
--R   [[ci= w b,tci= 4,cj= t b,tcj= 6,c= w ,tc= 7,rc= 0,trc= 0,tF= 6,tD= 8]]
--R
--R
--R                                    2
--R   [[ci= s b,tci= 3,cj= s,tcj= 3,c= b ,tc= 3,rc= 0,trc= 0,tF= 6,tD= 7]]
--R
--R
--R   [[ci= t b,tci= 6,cj= t,tcj= 4,c= s b,tc= 7,rc= 0,trc= 0,tF= 6,tD= 6]]
--R
--R
--R   [[ci= w b,tci= 4,cj= w,tcj= 5,c= t b,tc= 6,rc= 0,trc= 0,tF= 6,tD= 5]]
```

```
--R
--R
--R          2
--R   [[ci= s ,tci= 6,cj= s,tcj= 3,c= s b,tc= 6,rc= 0,trc= 0,tF= 6,tD= 4]]
--R
--R
--R                                      2
--R   [[ci= t s,tci= 7,cj= t,tcj= 4,c= s ,tc= 8,rc= 0,trc= 0,tF= 6,tD= 3]]
--R
--R
--R   [[ci= w s,tci= 6,cj= w,tcj= 5,c= t s,tc= 8,rc= 0,trc= 0,tF= 6,tD= 2]]
--R
--R
--R          2
--R   [[ci= t ,tci= 9,cj= t,tcj= 4,c= w s,tc= 9,rc= 0,trc= 0,tF= 6,tD= 1]]
--R
--R
--R                                      2
--R   [[ci= w t,tci= 7,cj= w,tcj= 5,c= t ,tc= 8,rc= 0,trc= 0,tF= 6,tD= 0]]
--R
--R
--R     There are
--R
--R   6
--R
--R     Groebner Basis Polynomials.
--R
--R
--R       THE GROEBNER BASIS POLYNOMIALS
--R
--R   (23)
--R     2   33      2673       19      1323      31     153      49     1143
--R   [b  + -- b + -----, w + --- b + -----, p - -- b - ---, z + -- b + ----,
--R          50     10000      120     20000      18     200      36     2000
--R        37      27      5      9
--R    t - -- b + ---, s - - b - ---]
--R        15     250      2     200
--RType: List HomogeneousDistributedMultivariatePolynomial([w,p,z,t,s,b],Fraction
--E 23

--S 24 of 24
groebner(hsn7,"redcrit","info")
--R
--R
--R    reduced Critpair - Polynom :
--R
```

```
--R
--R       5     61     77      7
--R   z + - t - -- s + -- b + --
--R       8     45     24     10
--R
--R
--R
--R   you choose option  -info-
--R   abbrev. for the following information strings are
--R     ci  =>  Leading monomial  for critpair calculation
--R     tci =>  Number of terms of polynomial i
--R     cj  =>  Leading monomial  for critpair calculation
--R     tcj =>  Number of terms of polynomial j
--R     c   =>  Leading monomial of critpair polynomial
--R     tc  =>  Number of terms of critpair polynomial
--R     rc  =>  Leading monomial of redcritpair polynomial
--R     trc =>  Number of terms of redcritpair polynomial
--R     tF  =>  Number of polynomials in reduction list F
--R     tD  =>  Number of critpairs still to do
--R
--R
--R
--R
--R   [[ci= p,tci= 4,cj= p,tcj= 4,c= z,tc= 5,rc= z,trc= 5,tF= 4,tD= 5]]
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R    2   216     189     78      99     10557
--R   s  - --- w + --- t - -- s + --- b - -----
--R         5      100     25     500     12500
--R
--R
--R
--R                                                      2
--R   [[ci= p s,tci= 5,cj= p,tcj= 4,c= z s,tc= 7,rc= s ,trc= 6,tF= 5,tD= 5]]
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R         66       17541     5886     10588      9273     8272413
```

```
--R   t s - -- t b - ----- w + ---- t - ----- s - ----- b - -------
--R         29         725       3625       3625      36250      7250000
--R
--R
--R
--R   [[ci= p t,tci= 3,cj= p,tcj= 4,c= z t,tc= 5,rc= t s,trc= 7,tF= 6,tD= 6]]
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R      2   28       44       143       962712     420652     5166944
--R     t  + -- w s - -- w b + --- t b - ------ w + ------ t - ------- s
--R          45       15       725        18125      90625      815625
--R   +
--R     5036339     83580953
--R     ------- b - --------
--R     5437500     90625000
--R
--R
--R
--R          3             2                          2
--R   [[ci= b ,tci= 3,cj= b ,tcj= 3,c= w p,tc= 4,rc= t ,trc= 9,tF= 7,tD= 6]]
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R         33      297        81
--R   w b + -- w + ----- s - ----- b
--R         50     10000      10000
--R
--R
--R
--R                         2            3
--R   [[ci= s b,tci= 3,cj= b ,tcj= 3,c= b ,tc= 4,rc= w b,trc= 4,tF= 8,tD= 7]]
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R          21        33      6723      2031      104247
--R   w s + --- t b - --- w + ----- s - ----- b + -------
```

```
--R         100       250      50000      25000      5000000
--R
--R
--R
--R                          2              2
--R   [[ci= s b,tci= 3,cj= s ,tcj= 6,c= s b ,tc= 7,rc= w s,trc= 6,tF= 9,tD= 9]]
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R         2373       41563       17253       578853       258751       11330361
--R   w t + ---- t b - ----- w + ------ t + ------- s - ------- b + ---------
--R         7250       36250      290000      7250000      3625000      362500000
--R
--R
--R
--R                                         2
--R   [[ci= s b,tci= 3,cj= t s,tcj= 7,c= t b ,tc= 7,rc= w t,trc= 7,tF= 10,tD= 11]]
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R   0
--R
--R
--R
--R                                         2
--R   [[ci= p s,tci= 5,cj= s b,tcj= 3,c= p b ,tc= 6,rc= 0,trc= 0,tF= 10,tD= 10]]
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R         51061712      91248294      1516761889      481096937      5789482077
--R   t b - -------- w + --------- t - ---------- s + ---------- b + -----------
--R          5127061     128176525      1922647875     1281765250     51270610000
--R
--R
--R
--R          2
--R   [[ci= s ,tci= 6,cj= t s,tcj= 7,c= t s b,tc= 10,rc= t b,trc= 6,tF= 11,tD= 13]]
```

```
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R          2962071220563579     1229379913128787     4524811449715289
--R     w + ---------------- t - ---------------- s + ---------------- b
--R           98138188260880       36801820597830       490690941304400
--R   +
--R     59240140318722273
--R     -----------------
--R     12267273532610000
--R
--R
--R
--R          2
--R   [[ci= b ,tci= 3,cj= t b,tcj= 6,c= w b,tc= 6,rc= w,trc= 5,tF= 9,tD= 14]]
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R   0
--R
--R
--R
--R          2
--R   [[ci= b ,tci= 3,cj= w b,tcj= 4,c= s b,tc= 3,rc= 0,trc= 0,tF= 9,tD= 13]]
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R       172832706542351932      47302810289036749      2736061156820726
--R   t - ------------------ s + ------------------ b + ------------------
--R       155991468675747195     155991468675747195     17332385408416355
--R
--R
--R
--R                                          2
--R   [[ci= s b,tci= 3,cj= t b,tcj= 6,c= t b ,tc= 7,rc= t,trc= 4,tF= 7,tD= 11]]
--R
--R
```

```
--R
--R    reduced Critpair - Polynom :
--R
--R
--R       5      9
--R   s - - b - ---
--R       2     200
--R
--R
--R
--R                                          2
--R   [[ci= s b,tci= 3,cj= w b,tcj= 4,c= w b ,tc= 5,rc= s,trc= 3,tF= 6,tD= 9]]
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R   0
--R
--R
--R
--R                                        2
--R   [[ci= w b,tci= 4,cj= t b,tcj= 6,c= w ,tc= 7,rc= 0,trc= 0,tF= 6,tD= 8]]
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R   0
--R
--R
--R
--R                                      2
--R   [[ci= s b,tci= 3,cj= s,tcj= 3,c= b ,tc= 3,rc= 0,trc= 0,tF= 6,tD= 7]]
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R   0
--R
--R
--R
```

```
--R   [[ci= t b,tci= 6,cj= t,tcj= 4,c= s b,tc= 7,rc= 0,trc= 0,tF= 6,tD= 6]]
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R   0
--R
--R
--R
--R   [[ci= w b,tci= 4,cj= w,tcj= 5,c= t b,tc= 6,rc= 0,trc= 0,tF= 6,tD= 5]]
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R   0
--R
--R
--R
--R          2
--R   [[ci= s ,tci= 6,cj= s,tcj= 3,c= s b,tc= 6,rc= 0,trc= 0,tF= 6,tD= 4]]
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R   0
--R
--R
--R
--R                                      2
--R   [[ci= t s,tci= 7,cj= t,tcj= 4,c= s ,tc= 8,rc= 0,trc= 0,tF= 6,tD= 3]]
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R   0
--R
--R
--R
```

```
--R   [[ci= w s,tci= 6,cj= w,tcj= 5,c= t s,tc= 8,rc= 0,trc= 0,tF= 6,tD= 2]]
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R   0
--R
--R
--R
--R          2
--R   [[ci= t ,tci= 9,cj= t,tcj= 4,c= w s,tc= 9,rc= 0,trc= 0,tF= 6,tD= 1]]
--R
--R
--R
--R    reduced Critpair - Polynom :
--R
--R
--R   0
--R
--R
--R
--R                                      2
--R   [[ci= w t,tci= 7,cj= w,tcj= 5,c= t ,tc= 8,rc= 0,trc= 0,tF= 6,tD= 0]]
--R
--R
--R     There are
--R
--R   6
--R
--R     Groebner Basis Polynomials.
--R
--R
--R       THE GROEBNER BASIS POLYNOMIALS
--R
--R   (24)
--R     2   33      2673       19      1323      31     153      49     1143
--R   [b  + -- b + -----, w + --- b + -----, p - -- b - ---, z + -- b + ----,
--R         50     10000      120     20000      18     200      36     2000
--R        37      27      5      9
--R    t - -- b + ---, s - - b - ---]
--R        15     250      2     200
--RType: List HomogeneousDistributedMultivariatePolynomial([w,p,z,t,s,b],Fraction Integer)
--E 24
```

```
)spool
)lisp (bye)
```

⟨*GroebnerPackage.help*⟩≡

```
==========================================================================
groebner examples
==========================================================================

Example to call groebner:

  s1:DMP[w,p,z,t,s,b]RN:= 45*p + 35*s - 165*b - 36
  s2:DMP[w,p,z,t,s,b]RN:= 35*p + 40*z + 25*t - 27*s
  s3:DMP[w,p,z,t,s,b]RN:= 15*w + 25*p*s + 30*z - 18*t - 165*b**2
  s4:DMP[w,p,z,t,s,b]RN:= -9*w + 15*p*t + 20*z*s
  s5:DMP[w,p,z,t,s,b]RN:= w*p + 2*z*t - 11*b**3
  s6:DMP[w,p,z,t,s,b]RN:= 99*w - 11*b*s + 3*b**2
  s7:DMP[w,p,z,t,s,b]RN:= b**2 + 33/50*b + 2673/10000

  sn7:=[s1,s2,s3,s4,s5,s6,s7]

  groebner(sn7,info)

groebner calculates a minimal Groebner Basis
all reductions are TOTAL reductions

To get the reduced critical pairs do:

  groebner(sn7,"redcrit")

You can get other information by calling:

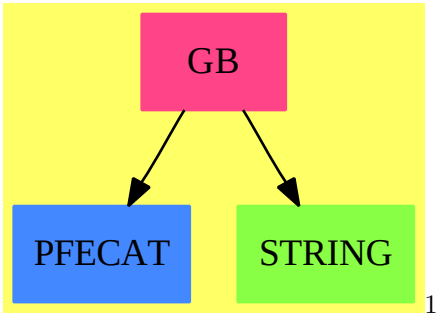
  groebner(sn7,"info")

which returns:
      ci  =>  Leading monomial  for critpair calculation
      tci =>  Number of terms of polynomial i
      cj  =>  Leading monomial  for critpair calculation
      tcj =>  Number of terms of polynomial j
      c   =>  Leading monomial of critpair polynomial
      tc  =>  Number of terms of critpair polynomial
      rc  =>  Leading monomial of redcritpair polynomial
      trc =>  Number of terms of redcritpair polynomial
      tF  =>  Number of polynomials in reduction list F
      tD  =>  Number of critpairs still to do

See Also:
o )display operations groebner
o )show GroebnerPackage
o )show DistributedMultivariatePolynomial
```

```
o )show HomogeneousDistributedMultivariatePolynomial
o )show EuclideanGroebnerBasisPackage
```

8.36 GroebnerPackage



1.00

Exports:
groebner normalForm

⟨*package GB GroebnerPackage*⟩≡
```
)abbrev package GB GroebnerPackage
++ Authors: Gebauer, Trager
++ Date Created: 12-1-86
++ Date Last Updated: 2-28-91
++ Basic Functions: groebner normalForm
++ Related Constructors: Ideal, IdealDecompositionPackage
++ Also See:
++ AMS Classifications:
++ Keywords: groebner basis, polynomial ideal
++ References:
++ Description: \spadtype{GroebnerPackage} computes groebner
++ bases for polynomial ideals. The basic computation provides
++ a distinguished set of generators for polynomial ideals over fields.
++ This basis allows an easy test for membership: the operation \spadfun{normalForm}
++ returns zero on ideal members. When the provided coefficient domain, Dom,
++ is not a field, the result is equivalent to considering the extended
++ ideal with \spadtype{Fraction(Dom)} as coefficients, but considerably more efficient
++ since all calculations are performed in Dom. Additional argument "info" and "redcrit"
++ can be given to provide incremental information during
++ computation. Argument "info" produces a computational summary for each s-polynomial.
++ Argument "redcrit" prints out the reduced critical pairs. The term ordering
++ is determined by the polynomial type used. Suggested types include
++ \spadtype{DistributedMultivariatePolynomial},
++ \spadtype{HomogeneousDistributedMultivariatePolynomial},
++ \spadtype{GeneralDistributedMultivariatePolynomial}.

GroebnerPackage(Dom, Expon, VarSet, Dpol): T == C where

 Dom:   GcdDomain
 Expon: OrderedAbelianMonoidSup
```

```
VarSet: OrderedSet
Dpol:  PolynomialCategory(Dom, Expon, VarSet)

T== with

 groebner: List(Dpol) -> List(Dpol)
  ++ groebner(lp) computes a groebner basis for a polynomial ideal
  ++ generated by the list of polynomials lp.
  ++
  ++X s1:DMP([w,p,z,t,s,b],FRAC(INT)):= 45*p + 35*s - 165*b - 36
  ++X s2:DMP([w,p,z,t,s,b],FRAC(INT)):= 35*p + 40*z + 25*t - 27*s
  ++X s3:DMP([w,p,z,t,s,b],FRAC(INT)):= 15*w + 25*p*s + 30*z - 18*t - 165*b**2
  ++X s4:DMP([w,p,z,t,s,b],FRAC(INT)):= -9*w + 15*p*t + 20*z*s
  ++X s5:DMP([w,p,z,t,s,b],FRAC(INT)):= w*p + 2*z*t - 11*b**3
  ++X s6:DMP([w,p,z,t,s,b],FRAC(INT)):= 99*w - 11*b*s + 3*b**2
  ++X s7:DMP([w,p,z,t,s,b],FRAC(INT)):= b**2 + 33/50*b + 2673/10000
  ++X sn7:=[s1,s2,s3,s4,s5,s6,s7]
  ++X groebner(sn7)

 groebner: ( List(Dpol), String ) -> List(Dpol)
  ++ groebner(lp, infoflag) computes a groebner basis
  ++ for a polynomial ideal
  ++ generated by the list of polynomials lp.
  ++ Argument infoflag is used to get information on the computation.
  ++ If infoflag is "info", then summary information
  ++ is displayed for each s-polynomial generated.
  ++ If infoflag is "redcrit", the reduced critical pairs are displayed.
  ++ If infoflag is any other string,
  ++ no information is printed during computation.
  ++
  ++X s1:DMP([w,p,z,t,s,b],FRAC(INT)):= 45*p + 35*s - 165*b - 36
  ++X s2:DMP([w,p,z,t,s,b],FRAC(INT)):= 35*p + 40*z + 25*t - 27*s
  ++X s3:DMP([w,p,z,t,s,b],FRAC(INT)):= 15*w + 25*p*s + 30*z - 18*t - 165*b**2
  ++X s4:DMP([w,p,z,t,s,b],FRAC(INT)):= -9*w + 15*p*t + 20*z*s
  ++X s5:DMP([w,p,z,t,s,b],FRAC(INT)):= w*p + 2*z*t - 11*b**3
  ++X s6:DMP([w,p,z,t,s,b],FRAC(INT)):= 99*w - 11*b*s + 3*b**2
  ++X s7:DMP([w,p,z,t,s,b],FRAC(INT)):= b**2 + 33/50*b + 2673/10000
  ++X sn7:=[s1,s2,s3,s4,s5,s6,s7]
  ++X groebner(sn7,"info")
  ++X groebner(sn7,"redcrit")

 groebner: ( List(Dpol), String, String ) -> List(Dpol)
  ++ groebner(lp, "info", "redcrit") computes a groebner basis
  ++ for a polynomial ideal generated by the list of polynomials lp,
  ++ displaying both a summary of the critical pairs considered ("info")
  ++ and the result of reducing each critical pair ("redcrit").
```

```
    ++ If the second or third arguments have any other string value,
    ++ the indicated information is suppressed.
    ++
    ++X s1:DMP([w,p,z,t,s,b],FRAC(INT)):= 45*p + 35*s - 165*b - 36
    ++X s2:DMP([w,p,z,t,s,b],FRAC(INT)):= 35*p + 40*z + 25*t - 27*s
    ++X s3:DMP([w,p,z,t,s,b],FRAC(INT)):= 15*w + 25*p*s + 30*z - 18*t - 165*b**2
    ++X s4:DMP([w,p,z,t,s,b],FRAC(INT)):= -9*w + 15*p*t + 20*z*s
    ++X s5:DMP([w,p,z,t,s,b],FRAC(INT)):= w*p + 2*z*t - 11*b**3
    ++X s6:DMP([w,p,z,t,s,b],FRAC(INT)):= 99*w - 11*b*s + 3*b**2
    ++X s7:DMP([w,p,z,t,s,b],FRAC(INT)):= b**2 + 33/50*b + 2673/10000
    ++X sn7:=[s1,s2,s3,s4,s5,s6,s7]
    ++X groebner(sn7,"info","redcrit")

   if Dom has Field then
     normalForm: (Dpol, List(Dpol))  -> Dpol
       ++ normalForm(poly,gb) reduces the polynomial poly modulo the
       ++ precomputed groebner basis gb giving a canonical representative
       ++ of the residue class.
  C== add
    import OutputForm
    import GroebnerInternalPackage(Dom,Expon,VarSet,Dpol)

    if Dom has Field then
      monicize(p: Dpol):Dpol ==
--       one?(lc := leadingCoefficient p) => p
       ((lc := leadingCoefficient p) = 1) => p
       inv(lc)*p

      normalForm(p : Dpol, l : List(Dpol)) : Dpol ==
        redPol(p,map(monicize,l))

    ------    MAIN ALGORITHM GROEBNER ------------------------

    groebner( Pol: List(Dpol) ) ==
      Pol=[] => Pol
      Pol:=[x for x in Pol | x ^= 0]
      Pol=[] => [0]
      minGbasis(sort((x,y) +->  degree x > degree y, gbasis(Pol,0,0)))

    groebner( Pol: List(Dpol), xx1: String) ==
      Pol=[] => Pol
      Pol:=[x for x in Pol | x ^= 0]
      Pol=[] => [0]
      xx1 = "redcrit" =>
        minGbasis(sort((x,y) +->  degree x > degree y, gbasis(Pol,1,0)))
      xx1 = "info" =>
```

```
      minGbasis(sort((x,y) +-> degree x > degree y, gbasis(Pol,2,1)))
    messagePrint("   ")
    messagePrint("WARNING: options are - redcrit and/or info - ")
    messagePrint("         you didn't type them correct")
    messagePrint("         please try again")
    messagePrint("   ")
    []

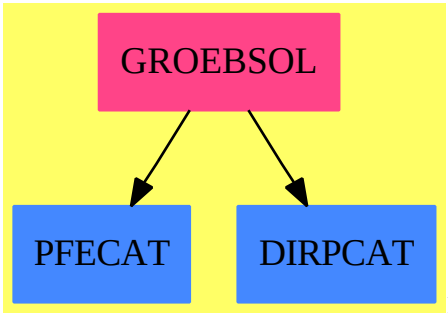
  groebner( Pol: List(Dpol), xx1: String, xx2: String) ==
    Pol=[] => Pol
    Pol:=[x for x in Pol | x ^= 0]
    Pol=[] => [0]
    (xx1 = "redcrit" and xx2 = "info") or
     (xx1 = "info" and xx2 = "redcrit")   =>
      minGbasis(sort((x,y) +-> degree x > degree y, gbasis(Pol,1,1)))
    xx1 = "redcrit" and xx2 = "redcrit" =>
      minGbasis(sort((x,y) +-> degree x > degree y, gbasis(Pol,1,0)))
    xx1 = "info" and xx2 = "info" =>
      minGbasis(sort((x,y) +-> degree x > degree y, gbasis(Pol,2,1)))
    messagePrint("   ")
    messagePrint("WARNING:  options are - redcrit and/or info - ")
    messagePrint("          you didn't type them correctly")
    messagePrint("          please try again ")
    messagePrint("   ")
    []
```

⟨*GB.dotabb*⟩≡

```
"GB" [color="#FF4488",href="bookvol10.4.pdf#nameddest=GB"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"STRING" [color="#88FF44",href="bookvol10.3.pdf#nameddest=STRING"]
"GB" -> "PFECAT"
"GB" -> "STRING"
```

8.37 package GROEBSOL GroebnerSolve

8.38 GroebnerSolve



1.00

Exports:
genericPosition groebSolve testDim

⟨*package GROEBSOL GroebnerSolve*⟩≡

```
)abbrev package GROEBSOL GroebnerSolve
++ Author : P.Gianni, Summer '88, revised November '89
++ Solve systems of polynomial equations using Groebner bases
++ Total order Groebner bases are computed and then converted to lex ones
++ This package is mostly intended for internal use.
GroebnerSolve(lv,F,R) : C == T

  where
   R       :   GcdDomain
   F       :   GcdDomain
   lv      :   List Symbol

   NNI     ==>  NonNegativeInteger
   I       ==>  Integer
   S       ==>  Symbol

   OV      ==>  OrderedVariableList(lv)
   IES     ==>  IndexedExponents Symbol

   DP      ==>  DirectProduct(#lv,NonNegativeInteger)
   DPoly   ==>  DistributedMultivariatePolynomial(lv,F)

   HDP     ==>  HomogeneousDirectProduct(#lv,NonNegativeInteger)
   HDPoly ==>  HomogeneousDistributedMultivariatePolynomial(lv,F)

   SUP     ==>  SparseUnivariatePolynomial(DPoly)
   L       ==>  List
```

```
P      ==>  Polynomial

C == with
   groebSolve   : (L DPoly,L OV)  -> L L DPoly
     ++ groebSolve(lp,lv) reduces the polynomial system lp in variables lv
     ++ to triangular form. Algorithm based on groebner bases algorithm
     ++ with linear algebra for change of ordering.
     ++ Preprocessing for the general solver.
     ++ The polynomials in input are of type \spadtype{DMP}.

   testDim     : (L HDPoly,L OV)  -> Union(L HDPoly,"failed")
     ++ testDim(lp,lv) tests if the polynomial system lp
     ++ in variables lv is zero dimensional.

   genericPosition : (L DPoly, L OV) -> Record(dpolys:L DPoly, coords: L I)
     ++ genericPosition(lp,lv) puts a radical zero dimensional ideal
     ++ in general position, for system lp in variables lv.

T == add
  import PolToPol(lv,F)
  import GroebnerPackage(F,DP,OV,DPoly)
  import GroebnerInternalPackage(F,DP,OV,DPoly)
  import GroebnerPackage(F,HDP,OV,HDPoly)
  import LinGroebnerPackage(lv,F)

  nv:NNI:=#lv

       ---- test if f is power of a linear mod (rad lpol) ----
                  ----  f is monic  ----
  testPower(uf:SUP,x:OV,lpol:L DPoly) : Union(DPoly,"failed") ==
    df:=degree(uf)
    trailp:DPoly := coefficient(uf,(df-1)::NNI)
    (testquo := trailp exquo (df::F)) case "failed" => "failed"
    trailp := testquo::DPoly
    gg:=gcd(lc:=leadingCoefficient(uf),trailp)
    trailp := (trailp exquo gg)::DPoly
    lc := (lc exquo gg)::DPoly
    linp:SUP:=monomial(lc,1$NNI)$SUP + monomial(trailp,0$NNI)$SUP
    g:DPoly:=multivariate(uf-linp**df,x)
    redPol(g,lpol) ^= 0 => "failed"
    multivariate(linp,x)

         -- is the 0-dimensional ideal I in general position ?  --
                  ----  internal function  ----
  testGenPos(lpol:L DPoly,lvar:L OV):Union(L DPoly,"failed") ==
    rlpol:=reverse lpol
```

```
  f:=rlpol.first
  #lvar=1 => [f]
  rlvar:=rest reverse lvar
  newlpol:List(DPoly):=[f]
  for f in rlpol.rest repeat
    x:=first rlvar
    fi:= univariate(f,x)
    if (mainVariable leadingCoefficient fi case "failed") then
      if ((g:= testPower(fi,x,newlpol)) case "failed")
      then return "failed"
      newlpol :=concat(redPol(g::DPoly,newlpol),newlpol)
      rlvar:=rest rlvar
    else if redPol(f,newlpol)^=0 then return"failed"
  newlpol


   -- change coordinates and out the ideal in general position  ----
genPos(lp:L DPoly,lvar:L OV): Record(polys:L HDPoly, lpolys:L DPoly,
                                     coord:L I, univp:HDPoly) ==
      rlvar:=reverse lvar
      lnp:=[dmpToHdmp(f) for f in lp]
      x := first rlvar;rlvar:=rest rlvar
      testfail:=true
      for count in 1.. while testfail repeat
        ranvals:L I:=[1+(random()$I rem (count*(# lvar))) for vv in rlvar]
        val:=+/[rv*(vv::HDPoly)
                   for vv in rlvar for rv in ranvals]
        val:=val+x::HDPoly
        gb:L HDPoly:= [elt(univariate(p,x),val) for p in lnp]
        gb:=groebner gb
        gbt:=totolex gb
        (gb1:=testGenPos(gbt,lvar)) case "failed"=>"try again"
        testfail:=false
      [gb,gbt,ranvals,dmpToHdmp(last (gb1::L DPoly))]

genericPosition(lp:L DPoly,lvar:L OV) ==
   nans:=genPos(lp,lvar)
   [nans.lpolys, nans.coord]

   ---- select  the univariate factors
select(lup:L L HDPoly) : L L HDPoly ==
  lup=[] => list []
  [:[cons(f,lsel) for lsel in select lup.rest] for f in lup.first]

   ---- in the non generic case, we compute the prime ideals ----
      ---- associated to leq, basis is the algebra basis  ----
```

```
findCompon(leq:L HDPoly,lvar:L OV):L L DPoly ==
  teq:=totolex(leq)
  #teq = #lvar => [teq]
 -- ^((teq1:=testGenPos(teq,lvar)) case "failed") => [teq1::L DPoly]
  gp:=genPos(teq,lvar)
  lgp:= gp.polys
  g:HDPoly:=gp.univp
  fg:=(factor g)$GeneralizedMultivariateFactorize(OV,HDP,R,F,HDPoly)
  lfact:=[ff.factor for ff in factors(fg::Factored(HDPoly))]
  result: L L HDPoly := []
  #lfact=1 => [teq]
  for tfact in lfact repeat
    tlfact:=concat(tfact,lgp)
    result:=concat(tlfact,result)
  ranvals:L I:=gp.coord
  rlvar:=reverse lvar
  x:=first rlvar
  rlvar:=rest rlvar
  val:=+/[rv*(vv::HDPoly) for vv in rlvar for rv in ranvals]
  val:=(x::HDPoly)-val
  ans:=[totolex groebner [elt(univariate(p,x),val) for p in lp]
                      for lp in result]
  [ll for ll in ans | ll^=[1]]

zeroDim?(lp: List HDPoly,lvar:L OV) : Boolean ==
  empty? lp => false
  n:NNI := #lvar
  #lp < n => false
  lvint1 := lvar
  for f in lp while not empty?(lvint1) repeat
     g:= f - reductum f
     x:=mainVariable(g)::OV
     if ground?(leadingCoefficient(univariate(g,x))) then
          lvint1 := remove(x, lvint1)
  empty? lvint1

-- general solve, gives an error if the system not 0-dimensional
groebSolve(leq: L DPoly,lvar:L OV) : L L DPoly ==
  lnp:=[dmpToHdmp(f) for f in leq]
  leq1:=groebner lnp
  #(leq1) = 1 and first(leq1) = 1 => list empty()
  ^(zeroDim?(leq1,lvar)) =>
    error "system does not have a finite number of solutions"
  -- add computation of dimension, for a more useful error
  basis:=computeBasis(leq1)
  lup:L HDPoly:=[]
```

```
      llfact:L Factored(HDPoly):=[]
      for x in lvar repeat
        g:=minPol(leq1,basis,x)
        fg:=(factor g)$GeneralizedMultivariateFactorize(OV,HDP,R,F,HDPoly)
        llfact:=concat(fg::Factored(HDPoly),llfact)
        if degree(g,x) = #basis then leave "stop factoring"
      result: L L DPoly := []
      -- selecting a factor from the lists of the univariate factors
      lfact:=select [[ff.factor for ff in factors llf]
                     for llf in llfact]
      for tfact in lfact repeat
        tfact:=groebner concat(tfact,leq1)
        tfact=[1] => "next value"
        result:=concat(result,findCompon(tfact,lvar))
      result

    -- test if the system is zero dimensional
    testDim(leq : L HDPoly,lvar : L OV) : Union(L HDPoly,"failed") ==
      leq1:=groebner leq
      #(leq1) = 1 and first(leq1) = 1 => empty()
      ^(zeroDim?(leq1,lvar)) => "failed"
      leq1
```

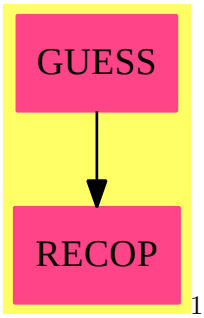
⟨*GROEBSOL.dotabb*⟩≡

```
"GROEBSOL" [color="#FF4488",href="bookvol10.4.pdf#nameddest=GROEBSOL"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"DIRPCAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=DIRPCAT"]
"GROEBSOL" -> "PFECAT"
"GROEBSOL" -> "DIRPCAT"
```

8.39 package GUESS Guess

The packages defined in this file enable Axiom to guess formulas for sequences of, for example, rational numbers or rational functions, given the first few terms. It extends and complements Christian Krattenthaler's program Rate and the relevant parts of Bruno Salvy and Paul Zimmermann's GFUN.

8.40 Guess



1.00

Exports:

diffHP	guess	guessADE	guessAlg	guessBinRat
guessExpRat	guessHP	guessHolo	guessPRec	guessPade
guessRat	guessRec	shiftHP		

⟨*package GUESS Guess*⟩≡

```
)abbrev package GUESS Guess
++ Author: Martin Rubey
++ Description: This package implements guessing of sequences. Packages for the
++ most common cases are provided as \spadtype{GuessInteger},
++ \spadtype{GuessPolynomial}, etc.
Guess(F, S, EXPRR, R, retract, coerce): Exports == Implementation where
    F: Field                                  -- zB.: FRAC POLY PF 5
-- in F we interpolate und check

    S: GcdDomain

-- in guessExpRat I would like to determine the roots of polynomials in F. When
-- F is a quotientfield, I can get rid of the denominator.  In this case F is
-- roughly QFCAT S

    R: Join(OrderedSet, IntegralDomain)       -- zB.: FRAC POLY INT

-- results are given as elements of EXPRR
--    EXPRR: Join(ExpressionSpace, IntegralDomain,
    EXPRR: Join(FunctionSpace Integer, IntegralDomain,
```

```
                RetractableTo R, RetractableTo Symbol,
                RetractableTo Integer, CombinatorialOpsCategory,
                PartialDifferentialRing Symbol) with
              _* : (%,%) -> %
              _/ : (%,%) -> %
              _*_* : (%,%) -> %
              numerator : % -> %
              denominator : % -> %
              ground? : % -> Boolean

                                                -- zB.: EXPR INT
  -- EXPR FRAC POLY INT is forbidden. Thus i cannot just use EXPR R

  -- EXPRR exists, in case at some point there is support for EXPR PF 5.


  -- the following I really would like to get rid of

      retract: R -> F                           -- zB.: i+->i
      coerce: F -> EXPRR                        -- zB.: i+->i
  -- attention: EXPRR ~= EXPR R

      LGOPT ==> List GuessOption
      GOPT0 ==> GuessOptionFunctions0

      NNI ==> NonNegativeInteger
      PI ==> PositiveInteger
      EXPRI ==> Expression Integer
      GUESSRESULT ==> List Record(function: EXPRR, order: NNI)

      UFPSF ==> UnivariateFormalPowerSeries F
      UFPS1 ==> UnivariateFormalPowerSeriesFunctions

      UFPSS ==> UnivariateFormalPowerSeries S

      SUP ==> SparseUnivariatePolynomial

      UFPSSUPF ==> UnivariateFormalPowerSeries SUP F

      FFFG ==> FractionFreeFastGaussian
      FFFGF ==> FractionFreeFastGaussianFractions

  -- CoeffAction
      DIFFSPECA ==> (NNI, NNI, SUP S) -> S

      DIFFSPECAF ==> (NNI, NNI, UFPSSUPF) -> SUP F
```

```
    DIFFSPECAX ==> (NNI, Symbol, EXPRR) -> EXPRR

  -- the diagonal of the C-matrix
    DIFFSPECC ==> NNI -> List S


    HPSPEC ==> Record(guessStream:  UFPSF -> Stream UFPSF,
                      degreeStream: Stream NNI,
                      testStream:   UFPSSUPF -> Stream UFPSSUPF,
                      exprStream:   (EXPRR, Symbol) -> Stream EXPRR,
                      A:  DIFFSPECA,
                      AF: DIFFSPECAF,
                      AX: DIFFSPECAX,
                      C:  DIFFSPECC)

  -- note that empty?(guessStream.o) has to return always. In other words, if the
  -- stream is finite, empty? should recognize it.

    DIFFSPECN ==> EXPRR -> EXPRR             -- eg.: i+->q^i

    GUESSER ==> (List F, LGOPT) -> GUESSRESULT

    FSUPS ==> Fraction SUP S
    FSUPF ==> Fraction SUP F

    V ==> OrderedVariableList(['a1, 'A])
    POLYF ==> SparseMultivariatePolynomial(F, V)
    FPOLYF ==> Fraction POLYF
    FSUPFPOLYF ==> Fraction SUP FPOLYF
    POLYS ==> SparseMultivariatePolynomial(S, V)
    FPOLYS ==> Fraction POLYS
    FSUPFPOLYS ==> Fraction SUP FPOLYS
```

The following should be commented out when not debugging, see also Section 8.40.4.

⟨*package GUESS Guess*⟩+≡

```
    --<<implementation: Guess - Hermite-Pade - Types for Operators>>
    -- EXT ==> (Integer, V, V) -> FPOLYS
    -- EXTEXPR ==> (Symbol, F, F) -> EXPRR
```

⟨*package GUESS Guess*⟩+≡

```
Exports == with

    guess: List F -> GUESSRESULT
      ++ \spad{guess l} applies recursively \spadfun{guessRec} and
      ++ \spadfun{guessADE} to the successive differences and quotients of
      ++ the list. Default options as described in
      ++ \spadtype{GuessOptionFunctions0} are used.

    guess: (List F, LGOPT) -> GUESSRESULT
      ++ \spad{guess(l, options)} applies recursively \spadfun{guessRec}
      ++ and \spadfun{guessADE} to the successive differences and quotients
      ++ of the list. The given options are used.

    guess: (List F, List GUESSER, List Symbol) -> GUESSRESULT
      ++ \spad{guess(l, guessers, ops)} applies recursively the given
      ++ guessers to the successive differences if ops contains the symbol
      ++ guessSum and quotients if ops contains the symbol guessProduct to
      ++ the list. Default options as described in
      ++ \spadtype{GuessOptionFunctions0} are used.

    guess: (List F, List GUESSER, List Symbol, LGOPT) -> GUESSRESULT
      ++ \spad{guess(l, guessers, ops)} applies recursively the given
      ++ guessers to the successive differences if ops contains the symbol
      ++ \spad{guessSum} and quotients if ops contains the symbol
      ++ \spad{guessProduct} to the list. The given options are used.

    guessExpRat: List F -> GUESSRESULT
      ++ \spad{guessExpRat l} tries to find a function of the form
      ++ n+->(a+b n)^n r(n), where r(n) is a rational function, that fits
      ++ l.

    guessExpRat: (List F, LGOPT) -> GUESSRESULT
      ++ \spad{guessExpRat(l, options)} tries to find a function of the
      ++ form n+->(a+b n)^n r(n), where r(n) is a rational function, that
      ++ fits l.

    guessBinRat: List F -> GUESSRESULT
      ++ \spad{guessBinRat(l, options)} tries to find a function of the
      ++ form n+->binomial(a+b n, n) r(n), where r(n) is a rational
      ++ function, that fits l.

    guessBinRat: (List F, LGOPT) -> GUESSRESULT
      ++ \spad{guessBinRat(l, options)} tries to find a function of the
      ++ form n+->binomial(a+b n, n) r(n), where r(n) is a rational
      ++ function, that fits l.
```

```
if F has RetractableTo Symbol and S has RetractableTo Symbol then

    guessExpRat: Symbol -> GUESSER
      ++ \spad{guessExpRat q} returns a guesser that tries to find a
      ++ function of the form n+->(a+b q^n)^n r(q^n), where r(q^n) is a
      ++ q-rational function, that fits l.

    guessBinRat: Symbol -> GUESSER
      ++ \spad{guessBinRat q} returns a guesser that tries to find a
      ++ function of the form n+->qbinomial(a+b n, n) r(n), where r(q^n)
      ++ q-rational function, that fits l.

guessHP: (LGOPT -> HPSPEC) -> GUESSER
  ++ \spad{guessHP f} constructs an operation that applies Hermite-Pade
  ++ approximation to the series generated by the given function f.

guessADE: List F -> GUESSRESULT
  ++ \spad{guessADE l} tries to find an algebraic differential equation
  ++ for a generating function whose first Taylor coefficients are
  ++ given by l, using the default options described in
  ++ \spadtype{GuessOptionFunctions0}.

guessADE: (List F, LGOPT) -> GUESSRESULT
  ++ \spad{guessADE(l, options)} tries to find an algebraic
  ++ differential equation for a generating function whose first Taylor
  ++ coefficients are given by l, using the given options.

guessAlg: List F -> GUESSRESULT
  ++ \spad{guessAlg l} tries to find an algebraic equation for a
  ++ generating function whose first Taylor coefficients are given by
  ++ l, using the default options described in
  ++ \spadtype{GuessOptionFunctions0}. It is equivalent to
  ++ \spadfun{guessADE}(l, maxDerivative == 0).

guessAlg: (List F, LGOPT) -> GUESSRESULT
  ++ \spad{guessAlg(l, options)} tries to find an algebraic equation
  ++ for a generating function whose first Taylor coefficients are
  ++ given by l, using the given options. It is equivalent to
  ++ \spadfun{guessADE}(l, options) with \spad{maxDerivative == 0}.

guessHolo: List F -> GUESSRESULT
  ++ \spad{guessHolo l} tries to find an ordinary linear differential
  ++ equation for a generating function whose first Taylor coefficients
  ++ are given by l, using the default options described in
  ++ \spadtype{GuessOptionFunctions0}. It is equivalent to
```

```
    ++ \spadfun{guessADE}\spad{(l, maxPower == 1)}.

  guessHolo: (List F, LGOPT) -> GUESSRESULT
    ++ \spad{guessHolo(l, options)} tries to find an ordinary linear
    ++ differential equation for a generating function whose first Taylor
    ++ coefficients are given by l, using the given options. It is
    ++ equivalent to \spadfun{guessADE}\spad{(l, options)} with
    ++ \spad{maxPower == 1}.

  guessPade: (List F, LGOPT) -> GUESSRESULT
    ++ \spad{guessPade(l, options)} tries to find a rational function
    ++ whose first Taylor coefficients are given by l, using the given
    ++ options. It is equivalent to \spadfun{guessADE}\spad{(l,
    ++ maxDerivative == 0, maxPower == 1, allDegrees == true)}.

  guessPade: List F -> GUESSRESULT
    ++ \spad{guessPade(l, options)} tries to find a rational function
    ++ whose first Taylor coefficients are given by l, using the default
    ++ options described in \spadtype{GuessOptionFunctions0}. It is
    ++ equivalent to \spadfun{guessADE}\spad{(l, options)} with
    ++ \spad{maxDerivative == 0, maxPower == 1, allDegrees == true}.

  guessRec: List F -> GUESSRESULT
    ++ \spad{guessRec l} tries to find an ordinary difference equation
    ++ whose first values are given by l, using the default options
    ++ described in \spadtype{GuessOptionFunctions0}.

  guessRec: (List F, LGOPT) -> GUESSRESULT
    ++ \spad{guessRec(l, options)} tries to find an ordinary difference
    ++ equation whose first values are given by l, using the given
    ++ options.

  guessPRec: (List F, LGOPT) -> GUESSRESULT
    ++ \spad{guessPRec(l, options)} tries to find a linear recurrence
    ++ with polynomial coefficients whose first values are given by l,
    ++ using the given options. It is equivalent to
    ++ \spadfun{guessRec}\spad{(l, options)} with \spad{maxPower == 1}.

  guessPRec: List F -> GUESSRESULT
    ++ \spad{guessPRec l} tries to find a linear recurrence with
    ++ polynomial coefficients whose first values are given by l, using
    ++ the default options described in
    ++ \spadtype{GuessOptionFunctions0}. It is equivalent to
    ++ \spadfun{guessRec}\spad{(l, maxPower == 1)}.

  guessRat: (List F, LGOPT) -> GUESSRESULT
```

```
    ++ \spad{guessRat(l, options)} tries to find a rational function
    ++ whose first values are given by l, using the given options. It is
    ++ equivalent to \spadfun{guessRec}\spad{(l, maxShift == 0, maxPower
    ++ == 1, allDegrees == true)}.

  guessRat: List F -> GUESSRESULT
    ++ \spad{guessRat l} tries to find a rational function whose first
    ++ values are given by l, using the default options described in
    ++ \spadtype{GuessOptionFunctions0}. It is equivalent to
    ++ \spadfun{guessRec}\spad{(l, maxShift == 0, maxPower == 1,
    ++ allDegrees == true)}.

  diffHP: LGOPT -> HPSPEC
    ++ \spad{diffHP options} returns a specification for Hermite-Pade
    ++ approximation with the differential operator

  shiftHP: LGOPT -> HPSPEC
    ++ \spad{shiftHP options} returns a specification for Hermite-Pade
    ++ approximation with the shift operator

  if F has RetractableTo Symbol and S has RetractableTo Symbol then

      shiftHP: Symbol -> (LGOPT -> HPSPEC)
        ++ \spad{shiftHP options} returns a specification for
        ++ Hermite-Pade approximation with the $q$-shift operator

      diffHP: Symbol -> (LGOPT -> HPSPEC)
        ++ \spad{diffHP options} returns a specification for Hermite-Pade
        ++ approximation with the  $q$-dilation operator

      guessRec: Symbol -> GUESSER
        ++ \spad{guessRec q} returns a guesser that finds an ordinary
        ++ q-difference equation whose first values are given by l, using
        ++ the given options.

      guessPRec: Symbol -> GUESSER
        ++ \spad{guessPRec q} returns a guesser that tries to find
        ++ a linear q-recurrence with polynomial coefficients whose first
        ++ values are given by l, using the given options. It is
        ++ equivalent to \spadfun{guessRec}\spad{(q)} with
        ++ \spad{maxPower == 1}.

      guessRat: Symbol -> GUESSER
        ++ \spad{guessRat q} returns a guesser that tries to find a
        ++ q-rational function whose first values are given by l, using
        ++ the given options. It is equivalent to \spadfun{guessRec} with
```

```
              ++ \spad{(l, maxShift == 0, maxPower == 1, allDegrees == true)}.

            guessADE: Symbol -> GUESSER
              ++ \spad{guessADE q} returns a guesser that tries to find an
              ++ algebraic differential equation for a generating function whose
              ++ first Taylor coefficients are given by l, using the given
              ++ options.

        --<<debug exports: Guess>>
```

⟨*debug exports: Guess*⟩≡

```
  --termAsUFPSF: (UFPSF, List Integer, DIFFSPECS, DIFFSPEC1) -> UFPSF
  --termAsUFPSF2: (UFPSF, List Integer, DIFFSPECS, DIFFSPEC1) -> UFPSF
  --termAsEXPRR: (EXPRR, Symbol, List Integer, DIFFSPECX, DIFFSPEC1X) -> EXPRR
  --termAsUFPSSUPF: (UFPSSUPF, List Integer, DIFFSPECSF, DIFFSPEC1F) -> UFPSSUPF
  --termAsUFPSSUPF2: (UFPSSUPF, List Integer, DIFFSPECSF, DIFFSPEC1F) -> UFPSSUPF
  --
  --ShiftSXGF: (EXPRR, Symbol, NNI) -> EXPRR
  --ShiftAXGF: (NNI, Symbol, EXPRR) -> EXPRR
  --
  --FilteredPartitionStream: LGOPT -> Stream List Integer
  --
  --guessInterpolate: (List SUP F, List NNI, HPSPEC) -> Matrix SUP S
  testInterpolant: (List SUP S, List F, List UFPSSUPF, List EXPRR, List EXPRR, _
                    NNI, HPSPEC, Symbol, BasicOperator, LGOPT) _
                    -> Union("failed", Record(function: EXPRR, order: NNI))

  --checkResult: (EXPRR, Symbol, Integer, List F, LGOPT) -> NNI
  --
  --guessBinRatAux: (Symbol, List F, DIFFSPECN, EXT, EXTEXPR,
  --                  List Integer, LGOPT) -> List EXPRR
  --guessBinRatAux0: (List F, DIFFSPECN, EXT, EXTEXPR,
  --                   LGOPT) -> GUESSRESULT
  --binExt: EXT
  --binExtEXPR: EXTEXPR
  --defaultD: DIFFSPECN
```

⟨*package GUESS Guess*⟩+≡

```
      Implementation == add
```

⟨*implementation: Guess*⟩

⟨*implementation: Guess*⟩≡

```
-- We have to put this chunk at the beginning, because otherwise it will take
-- very long to compile.
```

⟨*implementation: Guess - guessExpRat - Order and Degree*⟩

⟨*implementation: Guess - Utilities*⟩
⟨*implementation: Guess - guessExpRat*⟩
⟨*implementation: Guess - guessBinRat*⟩
⟨*implementation: Guess - Hermite-Pade*⟩
⟨*implementation: Guess - guess*⟩

8.40.1 general utilities

⟨*implementation: Guess - Utilities*⟩≡

```
checkResult(res: EXPRR, n: Symbol, l: Integer, list: List F,
            options: LGOPT): NNI ==
    for i in l..1 by -1 repeat
        den := eval(denominator res, n::EXPRR, (i-1)::EXPRR)
        if den = 0 then return i::NNI
        num := eval(numerator res, n::EXPRR, (i-1)::EXPRR)
        if list.i ~= retract(retract(num/den)@R)
        then return i::NNI
    0$NNI

SUPS2SUPF(p: SUP S): SUP F ==
  if F is S then
    p pretend SUP(F)
  else if F is Fraction S then
    map(coerce(#1)$Fraction(S), p)
      $SparseUnivariatePolynomialFunctions2(S, F)
  else error "Type parameter F should be either equal to S or equal _
              to Fraction S"
```

8.40.2 guessing rational functions with an exponential term

⟨*implementation: Guess - guessExpRat*⟩≡
⟨*implementation: Guess - guessExpRat - Utilities*⟩
⟨*implementation: Guess - guessExpRat - Main*⟩

Utilities

conversion routines

⟨*implementation: Guess - guessExpRat - Utilities*⟩≡

```
F2FPOLYS(p: F): FPOLYS ==
  if F is S then
    p::POLYF::FPOLYF pretend FPOLYS
  else if F is Fraction S then
    numer(p)$Fraction(S)::POLYS/denom(p)$Fraction(S)::POLYS
  else error "Type parameter F should be either equal to S or equal _
              to Fraction S"

MPCSF ==> MPolyCatFunctions2(V, IndexedExponents V,
                                IndexedExponents V, S, F,
                                POLYS, POLYF)

SUPF2EXPRR(xx: Symbol, p: SUP F): EXPRR ==
  zero? p => 0
  (coerce(leadingCoefficient p))::EXPRR * (xx::EXPRR)**degree p
     + SUPF2EXPRR(xx, reductum p)

FSUPF2EXPRR(xx: Symbol, p: FSUPF): EXPRR ==
  (SUPF2EXPRR(xx, numer p)) / (SUPF2EXPRR(xx, denom p))


POLYS2POLYF(p: POLYS): POLYF ==
  if F is S then
    p pretend POLYF
  else if F is Fraction S then
    map(coerce(#1)$Fraction(S), p)$MPCSF
  else error "Type parameter F should be either equal to S or equal _
              to Fraction S"

SUPPOLYS2SUPF(p: SUP POLYS, a1v: F, Av: F): SUP F ==
  zero? p => 0
  lc: POLYF := POLYS2POLYF leadingCoefficient(p)
  monomial(retract(eval(lc, [index(1)$V, index(2)$V]::List V,
                            [a1v, Av])),
           degree p)
    + SUPPOLYS2SUPF(reductum p, a1v, Av)


SUPFPOLYS2FSUPPOLYS(p: SUP FPOLYS): Fraction SUP POLYS  ==
  cden := splitDenominator(p)
         $UnivariatePolynomialCommonDenominator(POLYS, FPOLYS,SUP FPOLYS)
```

```
    pnum: SUP POLYS
         := map(retract(#1 * cden.den)$FPOLYS, p)
               $SparseUnivariatePolynomialFunctions2(FPOLYS, POLYS)
    pden: SUP POLYS := (cden.den)::SUP POLYS

    pnum/pden


  POLYF2EXPRR(p: POLYF): EXPRR ==
    map(convert(#1)@Symbol::EXPRR, coerce(#1)@EXPRR, p)
       $PolynomialCategoryLifting(IndexedExponents V, V,
                                  F, POLYF, EXPRR)

  -- this needs documentation. In particular, why is V appearing here?
  GF ==> GeneralizedMultivariateFactorize(SingletonAsOrderedSet,
                                          IndexedExponents V, F, F,
                                          SUP F)

  -- does not work:
  --                                                          6
  --   WARNING (genufact): No known algorithm to factor ? , trying square-free.

  -- GF ==> GenUFactorize F
```

mimicking q-analoga

⟨*implementation: Guess - guessExpRat - Utilities*⟩+≡

```
  defaultD: DIFFSPECN
  defaultD(expr: EXPRR): EXPRR == expr

  -- applies n+->q^n or whatever DN is to i
  DN2DL: (DIFFSPECN, Integer) -> F
  DN2DL(DN, i) == retract(retract(DN(i::EXPRR))@R)
```

⟨*implementation: Guess - guessExpRat - evalResultant*⟩

⟨*implementation: Guess - guessExpRat - substitute*⟩

reducing the degree of the polynomials

The degree of `poly3` is governed by $(a_0 + x_m a_1)^{x_m}$. Therefore, we substitute $A - x_m a1$ for a_0, which reduces the degree in a_1 by $x_m - x_{i+1}$.

⟨*implementation: Guess - guessExpRat - substitute*⟩≡

```
p(xm: Integer, i: Integer, va1: V, vA: V, basis: DIFFSPECN): FPOLYS ==
    vA::POLYS::FPOLYS + va1::POLYS::FPOLYS _
                      * F2FPOLYS(DN2DL(basis, i) - DN2DL(basis, xm))

p2(xm: Integer, i: Symbol, a1v: F, Av: F, basis: DIFFSPECN): EXPRR ==
    coerce(Av) + coerce(a1v)*(basis(i::EXPRR) - basis(xm::EXPRR))
```

⟨*not interesting implementation: Guess - guessExpRat - substitute*⟩≡

```
p(xm: Integer, i: Integer, va1: V, vA: V, basis: DIFFSPECN): FPOLYS ==
    vA::POLYS::FPOLYS + (i-xm)*va1::POLYS::FPOLYS

p2(xm: Integer, i: Symbol, a1v: F, Av: F, basis: DIFFSPECN): EXPRR ==
    coerce(Av) + coerce(a1v)*(i::EXPRR - xm::EXPRR)
```

Order and Degree

The following expressions for order and degree of the resultants `res1` and `res2` in `guessExpRatAux` were first guessed – partially with the aid of `guessRat`, and then proven to be correct.

⟨*implementation: Guess - guessExpRat - Order and Degree*⟩≡

```
ord1(x: List Integer, i: Integer): Integer ==
    n := #x - 3 - i
    x.(n+1)*reduce(_+, [x.j for j in 1..n], 0) + _
        2*reduce(_+, [reduce(_+, [x.k*x.j for k in 1..j-1], 0) _
                      for j in 1..n], 0)

ord2(x: List Integer, i: Integer): Integer ==
    if zero? i then
        n := #x - 3 - i
        ord1(x, i) + reduce(_+, [x.j for j in 1..n], 0)*(x.(n+2)-x.(n+1))
    else
        ord1(x, i)

deg1(x: List Integer, i: Integer): Integer ==
    m := #x - 3
    (x.(m+3)+x.(m+1)+x.(1+i))*reduce(_+, [x.j for j in 2+i..m], 0) + _
        x.(m+3)*x.(m+1) + _
        2*reduce(_+, [reduce(_+, [x.k*x.j for k in 2+i..j-1], 0) _
                      for j in 2+i..m], 0)

deg2(x: List Integer, i: Integer): Integer ==
    m := #x - 3
    deg1(x, i) + _
        (x.(m+3) + reduce(_+, [x.j for j in 2+i..m], 0)) * _
        (x.(m+2)-x.(m+1))
```

`evalResultant` evaluates the resultant of `p1` and `p2` at `d-o+1` points, so that we can recover it by interpolation.

⟨*implementation: Guess - guessExpRat - evalResultant*⟩≡

```
evalResultant(p1: POLYS, p2: POLYS, o: Integer, d: Integer, va1: V, vA: V)_
: List S ==
    res: List S := []
    d1 := degree(p1, va1)
    d2 := degree(p2, va1)
    lead: S
    for k in 1..d-o+1 repeat
        p1atk := univariate eval(p1, vA, k::S)
        p2atk := univariate eval(p2, vA, k::S)
```

It may happen, that the leading coefficients of one or both of the polynomials changes, when we evaluate it at k. In this case, we need to correct this by multiplying with the corresponding power of the leading coefficient of the other polynomial.

Consider the Sylvester matrix of the original polynomials. We want to evaluate it at $A = k$. If the first few leading coefficients of $p2$ vanish, the first few columns of the Sylvester matrix have triangular shape, with the leading coefficient of $p1$ on the diagonal. The same thing happens, if we exchange the roles of $p1$ and $p2$, only that we have to take care of the sign, too.

⟨*implementation: Guess - guessExpRat - evalResultant*⟩+≡

```
        d1atk := degree p1atk
        d2atk := degree p2atk

--      output("k: " string(k))$OutputPackage
--      output("d1: " string(d1) " d1atk: " string(d1atk))$OutputPackage
--      output("d2: " string(d2) " d2atk: " string(d2atk))$OutputPackage


        if d2atk < d2 then
            if  d1atk < d1
            then lead := 0$S
            else lead := (leadingCoefficient p1atk)**((d2-d2atk)::NNI)
        else
            if  d1atk < d1
            then lead := (-1$S)**d2 * (leadingCoefficient p2atk)**((d1-d1atk)::NN
            else lead := 1$S

        if zero? lead
        then res := cons(0, res)
        else res := cons(lead * (resultant(p1atk, p2atk)$SUP(S) exquo _
                                 (k::S)**(o::NNI))::S,
                         res)
```

Since we also have an lower bound for the order of the resultant, we need to evaluate it only at $d - o + 1$ points. Furthermore, we can divide by k^o and still obtain a polynomial.

⟨*implementation: Guess - guessExpRat - evalResultant*⟩+≡

```
      reverse res
```

The main routine

⟨*implementation: Guess - guessExpRat - Main*⟩≡

```
guessExpRatAux(xx: Symbol, list: List F, basis: DIFFSPECN,
               xValues: List Integer, options: LGOPT): List EXPRR ==

    a1: V := index(1)$V
    A: V := index(2)$V

    len: NNI := #list
    if len < 4 then return []
               else len := (len-3)::NNI

    xlist := [F2FPOLYS DN2DL(basis, xValues.i) for i in 1..len]
    x1 := F2FPOLYS DN2DL(basis, xValues.(len+1))
    x2 := F2FPOLYS DN2DL(basis, xValues.(len+2))
    x3 := F2FPOLYS DN2DL(basis, xValues.(len+3))
```

We try to fit the data $(s1, s2, \ldots)$ to the model $(a+bn)^n y(n)$, r being a rational function. To obtain y, we compute $y(n) = s_n * (a+bn)^{-n}$.

⟨*implementation: Guess - guessExpRat - Main*⟩+≡

```
    y: NNI -> FPOLYS :=
        F2FPOLYS(list.#1) * _
        p(last xValues, (xValues.#1)::Integer, a1, A, basis)**_
            (-(xValues.#1)::Integer)

    ylist: List FPOLYS := [y i for i in 1..len]

    y1 := y(len+1)
    y2 := y(len+2)
    y3 := y(len+3)

    res := []::List EXPRR
    if maxDegree(options)$GOPT0 = -1
    then maxDeg := len-1
    else maxDeg := min(maxDegree(options)$GOPT0, len-1)

    for i in 0..maxDeg repeat
        if debug(options)$GOPT0 then
            output(hconcat("degree ExpRat "::OutputForm, i::OutputForm))
                $OutputPackage
```

```
Shouldn't we use the algorithm over {\tt{}POLYS} here? Strange enough, for
polynomial interpolation, it is faster, but for rational interpolation
\emph{much} slower. This should be investigated.

It seems that {\tt{}maxDeg} bounds the degree of the denominator, rather than
the numerator? This is now added to the documentation of {\tt{}maxDegree}, it
should make sense.
```

⟨*implementation: Guess - guessExpRat - Main*⟩+≡

```
        if debug(options)$GOPT0 then
            systemCommand("sys date +%s")$MoreSystemCommands
            output("interpolating..."::OutputForm)$OutputPackage

        ri: FSUPFPOLYS
           := interpolate(xlist, ylist, (len-1-i)::NNI) _
                        $FFFG(FPOLYS, SUP FPOLYS)

-- for experimental fraction free interpolation
--        ri: Fraction SUP POLYS
--           := interpolate(xlist, ylist, (len-1-i)::NNI) _
--                         $FFFG(POLYS, SUP POLYS)

        if debug(options)$GOPT0 then
--            output(hconcat("xlist: ", xlist::OutputForm))$OutputPackage
--            output(hconcat("ylist: ", ylist::OutputForm))$OutputPackage
--            output(hconcat("ri: ", ri::OutputForm))$OutputPackage
            systemCommand("sys date +%s")$MoreSystemCommands
            output("polynomials..."::OutputForm)$OutputPackage

        poly1: POLYS := numer(elt(ri, x1)$SUP(FPOLYS) - y1)
        poly2: POLYS := numer(elt(ri, x2)$SUP(FPOLYS) - y2)
        poly3: POLYS := numer(elt(ri, x3)$SUP(FPOLYS) - y3)

-- for experimental fraction free interpolation
--        ri2: FSUPFPOLYS := map(#1::FPOLYS, numer ri)
--                           $SparseUnivariatePolynomialFunctions2(POLYS, FPOLYS)
--                          /map(#1::FPOLYS, denom ri)
--                           $SparseUnivariatePolynomialFunctions2(POLYS, FPOLYS)
--
--        poly1: POLYS := numer(elt(ri2, x1)$SUP(FPOLYS) - y1)
--        poly2: POLYS := numer(elt(ri2, x2)$SUP(FPOLYS) - y2)
--        poly3: POLYS := numer(elt(ri2, x3)$SUP(FPOLYS) - y3)

        n:Integer := len - i
        o1: Integer := ord1(xValues, i)
        d1: Integer := deg1(xValues, i)
        o2: Integer := ord2(xValues, i)
```

```
        d2: Integer := deg2(xValues, i)

  -- another compiler bug: using i as iterator here makes the loop break

        if debug(options)$GOPT0 then
            systemCommand("sys date +%s")$MoreSystemCommands
            output("interpolating resultants..."::OutputForm)$OutputPackage

        res1: SUP S
             := newton(evalResultant(poly1, poly3, o1, d1, a1, A))
                      $NewtonInterpolation(S)

        res2: SUP S
             := newton(evalResultant(poly2, poly3, o2, d2, a1, A))
                      $NewtonInterpolation(S)

        if debug(options)$GOPT0 then
  --             res1: SUP S := univariate(resultant(poly1, poly3, a1))
  --             res2: SUP S := univariate(resultant(poly2, poly3, a1))
  --             if res1 ~= res1res or res2 ~= res2res then
  --             output(hconcat("poly1 ", poly1::OutputForm))$OutputPackage
  --                 output(hconcat("poly2 ", poly2::OutputForm))$OutputPackage
  --             output(hconcat("poly3 ", poly3::OutputForm))$OutputPackage
  --                 output(hconcat("res1 ", res1::OutputForm))$OutputPackage
  --                 output(hconcat("res2 ", res2::OutputForm))$OutputPackage
              output("n/i: " string(n) " " string(i))$OutputPackage
              output("res1 ord: " string(o1) " " string(minimumDegree res1))
                    $OutputPackage
              output("res1 deg: " string(d1) " " string(degree res1))
                    $OutputPackage
              output("res2 ord: " string(o2) " " string(minimumDegree res2))
                    $OutputPackage
              output("res2 deg: " string(d2) " " string(degree res2))
                    $OutputPackage

        if debug(options)$GOPT0 then
            systemCommand("sys date +%s")$MoreSystemCommands
            output("computing gcd..."::OutputForm)$OutputPackage

  -- we want to solve over F
  -- for polynomial domains S this seems to be very costly!
        res3: SUP F := SUPS2SUPF(primitivePart(gcd(res1, res2)))

        if debug(options)$GOPT0 then
            systemCommand("sys date +%s")$MoreSystemCommands
            output("solving..."::OutputForm)$OutputPackage
```

```
-- res3 is a polynomial in A=a0+(len+3)*a1
-- now we have to find the roots of res3

        for f in factors factor(res3)$GF | degree f.factor = 1 repeat
-- we are only interested in the linear factors
            if debug(options)$GOPT0 then
                output(hconcat("f: ", f::OutputForm))$OutputPackage

            Av: F := -coefficient(f.factor, 0)
                    / leadingCoefficient f.factor

-- FIXME: in an earlier version, we disregarded vanishing Av
--        maybe we intended to disregard vanishing a1v? Either doesn't really
--        make sense to me right now.

            evalPoly := eval(POLYS2POLYF poly3, A, Av)
            if zero? evalPoly
            then evalPoly := eval(POLYS2POLYF poly1, A, Av)
-- Note that it really may happen that poly3 vanishes when specializing
-- A. Consider for example guessExpRat([1,1,1,1]).

-- FIXME: We check poly1 below, too. I should work out in what cases poly3
-- vanishes.

            for g in factors factor(univariate evalPoly)$GF
                     | degree g.factor = 1 repeat
                if debug(options)$GOPT0 then
                    output(hconcat("g: ", g::OutputForm))$OutputPackage

                a1v: F := -coefficient(g.factor, 0)
                         / leadingCoefficient g.factor

-- check whether poly1 and poly2 really vanish. Note that we could have found
-- an extraneous solution, since we only computed the gcd of the two
-- resultants.

                t1 := eval(POLYS2POLYF poly1, [a1, A]::List V,
                                              [a1v, Av]::List F)
                if zero? t1 then
                    t2 := eval(POLYS2POLYF poly2, [a1, A]::List V,
                                                  [a1v, Av]::List F)
                    if zero? t2 then

                        ri1: Fraction SUP POLYS
                            := SUPFPOLYS2FSUPPOLYS(numer ri)
```

```
                                / SUPFPOLYS2FSUPPOLYS(denom ri)

    -- for experimental fraction free interpolation
    --                          ri1: Fraction SUP POLYS := ri

                               numr: SUP F := SUPPOLYS2SUPF(numer ri1, a1v, Av)
                               denr: SUP F := SUPPOLYS2SUPF(denom ri1, a1v, Av)

                               if not zero? denr then
                                   res4: EXPRR := eval(FSUPF2EXPRR(xx, numr/denr),
                                                       kernel(xx),
                                                       basis(xx::EXPRR))
                                                  *p2(last xValues, _
                                                      xx, a1v, Av, basis)_
                                                   **xx::EXPRR
                                   res := cons(res4, res)
                               else if zero? numr and debug(options)$GOPT0 then
                                   output("numerator and denominator vanish!")
                                         $OutputPackage

    -- If we are only interested in one solution, we do not try other degrees if we
    -- have found already some solutions. I.e., the indentation here is correct.

            if not null(res) and one(options)$GOPT0 then return res

        res

    guessExpRatAux0(list: List F, basis: DIFFSPECN, options: LGOPT): GUESSRESULT ==
        if zero? safety(options)$GOPT0 then
            error "Guess: guessExpRat does not support zero safety"
    -- guesses Functions of the Form (a1*n+a0)^n*rat(n)
        xx := indexName(options)$GOPT0

    -- restrict to safety

        len: Integer := #list
        if len-safety(options)$GOPT0+1 < 0 then return []

        shortlist: List F := first(list, (len-safety(options)$GOPT0+1)::NNI)

    -- remove zeros from list

        zeros: EXPRR := 1
        newlist: List F
        xValues: List Integer
```

```
        i: Integer := -1
        for x in shortlist repeat
            i := i+1
            if x = 0 then
                zeros := zeros * (basis(xx::EXPRR) - basis(i::EXPRR))

        i := -1
        for x in shortlist repeat
            i := i+1
            if x ~= 0 then
                newlist := cons(x/retract(retract(eval(zeros, xx::EXPRR,
                                                       i::EXPRR))@R),
                                newlist)
                xValues := cons(i, xValues)

        newlist := reverse newlist
        xValues := reverse xValues

        res: List EXPRR
            := [eval(zeros * f, xx::EXPRR, xx::EXPRR) _
                for f in guessExpRatAux(xx, newlist, basis, xValues, options)]

        reslist := map([#1, checkResult(#1, xx, len, list, options)], res)
                     $ListFunctions2(EXPRR, Record(function: EXPRR, order: NNI))

        select(#1.order < len-safety(options)$GOPT0, reslist)

    guessExpRat(list : List F): GUESSRESULT ==
        guessExpRatAux0(list, defaultD, [])

    guessExpRat(list: List F, options: LGOPT): GUESSRESULT ==
        guessExpRatAux0(list, defaultD, options)

    if F has RetractableTo Symbol and S has RetractableTo Symbol then

        guessExpRat(q: Symbol): GUESSER ==
            guessExpRatAux0(#1, q::EXPRR**#1, #2)
```

8.40.3 guessing rational functions with a binomial term

```
It is not clear whether one should take the model
\begin{equation*}
  \binom{a+bn}{n}q(n),
\end{equation*}
which includes rational functions, or
\begin{equation*}
  (a+bn)(a+bn+1)\dots (a+bn+n)q(n).
\end{equation*}
which includes rational functions times $n!$. We choose the former, since
dividing by $n!$ is a common normalisation. The question remains whether we
should do the same for {\tt{}guessExpRat}.
```

⟨*implementation: Guess - guessBinRat*⟩≡

```
EXT ==> (Integer, V, V) -> FPOLYS
EXTEXPR ==> (Symbol, F, F) -> EXPRR

binExt: EXT
binExt(i: Integer, va1: V, vA: V): FPOLYS ==
    numl: List POLYS := [(vA::POLYS) + i * (va1::POLYS) - (l::POLYS) _
                         for l in 0..i-1]
    num: POLYS := reduce(_*, numl, 1)

    num/(factorial(i)::POLYS)

binExtEXPR: EXTEXPR
binExtEXPR(i: Symbol, a1v: F, Av: F): EXPRR ==
    binomial(coerce Av + coerce a1v * (i::EXPRR), i::EXPRR)


guessBinRatAux(xx: Symbol, list: List F,
               basis: DIFFSPECN, ext: EXT, extEXPR: EXTEXPR,
               xValues: List Integer, options: LGOPT): List EXPRR ==

    a1: V := index(1)$V
    A: V := index(2)$V

    len: NNI := #list
    if len < 4 then return []
               else len := (len-3)::NNI

    xlist := [F2FPOLYS DN2DL(basis, xValues.i) for i in 1..len]
    x1 := F2FPOLYS DN2DL(basis, xValues.(len+1))
    x2 := F2FPOLYS DN2DL(basis, xValues.(len+2))
    x3 := F2FPOLYS DN2DL(basis, xValues.(len+3))
```

We try to fit the data $(s1, s2, \ldots)$ to the model $\binom{a+bn}{n} y(n)$, r being a rational function. To obtain y, we compute $y(n) = s_n * \binom{a+bn}{n}^{-} 1$.

⟨*implementation: Guess - guessBinRat*⟩+≡

```
      y: NNI -> FPOLYS :=
          F2FPOLYS(list.#1) / _
          ext((xValues.#1)::Integer, a1, A)

      ylist: List FPOLYS := [y i for i in 1..len]

      y1 := y(len+1)
      y2 := y(len+2)
      y3 := y(len+3)

      res := []::List EXPRR
      if maxDegree(options)$GOPT0 = -1
      then maxDeg := len-1
      else maxDeg := min(maxDegree(options)$GOPT0, len-1)

      for i in 0..maxDeg repeat
  --        if debug(options)$GOPT0 then
  --            output(hconcat("degree BinRat "::OutputForm, i::OutputForm))
  --                $OutputPackage
```

```
Shouldn't we use the algorithm over {\tt{}POLYS} here? Strange enough, for
polynomial interpolation, it is faster, but for rational interpolation
\emph{much} slower. This should be investigated.

It seems that {\tt{}maxDeg} bounds the degree of the denominator, rather than
the numerator? This is now added to the documentation of {\tt{}maxDegree}, it
should make sense.
```

⟨*implementation: Guess - guessBinRat*⟩+≡

```
--        if debug(options)$GOPT0 then
--            output("interpolating..."::OutputForm)$OutputPackage

        ri: FSUPFPOLYS
           := interpolate(xlist, ylist, (len-1-i)::NNI) _
                         $FFFG(FPOLYS, SUP FPOLYS)

--        if debug(options)$GOPT0 then
--            output(hconcat("ri ", ri::OutputForm))$OutputPackage

        poly1: POLYS := numer(elt(ri, x1)$SUP(FPOLYS) - y1)
        poly2: POLYS := numer(elt(ri, x2)$SUP(FPOLYS) - y2)
        poly3: POLYS := numer(elt(ri, x3)$SUP(FPOLYS) - y3)

--        if debug(options)$GOPT0 then
--            output(hconcat("poly1 ", poly1::OutputForm))$OutputPackage
--            output(hconcat("poly2 ", poly2::OutputForm))$OutputPackage
--            output(hconcat("poly3 ", poly3::OutputForm))$OutputPackage


        n:Integer := len - i
        res1: SUP S := univariate(resultant(poly1, poly3, a1))
        res2: SUP S := univariate(resultant(poly2, poly3, a1))
        if debug(options)$GOPT0 then
--            output(hconcat("res1 ", res1::OutputForm))$OutputPackage
--            output(hconcat("res2 ", res2::OutputForm))$OutputPackage

--            if res1 ~= res1res or res2 ~= res2res then
--            output(hconcat("poly1 ", poly1::OutputForm))$OutputPackage
--                output(hconcat("poly2 ", poly2::OutputForm))$OutputPackage
--            output(hconcat("poly3 ", poly3::OutputForm))$OutputPackage
--                output(hconcat("res1 ", res1::OutputForm))$OutputPackage
--                output(hconcat("res2 ", res2::OutputForm))$OutputPackage
--            output("n/i: " string(n) " " string(i))$OutputPackage
              output("res1 ord: " string(minimumDegree res1))
                    $OutputPackage
              output("res1 deg: " string(degree res1))
                    $OutputPackage
```

```
            output("res2 ord: " string(minimumDegree res2))
                  $OutputPackage
            output("res2 deg: " string(degree res2))
                  $OutputPackage

        if debug(options)$GOPT0 then
            output("computing gcd..."::OutputForm)$OutputPackage

-- we want to solve over F
        res3: SUP F := SUPS2SUPF(primitivePart(gcd(res1, res2)))

--        if debug(options)$GOPT0 then
--            output(hconcat("res3 ", res3::OutputForm))$OutputPackage

-- res3 is a polynomial in A=a0+(len+3)*a1
-- now we have to find the roots of res3

        for f in factors factor(res3)$GF | degree f.factor = 1 repeat
-- we are only interested in the linear factors
--             if debug(options)$GOPT0 then
--                 output(hconcat("f: ", f::OutputForm))$OutputPackage

            Av: F := -coefficient(f.factor, 0)
                    / leadingCoefficient f.factor

--             if debug(options)$GOPT0 then
--                 output(hconcat("Av: ", Av::OutputForm))$OutputPackage

-- FIXME: in an earlier version, we disregarded vanishing Av
--        maybe we intended to disregard vanishing a1v? Either doesn't really
--        make sense to me right now.

            evalPoly := eval(POLYS2POLYF poly3, A, Av)
            if zero? evalPoly
            then evalPoly := eval(POLYS2POLYF poly1, A, Av)
-- Note that it really may happen that poly3 vanishes when specializing
-- A. Consider for example guessExpRat([1,1,1,1]).

-- FIXME: We check poly1 below, too. I should work out in what cases poly3
-- vanishes.

            for g in factors factor(univariate evalPoly)$GF
                     | degree g.factor = 1 repeat
--                 if debug(options)$GOPT0 then
--                     output(hconcat("g: ", g::OutputForm))$OutputPackage
```

```
                a1v: F := -coefficient(g.factor, 0)
                        / leadingCoefficient g.factor

--                if debug(options)$GOPT0 then
--                    output(hconcat("a1v: ", a1v::OutputForm))$OutputPackage

-- check whether poly1 and poly2 really vanish. Note that we could have found
-- an extraneous solution, since we only computed the gcd of the two
-- resultants.

                t1 := eval(POLYS2POLYF poly1, [a1, A]::List V,
                                              [a1v, Av]::List F)

--                if debug(options)$GOPT0 then
--                    output(hconcat("t1: ", t1::OutputForm))$OutputPackage

                if zero? t1 then
                    t2 := eval(POLYS2POLYF poly2, [a1, A]::List V,
                                                  [a1v, Av]::List F)

--                      if debug(options)$GOPT0 then
--                          output(hconcat("t2: ", t2::OutputForm))$OutputPackage

                      if zero? t2 then

                          ri1: Fraction SUP POLYS
                              := SUPFPOLYS2FSUPPOLYS(numer ri)
                               / SUPFPOLYS2FSUPPOLYS(denom ri)

--                          if debug(options)$GOPT0 then
--                              output(hconcat("ri1: ", ri1::OutputForm))$OutputPackage

                          numr: SUP F := SUPPOLYS2SUPF(numer ri1, a1v, Av)
                          denr: SUP F := SUPPOLYS2SUPF(denom ri1, a1v, Av)

--                          if debug(options)$GOPT0 then
--                              output(hconcat("numr: ", numr::OutputForm))$OutputPackage
--                              output(hconcat("denr: ", denr::OutputForm))$OutputPackage

                          if not zero? denr then
                              res4: EXPRR := eval(FSUPF2EXPRR(xx, numr/denr),
                                                  kernel(xx),
                                                  basis(xx::EXPRR))
                                             * extEXPR(xx, a1v, Av)

--                              if debug(options)$GOPT0 then
```

```
--                                  output(hconcat("res4: ", res4::OutputForm))$Ou

                                res := cons(res4, res)
                            else if zero? numr and debug(options)$GOPT0 then
                                output("numerator and denominator vanish!")
                                      $OutputPackage

-- If we are only interested in one solution, we do not try other degrees if we
-- have found already some solutions. I.e., the indentation here is correct.

        if not null(res) and one(options)$GOPT0 then return res

    res

guessBinRatAux0(list: List F,
                basis: DIFFSPECN, ext: EXT, extEXPR: EXTEXPR,
                options: LGOPT): GUESSRESULT ==

    if zero? safety(options)$GOPT0 then
        error "Guess: guessBinRat does not support zero safety"
-- guesses Functions of the form binomial(a+b*n, n)*rat(n)
    xx := indexName(options)$GOPT0

-- restrict to safety

    len: Integer := #list
    if len-safety(options)$GOPT0+1 < 0 then return []

    shortlist: List F := first(list, (len-safety(options)$GOPT0+1)::NNI)

-- remove zeros from list

    zeros: EXPRR := 1
    newlist: List F
    xValues: List Integer

    i: Integer := -1
    for x in shortlist repeat
        i := i+1
        if x = 0 then
            zeros := zeros * (basis(xx::EXPRR) - basis(i::EXPRR))

    i := -1
    for x in shortlist repeat
        i := i+1
        if x ~= 0 then
```

```
            newlist := cons(x/retract(retract(eval(zeros, xx::EXPRR,
                                                   i::EXPRR))@R),
                            newlist)
            xValues := cons(i, xValues)

        newlist := reverse newlist
        xValues := reverse xValues

        res: List EXPRR
            := [eval(zeros * f, xx::EXPRR, xx::EXPRR) _
                for f in guessBinRatAux(xx, newlist, basis, ext, extEXPR, xValues, _
                                        options)]

        reslist := map([#1, checkResult(#1, xx, len, list, options)], res)
                     $ListFunctions2(EXPRR, Record(function: EXPRR, order: NNI))

        select(#1.order < len-safety(options)$GOPT0, reslist)

    guessBinRat(list : List F): GUESSRESULT ==
        guessBinRatAux0(list, defaultD, binExt, binExtEXPR, [])

    guessBinRat(list: List F, options: LGOPT): GUESSRESULT ==
        guessBinRatAux0(list, defaultD, binExt, binExtEXPR, options)


    if F has RetractableTo Symbol and S has RetractableTo Symbol then

        qD: Symbol -> DIFFSPECN
        qD q == (q::EXPRR)**#1


        qBinExtAux(q: Symbol, i: Integer, va1: V, vA: V): FPOLYS ==
            fl: List FPOLYS
                 := [(1$FPOLYS - _
                      va1::POLYS::FPOLYS * (vA::POLYS::FPOLYS)**(i-1) * _
                      F2FPOLYS(q::F)**l) / (1-F2FPOLYS(q::F)**l) _
                     for l in 1..i]
            reduce(_*, fl, 1)

        qBinExt: Symbol -> EXT
        qBinExt q == qBinExtAux(q, #1, #2, #3)

        qBinExtEXPRaux(q: Symbol, i: Symbol, a1v: F, Av: F): EXPRR ==
            l: Symbol := 'l
            product((1$EXPRR - _
                     coerce a1v * (coerce Av) ** (coerce i - 1$EXPRR) * _
```

```
                (q::EXPRR) ** coerce(l)) / _
               (1$EXPRR - (q::EXPRR) ** coerce(l)), _
               equation(l, 1$EXPRR..i::EXPRR))

    qBinExtEXPR: Symbol -> EXTEXPR
    qBinExtEXPR q == qBinExtEXPRaux(q, #1, #2, #3)

    guessBinRat(q: Symbol): GUESSER ==
        guessBinRatAux0(#1, qD q, qBinExt q, qBinExtEXPR q, #2)_
```

8.40.4 Hermite Padé interpolation

⟨*implementation: Guess - Hermite-Pade*⟩≡
  ⟨*implementation: Guess - Hermite-Pade - Types for Operators*⟩
  ⟨*implementation: Guess - Hermite-Pade - Streams*⟩
  ⟨*implementation: Guess - Hermite-Pade - Operators*⟩
  ⟨*implementation: Guess - Hermite-Pade - Utilities*⟩
  ⟨*implementation: Guess - guessHPaux*⟩
  ⟨*implementation: Guess - guessHP*⟩

Types for Operators

⟨*implementation: Guess - Hermite-Pade - Types for Operators*⟩≡

```
  -- some useful types for Ore operators that work on series

  -- the differentiation operator
  DIFFSPECX ==> (EXPRR, Symbol, NonNegativeInteger) -> EXPRR
                                                  -- eg.: f(x)+->f(q*x)
                                                  --      f(x)+->D(f, x)
  DIFFSPECS ==> (UFPSF, NonNegativeInteger) -> UFPSF
                                                  -- eg.: f(x)+->f(q*x)

  DIFFSPECSF ==> (UFPSSUPF, NonNegativeInteger) -> UFPSSUPF
                                                  -- eg.: f(x)+->f(q*x)

  -- the constant term for the inhomogeneous case

  DIFFSPEC1 ==> UFPSF

  DIFFSPEC1F ==> UFPSSUPF

  DIFFSPEC1X ==> Symbol -> EXPRR
```

Streams

In this section we define some functions that provide streams for `HermitePade`.

The following three functions transform a partition `l` into a product of derivatives of `f`, using the given operators. We need to provide the same functionality for expressions, series and series with a transcendental element. Only for expressions we do not provide a version using the Hadamard product, although it would be quite easy to define an appropriate operator on expressions.

A partition $(\lambda_1^{p_1}, \lambda_2^{p_2}, \ldots)$ is transformed into the expression $(f^{(\lambda_1-1)})^{p_1}(f^{(\lambda_2-1)})^{p_2}\cdots$, i.e., the size of the part is interpreted as derivative, the exponent as power.

⟨*implementation: Guess - Hermite-Pade - Streams*⟩≡

```
termAsEXPRR(f: EXPRR, xx: Symbol, l: List Integer,
            DX: DIFFSPECX, D1X: DIFFSPEC1X): EXPRR ==
    if empty? l then D1X(xx)
    else
        ll: List List Integer := powers(l)$Partition

        fl: List EXPRR := [DX(f, xx, (first part-1)::NonNegativeInteger)
                           ** second(part)::NNI for part in ll]
        reduce(_*, fl)

termAsUFPSF(f: UFPSF, l: List Integer, DS: DIFFSPECS, D1: DIFFSPEC1): UFPSF ==
    if empty? l then D1
    else
        ll: List List Integer := powers(l)$Partition

-- first of each element of ll is the derivative, second is the power

        fl: List UFPSF := [DS(f, (first part -1)::NonNegativeInteger) _
                           ** second(part)::NNI for part in ll]

        reduce(_*, fl)

-- returns \prod f^(l.i), but using the Hadamard product
termAsUFPSF2(f: UFPSF, l: List Integer,
             DS: DIFFSPECS, D1: DIFFSPEC1): UFPSF ==
    if empty? l then D1
    else
        ll: List List Integer := powers(l)$Partition

-- first of each element of ll is the derivative, second is the power

        fl: List UFPSF
            := [map(#1** second(part)::NNI, DS(f, (first part -1)::NNI)) _
                for part in ll]
```

```
        reduce(hadamard$UFPS1(F), fl)


termAsUFPSSUPF(f: UFPSSUPF, l: List Integer,
                    DSF: DIFFSPECSF, D1F: DIFFSPEC1F): UFPSSUPF ==
    if empty? l then D1F
    else
        ll: List List Integer := powers(l)$Partition

-- first of each element of ll is the derivative, second is the power

        fl: List UFPSSUPF
           := [DSF(f, (first part -1)::NonNegativeInteger)
               ** second(part)::NNI for part in ll]

        reduce(_*, fl)


-- returns \prod f^(l.i), but using the Hadamard product
termAsUFPSSUPF2(f: UFPSSUPF, l: List Integer,
                DSF: DIFFSPECSF, D1F: DIFFSPEC1F): UFPSSUPF ==
    if empty? l then D1F
    else
        ll: List List Integer := powers(l)$Partition

-- first of each element of ll is the derivative, second is the power

        fl: List UFPSSUPF
           := [map(#1 ** second(part)::NNI, DSF(f, (first part -1)::NNI)) _
               for part in ll]

        reduce(hadamard$UFPS1(SUP F), fl)
```

It is not clear whether we should 'prefer' shifting and differentiation over powering. Currently, we produce the stream

$$\begin{array}{cccccccccc} \emptyset & 1 & 11 & 2 & 111 & 21 & 3 & 1111 & & \\ 1 & f & f^2 & f' & f^3 & ff' & f'' & f^4 & \ldots \end{array}$$

Maybe it would be better to produce

$$\begin{array}{cccccccccc} \emptyset & 1 & 2 & 11 & 3 & 21 & 111 & 4 & & \\ 1 & f & f' & f^2 & f'' & ff' & f^3 & f''' & \ldots \end{array}$$

instead, i.e., to leave the partitions unconjugated. Note however, that shifting and differentiation decrease the number of valid terms, while powering does not.

Note that we conjugate all partitions at the very end of the following procedure...

⟨*implementation: Guess - Hermite-Pade - Streams*⟩+≡

```
FilteredPartitionStream(options: LGOPT): Stream List Integer ==
    maxD := 1+maxDerivative(options)$GOPT0
    maxP := maxPower(options)$GOPT0

    if maxD > 0 and maxP > -1 then
        s := partitions(maxD, maxP)$PartitionsAndPermutations
    else
        s1: Stream Integer := generate(inc, 1)$Stream(Integer)
        s2: Stream Stream List Integer
           := map(partitions(#1)$PartitionsAndPermutations, s1)
                 $StreamFunctions2(Integer, Stream List Integer)
        s3: Stream List Integer
           := concat(s2)$StreamFunctions1(List Integer)

--        s := cons([],
--                  select(((maxD = 0) or (first #1 <= maxD)) _
--                     and ((maxP = -1) or (# #1 <= maxP)), s3))

        s := cons([],
                  select(((maxD = 0) or (# #1 <= maxD)) _
                     and ((maxP = -1) or (first #1 <= maxP)), s3))

    s := conjugates(s)$PartitionsAndPermutations
    if homogeneous(options)$GOPT0 then rest s else s

-- for functions
ADEguessStream(f: UFPSF, partitions: Stream List Integer,
               DS: DIFFSPECS, D1: DIFFSPEC1): Stream UFPSF ==
    map(termAsUFPSF(f, #1, DS, D1), partitions)
       $StreamFunctions2(List Integer, UFPSF)
```

```
-- for coefficients, i.e., using the Hadamard product
ADEguessStream2(f: UFPSF, partitions: Stream List Integer,
                DS: DIFFSPECS, D1: DIFFSPEC1): Stream UFPSF ==
    map(termAsUFPSF2(f, #1, DS, D1), partitions)
       $StreamFunctions2(List Integer, UFPSF)
```

The entries of the following stream indicate how many terms we loose when applying one of the power and shift or differentiation operators. More precisely, the n^{th} entry of the stream takes into account all partitions up to index n. Thus, the entries of the stream are weakly increasing.

⟨*implementation: Guess - Hermite-Pade - Streams*⟩+≡

```
ADEdegreeStream(partitions: Stream List Integer): Stream NNI ==
    scan(0, max((if empty? #1 then 0 else (first #1 - 1)::NNI), #2),
         partitions)$StreamFunctions2(List Integer, NNI)

ADEtestStream(f: UFPSSUPF, partitions: Stream List Integer,
              DSF: DIFFSPECSF, D1F: DIFFSPEC1F): Stream UFPSSUPF ==
    map(termAsUFPSSUPF(f, #1, DSF, D1F), partitions)
       $StreamFunctions2(List Integer, UFPSSUPF)

ADEtestStream2(f: UFPSSUPF, partitions: Stream List Integer,
              DSF: DIFFSPECSF, D1F: DIFFSPEC1F): Stream UFPSSUPF ==
    map(termAsUFPSSUPF2(f, #1, DSF, D1F), partitions)
       $StreamFunctions2(List Integer, UFPSSUPF)

ADEEXPRRStream(f: EXPRR, xx: Symbol, partitions: Stream List Integer,
               DX: DIFFSPECX, D1X: DIFFSPEC1X): Stream EXPRR ==
    map(termAsEXPRR(f, xx, #1, DX, D1X), partitions)
       $StreamFunctions2(List Integer, EXPRR)
```

Operators

We need to define operators that transform series for differentiation and shifting. We also provide operators for q-analogs. The functionality corresponding to powering and taking the Hadamard product if provided by the streams, see Section 8.40.4.

We have to provide each operator in three versions:

- for expressions,
- for series, and
- for series with an additional transcendental element.

The latter makes it possible to detect lazily whether a computed coefficient of a series is valid or not.

Furthermore, we have to define for each operator how to extract the coefficient of x^k in $z^l f(x)$, where multiplication with z is defined depending on the operator. Again, it is necessary to provide this functionality for expressions, series and series with a transcendental element.

Finally, we define a function that returns the diagonal elements $c_{k,k}$ in the expansion $\langle x^k \rangle z f(x) = \sum_{i=0}^{k} c_{k,i} \langle x^i \rangle f(x)$, and an expression that represents the constant term for the inhomogeneous case.

The Differentiation Setting In this setting, we have $zf(x) := xf(x)$.

⟨*implementation: Guess - Hermite-Pade - Operators*⟩≡

```
diffDX: DIFFSPECX
diffDX(expr, x, n) == D(expr, x, n)

diffDS: DIFFSPECS
diffDS(s, n) == D(s, n)

diffDSF: DIFFSPECSF
diffDSF(s, n) ==
-- I have to help the compiler here a little to choose the right signature...
    if SUP F has _*: (NonNegativeInteger, SUP F) -> SUP F
    then D(s, n)
```

The next three functions extract the coefficient of x^k in $z^l f(x)$. Only, for expressions, we rather need $\sum_{k\geq 0}\langle x^k\rangle z^l f(x)$, i.e., the function itself, which is by definition equal to $x^l f(x)$.

⟨*implementation: Guess - Hermite-Pade - Operators*⟩+≡

```
diffAX: DIFFSPECAX
diffAX(l: NNI, x: Symbol, f: EXPRR): EXPRR ==
    (x::EXPRR)**l * f

diffA: DIFFSPECA
diffA(k: NNI, l: NNI, f: SUP S): S ==
    DiffAction(k, l, f)$FFFG(S, SUP S)

diffAF: DIFFSPECAF
diffAF(k: NNI, l: NNI, f: UFPSSUPF): SUP F ==
    DiffAction(k, l, f)$FFFG(SUP F, UFPSSUPF)

diffC: DIFFSPECC
diffC(total: NNI): List S == DiffC(total)$FFFG(S, SUP S)

diff1X: DIFFSPEC1X
diff1X(x: Symbol)== 1$EXPRR

diffHP options ==
    if displayAsGF(options)$GOPT0 then
        partitions := FilteredPartitionStream options
        [ADEguessStream(#1, partitions, diffDS, 1$UFPSF), _
         ADEdegreeStream partitions, _
         ADEtestStream(#1, partitions, diffDSF, 1$UFPSSUPF), _
         ADEEXPRRStream(#1, #2, partitions, diffDX, diff1X), _
         diffA, diffAF, diffAX, diffC]$HPSPEC
    else
        error "Guess: guessADE supports only displayAsGF"
```

q-dilation In this setting, we also have $zf(x) := xf(x)$, therefore we can reuse some of the functions of the previous paragraph. Differentiation is defined by $D_q f(x,q) = f(qx,q)$.

⟨*implementation: Guess - Hermite-Pade - Operators*⟩+≡

```
if F has RetractableTo Symbol and S has RetractableTo Symbol then

    qDiffDX(q: Symbol, expr: EXPRR, x: Symbol, n: NonNegativeInteger): EXPRR ==
        eval(expr, x::EXPRR, (q::EXPRR)**n*x::EXPRR)

    qDiffDS(q: Symbol, s: UFPSF, n: NonNegativeInteger): UFPSF ==
        multiplyCoefficients((q::F)**((n*#1)::NonNegativeInteger), s)

    qDiffDSF(q: Symbol, s: UFPSSUPF, n: NonNegativeInteger): UFPSSUPF ==
        multiplyCoefficients((q::F::SUP F)**((n*#1)::NonNegativeInteger), s)

    diffHP(q: Symbol): (LGOPT -> HPSPEC) ==
        if displayAsGF(#1)$GOPT0 then
            partitions := FilteredPartitionStream #1
            [ADEguessStream(#1, partitions, qDiffDS(q, #1, #2), 1$UFPSF), _
             repeating([0$NNI])$Stream(NNI), _
             ADEtestStream(#1, partitions, qDiffDSF(q, #1, #2), 1$UFPSSUPF), _
             ADEEXPRRStream(#1, #2, partitions, qDiffDX(q, #1, #2, #3), diff1X), _
             diffA, diffAF, diffAX, diffC]$HPSPEC
        else
            error "Guess: guessADE supports only displayAsGF"
```

Shifting The shift operator transforms a sequence $u(k)$ into $u(k+1)$. We also provide operators `ShiftSXGF`, `ShiftAXGF` that act on the power series, as long as no powering is involved. In this case, shifting transforms $f(x)$ into $\frac{f(x)-f(0)}{x}$. Multiplication with z transforms the coefficients $u(n)$ of the series into $zu(n) := nu(n)$. The description in terms of power series is given by $xDf(x)$.

⟨*implementation: Guess - Hermite-Pade - Operators*⟩+≡

```
ShiftSX(expr: EXPRR, x: Symbol, n: NNI): EXPRR ==
    eval(expr, x::EXPRR, x::EXPRR+n::EXPRR)

ShiftSXGF(expr: EXPRR, x: Symbol, n: NNI): EXPRR ==
    if zero? n then expr
    else
        l := [eval(D(expr, x, i)/factorial(i)::EXPRR, x::EXPRR, 0$EXPRR)_
              *(x::EXPRR)**i for i in 0..n-1]
        (expr-reduce(_+, l))/(x::EXPRR**n)

ShiftSS(s:UFPSF, n:NNI): UFPSF ==
    ((quoByVar #1)**n)$MappingPackage1(UFPSF) (s)

ShiftSF(s:UFPSSUPF, n: NNI):UFPSSUPF ==
    ((quoByVar #1)**n)$MappingPackage1(UFPSSUPF) (s)
```

As before, next three functions extract the coefficient of x^k in $z^l f(x)$.

⟨*implementation: Guess - Hermite-Pade - Operators*⟩+≡

```
ShiftAX(l: NNI, n: Symbol, f: EXPRR): EXPRR ==
    n::EXPRR**l * f

ShiftAXGF(l: NNI, x: Symbol, f: EXPRR): EXPRR ==
-- I need to help the compiler here, unfortunately
      if zero? l then f
      else
          s := [stirling2(l, i)$IntegerCombinatoricFunctions(Integer)::EXPRR _
                * (x::EXPRR)**i*D(f, x, i) for i in 1..l]
          reduce(_+, s)

ShiftA(k: NNI, l: NNI, f: SUP S): S ==
    ShiftAction(k, l, f)$FFFG(S, SUP S)

ShiftAF(k: NNI, l: NNI, f: UFPSSUPF): SUP F ==
    ShiftAction(k, l, f)$FFFG(SUP F, UFPSSUPF)

ShiftC(total: NNI): List S ==
    ShiftC(total)$FFFG(S, SUP S)

shiftHP options ==
    partitions := FilteredPartitionStream options
    if displayAsGF(options)$GOPT0 then
        if maxPower(options)$GOPT0 = 1 then
            [ADEguessStream(#1, partitions, ShiftSS, (1-monomial(1,1))**(-1)),_
             ADEdegreeStream partitions, _
             ADEtestStream(#1, partitions, ShiftSF, (1-monomial(1,1))**(-1)), _
             ADEEXPRRStream(#1, #2, partitions, ShiftSXGF, 1/(1-#1::EXPRR)), _
             ShiftA, ShiftAF, ShiftAXGF, ShiftC]$HPSPEC
       else
            error "Guess: no support for the Shift operator with displayAsGF _
                   and maxPower>1"
    else
        [ADEguessStream2(#1, partitions, ShiftSS, (1-monomial(1,1))**(-1)), _
         ADEdegreeStream partitions, _
         ADEtestStream2(#1, partitions, ShiftSF, (1-monomial(1,1))**(-1)), _
         ADEEXPRRStream(#1, #2, partitions, ShiftSX, diff1X), _
         ShiftA, ShiftAF, ShiftAX, ShiftC]$HPSPEC
```

q-Shifting The q-shift also transforms $u(n)$ into $u(n+1)$, and we can reuse the corresponding functions of the previous paragraph. However, this time multiplication with z is defined differently: the coefficient of x^k in $zu(n)$ is $q^n u(n)$. We do not define the corresponding functionality for power series.

⟨*implementation: Guess - Hermite-Pade - Operators*⟩+≡
```
if F has RetractableTo Symbol and S has RetractableTo Symbol then

    qShiftAX(q: Symbol, l: NNI, n: Symbol, f: EXPRR): EXPRR ==
        (q::EXPRR)**(l*n::EXPRR) * f

    qShiftA(q: Symbol, k: NNI, l: NNI, f: SUP S): S ==
        qShiftAction(q::S, k, l, f)$FFFG(S, SUP S)

    qShiftAF(q: Symbol, k: NNI, l: NNI, f: UFPSSUPF): SUP F ==
        qShiftAction(q::F::SUP(F), k, l, f)$FFFG(SUP F, UFPSSUPF)

    qShiftC(q: Symbol, total: NNI): List S ==
        qShiftC(q::S, total)$FFFG(S, SUP S)

    shiftHP(q: Symbol): (LGOPT -> HPSPEC) ==
        partitions := FilteredPartitionStream #1
        if displayAsGF(#1)$GOPT0 then
            error "Guess: no support for the qShift operator with displayAsGF"
        else
            [ADEguessStream2(#1, partitions, ShiftSS, _
                             (1-monomial(1,1))**(-1)), _
             ADEdegreeStream partitions, _
             ADEtestStream2(#1, partitions, ShiftSF, _
                            (1-monomial(1,1))**(-1)), _
             ADEEXPRRStream(#1, #2, partitions, ShiftSX, diff1X), _
             qShiftA(q, #1, #2, #3), qShiftAF(q, #1, #2, #3), _
             qShiftAX(q, #1, #2, #3), qShiftC(q, #1)]$HPSPEC
```

Extend the action to polynomials The following operation uses the given action of z on a function to multiply a f with a polynomial.

⟨*implementation: Guess - Hermite-Pade - Operators*⟩+≡
```
makeEXPRR(DAX: DIFFSPECAX, x: Symbol, p: SUP F, expr: EXPRR): EXPRR ==
    if zero? p then 0$EXPRR
    else
        coerce(leadingCoefficient p)::EXPRR * DAX(degree p, x, expr) _
        + makeEXPRR(DAX, x, reductum p, expr)
```

Utilities

`list2UFPSF` and `list2UFPSSUPF` transform the list passed to the guessing functions into a series. One might be tempted to transform the list into a polynomial instead, but the present approach makes computing powers and derivatives much cheaper, since, because of laziness, only coefficients that are actually used are computed.

The second of the two procedures augments the list with a transcendental element. This way we can easily check whether a coefficient is valid or not: if it contains the transcendental element, it depends on data we do not have. In other words, this transcendental element simply represents the $O(x^d)$, when d is the number of elements in the list.

⟨*implementation: Guess - Hermite-Pade - Utilities*⟩≡
```
list2UFPSF(list: List F): UFPSF == series(list::Stream F)$UFPSF

list2UFPSSUPF(list: List F): UFPSSUPF ==
    l := [e::SUP(F) for e in list for i in 0..]::Stream SUP F
    series(l)$UFPSSUPF + monomial(monomial(1,1)$SUP(F), #list)$UFPSSUPF
```

`SUPF2SUPSUPF` interprets each coefficient as a univariate polynomial.

⟨*implementation: Guess - Hermite-Pade - Utilities*⟩+≡
```
SUPF2SUPSUPF(p: SUP F): SUP SUP F ==
    map(#1::SUP F, p)$SparseUnivariatePolynomialFunctions2(F, SUP F)
```

getListSUPF returns the first o elements of the stream, each truncated after degree deg.

⟨*implementation: Guess - Hermite-Pade - Utilities*⟩+≡
```
UFPSF2SUPF(f: UFPSF, deg: NNI): SUP F ==
    makeSUP univariatePolynomial(f, deg)

getListSUPF(s: Stream UFPSF, o: NNI, deg: NNI): List SUP F ==
    map(UFPSF2SUPF(#1, deg), entries complete first(s, o))
       $ListFunctions2(UFPSF, SUP F)

S2EXPRR(s: S): EXPRR ==
    if F is S then
        coerce(s pretend F)@EXPRR
    else if F is Fraction S then
        coerce(s::Fraction(S))@EXPRR
    else error "Type parameter F should be either equal to S or equal _
                to Fraction S"
```

⟨*implementation: Guess - guessInterpolate*⟩
⟨*implementation: Guess - guessInterpolate2*⟩
⟨*implementation: Guess - testInterpolant*⟩

guessInterpolate calls the appropriate generalInterpolation from FFFG, for one vector of degrees, namely eta.

⟨*implementation: Guess - guessInterpolate*⟩≡
```
guessInterpolate(guessList: List SUP F, eta: List NNI, D: HPSPEC)
                : Matrix SUP S ==
    if F is S then
        vguessList: Vector SUP S := vector(guessList pretend List(SUP(S)))
        generalInterpolation((D.C)(reduce(_+, eta)), D.A,
                             vguessList, eta)$FFFG(S, SUP S)
    else if F is Fraction S then
        vguessListF: Vector SUP F := vector(guessList)
        generalInterpolation((D.C)(reduce(_+, eta)), D.A,
                             vguessListF, eta)$FFFGF(S, SUP S, SUP F)

    else error "Type parameter F should be either equal to S or equal _
                to Fraction S"
```

guessInterpolate2 calls the appropriate generalInterpolation from FFFG, for all degree vectors with given sumEta and maxEta.

⟨*implementation: Guess - guessInterpolate2*⟩≡
```
  guessInterpolate2(guessList: List SUP F,
                    sumEta: NNI, maxEta: NNI,
                    D: HPSPEC): Stream Matrix SUP S ==
      if F is S then
          vguessList: Vector SUP S := vector(guessList pretend List(SUP(S)))
          generalInterpolation((D.C)(sumEta), D.A,
                               vguessList, sumEta, maxEta)
                              $FFFG(S, SUP S)
      else if F is Fraction S then
          vguessListF: Vector SUP F := vector(guessList)
          generalInterpolation((D.C)(sumEta), D.A,
                               vguessListF, sumEta, maxEta)
                              $FFFGF(S, SUP S, SUP F)

      else error "Type parameter F should be either equal to S or equal _
                  to Fraction S"
```

testInterpolant checks whether p is really a solution.

⟨*implementation: Guess - testInterpolant*⟩≡
```
  testInterpolant(resi: List SUP S,
                  list: List F,
                  testList: List UFPSSUPF,
                  exprList: List EXPRR,
                  initials: List EXPRR,
                  guessDegree: NNI,
                  D: HPSPEC,
                  dummy: Symbol, op: BasicOperator, options: LGOPT)
                 : Union("failed", Record(function: EXPRR, order: NNI)) ==
```

First we make sure it is not a solution we should have found already. Note that we cannot check this if maxDegree is set, in which case some initial solutions may have been overlooked.

⟨*implementation: Guess - testInterpolant*⟩+≡
```
      ((maxDegree(options)$GOPT0 = -1) and
       (allDegrees(options)$GOPT0 = false) and
       zero?(last resi))
       => return "failed"
```

Then we check all the coefficients that should be valid. We want the zero solution only, if the function is really zero. Without this test, every sequence ending with zero is interpreted as the zero sequence, since the factor in front of the only non-vanishing term can cancel everything else.

⟨*implementation: Guess - testInterpolant*⟩+≡
```
nonZeroCoefficient: Integer := 0

for i in 1..#resi repeat
    if not zero? resi.i then
        if zero? nonZeroCoefficient then
            nonZeroCoefficient := i
        else
            nonZeroCoefficient := 0
            break
```

We set `nonZeroCoefficient` to the only non zero coefficient or, if there are several, it is 0. It should not happen that all coefficients in `resi` vanish.

```
Check that not all coefficients in {\tt{}resi} can vanish simultaneously.
```

⟨*implementation: Guess - testInterpolant*⟩+≡
```
if not zero? nonZeroCoefficient then
    (freeOf?(exprList.nonZeroCoefficient, name op)) => return "failed"

    for e in list repeat
        if not zero? e then return "failed"
```

We first deal with the case that there is only one non-vanishing coefficient in `resi`. If the only expression in `exprList` whose coefficient does not vanish does not contain the name of the generating function, or if there is a non-zero term in `list`, we reject the proposed solution.

⟨*implementation: Guess - testInterpolant*⟩+≡

```
      else
          resiSUPF := map(SUPF2SUPSUPF SUPS2SUPF #1, resi)
                        $ListFunctions2(SUP S, SUP SUP F)

          iterate? := true;
          for d in guessDegree+1.. repeat
              c: SUP F := generalCoefficient(D.AF, vector testList,
                                              d, vector resiSUPF)
                                             $FFFG(SUP F, UFPSSUPF)

              if not zero? c then
                  iterate? := ground? c
                  break

          iterate? => return "failed"
```

Here we check for each degree `d` larger than `guessDegree` whether the proposed linear combination vanishes. When the first coefficient that does not vanish contains the transcendental element we accept the linear combination as a solution.

Finally, it seems that we have found a solution. Now we cancel the greatest common divisor of the equation. Note that this may take quite some time, it seems to be quicker to check `generalCoefficient` with the original `resi`.

If `S` is a `Field`, the `gcd` will always be 1. Thus, in this case we make the result monic.

⟨*implementation: Guess - testInterpolant*⟩+≡

```
g: SUP S
if S has Field
then g := leadingCoefficient(find(not zero? #1, reverse resi)::SUP(S))::SUP(S
else g := gcd resi
resiF := map(SUPS2SUPF((#1 exquo g)::SUP(S)), resi)
            $ListFunctions2(SUP S, SUP F)


if debug(options)$GOPT0 then
    output(hconcat("trying possible solution ", resiF::OutputForm))
          $OutputPackage

-- transform each term into an expression

    ex: List EXPRR := [makeEXPRR(D.AX, dummy, p, e) _
                       for p in resiF for e in exprList]

-- transform the list of expressions into a sum of expressions

    res: EXPRR
    if displayAsGF(options)$GOPT0 then
        res := evalADE(op, dummy, variableName(options)$GOPT0::EXPRR,
                       indexName(options)$GOPT0::EXPRR,
                       numerator reduce(_+, ex),
                       reverse initials)
                      $RecurrenceOperator(Integer, EXPRR)
        ord: NNI := 0
-- FIXME: checkResult doesn't really work yet for generating functions
    else
        res := evalRec(op, dummy, indexName(options)$GOPT0::EXPRR,
                       indexName(options)$GOPT0::EXPRR,
                       numerator reduce(_+, ex),
                       reverse initials)
                      $RecurrenceOperator(Integer, EXPRR)
        ord: NNI := checkResult(res, indexName(options)$GOPT0, _
```

```
                                    #list, list, options)

      [res, ord]$Record(function: EXPRR, order: NNI)
```

The main routine

The following is the main guessing routine, called by all others – except `guessExpRat`.

⟨*implementation: Guess - guessHPaux*⟩≡

```
  guessHPaux(list: List F, D: HPSPEC, options: LGOPT): GUESSRESULT ==
      reslist: GUESSRESULT := []

      listDegree := #list-1-safety(options)$GOPTO
      if listDegree < 0 then return reslist
```

We do as if we knew only the coefficients up to and including degree `listDegree-safety`. Thus, if we have less elements of the sequence than `safety`, there remain no elements for guessing. Originally we demanded to have at least two elements for guessing. However, we want to be able to induce from $[c, c]$ that the function equals c with `safety` one. This is important if we apply, for example, `guessRat` recursively. In this case, `listDegree` equals zero.

⟨*implementation: Guess - guessHPaux*⟩+≡

```
      a := functionName(options)$GOPTO
      op := operator a
      x := variableName(options)$GOPTO
      dummy := new$Symbol

      initials: List EXPRR := [coerce(e)@EXPRR for e in list]
```

We need to create several streams. Let P be the univariate power series whose first few elements are given by `list`. As an example, consider the differentiation setting. In this case, the elements of `guessS` consist of P differentiated and taken to some power. The elements of `degreeS` are integers, that tell us how many terms less than in `list` are valid in the corresponding element of `guessS`. The elements of `testS` are very similar to those of `guessS`, with the difference that they are derived from P with an transcendental element added, which corresponds to $O(x^d)$. Finally, the elements of `exprS` contain representations of the transformations applied to P as expressions.

```
I am not sure whether it is better to get rid of denominators in {\tt{}list}
here or, as I currently do it, only in {\tt{}generalInterpolation{\char36}FFFG}. %$
If we clear them at here, we cannot take advantage of factors that may appear
only after differentiating or powering.
```

⟨*implementation: Guess - guessHPaux*⟩+≡

```
guessS  := (D.guessStream)(list2UFPSF list)
degreeS := D.degreeStream
testS   := (D.testStream)(list2UFPSSUPF list)
exprS   := (D.exprStream)(op(dummy::EXPRR)::EXPRR, dummy)
```

We call the number of terms of the linear combination its *order*. We consider linear combinations of at least two terms, since otherwise the function would have to vanish identically...

When proceeding in the stream `guessS`, it may happen that we loose valid terms. For example, when trying to find a linear ordinary differential equation, we are looking for a linear combination of $f, f', f'', \ldots$. Suppose `listDegree` equals 2, i.e. we have

```
f               &= l_0 + l_1 x + l_2 x^2\\
f^\prime        &= l_1 + 2l_2 x\\
f^{\prime\prime} &= 2l_2.
```

Thus, each time we differentiate the series, we loose one term for guessing. Therefore, we cannot use the coefficient of x^2 of f'' for generating a linear combination. `guessDegree` contains the degree up to which all the generated series are correct, taking into account `safety`.

⟨*implementation: Guess - guessHPaux*⟩+≡

```
      iterate?: Boolean := false -- this is necessary because the compiler
                                 -- doesn't understand => "iterate" properly
                                 -- the latter just leaves the current block, it
                                 -- seems
      for o in 2.. repeat
          empty? rest(guessS, (o-1)::NNI) => break
          guessDegree: Integer := listDegree-(degreeS.o)::Integer
          guessDegree < 0 => break
          if debug(options)$GOPT0 then
              output(hconcat("Trying order ", o::OutputForm))$OutputPackage
              output(hconcat("guessDegree is ", guessDegree::OutputForm))
                    $OutputPackage
```

We now have to distinguish between the case where we try all combination of degrees and the case where we try only an (nearly) evenly distributed vector of degrees.

In the first case, `guessInterpolate2` is going to look at all degree vectors with `o` elements with total degree `guessDegree+1`. We give up as soon as the order `o` is greater than the number of available terms plus one. In the extreme case, i.e., when `o=guessDegree+2`, we allow for example constant coefficients for all terms of the linear combination. It seems that it is a bit arbitrary at what point we stop, however, we would like to be consistent with the evenly distributed case.

⟨*implementation: Guess - guessHPaux*⟩+≡

```
        if allDegrees(options)$GOPT0 then
            (o > guessDegree+2) => return reslist

            maxEta: Integer := 1+maxDegree(options)$GOPT0
            if maxEta = 0
            then maxEta := guessDegree+1
        else
```

In the second case, we first compute the number of parameters available for determining the coefficient polynomials. We have to take care of the fact, that HermitePade produces solutions with sum of degrees being one more than the sum of elements in `eta`.

⟨*implementation: Guess - guessHPaux*⟩+≡

```
            maxParams := divide(guessDegree::NNI+1, o)
            if debug(options)$GOPT0
            then output(hconcat("maxParams: ", maxParams::OutputForm))
                        $OutputPackage
```

If we do not have enough parameters, we give up. We allow for at most one zero entry in the degree vector, because then there is one column that allows at least a constant term in each entry.

⟨*implementation: Guess - guessHPaux*⟩+≡

```
            if maxParams.quotient = 0 and maxParams.remainder < o-1
            then return reslist
```

If `maxDegree` is set, we skip the first few partitions, unless we cannot increase the order anymore.

```
I have no satisfactory way to determine whether we can increase the order or
not.
```

⟨*implementation: Guess - guessHPaux*⟩+≡

```
            if ((maxDegree(options)$GOPT0 ~= -1) and
                (maxDegree(options)$GOPT0 < maxParams.quotient)) and
                not (empty? rest(guessS, o) or
                     ((newGuessDegree := listDegree-(degreeS.(o+1))::Integer)
                          < 0) or
                      (((newMaxParams := divide(newGuessDegree::NNI+1, o+1))
                          .quotient = 0) and
                       (newMaxParams.remainder < o)))
            then iterate? := true
            else if ((maxDegree(options)$GOPT0 ~= -1) and
                     (maxParams.quotient > maxDegree(options)$GOPT0))
                 then
                     guessDegree := o*(1+maxDegree(options)$GOPT0)-2
                     eta: List NNI
                         := [(if i < o    _
                                then maxDegree(options)$GOPT0 + 1   _
                                else maxDegree(options)$GOPT0)::NNI _
                            for i in 1..o]
                 else eta: List NNI
                          := [(if i <= maxParams.remainder   _
                                then maxParams.quotient + 1   _
                                else maxParams.quotient)::NNI for i in 1..o]
```

We distribute the parameters as evenly as possible. Is it better to have higher degrees at the end or at the beginning?

It remains to prepare `guessList`, which is the list of `o` power series polynomials truncated after degree `guessDegree`. Then we can call HermitePade.

```
{\tt{}maxDegree} should be handled differently, maybe: we should only pass as
many coefficients to {\tt{}FFFG} as {\tt{}maxDegree} implies! That is, if we cannot
increase the order anymore, we should decrease {\tt{}guessDegree} to %
$o\cdot {\tt{}maxDegree} - 2$ and set {\tt{}eta} accordingly.  I might have to take
care of {\tt{}allDegrees}, too.
```

⟨*implementation: Guess - guessHPaux*⟩+≡

```
        if iterate?
        then
            iterate? := false
            if debug(options)$GOPT0 then output("iterating")$OutputPackage
        else
            guessList: List SUP F    := getListSUPF(guessS, o, guessDegree::NNI)
            testList:  List UFPSSUPF := entries complete first(testS, o)
            exprList:  List EXPRR    := entries complete first(exprS, o)

            if debug(options)$GOPT0 then
                output("The list of expressions is")$OutputPackage
                output(exprList::OutputForm)$OutputPackage

            if allDegrees(options)$GOPT0 then
                MS: Stream Matrix SUP S := guessInterpolate2(guessList,
                                                             guessDegree::NNI+1,
                                                             maxEta::NNI, D)
                repeat
                    (empty? MS) => break
                    M := first MS

                    for i in 1..o repeat
                        res := testInterpolant(entries column(M, i),
                                               list,
                                               testList,
                                               exprList,
                                               initials,
                                               guessDegree::NNI,
                                               D, dummy, op, options)

                        (res case "failed") => "iterate"

                        if not member?(res, reslist)
                        then reslist := cons(res, reslist)
```

```
                        if one(options)$GOPT0 then return reslist

                    MS := rest MS
            else
                M: Matrix SUP S := guessInterpolate(guessList, eta, D)

                for i in 1..o repeat
                    res := testInterpolant(entries column(M, i),
                                           list,
                                           testList,
                                           exprList,
                                           initials,
                                           guessDegree::NNI,
                                           D, dummy, op, options)
                    (res case "failed") => "iterate"

                    if not member?(res, reslist)
                    then reslist := cons(res, reslist)

                    if one(options)$GOPT0 then return reslist

    reslist
```

```
The duplicated block at the end should really go into {\tt{}testInterpolant}, I
guess. Furthermore, it would be better to remove duplicates already there.
```

Specialisations

For convenience we provide some specialisations that cover the most common situations.

generating functions

⟨*implementation: Guess - guessHP*⟩≡

```
guessHP(D: LGOPT -> HPSPEC): GUESSER == guessHPaux(#1, D #2, #2)

guessADE(list: List F, options: LGOPT): GUESSRESULT ==
    opts: LGOPT := cons(displayAsGF(true)$GuessOption, options)
    guessHPaux(list, diffHP opts, opts)

guessADE(list: List F): GUESSRESULT == guessADE(list, [])

guessAlg(list: List F, options: LGOPT) ==
    guessADE(list, cons(maxDerivative(0)$GuessOption, options))

guessAlg(list: List F): GUESSRESULT == guessAlg(list, [])

guessHolo(list: List F, options: LGOPT): GUESSRESULT ==
    guessADE(list, cons(maxPower(1)$GuessOption, options))

guessHolo(list: List F): GUESSRESULT == guessHolo(list, [])

guessPade(list: List F, options: LGOPT): GUESSRESULT ==
    opts := append(options, [maxDerivative(0)$GuessOption,
                             maxPower(1)$GuessOption,
                             allDegrees(true)$GuessOption])
    guessADE(list, opts)

guessPade(list: List F): GUESSRESULT == guessPade(list, [])
```

q-generating functions

⟨*implementation: Guess - guessHP*⟩+≡

```
if F has RetractableTo Symbol and S has RetractableTo Symbol then

    guessADE(q: Symbol): GUESSER ==
        opts: LGOPT := cons(displayAsGF(true)$GuessOption, #2)
        guessHPaux(#1, (diffHP q)(opts), opts)
```

coefficients

⟨*implementation: Guess - guessHP*⟩+≡

```
guessRec(list: List F, options: LGOPT): GUESSRESULT ==
      opts: LGOPT := cons(displayAsGF(false)$GuessOption, options)
      guessHPaux(list, shiftHP opts, opts)

guessRec(list: List F): GUESSRESULT == guessRec(list, [])

guessPRec(list: List F, options: LGOPT): GUESSRESULT ==
      guessRec(list, cons(maxPower(1)$GuessOption, options))

guessPRec(list: List F): GUESSRESULT == guessPRec(list, [])

guessRat(list: List F, options: LGOPT): GUESSRESULT ==
      opts := append(options, [maxShift(0)$GuessOption,
                               maxPower(1)$GuessOption,
                               allDegrees(true)$GuessOption])
      guessRec(list, opts)

guessRat(list: List F): GUESSRESULT == guessRat(list, [])
```

q-coefficients

⟨*implementation: Guess - guessHP*⟩+≡

```
if F has RetractableTo Symbol and S has RetractableTo Symbol then

    guessRec(q: Symbol): GUESSER ==
        opts: LGOPT := cons(displayAsGF(false)$GuessOption, #2)
        guessHPaux(#1, (shiftHP q)(opts), opts)

    guessPRec(q: Symbol): GUESSER ==
        opts: LGOPT := append([displayAsGF(false)$GuessOption,
                               maxPower(1)$GuessOption], #2)
        guessHPaux(#1, (shiftHP q)(opts), opts)

    guessRat(q: Symbol): GUESSER ==
        opts := append(#2, [displayAsGF(false)$GuessOption,
                            maxShift(0)$GuessOption,
                            maxPower(1)$GuessOption,
                            allDegrees(true)$GuessOption])
        guessHPaux(#1, (shiftHP q)(opts), opts)
```

8.40.5 guess – applying operators recursively

The main observation made by Christian Krattenthaler in designing his program Rate is the following: it occurs frequently that although a sequence of numbers is not generated by a rational function, the sequence of successive quotients is.

We slightly extend upon this idea, and apply recursively one or both of the two following operators:

Δ_n the differencing operator, transforming $f(n)$ into $f(n) - f(n-1)$, and

Q_n the operator that transforms $f(n)$ into $f(n)/f(n-1)$.

⟨*implementation: Guess - guess*⟩≡

```
guess(list: List F, guessers: List GUESSER, ops: List Symbol,
      options: LGOPT): GUESSRESULT ==
    maxLevel := maxLevel(options)$GOPT0
    xx := indexName(options)$GOPT0
    if debug(options)$GOPT0 then
        output(hconcat("guessing level ", maxLevel::OutputForm))
              $OutputPackage
        output(hconcat("guessing ", list::OutputForm))$OutputPackage
    res: GUESSRESULT := []
    len := #list :: PositiveInteger
    if len <= 1 then return res

    for guesser in guessers repeat
        res := append(guesser(list, options), res)

        if debug(options)$GOPT0 then
            output(hconcat("res ", res::OutputForm))$OutputPackage

        if one(options)$GOPT0 and not empty? res then return res

    if (maxLevel = 0) then return res

    if member?('guessProduct, ops) and not member?(0$F, list) then
        prodList: List F := [(list.(i+1))/(list.i) for i in 1..(len-1)]

        if not every?(one?, prodList) then
            var: Symbol := subscript('p, [len::OutputForm])
            prodGuess :=
                [[coerce(list.(guess.order+1))
                  * product(guess.function, _
                            equation(var, _
                                     (guess.order)::EXPRR..xx::EXPRR-1)), _
                  guess.order] _
                 for guess in guess(prodList, guessers, ops,
```

```
                                        append([indexName(var)$GuessOption,
                                                maxLevel(maxLevel-1)
                                                        $GuessOption],
                                               options))$%]

                if debug(options)$GOPTO then
                    output(hconcat("prodGuess "::OutputForm,
                                   prodGuess::OutputForm))
                          $OutputPackage

                for guess in prodGuess
                        | not any?(guess.function = #1.function, res) repeat
                    res := cons(guess, res)

        if one(options)$GOPTO and not empty? res then return res

        if member?('guessSum, ops) then
            sumList: List F := [(list.(i+1))-(list.i) for i in 1..(len-1)]

            if not every?(#1=sumList.1, sumList) then
                var: Symbol := subscript('s, [len::OutputForm])
                sumGuess :=
                    [[coerce(list.(guess.order+1)) _
                      + summation(guess.function, _
                                  equation(var, _
                                           (guess.order)::EXPRR..xx::EXPRR-1)),_
                      guess.order] _
                     for guess in guess(sumList, guessers, ops,
                                        append([indexName(var)$GuessOption,
                                                maxLevel(maxLevel-1)
                                                        $GuessOption],
                                               options))$%]

                for guess in sumGuess
                        | not any?(guess.function = #1.function, res) repeat
                    res := cons(guess, res)

        res

    guess(list: List F): GUESSRESULT ==
        guess(list, [guessADE$%, guessRec$%], ['guessProduct, 'guessSum], [])

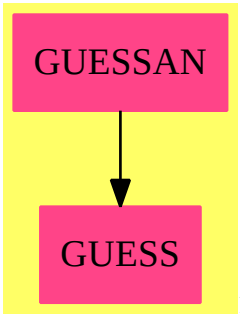
    guess(list: List F, options: LGOPT): GUESSRESULT ==
        guess(list, [guessADE$%, guessRat$%], ['guessProduct, 'guessSum],
              options)
```

```
guess(list: List F, guessers: List GUESSER, ops: List Symbol)
     : GUESSRESULT ==
    guess(list, guessers, ops, [])
```

⟨*GUESS.dotabb*⟩≡
```
"GUESS" [color="#FF4488",href="bookvol10.4.pdf#nameddest=GUESS"]
"RECOP" [color="#FF4488",href="bookvol10.4.pdf#nameddest=RECOP"]
"GUESS" -> "RECOP"
```

8.41 package GUESSAN GuessAlgebraicNumber

8.42 GuessAlgebraicNumber



1.00

Exports:

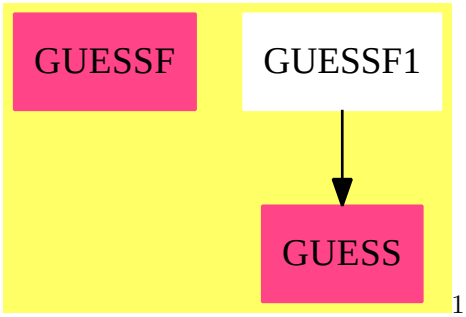
diffHP	guess	guessADE	guessAlg	guessBinRat
guessExpRat	guessHP	guessHolo	guessPRec	guessPade
guessRat	guessRec	shiftHP		

⟨*package GUESSAN GuessAlgebraicNumber*⟩≡
```
)abbrev package GUESSAN GuessAlgebraicNumber
++ Description:
++ This package exports guessing of sequences of rational functions
GuessAlgebraicNumber() == Guess(AlgebraicNumber, AlgebraicNumber,
                            Expression Integer,
                            AlgebraicNumber,
                            id$MappingPackage1(AlgebraicNumber),
                            coerce$Expression(Integer))
```

⟨*GUESSAN.dotabb*⟩≡
```
"GUESSAN" [color="#FF4488",href="bookvol10.4.pdf#nameddest=GUESSAN"]
"GUESS" [color="#FF4488",href="bookvol10.4.pdf#nameddest=GUESS"]
"GUESSAN" -> "GUESS"
```

8.43 package GUESSF GuessFinite

8.44 GuessFinite



1.00

Exports:

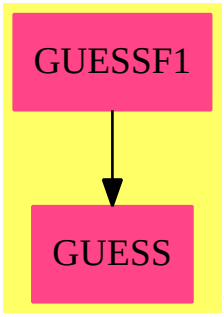
diffHP	guess	guessADE	guessAlg	guessBinRat
guessExpRat	guessHP	guessHolo	guessPRec	guessPade
guessRat	guessRec	shiftHP		

⟨*package GUESSF GuessFinite*⟩≡
```
)abbrev package GUESSF GuessFinite
++ Description:
++ This package exports guessing of sequences of numbers in a finite field
GuessFinite(F:Join(FiniteFieldCategory, ConvertibleTo Integer)) ==
  Guess(F, F, Expression Integer, Integer, coerce$F,
        F2EXPRR$GuessFiniteFunctions(F))
```

⟨*GUESSF.dotabb*⟩≡
```
"GUESSF" [color="#FF4488",href="bookvol10.4.pdf#nameddest=GUESSF"]
"GUESS" [color="#FF4488",href="bookvol10.4.pdf#nameddest=GUESS"]
"GUESSF1" -> "GUESS"
```

8.45 package GUESSF1 GuessFiniteFunctions

8.46 GuessFiniteFunctions



1.00

Exports:
F2EXPRR

⟨*package GUESSF1 GuessFiniteFunctions*⟩≡
```
)abbrev package GUESSF1 GuessFiniteFunctions
++ Description:
++ This package exports guessing of sequences of numbers in a finite field
GuessFiniteFunctions(F:Join(FiniteFieldCategory, ConvertibleTo Integer)):
  Exports == Implementation where

    EXPRR ==> Expression Integer

    Exports == with

      F2EXPRR: F -> EXPRR

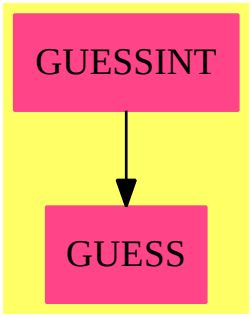
    Implementation == add

      F2EXPRR(p: F): EXPRR == convert(p)@Integer::EXPRR
```

⟨*GUESSF1.dotabb*⟩≡
```
"GUESSF1" [color="#FF4488",href="bookvol10.4.pdf#nameddest=GUESSF1"]
"GUESS" [color="#FF4488",href="bookvol10.4.pdf#nameddest=GUESS"]
"GUESSF1" -> "GUESS"
```

8.47 package GUESSINT GuessInteger

8.48 GuessInteger



1.00

Exports:

diffHP	guess	guessADE	guessAlg	guessBinRat
guessExpRat	guessHP	guessHolo	guessPRec	guessPade
guessRat	guessRec	shiftHP		

⟨*package GUESSINT GuessInteger*⟩≡

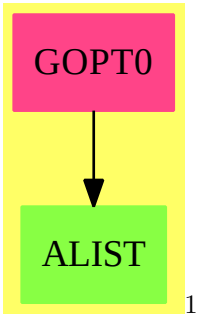
```
-- concerning algebraic functions, it may make sense to look at A.J.van der
-- Poorten, Power Series Representing Algebraic Functions, Section 6.
)abbrev package GUESSINT GuessInteger
++ Description:
++ This package exports guessing of sequences of rational numbers
GuessInteger() == Guess(Fraction Integer, Integer, Expression Integer,
                    Fraction Integer,
                    id$MappingPackage1(Fraction Integer),
                    coerce$Expression(Integer))
```

⟨*GUESSINT.dotabb*⟩≡

```
"GUESSINT" [color="#FF4488",href="bookvol10.4.pdf#nameddest=GUESSINT"]
"GUESS" [color="#FF4488",href="bookvol10.4.pdf#nameddest=GUESS"]
"GUESSINT" -> "GUESS"
```

8.49 package GOPT0 GuessOptionFunctions0

8.50 GuessOptionFunctions0



1.00

Exports:

allDegrees	coerce	debug	displayAsGF	functionName
hash	homogeneous	indexName	latex	maxDegree
maxDerivative	maxLevel	maxPower	maxShift	one
safety	variableName	?~=?	?=?	

⟨*package GOPT0 GuessOptionFunctions0*⟩≡

```
)abbrev package GOPT0 GuessOptionFunctions0
++ Author: Martin Rubey
++ Description: GuessOptionFunctions0 provides operations that extract the
++ values of options for \spadtype{Guess}.
GuessOptionFunctions0(): Exports == Implementation where

  LGOPT ==> List GuessOption

  Exports == SetCategory with

    maxLevel: LGOPT -> Integer
      ++ maxLevel returns the specified maxLevel or -1 as default.

    maxPower: LGOPT -> Integer
      ++ maxPower returns the specified maxPower or -1 as default.

    maxDerivative: LGOPT -> Integer
      ++ maxDerivative returns the specified maxDerivative or -1 as default.

    maxShift: LGOPT -> Integer
      ++ maxShift returns the specified maxShift or -1 as default.

    maxDegree: LGOPT -> Integer
      ++ maxDegree returns the specified maxDegree or -1 as default.
```

```
    allDegrees: LGOPT -> Boolean
      ++ allDegrees returns whether all possibilities of the degree vector
      ++ should be tried, the default being false.

    safety: LGOPT -> NonNegativeInteger
      ++ safety returns the specified safety or 1 as default.

    one: LGOPT -> Boolean
      ++ one returns whether we need only one solution, default being true.

    homogeneous: LGOPT -> Boolean
      ++ homogeneous returns whether we allow only homogeneous algebraic
      ++ differential equations, default being false

    functionName: LGOPT -> Symbol
      ++ functionName returns the name of the function given by the algebraic
      ++ differential equation, default being f

    variableName: LGOPT -> Symbol
      ++ variableName returns the name of the variable used in by the
      ++ algebraic differential equation, default being x

    indexName: LGOPT -> Symbol
      ++ indexName returns the name of the index variable used for the
      ++ formulas, default being n

    displayAsGF: LGOPT -> Boolean
      ++ displayAsGF specifies whether the result is a generating function
      ++ or a recurrence. This option should not be set by the user, but rather
      ++ by the HP-specification, therefore, there is no default.

    debug: LGOPT -> Boolean
      ++ debug returns whether we want additional output on the progress,
      ++ default being false

  Implementation == add

    maxLevel l ==
      if (opt := option(l, "maxLevel" :: Symbol)) case "failed" then
        -1
      else
        retract(opt :: Any)$AnyFunctions1(Integer)

    maxDerivative l ==
      if (opt := option(l, "maxDerivative" :: Symbol)) case "failed" then
```

```
    -1
  else
    retract(opt :: Any)$AnyFunctions1(Integer)

maxShift l == maxDerivative l

maxDegree l ==
  if (opt := option(l, "maxDegree" :: Symbol)) case "failed" then
    -1
  else
    retract(opt :: Any)$AnyFunctions1(Integer)

allDegrees l ==
  if (opt := option(l, "allDegrees" :: Symbol)) case "failed" then
    false
  else
    retract(opt :: Any)$AnyFunctions1(Boolean)

maxPower l ==
  if (opt := option(l, "maxPower" :: Symbol)) case "failed" then
    -1
  else
    retract(opt :: Any)$AnyFunctions1(Integer)

safety l ==
  if (opt := option(l, "safety" :: Symbol)) case "failed" then
    1$NonNegativeInteger
  else
    retract(opt :: Any)$AnyFunctions1(Integer)::NonNegativeInteger

one l ==
  if (opt := option(l, "one" :: Symbol)) case "failed" then
    true
  else
    retract(opt :: Any)$AnyFunctions1(Boolean)

debug l ==
  if (opt := option(l, "debug" :: Symbol)) case "failed" then
    false
  else
    retract(opt :: Any)$AnyFunctions1(Boolean)

homogeneous l ==
  if (opt := option(l, "homogeneous" :: Symbol)) case "failed" then
    false
  else
```

```
      retract(opt :: Any)$AnyFunctions1(Boolean)

  variableName l ==
    if (opt := option(l, "variableName" :: Symbol)) case "failed" then
      "x" :: Symbol
    else
      retract(opt :: Any)$AnyFunctions1(Symbol)

  functionName l ==
    if (opt := option(l, "functionName" :: Symbol)) case "failed" then
      "f" :: Symbol
    else
      retract(opt :: Any)$AnyFunctions1(Symbol)

  indexName l ==
    if (opt := option(l, "indexName" :: Symbol)) case "failed" then
      "n" :: Symbol
    else
      retract(opt :: Any)$AnyFunctions1(Symbol)

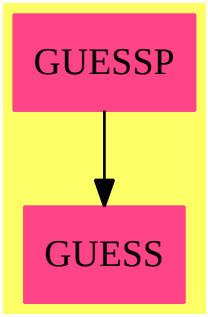
  displayAsGF l ==
    if (opt := option(l, "displayAsGF" :: Symbol)) case "failed" then
      error "GuessOption: displayAsGF not set"
    else
      retract(opt :: Any)$AnyFunctions1(Boolean)
```

⟨*GOPT0.dotabb*⟩≡

```
"GOPT0" [color="#FF4488",href="bookvol10.4.pdf#nameddest=GOPT0"]
"ALIST" [color="#88FF44",href="bookvol10.3.pdf#nameddest=ALIST"]
"GOPT0" -> "ALIST"
```

8.51 package GUESSP GuessPolynomial

8.52 GuessPolynomial



1.00

Exports:

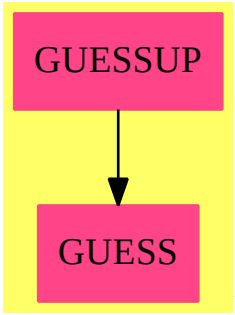
diffHP	guess	guessADE	guessAlg	guessBinRat
guessExpRat	guessHP	guessHolo	guessPRec	guessPade
guessRat	guessRec	shiftHP		

⟨*package GUESSP GuessPolynomial*⟩≡
```
)abbrev package GUESSP GuessPolynomial
++ Description:
++ This package exports guessing of sequences of rational functions
GuessPolynomial() == Guess(Fraction Polynomial Integer, Polynomial Integer,
                           Expression Integer,
                           Fraction Polynomial Integer,
                           id$MappingPackage1(Fraction Polynomial Integer),
                           coerce$Expression(Integer))
```

⟨*GUESSP.dotabb*⟩≡
```
"GUESSP" [color="#FF4488",href="bookvol10.4.pdf#nameddest=GUESSP"]
"GUESS" [color="#FF4488",href="bookvol10.4.pdf#nameddest=GUESS"]
"GUESSP" -> "GUESS"
```

8.53 package GUESSUP GuessUnivariatePolynomial

8.54 GuessUnivariatePolynomial



1.00

Exports:

diffHP	guess	guessADE	guessAlg	guessBinRat
guessExpRat	guessHP	guessHolo	guessPRec	guessPade
guessRat	guessRec	shiftHP		

⟨*package GUESSUP GuessUnivariatePolynomial*⟩≡

```
)abbrev package GUESSUP GuessUnivariatePolynomial
++ Description:
++ This package exports guessing of sequences of univariate rational functions
GuessUnivariatePolynomial(q: Symbol): Exports == Implementation where

    UP ==> MyUnivariatePolynomial(q, Integer)
    EXPRR ==> MyExpression(q, Integer)
    F ==> Fraction UP
    S ==> UP
    NNI ==> NonNegativeInteger
    LGOPT ==> List GuessOption
    GUESSRESULT ==> List Record(function: EXPRR, order: NNI)
    GUESSER ==> (List F, LGOPT) -> GUESSRESULT

    Exports == with

        guess: List F -> GUESSRESULT
          ++ \spad{guess l} applies recursively \spadfun{guessRec} and
          ++ \spadfun{guessADE} to the successive differences and quotients of
          ++ the list. Default options as described in
          ++ \spadtype{GuessOptionFunctions0} are used.

        guess: (List F, LGOPT) -> GUESSRESULT
          ++ \spad{guess(l, options)} applies recursively \spadfun{guessRec}
```

```
          ++ and \spadfun{guessADE} to the successive differences and quotients
          ++ of the list. The given options are used.

        guess: (List F, List GUESSER, List Symbol) -> GUESSRESULT
          ++ \spad{guess(l, guessers, ops)} applies recursively the given
          ++ guessers to the successive differences if ops contains the symbol
          ++ guessSum and quotients if ops contains the symbol guessProduct to
          ++ the list. Default options as described in
          ++ \spadtype{GuessOptionFunctions0} are used.

        guess: (List F, List GUESSER, List Symbol, LGOPT) -> GUESSRESULT
          ++ \spad{guess(l, guessers, ops)} applies recursively the given
          ++ guessers to the successive differences if ops contains the symbol
          ++ \spad{guessSum} and quotients if ops contains the symbol
          ++ \spad{guessProduct} to the list. The given options are used.

        guessExpRat: List F -> GUESSRESULT
          ++ \spad{guessExpRat l} tries to find a function of the form
          ++ n+->(a+b n)^n r(n), where r(n) is a rational function, that fits
          ++ l.

        guessExpRat: (List F, LGOPT) -> GUESSRESULT
          ++ \spad{guessExpRat(l, options)} tries to find a function of the
          ++ form n+->(a+b n)^n r(n), where r(n) is a rational function, that
          ++ fits l.

        guessBinRat: List F -> GUESSRESULT
          ++ \spad{guessBinRat(l, options)} tries to find a function of the
          ++ form n+->binomial(a+b n, n) r(n), where r(n) is a rational
          ++ function, that fits l.

        guessBinRat: (List F, LGOPT) -> GUESSRESULT
          ++ \spad{guessBinRat(l, options)} tries to find a function of the
          ++ form n+->binomial(a+b n, n) r(n), where r(n) is a rational
          ++ function, that fits l.

--        if F has RetractableTo Symbol and S has RetractableTo Symbol then

        guessExpRat: Symbol -> GUESSER
          ++ \spad{guessExpRat q} returns a guesser that tries to find a
          ++ function of the form n+->(a+b q^n)^n r(q^n), where r(q^n) is a
          ++ q-rational function, that fits l.

        guessBinRat: Symbol -> GUESSER
          ++ \spad{guessBinRat q} returns a guesser that tries to find a
          ++ function of the form n+->qbinomial(a+b n, n) r(n), where r(q^n) is a
```

```
    ++ q-rational function, that fits l.

------------------------------------------------------------------------

  guessHP: (LGOPT -> HPSPEC) -> GUESSER
    ++ \spad{guessHP f} constructs an operation that applies Hermite-Pade
    ++ approximation to the series generated by the given function f.

  guessADE: List F -> GUESSRESULT
    ++ \spad{guessADE l} tries to find an algebraic differential equation
    ++ for a generating function whose first Taylor coefficients are
    ++ given by l, using the default options described in
    ++ \spadtype{GuessOptionFunctions0}.

  guessADE: (List F, LGOPT) -> GUESSRESULT
    ++ \spad{guessADE(l, options)} tries to find an algebraic
    ++ differential equation for a generating function whose first Taylor
    ++ coefficients are given by l, using the given options.

  guessAlg: List F -> GUESSRESULT
    ++ \spad{guessAlg l} tries to find an algebraic equation for a
    ++ generating function whose first Taylor coefficients are given by
    ++ l, using the default options described in
    ++ \spadtype{GuessOptionFunctions0}. It is equivalent to
    ++ \spadfun{guessADE}(l, maxDerivative == 0).

  guessAlg: (List F, LGOPT) -> GUESSRESULT
    ++ \spad{guessAlg(l, options)} tries to find an algebraic equation
    ++ for a generating function whose first Taylor coefficients are
    ++ given by l, using the given options. It is equivalent to
    ++ \spadfun{guessADE}(l, options) with \spad{maxDerivative == 0}.

  guessHolo: List F -> GUESSRESULT
    ++ \spad{guessHolo l} tries to find an ordinary linear differential
    ++ equation for a generating function whose first Taylor coefficients
    ++ are given by l, using the default options described in
    ++ \spadtype{GuessOptionFunctions0}. It is equivalent to
    ++ \spadfun{guessADE}\spad{(l, maxPower == 1)}.

  guessHolo: (List F, LGOPT) -> GUESSRESULT
    ++ \spad{guessHolo(l, options)} tries to find an ordinary linear
    ++ differential equation for a generating function whose first Taylor
    ++ coefficients are given by l, using the given options. It is
    ++ equivalent to \spadfun{guessADE}\spad{(l, options)} with
    ++ \spad{maxPower == 1}.
```

```
guessPade: (List F, LGOPT) -> GUESSRESULT
  ++ \spad{guessPade(l, options)} tries to find a rational function
  ++ whose first Taylor coefficients are given by l, using the given
  ++ options. It is equivalent to \spadfun{guessADE}\spad{(l,
  ++ maxDerivative == 0, maxPower == 1, allDegrees == true)}.

guessPade: List F -> GUESSRESULT
  ++ \spad{guessPade(l, options)} tries to find a rational function
  ++ whose first Taylor coefficients are given by l, using the default
  ++ options described in \spadtype{GuessOptionFunctions0}. It is
  ++ equivalent to \spadfun{guessADE}\spad{(l, options)} with
  ++ \spad{maxDerivative == 0, maxPower == 1, allDegrees == true}.

guessRec: List F -> GUESSRESULT
  ++ \spad{guessRec l} tries to find an ordinary difference equation
  ++ whose first values are given by l, using the default options
  ++ described in \spadtype{GuessOptionFunctions0}.

guessRec: (List F, LGOPT) -> GUESSRESULT
  ++ \spad{guessRec(l, options)} tries to find an ordinary difference
  ++ equation whose first values are given by l, using the given
  ++ options.

guessPRec: (List F, LGOPT) -> GUESSRESULT
  ++ \spad{guessPRec(l, options)} tries to find a linear recurrence
  ++ with polynomial coefficients whose first values are given by l,
  ++ using the given options. It is equivalent to
  ++ \spadfun{guessRec}\spad{(l, options)} with \spad{maxPower == 1}.

guessPRec: List F -> GUESSRESULT
  ++ \spad{guessPRec l} tries to find a linear recurrence with
  ++ polynomial coefficients whose first values are given by l, using
  ++ the default options described in
  ++ \spadtype{GuessOptionFunctions0}. It is equivalent to
  ++ \spadfun{guessRec}\spad{(l, maxPower == 1)}.

guessRat: (List F, LGOPT) -> GUESSRESULT
  ++ \spad{guessRat(l, options)} tries to find a rational function
  ++ whose first values are given by l, using the given options. It is
  ++ equivalent to \spadfun{guessRec}\spad{(l, maxShift == 0, maxPower
  ++ == 1, allDegrees == true)}.

guessRat: List F -> GUESSRESULT
  ++ \spad{guessRat l} tries to find a rational function whose first
  ++ values are given by l, using the default options described in
  ++ \spadtype{GuessOptionFunctions0}. It is equivalent to
```

```
          ++ \spadfun{guessRec}\spad{(l, maxShift == 0, maxPower == 1,
          ++ allDegrees == true)}.

        diffHP: LGOPT -> HPSPEC
          ++ \spad{diffHP options} returns a specification for Hermite-Pade
          ++ approximation with the differential operator

        shiftHP: LGOPT -> HPSPEC
          ++ \spad{shiftHP options} returns a specification for Hermite-Pade
          ++ approximation with the shift operator

--        if F has RetractableTo Symbol and S has RetractableTo Symbol then

        shiftHP: Symbol -> (LGOPT -> HPSPEC)
          ++ \spad{shiftHP options} returns a specification for
          ++ Hermite-Pade approximation with the $q$-shift operator

        diffHP: Symbol -> (LGOPT -> HPSPEC)
          ++ \spad{diffHP options} returns a specification for Hermite-Pade
          ++ approximation with the  $q$-dilation operator

        guessRec: Symbol -> GUESSER
          ++ \spad{guessRec q} returns a guesser that finds an ordinary
          ++ q-difference equation whose first values are given by l, using
          ++ the given options.

        guessPRec: Symbol -> GUESSER
          ++ \spad{guessPRec q} returns a guesser that tries to find
          ++ a linear q-recurrence with polynomial coefficients whose first
          ++ values are given by l, using the given options. It is
          ++ equivalent to \spadfun{guessRec}\spad{(q)} with
          ++ \spad{maxPower == 1}.

        guessRat: Symbol -> GUESSER
          ++ \spad{guessRat q} returns a guesser that tries to find a
          ++ q-rational function whose first values are given by l, using
          ++ the given options. It is equivalent to \spadfun{guessRec} with
          ++ \spad{(l, maxShift == 0, maxPower == 1, allDegrees == true)}.

        guessADE: Symbol -> GUESSER
          ++ \spad{guessADE q} returns a guesser that tries to find an
          ++ algebraic differential equation for a generating function whose
          ++ first Taylor coefficients are given by l, using the given
          ++ options.

-----------------------------------------------------------------------------
```

```
Implementation == Guess(Fraction UP, UP,
                        MyExpression(q, Integer),
                        Fraction UP,
                        id$MappingPackage1(Fraction UP),
                        coerce$MyExpression(q, Integer))
```

⟨*GUESSUP.dotabb*⟩≡

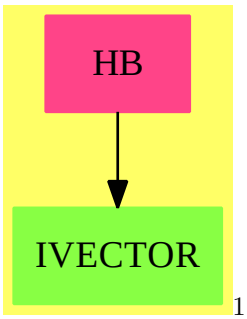
```
"GUESSUP" [color="#FF4488",href="bookvol10.4.pdf#nameddest=GUESSUP"]
"GUESS" [color="#FF4488",href="bookvol10.4.pdf#nameddest=GUESS"]
"GUESSUP" -> "GUESS"
```

Chapter 9

Chapter H

9.1 package HB HallBasis

9.2 HallBasis



1.00

Exports:
generate inHallBasis? lfunc

⟨*package HB HallBasis*⟩≡
```
)abbrev package HB HallBasis
++ Author : Larry Lambe
++ Date Created : August  1988
++ Date Last Updated : March 9 1990
++ Related Constructors: OrderedSetInts, Commutator, FreeNilpotentLie
++ AMS Classification: Primary 17B05, 17B30; Secondary 17A50
++ Keywords: free Lie algebra, Hall basis, basic commutators
++ Description : Generate a basis for the free Lie algebra on n
++ generators over a ring R with identity up to basic commutators
++ of length c using the algorithm of P. Hall as given in Serre's
++ book Lie Groups -- Lie Algebras
```

```
HallBasis() : Export == Implement where
   B   ==> Boolean
   I   ==> Integer
   NNI ==> NonNegativeInteger
   VI  ==> Vector Integer
   VLI ==> Vector List Integer

   Export  ==> with
     lfunc : (I,I) -> I
       ++ lfunc(d,n) computes the rank of the nth factor in the
       ++ lower central series of the free d-generated free Lie
       ++ algebra;  This rank is d if n = 1 and binom(d,2) if
       ++ n = 2
     inHallBasis? : (I,I,I,I) -> B
       ++ inHallBasis?(numberOfGens, leftCandidate, rightCandidate, left)
       ++ tests to see if a new element should be added to the P. Hall
       ++ basis being constructed.
       ++ The list \spad{[leftCandidate,wt,rightCandidate]}
       ++ is included in the basis if in the unique factorization of
       ++ rightCandidate, we have left factor leftOfRight, and
       ++ leftOfRight <= leftCandidate
     generate : (NNI,NNI) -> VLI
       ++ generate(numberOfGens, maximalWeight) generates a vector of
       ++ elements of the form [left,weight,right] which represents a
       ++ P. Hall basis element for the free lie algebra on numberOfGens
       ++ generators.  We only generate those basis elements of weight
       ++ less than or equal to maximalWeight

   Implement ==> add

     lfunc(d,n) ==
        n < 0 => 0
        n = 0 => 1
        n = 1 => d
        sum:I := 0
        m:I
        for m in 1..(n-1) repeat
          if n rem m = 0 then
            sum := sum + m * lfunc(d,m)
        res := (d**(n::NNI) - sum) quo n

     inHallBasis?(n,i,j,l) ==
        i >= j => false
        j <= n => true
        l <= i => true
```

```
      false

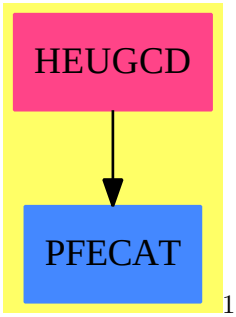
    generate(n:NNI,c:NNI) ==
      gens:=n
      maxweight:=c
      siz:I := 0
      for i in 1 .. maxweight repeat siz := siz + lfunc(gens,i)
      v:VLI:= new(siz::NNI,[])
      for i in 1..gens repeat v(i) := [0, 1, i]
      firstindex:VI := new(maxweight::NNI,0)
      wt:I := 1
      firstindex(1) := 1
      numComms:I := gens
      newNumComms:I := numComms
      done:B := false
      while not done repeat
        wt := wt + 1
        if wt > maxweight then done := true
        else
          firstindex(wt) := newNumComms + 1
          leftIndex := 1
          -- cW == complimentaryWeight
          cW:I := wt - 1
          while (leftIndex <= numComms) and (v(leftIndex).2 <= cW) repeat
            for rightIndex in firstindex(cW)..(firstindex(cW+1) - 1) repeat
              if inHallBasis?(gens,leftIndex,rightIndex,v(rightIndex).1) then
                newNumComms := newNumComms + 1
                v(newNumComms) := [leftIndex,wt,rightIndex]
            leftIndex := leftIndex + 1
            cW := wt - v(leftIndex).2
          numComms := newNumComms
      v
```

⟨*HB.dotabb*⟩≡

```
"HB" [color="#FF4488",href="bookvol10.4.pdf#nameddest=HB"]
"IVECTOR" [color="#88FF44",href="bookvol10.3.pdf#nameddest=IVECTOR"]
"HB" -> "IVECTOR"
```

9.3 package HEUGCD HeuGcd

9.4 HeuGcd



1.00

Exports:
content gcd gcdcofact gcdcofactprim gcdprim lintgcd

⟨*package HEUGCD HeuGcd*⟩≡
```
)abbrev package HEUGCD HeuGcd
++ Author: P.Gianni
++ Date Created:
++ Date Last Updated: 13 September 94
++ Basic Functions:
++ Related Constructors:
++ Also See:
++ AMS Classifications:
++ Keywords:
++ References:
++ Description:
++ This package provides the functions for the heuristic integer gcd.
++ Geddes's algorithm,for univariate polynomials with integer coefficients
HeuGcd (BP):C == T
 where
  BP       :   UnivariatePolynomialCategory Integer
  Z        ==> Integer
  ContPrim ==> Record(cont:Z,prim:BP)


  C == with
     gcd          : List BP  -> BP
       ++ gcd([f1,..,fk]) = gcd of the polynomials fi.
       ++
       ++X gcd([671*671*x^2-1,671*671*x^2+2*671*x+1])
       ++X gcd([7*x^2+1,(7*x^2+1)^2])
```

```
    gcdprim      : List BP  -> BP
      ++ gcdprim([f1,..,fk]) = gcd of k PRIMITIVE univariate polynomials
    gcdcofact    : List BP  -> List BP
      ++ gcdcofact([f1,..fk]) = gcd and cofactors of k univariate polynomials.
    gcdcofactprim: List BP  -> List BP
      ++ gcdcofactprim([f1,..fk]) = gcd and cofactors of k
      ++ primitive polynomials.
    content      : List BP  -> List Z
      ++ content([f1,..,fk]) = content of a list of univariate polynonials
    lintgcd      : List  Z  -> Z
      ++ lintgcd([a1,..,ak]) = gcd of a list of integers

  T == add

    PI    ==> PositiveInteger
    NNI   ==> NonNegativeInteger
    Cases ==> Union("gcdprim","gcd","gcdcofactprim","gcdcofact")
    import ModularDistinctDegreeFactorizer BP

    --local functions
    localgcd     :        List BP       -> List BP
    constNotZero :           BP         -> Boolean
    height       :           BP         -> PI
    genpoly      :         (Z,PI)       -> BP
    negShiftz    :         (Z,PI)       -> Z
    internal     :     (Cases,List BP ) -> List BP
    constcase    : (List NNI ,List BP ) -> List BP
    lincase      : (List NNI ,List BP ) -> List BP
    myNextPrime  :        ( Z , NNI )   -> Z

    bigPrime:= prevPrime(2**26)$IntegerPrimesPackage(Integer)

    myNextPrime(val:Z,bound:NNI) : Z == nextPrime(val)$IntegerPrimesPackage(Z)

    constNotZero(f : BP ) : Boolean == (degree f = 0) and ^(zero? f)

    negShiftz(n:Z,Modulus:PI):Z ==
      n < 0 => n:= n+Modulus
      n > (Modulus quo 2) => n-Modulus
      n

    --compute the height of a polynomial
    height(f:BP):PI ==
      k:PI:=1
      while f^=0 repeat
           k:=max(k,abs(leadingCoefficient(f)@Z)::PI)
```

```
        f:=reductum f
  k

--reconstruct the polynomial from the value-adic representation of
--dval.
genpoly(dval:Z,value:PI):BP ==
  d:=0$BP
  val:=dval
  for i in 0..  while (val^=0) repeat
    val1:=negShiftz(val rem value,value)
    d:= d+monomial(val1,i)
    val:=(val-val1) quo value
  d

--gcd of a list of integers
lintgcd(lval:List(Z)):Z ==
  empty? lval => 0$Z
  member?(1,lval) => 1$Z
  lval:=sort((z1,z2) +-> z1<z2,lval)
  val:=lval.first
  for val1 in lval.rest while ^(val=1) repeat val:=gcd(val,val1)
  val

--content for a list of univariate polynomials
content(listf:List BP ):List(Z) ==
  [lintgcd coefficients f for f in listf]

--content of a list of polynomials with the relative primitive parts
contprim(listf:List BP ):List(ContPrim) ==
   [[c:=lintgcd coefficients f,(f exquo c)::BP]$ContPrim  for f in listf]

-- one polynomial is constant, remark that they are primitive
-- but listf can contain the zero polynomial
constcase(listdeg:List NNI ,listf:List BP ): List BP  ==
  lind:=select(constNotZero,listf)
  empty? lind =>
    member?(1,listdeg) => lincase(listdeg,listf)
    localgcd listf
  or/[n>0 for n in listdeg] => cons(1$BP,listf)
  lclistf:List(Z):= [leadingCoefficient f for f in listf]
  d:=lintgcd(lclistf)
  d=1 =>  cons(1$BP,listf)
  cons(d::BP,[(lcf quo d)::BP for lcf in lclistf])

testDivide(listf: List BP, g:BP):Union(List BP, "failed") ==
  result:List BP := []
```

```
  for f in listf repeat
    if (f1:=f exquo g) case "failed" then return "failed"
    result := cons(f1::BP,result)
  reverse!(result)

--one polynomial is linear, remark that they are primitive
lincase(listdeg:List NNI ,listf:List BP ):List BP  ==
  n:= position(1,listdeg)
  g:=listf.n
  result:=[g]
  for f in listf repeat
    if (f1:=f exquo g) case "failed" then return cons(1$BP,listf)
    result := cons(f1::BP,result)
  reverse(result)

IMG := InnerModularGcd(Z,BP,67108859,myNextPrime)

mindegpol(f:BP, g:BP):BP ==
  degree(g) < degree (f) => g
  f

--local function for the gcd among n PRIMITIVE univariate polynomials
localgcd(listf:List BP ):List BP  ==
  hgt:="min"/[height(f) for f in listf|^zero? f]
  answr:=2+2*hgt
  minf := "mindegpol"/[f for f in listf|^zero? f]
  (result := testDivide(listf, minf)) case List(BP) =>
       cons(minf, result::List BP)
  if degree minf < 100 then for k in 1..10 repeat
    listval:=[f answr for f in listf]
    dval:=lintgcd(listval)
    dd:=genpoly(dval,answr)
    contd:=content(dd)
    d:=(dd exquo contd)::BP
    result:List BP :=[d]
    flag : Boolean := true
    for f in listf while flag repeat
      (f1:=f exquo d) case "failed" => flag:=false
      result := cons (f1::BP,result)
    if flag then return reverse(result)
    nvalue:= answr*832040 quo 317811
    if ((nvalue + answr) rem 2) = 0 then nvalue:=nvalue+1
    answr:=nvalue::PI
  gg:=modularGcdPrimitive(listf)$IMG
  cons(gg,[(f exquo gg) :: BP for f in listf])
```

```
--internal function:it evaluates the gcd and avoids duplication of
--code.
internal(flag:Cases,listf:List BP ):List BP  ==
  --special cases
  listf=[] => [1$BP]
  (nlf:=#listf)=1 => [first listf,1$BP]
  minpol:=1$BP
  -- extract a monomial gcd
  mdeg:= "min"/[minimumDegree f for f in listf]
  if mdeg>0 then
    minpol1:= monomial(1,mdeg)
    listf:= [(f exquo minpol1)::BP for f in listf]
    minpol:=minpol*minpol1
  -- make the polynomials primitive
  Cgcd:List(Z):=[]
  contgcd : Z := 1
  if (flag case "gcd") or (flag case "gcdcofact") then
    contlistf:List(ContPrim):=contprim(listf)
    Cgcd:= [term.cont for term in contlistf]
    contgcd:=lintgcd(Cgcd)
    listf:List BP :=[term.prim for term in contlistf]
    minpol:=contgcd*minpol
  listdeg:=[degree f for f in listf ]
  f:= first listf
  if positiveRemainder(leadingCoefficient(f), bigPrime) ~= 0 then
    for g in rest listf repeat
      lcg := leadingCoefficient(g)
      if positiveRemainder(lcg, bigPrime) = 0 then
        leave
      f:=gcd(f,g,bigPrime)
      if degree f = 0 then return cons(minpol,listf)
  ans:List BP :=
     --one polynomial is constant
     member?(0,listdeg) => constcase(listdeg,listf)
     --one polynomial is linear
     member?(1,listdeg) => lincase(listdeg,listf)
     localgcd(listf)
  (result,ans):=(first ans*minpol,rest ans)
  if (flag case "gcdcofact") then
    ans:= [(p quo contgcd)*q for p in Cgcd for q in ans]
  cons(result,ans)

--gcd among n PRIMITIVE univariate polynomials
gcdprim (listf:List BP ):BP == first internal("gcdprim",listf)

--gcd and cofactors for n PRIMITIVE univariate polynomials
```

```
gcdcofactprim(listf:List BP ):List BP  == internal("gcdcofactprim",listf)

--gcd for n generic univariate polynomials.
gcd(listf:List BP ): BP  ==  first internal("gcd",listf)

--gcd and cofactors for n generic univariate polynomials.
gcdcofact (listf:List BP ):List BP == internal("gcdcofact",listf)
```

⟨*HEUGCD.dotabb*⟩≡

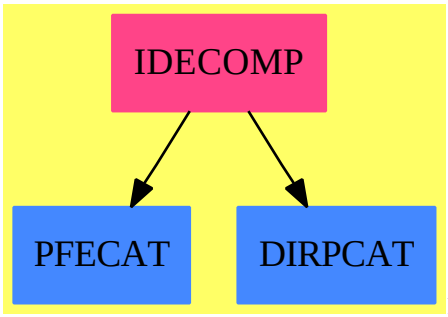
```
"HEUGCD" [color="#FF4488",href="bookvol10.4.pdf#nameddest=HEUGCD"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"HEUGCD" -> "PFECAT"
```

Chapter 10

Chapter I

10.1 package IDECOMP IdealDecomposition-Package

10.2 IdealDecompositionPackage



1.00

Exports:
contract primaryDecomp prime? radical zeroDimPrimary? zeroDimPrime?

⟨*package IDECOMP IdealDecompositionPackage*⟩≡
```
)abbrev package IDECOMP IdealDecompositionPackage
++ Author: P. Gianni
++ Date Created: summer 1986
++ Date Last Updated:
++ Basic Functions:
++ Related Constructors: PolynomialIdeals
++ Also See:
++ AMS Classifications:
++ Keywords:
```

```
++ References:
++ Description:
++   This package provides functions for the primary decomposition of
++ polynomial ideals over the rational numbers. The ideals are members
++ of the \spadtype{PolynomialIdeals} domain, and the polynomial generators are
++ required to be from the \spadtype{DistributedMultivariatePolynomial} domain.

IdealDecompositionPackage(vl,nv) : C == T -- take away nv, now doesn't
                                          -- compile if it isn't there
 where
   vl      :  List Symbol
   nv      :  NonNegativeInteger
   Z       ==>  Integer  -- substitute with PFE cat
   Q       ==>  Fraction Z
   F       ==>  Fraction P
   P       ==>  Polynomial Z
   UP      ==>  SparseUnivariatePolynomial P
   Expon   ==>  DirectProduct(nv,NNI)
   OV      ==>  OrderedVariableList(vl)
   SE      ==>  Symbol
   SUP     ==>  SparseUnivariatePolynomial(DPoly)

   DPoly1 ==>  DistributedMultivariatePolynomial(vl,Q)
   DPoly  ==>  DistributedMultivariatePolynomial(vl,F)
   NNI    ==>  NonNegativeInteger

   Ideal  ==  PolynomialIdeals(Q,Expon,OV,DPoly1)
   FIdeal ==  PolynomialIdeals(F,Expon,OV,DPoly)
   Fun0   ==  Union("zeroPrimDecomp","zeroRadComp")
   GenPos ==  Record(changeval:List Z,genideal:FIdeal)

   C == with


     zeroDimPrime?       :        Ideal         -> Boolean
       ++ zeroDimPrime?(I) tests if the ideal I is a 0-dimensional prime.

     zeroDimPrimary?     :        Ideal         -> Boolean
       ++ zeroDimPrimary?(I) tests if the ideal I is 0-dimensional primary.
     prime?              :        Ideal         -> Boolean
       ++ prime?(I) tests if the ideal I is prime.
     radical             :        Ideal         -> Ideal
       ++ radical(I) returns the radical of the ideal I.
     primaryDecomp       :        Ideal         -> List(Ideal)
       ++ primaryDecomp(I) returns a list of primary ideals such that their
       ++ intersection is the ideal I.
```

```
  contract        : (Ideal,List OV   )       -> Ideal
    ++ contract(I,lvar) contracts the ideal I to the polynomial ring
    ++ \spad{F[lvar]}.

T  == add

  import MPolyCatRationalFunctionFactorizer(Expon,OV,Z,DPoly)
  import GroebnerPackage(F,Expon,OV,DPoly)
  import GroebnerPackage(Q,Expon,OV,DPoly1)

              ----  Local  Functions  -----
  genPosLastVar       :    (FIdeal,List OV)     -> GenPos
  zeroPrimDecomp      :    (FIdeal,List OV)     -> List(FIdeal)
  zeroRadComp         :    (FIdeal,List OV)     -> FIdeal
  zerodimcase         :    (FIdeal,List OV)     -> Boolean
  is0dimprimary       :    (FIdeal,List OV)     -> Boolean
  backGenPos          : (FIdeal,List Z,List OV) -> FIdeal
  reduceDim           : (Fun0,FIdeal,List OV)   -> List FIdeal
  findvar             :   (FIdeal,List OV)      -> OV
  testPower           :    (SUP,OV,FIdeal)      -> Boolean
  goodPower           :     (DPoly,FIdeal)  -> Record(spol:DPoly,id:FIdeal)
  pushdown            :      (DPoly,OV)        -> DPoly
  pushdterm           :     (DPoly,OV,Z)       -> DPoly
  pushup              :      (DPoly,OV)        -> DPoly
  pushuterm           :    (DPoly,SE,OV)       -> DPoly
  pushucoef           :       (UP,OV)          -> DPoly
  trueden             :        (P,SE)          -> P
  rearrange           :       (List OV)        -> List OV
  deleteunit          :      List FIdeal        -> List FIdeal
  ismonic             :      (DPoly,OV)        -> Boolean


  MPCFQF ==> MPolyCatFunctions2(OV,Expon,Expon,Q,F,DPoly1,DPoly)
  MPCFFQ ==> MPolyCatFunctions2(OV,Expon,Expon,F,Q,DPoly,DPoly1)

  convertQF(a:Q) : F == ((numer a):: F)/((denom a)::F)
  convertFQ(a:F) : Q == (ground numer a)/(ground denom a)

  internalForm(I:Ideal) : FIdeal ==
    Id:=generators I
    nId:=[map(convertQF,poly)$MPCFQF for poly in Id]
    groebner? I => groebnerIdeal nId
    ideal nId

  externalForm(I:FIdeal) : Ideal ==
```

```
  Id:=generators I
  nId:=[map(convertFQ,poly)$MPCFFQ for poly in Id]
  groebner? I => groebnerIdeal nId
  ideal nId

lvint:=[variable(xx)::OV for xx in vl]
nvint1:=(#lvint-1)::NNI

deleteunit(lI: List FIdeal) : List FIdeal ==
  [I for I in lI | _^ element?(1$DPoly,I)]

rearrange(vlist:List OV) :List OV ==
  vlist=[] => vlist
  sort((z1,z2)+->z1>z2,setDifference(lvint,setDifference(lvint,vlist)))

      ---- radical of a 0-dimensional ideal  ----
zeroRadComp(I:FIdeal,truelist:List OV) : FIdeal ==
  truelist=[] => I
  Id:=generators I
  x:OV:=truelist.last
  #Id=1 =>
    f:=Id.first
    g:= (f exquo (gcd (f,differentiate(f,x))))::DPoly
    groebnerIdeal([g])
  y:=truelist.first
  px:DPoly:=x::DPoly
  py:DPoly:=y::DPoly
  f:=Id.last
  g:= (f exquo (gcd (f,differentiate(f,x))))::DPoly
  Id:=groebner(cons(g,remove(f,Id)))
  lf:=Id.first
  pv:DPoly:=0
  pw:DPoly:=0
  while degree(lf,y)^=1 repeat
    val:=random()$Z rem 23
    pv:=px+val*py
    pw:=px-val*py
    Id:=groebner([(univariate(h,x)).pv for h in Id])
    lf:=Id.first
  ris:= generators(zeroRadComp(groebnerIdeal(Id.rest),truelist.rest))
  ris:=cons(lf,ris)
  if pv^=0 then
    ris:=[(univariate(h,x)).pw for h in ris]
  groebnerIdeal(groebner ris)

     ----  find the power that stabilizes (I:s)  ----
```

```
goodPower(s:DPoly,I:FIdeal) : Record(spol:DPoly,id:FIdeal) ==
  f:DPoly:=s
  I:=groebner I
  J:=generators(JJ:= (saturate(I,s)))
  while _^ in?(ideal([f*g for g in J]),I) repeat f:=s*f
  [f,JJ]

        ----  is the ideal zerodimensional?  ----
  ----     the "true variables" are  in truelist         ----
zerodimcase(J:FIdeal,truelist:List OV) : Boolean ==
  element?(1,J) => true
  truelist=[] => true
  n:=#truelist
  Jd:=groebner generators J
  for x in truelist while Jd^=[] repeat
    f := Jd.first
    Jd:=Jd.rest
    if ((y:=mainVariable f) case "failed") or (y::OV ^=x )
         or _^ (ismonic (f,x)) then return false
    while Jd^=[] and (mainVariable Jd.first)::OV=x repeat Jd:=Jd.rest
    if Jd=[] and position(x,truelist)<n then return false
  true

    ----  choose the variable for the reduction step  ----
              --- J groebnerner in gen pos  ---
findvar(J:FIdeal,truelist:List OV) : OV ==
  lmonicvar:List OV :=[]
  for f in generators J repeat
    t:=f - reductum f
    vt:List OV :=variables t
    if #vt=1 then lmonicvar:=setUnion(vt,lmonicvar)
  badvar:=setDifference(truelist,lmonicvar)
  badvar.first

       ---- function for the "reduction step  ----
reduceDim(flag:Fun0,J:FIdeal,truelist:List OV) : List(FIdeal) ==
  element?(1,J) => [J]
  zerodimcase(J,truelist) =>
    (flag case "zeroPrimDecomp") => zeroPrimDecomp(J,truelist)
    (flag case "zeroRadComp") => [zeroRadComp(J,truelist)]
  x:OV:=findvar(J,truelist)
  Jnew:=[pushdown(f,x) for f in generators J]
  Jc: List FIdeal :=[]
  Jc:=reduceDim(flag,groebnerIdeal Jnew,remove(x,truelist))
  res1:=[ideal([pushup(f,x) for f in generators idp]) for idp in Jc]
  s:=pushup((_*/[leadingCoefficient f for f in Jnew])::DPoly,x)
```

```
        degree(s,x)=0 => res1
        res1:=[saturate(II,s) for II in res1]
        good:=goodPower(s,J)
        sideal := groebnerIdeal(groebner(cons(good.spol,generators J)))
        in?(good.id, sideal) => res1
        sresult:=reduceDim(flag,sideal,truelist)
        for JJ in sresult repeat
           if not(in?(good.id,JJ)) then res1:=cons(JJ,res1)
        res1

       ----  Primary Decomposition for 0-dimensional ideals  ----
      zeroPrimDecomp(I:FIdeal,truelist:List OV): List(FIdeal) ==
        truelist=[] => list I
        newJ:=genPosLastVar(I,truelist);lval:=newJ.changeval;
        J:=groebner newJ.genideal
        x:=truelist.last
        Jd:=generators J
        g:=Jd.last
        lfact:= factors factor(g)
        ris:List FIdeal:=[]
        for ef in lfact repeat
          g:DPoly:=(ef.factor)**(ef.exponent::NNI)
          J1:= groebnerIdeal(groebner cons(g,Jd))
          if _^ (isOdimprimary (J1,truelist)) then
                                  return zeroPrimDecomp(I,truelist)
          ris:=cons(groebner backGenPos(J1,lval,truelist),ris)
        ris

              ----  radical of an Ideal  ----
      radical(I:Ideal) : Ideal ==
        J:=groebner(internalForm I)
        truelist:=rearrange("setUnion"/[variables f for f in generators J])
        truelist=[] => externalForm J
        externalForm("intersect"/reduceDim("zeroRadComp",J,truelist))


   -- the following functions are used to "push" x in the coefficient ring -

         ----  push x in the coefficient domain for a polynomial ----
      pushdown(g:DPoly,x:OV) : DPoly ==
        rf:DPoly:=0$DPoly
        i:=position(x,lvint)
        while g^=0 repeat
          g1:=reductum g
          rf:=rf+pushdterm(g-g1,x,i)
          g := g1
```

```
  rf

 ----  push x in the coefficient domain for a term ----
pushdterm(t:DPoly,x:OV,i:Z):DPoly ==
  n:=degree(t,x)
  xp:=convert(x)@SE
  cf:=monomial(1,xp,n)$P :: F
  newt := t exquo monomial(1,x,n)$DPoly
  cf * newt::DPoly

          ----  push back the variable  ----
pushup(f:DPoly,x:OV) :DPoly ==
  h:=1$P
  rf:DPoly:=0$DPoly
  g := f
  xp := convert(x)@SE
  while g^=0 repeat
    h:=lcm(trueden(denom leadingCoefficient g,xp),h)
    g:=reductum g
  f:=(h::F)*f
  while f^=0 repeat
    g:=reductum f
    rf:=rf+pushuterm(f-g,xp,x)
    f:=g
  rf

trueden(c:P,x:SE) : P ==
  degree(c,x) = 0 => 1
  c

 ----  push x back from the coefficient domain for a term ----
pushuterm(t:DPoly,xp:SE,x:OV):DPoly ==
  pushucoef((univariate(numer leadingCoefficient t,xp)$P), x)*
     monomial(inv((denom leadingCoefficient t)::F),degree t)$DPoly


pushucoef(c:UP,x:OV):DPoly ==
  c = 0 => 0
  monomial((leadingCoefficient c)::F::DPoly,x,degree c) +
            pushucoef(reductum c,x)

      -- is the 0-dimensional ideal I primary ?  --
          ----  internal function  ----
is0dimprimary(J:FIdeal,truelist:List OV) : Boolean ==
  element?(1,J) => true
  Jd:=generators(groebner J)
```

```
  #(factors factor Jd.last)^=1 => return false
  i:=subtractIfCan(#truelist,1)
  (i case "failed") => return true
  JR:=(reverse Jd);JM:=groebnerIdeal([JR.first]);JP:List(DPoly):=[]
  for f in JR.rest repeat
    if _^ ismonic(f,truelist.i) then
      if _^ inRadical?(f,JM) then return false
      JP:=cons(f,JP)
     else
      x:=truelist.i
      i:=(i-1)::NNI
      if _^ testPower(univariate(f,x),x,JM) then return false
      JM :=groebnerIdeal(append(cons(f,JP),generators JM))
  true

    ---- Functions for the General Position step  ----

  ----  put the ideal in general position  ----
genPosLastVar(J:FIdeal,truelist:List OV):GenPos ==
  x := last truelist ;lv1:List OV :=remove(x,truelist)
  ranvals:List(Z):=[(random()$Z rem 23) for vv in lv1]
  val:=_+/[rv*(vv::DPoly)  for vv in lv1 for rv in ranvals]
  val:=val+(x::DPoly)
  [ranvals,groebnerIdeal(groebner([(univariate(p,x)).val
                        for p in generators J]))]$GenPos


        ----  convert back the ideal  ----
backGenPos(I:FIdeal,lval:List Z,truelist:List OV) : FIdeal ==
  lval=[] => I
  x := last truelist ;lv1:List OV:=remove(x,truelist)
  val:=-(_+/[rv*(vv::DPoly) for vv in lv1 for rv in lval])
  val:=val+(x::DPoly)
  groebnerIdeal
      (groebner([(univariate(p,x)).val for p in generators I ]))

ismonic(f:DPoly,x:OV) : Boolean == ground? leadingCoefficient(univariate(f,x

    ---- test if f is power of a linear mod (rad J) ----
              ----  f is monic  ----
testPower(uf:SUP,x:OV,J:FIdeal) : Boolean ==
  df:=degree(uf)
  trailp:DPoly := inv(df:Z ::F) *coefficient(uf,(df-1)::NNI)
  linp:SUP:=(monomial(1$DPoly,1$NNI)$SUP +
            monomial(trailp,0$NNI)$SUP)**df
  g:DPoly:=multivariate(uf-linp,x)
```

```
  inRadical?(g,J)


              ----  Exported Functions  ----

      -- is the 0-dimensional ideal I prime ?  --
zeroDimPrime?(I:Ideal) : Boolean ==
  J:=groebner((genPosLastVar(internalForm I,lvint)).genideal)
  element?(1,J) => true
  n:NNI:=#vl;i:NNI:=1
  Jd:=generators J
  #Jd^=n => false
  for f in Jd repeat
    if _^ ismonic(f,lvint.i) then return false
    if i<n and (degree univariate(f,lvint.i))^=1 then return false
    i:=i+1
  g:=Jd.n
  #(lfact:=factors(factor g)) >1 => false
  lfact.1.exponent =1


      -- is the 0-dimensional ideal I primary ?  --
zeroDimPrimary?(J:Ideal):Boolean ==
  is0dimprimary(internalForm J,lvint)

        ----  Primary Decomposition of I  -----

primaryDecomp(I:Ideal) : List(Ideal) ==
  J:=groebner(internalForm I)
  truelist:=rearrange("setUnion"/[variables f for f in generators J])
  truelist=[] => [externalForm J]
  [externalForm II for II in reduceDim("zeroPrimDecomp",J,truelist)]

     ----  contract I to the ring with lvar variables  ----
contract(I:Ideal,lvar: List OV) : Ideal ==
  Id:= generators(groebner I)
  empty?(Id) => I
  fullVars:= "setUnion"/[variables g for g in Id]
  fullVars = lvar => I
  n:= # lvar
  #fullVars < n  => error "wrong vars"
  n=0 => I
  newVars:= append([vv for vv in fullVars| ^member?(vv,lvar)]$List(OV),lvar)
  subsVars := [monomial(1,vv,1)$DPoly1 for vv in newVars]
  lJ:= [eval(g,fullVars,subsVars) for g in Id]
  J := groebner(lJ)
```

```
        J=[1] => groebnerIdeal J
        J=[0] => groebnerIdeal empty()
        J:=[f for f in J| member?(mainVariable(f)::OV,newVars)]
        fullPol :=[monomial(1,vv,1)$DPoly1 for vv in fullVars]
        groebnerIdeal([eval(gg,newVars,fullPol) for gg in J])
```

⟨*IDECOMP.dotabb*⟩≡

```
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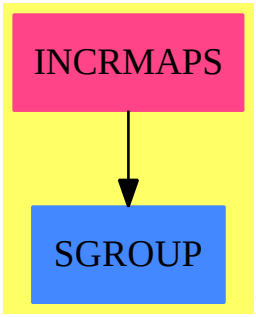
  "PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
  "DIRPCAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=DIRPCAT"]
  "IDECOMP" -> "PFECAT"
  "IDECOMP" -> "DIRPCAT"
```

10.3 package INCRMAPS IncrementingMaps

10.4 IncrementingMaps



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Exports:
increment incrementBy

⟨*package INCRMAPS IncrementingMaps*⟩≡
```
)abbrev package INCRMAPS IncrementingMaps
++ Author:
++ Date Created:
++ Date Last Updated: June 4, 1991
++ Basic Operations:
++ Related Domains: UniversalSegment
++ Also See:
++ AMS Classifications:
++ Keywords: equation
++ Examples:
++ References:
++ Description:
++   This package provides operations to create incrementing functions.

IncrementingMaps(R:Join(Monoid, AbelianSemiGroup)): with
    increment: () -> (R -> R)
        ++ increment() produces a function which adds \spad{1} to whatever
        ++ argument it is given.  For example, if {f := increment()} then
        ++ \spad{f x} is \spad{x+1}.
    incrementBy: R -> (R -> R)
        ++ incrementBy(n) produces a function which adds \spad{n} to whatever
        ++ argument it is given.  For example, if {f := increment(n)} then
        ++ \spad{f x} is \spad{x+n}.
  == add
    increment()   == x +-> 1 + x
    incrementBy n == x +-> n + x
```

⟨*INCRMAPS.dotabb*⟩≡

```
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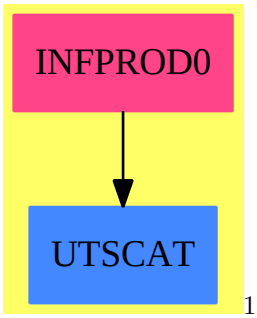
"SGROUP" [color="#4488FF",href="bookvol10.2.pdf#nameddest=SGROUP"]
"INCRMAPS" -> "SGROUP"
```

10.5 package INFPROD0 InfiniteProductCharacteristicZero

10.6 InfiniteProductCharacteristicZero



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Exports:
evenInfiniteProduct infiniteProduct oddInfiniteProduct generalInfiniteProduct

⟨*package INFPROD0 InfiniteProductCharacteristicZero*⟩≡

```
)abbrev package INFPROD0 InfiniteProductCharacteristicZero
++ Author: Clifton J. Williamson
++ Date Created: 22 February 1990
++ Date Last Updated: 23 February 1990
++ Basic Operations: infiniteProduct, evenInfiniteProduct, oddInfiniteProduct,
++   generalInfiniteProduct
++ Related Domains: UnivariateTaylorSeriesCategory
++ Also See:
++ AMS Classifications:
++ Keywords: Taylor series, infinite product
++ Examples:
++ References:
++ Description:
++   This package computes infinite products of univariate Taylor series
++   over an integral domain of characteristic 0.
InfiniteProductCharacteristicZero(Coef,UTS):_
 Exports == Implementation where
  Coef : Join(IntegralDomain,CharacteristicZero)
  UTS  : UnivariateTaylorSeriesCategory Coef
  I  ==> Integer

  Exports ==> with

    infiniteProduct: UTS -> UTS
      ++ infiniteProduct(f(x)) computes \spad{product(n=1,2,3...,f(x**n))}.
      ++ The series \spad{f(x)} should have constant coefficient 1.
```

```
    evenInfiniteProduct: UTS -> UTS
      ++ evenInfiniteProduct(f(x)) computes \spad{product(n=2,4,6...,f(x**n))}.
      ++ The series \spad{f(x)} should have constant coefficient 1.
    oddInfiniteProduct: UTS -> UTS
      ++ oddInfiniteProduct(f(x)) computes \spad{product(n=1,3,5...,f(x**n))}.
      ++ The series \spad{f(x)} should have constant coefficient 1.
    generalInfiniteProduct: (UTS,I,I) -> UTS
      ++ generalInfiniteProduct(f(x),a,d) computes
      ++ \spad{product(n=a,a+d,a+2*d,...,f(x**n))}.
      ++ The series \spad{f(x)} should have constant coefficient 1.

  Implementation ==> add

    import StreamInfiniteProduct Coef

    infiniteProduct x     == series infiniteProduct coefficients x
    evenInfiniteProduct x == series evenInfiniteProduct coefficients x
    oddInfiniteProduct x  == series oddInfiniteProduct coefficients x

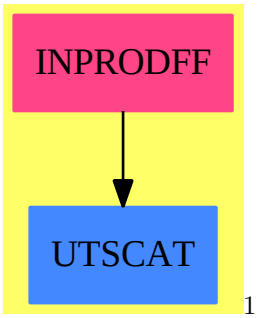
    generalInfiniteProduct(x,a,d) ==
      series generalInfiniteProduct(coefficients x,a,d)
```

⟨*INFPROD0.dotabb*⟩≡

```
"INFPROD0" [color="#FF4488",href="bookvol10.4.pdf#nameddest=INFPROD0"]
"UTSCAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=UTSCAT"]
"INFPROD0" -> "UTSCAT"
```

10.7 package INPRODFF InfiniteProductFiniteField

10.8 InfiniteProductFiniteField



1.00

Exports:
evenInfiniteProduct generalInfiniteProduct infiniteProduct oddInfiniteProduct

⟨*package INPRODFF InfiniteProductFiniteField*⟩≡

```
)abbrev package INPRODFF InfiniteProductFiniteField
++ Author: Clifton J. Williamson
++ Date Created: 22 February 1990
++ Date Last Updated: 23 February 1990
++ Basic Operations: infiniteProduct, evenInfiniteProduct, oddInfiniteProduct,
++   generalInfiniteProduct
++ Related Domains: UnivariateTaylorSeriesCategory
++ Also See:
++ AMS Classifications:
++ Keywords: Taylor series, infinite product
++ Examples:
++ References:
++ Description:
++   This package computes infinite products of univariate Taylor series
++   over an arbitrary finite field.
InfiniteProductFiniteField(K,UP,Coef,UTS):_
 Exports == Implementation where
  K    :  Join(Field,Finite,ConvertibleTo Integer)
  UP   :  UnivariatePolynomialCategory K
  Coef :  MonogenicAlgebra(K,UP)
  UTS  :  UnivariateTaylorSeriesCategory Coef
  I   ==> Integer
  RN  ==> Fraction Integer
  SAE ==> SimpleAlgebraicExtension
  ST  ==> Stream
  STF ==> StreamTranscendentalFunctions
```

```
STT ==> StreamTaylorSeriesOperations
ST2 ==> StreamFunctions2
SUP ==> SparseUnivariatePolynomial

Exports ==> with

  infiniteProduct: UTS -> UTS
    ++ infiniteProduct(f(x)) computes \spad{product(n=1,2,3...,f(x**n))}.
    ++ The series \spad{f(x)} should have constant coefficient 1.
  evenInfiniteProduct: UTS -> UTS
    ++ evenInfiniteProduct(f(x)) computes \spad{product(n=2,4,6...,f(x**n))}.
    ++ The series \spad{f(x)} should have constant coefficient 1.
  oddInfiniteProduct: UTS -> UTS
    ++ oddInfiniteProduct(f(x)) computes \spad{product(n=1,3,5...,f(x**n))}.
    ++ The series \spad{f(x)} should have constant coefficient 1.
  generalInfiniteProduct: (UTS,I,I) -> UTS
    ++ generalInfiniteProduct(f(x),a,d) computes
    ++ \spad{product(n=a,a+d,a+2*d,...,f(x**n))}.
    ++ The series \spad{f(x)} should have constant coefficient 1.

Implementation ==> add

  liftPoly: UP -> SUP RN
  liftPoly poly ==
    -- lift coefficients of 'poly' to integers
    ans : SUP RN := 0
    while not zero? poly repeat
      coef := convert(leadingCoefficient poly)@I :: RN
      ans := ans + monomial(coef,degree poly)
      poly := reductum poly
    ans

  reducePoly: SUP RN -> UP
  reducePoly poly ==
    -- reduce coefficients of 'poly' to elements of K
    ans : UP := 0
    while not zero? poly repeat
      coef := numer(leadingCoefficient(poly)) :: K
      ans := ans + monomial(coef,degree poly)
      poly := reductum poly
    ans

  POLY := liftPoly definingPolynomial()$Coef
  ALG  := SAE(RN,SUP RN,POLY)

  infiniteProduct x ==
```

```
      stUP := map(lift,coefficients x)$ST2(Coef,UP)
      stSUP := map(liftPoly,stUP)$ST2(UP,SUP RN)
      stALG := map(reduce,stSUP)$ST2(SUP RN,ALG)
      stALG := exp(lambert(log(stALG)$STF(ALG))$STT(ALG))$STF(ALG)
      stSUP := map(lift,stALG)$ST2(ALG,SUP RN)
      stUP := map(reducePoly,stSUP)$ST2(SUP RN,UP)
      series map(reduce,stUP)$ST2(UP,Coef)

    evenInfiniteProduct x ==
      stUP := map(lift,coefficients x)$ST2(Coef,UP)
      stSUP := map(liftPoly,stUP)$ST2(UP,SUP RN)
      stALG := map(reduce,stSUP)$ST2(SUP RN,ALG)
      stALG := exp(evenlambert(log(stALG)$STF(ALG))$STT(ALG))$STF(ALG)
      stSUP := map(lift,stALG)$ST2(ALG,SUP RN)
      stUP := map(reducePoly,stSUP)$ST2(SUP RN,UP)
      series map(reduce,stUP)$ST2(UP,Coef)

    oddInfiniteProduct x ==
      stUP := map(lift,coefficients x)$ST2(Coef,UP)
      stSUP := map(liftPoly,stUP)$ST2(UP,SUP RN)
      stALG := map(reduce,stSUP)$ST2(SUP RN,ALG)
      stALG := exp(oddlambert(log(stALG)$STF(ALG))$STT(ALG))$STF(ALG)
      stSUP := map(lift,stALG)$ST2(ALG,SUP RN)
      stUP := map(reducePoly,stSUP)$ST2(SUP RN,UP)
      series map(reduce,stUP)$ST2(UP,Coef)

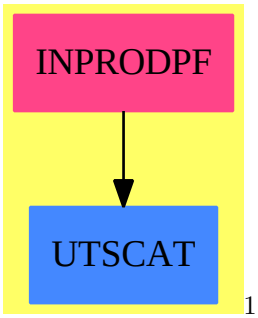
    generalInfiniteProduct(x,a,d) ==
      stUP := map(lift,coefficients x)$ST2(Coef,UP)
      stSUP := map(liftPoly,stUP)$ST2(UP,SUP RN)
      stALG := map(reduce,stSUP)$ST2(SUP RN,ALG)
      stALG := generalLambert(log(stALG)$STF(ALG),a,d)$STT(ALG)
      stALG := exp(stALG)$STF(ALG)
      stSUP := map(lift,stALG)$ST2(ALG,SUP RN)
      stUP := map(reducePoly,stSUP)$ST2(SUP RN,UP)
      series map(reduce,stUP)$ST2(UP,Coef)
```

⟨*INPRODFF.dotabb*⟩≡

```
"INPRODFF" [color="#FF4488",href="bookvol10.4.pdf#nameddest=INPRODFF"]
"UTSCAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=UTSCAT"]
"INPRODFF" -> "UTSCAT"
```

10.9 package INPRODPF InfiniteProduct-PrimeField

10.10 InfiniteProductPrimeField



1.00

Exports:
evenInfiniteProduct generalInfiniteProduct infiniteProduct oddInfiniteProduct

⟨*package INPRODPF InfiniteProductPrimeField*⟩≡

```
)abbrev package INPRODPF InfiniteProductPrimeField
++ Author: Clifton J. Williamson
++ Date Created: 22 February 1990
++ Date Last Updated: 23 February 1990
++ Basic Operations: infiniteProduct, evenInfiniteProduct, oddInfiniteProduct,
++   generalInfiniteProduct
++ Related Domains: UnivariateTaylorSeriesCategory
++ Also See:
++ AMS Classifications:
++ Keywords: Taylor series, infinite product
++ Examples:
++ References:
++ Description:
++    This package computes infinite products of univariate Taylor series
++    over a field of prime order.
InfiniteProductPrimeField(Coef,UTS): Exports == Implementation where
  Coef : Join(Field,Finite,ConvertibleTo Integer)
  UTS  : UnivariateTaylorSeriesCategory Coef
  I  ==> Integer
  ST ==> Stream

  Exports ==> with

    infiniteProduct: UTS -> UTS
      ++ infiniteProduct(f(x)) computes \spad{product(n=1,2,3...,f(x**n))}.
      ++ The series \spad{f(x)} should have constant coefficient 1.
```

```
    evenInfiniteProduct: UTS -> UTS
      ++ evenInfiniteProduct(f(x)) computes \spad{product(n=2,4,6...,f(x**n))}.
      ++ The series \spad{f(x)} should have constant coefficient 1.
    oddInfiniteProduct: UTS -> UTS
      ++ oddInfiniteProduct(f(x)) computes \spad{product(n=1,3,5...,f(x**n))}.
      ++ The series \spad{f(x)} should have constant coefficient 1.
    generalInfiniteProduct: (UTS,I,I) -> UTS
      ++ generalInfiniteProduct(f(x),a,d) computes
      ++ \spad{product(n=a,a+d,a+2*d,...,f(x**n))}.
      ++ The series \spad{f(x)} should have constant coefficient 1.

  Implementation ==> add

    import StreamInfiniteProduct Integer

    applyOverZ:(ST I -> ST I,ST Coef) -> ST Coef
    applyOverZ(f,st) ==
      stZ := map(z1 +-> convert(z1)@Integer,st)$StreamFunctions2(Coef,I)
      map(z1 +-> z1 :: Coef,f stZ)$StreamFunctions2(I,Coef)

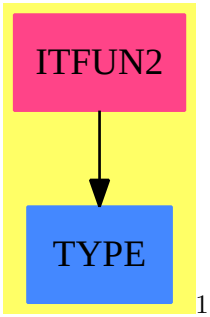
    infiniteProduct x ==
      series applyOverZ(infiniteProduct,coefficients x)
    evenInfiniteProduct x ==
      series applyOverZ(evenInfiniteProduct,coefficients x)
    oddInfiniteProduct x ==
      series applyOverZ(oddInfiniteProduct,coefficients x)
    generalInfiniteProduct(x,a,d) ==
      series
       applyOverZ(
        (z1:ST(I)):ST(I) +-> generalInfiniteProduct(z1,a,d),coefficients x)
```

⟨*INPRODPF.dotabb*⟩≡

```
"INPRODPF" [color="#FF4488",href="bookvol10.4.pdf#nameddest=INPRODPF"]
"UTSCAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=UTSCAT"]
"INPRODPF" -> "UTSCAT"
```

10.11 package ITFUN2 InfiniteTupleFunctions2

10.12 InfiniteTupleFunctions2



1.00

Exports:
map

⟨*package ITFUN2 InfiniteTupleFunctions2*⟩≡
```
)abbrev package ITFUN2 InfiniteTupleFunctions2
InfiniteTupleFunctions2(A:Type,B:Type): Exports == Implementation where
  ++ Functions defined on streams with entries in two sets.
  IT   ==> InfiniteTuple

  Exports ==> with
    map: ((A -> B),IT A) -> IT B
      ++ \spad{map(f,[x0,x1,x2,...])} returns \spad{[f(x0),f(x1),f(x2),..]}.

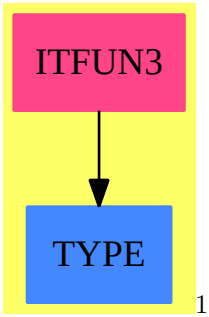
  Implementation ==> add

    map(f,x) ==
      map(f,x pretend Stream(A))$StreamFunctions2(A,B) pretend IT(B)
```

⟨*ITFUN2.dotabb*⟩≡
```
"ITFUN2" [color="#FF4488",href="bookvol10.4.pdf#nameddest=ITFUN2"]
"TYPE" [color="#4488FF",href="bookvol10.2.pdf#nameddest=TYPE"]
"ITFUN2" -> "TYPE"
```

10.13 package ITFUN3 InfiniteTupleFunctions3

10.14 InfiniteTupleFunctions3



1.00

Exports:
map

⟨*package ITFUN3 InfiniteTupleFunctions3*⟩≡

```
)abbrev package ITFUN3 InfiniteTupleFunctions3
InfiniteTupleFunctions3(A:Type, B:Type,C:Type): Exports
 == Implementation where
   ++ Functions defined on streams with entries in two sets.
   IT   ==> InfiniteTuple
   ST   ==> Stream
   SF3  ==> StreamFunctions3(A,B,C)
   FUN  ==> ((A,B)->C)
   Exports ==> with
     map: (((A,B)->C), IT A, IT B) -> IT C
        ++ map(f,a,b) \undocumented
     map: (((A,B)->C), ST A, IT B) -> ST C
        ++ map(f,a,b) \undocumented
     map: (((A,B)->C), IT A, ST B) -> ST C
        ++ map(f,a,b) \undocumented

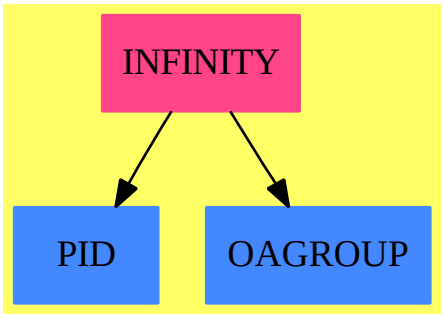
   Implementation ==> add

     map(f:FUN, s1:IT A, s2:IT B):IT C ==
       map(f, s1 pretend Stream(A), s2 pretend Stream(B))$SF3 pretend IT(C)
     map(f:FUN, s1:ST A, s2:IT B):ST C ==
       map(f, s1, s2 pretend Stream(B))$SF3
     map(f:FUN, s1:IT A, s2:ST B):ST C ==
       map(f, s1 pretend Stream(A), s2)$SF3
```

⟨*ITFUN3.dotabb*⟩≡
```
"ITFUN3" [color="#FF4488",href="bookvol10.4.pdf#nameddest=ITFUN3"]
"TYPE" [color="#4488FF",href="bookvol10.2.pdf#nameddest=TYPE"]
"ITFUN3" -> "TYPE"
```

10.15 package INFINITY Infinity

10.16 Infinity



1.00

Exports:
infinity minusInfinity plusInfinity

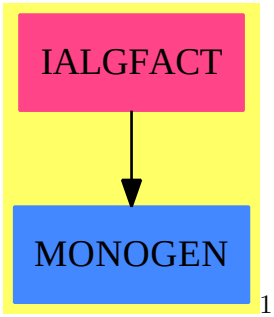
⟨*package INFINITY Infinity*⟩≡
```
)abbrev package INFINITY Infinity
++ Top-level infinity
++ Author: Manuel Bronstein
++ Description: Default infinity signatures for the interpreter;
++ Date Created: 4 Oct 1989
++ Date Last Updated: 4 Oct 1989
Infinity(): with
  infinity     : () -> OnePointCompletion Integer
    ++ infinity() returns infinity.
  plusInfinity : () -> OrderedCompletion  Integer
    ++ plusInfinity() returns plusIinfinity.
  minusInfinity: () -> OrderedCompletion  Integer
    ++ minusInfinity() returns minusInfinity.
 == add
  infinity()      == infinity()$OnePointCompletion(Integer)
  plusInfinity()  == plusInfinity()$OrderedCompletion(Integer)
  minusInfinity() == minusInfinity()$OrderedCompletion(Integer)
```

⟨*INFINITY.dotabb*⟩≡

```
"INFINITY" [color="#FF4488",href="bookvol10.4.pdf#nameddest=INFINITY"]
"PID" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PID"]
"OAGROUP" [color="#4488FF",href="bookvol10.2.pdf#nameddest=OAGROUP"]
"INFINITY" -> "PID"
"INFINITY" -> "OAGROUP"
```

10.17 package IALGFACT InnerAlgFactor

10.18 InnerAlgFactor



1.00

Exports:
factor

⟨*package IALGFACT InnerAlgFactor*⟩≡
```
)abbrev package IALGFACT InnerAlgFactor
++ Factorisation in a simple algebraic extension
++ Author: Patrizia Gianni
++ Date Created: ???
++ Date Last Updated: 20 Jul 1988
++ Description:
++ Factorization of univariate polynomials with coefficients in an
++ algebraic extension of a field over which we can factor UP's;
++ Keywords: factorization, algebraic extension, univariate polynomial

InnerAlgFactor(F, UP, AlExt, AlPol): Exports == Implementation where
  F    : Field
  UP   : UnivariatePolynomialCategory F
  AlPol: UnivariatePolynomialCategory AlExt
  AlExt : Join(Field, CharacteristicZero, MonogenicAlgebra(F,UP))
  NUP   ==> SparseUnivariatePolynomial UP
  N     ==> NonNegativeInteger
  Z     ==> Integer
  FR    ==> Factored UP
  UPCF2 ==> UnivariatePolynomialCategoryFunctions2


  Exports ==> with
    factor: (AlPol, UP -> FR)  ->  Factored AlPol
      ++ factor(p, f) returns a prime factorisation of p;
      ++ f is a factorisation map for elements of UP;
```

```
Implementation ==> add
  pnorm        : AlPol -> UP
  convrt       : AlPol -> NUP
  change       : UP    -> AlPol
  perturbfactor: (AlPol, Z, UP -> FR) -> List AlPol
  irrfactor    : (AlPol, Z, UP -> FR) -> List AlPol


  perturbfactor(f, k, fact) ==
    pol   := monomial(1$AlExt,1)-
             monomial(reduce monomial(k::F,1)$UP ,0)
    newf  := elt(f, pol)
    lsols := irrfactor(newf, k, fact)
    pol   := monomial(1, 1) +
             monomial(reduce monomial(k::F,1)$UP,0)
    [elt(pp, pol) for pp in lsols]

  ---  factorize the square-free parts of f  ---
  irrfactor(f, k, fact) ==
    degree(f) =$N 1 => [f]
    newf := f
    nn   := pnorm f
    --newval:RN:=1
    --pert:=false
    --if ^ SqFr? nn then
    --  pert:=true
    --  newterm:=perturb(f)
    --  newf:=newterm.ppol
    --  newval:=newterm.pval
    --  nn:=newterm.nnorm
    listfact := factors fact nn
    #listfact =$N 1 =>
      first(listfact).exponent =$Z 1 => [f]
      perturbfactor(f, k + 1, fact)
    listerm:List(AlPol):= []
    for pelt in listfact repeat
      g    := gcd(change(pelt.factor), newf)
      newf := (newf exquo g)::AlPol
      listerm :=
        pelt.exponent =$Z 1 => cons(g, listerm)
        append(perturbfactor(g, k + 1, fact), listerm)
    listerm

  factor(f, fact) ==
    sqf := squareFree f
    unit(sqf) * _*/[_*/[primeFactor(pol, sqterm.exponent)
```

```
                          for pol in irrfactor(sqterm.factor, 0, fact)]
                                        for sqterm in factors sqf]

      p := definingPolynomial()$AlExt
      newp := map(x +-> x::UP, p)$UPCF2(F, UP, UP, NUP)

      pnorm  q == resultant(convrt q, newp)
      change q == map(coerce, q)$UPCF2(F,UP,AlExt,AlPol)

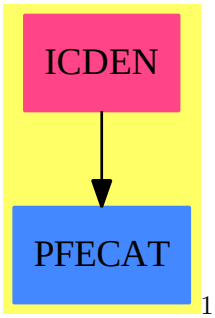
      convrt q ==
        swap(map(lift, q)$UPCF2(AlExt, AlPol,
             UP, NUP))$CommuteUnivariatePolynomialCategory(F, UP, NUP)
```

⟨*IALGFACT.dotabb*⟩≡

```
"IALGFACT" [color="#FF4488",href="bookvol10.4.pdf#nameddest=IALGFACT"]
"MONOGEN" [color="#4488FF",href="bookvol10.2.pdf#nameddest=MONOGEN"]
"IALGFACT" -> "MONOGEN"
```

10.19 package ICDEN InnerCommonDenominator

10.20 InnerCommonDenominator



1.00

Exports:
clearDenominator commonDenominator splitDenominator

⟨*package ICDEN InnerCommonDenominator*⟩≡

```
)abbrev package ICDEN InnerCommonDenominator
--% InnerCommonDenominator
++ Author: Manuel Bronstein
++ Date Created: 2 May 1988
++ Date Last Updated: 22 Nov 1989
++ Description: InnerCommonDenominator provides functions to compute
++ the common denominator of a finite linear aggregate of elements
++ of the quotient field of an integral domain.
++ Keywords: gcd, quotient, common, denominator.
InnerCommonDenominator(R, Q, A, B): Exports == Implementation where
  R: IntegralDomain
  Q: QuotientFieldCategory R
  A: FiniteLinearAggregate R
  B: FiniteLinearAggregate Q

  Exports ==> with
    commonDenominator: B -> R
      ++ commonDenominator([q1,...,qn]) returns a common denominator
      ++ d for q1,...,qn.
    clearDenominator : B -> A
      ++ clearDenominator([q1,...,qn]) returns \spad{[p1,...,pn]} such that
      ++ \spad{qi = pi/d} where d is a common denominator for the qi's.
    splitDenominator : B -> Record(num: A, den: R)
      ++ splitDenominator([q1,...,qn]) returns
      ++ \spad{[[p1,...,pn], d]} such that
      ++ \spad{qi = pi/d} and d is a common denominator for the qi's.
```

```
Implementation ==> add
  import FiniteLinearAggregateFunctions2(Q, B, R, A)

  clearDenominator l ==
    d := commonDenominator l
    map(x +-> numer(d*x), l)

  splitDenominator l ==
    d := commonDenominator l
    [map(x +-> numer(d*x), l), d]

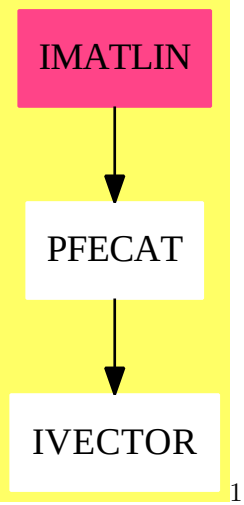
  if R has GcdDomain then
    commonDenominator l == reduce(lcm, map(denom, l),1)
  else
    commonDenominator l == reduce("*", map(denom, l), 1)
```

⟨*ICDEN.dotabb*⟩≡

```
"ICDEN" [color="#FF4488",href="bookvol10.4.pdf#nameddest=ICDEN"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"ICDEN" -> "PFECAT"
```

10.21 package IMATLIN InnerMatrixLinearAlgebraFunctions

10.22 InnerMatrixLinearAlgebraFunctions



1.00

Exports:

⟨*package IMATLIN InnerMatrixLinearAlgebraFunctions*⟩≡

```
)abbrev package IMATLIN InnerMatrixLinearAlgebraFunctions
++ Author: Clifton J. Williamson, P.Gianni
++ Date Created: 13 November 1989
++ Date Last Updated: September 1993
++ Basic Operations:
++ Related Domains: IndexedMatrix(R,minRow,minCol), Matrix(R),
++    RectangularMatrix(n,m,R), SquareMatrix(n,R)
++ Also See:
++ AMS Classifications:
++ Keywords: matrix, canonical forms, linear algebra
++ Examples:
++ References:
++ Description:
++   \spadtype{InnerMatrixLinearAlgebraFunctions} is an internal package
++   which provides standard linear algebra functions on domains in
++   \spad{MatrixCategory}
InnerMatrixLinearAlgebraFunctions(R,Row,Col,M):_
             Exports == Implementation where
  R   : Field
```

```
Row : FiniteLinearAggregate R
Col : FiniteLinearAggregate R
M   : MatrixCategory(R,Row,Col)
I ==> Integer

Exports ==> with
  rowEchelon: M -> M
    ++ \spad{rowEchelon(m)} returns the row echelon form of the matrix m.
  rank: M -> NonNegativeInteger
    ++ \spad{rank(m)} returns the rank of the matrix m.
  nullity: M -> NonNegativeInteger
    ++ \spad{nullity(m)} returns the mullity of the matrix m. This is the
    ++ dimension of the null space of the matrix m.
  if Col has shallowlyMutable then
    nullSpace: M -> List Col
      ++ \spad{nullSpace(m)} returns a basis for the null space of the
      ++ matrix m.
  determinant: M -> R
    ++ \spad{determinant(m)} returns the determinant of the matrix m.
    ++ an error message is returned if the matrix is not square.
  generalizedInverse: M -> M
    ++ \spad{generalizedInverse(m)} returns the generalized (Moore--Penrose)
    ++ inverse of the matrix m, i.e. the matrix h such that
    ++ m*h*m=h, h*m*h=m, m*h and h*m are both symmetric matrices.
  inverse: M -> Union(M,"failed")
    ++ \spad{inverse(m)} returns the inverse of the matrix m.
    ++ If the matrix is not invertible, "failed" is returned.
    ++ Error: if the matrix is not square.

Implementation ==> add

  rowAllZeroes?: (M,I) -> Boolean
  rowAllZeroes?(x,i) ==
    -- determines if the ith row of x consists only of zeroes
    -- internal function: no check on index i
    for j in minColIndex(x)..maxColIndex(x) repeat
      qelt(x,i,j) ^= 0 => return false
    true

  colAllZeroes?: (M,I) -> Boolean
  colAllZeroes?(x,j) ==
    -- determines if the ith column of x consists only of zeroes
    -- internal function: no check on index j
    for i in minRowIndex(x)..maxRowIndex(x) repeat
      qelt(x,i,j) ^= 0 => return false
    true
```

```
rowEchelon y ==
  -- row echelon form via Gaussian elimination
  x := copy y
  minR := minRowIndex x; maxR := maxRowIndex x
  minC := minColIndex x; maxC := maxColIndex x
  i := minR
  n: I := minR - 1
  for j in minC..maxC repeat
    i > maxR => return x
    n := minR - 1
    -- n = smallest k such that k >= i and x(k,j) ^= 0
    for k in i..maxR repeat
      if qelt(x,k,j) ^= 0 then leave (n := k)
    n = minR - 1 => "no non-zeroes"
    -- put nth row in ith position
    if i ^= n then swapRows_!(x,i,n)
    -- divide ith row by its first non-zero entry
    b := inv qelt(x,i,j)
    qsetelt_!(x,i,j,1)
    for k in (j+1)..maxC repeat qsetelt_!(x,i,k,b * qelt(x,i,k))
    -- perform row operations so that jth column has only one 1
    for k in minR..maxR repeat
      if k ^= i and qelt(x,k,j) ^= 0 then
        for k1 in (j+1)..maxC repeat
          qsetelt_!(x,k,k1,qelt(x,k,k1) - qelt(x,k,j) * qelt(x,i,k1))
        qsetelt_!(x,k,j,0)
    -- increment i
    i := i + 1
  x

rank x ==
  y :=
    (rk := nrows x) > (rh := ncols x) =>
      rk := rh
      transpose x
    copy x
  y := rowEchelon y; i := maxRowIndex y
  while rk > 0 and rowAllZeroes?(y,i) repeat
    i := i - 1
    rk := (rk - 1) :: NonNegativeInteger
  rk :: NonNegativeInteger

nullity x == (ncols x - rank x) :: NonNegativeInteger

if Col has shallowlyMutable then
```

```
  nullSpace y ==
    x := rowEchelon y
    minR := minRowIndex x; maxR := maxRowIndex x
    minC := minColIndex x; maxC := maxColIndex x
    nrow := nrows x; ncol := ncols x
    basis : List Col := nil()
    rk := nrow; row := maxR
    -- compute rank = # rows - # rows of all zeroes
    while rk > 0 and rowAllZeroes?(x,row) repeat
      rk := (rk - 1) :: NonNegativeInteger
      row := (row - 1) :: NonNegativeInteger
    -- if maximal rank, return zero vector
    ncol <= nrow and rk = ncol => [new(ncol,0)]
    -- if rank = 0, return standard basis vectors
    rk = 0 =>
      for j in minC..maxC repeat
        w : Col := new(ncol,0)
        qsetelt_!(w,j,1)
        basis := cons(w,basis)
      basis
    -- v contains information about initial 1's in the rows of x
    -- if the ith row has an initial 1 in the jth column, then
    -- v.j = i; v.j = minR - 1, otherwise
    v : IndexedOneDimensionalArray(I,minC) := new(ncol,minR - 1)
    for i in minR..(minR + rk - 1) repeat
      for j in minC.. while qelt(x,i,j) = 0 repeat j
      qsetelt_!(v,j,i)
    j := maxC; l := minR + ncol - 1
    while j >= minC repeat
      w : Col := new(ncol,0)
      -- if there is no row with an initial 1 in the jth column,
      -- create a basis vector with a 1 in the jth row
      if qelt(v,j) = minR - 1 then
        colAllZeroes?(x,j) =>
          qsetelt_!(w,l,1)
          basis := cons(w,basis)
        for k in minC..(j-1) for ll in minR..(l-1) repeat
          if qelt(v,k) ^= minR - 1 then
            qsetelt_!(w,ll,-qelt(x,qelt(v,k),j))
        qsetelt_!(w,l,1)
        basis := cons(w,basis)
      j := j - 1; l := l - 1
    basis

determinant y ==
```

```
    (ndim := nrows y) ^= (ncols y) =>
      error "determinant: matrix must be square"
    -- Gaussian Elimination
    ndim = 1 => qelt(y,minRowIndex y,minColIndex y)
    x := copy y
    minR := minRowIndex x; maxR := maxRowIndex x
    minC := minColIndex x; maxC := maxColIndex x
    ans : R := 1
    for i in minR..(maxR - 1) for j in minC..(maxC - 1) repeat
      if qelt(x,i,j) = 0 then
        rown := minR - 1
        for k in (i+1)..maxR repeat
          qelt(x,k,j) ^= 0 => leave (rown := k)
        if rown = minR - 1 then return 0
        swapRows_!(x,i,rown); ans := -ans
      ans := qelt(x,i,j) * ans; b := -inv qelt(x,i,j)
      for l in (j+1)..maxC repeat qsetelt_!(x,i,l,b * qelt(x,i,l))
      for k in (i+1)..maxR repeat
        if (b := qelt(x,k,j)) ^= 0 then
          for l in (j+1)..maxC repeat
            qsetelt_!(x,k,l,qelt(x,k,l) + b * qelt(x,i,l))
    qelt(x,maxR,maxC) * ans

  generalizedInverse(x) ==
    SUP:=SparseUnivariatePolynomial R
    FSUP := Fraction SUP
    VFSUP := Vector FSUP
    MATCAT2 := MatrixCategoryFunctions2(R, Row, Col, M,
                FSUP, VFSUP, VFSUP, Matrix FSUP)
    MATCAT22 := MatrixCategoryFunctions2(FSUP, VFSUP, VFSUP, Matrix FSUP,
                R, Row, Col, M)
    y:= map((r1:R):FSUP +-> coerce(coerce(r1)$SUP)$(Fraction SUP),x)$MATCAT2
    ty:=transpose y
    yy:=ty*y
    nc:=ncols yy
    var:=monomial(1,1)$SUP ::(Fraction SUP)
    yy:=inverse(yy+scalarMatrix(ncols yy,var))::Matrix(FSUP)*ty
    map((z1:FSUP):R +-> elt(z1,0),yy)$MATCAT22

  inverse x ==
    (ndim := nrows x) ^= (ncols x) =>
      error "inverse: matrix must be square"
    ndim = 2 =>
       ans2 : M := zero(ndim, ndim)
       zero?(det :=  x(1,1)*x(2,2)-x(1,2)*x(2,1)) => "failed"
       detinv := inv det
```

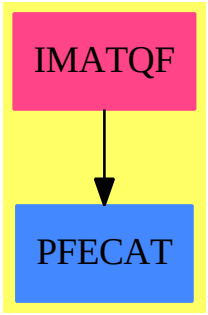
```
      ans2(1,1) := x(2,2)*detinv
      ans2(1,2) := -x(1,2)*detinv
      ans2(2,1) := -x(2,1)*detinv
      ans2(2,2) := x(1,1)*detinv
      ans2
    AB : M := zero(ndim,ndim + ndim)
    minR := minRowIndex x; maxR := maxRowIndex x
    minC := minColIndex x; maxC := maxColIndex x
    kmin := minRowIndex AB; kmax := kmin + ndim - 1
    lmin := minColIndex AB; lmax := lmin + ndim - 1
    for i in minR..maxR for k in kmin..kmax repeat
      for j in minC..maxC for l in lmin..lmax repeat
        qsetelt_!(AB,k,l,qelt(x,i,j))
      qsetelt_!(AB,k,lmin + ndim + k - kmin,1)
    AB := rowEchelon AB
    elt(AB,kmax,lmax) = 0 => "failed"
    subMatrix(AB,kmin,kmax,lmin + ndim,lmax + ndim)
```

⟨*IMATLIN.dotabb*⟩≡

```
"IMATLIN" [color="#FF4488",href="bookvol10.4.pdf#nameddest=IMATLIN"]
"PFECAT" -> "IVECTOR"
"IMATLIN" -> "PFECAT"
```

10.23 package IMATQF InnerMatrixQuotient-FieldFunctions

10.24 InnerMatrixQuotientFieldFunctions



1.00

Exports:
inverse rowEchelon nullSpace

⟨*package IMATQF InnerMatrixQuotientFieldFunctions*⟩≡

```
)abbrev package IMATQF InnerMatrixQuotientFieldFunctions
++ Author: Clifton J. Williamson
++ Date Created: 22 November 1989
++ Date Last Updated: 22 November 1989
++ Basic Operations:
++ Related Domains: IndexedMatrix(R,minRow,minCol), Matrix(R), RectangularMatrix(n,m,R), Squ
++ Also See:
++ AMS Classifications:
++ Keywords: matrix, inverse, integral domain
++ Examples:
++ References:
++ Description:
++   \spadtype{InnerMatrixQuotientFieldFunctions} provides functions on matrices
++   over an integral domain which involve the quotient field of that integral
++   domain.  The functions rowEchelon and inverse return matrices with
++   entries in the quotient field.
InnerMatrixQuotientFieldFunctions(R,Row,Col,M,QF,Row2,Col2,M2):_
         Exports == Implementation where
  R    : IntegralDomain
  Row  : FiniteLinearAggregate R
  Col  : FiniteLinearAggregate R
  M    : MatrixCategory(R,Row,Col)
  QF   : QuotientFieldCategory R
  Row2 : FiniteLinearAggregate QF
  Col2 : FiniteLinearAggregate QF
  M2   : MatrixCategory(QF,Row2,Col2)
```

```
IMATLIN ==> InnerMatrixLinearAlgebraFunctions(QF,Row2,Col2,M2)
MATCAT2 ==> MatrixCategoryFunctions2(R,Row,Col,M,QF,Row2,Col2,M2)
CDEN    ==> InnerCommonDenominator(R,QF,Col,Col2)

Exports ==> with
    rowEchelon: M -> M2
      ++ \spad{rowEchelon(m)} returns the row echelon form of the matrix m.
      ++ the result will have entries in the quotient field.
    inverse: M -> Union(M2,"failed")
      ++ \spad{inverse(m)} returns the inverse of the matrix m.
      ++ If the matrix is not invertible, "failed" is returned.
      ++ Error: if the matrix is not square.
      ++ Note: the result will have entries in the quotient field.
    if Col2 has shallowlyMutable then
      nullSpace : M -> List Col
        ++ \spad{nullSpace(m)} returns a basis for the null space of the
        ++ matrix m.
Implementation ==> add

  qfMat: M -> M2
  qfMat m == map((r1:R):QF +-> r1::QF,m)$MATCAT2

  rowEchelon m == rowEchelon(qfMat m)$IMATLIN
  inverse m ==
    (inv := inverse(qfMat m)$IMATLIN) case "failed" => "failed"
    inv :: M2

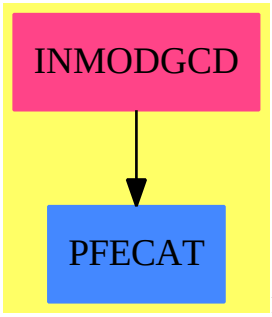
  if Col2 has shallowlyMutable then
    nullSpace m ==
      [clearDenominator(v)$CDEN for v in nullSpace(qfMat m)$IMATLIN]
```

⟨*IMATQF.dotabb*⟩≡

```
"IMATQF" [color="#FF4488",href="bookvol10.4.pdf#nameddest=IMATQF"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"IMATQF" -> "PFECAT"
```

10.25 package INMODGCD InnerModularGcd

10.26 InnerModularGcd



1.00

Exports:
modularGcd modularGcdPrimitive reduction

⟨*package INMODGCD InnerModularGcd*⟩≡
```
)abbrev package INMODGCD InnerModularGcd
++ Author: J.H. Davenport and P. Gianni
++ Date Created: July 1990
++ Date Last Updated: November 1991
++ Description:
++ This file contains the functions for modular gcd algorithm
++ for univariate polynomials with coefficients in a
++ non-trivial euclidean domain (i.e. not a field).
++ The package parametrised by the coefficient domain,
++ the polynomial domain, a prime,
++ and a function for choosing the next prime

Z    ==> Integer
NNI  ==> NonNegativeInteger

InnerModularGcd(R,BP,pMod,nextMod):C == T
 where
  R       :  EuclideanDomain
  BP      :  UnivariatePolynomialCategory(R)
  pMod    :  R
  nextMod :  (R,NNI) -> R

  C == with
     modularGcdPrimitive    : List BP  -> BP
         ++ modularGcdPrimitive(f1,f2) computes the gcd of the two polynomials
         ++ f1 and f2 by modular methods.
     modularGcd    : List BP  -> BP
```

```
        ++ modularGcd(listf) computes the gcd of the list of polynomials
        ++ listf  by modular methods.
     reduction :       (BP,R)              ->   BP
        ++ reduction(f,p) reduces the coefficients of the polynomial f
        ++ modulo the prime p.

  T == add

                  -- local functions --
    height    :         BP                ->   NNI
    mbound    :       (BP,BP)             ->   NNI
    modGcdPrimitive    :   (BP,BP)        ->   BP
    test         :     (BP,BP,BP)         ->   Boolean
    merge        :        (R,R)           ->   Union(R,"failed")
    modInverse   :      (R,R)             ->   R
    exactquo     :       (BP,BP,R)        ->   Union(BP,"failed")
    constNotZero :           BP           ->   Boolean
    constcase    : (List NNI ,List BP )   ->   BP
    lincase      : (List NNI ,List BP )   ->   BP


    if R has IntegerNumberSystem then
        reduction(u:BP,p:R):BP ==
            p = 0 => u
            map((r1:R):R +-> symmetricRemainder(r1,p),u)
      else
        reduction(u:BP,p:R):BP ==
            p = 0 => u
            map((r1:R):R +-> r1 rem p,u)

    FP:=EuclideanModularRing(R,BP,R,reduction,merge,exactquo)
    zeroChar : Boolean := R has CharacteristicZero

                --  exported functions --

    -- modular Gcd for a list of primitive polynomials
    modularGcdPrimitive(listf : List BP) :BP ==
      empty? listf => 0$BP
      g := first listf
      for f in rest listf | ^zero? f  while degree g > 0 repeat
        g:=modGcdPrimitive(g,f)
      g

    -- gcd for univariate polynomials
    modularGcd(listf : List BP): BP  ==
      listf:=remove!(0$BP,listf)
```

```
  empty? listf => 0$BP
  # listf = 1 => first listf
  minpol:=1$BP
  -- extract a monomial gcd
  mdeg:= "min"/[minimumDegree f for f in listf]
  if mdeg>0 then
    minpol1:= monomial(1,mdeg)
    listf:= [(f exquo minpol1)::BP for f in listf]
    minpol:=minpol*minpol1
  listdeg:=[degree f for f in listf ]
-- make the polynomials primitive
  listCont := [content f for f in listf]
  contgcd:= gcd listCont
  -- make the polynomials primitive
  listf :=[(f exquo cf)::BP for f in listf for cf in listCont]
  minpol:=contgcd*minpol
  ans:BP :=
     --one polynomial is constant
     member?(1,listf) => 1
     --one polynomial is linear
     member?(1,listdeg) => lincase(listdeg,listf)
     modularGcdPrimitive listf
  minpol*ans

              --  local functions --

--one polynomial is linear, remark that they are primitive
lincase(listdeg:List NNI ,listf:List BP ): BP  ==
  n:= position(1,listdeg)
  g:=listf.n
  for f in listf repeat
    if (f1:=f exquo g) case "failed" then return 1$BP
  g

-- test if d is the gcd
test(f:BP,g:BP,d:BP):Boolean ==
  d0:=coefficient(d,0)
  coefficient(f,0) exquo d0 case "failed" => false
  coefficient(g,0) exquo d0 case "failed" => false
  f exquo d case "failed" => false
  g exquo d case "failed" => false
  true

-- gcd and cofactors for PRIMITIVE univariate polynomials
-- also assumes that constant terms are non zero
modGcdPrimitive(f:BP,g:BP): BP ==
```

```
df:=degree f
dg:=degree g
dp:FP
lcf:=leadingCoefficient f
lcg:=leadingCoefficient g
testdeg:NNI
lcd:R:=gcd(lcf,lcg)
prime:=pMod
bound:=mbound(f,g)
while zero? (lcd rem prime) repeat
  prime := nextMod(prime,bound)
soFar:=gcd(reduce(f,prime),reduce(g,prime))::BP
testdeg:=degree soFar
zero? testdeg => return 1$BP
ldp:FP:=
  ((lcdp:=leadingCoefficient(soFar::BP)) = 1) =>
                                  reduce(lcd::BP,prime)
  reduce((modInverse(lcdp,prime)*lcd)::BP,prime)
soFar:=reduce(ldp::BP *soFar,prime)::BP
soFarModulus:=prime
-- choose the prime
while true repeat
  prime := nextMod(prime,bound)
  lcd rem prime =0 => "next prime"
  fp:=reduce(f,prime)
  gp:=reduce(g,prime)
  dp:=gcd(fp,gp)
  dgp :=euclideanSize dp
  if dgp =0 then return 1$BP
  if dgp=dg and ^(f exquo g case "failed") then return g
  if dgp=df and ^(g exquo f case "failed") then return f
  dgp > testdeg => "next prime"
  ldp:FP:=
    ((lcdp:=leadingCoefficient(dp::BP)) = 1) =>
                                  reduce(lcd::BP,prime)
    reduce((modInverse(lcdp,prime)*lcd)::BP,prime)
  dp:=ldp *dp
  dgp=testdeg  =>
     correction:=reduce(dp::BP-soFar,prime)::BP
     zero? correction =>
        ans:=reduce(lcd::BP*soFar,soFarModulus)::BP
        cont:=content ans
        ans:=(ans exquo cont)::BP
        test(f,g,ans) => return ans
        soFarModulus:=soFarModulus*prime
     correctionFactor:=modInverse(soFarModulus rem prime,prime)
```

```
                          -- the initial rem is just for efficiency
            soFar:=soFar+soFarModulus*(correctionFactor*correction)
            soFarModulus:=soFarModulus*prime
            soFar:=reduce(soFar,soFarModulus)::BP
          dgp<testdeg =>
            soFarModulus:=prime
            soFar:=dp::BP
            testdeg:=dgp
          if ^zeroChar and euclideanSize(prime)>1 then
             result:=dp::BP
             test(f,g,result) => return result
          -- this is based on the assumption that the caller of this package,
          -- in non-zero characteristic, will use primes of the form
          -- x-alpha as long as possible, but, if these are exhausted,
          -- will switch to a prime of degree larger than the answer
          -- so the result can be used directly.

      merge(p:R,q:R):Union(R,"failed") ==
           p = q => p
           p = 0 => q
           q = 0 => p
           "failed"

      modInverse(c:R,p:R):R ==
          (extendedEuclidean(c,p,1)::Record(coef1:R,coef2:R)).coef1

      exactquo(u:BP,v:BP,p:R):Union(BP,"failed") ==
          invlcv:=modInverse(leadingCoefficient v,p)
          r:=monicDivide(u,reduction(invlcv*v,p))
          reduction(r.remainder,p) ^=0 => "failed"
          reduction(invlcv*r.quotient,p)


      -- compute the height of a polynomial --
      height(f:BP):NNI ==
        degf:=degree f
        "max"/[euclideanSize cc for cc in coefficients f]

      -- compute the bound
      mbound(f:BP,g:BP):NNI ==
        hf:=height f
        hg:=height g
        2*min(hf,hg)

\section{package FOMOGCD ForModularGcd}
-- ForModularGcd(R,BP) : C == T
```

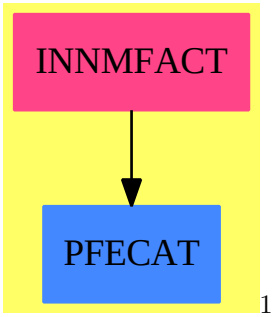
```
--  where
--   R          :   EuclideanDomain  -- characteristic 0
--   BP         :   UnivariatePolynomialCategory(R)
--
--   C == with
--     nextMod :  (R,NNI) -> R
--
--   T == add
--     nextMod(val:R,bound:NNI) : R  ==
--       ival:Z:= val pretend Z
--       (nextPrime(ival)$IntegerPrimesPackage(Z))::R
--
-- ForTwoGcd(F) : C == T
--  where
--   F          :   Join(Finite,Field)
--   SUP       ==>  SparseUnivariatePolynomial
--   R         ==>  SUP F
--   P         ==>  SUP R
--   UPCF2     ==>  UnivariatePolynomialCategoryFunctions2
--
--   C == with
--     nextMod :  (R,NNI) -> R
--
--   T == add
--     nextMod(val:R,bound:NNI) : R ==
--       ris:R:= nextItem(val) :: R
--       euclideanSize ris < 2 => ris
--       generateIrredPoly(
--             (bound+1)::PositiveInteger)$IrredPolyOverFiniteField(F)
--
--
-- ModularGcd(R,BP) == T
--  where
--   R  : EuclideanDomain -- characteristic 0
--   BP : UnivariatePolynomialCategory(R)
--   T ==> InnerModularGcd(R,BP,67108859::R,nextMod$ForModularGcd(R,BP))
--
-- TwoGcd(F) : C == T
--  where
--   F          :   Join(Finite,Field)
--   SUP       ==>  SparseUnivariatePolynomial
--   R         ==>  SUP F
--   P         ==>  SUP R
--
--   T ==> InnerModularGcd(R,P,nextMod(monomial(1,1)$R)$ForTwoGcd(F),
--                         nextMod$ForTwoGcd(F))
```

⟨*INMODGCD.dotabb*⟩≡

```
"INMODGCD" [color="#FF4488",href="bookvol10.4.pdf#nameddest=INMODGCD"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"INMODGCD" -> "PFECAT"
```

10.27 package INNMFACT InnerMultFact

10.28 InnerMultFact



1.00

Exports:
factor

⟨*package INNMFACT InnerMultFact*⟩≡
```
)abbrev package INNMFACT InnerMultFact
++ Author: P. Gianni
++ Date Created: 1983
++ Date Last Updated: Sept. 1990
++ Additional Comments: JHD Aug 1997
++ Basic Functions:
++ Related Constructors: MultivariateFactorize, AlgebraicMultFact
++ Also See:
++ AMS Classifications:
++ Keywords:
++ References:
++ Description:
++   This is an inner package for factoring multivariate polynomials
++ over various coefficient domains in characteristic 0.
++ The univariate factor operation is passed as a parameter.
++ Multivariate hensel lifting is used to lift the univariate
++ factorization

-- Both exposed functions call mFactor. This deals with issues such as
-- monomial factors, contents, square-freeness etc., then calls intfact.
-- This uses intChoose to find a "good" evaluation and factorise the
-- corresponding univariate, and then uses MultivariateLifting to find
-- the multivariate factors.

InnerMultFact(OV,E,R,P) : C == T
 where
  R          :   Join(EuclideanDomain, CharacteristicZero)
```

```
                    -- with factor on R[x]
OV          :   OrderedSet
E           :   OrderedAbelianMonoidSup
P           :   PolynomialCategory(R,E,OV)
BP          ==> SparseUnivariatePolynomial R
UFactor     ==> (BP -> Factored BP)
Z             ==> Integer
MParFact      ==> Record(irr:P,pow:Z)
USP           ==> SparseUnivariatePolynomial P
SUParFact     ==> Record(irr:USP,pow:Z)
SUPFinalFact ==> Record(contp:R,factors:List SUParFact)
MFinalFact    ==> Record(contp:R,factors:List MParFact)

              --  contp   =  content,
              --  factors =  List of irreducible factors with exponent
L           ==> List

C == with
  factor      :      (P,UFactor)    ->  Factored P
    ++ factor(p,ufact) factors the multivariate polynomial p
    ++ by specializing variables and calling the univariate
    ++ factorizer ufact.
  factor      :      (USP,UFactor)    ->  Factored USP
    ++ factor(p,ufact) factors the multivariate polynomial p
    ++ by specializing variables and calling the univariate
    ++ factorizer ufact. p is represented as a univariate
    ++ polynomial with multivariate coefficients.

T == add

  NNI       ==> NonNegativeInteger

  LeadFact  ==> Record(polfac:L P,correct:R,corrfact:L BP)
  ContPrim  ==> Record(cont:P,prim:P)
  ParFact   ==> Record(irr:BP,pow:Z)
  FinalFact ==> Record(contp:R,factors:L ParFact)
  NewOrd    ==> Record(npol:USP,nvar:L OV,newdeg:L NNI)
  pmod:R   :=  (prevPrime(2**26)$IntegerPrimesPackage(Integer))::R

  import GenExEuclid(R,BP)
  import MultivariateLifting(E,OV,R,P)
  import FactoringUtilities(E,OV,R,P)
  import LeadingCoefDetermination(OV,E,R,P)
  Valuf ==> Record(inval:L L R,unvfact:L BP,lu:R,complead:L R)
  UPCF2 ==> UnivariatePolynomialCategoryFunctions2
```

```
              ----  Local Functions  ----
mFactor   :            (P,UFactor)           ->  MFinalFact
supFactor :           (USP,UFactor)          ->  SUPFinalFact
mfconst   :      (USP,L OV,L NNI,UFactor)    -> L USP
mfpol     :      (USP,L OV,L NNI,UFactor)    -> L USP
monicMfpol:      (USP,L OV,L NNI,UFactor)    -> L USP
varChoose :           (P,L OV,L NNI)         -> NewOrd
simplify  :       (P,L OV,L NNI,UFactor)     -> MFinalFact
intChoose :  (USP,L OV,R,L P,L L R,UFactor)  -> Union(Valuf,"failed")
intfact   : (USP,L OV,L NNI,MFinalFact,L L R,UFactor) -> L USP
pretest   :         (P,NNI,L OV,L R)         -> FinalFact
checkzero :            (USP,BP)              -> Boolean
localNorm :               L BP               -> Z

convertPUP(lfg:MFinalFact): SUPFinalFact ==
  [lfg.contp,[[lff.irr ::USP,lff.pow]$SUParFact
               for lff in lfg.factors]]$SUPFinalFact

-- intermediate routine if an SUP was passed in.
supFactor(um:USP,ufactor:UFactor) : SUPFinalFact ==
  ground?(um) => convertPUP(mFactor(ground um,ufactor))
  lvar:L OV:= "setUnion"/[variables cf for cf in coefficients um]
  empty? lvar => -- the polynomial is univariate
    umv:= map(ground,um)$UPCF2(P,USP,R,BP)
    lfact:=ufactor umv
    [retract unit lfact,[[map(coerce,ff.factor)$UPCF2(R,BP,P,USP),
       ff.exponent] for ff in factors lfact]]$SUPFinalFact
  lcont:P
  lf:L USP
  flead : SUPFinalFact:=[0,empty()]$SUPFinalFact
  factorlist:L SUParFact :=empty()

  mdeg :=minimumDegree um     ---- is the Mindeg > 0? ----
  if mdeg>0 then
    f1:USP:=monomial(1,mdeg)
    um:=(um exquo f1)::USP
    factorlist:=cons([monomial(1,1),mdeg],factorlist)
    if degree um=0 then return
      lfg:=convertPUP mFactor(ground um, ufactor)
      [lfg.contp,append(factorlist,lfg.factors)]
  uum:=unitNormal um
  um :=uum.canonical
  sqfacs := squareFree(um)$MultivariateSquareFree(E,OV,R,P)
  lcont :=  ground(uum.unit * unit sqfacs)
                                ----  Factorize the content  ----
  flead:=convertPUP mFactor(lcont,ufactor)
```

```
    factorlist:=append(flead.factors,factorlist)
                              ----  Make the polynomial square-free  ----
    sqqfact:=factors sqfacs
                     ---  Factorize the primitive square-free terms ---
    for fact in sqqfact repeat
      ffactor:USP:=fact.factor
      ffexp:=fact.exponent
      zero? degree ffactor =>
        lfg:=mFactor(ground ffactor,ufactor)
        lcont:=lfg.contp * lcont
        factorlist := append(factorlist,
           [[lff.irr ::USP,lff.pow * ffexp]$SUParFact
                     for lff in lfg.factors])
      coefs := coefficients ffactor
      ldeg:= ["max"/[degree(fc,xx) for fc in coefs] for xx in lvar]
      lf :=
        ground?(leadingCoefficient ffactor) =>
           mfconst(ffactor,lvar,ldeg,ufactor)
        mfpol(ffactor,lvar,ldeg,ufactor)
      auxfl:=[[lfp,ffexp]$SUParFact  for lfp in lf]
      factorlist:=append(factorlist,auxfl)
    lcfacs := */[leadingCoefficient leadingCoefficient(f.irr)**((f.pow)::NNI)
                         for f in factorlist]
    [(leadingCoefficient leadingCoefficient(um) exquo lcfacs)::R,
                  factorlist]$SUPFinalFact

  factor(um:USP,ufactor:UFactor):Factored USP ==
    flist := supFactor(um,ufactor)
    (flist.contp):: P :: USP *
      (*/[primeFactor(u.irr,u.pow) for u in flist.factors])

  checkzero(u:USP,um:BP) : Boolean ==
    u=0 => um =0
    um = 0 => false
    degree u = degree um => checkzero(reductum u, reductum um)
    false
           ---  Choose the variable of less degree  ---
  varChoose(m:P,lvar:L OV,ldeg:L NNI) : NewOrd ==
    k:="min"/[d for d in ldeg]
    k=degree(m,first lvar) =>
                           [univariate(m,first lvar),lvar,ldeg]$NewOrd
    i:=position(k,ldeg)
    x:OV:=lvar.i
    ldeg:=cons(k,delete(ldeg,i))
    lvar:=cons(x,delete(lvar,i))
    [univariate(m,x),lvar,ldeg]$NewOrd
```

```
localNorm(lum: L BP): Z ==
  R is AlgebraicNumber =>
    "max"/[numberOfMonomials ff for ff in lum]

  "max"/[+/[euclideanSize cc for i in 0..degree ff|
            (cc:= coefficient(ff,i))^=0] for ff in lum]

      ---  Choose the integer to reduce to univariate case  ---
intChoose(um:USP,lvar:L OV,clc:R,plist:L P,ltry:L L R,
                              ufactor:UFactor) : Union(Valuf,"failed") ==
  -- declarations
  degum:NNI := degree um
  nvar1:=#lvar
  range:NNI:=5
  unifact:L BP
  ctf1 : R := 1
  testp:Boolean :=             -- polynomial leading coefficient
    empty? plist => false
    true
  leadcomp,leadcomp1 : L R
  leadcomp:=leadcomp1:=empty()
  nfatt:NNI := degum+1
  lffc:R:=1
  lffc1:=lffc
  newunifact : L BP:=empty()
  leadtest:=true --- the lc test with polCase has to be performed
  int:L R:=empty()

--  New sets of integers are chosen to reduce the multivariate problem to
--  a univariate one, until we find twice the
--  same (and minimal) number of "univariate" factors:
--  the set smaller in modulo is chosen.
--  Note that there is no guarantee that this is the truth:
--  merely the closest approximation we have found!

  while true repeat
   testp and #ltry>10 => return "failed"
   lval := [ ran(range) for i in 1..nvar1]
   member?(lval,ltry) => range:=2*range
   ltry := cons(lval,ltry)
   leadcomp1:=[retract eval(pol,lvar,lval) for pol in plist]
   testp and or/[unit? epl for epl in leadcomp1] => range:=2*range
   newm:BP:=completeEval(um,lvar,lval)
   degum ^= degree newm or minimumDegree newm ^=0 => range:=2*range
   lffc1:=content newm
```

```
        newm:=(newm exquo lffc1)::BP
        testp and leadtest and ^ polCase(lffc1*clc,#plist,leadcomp1)
                              => range:=2*range
        degree(gcd [newm,differentiate(newm)])^=0 => range:=2*range
        luniv:=ufactor(newm)
        lunivf:= factors luniv
        lffc1:R:=retract(unit luniv)@R * lffc1
        nf:= #lunivf

        nf=0 or nf>nfatt => "next values"      ---  pretest failed ---

                         --- the univariate polynomial is irreducible ---
        if nf=1 then leave (unifact:=[newm])

    --  the new integer give the same number of factors
        nfatt = nf =>
        -- if this is the first univariate factorization with polCase=true
        -- or if the last factorization has smaller norm and satisfies
        -- polCase
          if leadtest or
            ((localNorm unifact > localNorm [ff.factor for ff in lunivf])
              and (^testp or polCase(lffc1*clc,#plist,leadcomp1))) then
                 unifact:=[uf.factor for uf in lunivf]
                 int:=lval
                 lffc:=lffc1
                 if testp then leadcomp:=leadcomp1
          leave "foundit"

    --  the first univariate factorization, inizialize
        nfatt > degum =>
          unifact:=[uf.factor for uf in lunivf]
          lffc:=lffc1
          if testp then leadcomp:=leadcomp1
          int:=lval
          leadtest := false
          nfatt := nf

        nfatt>nf =>  -- for the previous values there were more factors
          if testp then leadtest:=^polCase(lffc*clc,#plist,leadcomp)
          else leadtest:= false
          -- if polCase=true we can consider the univariate decomposition
          if ^leadtest then
            unifact:=[uf.factor for uf in lunivf]
            lffc:=lffc1
            if testp then leadcomp:=leadcomp1
            int:=lval
```

```
      nfatt := nf
    [cons(int,ltry),unifact,lffc,leadcomp]$Valuf


              ----  The polynomial has mindeg>0   ----

  simplify(m:P,lvar:L OV,lmdeg:L NNI,ufactor:UFactor):MFinalFact ==
    factorlist:L MParFact:=[]
    pol1:P:= 1$P
    for x in lvar repeat
      i := lmdeg.(position(x,lvar))
      i=0 => "next value"
      pol1:=pol1*monomial(1$P,x,i)
      factorlist:=cons([x::P,i]$MParFact,factorlist)
    m := (m exquo pol1)::P
    ground? m => [retract m,factorlist]$MFinalFact
    flead:=mFactor(m,ufactor)
    flead.factors:=append(factorlist,flead.factors)
    flead

  -- This is the key internal function
  -- We now know that the polynomial is square-free etc.,
  -- We use intChoose to find a set of integer values to reduce the
  -- problem to univariate (and for efficiency, intChoose returns
  -- the univariate factors).
  -- In the case of a polynomial leading coefficient, we check that this
  -- is consistent with leading coefficient determination (else try again)
  -- We then lift the univariate factors to multivariate factors, and
  -- return the result
  intfact(um:USP,lvar: L OV,ldeg:L NNI,tleadpol:MFinalFact,
                                  ltry:L L R,ufactor:UFactor) :  L USP ==
    polcase:Boolean:=(not empty? tleadpol.factors)
    vfchoo:Valuf:=
      polcase =>
        leadpol:L P:=[ff.irr for ff in tleadpol.factors]
        check:=intChoose(um,lvar,tleadpol.contp,leadpol,ltry,ufactor)
        check case "failed" => return monicMfpol(um,lvar,ldeg,ufactor)
        check::Valuf
      intChoose(um,lvar,1,empty(),empty(),ufactor)::Valuf
    unifact:List BP := vfchoo.unvfact
    nfact:NNI := #unifact
    nfact=1 => [um]
    ltry:L L R:= vfchoo.inval
    lval:L R:=first ltry
    dd:= vfchoo.lu
    leadval:L R:=empty()
```

```
        lpol:List P:=empty()
        if polcase then
          leadval := vfchoo.complead
          distf := distFact(vfchoo.lu,unifact,tleadpol,leadval,lvar,lval)
          distf case "failed" =>
               return intfact(um,lvar,ldeg,tleadpol,ltry,ufactor)
          dist := distf :: LeadFact
            -- check the factorization of leading coefficient
          lpol:= dist.polfac
          dd := dist.correct
          unifact:=dist.corrfact
        if dd^=1 then
--        if polcase then lpol := [unitCanonical lp for lp in lpol]
--        dd:=unitCanonical(dd)
          unifact := [dd * unif for unif in unifact]
          umd := unitNormal(dd).unit * ((dd**(nfact-1)::NNI)::P)*um
        else umd := um
        (ffin:=lifting(umd,lvar,unifact,lval,lpol,ldeg,pmod))
          case "failed" => intfact(um,lvar,ldeg,tleadpol,ltry,ufactor)
        factfin: L USP:=ffin :: L USP
        if dd^=1 then
          factfin:=[primitivePart ff for ff in factfin]
        factfin

                  ----  m square-free,primitive,lc constant  ----
      mfconst(um:USP,lvar:L OV,ldeg:L NNI,ufactor:UFactor):L USP ==
        factfin:L USP:=empty()
        empty? lvar =>
          lum:=factors ufactor(map(ground,um)$UPCF2(P,USP,R,BP))
          [map(coerce,uf.factor)$UPCF2(R,BP,P,USP) for uf in lum]
        intfact(um,lvar,ldeg,[0,empty()]$MFinalFact,empty(),ufactor)

      monicize(um:USP,c:P):USP ==
        n:=degree(um)
        ans:USP := monomial(1,n)
        n:=(n-1)::NonNegativeInteger
        prod:P:=1
        while (um:=reductum(um)) ^= 0 repeat
          i := degree um
          lc := leadingCoefficient um
          prod := prod * c ** (n-(n:=i))::NonNegativeInteger
          ans := ans + monomial(prod*lc, i)
        ans

      unmonicize(m:USP,c:P):USP == primitivePart m(monomial(c,1))
```

```
          --- m is square-free,primitive,lc is a polynomial  ---
monicMfpol(um:USP,lvar:L OV,ldeg:L NNI,ufactor:UFactor):L USP ==
  l := leadingCoefficient um
  monpol := monicize(um,l)
  nldeg := degree(monpol,lvar)
  map((z1:USP):USP +-> unmonicize(z1,l),
           mfconst(monpol,lvar,nldeg,ufactor))

mfpol(um:USP,lvar:L OV,ldeg:L NNI,ufactor:UFactor):L USP ==
  R has Field =>
    monicMfpol(um,lvar,ldeg,ufactor)
  tleadpol:=mFactor(leadingCoefficient um,ufactor)
  intfact(um,lvar,ldeg,tleadpol,[],ufactor)

mFactor(m:P,ufactor:UFactor) : MFinalFact ==
  ground?(m) => [retract(m),empty()]$MFinalFact
  lvar:L OV:= variables m
  lcont:P
  lf:L USP
  flead : MFinalFact:=[0,empty()]$MFinalFact
  factorlist:L MParFact :=empty()

  lmdeg :=minimumDegree(m,lvar)     ---- is the Mindeg > 0? ----
  or/[n>0 for n in lmdeg] => simplify(m,lvar,lmdeg,ufactor)

  sqfacs := squareFree m
  lcont := unit sqfacs

                               ----  Factorize the content  ----
  if ground? lcont then flead.contp:=retract lcont
  else flead:=mFactor(lcont,ufactor)
  factorlist:=flead.factors



                           ----  Make the polynomial square-free  ----
  sqqfact:=factors sqfacs

                    ---  Factorize the primitive square-free terms ---
  for fact in sqqfact repeat
    ffactor:P:=fact.factor
    ffexp := fact.exponent
    lvar := variables ffactor
    x:OV :=lvar.first
    ldeg:=degree(ffactor,lvar)
         ---  Is the polynomial linear in one of the variables ? ---
```

```
          member?(1,ldeg) =>
            x:OV:=lvar.position(1,ldeg)
            lcont:= gcd coefficients(univariate(ffactor,x))
            ffactor:=(ffactor exquo lcont)::P
            factorlist:=cons([ffactor,ffexp]$MParFact,factorlist)
            for lcterm in mFactor(lcont,ufactor).factors repeat
             factorlist:=cons([lcterm.irr,lcterm.pow * ffexp], factorlist)

          varch:=varChoose(ffactor,lvar,ldeg)
          um:=varch.npol

          x:=lvar.first
          ldeg:=ldeg.rest
          lvar := lvar.rest
          if varch.nvar.first ^= x then
            lvar:= varch.nvar
            x := lvar.first
            lvar := lvar.rest
          pc:= gcd coefficients um
          if pc^=1 then
              um:=(um exquo pc)::USP
              ffactor:=multivariate(um,x)
              for lcterm in mFactor(pc,ufactor).factors repeat
                factorlist:=cons([lcterm.irr,lcterm.pow*ffexp],factorlist)
          ldeg:=degree(ffactor,lvar)
          um := unitCanonical um
          if ground?(leadingCoefficient um) then
             lf:= mfconst(um,lvar,ldeg,ufactor)
          else lf:=mfpol(um,lvar,ldeg,ufactor)
          auxfl:=[[unitCanonical multivariate(lfp,x),ffexp]$MParFact  for lfp in lf]
          factorlist:=append(factorlist,auxfl)
        lcfacs := */[leadingCoefficient(f.irr)**((f.pow)::NNI) for f in factorlist]
        [(leadingCoefficient(m) exquo lcfacs):: R,factorlist]$MFinalFact

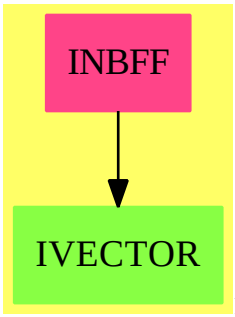
      factor(m:P,ufactor:UFactor):Factored P ==
        flist := mFactor(m,ufactor)
        (flist.contp):: P *
          (*/[primeFactor(u.irr,u.pow) for u in flist.factors])
```

⟨*INNMFACT.dotabb*⟩≡

```
"INNMFACT" [color="#FF4488",href="bookvol10.4.pdf#nameddest=INNMFACT"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"INNMFACT" -> "PFECAT"
```

10.29 package INBFF InnerNormalBasisField-Functions

10.30 InnerNormalBasisFieldFunctions



1.00

Exports:

basis	dAndcExp	expPot	index	inv
lookup	minimalPolynomial	norm	normalElement	normal?
pol	qPot	random	repSq	setFieldInfo
trace	xn	?*?	?**?	?/?

⟨*package INBFF InnerNormalBasisFieldFunctions*⟩≡

```
)abbrev package INBFF InnerNormalBasisFieldFunctions
++ Authors: J.Grabmeier, A.Scheerhorn
++ Date Created: 26.03.1991
++ Date Last Updated: 31 March 1991
++ Basic Operations:
++ Related Constructors:
++ Also See:
++ AMS Classifications:
++ Keywords: finite field, normal basis
++ References:
++  R.Lidl, H.Niederreiter: Finite Field, Encyclopedia of Mathematics and
++   Its Applications, Vol. 20, Cambridge Univ. Press, 1983, ISBN 0 521 30240 4
++  D.R.Stinson: Some observations on parallel Algorithms for fast
++   exponentiation in GF(2^n), Siam J. Comp., Vol.19, No.4, pp.711-717,
++   August 1990
++  T.Itoh, S.Tsujii: A fast algorithm for computing multiplicative inverses
++   in GF(2^m) using normal bases, Inf. and Comp. 78, pp.171-177, 1988
++  J. Grabmeier, A. Scheerhorn: Finite Fields in AXIOM.
++   AXIOM Technical Report Series, ATR/5 NP2522.
++ Description:
++  InnerNormalBasisFieldFunctions(GF) (unexposed):
++  This package has functions used by
++  every normal basis finite field extension domain.
```

```
InnerNormalBasisFieldFunctions(GF): Exports == Implementation where
  GF    : FiniteFieldCategory       -- the ground field

  PI   ==> PositiveInteger
  NNI  ==> NonNegativeInteger
  I    ==> Integer
  SI   ==> SingleInteger
  SUP  ==> SparseUnivariatePolynomial
  VGF  ==> Vector GF
  M    ==> Matrix
  V    ==> Vector
  L    ==> List
  OUT  ==> OutputForm
  TERM ==> Record(value:GF,index:SI)
  MM   ==> ModMonic(GF,SUP GF)

  Exports ==> with

      setFieldInfo: (V L TERM,GF) -> Void
        ++ setFieldInfo(m,p) initializes the field arithmetic, where m is
        ++ the multiplication table and p is the respective normal element
        ++ of the ground field GF.
      random    : PI           -> VGF
        ++ random(n) creates a vector over the ground field with random entries.
      index     : (PI,PI)      -> VGF
        ++ index(n,m) is a index function for vectors of length n over
        ++ the ground field.
      pol       : VGF          -> SUP GF
        ++ pol(v) turns the vector \spad{[v0,...,vn]} into the polynomial
        ++ \spad{v0+v1*x+ ... + vn*x**n}.
      xn         : NNI          -> SUP GF
        ++ xn(n) returns the polynomial \spad{x**n-1}.
      dAndcExp  : (VGF,NNI,SI) -> VGF
        ++ dAndcExp(v,n,k) computes \spad{v**e} interpreting v as an element of
        ++ normal basis field. A divide and conquer algorithm similar to the
        ++ one from D.R.Stinson,
        ++ "Some observations on parallel Algorithms for fast exponentiation in
        ++ GF(2^n)", Siam J. Computation, Vol.19, No.4, pp.711-717, August 1990
        ++ is used. Argument k is a parameter of this algorithm.
      repSq     : (VGF,NNI)    -> VGF
        ++ repSq(v,e) computes \spad{v**e} by repeated squaring,
        ++ interpreting v as an element of a normal basis field.
      expPot    : (VGF,SI,SI)  -> VGF
        ++ expPot(v,e,d) returns the sum from \spad{i = 0} to
        ++ \spad{e - 1} of \spad{v**(q**i*d)}, interpreting
```

```
        ++ v as an element of a normal basis field and where q is
        ++ the size of the ground field.
        ++ Note: for a description of the algorithm, see T.Itoh and S.Tsujii,
        ++ "A fast algorithm for computing multiplicative inverses in GF(2^m)
        ++ using normal bases",
        ++ Information and Computation 78, pp.171-177, 1988.
      qPot      : (VGF,I)      -> VGF
        ++ qPot(v,e) computes \spad{v**(q**e)}, interpreting v as an element of
        ++ normal basis field, q the size of the ground field.
        ++ This is done by a cyclic e-shift of the vector v.

  -- the semantic of the following functions is obvious from the finite field
  -- context, for description see category FAXF
      "**"      :(VGF,I)       -> VGF
        ++ x**n \undocumented{}
        ++ See \axiomFunFrom{**}{DivisionRing}
      "*"       :(VGF,VGF)     -> VGF
        ++ x*y \undocumented{}
        ++ See \axiomFunFrom{*}{SemiGroup}
      "/"       :(VGF,VGF)     -> VGF
        ++ x/y \undocumented{}
        ++ See \axiomFunFrom{/}{Field}
      norm      :(VGF,PI)      -> VGF
        ++ norm(x,n) \undocumented{}
        ++ See \axiomFunFrom{norm}{FiniteAlgebraicExtensionField}
      trace     :(VGF,PI)      -> VGF
        ++ trace(x,n) \undocumented{}
        ++ See \axiomFunFrom{trace}{FiniteAlgebraicExtensionField}
      inv       : VGF          -> VGF
        ++ inv x \undocumented{}
        ++ See \axiomFunFrom{inv}{DivisionRing}
      lookup    : VGF          -> PI
        ++ lookup(x) \undocumented{}
        ++ See \axiomFunFrom{lookup}{Finite}
      normal?   : VGF          -> Boolean
        ++ normal?(x) \undocumented{}
        ++ See \axiomFunFrom{normal?}{FiniteAlgebraicExtensionField}
      basis     : PI           -> V VGF
        ++ basis(n) \undocumented{}
        ++ See \axiomFunFrom{basis}{FiniteAlgebraicExtensionField}
      normalElement:PI         -> VGF
        ++ normalElement(n) \undocumented{}
        ++ See \axiomFunFrom{normalElement}{FiniteAlgebraicExtensionField}
      minimalPolynomial: VGF   -> SUP GF
        ++ minimalPolynomial(x) \undocumented{}
        ++ See \axiomFunFrom{minimalPolynomial}{FiniteAlgebraicExtensionField}
```

```
  Implementation ==> add

-- global variables ====================================================

    sizeGF:NNI:=size()$GF
    -- the size of the ground field

    multTable:V L TERM:=new(1,nil()$(L TERM))$(V L TERM)
    -- global variable containing the multiplication table

    trGen:GF:=1$GF
    -- controls the imbedding of the ground field

    logq:List SI:=[0,10::SI,16::SI,20::SI,23::SI,0,28::SI,_
                             30::SI,32::SI,0,35::SI]
    -- logq.i is about 10*log2(i) for the values <12 which
    -- can match sizeGF. It's used by "**"

    expTable:L L SI:=[[],_
        [4::SI,12::SI,48::SI,160::SI,480::SI,0],_
        [8::SI,72::SI,432::SI,0],_
        [18::SI,216::SI,0],_
        [32::SI,480::SI,0],[],_
        [72::SI,0],[98::SI,0],[128::SI,0],[],[200::SI,0]]
    -- expT is used by "**" to optimize the parameter k
    -- before calling dAndcExp(..,..,k)

-- functions ===========================================================

--  computes a**(-1) = a**((q**extDeg)-2)
--  see reference of function expPot
    inv(a) ==
      b:VGF:=qPot(expPot(a,(#a-1)::NNI::SI,1::SI)$$,1)$$
      erg:VGF:=inv((a *$$ b).1 *$GF trGen)$GF *$VGF b

-- "**" decides which exponentiation algorithm will be used, in order to
-- get the fastest computation. If dAndcExp is used, it chooses the
-- optimal parameter k for that algorithm.
    a ** ex  ==
      e:NNI:=positiveRemainder(ex,sizeGF**((#a)::PI)-1)$I :: NNI
      zero?(e)$NNI => new(#a,trGen)$VGF
--      one?(e)$NNI  => copy(a)$VGF
      (e = 1)$NNI  => copy(a)$VGF
--    inGroundField?(a) => new(#a,((a.1*trGen) **$GF e))$VGF
      e1:SI:=(length(e)$I)::SI
```

```
      sizeGF >$I 11 =>
        q1:SI:=(length(sizeGF)$I)::SI
        logqe:SI:=(e1 quo$SI q1) +$SI 1$SI
        10::SI * (logqe + sizeGF-2) > 15::SI * e1 =>
--        print("repeatedSquaring"::OUT)
          repSq(a,e)
--      print("divAndConquer(a,e,1)"::OUT)
        dAndcExp(a,e,1)
      logqe:SI:=((10::SI *$SI e1) quo$SI (logq.sizeGF)) +$SI 1$SI
      k:SI:=1$SI
      expT:List SI:=expTable.sizeGF
      while (logqe >= expT.k) and not zero? expT.k repeat k:=k +$SI 1$SI
      mult:I:=(sizeGF-1) *$I sizeGF **$I ((k-1)pretend NNI) +$I_
              ((logqe +$SI k -$SI 1$SI) quo$SI k)::I -$I 2
      (10*mult) >= (15 * (e1::I)) =>
--      print("repeatedSquaring(a,e)"::OUT)
        repSq(a,e)
--    print(hconcat(["divAndConquer(a,e,"::OUT,k::OUT,")"::OUT])$OUT)
      dAndcExp(a,e,k)

-- computes a**e by repeated squaring
    repSq(b,e) ==
      a:=copy(b)$VGF
--      one? e => a
      (e = 1) => a
      odd?(e)$I => a * repSq(a*a,(e quo 2) pretend NNI)
      repSq(a*a,(e quo 2) pretend NNI)

-- computes a**e using the divide and conquer algorithm similar to the
-- one from D.R.Stinson,
-- "Some observations on parallel Algorithms for fast exponentiation in
-- GF(2^n)", Siam J. Computation, Vol.19, No.4, pp.711-717, August 1990
    dAndcExp(a,e,k) ==
      plist:List VGF:=[copy(a)$VGF]
      qk:I:=sizeGF**(k pretend NNI)
      for j in 2..(qk-1) repeat
        if positiveRemainder(j,sizeGF)=0 then b:=qPot(plist.(j quo sizeGF),1)$$
                          else b:=a *$$ last(plist)$(List VGF)
        plist:=concat(plist,b)
      l:List NNI:=nil()
      ex:I:=e
      while not(ex = 0) repeat
        l:=concat(l,positiveRemainder(ex,qk) pretend NNI)
        ex:=ex quo qk
      if first(l)=0 then erg:VGF:=new(#a,trGen)$VGF
                    else erg:VGF:=plist.(first(l))
```

```
      i:SI:=k
      for j in rest(l) repeat
        if j^=0 then erg:=erg *$$ qPot(plist.j,i)$$
        i:=i+k
      erg

    a * b ==
      e:SI:=(#a)::SI
      erg:=zero(#a)$VGF
      for t in multTable.1 repeat
        for j in 1..e repeat
          y:=t.value  -- didn't work without defining x and y
          x:=t.index
          k:SI:=addmod(x,j::SI,e)$SI +$SI 1$SI
          erg.k:=erg.k +$GF a.j *$GF b.j *$GF y
      for i in 1..e-1 repeat
        for j in i+1..e repeat
          for t in multTable.(j-i+1) repeat
            y:=t.value   -- didn't work without defining x and y
            x:=t.index
            k:SI:=addmod(x,i::SI,e)$SI +$SI 1$SI
            erg.k:GF:=erg.k +$GF (a.i *$GF b.j +$GF a.j *$GF b.i) *$GF y
      erg

    lookup(x) ==
      erg:I:=0
      for j in (#x)..1 by -1 repeat
        erg:=(erg * sizeGF) + (lookup(x.j)$GF rem sizeGF)
      erg=0 => (sizeGF**(#x)) :: PI
      erg :: PI

--  computes the norm of a over GF**d, d must devide extdeg
--  see reference of function expPot below
    norm(a,d) ==
      dSI:=d::SI
      r:=divide((#a)::SI,dSI)
      not(r.remainder = 0) => error "norm: 2.arg must divide extdeg"
      expPot(a,r.quotient,dSI)$$

--  computes expPot(a,e,d) = sum form i=0 to e-1 over a**(q**id))
--  see T.Itoh and S.Tsujii,
--  "A fast algorithm for computing multiplicative inverses in GF(2^m)
--   using normal bases",
--  Information and Computation 78, pp.171-177, 1988
    expPot(a,e,d) ==
      deg:SI:=(#a)::SI
```

```
      e=1 => copy(a)$VGF
      k2:SI:=d
      y:=copy(a)
      if bit?(e,0) then
        erg:=copy(y)
        qpot:SI:=k2
      else
        erg:=new(#a,inv(trGen)$GF)$VGF
        qpot:SI:=0
      for k in 1..length(e) repeat
        y:= y *$$ qPot(y,k2)
        k2:=addmod(k2,k2,deg)$SI
        if bit?(e,k) then
          erg:=erg *$$ qPot(y,qpot)
          qpot:=addmod(qpot,k2,deg)$SI
      erg

  -- computes qPot(a,n) = a**(q**n), q=size of GF
    qPot(e,n) ==
      ei:=(#e)::SI
      m:SI:= positiveRemainder(n::SI,ei)$SI
      zero?(m) => e
      e1:=zero(#e)$VGF
      for i in m+1..ei repeat e1.i:=e.(i-m)
      for i in 1..m    repeat e1.i:=e.(ei+i-m)
      e1

    trace(a,d) ==
      dSI:=d::SI
      r:=divide((#a)::SI,dSI)$SI
      not(r.remainder = 0) => error "trace: 2.arg must divide extdeg"
      v:=copy(a.(1..dSI))$VGF
      sSI:SI:=r.quotient
      for i in 1..dSI repeat
        for j in 1..sSI-1 repeat
          v.i:=v.i+a.(i+j::SI*dSI)
      v

    random(n) ==
      v:=zero(n)$VGF
      for i in 1..n repeat v.i:=random()$GF
      v


    xn(m) == monomial(1,m)$(SUP GF) - 1$(SUP GF)
```

```
    normal?(x) ==
      gcd(xn(#x),pol(x))$(SUP GF) = 1 => true
      false

    x:VGF / y:VGF == x *$$ inv(y)$$


    setFieldInfo(m,n) ==
      multTable:=m
      trGen:=n
      void()$Void

    minimalPolynomial(x) ==
      dx:=#x
      y:=new(#x,inv(trGen)$GF)$VGF
      m:=zero(dx,dx+1)$(M GF)
      for i in 1..dx+1 repeat
        dy:=#y
        for j in 1..dy repeat
          for k in 0..((dx quo dy)-1) repeat
            qsetelt_!(m,j+k*dy,i,y.j)$(M GF)
        y:=y *$$ x
      v:=first nullSpace(m)$(M GF)
      pol(v)$$

    basis(n) ==
      bas:(V VGF):=new(n,zero(n)$VGF)$(V VGF)
      for i in 1..n repeat
        uniti:=zero(n)$VGF
        qsetelt_!(uniti,i,1$GF)$VGF
        qsetelt_!(bas,i,uniti)$(V VGF)
      bas

    normalElement(n) ==
       v:=zero(n)$VGF
       qsetelt_!(v,1,1$GF)
       v
--    normalElement(n) == index(n,1)$$

    index(degm,n) ==
      m:I:=n rem$I (sizeGF ** degm)
      erg:=zero(degm)$VGF
      for j in 1..degm repeat
        erg.j:=index((sizeGF+(m rem sizeGF)) pretend PI)$GF
        m:=m quo sizeGF
      erg
```

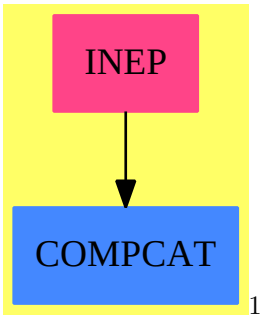
```
      pol(x) ==
        +/[monomial(x.i,(i-1)::NNI)$(SUP GF) for i in 1..(#x)::I]
```

⟨*INBFF.dotabb*⟩≡

```
"INBFF" [color="#FF4488",href="bookvol10.4.pdf#nameddest=INBFF"]
"IVECTOR" [color="#88FF44",href="bookvol10.3.pdf#nameddest=IVECTOR"]
"INBFF" -> "IVECTOR"
```

10.31 package INEP InnerNumericEigenPackage

10.32 InnerNumericEigenPackage



1.00

Exports:
charpol innerEigenvectors solve1

⟨*package INEP InnerNumericEigenPackage*⟩≡

```
)abbrev package INEP InnerNumericEigenPackage
++ Author:P. Gianni
++ Date Created: Summer 1990
++ Date Last Updated:Spring 1991
++ Basic Functions:
++ Related Constructors: ModularField
++ Also See:
++ AMS Classifications:
++ Keywords:
++ References:
++ Description:
++ This package is the inner package to be used by NumericRealEigenPackage
++ and NumericComplexEigenPackage for the computation of numeric
++ eigenvalues and eigenvectors.
InnerNumericEigenPackage(K,F,Par) : C == T
 where
   F    :   Field  -- this is the field where the answer will be
                   -- for dealing with the complex case
   K    :   Field  -- type of the  input
   Par  :   Join(Field,OrderedRing)  -- it will be NF or RN

   SE    ==> Symbol()
   RN    ==> Fraction Integer
   I     ==> Integer
   NF    ==> Float
   CF    ==> Complex Float
```

```
GRN   ==> Complex RN
GI    ==> Complex Integer
PI    ==> PositiveInteger
NNI   ==> NonNegativeInteger
MRN   ==> Matrix RN

MK          ==> Matrix K
PK          ==> Polynomial K
MF          ==> Matrix F
SUK         ==> SparseUnivariatePolynomial K
SUF         ==> SparseUnivariatePolynomial F
SUP         ==> SparseUnivariatePolynomial
MSUK        ==> Matrix SUK

PEigenForm  ==> Record(algpol:SUK,almult:Integer,poleigen:List(MSUK))

outForm     ==> Record(outval:F,outmult:Integer,outvect:List MF)

IntForm     ==> Union(outForm,PEigenForm)
UFactor     ==> (SUK -> Factored SUK)
C == with

  charpol  :  MK   ->  SUK
    ++ charpol(m) computes the characteristic polynomial of a matrix
    ++ m with entries in K.
    ++ This function returns a polynomial
    ++ over K, while the general one (that is in EiegenPackage) returns
    ++ Fraction P K

  solve1   : (SUK, Par) -> List F
    ++ solve1(pol, eps) finds the roots of the univariate polynomial
    ++ polynomial pol to precision eps. If K is \spad{Fraction Integer}
    ++ then only the real roots are returned, if K is
    ++ \spad{Complex Fraction Integer} then all roots are found.

  innerEigenvectors    : (MK,Par,UFactor)   ->  List(outForm)
    ++ innerEigenvectors(m,eps,factor) computes explicitly
    ++ the eigenvalues and the correspondent eigenvectors
    ++ of the matrix m. The parameter eps determines the type of
    ++ the output, factor is the univariate factorizer to br used
    ++ to reduce the characteristic polynomial into irreducible factors.

T == add

  numeric(r:K):F ==
    K is RN =>
```

```
      F is NF => convert(r)$RN
      F is RN    => r
      F is CF    => r :: RN :: CF
      F is GRN   => r::RN::GRN
    K is GRN =>
      F is GRN => r
      F is CF  => convert(convert r)
    error "unsupported coefficient type"

  ---- next functions neeeded for defining  ModularField ----

  monicize(f:SUK) : SUK ==
    (a:=leadingCoefficient f) =1 => f
    inv(a)*f

  reduction(u:SUK,p:SUK):SUK == u rem p

  merge(p:SUK,q:SUK):Union(SUK,"failed") ==
      p = q => p
      p = 0 => q
      q = 0 => p
      "failed"

  exactquo(u:SUK,v:SUK,p:SUK):Union(SUK,"failed") ==
     val:=extendedEuclidean(v,p,u)
     val case "failed" => "failed"
     val.coef1

      ----  eval a vector of F in a radical expression  ----
  evalvect(vect:MSUK,alg:F) : MF ==
    n:=nrows vect
    w:MF:=zero(n,1)$MF
    for i in 1..n repeat
      polf:=map(numeric,
        vect(i,1))$UnivariatePolynomialCategoryFunctions2(K,SUK,F,SUF)
      v:F:=elt(polf,alg)
      setelt(w,i,1,v)
    w

    ---- internal function for the computation of eigenvectors  ----
  inteigen(A:MK,p:SUK,fact:UFactor) : List(IntForm) ==
    dimA:NNI:=  nrows A
    MM:=ModularField(SUK,SUK,reduction,merge,exactquo)
    AM:=Matrix(MM)
    lff:=factors fact(p)
    res: List IntForm  :=[]
```

```
  lr : List MF:=[]
  for ff in lff repeat
    pol:SUK:= ff.factor
    if (degree pol)=1 then
      alpha:K:=-coefficient(pol,0)/leadingCoefficient pol
      -- compute the eigenvectors, rational case
      B1:MK := zero(dimA,dimA)$MK
      for i in 1..dimA repeat
        for j in 1..dimA repeat B1(i,j):=A(i,j)
        B1(i,i):= B1(i,i) - alpha
      lr:=[]
      for vecr in nullSpace B1 repeat
        wf:MF:=zero(dimA,1)
        for i in 1..dimA repeat wf(i,1):=numeric vecr.i
        lr:=cons(wf,lr)
      res:=cons([numeric alpha,ff.exponent,lr]$outForm,res)
    else
      ppol:=monicize pol
      alg:MM:= reduce(monomial(1,1),ppol)
      B:AM:= zero(dimA,dimA)$AM
      for i in 1..dimA  repeat
        for j in 1..dimA repeat B(i,j):=reduce(A(i,j) ::SUK,ppol)
        B(i,i):=B(i,i) - alg
      sln2:=nullSpace B
      soln:List MSUK :=[]
      for vec in sln2 repeat
        wk:MSUK:=zero(dimA,1)
        for i in 1..dimA repeat wk(i,1):=(vec.i)::SUK
        soln:=cons(wk,soln)
      res:=cons([ff.factor,ff.exponent,soln]$PEigenForm,
                      res)
  res

if K is RN then
    solve1(up:SUK, eps:Par) : List(F) ==
      denom := "lcm"/[denom(c::RN) for c in coefficients up]
      up:=denom*up
      upi:=map(numer,up)_
        $UnivariatePolynomialCategoryFunctions2(RN,SUP RN,I,SUP I)
      innerSolve1(upi, eps)$InnerNumericFloatSolvePackage(I,F,Par)
else if K is GRN then
    solve1(up:SUK, eps:Par) : List(F) ==
      denom := "lcm"/[lcm(denom real(c::GRN), denom imag(c::GRN))
                          for c in coefficients up]
      up:=denom*up
      upgi := map((c:GRN):GI+->complex(numer(real c), numer(imag c)),up)_
```

```
          $UnivariatePolynomialCategoryFunctions2(GRN,SUP GRN,GI,SUP GI)
        innerSolve1(upgi, eps)$InnerNumericFloatSolvePackage(GI,F,Par)
    else error "unsupported matrix type"

        ----  the real eigenvectors expressed as floats  ----

    innerEigenvectors(A:MK,eps:Par,fact:UFactor) : List outForm ==
      pol:= charpol A
      sln1:List(IntForm):=inteigen(A,pol,fact)
      n:=nrows A
      sln:List(outForm):=[]
      for lev in sln1 repeat
        lev case outForm => sln:=cons(lev,sln)
        leva:=lev::PEigenForm
        lval:List(F):= solve1(leva.algpol,eps)
        lvect:=leva.poleigen
        lmult:=leva.almult
        for alg in lval repeat
          nsl:=[alg,lmult,[evalvect(ep,alg) for ep in lvect]]$outForm
          sln:=cons(nsl,sln)
      sln

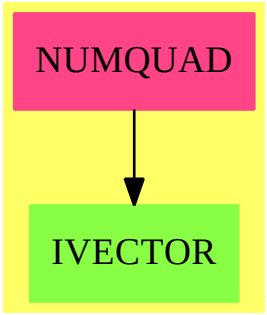
    charpol(A:MK) : SUK ==
      dimA :PI := (nrows A):PI
      dimA ^= ncols A => error " The matrix is not square"
      B:Matrix SUK :=zero(dimA,dimA)
      for i in 1..dimA repeat
        for j in 1..dimA repeat  B(i,j):=A(i,j)::SUK
        B(i,i) := B(i,i) - monomial(1,1)$SUK
      determinant B
```

⟨*INEP.dotabb*⟩≡

```
"INEP" [color="#FF4488",href="bookvol10.4.pdf#nameddest=INEP"]
"COMPCAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=COMPCAT"]
"INEP" -> "COMPCAT"
```

10.33 package INFSP InnerNumericFloatSolvePackage

10.34 InnerNumericFloatSolvePackage



1.00

Exports:
innerSolve innerSolve1 makeEq

⟨*package INFSP InnerNumericFloatSolvePackage*⟩≡

```
)abbrev package INFSP InnerNumericFloatSolvePackage
++ Author: P. Gianni
++ Date Created: January 1990
++ Date Last Updated:
++ Basic Functions:
++ Related Constructors:
++ Also See:
++ AMS Classifications:
++ Keywords:
++ References:
++ Description:
++ This is an internal package
++ for computing approximate solutions to systems of polynomial equations.
++ The parameter K specifies the coefficient field of the input polynomials
++ and must be either \spad{Fraction(Integer)} or \spad{Complex(Fraction Integer)
++ The parameter F specifies where the solutions must lie and can
++ be one of the following: \spad{Float}, \spad{Fraction(Integer)}, \spad{Complex
++ \spad{Complex(Fraction Integer)}. The last parameter specifies the type
++ of the precision operand and must be either \spad{Fraction(Integer)} or \spad{
InnerNumericFloatSolvePackage(K,F,Par): Cat == Cap where
    F    :   Field  -- this is the field where the answer will be
    K    :   GcdDomain  -- type of the  input
    Par  :   Join(Field, OrderedRing ) -- it will be NF or RN

    I        ==> Integer
    NNI      ==> NonNegativeInteger
```

```
P        ==> Polynomial
EQ       ==> Equation
L        ==> List
SUP      ==> SparseUnivariatePolynomial
RN       ==> Fraction Integer
NF       ==> Float
CF       ==> Complex Float
GI       ==> Complex Integer
GRN      ==> Complex RN
SE       ==> Symbol
RFI      ==> Fraction P I

Cat == with

   innerSolve1   :  (SUP K,Par)  -> L F
      ++ innerSolve1(up,eps) returns the list of the zeros
      ++ of the univariate polynomial up with precision eps.
   innerSolve1   :  (P K,Par)  -> L F
      ++ innerSolve1(p,eps) returns the list of the zeros
      ++ of the polynomial p with precision eps.
   innerSolve    : (L P K,L P K,L SE,Par) -> L L F
      ++ innerSolve(lnum,lden,lvar,eps) returns a list of
      ++ solutions of the system of polynomials lnum, with
      ++ the side condition that none of the members of lden
      ++ vanish identically on any solution. Each solution
      ++ is expressed as a list corresponding to the list of
      ++ variables in lvar and with precision specified by eps.
   makeEq        : (L F,L SE)     -> L EQ P F
      ++ makeEq(lsol,lvar) returns a list of equations formed
      ++ by corresponding members of lvar and lsol.

Cap == add

            ------  Local Functions  ------
   isGeneric? : (L P K,L SE) -> Boolean
   evaluate  : (P K,SE,SE,F) ->  F
   numeric   :     K          -> F
   oldCoord      : (L F,L I) -> L F
   findGenZeros  : (L P K,L SE,Par) -> L L F
   failPolSolve  : (L P K,L SE)  -> Union(L L P K,"failed")

   numeric(r:K):F ==
     K is I =>
       F is Float => r::I::Float
       F is RN    => r::I::RN
       F is CF    => r::I::CF
```

```
    F is GRN   => r::I::GRN
  K is GI =>
    gr:GI := r::GI
    F is GRN => complex(real(gr)::RN,imag(gr)::RN)$GRN
    F is CF  => convert(gr)
  error "case not handled"

-- construct the equation
makeEq(nres:L F,lv:L SE) : L EQ P F ==
    [equation(x::(P F),r::(P F)) for x in lv for r in nres]

evaluate(pol:P K,xvar:SE,zvar:SE,z:F):F ==
  rpp:=map(numeric,pol)$PolynomialFunctions2(K,F)
  rpp := eval(rpp,zvar,z)
  upol:=univariate(rpp,xvar)
  retract(-coefficient(upol,0))/retract(leadingCoefficient upol)

myConvert(eps:Par) : RN ==
  Par is RN => eps
  Par is NF => retract(eps)$NF

innerSolve1(pol:P K,eps:Par) : L F == innerSolve1(univariate pol,eps)

innerSolve1(upol:SUP K,eps:Par) : L F ==
  K is GI and (Par is RN or Par is NF) =>
      (complexZeros(upol,
                  eps)$ComplexRootPackage(SUP K,Par)) pretend L(F)
  K is I =>
    F is Float =>
      z:= realZeros(upol,myConvert eps)$RealZeroPackage(SUP I)
      [convert((1/2)*(x.left+x.right))@Float for x in z] pretend L(F)

    F is RN =>
      z:= realZeros(upol,myConvert eps)$RealZeroPackage(SUP I)
      [(1/2)*(x.left + x.right) for x in z] pretend L(F)
    error "improper arguments to INFSP"
  error "improper arguments to INFSP"


-- find the zeros of components in "generic" position --
findGenZeros(lp:L P K,rlvar:L SE,eps:Par) : L L F ==
  rlp:=reverse lp
  f:=rlp.first
  zvar:= rlvar.first
  rlp:=rlp.rest
  lz:=innerSolve1(f,eps)
```

```
    [reverse cons(z,[evaluate(pol,xvar,zvar,z) for pol in rlp
                    for xvar in rlvar.rest]) for z in lz]

  -- convert to the old coordinates --
  oldCoord(numres:L F,lval:L I) : L F ==
    rnumres:=reverse numres
    rnumres.first:= rnumres.first +
       (+/[n*nr for n in lval for nr in rnumres.rest])
    reverse rnumres

  -- real zeros of a system of 2 polynomials lp (incomplete)
  innerSolve2(lp:L P K,lv:L SE,eps: Par):L L F ==
     mainvar := first lv
     up1:=univariate(lp.1, mainvar)
     up2:=univariate(lp.2, mainvar)
     vec := subresultantVector(up1,up2)$SubResultantPackage(P K,SUP P K)
     p0 := primitivePart multivariate(vec.0, mainvar)
     p1 := primitivePart(multivariate(vec.1, mainvar),mainvar)
     zero? p1 or
       gcd(p0, leadingCoefficient(univariate(p1,mainvar))) ^=1 =>
         innerSolve(cons(0,lp),empty(),lv,eps)
     findGenZeros([p1, p0], reverse lv, eps)

  -- real zeros of the system of polynomial lp --
  innerSolve(lp:L P K,ld:L P K,lv:L SE,eps: Par) : L L F ==
     -- empty?(ld) and (#lv = 2) and (# lp = 2) => innerSolve2(lp, lv, eps)
      lnp:= [pToDmp(p)$PolToPol(lv,K) for p in lp]
      OV:=OrderedVariableList(lv)
      lvv:L OV:= [variable(vv)::OV for vv in lv]
      DP:=DirectProduct(#lv,NonNegativeInteger)
      dmp:=DistributedMultivariatePolynomial(lv,K)
      lq:L dmp:=[]
      if ld^=[] then
        lq:= [(pToDmp(q1)$PolToPol(lv,K)) pretend dmp for q1 in ld]
      partRes:=groebSolve(lnp,lvv)$GroebnerSolve(lv,K,K) pretend (L L dmp)
      partRes=list [] => []
      -- remove components where denominators vanish
      if lq^=[] then
        gb:=GroebnerInternalPackage(K,DirectProduct(#lv,NNI),OV,dmp)
        partRes:=[pr for pr in partRes|
                  and/[(redPol(fq,pr pretend List(dmp))$gb) ^=0
                    for fq in lq]]

      -- select the components in "generic" form
      rlv:=reverse lv
      rrlvv:= rest reverse lvv
```

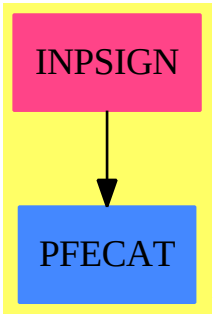
```
listGen:L L dmp:=[]
for res in partRes repeat
  res1:=rest reverse res
  "and"/[("max"/degree(f,rrlvv))=1  for f in res1] =>
           listGen:=concat(res pretend (L dmp),listGen)
result:L L F := []
if listGen^=[] then
  listG :L L P K:=
    [[dmpToP(pf)$PolToPol(lv,K) for pf in pr] for pr in listGen]
  result:=
    "append"/[findGenZeros(res,rlv,eps) for res in listG]
  for gres in listGen repeat
     partRes:=delete(partRes,position(gres,partRes))
-- adjust the non-generic components
for gres in partRes repeat
    genRecord := genericPosition(gres,lvv)$GroebnerSolve(lv,K,K)
    lgen := genRecord.dpolys
    lval := genRecord.coords
    lgen1:=[dmpToP(pf)$PolToPol(lv,K) for pf in lgen]
    lris:=findGenZeros(lgen1,rlv,eps)
    result:= append([oldCoord(r,lval) for r in lris],result)
result
```

⟨*INFSP.dotabb*⟩≡

```
"INFSP" [color="#FF4488",href="bookvol10.4.pdf#nameddest=INFSP"]
"COMPCAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=COMPCAT"]
"INFSP" -> "COMPCAT"
```

10.35 package INPSIGN InnerPolySign

10.36 InnerPolySign



1.00

Exports:
signAround

⟨*package INPSIGN InnerPolySign*⟩≡

```
)abbrev package INPSIGN InnerPolySign
--%% InnerPolySign
++ Author: Manuel Bronstein
++ Date Created: 23 Aug 1989
++ Date Last Updated: 19 Feb 1990
++ Description:
++ Find the sign of a polynomial around a point or infinity.
InnerPolySign(R, UP): Exports == Implementation where
  R : Ring
  UP: UnivariatePolynomialCategory R

  U ==> Union(Integer, "failed")

  Exports ==> with
    signAround: (UP,    Integer, R -> U) -> U
        ++ signAround(u,i,f) \undocumented
    signAround: (UP, R, Integer, R -> U) -> U
        ++ signAround(u,r,i,f) \undocumented
    signAround: (UP, R,          R -> U) -> U
        ++ signAround(u,r,f) \undocumented

  Implementation ==> add
    signAround(p:UP, x:R, rsign:R -> U) ==
      (ur := signAround(p, x,  1, rsign)) case "failed" => "failed"
      (ul := signAround(p, x, -1, rsign)) case "failed" => "failed"
      (ur::Integer) = (ul::Integer) => ur
      "failed"
```

```
signAround(p, x, dir, rsign) ==
  zero? p => 0
  zero?(r := p x) =>
    (u := signAround(differentiate p, x, dir, rsign)) case "failed"
      => "failed"
    dir * u::Integer
  rsign r

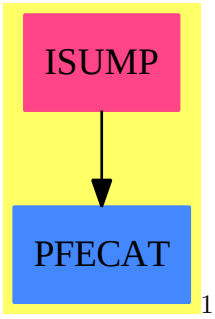
signAround(p:UP, dir:Integer, rsign:R -> U) ==
  zero? p => 0
  (u := rsign leadingCoefficient p) case "failed" => "failed"
  (dir > 0) or (even? degree p) => u::Integer
  - (u::Integer)
```

⟨*INPSIGN.dotabb*⟩≡

```
"INPSIGN" [color="#FF4488",href="bookvol10.4.pdf#nameddest=INPSIGN"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"INPSIGN" -> "PFECAT"
```

10.37 package ISUMP InnerPolySum

10.38 InnerPolySum



1.00

Exports:
sum

⟨*package ISUMP InnerPolySum*⟩≡

```
)abbrev package ISUMP InnerPolySum
++ Summation of polynomials
++ Author: SMW
++ Date Created: ???
++ Date Last Updated: 19 April 1991
++ Description: tools for the summation packages.
InnerPolySum(E, V, R, P): Exports == Impl where
    E: OrderedAbelianMonoidSup
    V: OrderedSet
    R: IntegralDomain
    P: PolynomialCategory(R, E, V)

    Z   ==> Integer
    Q   ==> Fraction Z
    SUP ==> SparseUnivariatePolynomial

    Exports ==> with
        sum: (P, V, Segment P) -> Record(num:P, den:Z)
            ++ sum(p(n), n = a..b) returns \spad{p(a) + p(a+1) + ... + p(b)}.
        sum: (P, V) -> Record(num:P, den: Z)
            ++ sum(p(n), n) returns \spad{P(n)},
            ++ the indefinite sum of \spad{p(n)} with respect to
            ++ upward difference on n, i.e. \spad{P(n+1) - P(n) = a(n)};

    Impl ==> add
        import PolynomialNumberTheoryFunctions()
        import UnivariatePolynomialCommonDenominator(Z, Q, SUP Q)
```

```
pmul: (P, SUP Q) -> Record(num:SUP P, den:Z)

pmul(c, p) ==
    pn := (rec := splitDenominator p).num
    [map(x +-> numer(x) * c, pn)_
       $SparseUnivariatePolynomialFunctions2(Q, P), rec.den]

sum(p, v, s) ==
  indef := sum(p, v)
  [eval(indef.num, v, 1 + hi s) - eval(indef.num, v, lo s),
   indef.den]

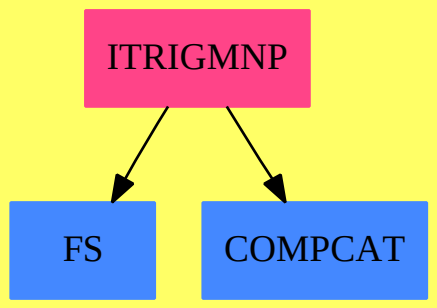
sum(p, v) ==
    up := univariate(p, v)
    lp := nil()$List(SUP P)
    ld := nil()$List(Z)
    while up ^= 0 repeat
        ud  := degree up; uc := leadingCoefficient up
        up  := reductum up
        rec := pmul(uc, 1 / (ud+1) * bernoulli(ud+1))
        lp  := concat(rec.num, lp)
        ld  := concat(rec.den, ld)
    d := lcm ld
    vp := +/[(d exquo di)::Z * pi for di in ld for pi in lp]
    [multivariate(vp, v), d]
```

⟨*ISUMP.dotabb*⟩≡

```
"ISUMP" [color="#FF4488",href="bookvol10.4.pdf#nameddest=ISUMP"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"ISUMP" -> "PFECAT"
```

10.39 package ITRIGMNP InnerTrigonometricManipulations

10.40 InnerTrigonometricManipulations



1.00

Exports:
F2FG FG2F GF2FG explogs2trigs trigs2explogs

⟨*package ITRIGMNP InnerTrigonometricManipulations*⟩≡
```
)abbrev package ITRIGMNP InnerTrigonometricManipulations
++ Trigs to/from exps and logs
++ Author: Manuel Bronstein
++ Date Created: 4 April 1988
++ Date Last Updated: 9 October 1993
++ Description:
++   This package provides transformations from trigonometric functions
++   to exponentials and logarithms, and back.
++   F and FG should be the same type of function space.
++ Keywords: trigonometric, function, manipulation.
InnerTrigonometricManipulations(R,F,FG): Exports == Implementation where
  R  : Join(IntegralDomain, OrderedSet)
  F  : Join(FunctionSpace R, RadicalCategory,
            TranscendentalFunctionCategory)
  FG : Join(FunctionSpace Complex R, RadicalCategory,
            TranscendentalFunctionCategory)

  Z   ==> Integer
  SY  ==> Symbol
  OP  ==> BasicOperator
  GR  ==> Complex R
  GF  ==> Complex F
  KG  ==> Kernel FG
  PG  ==> SparseMultivariatePolynomial(GR, KG)
  UP  ==> SparseUnivariatePolynomial PG
  NTHR  ==> "nthRoot"::SY
```

```
  Exports ==> with
    GF2FG        : GF -> FG
      ++ GF2FG(a + i b) returns \spad{a + i b} viewed as a function with
      ++ the \spad{i} pushed down into the coefficient domain.
    FG2F         : FG -> F
      ++ FG2F(a + i b) returns \spad{a + sqrt(-1) b}.
    F2FG         : F  -> FG
      ++ F2FG(a + sqrt(-1) b) returns \spad{a + i b}.
    explogs2trigs: FG -> GF
      ++ explogs2trigs(f) rewrites all the complex logs and
      ++ exponentials appearing in \spad{f} in terms of trigonometric
      ++ functions.
    trigs2explogs: (FG, List KG, List SY) -> FG
      ++ trigs2explogs(f, [k1,...,kn], [x1,...,xm]) rewrites
      ++ all the trigonometric functions appearing in \spad{f} and involving
      ++ one of the \spad{xi's} in terms of complex logarithms and
      ++ exponentials. A kernel of the form \spad{tan(u)} is expressed
      ++ using \spad{exp(u)**2} if it is one of the \spad{ki's}, in terms of
      ++ \spad{exp(2*u)} otherwise.

  Implementation ==> add
    ker2explogs: (KG, List KG, List SY) -> FG
    smp2explogs: (PG, List KG, List SY) -> FG
    supexp     : (UP, GF, GF, Z) -> GF
    GR2GF      : GR -> GF
    GR2F       : GR -> F
    KG2F       : KG -> F
    PG2F       : PG -> F
    ker2trigs  : (OP, List GF) -> GF
    smp2trigs  : PG -> GF
    sup2trigs  : (UP, GF) -> GF

    nth := R has RetractableTo(Integer) and F has RadicalCategory

    GR2F g        == real(g)::F + sqrt(-(1::F)) * imag(g)::F
    KG2F k        == map(FG2F, k)$ExpressionSpaceFunctions2(FG, F)
    FG2F f        == (PG2F numer f) / (PG2F denom f)
    F2FG f        == map(x +-> x::GR, f)$FunctionSpaceFunctions2(R,F,GR,FG)
    GF2FG f       == (F2FG real f) + complex(0, 1)$GR ::FG * F2FG imag f
    GR2GF gr      == complex(real(gr)::F, imag(gr)::F)

-- This expects the argument to have only tan and atans left.
-- Does a half-angle correction if k is not in the initial kernel list.
    ker2explogs(k, l, lx) ==
      empty?([v for v in variables(kf := k::FG) |
```

```
                                        member?(v, lx)]$List(SY)) => kf
  empty?(args := [trigs2explogs(a, l, lx)
                                  for a in argument k]$List(FG)) => kf
  im := complex(0, 1)$GR :: FG
  z  := first args
  is?(k, "tan"::Symbol)  =>
    e := (member?(k, l) => exp(im * z) ** 2;  exp(2 * im * z))
    - im * (e - 1) /$FG (e + 1)
  is?(k, "atan"::Symbol) =>
    im * log((1 -$FG im *$FG z)/$FG (1 +$FG im *$FG z))$FG / (2::FG)
  (operator k) args

trigs2explogs(f, l, lx) ==
  smp2explogs(numer f, l, lx) / smp2explogs(denom f, l, lx)

-- return op(arg) as f + %i g
-- op is already an operator with semantics over R, not GR
ker2trigs(op, arg) ==
  "and"/[zero? imag x for x in arg] =>
    complex(op [real x for x in arg]$List(F), 0)
  a := first arg
  is?(op, "exp"::Symbol)  => exp a
  is?(op, "log"::Symbol)  => log a
  is?(op, "sin"::Symbol)  => sin a
  is?(op, "cos"::Symbol)  => cos a
  is?(op, "tan"::Symbol)  => tan a
  is?(op, "cot"::Symbol)  => cot a
  is?(op, "sec"::Symbol)  => sec a
  is?(op, "csc"::Symbol)  => csc a
  is?(op, "asin"::Symbol)  => asin a
  is?(op, "acos"::Symbol)  => acos a
  is?(op, "atan"::Symbol)  => atan a
  is?(op, "acot"::Symbol)  => acot a
  is?(op, "asec"::Symbol)  => asec a
  is?(op, "acsc"::Symbol)  => acsc a
  is?(op, "sinh"::Symbol)  => sinh a
  is?(op, "cosh"::Symbol)  => cosh a
  is?(op, "tanh"::Symbol)  => tanh a
  is?(op, "coth"::Symbol)  => coth a
  is?(op, "sech"::Symbol)  => sech a
  is?(op, "csch"::Symbol)  => csch a
  is?(op, "asinh"::Symbol)  => asinh a
  is?(op, "acosh"::Symbol)  => acosh a
  is?(op, "atanh"::Symbol)  => atanh a
  is?(op, "acoth"::Symbol)  => acoth a
  is?(op, "asech"::Symbol)  => asech a
```

```
  is?(op, "acsch"::Symbol)  => acsch a
  is?(op, "abs"::Symbol)    => sqrt(norm a)::GF
  nth and is?(op, NTHR) => nthRoot(a, retract(second arg)@Z)
  error "ker2trigs: cannot convert kernel to gaussian function"

sup2trigs(p, f) ==
  map(smp2trigs, p)$SparseUnivariatePolynomialFunctions2(PG, GF) f

smp2trigs p ==
  map(x +-> explogs2trigs(x::FG),GR2GF, p)_
     $PolynomialCategoryLifting(IndexedExponents KG, KG, GR, PG, GF)

explogs2trigs f ==
  (m := mainKernel f) case "failed" =>
    GR2GF(retract(numer f)@GR) / GR2GF(retract(denom f)@GR)
  op  := operator(operator(k := m::KG))$F
  arg := [explogs2trigs x for x in argument k]
  num := univariate(numer f, k)
  den := univariate(denom f, k)
  is?(op, "exp"::Symbol) =>
    e  := exp real first arg
    y  := imag first arg
    g  := complex(e *  cos y, e * sin y)$GF
    gi := complex(cos(y) / e, - sin(y) / e)$GF
    supexp(num,g,gi,b := (degree num)::Z quo 2)/supexp(den,g,gi,b)
  sup2trigs(num, g := ker2trigs(op, arg)) / sup2trigs(den, g)

supexp(p, f1, f2, bse) ==
  ans:GF := 0
  while p ^= 0 repeat
    g := explogs2trigs(leadingCoefficient(p)::FG)
    if ((d := degree(p)::Z - bse) >= 0) then
         ans := ans + g * f1 ** d
    else ans := ans + g * f2 ** (-d)
    p := reductum p
  ans

PG2F p ==
  map(KG2F, GR2F, p)$PolynomialCategoryLifting(IndexedExponents KG,
                                                 KG, GR, PG, F)

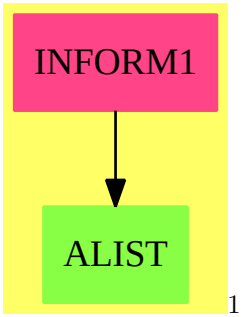
smp2explogs(p, l, lx) ==
  map(x +-> ker2explogs(x, l, lx), y +-> y::FG, p)_
   $PolynomialCategoryLifting(IndexedExponents KG, KG, GR, PG, FG)
```

⟨*ITRIGMNP.dotabb*⟩≡

```
"ITRIGMNP" [color="#FF4488",href="bookvol10.4.pdf#nameddest=ITRIGMNP"]
"FS" [color="#4488FF",href="bookvol10.2.pdf#nameddest=FS"]
"COMPCAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=COMPCAT"]
"ITRIGMNP" -> "FS"
"ITRIGMNP" -> "COMPCAT"
```

10.41 package INFORM1 InputFormFunctions1

10.42 InputFormFunctions1



1.00

Exports:
interpret packageCall

⟨*package INFORM1 InputFormFunctions1*⟩≡
```
)abbrev package INFORM1 InputFormFunctions1
--)boot $noSubsumption := false

++ Tools for manipulating input forms
++ Author: Manuel Bronstein
++ Date Created: ???
++ Date Last Updated: 19 April 1991
++ Description: Tools for manipulating input forms.

InputFormFunctions1(R:Type):with
  packageCall: Symbol -> InputForm
    ++ packageCall(f) returns the input form corresponding to f$R.
  interpret  : InputForm -> R
    ++ interpret(f) passes f to the interpreter, and transforms
    ++ the result into an object of type R.
 == add
  Rname := devaluate(R)$Lisp :: InputForm

  packageCall name ==
    convert([convert("$elt"::Symbol), Rname,
                                  convert name]$List(InputForm))@InputForm

  interpret form ==
    retract(interpret(convert([convert("@"::Symbol), form,
          Rname]$List(InputForm))@InputForm)$InputForm)$AnyFunctions1(R)
```

⟨*INFORM1.dotabb*⟩≡

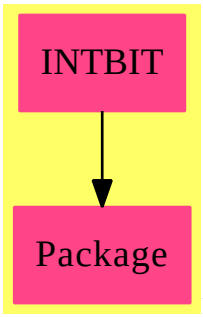
```
"INFORM1" [color="#FF4488",href="bookvol10.4.pdf#nameddest=INFORM1"]
"ALIST" [color="#88FF44",href="bookvol10.3.pdf#nameddest=ALIST"]
"INFORM1" -> "ALIST"
```

10.43 package INTBIT IntegerBits

Bug! Cannot precompute params and return a function which simply computes the last call. e.g. ridHack1, below.

Functions related to the binary representation of integers. These functions directly access the bits in the big integer representation and so are much facter than using a quotient loop.

10.44 IntegerBits



1.00

Exports:
bitLength bitCoef bitTruth

⟨*package INTBIT IntegerBits*⟩≡

```
)abbrev package INTBIT IntegerBits
++ Description:
++ This  package provides functions to lookup bits in integers
IntegerBits: with
    --  bitLength(n)  == # of bits to represent abs(n)
    --  bitCoef (n,i) == coef of 2**i in abs(n)
    --  bitTruth(n,i) == true if coef of 2**i in abs(n) is 1

        bitLength: Integer -> Integer
          ++ bitLength(n) returns the number of bits to represent abs(n)
        bitCoef:   (Integer, Integer) -> Integer
          ++ bitCoef(n,m) returns the coefficient of 2**m  in abs(n)
        bitTruth:  (Integer, Integer) -> Boolean
          ++ bitTruth(n,m) returns true if coefficient of 2**m in abs(n) is 1

    == add
        bitLength n   == INTEGER_-LENGTH(n)$Lisp
        bitCoef (n,i) == if INTEGER_-BIT(n,i)$Lisp then 1 else 0
        bitTruth(n,i) == INTEGER_-BIT(n,i)$Lisp
```

⟨*INTBIT.dotabb*⟩≡

```
"INTBIT" [color="#FF4488",href="bookvol10.4.pdf#nameddest=INTBIT"]
"Package" [color="#FF4488"]
"INTBIT" -> "Package"
```

10.45 package COMBINAT IntegerCombinatoricFunctions

⟨*IntegerCombinatoricFunctions.input*⟩≡

```
)set break resume
)sys rm -f IntegerCombinatoricFunctions.output
)spool IntegerCombinatoricFunctions.output
)set message test on
)set message auto off
)clear all
--S 1 of 4
)set expose add constructor OutputForm
--R
--I   OutputForm is already explicitly exposed in frame frame0
--E 1

--S 2 of 4
pascalRow(n) == [right(binomial(n,i),4) for i in 0..n]
--R
--R                                                                    Type: Void
--E 2

--S 3 of 4
displayRow(n)==output center blankSeparate pascalRow(n)
--R
--R                                                                    Type: Void
--E 3

--S 4 of 4
for i in 0..7 repeat displayRow i
--R
--R   Compiling function pascalRow with type NonNegativeInteger -> List
--R      OutputForm
--R   Compiling function displayRow with type NonNegativeInteger -> Void
--R                                      1
--R                                    1   1
--R                                  1   2   1
--R                                1   3   3   1
--R                              1   4   6   4   1
--R                            1   5   10  10   5   1
--R                          1   6   15  20  15   6   1
--R                        1   7   21  35  35  21   7   1
--R                                                                    Type: Void
--E 4
```

```
)spool
)lisp (bye)
```

⟨*IntegerCombinatoricFunctions.help*⟩≡

```
==========================================================================
IntegerCombinatoricFunctions examples
==========================================================================

The binomial(n, r) returns the number of subsets of r objects
taken among n objects, i.e. n!/(r! * (n-r)!)

The binomial coefficients are the coefficients of the series expansion
of a power of a binomial, that is

    n
   --+    / n \   k             n
   >     |     | x    =  (1 + x)
   --+    \ k /
   k= 0

This leads to the famous pascal triangle. First we expose the OutputForm
domain, which is normally hidden, so we can use it to format the lines.

  )set expose add constructor OutputForm

Next we define a function that will output the list of binomial coefficients
right justified with proper spacing:

  pascalRow(n) == [right(binomial(n,i),4) for i in 0..n]

and now we format the whole line so that it looks centered:

  displayRow(n)==output center blankSeparate pascalRow(n)

and we compute the triangle

  for i in 0..7 repeat displayRow i

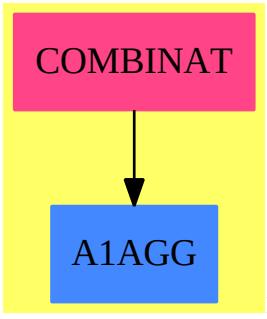
giving the pretty result:

   Compiling function pascalRow with type NonNegativeInteger -> List
      OutputForm
   Compiling function displayRow with type NonNegativeInteger -> Void
                                     1
                                   1   1
                                 1   2   1
                               1   3   3   1
                             1   4   6   4   1
                           1   5  10  10   5   1
```

```
                    1     6    15    20    15     6     1
                1     7    21    35    35    21     7     1
```

See Also:
o)show IntegerCombinatoricFunctions
o)d op binomial
o)show OutputForm
o)help set

10.46 IntegerCombinatoricFunctions



1.00

Exports:
binomial factorial multinomial partition permutation
stirling1 stirling2

⟨*package COMBINAT IntegerCombinatoricFunctions*⟩≡

```
)abbrev package COMBINAT IntegerCombinatoricFunctions
++ Authors: Martin Brock, Robert Sutor, Michael Monagan
++ Date Created: June 1987
++ Date Last Updated:
++ Basic Operations:
++ Related Domains:
++ Also See:
++ AMS Classifications:
++ Keywords: integer, combinatoric function
++ Examples:
++ References:
++ Description:
++   The \spadtype{IntegerCombinatoricFunctions} package provides some
++   standard functions in combinatorics.
Z   ==> Integer
N   ==> NonNegativeInteger
SUP ==> SparseUnivariatePolynomial

IntegerCombinatoricFunctions(I:IntegerNumberSystem): with
   binomial: (I, I) -> I
      ++ \spad{binomial(n,r)} returns the binomial coefficient
      ++ \spad{C(n,r) = n!/(r! (n-r)!)}, where \spad{n >= r >= 0}.
      ++ This is the number of combinations of n objects taken r at a time.
      ++
      ++X [binomial(5,i) for i in 0..5]
   factorial: I -> I
      ++ \spad{factorial(n)} returns \spad{n!}. this is the product of all
      ++ integers between 1 and n (inclusive).
      ++ Note: \spad{0!} is defined to be 1.
```

```
    multinomial: (I, List I) -> I
       ++ \spad{multinomial(n,[m1,m2,...,mk])} returns the multinomial
       ++ coefficient \spad{n!/(m1! m2! ... mk!)}.
    partition: I -> I
       ++ \spad{partition(n)} returns the number of partitions of the integer n.
       ++ This is the number of distinct ways that n can be written as
       ++ a sum of positive integers.
    permutation: (I, I) -> I
       ++ \spad{permutation(n)} returns \spad{!P(n,r) = n!/(n-r)!}. This is
       ++ the number of permutations of n objects taken r at a time.
    stirling1: (I, I) -> I
       ++ \spad{stirling1(n,m)} returns the Stirling number of the first kind
       ++ denoted \spad{S[n,m]}.
    stirling2: (I, I) -> I
       ++ \spad{stirling2(n,m)} returns the Stirling number of the second kind
       ++ denoted \spad{SS[n,m]}.
  == add
    F : Record(Fn:I, Fv:I) := [0,1]
    B : Record(Bn:I, Bm:I, Bv:I) := [0,0,0]
    S : Record(Sn:I, Sp:SUP I) := [0,0]
    P : IndexedFlexibleArray(I,0) := new(1,1)$IndexedFlexibleArray(I,0)

    partition n ==
       -- This is the number of ways of expressing n as a sum of positive
       -- integers, without regard to order.  For example partition 5 = 7
       -- since 5 = 1+1+1+1+1 = 1+1+1+2 = 1+2+2 = 1+1+3 = 1+4 = 2+3 = 5 .
       -- Uses O(sqrt n) term recurrence from Abramowitz & Stegun pp. 825
       -- p(n) = sum (-1)**k p(n-j) where 0 < j := (3*k**2+-k) quo 2 <= n
       minIndex(P) ^= 0 => error "Partition: must have minIndex of 0"
       m := #P
       n < 0 => error "partition is not defined for negative integers"
       n < m::I => P(convert(n)@Z)
       concat_!(P, new((convert(n+1)@Z - m)::N,0)$IndexedFlexibleArray(I,0))
       for i in m..convert(n)@Z repeat
          s:I := 1
          t:I := 0
          for k in 1.. repeat
             l := (3*k*k-k) quo 2
             l > i => leave
             u := l+k
             t := t + s * P(convert(i-l)@Z)
             u > i => leave
             t := t + s * P(convert(i-u)@Z)
             s := -s
          P.i := t
       P(convert(n)@Z)
```

```
factorial n ==
   s,f,t : I
   n < 0 => error "factorial not defined for negative integers"
   if n <= F.Fn then s := f := 1 else (s, f) := F
   for k in convert(s+1)@Z .. convert(n)@Z by 2 repeat
      if k::I = n then t := n else t := k::I * (k+1)::I
      f := t * f
   F.Fn := n
   F.Fv := f

binomial(n, m) ==
   s,b:I
   n < 0 or m < 0 or m > n => 0
   m = 0 => 1
   n < 2*m => binomial(n, n-m)
   (s,b) := (0,1)
   if B.Bn = n then
      B.Bm = m+1 =>
         b := (B.Bv * (m+1)) quo (n-m)
         B.Bn := n
         B.Bm := m
         return(B.Bv := b)
      if m >= B.Bm then (s := B.Bm; b := B.Bv) else (s,b) := (0,1)
   for k in convert(s+1)@Z .. convert(m)@Z repeat
     b := (b*(n-k::I+1)) quo k::I
   B.Bn := n
   B.Bm := m
   B.Bv := b

multinomial(n, m) ==
   for t in m repeat t < 0 => return 0
   n < _+/m => 0
   s:I := 1
   for t in m repeat s := s * factorial t
   factorial n quo s

permutation(n, m) ==
   t:I
   m < 0 or n < m => 0
   m := n-m
   p:I := 1
   for k in convert(m+1)@Z .. convert(n)@Z by 2 repeat
      if k::I = n then t := n else t := (k*(k+1))::I
      p := p * t
   p
```

```
stirling1(n, m) ==
   -- Definition: (-1)**(n-m) S[n,m] is the number of
   -- permutations of n symbols which have m cycles.
   n < 0 or m < 1 or m > n => 0
   m = n => 1
   S.Sn = n => coefficient(S.Sp, convert(m)@Z :: N)
   x := monomial(1, 1)$SUP(I)
   S.Sn := n
   S.Sp := x
   for k in 1 .. convert(n-1)@Z repeat S.Sp := S.Sp * (x - k::SUP(I))
   coefficient(S.Sp, convert(m)@Z :: N)

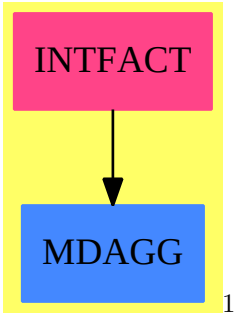
stirling2(n, m) ==
   -- definition: SS[n,m] is the number of ways of partitioning
   -- a set of n elements into m non-empty subsets
   n < 0 or m < 1 or m > n => 0
   m = 1 or n = m => 1
   s:I := if odd? m then -1 else 1
   t:I := 0
   for k in 1..convert(m)@Z repeat
      s := -s
      t := t + s * binomial(m, k::I) * k::I ** (convert(n)@Z :: N)
   t quo factorial m
```

⟨*COMBINAT.dotabb*⟩≡

```
"COMBINAT" [color="#FF4488",href="bookvol10.4.pdf#nameddest=COMBINAT"]
"A1AGG" [color="#4488FF",href="bookvol10.2.pdf#nameddest=A1AGG"]
"COMBINAT" -> "A1AGG"
```

10.47 package INTFACT IntegerFactorization-Package

10.48 IntegerFactorizationPackage



1.00

Exports:
BasicMethod factor squareFree PollardSmallFactor

⟨*package INTFACT IntegerFactorizationPackage*⟩≡
```
)abbrev package INTFACT IntegerFactorizationPackage
++ This Package contains basic methods for integer factorization.
++ The factor operation employs trial division up to 10,000.  It
++ then tests to see if n is a perfect power before using Pollards
++ rho method.  Because Pollards method may fail, the result
++ of factor may contain composite factors.  We should also employ
++ Lenstra's eliptic curve method.

IntegerFactorizationPackage(I): Exports == Implementation where
  I: IntegerNumberSystem

  B      ==> Boolean
  FF     ==> Factored I
  NNI    ==> NonNegativeInteger
  LMI    ==> ListMultiDictionary I
  FFE    ==> Record(flg:Union("nil","sqfr","irred","prime"),
                                                fctr:I, xpnt:Integer)

  Exports ==  with
    factor : I -> FF
      ++ factor(n) returns the full factorization of integer n
    squareFree   : I -> FF
      ++ squareFree(n) returns the square free factorization of integer n
    BasicMethod : I -> FF
      ++ BasicMethod(n) returns the factorization
      ++ of integer n by trial division
```

```
      PollardSmallFactor: I -> Union(I,"failed")
         ++ PollardSmallFactor(n) returns a factor
         ++ of n or "failed" if no one is found

    Implementation ==> add
      import IntegerRoots(I)

      BasicSieve: (I, I) -> FF
```

10.48.1 squareFree

⟨*package INTFACT IntegerFactorizationPackage*⟩+≡

```
      squareFree(n:I):FF ==
         u:I
         if n<0 then (m := -n; u := -1)
                else (m := n; u := 1)
         (m > 1) and ((v := perfectSqrt m) case I) =>
            for rec in (l := factorList(sv := squareFree(v::I))) repeat
              rec.xpnt := 2 * rec.xpnt
            makeFR(u * unit sv, l)
      -- avoid using basic sieve when the lim is too big
      -- we know the sieve constants up to sqrt(100000000)
         lim := 1 + approxSqrt(m)
         lim > (100000000::I) => makeFR(u, factorList factor m)
         x := BasicSieve(m, lim)
         y :=
           ((m:= unit x) = 1) => factorList x
           (v := perfectSqrt m) case I =>
              concat_!(factorList x, ["sqfr",v,2]$FFE)
           concat_!(factorList x, ["sqfr",m,1]$FFE)
         makeFR(u, y)
```

10.48.2 PollardSmallFactor

This is Brent's[**?**] optimization of Pollard's[**?**] rho factoring. Brent's algorithm is about 24 percent faster than Pollard's. Pollard;s algorithm has complexity $O(p^{1/2})$ where p is the smallest prime factor of the composite number N.

Pollard's idea is based on the observation that two numbers x and y are congruent modulo p with probability 0.5 after $1.177 * \sqrt{p}$ numbers have been randomly chosen. If we try to factor n and p is a factor of n, then

$$1 < gcd(|x - y|, n) \leq n$$

since p divides both $|x - y|$ and n.

Given a function f which generates a pseudo-random sequence of numbers we allow x to walk the sequence in order and y to walk the sequence at twice the rate. At each cycle we compute $gcd(|x - y|, n)$. If this GCD ever equals n then $x = y$ which means that we have walked "all the way around the pseudo-random cycle" and we terminate with failure.

This algorithm returns failure on all primes but also fails on some composite numbers.

Quoting Brent's back-tracking idea:

> The best-known algorithm for finding GCDs is the Euclidean algorithm which takes $O(\log N)$ times as long as one multiplication mod N. Pollard showed that most of the GCD computations in Floyd's algorithm could be dispensed with. ... The idea is simple: if P_F computes $GCD(z_1, N)$, $GCD(z_2, N)$,..., then we compute
>
> $$q_i = \prod_{j=1}^{i} z_j (\text{mod } N)$$
>
> and only compute $GCD(q_i, N)$ when i is a multiple of m, where $\log N << m << N^{1/4}$. Since $q_{i+1} = q_i \times z_{i+1} (\text{mod } N)$, the work required for each GCD computation in algorithm P_F is effectively reduced to that for a multiplication mod N in the modified algorithm. The probability of the algorithm failing because $q_i = 0$ increases, so it is best not to choose m too large. This problem can be minimized by backtracking to the state after the previous GCD computation and setting $m = 1$.

Brent incorporates back-tracking, omits the random choice of u, and makes some minor modifications. His algorithm (p192-183) reads:

$y := x_0; r := 1; q := 1;$

repeat $x := y;$

for $i := 1$ **to** r **do** $y := f(y); k := 0;$

repeat $ys := y$;
for $i := 1$ **to** $min(m, r-k)$ **do**
begin $y := f(y); q := q * |x - y| mod N$
end;
$G := GCD(q, N); k := k + m$
until $(k \geq r)$ **or** $(G > 1); r := 2 * r$
until $G > 1$;
if $G = N$ **then**
repeat $ys := f(ys); G := GCD(|y - yx|, N)$
until $G > 1$;
if $G = N$ **then** failure **else** success

Here we use the function

$$(y * y + 5 :: I) \; rem \; n$$

as our pseudo-random sequence with a random starting value for y.

On possible optimization to explore is to keep a hash table for the computed values of the function $y_{i+1} := f(y_i)$ since we effectively walk the sequence several times. And we walk the sequence in a loop many times. But because we are generating a very large number of numbers the array can be a simple array of fixed size that captures the last n values. So if we make a fixed array F of, say 2^q elements we can store $f(y_i)$ in F[y_i mod 2^q].

One property that this algorithm assumes is that the function used to generate the numbers has a long, hopefully complete, period. It is not clear that the recommended function has that property.

⟨*package INTFACT IntegerFactorizationPackage*⟩+≡

```
PollardSmallFactor(n:I):Union(I,"failed") ==
   -- Use the Brent variation
   x0 := random()$I
   m := 100::I
   y := x0 rem n
   r:I := 1
   q:I := 1
   G:I := 1
   until G > 1 repeat
      x := y
      for i in 1..convert(r)@Integer repeat
         y := (y*y+5::I) rem n
         k:I := 0
      until (k>=r) or (G>1) repeat
         ys := y
```

```
      for i in 1..convert(min(m,r-k))@Integer repeat
         y := (y*y+5::I) rem n
         q := q*abs(x-y) rem n
      G := gcd(q,n)
      k := k+m
   r := 2*r
if G=n then
   until G>1 repeat
      ys := (ys*ys+5::I) rem n
      G := gcd(abs(x-ys),n)
G=n => "failed"
G
```

10.48.3 BasicSieve

We create a list of prime numbers up to the limit given. The prior code used a circular list but tests of that list show that on average more than 50the required prime numbers. Overall this is a small percentage of the time needed to factor.

This loop uses three pieces of information

1. n which is the number we are testing
2. d which is the current prime to test
3. lim which is the upper limit of the primes to test

We loop d over the list of primes. If the remaining number n is smaller than the square of d then n must be prime and if it is not one, we add it to the list of primes. If the remaining number is larger than the square of d we remove all factors of d, reducing n each time. Then we add a record of the new factor and its multiplicity, m. We continue the loop until we run out of primes.

Annoyingly enough, primes does not return an ordered list so we fix this.

The sieve works up to a given limit, reducing out the factors that it finds. If it can find all of the factors than it returns a factored result where the first element is the unit 1. If there is still a part of the number unfactored it returns the number and a list of the factors found and their multiplicity.

Basically we just loop thru the prime factors checking to see if they are a component of the number, n. If so, we remove the factor from the number n (possibly m times) and continue thru the list of primes.

⟨*package INTFACT IntegerFactorizationPackage*⟩+≡

```
BasicSieve(n, lim) ==
   p:=primes(1::I,lim::I)$IntegerPrimesPackage(I)
   l:List(I) := append([first p],reverse rest p)
   ls := empty()$List(FFE)
   for d in l repeat
      if n<d*d then
         if n>1 then ls := concat_!(ls, ["prime",n,1]$FFE)
         return makeFR(1, ls)
      for m in 0.. while zero?(n rem d) repeat n := n quo d
      if m>0 then ls := concat_!(ls, ["prime",d,convert m]$FFE)
   makeFR(n,ls)
```

10.48.4 BasicMethod

⟨*package INTFACT IntegerFactorizationPackage*⟩+≡

```
BasicMethod n ==
   u:I
   if n<0 then (m := -n; u := -1)
          else (m := n; u := 1)
   x := BasicSieve(m, 1 + approxSqrt m)
   makeFR(u, factorList x)
```

10.48.5 factor

The factor function is many orders of magnitude slower than the results of other systems. A posting on sci.math.symbolic showed that NTL could factor the final value (t6) in about 11 seconds. Axiom takes about 8 hours.

```
a1:=101
a2:=109
t1:=a1*a2
factor t1

a3:=21525175387
t2:=t1*a3
factor t2

a4:=218301576858349
t3:=t2*a4
factor t3

a5:=13731482973783137
t4:=t3*a5
factor t4

a6:=23326138687706820109
t5:=t4*a6
factor t5

a7:=432824080117318843825281371694451836916l
t6:=t5*a7
factor t6
```

⟨*package INTFACT IntegerFactorizationPackage*⟩+≡

```
      factor m ==
         u:I
         zero? m => 0
         if negative? m then (n := -m; u := -1)
                        else (n := m; u := 1)
         b := BasicSieve(n, 10000::I)
         flb := factorList b
         ((n := unit b) = 1) => makeFR(u, flb)
         a:LMI := dictionary() -- numbers yet to be factored
         b:LMI := dictionary() -- prime factors found
         f:LMI := dictionary() -- number which could not be factored
         insert_!(n, a)
         while not empty? a repeat
            n := inspect a; c := count(n, a); remove_!(n, a)
            prime?(n)$IntegerPrimesPackage(I) => insert_!(n, b, c)
            -- test for a perfect power
            (s := perfectNthRoot n).exponent > 1 =>
```

```
            insert_!(s.base, a, c * s.exponent)
         -- test for a difference of square
         x:=approxSqrt n
         if (x**2<n) then x:=x+1
         (y:=perfectSqrt (x**2-n)) case I =>
               insert_!(x+y,a,c)
               insert_!(x-y,a,c)
         (d := PollardSmallFactor n) case I =>
            for m in 0.. while zero?(n rem d) repeat n := n quo d
            insert_!(d, a, m * c)
            if n > 1 then insert_!(n, a, c)
         -- an elliptic curve factorization attempt should be made here
         insert_!(n, f, c)
      -- insert prime factors found
      while not empty? b repeat
         n := inspect b; c := count(n, b); remove_!(n, b)
         flb := concat_!(flb, ["prime",n,convert c]$FFE)
      -- insert non-prime factors found
      while not empty? f repeat
         n := inspect f; c := count(n, f); remove_!(n, f)
         flb := concat_!(flb, ["nil",n,convert c]$FFE)
      makeFR(u, flb)
```

⟨*INTFACT.dotabb*⟩≡

```
"INTFACT" [color="#FF4488",href="bookvol10.4.pdf#nameddest=INTFACT"]
"MDAGG" [color="#4488FF",href="bookvol10.2.pdf#nameddest=MDAGG"]
"INTFACT" -> "MDAGG"
```

10.49 package ZLINDEP IntegerLinearDependence

⟨*IntegerLinearDependence.input*⟩≡

```
)set break resume
)spool IntegerLinearDependence.output
)set message test on
)set message auto off
)clear all

--S 1 of 8
M := SQMATRIX(2,INT)
--R
--R
--R   (1)  SquareMatrix(2,Integer)
--R                                                          Type: Domain
--E 1

--S 2 of 8
m1: M := squareMatrix matrix [ [1, 2], [0, -1] ]
--R
--R
--R        +1   2 +
--R   (2)  |      |
--R        +0  - 1+
--R                                          Type: SquareMatrix(2,Integer)
--E 2

--S 3 of 8
m2: M := squareMatrix matrix [ [2, 3], [1, -2] ]
--R
--R
--R        +2   3 +
--R   (3)  |      |
--R        +1  - 2+
--R                                          Type: SquareMatrix(2,Integer)
--E 3

--S 4 of 8
m3: M := squareMatrix matrix [ [3, 4], [2, -3] ]
--R
--R
--R        +3   4 +
--R   (4)  |      |
--R        +2  - 3+
```

```
--R                                                  Type: SquareMatrix(2,Integer)
--E 4

--S 5 of 8
linearlyDependentOverZ? vector [m1, m2, m3]
--R
--R
--R   (5)  true
--R                                                                Type: Boolean
--E 5

--S 6 of 8
c := linearDependenceOverZ vector [m1, m2, m3]
--R
--R
--R   (6)  [1,- 2,1]
--R                                              Type: Union(Vector Integer,...)
--E 6

--S 7 of 8
c.1 * m1 + c.2 * m2 + c.3 * m3
--R
--R
--R        +0  0+
--R   (7)  |    |
--R        +0  0+
--R                                                Type: SquareMatrix(2,Integer)
--E 7

--S 8 of 8
solveLinearlyOverQ(vector [m1, m3], m2)
--R
--R
--R         1 1
--R   (8)  [-,-]
--R         2 2
--R                                    Type: Union(Vector Fraction Integer,...)
--E 8
)spool
)lisp (bye)
```

⟨*IntegerLinearDependence.help*⟩≡

```
==========================================================================
IntegerLinearDependence examples
==========================================================================

The elements v1,...,vN of a module M over a ring R are said to be
linearly dependent over R if there exist c1,...,cN in R, not all 0,
such that c1 v1 + ...  cN vNn = 0.  If such ci's exist, they form
what is called a linear dependence relation over R for the vi's.

The package IntegerLinearDependence provides functions for testing
whether some elements of a module over the integers are linearly
dependent over the integers, and to find the linear dependence
relations, if any.

Consider the domain of two by two square matrices with integer entries.

  M := SQMATRIX(2,INT)
    SquareMatrix(2,Integer)
                        Type: Domain

Now create three such matrices.

  m1: M := squareMatrix matrix [ [1, 2], [0, -1] ]
     +1   2 +
     |      |
     +0  - 1+
                        Type: SquareMatrix(2,Integer)

  m2: M := squareMatrix matrix [ [2, 3], [1, -2] ]
    +2   3 +
    |      |
    +1  - 2+
                        Type: SquareMatrix(2,Integer)

  m3: M := squareMatrix matrix [ [3, 4], [2, -3] ]
    +3   4 +
    |      |
    +2  - 3+
                        Type: SquareMatrix(2,Integer)

This tells you whether m1, m2 and m3 are linearly dependent over the integers.

  linearlyDependentOverZ? vector [m1, m2, m3]
    true
                        Type: Boolean
```

```
Since they are linearly dependent, you can ask for the dependence relation.

  c := linearDependenceOverZ vector [m1, m2, m3]
    [1,- 2,1]
                        Type: Union(Vector Integer,...)

This means that the following linear combination should be 0.

  c.1 * m1 + c.2 * m2 + c.3 * m3
    +0  0+
    |    |
    +0  0+
                        Type: SquareMatrix(2,Integer)

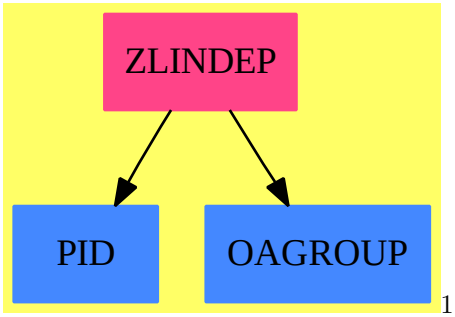
When a given set of elements are linearly dependent over R, this also
means that at least one of them can be rewritten as a linear combination
of the others with coefficients in the quotient field of R.

To express a given element in terms of other elements, use the operation
solveLinearlyOverQ.

  solveLinearlyOverQ(vector [m1, m3], m2)
     1 1
    [-,-]
     2 2
                        Type: Union(Vector Fraction Integer,...)

See Also:
o )show IntegerLinearDependence
```

10.50 IntegerLinearDependence



1.00

Exports:
linearDependenceOverZ linearlyDependentOverZ? solveLinearlyOverQ

⟨*package ZLINDEP IntegerLinearDependence*⟩≡
```
)abbrev package ZLINDEP IntegerLinearDependence
++ Test for linear dependence over the integers
++ Author: Manuel Bronstein
++ Date Created: ???
++ Date Last Updated: 14 May 1991
++ Description: Test for linear dependence over the integers.
IntegerLinearDependence(R): Exports == Implementation where
  R: LinearlyExplicitRingOver Integer

  Z ==> Integer

  Exports ==> with
    linearlyDependentOverZ?: Vector R -> Boolean
      ++ \spad{linearlyDependentOverZ?([v1,...,vn])} returns true if the
      ++ vi's are linearly dependent over the integers, false otherwise.
    linearDependenceOverZ  : Vector R -> Union(Vector Z, "failed")
      ++ \spad{linearlyDependenceOverZ([v1,...,vn])} returns
      ++ \spad{[c1,...,cn]} if
      ++ \spad{c1*v1 + ... + cn*vn = 0} and not all the ci's are 0, "failed"
      ++ if the vi's are linearly independent over the integers.
    solveLinearlyOverQ     : (Vector R, R) ->
                                    Union(Vector Fraction Z, "failed")
      ++ \spad{solveLinearlyOverQ([v1,...,vn], u)} returns \spad{[c1,...,cn]}
      ++ such that \spad{c1*v1 + ... + cn*vn = u},
      ++ "failed" if no such rational numbers ci's exist.

  Implementation ==> add
    import LinearDependence(Z, R)

    linearlyDependentOverZ? v == linearlyDependent? v
```

```
linearDependenceOverZ   v == linearDependence v
solveLinearlyOverQ(v, c)  == solveLinear(v, c)
```

⟨*ZLINDEP.dotabb*⟩≡

```
"ZLINDEP" [color="#FF4488",href="bookvol10.4.pdf#nameddest=ZLINDEP"]
"PID" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PID"]
"OAGROUP" [color="#4488FF",href="bookvol10.2.pdf#nameddest=OAGROUP"]
"ZLINDEP" -> "PID"
"ZLINDEP" -> "OAGROUP"
```

10.51 package INTHEORY IntegerNumberTheoryFunctions

⟨*IntegerNumberTheoryFunctions.input*⟩≡

```
)set break resume
)spool IntegerNumberTheoryFunctions.output
)set message test on
)set message auto off
)clear all

--S 1 of 30
div144 := divisors(144)
--R
--R
--R   (1)  [1,2,3,4,6,8,9,12,16,18,24,36,48,72,144]
--R                                                    Type: List Integer
--E 1

--S 2 of 30
#(div144)
--R
--R
--R   (2)  15
--R                                                 Type: PositiveInteger
--E 2

--S 3 of 30
reduce(+,div144)
--R
--R
--R   (3)  403
--R                                                 Type: PositiveInteger
--E 3

--S 4 of 30
numberOfDivisors(144)
--R
--R
--R   (4)  15
--R                                                 Type: PositiveInteger
--E 4

--S 5 of 30
sumOfDivisors(144)
--R
```

```
--R
--R   (5)  403
--R                                                    Type: PositiveInteger
--E 5

--S 6 of 30
f1(n)==reduce(+,[moebiusMu(d)*numberOfDivisors(quo(n,d))_
     for d in divisors(n)])
--R
--R                                                                Type: Void
--E 6

--S 7 of 30
f1(200)
--R
--R   Compiling function f1 with type PositiveInteger -> Integer
--R
--R   (7)  1
--R                                                    Type: PositiveInteger
--E 7

--S 8 of 30
f1(846)
--R
--R
--R   (8)  1
--R                                                    Type: PositiveInteger
--E 8

--S 9 of 30
f2(n) == reduce(+,[moebiusMu(d) * sumOfDivisors(quo(n,d))_
     for d in divisors(n)])
--R
--R                                                                Type: Void
--E 9

--S 10 of 30
f2(200)
--R
--R   Compiling function f2 with type PositiveInteger -> Integer
--R
--R   (10)  200
--R                                                    Type: PositiveInteger
--E 10

--S 11 of 30
```

```
f2(846)
--R
--R
--R   (11)  846
--R                                                    Type: PositiveInteger
--E 11

--S 12 of 30
fibonacci(25)
--R
--R
--R   (12)  75025
--R                                                    Type: PositiveInteger
--E 12

--S 13 of 30
[fibonacci(n) for n in 1..15]
--R
--R
--R   (13)  [1,1,2,3,5,8,13,21,34,55,89,144,233,377,610]
--R                                                       Type: List Integer
--E 13

--S 14 of 30
fib(n) == reduce(+,[binomial(n-1-k,k) for k in 0..quo(n-1,2)])
--R
--R                                                                 Type: Void
--E 14

--S 15 of 30
fib(25)
--R
--R   Compiling function fib with type PositiveInteger -> Integer
--R
--R   (15)  75025
--R                                                    Type: PositiveInteger
--E 15

--S 16 of 30
[fib(n) for n in 1..15]
--R
--R
--R   (16)  [1,1,2,3,5,8,13,21,34,55,89,144,233,377,610]
--R                                                       Type: List Integer
--E 16
```

```
--S 17 of 30
legendre(3,5)
--R
--R
--R   (17)  - 1
--R                                                          Type: Integer
--E 17

--S 18 of 30
legendre(23,691)
--R
--R
--R   (18)  - 1
--R                                                          Type: Integer
--E 18

--S 19 of 30
h(d) == quo(reduce(+,[jacobi(d,k) for k in 1..quo(-d, 2)]),2-jacobi(d,2))
--R
--R                                                             Type: Void
--E 19

--S 20 of 30
h(-163)
--R
--R   Compiling function h with type Integer -> Integer
--R
--R   (20)  1
--R                                                  Type: PositiveInteger
--E 20

--S 21 of 30
h(-499)
--R
--R
--R   (21)  3
--R                                                  Type: PositiveInteger
--E 21

--S 22 of 30
h(-1832)
--R
--R
--R   (22)  26
--R                                                  Type: PositiveInteger
--E 22
```

```
--S 23 of 30
inverse:(INT,INT)->INT
--R
--R                                                                    Type: Void
--E 23

--S 24 of 30
inverse(a,b) ==
  borg:INT:=b
  c1:INT := 1
  d1:INT := 0
  while b ~= 0 repeat
    q := a quo b
    r := a-q*b
    print [a, "=", q, "*(", b, ")+", r]
    (a,b):=(b,r)
    (c1,d1):=(d1,c1-q*d1)
  a ~= 1 => error("moduli are not relatively prime")
  positiveRemainder(c1,borg)
--R
--R                                                                    Type: Void
--E 24

--S 25 of 30
inverse(15,26)
--R
--R   Compiling function inverse with type (Integer,Integer) -> Integer
--R   [15,"=",0,"*(",26,")+",15]
--R   [26,"=",1,"*(",15,")+",11]
--R   [15,"=",1,"*(",11,")+",4]
--R   [11,"=",2,"*(",4,")+",3]
--R   [4,"=",1,"*(",3,")+",1]
--R   [3,"=",3,"*(",1,")+",0]
--R
--R   (25)  7
--R                                                  Type: PositiveInteger
--E 25

--S 26 of 30
x1:=4
--R
--R
--R   (26)  4
--R                                                  Type: PositiveInteger
--E 26
```

```
--S 27 of 30
m1:=5
--R
--R
--R   (27)  5
--R                                                    Type: PositiveInteger
--E 27

--S 28 of 30
x2:=2
--R
--R
--R   (28)  2
--R                                                    Type: PositiveInteger
--E 28

--S 29 of 30
m2:=3
--R
--R
--R   (29)  3
--R                                                    Type: PositiveInteger
--E 29

--S 30 of 30
result:=chineseRemainder(x1,m1,x2,m2)
--R
--R
--R   (30)  14
--R                                                    Type: PositiveInteger
--E 30

)spool
)lisp (bye)
```

⟨*IntegerNumberTheoryFunctions.help*⟩≡

```
==========================================================================
IntegerNumberTheoryFunctions examples
==========================================================================

The IntegerNumberTheoryFunctions package contains a variety of operations
of interest to number theorists.  Many of these operations deal with
divisibility properties of integers.  (Recall that an integer a divides
an integer b if there is an integer c such that b = a * c.)

The operation divisors returns a list of the divisors of an integer.

  div144 := divisors(144)
    [1,2,3,4,6,8,9,12,16,18,24,36,48,72,144]
                            Type: List Integer

You can now compute the number of divisors of 144 and the sum of the
divisors of 144 by counting and summing the elements of the list we
just created.

  #(div144)
    15
                            Type: PositiveInteger

  reduce(+,div144)
    403
                            Type: PositiveInteger

Of course, you can compute the number of divisors of an integer n,
usually denoted d(n), and the sum of the divisors of an integer n,
usually denoted sigma(n), without ever listing the divisors of n.

In Axiom, you can simply call the operations numberOfDivisors and
sumOfDivisors.

  numberOfDivisors(144)
    15
                            Type: PositiveInteger

  sumOfDivisors(144)
    403
                            Type: PositiveInteger

The key is that d(n) and sigma(n) are "multiplicative functions."
This means that when n and m are relatively prime, that is, when
n and m have no prime factor in common, then d(nm) = d(n) d(m) and
```

```
sigma(nm) = sigma(n) sigma(m).  Note that these functions are trivial to
compute when n is a prime power and are computed for general n from the
prime factorization of n.  Other examples of multiplicative functions
are sigma_k(n), the sum of the k-th powers of the divisors of n and
varphi(n), the number of integers between 1 and n which are prime to n.
The corresponding Axiom operations are called sumOfKthPowerDivisors and
eulerPhi.

An interesting function is mu(n), the Moebius mu function, defined as
follows: mu(1) = 1, mu(n) = 0, when n is divisible by a square, and
mu = (-1)^k, when n is the product of k distinct primes.  The corresponding
Axiom operation is moebiusMu. This function occurs in the following theorem:

Theorem: (Moebius Inversion Formula):
  Let f(n) be a function on the positive integers and let F(n)
  be defined by
    F(n) = \sum_{d | n} f(n)
  the sum of f(n) over d | n where the sum is taken over the
  positive divisors of n.  Then the values of f(n) can be recovered
  from the values of F(n): f(n) = sum_{d | n} \mu(n) F(n/d) where
  again the sum is taken over the positive divisors of n.

When f(n) = 1, then F(n) = d(n).  Thus, if you sum mu(d)..d(n/d) over
the positive divisors d of n, you should always get 1.

  f1(n)==reduce(+,[moebiusMu(d)*numberOfDivisors(quo(n,d))_
     for d in divisors(n)])
                              Type: Void

  f1(200)
    1
                              Type: PositiveInteger

  f1(846)
    1
                              Type: PositiveInteger

Similarly, when f(n) = n, then F(n) = sigma(n).  Thus, if you sum
mu(d)..sigma(n/d) over the positive divisors d of n, you should always get n.

  f2(n) == reduce(+,[moebiusMu(d) * sumOfDivisors(quo(n,d))_
     for d in divisors(n)])
                              Type: Void

  f2(200)
    200
```

```
                              Type: PositiveInteger

  f2(846)
    846
                              Type: PositiveInteger

The Fibonacci numbers are defined by
   F(1) = F(2) = 1 and
   F(n) = F(n-1) + F(n-2) for n = 3,4,...

The operation fibonacci computes the n-th Fibonacci number.

  fibonacci(25)
    75025
                              Type: PositiveInteger

  [fibonacci(n) for n in 1..15]
    [1,1,2,3,5,8,13,21,34,55,89,144,233,377,610]
                              Type: List Integer

Fibonacci numbers can also be expressed as sums of binomial coefficients.

  fib(n) == reduce(+,[binomial(n-1-k,k) for k in 0..quo(n-1,2)])
                               Type: Void

  fib(25)
    75025
                               Type: PositiveInteger

  [fib(n) for n in 1..15]
    [1,1,2,3,5,8,13,21,34,55,89,144,233,377,610]
                               Type: List Integer

Quadratic symbols can be computed with the operations legendre and
jacobi.  The Legendre symbol a/p is defined for integers a and p with
p an odd prime number.  By definition,

  (a/p) = +1, when a is a square (mod p),
  (a/p)= -1, when a is not a square (mod p), and
  (a/p) = 0, when a is divisible by p.

You compute (a/p) via the command legendre(a,p).

  legendre(3,5)
    - 1
                                Type: Integer
```

```
  legendre(23,691)
    - 1
                                Type: Integer
```

The Jacobi symbol (a/n) is the usual extension of the Legendre symbol, where n is an arbitrary integer. The most important property of the Jacobi symbol is the following: if K is a quadratic field with discriminant d and quadratic character chi, then chi(n) = (d/n). Thus, you can use the Jacobi symbol to compute, say, the class numbers of imaginary quadratic fields from a standard class number formula.

This function computes the class number of the imaginary quadratic field with discriminant d.

```
  h(d) == quo(reduce(+,[jacobi(d,k) for k in 1..quo(-d, 2)]),2-jacobi(d,2))
                                Type: Void

  h(-163)
    1
                                Type: PositiveInteger

  h(-499)
    3
                                Type: PositiveInteger

  h(-1832)
    26
                                Type: PositiveInteger
```

```
========================================================================
The inverse function
========================================================================
```

The inverse function is derived from the Extended Euclidean Algorithm. If we divide one integer by another nonzero integer we get an integer quotient plus a remainder which is, in general, a rational number. For instance,
  13/5 = 2 + 3/5
where 2 is the quotient and 3/5 is the remainder.

If we multiply thru by the denominator of the remainder we get an answer in integer terms which no longer involves division:
  13 = 2(5) + 3

This gives a method of dividing integers. Specifically, if a and b are

```
positive integers, there exist unique non-negative integers q and r so that

  a = qb + r , where 0 <= r < b

q is called the quotient and r the remainder.

The greatest common divisor of integers a and b, denoted by gcd(a,b),
is the largest integer that divides (without remainder) both a and
b. So, for example:
  gcd(15, 5)  = 5,
  gcd(7, 9)   = 1,
  gcd(12, 9)  = 3,
  gcd(81, 57) = 3.

The gcd of two integers can be found by repeated application of the
division algorithm, this is known as the Euclidean Algorithm. You
repeatedly divide the divisor by the remainder until the remainder is
0. The gcd is the last non-zero remainder in this algorithm. The
following example shows the algorithm.

Finding the gcd of 81 and 57 by the Euclidean Algorithm:
  81 = 1(57) + 24
  57 = 2(24) + 9
  24 = 2(9)  + 6
  9  = 1(6)  + 3
  6  = 2(3)  + 0

So the greatest commmon divisor, the GCD(81,51)=3.

If the gcd(a, b) = r then there exist integers s and t so that

  s(a) + t(b) = r

By back substitution in the steps in the Euclidean Algorithm, it is
possible to find these integers s and t. We shall do this with the
above example:

Starting with the next to last line, we have:

   3 = 9 -1(6)

From the line before that, we see that 6 = 24 - 2(9), so:

  3 = 9 - 1(24 - 2(9)) = 3(9) - 1(24)

From the line before that, we have 9 = 57 - 2(24), so:
```

```
  3 = 3( 57 - 2(24)) - 1(24) = 3(57) - 7(24)

And, from the line before that 24 = 81 - 1(57), giving us:

  3 = 3(57) - 7( 81 - 1(57)) = 10(57) -7(81)

So we have found s = -7 and t = 10.

The Extended Euclidean Algorithm computes the GCD(a,b) and
the values for s and t.

Suppose we were doing arithmetics modulo 26 and we needed to find the
inverse of a number mod 26. This turned out to be a difficult task (and
not always possible). We observed that a number x had an inverse mod 26
(i.e., a number y so that xy = 1 mod 26) if and only if gcd(x,26) = 1.
In the general case the inverse of x exists if and only if gcd(x, n) = 1
and if it exists then there exist integers s and t so that

  sx + tn = 1

But this says that sx = 1 + (-t)n, or in other words,

  sx == 1 mod n

So, s (reduced mod n if need be) is the inverse of x mod n.
The extended Euclidean algorithm calculates s efficiently.

======================================================================
Finding the inverse mod n
======================================================================

We will number the steps of the Euclidean algorithm starting with step 0.
The quotient obtained at step i will be denoted by qi and an auxillary
number, si. For the first two steps, the value of this number is given:
  s(0) = 0 and
  s(1) = 1.

For the remainder of the steps, we recursively calculate
  s(i) = s(i-2) - s(i-1) q(i-2) mod n

The algorithm starts by "dividing" n by x. If the last non-zero remainder
occurs at step k, then if this remainder is 1, x has an inverse and it is
s(k+2). If the remainder is not 1, then x does not have an inverse.

For example, find the inverse of 15 mod 26.
```

```
Step 0: 26 = 1(15) + 11 s(0) = 0
Step 1: 15 = 1(11) + 4  s(1) = 1
Step 2: 11 = 2(4) + 3   s(2) = 0  -  1( 1) mod 26 =  25
Step 3: 4  = 1(3) + 1   s(3) = 1  - 25( 1) mod 26 = -24 mod 26 = 2
Step 4: 3  = 3(1) + 0   s(4) = 25 -  2( 2) mod 26 =  21
                        s(5) = 2  - 21( 1) mod 26 = -19 mod 26 = 7

Notice that 15(7) = 105 = 1 + 4(26) == 1 (mod 26).

Using the half extended Euclidean algorithm we compute 1/a mod b.

  inverse:(INT,INT)->INT
                                Type: Void

  inverse(a,b) ==
    borg:INT:=b
    c1:INT := 1
    d1:INT := 0
    while b ~= 0 repeat
      q := a quo b
      r := a-q*b
      print [a, "=", q, "*(", b, ")+", r]
      (a,b):=(b,r)
      (c1,d1):=(d1,c1-q*d1)
    a ~= 1 => error("moduli are not relatively prime")
    positiveRemainder(c1,borg)
                                Type: Void

  inverse(15,26)
   [15,"=",0,"*(",26,")+",15]
   [26,"=",1,"*(",15,")+",11]
   [15,"=",1,"*(",11,")+",4]
   [11,"=",2,"*(",4,")+",3]
   [4,"=",1,"*(",3,")+",1]
   [3,"=",3,"*(",1,")+",0]

    7
                                Type: PositiveInteger


========================================================================
The Chinese Remainder Algorithm
========================================================================

Let m1,m2,...,mr be positive integers that are pairwise relatively prime.
```

```
Let x1,x2,..,xr be integers with 0 <= xi < mi. Then, there is exactly one
x in the interval 0 <= x < m1 ... m2 ... mr
that satisfies the remainder equations

  irem(x,mi) = xi, i=1,2,...,r

where irem is the positive integer remainder function.

For example, et x1 = 4, m1 = 5, x2 = 2, m2 = 3. We know that
  irem(x,m1) = x1
  irem(x,m2) = x2
where 0 <= x_ < m1 and 0 <= x2 < m2.

By the extended Euclidean Algorithm there are integers c and d such that
  c m1 + d m2 = 1

In this case we are looking for an integer such that
 irem(x,5) = 4,
 irem(x,3) = 2

The algorithm we use is to first compute the positive integer remainder of
x1 and m1 to get a new x1:

 x1 = positiveRemainder(x1,m1)
  4 = positiveRemainder(4,5)

Next compute the positive integer remainder of x2 and m2 to get a new x2:

 x2 = positiveRemainder(x2,m2)
  2 = positiveRemainder(2,3)

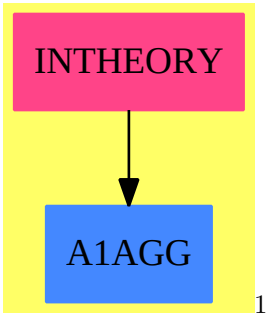
Then we compute x1 + m1 ... positiveRemainder(((x2-x1)*inverse(m1,m2)),m2)
or
  4+5*positiveRemainder(((2-4)*inverse(5,3)),3)
or
  4+5*positiveRemainder(-2*2),3)
or
  4+5*2
or
  14

This function has a restricted signature which only allows for
computing the chinese remainder of two numbers and two moduli.
  x1:=4
    4
                            Type: PositiveInteger
```

```
  m1:=5
    5
                          Type: PositiveInteger
  x2:=2
    2
                          Type: PositiveInteger
  m2:=3
    3
                          Type: PositiveInteger
  result:=chineseRemainder(x1,m1,x2,m2)
    14
                          Type: PositiveInteger

See Also:
o )show IntegerNumberTheoryFunctions
```

10.52 IntegerNumberTheoryFunctions



1.00

Exports:

bernoulli	chineseRemainder	divisors	euler	eulerPhi
fibonacci	harmonic	jacobi	legendre	moebiusMu
numberOfDivisors	sumOfDivisors	sumOfKthPowerDivisors		

⟨*package INTHEORY IntegerNumberTheoryFunctions*⟩≡

```
)abbrev package INTHEORY IntegerNumberTheoryFunctions
++ Author: Michael Monagan, Martin Brock, Robert Sutor, Timothy Daly
++ Date Created: June 1987
++ Date Last Updated:
++ Basic Operations:
++ Related Domains:
++ Also See:
++ AMS Classifications:
++ Keywords: number theory, integer
++ Examples:
++ References: Knuth, The Art of Computer Programming Vol.2
++ Description:
++ This package provides various number theoretic functions on the integers.
IntegerNumberTheoryFunctions(): Exports == Implementation where
 I ==> Integer
 RN ==> Fraction I
 SUP ==> SparseUnivariatePolynomial
 NNI ==> NonNegativeInteger

 Exports ==> with
  bernoulli : I -> RN
    ++ \spad{bernoulli(n)} returns the nth Bernoulli number.
    ++ this is \spad{B(n,0)}, where \spad{B(n,x)} is the \spad{n}th Bernoulli
    ++ polynomial.
  chineseRemainder: (I,I,I,I) -> I
    ++ \spad{chineseRemainder(x1,m1,x2,m2)} returns w, where w is such that
    ++ \spad{w = x1 mod m1} and \spad{w = x2 mod m2}. Note: \spad{m1} and
    ++ \spad{m2} must be relatively prime.
```

```
divisors : I -> List I
  ++ \spad{divisors(n)} returns a list of the divisors of n.
euler : I -> I
  ++ \spad{euler(n)} returns the \spad{n}th Euler number. This is
  ++ \spad{2^n E(n,1/2)}, where \spad{E(n,x)} is the nth Euler polynomial.
eulerPhi : I -> I
  ++ \spad{eulerPhi(n)} returns the number of integers between 1 and n
  ++ (including 1) which are relatively prime to n. This is the Euler phi
  ++ function \spad{\phi(n)} is also called the totient function.
fibonacci : I -> I
  ++ \spad{fibonacci(n)} returns the nth Fibonacci number. the Fibonacci
  ++ numbers \spad{F[n]} are defined by \spad{F[0] = F[1] = 1} and
  ++ \spad{F[n] = F[n-1] + F[n-2]}.
  ++ The algorithm has running time \spad{O(log(n)^3)}.
  ++ Reference: Knuth, The Art of Computer Programming
  ++ Vol 2, Semi-Numerical Algorithms.
harmonic : I -> RN
  ++ \spad{harmonic(n)} returns the nth harmonic number. This is
  ++ \spad{H[n] = sum(1/k,k=1..n)}.
jacobi : (I,I) -> I
  ++ \spad{jacobi(a,b)} returns the Jacobi symbol \spad{J(a/b)}.
  ++ When b is odd, \spad{J(a/b) = product(L(a/p) for p in factor b )}.
  ++ Note: by convention, 0 is returned if \spad{gcd(a,b) ^= 1}.
  ++ Iterative \spad{O(log(b)^2)} version coded by Michael Monagan June 1987.
legendre : (I,I) -> I
  ++ \spad{legendre(a,p)} returns the Legendre symbol \spad{L(a/p)}.
  ++ \spad{L(a/p) = (-1)**((p-1)/2) mod p} (p prime), which is 0 if \spad{a}
  ++ is 0, 1 if \spad{a} is a quadratic residue \spad{mod p} and -1 otherwise.
  ++ Note: because the primality test is expensive,
  ++ if it is known that p is prime then use \spad{jacobi(a,p)}.
moebiusMu : I -> I
  ++ \spad{moebiusMu(n)} returns the Moebius function \spad{mu(n)}.
  ++ \spad{mu(n)} is either -1,0 or 1 as follows:
  ++ \spad{mu(n) = 0} if n is divisible by a square > 1,
  ++ \spad{mu(n) = (-1)^k} if n is square-free and has k distinct
  ++ prime divisors.
numberOfDivisors: I -> I
  ++ \spad{numberOfDivisors(n)} returns the number of integers between 1 and n
  ++ (inclusive) which divide n. The number of divisors of n is often
  ++ denoted by \spad{tau(n)}.
sumOfDivisors : I -> I
  ++ \spad{sumOfDivisors(n)} returns the sum of the integers between 1 and n
  ++ (inclusive) which divide n. The sum of the divisors of n is often
  ++ denoted by \spad{sigma(n)}.
sumOfKthPowerDivisors: (I,NNI) -> I
  ++ \spad{sumOfKthPowerDivisors(n,k)} returns the sum of the \spad{k}th
```

```
    ++ powers of the integers between 1 and n (inclusive) which divide n.
    ++ the sum of the \spad{k}th powers of the divisors of n is often denoted
    ++ by \spad{sigma_k(n)}.
  Implementation ==> add
   import IntegerPrimesPackage(I)

   -- we store the euler and bernoulli numbers computed so far in
   -- a Vector because they are computed from an n-term recurrence
   E: IndexedFlexibleArray(I,0)   := new(1, 1)
   B: IndexedFlexibleArray(RN,0)  := new(1, 1)
   H: Record(Hn:I,Hv:RN) := [1, 1]

   harmonic n ==
     s:I; h:RN
     n < 0 => error("harmonic not defined for negative integers")
     if n >= H.Hn then (s,h) := H else (s := 0; h := 0)
     for k in s+1..n repeat h := h + 1/k
     H.Hn := n
     H.Hv := h
     h

   fibonacci n ==
     n = 0 => 0
     n < 0 => (odd? n => 1; -1) * fibonacci(-n)
     f1, f2 : I
     (f1,f2) := (0,1)
     for k in length(n)-2 .. 0 by -1 repeat
       t := f2**2
       (f1,f2) := (t+f1**2,t+2*f1*f2)
       if bit?(n,k) then (f1,f2) := (f2,f1+f2)
     f2

   euler n ==
     n < 0 => error "euler not defined for negative integers"
     odd? n => 0
     l := (#E) :: I
     n < l => E(n)
     concat_!(E, new((n+1-l)::NNI, 0)$IndexedFlexibleArray(I,0))
     for i in 1 .. l by 2 repeat E(i) := 0
     -- compute E(i) i = l+2,l+4,...,n given E(j) j = 0,2,...,i-2
     t,e : I
     for i in l+1 .. n by 2 repeat
       t := e := 1
       for j in 2 .. i-2 by 2 repeat
         t := (t*(i-j+1)*(i-j+2)) quo (j*(j-1))
         e := e + t*E(j)
```

```
      E(i) := -e
    E(n)

  bernoulli n ==
    n < 0 => error "bernoulli not defined for negative integers"
    odd? n =>
      n = 1 => -1/2
      0
    l := (#B) :: I
    n < l => B(n)
    concat_!(B, new((n+1-l)::NNI, 0)$IndexedFlexibleArray(RN,0))
    for i in 1 .. l by 2 repeat B(i) := 0
    -- compute B(i) i = l+2,l+4,...,n given B(j) j = 0,2,...,i-2
    for i in l+1 .. n by 2 repeat
      t:I := 1
      b := (1-i)/2
      for j in 2 .. i-2 by 2 repeat
        t := (t*(i-j+2)*(i-j+3)) quo (j*(j-1))
        b := b + (t::RN) * B(j)
      B(i) := -b/((i+1)::RN)
    B(n)

  inverse : (I,I) -> I

  inverse(a,b) ==
    borg:I:=b
    c1:I := 1
    d1:I := 0
    while b ^= 0 repeat
      q:I := a quo b
      r:I := a-q*b
      (a,b):=(b,r)
      (c1,d1):=(d1,c1-q*d1)
    a ^= 1 => error("moduli are not relatively prime")
    positiveRemainder(c1,borg)

  chineseRemainder(x1,m1,x2,m2) ==
    m1 < 0 or m2 < 0 => error "moduli must be positive"
    x1 := positiveRemainder(x1,m1)
    x2 := positiveRemainder(x2,m2)
    x1 + m1 * positiveRemainder(((x2-x1) * inverse(m1,m2)),m2)

  jacobi(a,b) ==
    -- Revised by Clifton Williamson January 1989.
    -- Previous version returned incorrect answers when b was even.
    -- The formula J(a/b) = product ( L(a/p) for p in factor b) is only
```

```
  -- valid when b is odd (the Legendre symbol L(a/p) is not defined
  -- for p = 2).  When b is even, the Jacobi symbol J(a/b) is only
  -- defined for a = 0 or 1 (mod 4).  When a = 1 (mod 8),
  -- J(a/2) = +1 and when a = 5 (mod 8), we define J(a/2) = -1.
  -- Extending by multiplicativity, we have J(a/b) for even b and
  -- appropriate a.
  -- We also define J(a/1) = 1.
  -- The point of this is the following: if d is the discriminant of
  -- a quadratic field K and chi is the quadratic character for K,
  -- then J(d/n) = chi(n) for n > 0.
  -- Reference: Hecke, Vorlesungen ueber die Theorie der Algebraischen
  -- Zahlen.
  if b < 0 then b := -b
  b = 0 => error "second argument of jacobi may not be 0"
  b = 1 => 1
  even? b and positiveRemainder(a,4) > 1 =>
    error "J(a/b) not defined for b even and a = 2 or 3 (mod 4)"
  even? b and even? a => 0
  for k in 0.. while even? b repeat b := b quo 2
  j:I := (odd? k and positiveRemainder(a,8) = 5 => -1; 1)
  b = 1 => j
  a := positiveRemainder(a,b)
  -- assertion: 0 < a < b and odd? b
  while a > 1 repeat
    if odd? a then
      -- J(a/b) = J(b/a) (-1) ** (a-1)/2 (b-1)/2
      if a rem 4 = 3 and b rem 4 = 3 then j := -j
      (a,b) := (b rem a,a)
    else
      -- J(2*a/b) = J(a/b) (-1) (b**2-1)/8
      for k in 0.. until odd? a repeat a := a quo 2
      if odd? k and (b+2) rem 8 > 4 then j := -j
  a = 0 => 0
  j

legendre(a,p) ==
  prime? p => jacobi(a,p)
  error "characteristic of legendre must be prime"

eulerPhi n ==
  n = 0 => 0
  r : RN := 1
  for entry in factors factor n repeat
    r := ((entry.factor - 1) /$RN entry.factor) * r
  numer(n * r)
```

```
divisors n ==
  oldList : List Integer := [1]
  for f in factors factor n repeat
    newList : List Integer := oldList
    for k in 1..f.exponent repeat
      pow := f.factor ** k
      for m in oldList repeat
        newList := concat(pow * m,newList)
    oldList := newList
  sort((i1:Integer,i2:Integer):Boolean +-> i1 < i2,oldList)

numberOfDivisors n ==
  n = 0 => 0
  */[1+entry.exponent for entry in factors factor n]

sumOfDivisors n ==
  n = 0 => 0
  r : RN := */[(entry.factor**(entry.exponent::NNI + 1)-1)/
    (entry.factor-1) for entry in factors factor n]
  numer r

sumOfKthPowerDivisors(n,k) ==
  n = 0 => 0
  r : RN := */[(entry.factor**(k*entry.exponent::NNI+k)-1)/
    (entry.factor**k-1) for entry in factors factor n]
  numer r

moebiusMu n ==
  n = 1 => 1
  t := factor n
  for k in factors t repeat
    k.exponent > 1 => return 0
  odd? numberOfFactors t => -1
  1
```

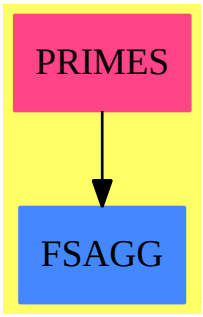
⟨*INTHEORY.dotabb*⟩≡

```
"INTHEORY" [color="#FF4488",href="bookvol10.4.pdf#nameddest=INTHEORY"]
"A1AGG" [color="#4488FF",href="bookvol10.2.pdf#nameddest=A1AGG"]
"INTHEORY" -> "A1AGG"
```

10.53 package PRIMES IntegerPrimesPackage

We've expanded the list of small primes to include those between 1 and 10000.

10.54 IntegerPrimesPackage



1.00

Exports:
nextPrime prevPrime prime? primes

⟨*package PRIMES IntegerPrimesPackage*⟩≡

```
)abbrev package PRIMES IntegerPrimesPackage
++ Author: Michael Monagan
++ Date Created: August 1987
++ Date Last Updated: 31 May 1993
++ Updated by: James Davenport
++ Updated Because: of problems with strong pseudo-primes
++   and for some efficiency reasons.
++ Basic Operations:
++ Related Domains:
++ Also See:
++ AMS Classifications:
++ Keywords: integer, prime
++ Examples:
++ References: Davenport's paper in ISSAC 1992
++             AXIOM Technical Report ATR/6
++ Description:
++   The \spadtype{IntegerPrimesPackage} implements a modification of
++   Rabin's probabilistic
++   primality test and the utility functions \spadfun{nextPrime},
++   \spadfun{prevPrime} and \spadfun{primes}.
IntegerPrimesPackage(I:IntegerNumberSystem): with
   prime?: I -> Boolean
     ++ \spad{prime?(n)} returns true if n is prime and false if not.
     ++ The algorithm used is Rabin's probabilistic primality test
```

```
    ++ (reference: Knuth Volume 2 Semi Numerical Algorithms).
    ++ If \spad{prime? n} returns false, n is proven composite.
    ++ If \spad{prime? n} returns true, prime? may be in error
    ++ however, the probability of error is very low.
    ++ and is zero below 25*10**9 (due to a result of Pomerance et al),
    ++ below 10**12 and 10**13 due to results of Pinch,
    ++ and below 341550071728321 due to a result of Jaeschke.
    ++ Specifically, this implementation does at least 10 pseudo prime
    ++ tests and so the probability of error is \spad{< 4**(-10)}.
    ++ The running time of this method is cubic in the length
    ++ of the input n, that is \spad{O( (log n)**3 )}, for n<10**20.
    ++ beyond that, the algorithm is quartic, \spad{O( (log n)**4 )}.
    ++ Two improvements due to Davenport have been incorporated
    ++ which catches some trivial strong pseudo-primes, such as
    ++ [Jaeschke, 1991] 1377161253229053 * 413148375987157, which
    ++ the original algorithm regards as prime
  nextPrime: I -> I
    ++ \spad{nextPrime(n)} returns the smallest prime strictly larger than n
  prevPrime: I -> I
    ++ \spad{prevPrime(n)} returns the largest prime strictly smaller than n
  primes: (I,I) -> List I
    ++ \spad{primes(a,b)} returns a list of all primes p with
    ++ \spad{a <= p <= b}
== add
```

10.54.1 smallPrimes

This is a table of all of the primes in [2..10000]. It is used by the prime? function to check for primality. It is used by the primes function to generate arrays of primes in a given range. Changing the range included in this table implies changing the value of the nextSmallPrime variable. There is a constant in the function squareFree from IntegerFactorizationPackage that is the square of the upper bound of the table range, in this case 10000000.

⟨*package PRIMES IntegerPrimesPackage*⟩+≡

```
smallPrimes: List I :=
  [2::I, 3::I, 5::I, 7::I, 11::I, 13::I, 17::I, 19::I,_
   23::I, 29::I, 31::I, 37::I, 41::I, 43::I, 47::I, 53::I,_
   59::I, 61::I, 67::I, 71::I, 73::I, 79::I, 83::I, 89::I,_
   97::I, 101::I, 103::I, 107::I, 109::I, 113::I, 127::I,_
   131::I, 137::I, 139::I, 149::I, 151::I, 157::I, 163::I,_
   167::I, 173::I, 179::I, 181::I, 191::I, 193::I, 197::I,_
   199::I, 211::I, 223::I, 227::I, 229::I, 233::I, 239::I,_
   241::I, 251::I, 257::I, 263::I, 269::I, 271::I, 277::I,_
   281::I, 283::I, 293::I, 307::I, 311::I, 313::I, 317::I,_
   331::I, 337::I, 347::I, 349::I, 353::I, 359::I, 367::I,_
   373::I, 379::I, 383::I, 389::I, 397::I, 401::I, 409::I,_
   419::I, 421::I, 431::I, 433::I, 439::I, 443::I, 449::I,_
   457::I, 461::I, 463::I, 467::I, 479::I, 487::I, 491::I,_
   499::I, 503::I, 509::I, 521::I, 523::I, 541::I, 547::I,_
   557::I, 563::I, 569::I, 571::I, 577::I, 587::I, 593::I,_
   599::I, 601::I, 607::I, 613::I, 617::I, 619::I, 631::I,_
   641::I, 643::I, 647::I, 653::I, 659::I, 661::I, 673::I,_
   677::I, 683::I, 691::I, 701::I, 709::I, 719::I, 727::I,_
   733::I, 739::I, 743::I, 751::I, 757::I, 761::I, 769::I,_
   773::I, 787::I, 797::I, 809::I, 811::I, 821::I, 823::I,_
   827::I, 829::I, 839::I, 853::I, 857::I, 859::I, 863::I,_
   877::I, 881::I, 883::I, 887::I, 907::I, 911::I, 919::I,_
   929::I, 937::I, 941::I, 947::I, 953::I, 967::I, 971::I,_
   977::I, 983::I, 991::I, 997::I, 1009::I, 1013::I,_
   1019::I, 1021::I, 1031::I, 1033::I, 1039::I, 1049::I,_
   1051::I, 1061::I, 1063::I, 1069::I, 1087::I, 1091::I,_
   1093::I, 1097::I, 1103::I, 1109::I, 1117::I, 1123::I,_
   1129::I, 1151::I, 1153::I, 1163::I, 1171::I, 1181::I,_
   1187::I, 1193::I, 1201::I, 1213::I, 1217::I, 1223::I,_
   1229::I, 1231::I, 1237::I, 1249::I, 1259::I, 1277::I,_
   1279::I, 1283::I, 1289::I, 1291::I, 1297::I, 1301::I,_
   1303::I, 1307::I, 1319::I, 1321::I, 1327::I, 1361::I,_
   1367::I, 1373::I, 1381::I, 1399::I, 1409::I, 1423::I,_
   1427::I, 1429::I, 1433::I, 1439::I, 1447::I, 1451::I,_
   1453::I, 1459::I, 1471::I, 1481::I, 1483::I, 1487::I,_
```

```
1489::I, 1493::I, 1499::I, 1511::I, 1523::I, 1531::I,_
1543::I, 1549::I, 1553::I, 1559::I, 1567::I, 1571::I,_
1579::I, 1583::I, 1597::I, 1601::I, 1607::I, 1609::I,_
1613::I, 1619::I, 1621::I, 1627::I, 1637::I, 1657::I,_
1663::I, 1667::I, 1669::I, 1693::I, 1697::I, 1699::I,_
1709::I, 1721::I, 1723::I, 1733::I, 1741::I, 1747::I,_
1753::I, 1759::I, 1777::I, 1783::I, 1787::I, 1789::I,_
1801::I, 1811::I, 1823::I, 1831::I, 1847::I, 1861::I,_
1867::I, 1871::I, 1873::I, 1877::I, 1879::I, 1889::I,_
1901::I, 1907::I, 1913::I, 1931::I, 1933::I, 1949::I,_
1951::I, 1973::I, 1979::I, 1987::I, 1993::I, 1997::I,_
1999::I, 2003::I, 2011::I, 2017::I, 2027::I, 2029::I,_
2039::I, 2053::I, 2063::I, 2069::I, 2081::I, 2083::I,_
2087::I, 2089::I, 2099::I, 2111::I, 2113::I, 2129::I,_
2131::I, 2137::I, 2141::I, 2143::I, 2153::I, 2161::I,_
2179::I, 2203::I, 2207::I, 2213::I, 2221::I, 2237::I,_
2239::I, 2243::I, 2251::I, 2267::I, 2269::I, 2273::I,_
2281::I, 2287::I, 2293::I, 2297::I, 2309::I, 2311::I,_
2333::I, 2339::I, 2341::I, 2347::I, 2351::I, 2357::I,_
2371::I, 2377::I, 2381::I, 2383::I, 2389::I, 2393::I,_
2399::I, 2411::I, 2417::I, 2423::I, 2437::I, 2441::I,_
2447::I, 2459::I, 2467::I, 2473::I, 2477::I, 2503::I,_
2521::I, 2531::I, 2539::I, 2543::I, 2549::I, 2551::I,_
2557::I, 2579::I, 2591::I, 2593::I, 2609::I, 2617::I,_
2621::I, 2633::I, 2647::I, 2657::I, 2659::I, 2663::I,_
2671::I, 2677::I, 2683::I, 2687::I, 2689::I, 2693::I,_
2699::I, 2707::I, 2711::I, 2713::I, 2719::I, 2729::I,_
2731::I, 2741::I, 2749::I, 2753::I, 2767::I, 2777::I,_
2789::I, 2791::I, 2797::I, 2801::I, 2803::I, 2819::I,_
2833::I, 2837::I, 2843::I, 2851::I, 2857::I, 2861::I,_
2879::I, 2887::I, 2897::I, 2903::I, 2909::I, 2917::I,_
2927::I, 2939::I, 2953::I, 2957::I, 2963::I, 2969::I,_
2971::I, 2999::I, 3001::I, 3011::I, 3019::I, 3023::I,_
3037::I, 3041::I, 3049::I, 3061::I, 3067::I, 3079::I,_
3083::I, 3089::I, 3109::I, 3119::I, 3121::I, 3137::I,_
3163::I, 3167::I, 3169::I, 3181::I, 3187::I, 3191::I,_
3203::I, 3209::I, 3217::I, 3221::I, 3229::I, 3251::I,_
3253::I, 3257::I, 3259::I, 3271::I, 3299::I, 3301::I,_
3307::I, 3313::I, 3319::I, 3323::I, 3329::I, 3331::I,_
3343::I, 3347::I, 3359::I, 3361::I, 3371::I, 3373::I,_
3389::I, 3391::I, 3407::I, 3413::I, 3433::I, 3449::I,_
3457::I, 3461::I, 3463::I, 3467::I, 3469::I, 3491::I,_
3499::I, 3511::I, 3517::I, 3527::I, 3529::I, 3533::I,_
3539::I, 3541::I, 3547::I, 3557::I, 3559::I, 3571::I,_
3581::I, 3583::I, 3593::I, 3607::I, 3613::I, 3617::I,_
3623::I, 3631::I, 3637::I, 3643::I, 3659::I, 3671::I,_
```

```
3673::I, 3677::I, 3691::I, 3697::I, 3701::I, 3709::I,_
3719::I, 3727::I, 3733::I, 3739::I, 3761::I, 3767::I,_
3769::I, 3779::I, 3793::I, 3797::I, 3803::I, 3821::I,_
3823::I, 3833::I, 3847::I, 3851::I, 3853::I, 3863::I,_
3877::I, 3881::I, 3889::I, 3907::I, 3911::I, 3917::I,_
3919::I, 3923::I, 3929::I, 3931::I, 3943::I, 3947::I,_
3967::I, 3989::I, 4001::I, 4003::I, 4007::I, 4013::I,_
4019::I, 4021::I, 4027::I, 4049::I, 4051::I, 4057::I,_
4073::I, 4079::I, 4091::I, 4093::I, 4099::I, 4111::I,_
4127::I, 4129::I, 4133::I, 4139::I, 4153::I, 4157::I,_
4159::I, 4177::I, 4201::I, 4211::I, 4217::I, 4219::I,_
4229::I, 4231::I, 4241::I, 4243::I, 4253::I, 4259::I,_
4261::I, 4271::I, 4273::I, 4283::I, 4289::I, 4297::I,_
4327::I, 4337::I, 4339::I, 4349::I, 4357::I, 4363::I,_
4373::I, 4391::I, 4397::I, 4409::I, 4421::I, 4423::I,_
4441::I, 4447::I, 4451::I, 4457::I, 4463::I, 4481::I,_
4483::I, 4493::I, 4507::I, 4513::I, 4517::I, 4519::I,_
4523::I, 4547::I, 4549::I, 4561::I, 4567::I, 4583::I,_
4591::I, 4597::I, 4603::I, 4621::I, 4637::I, 4639::I,_
4643::I, 4649::I, 4651::I, 4657::I, 4663::I, 4673::I,_
4679::I, 4691::I, 4703::I, 4721::I, 4723::I, 4729::I,_
4733::I, 4751::I, 4759::I, 4783::I, 4787::I, 4789::I,_
4793::I, 4799::I, 4801::I, 4813::I, 4817::I, 4831::I,_
4861::I, 4871::I, 4877::I, 4889::I, 4903::I, 4909::I,_
4919::I, 4931::I, 4933::I, 4937::I, 4943::I, 4951::I,_
4957::I, 4967::I, 4969::I, 4973::I, 4987::I, 4993::I,_
4999::I, 5003::I, 5009::I, 5011::I, 5021::I, 5023::I,_
5039::I, 5051::I, 5059::I, 5077::I, 5081::I, 5087::I,_
5099::I, 5101::I, 5107::I, 5113::I, 5119::I, 5147::I,_
5153::I, 5167::I, 5171::I, 5179::I, 5189::I, 5197::I,_
5209::I, 5227::I, 5231::I, 5233::I, 5237::I, 5261::I,_
5273::I, 5279::I, 5281::I, 5297::I, 5303::I, 5309::I,_
5323::I, 5333::I, 5347::I, 5351::I, 5381::I, 5387::I,_
5393::I, 5399::I, 5407::I, 5413::I, 5417::I, 5419::I,_
5431::I, 5437::I, 5441::I, 5443::I, 5449::I, 5471::I,_
5477::I, 5479::I, 5483::I, 5501::I, 5503::I, 5507::I,_
5519::I, 5521::I, 5527::I, 5531::I, 5557::I, 5563::I,_
5569::I, 5573::I, 5581::I, 5591::I, 5623::I, 5639::I,_
5641::I, 5647::I, 5651::I, 5653::I, 5657::I, 5659::I,_
5669::I, 5683::I, 5689::I, 5693::I, 5701::I, 5711::I,_
5717::I, 5737::I, 5741::I, 5743::I, 5749::I, 5779::I,_
5783::I, 5791::I, 5801::I, 5807::I, 5813::I, 5821::I,_
5827::I, 5839::I, 5843::I, 5849::I, 5851::I, 5857::I,_
5861::I, 5867::I, 5869::I, 5879::I, 5881::I, 5897::I,_
5903::I, 5923::I, 5927::I, 5939::I, 5953::I, 5981::I,_
5987::I, 6007::I, 6011::I, 6029::I, 6037::I, 6043::I,_
```

```
6047::I, 6053::I, 6067::I, 6073::I, 6079::I, 6089::I,_
6091::I, 6101::I, 6113::I, 6121::I, 6131::I, 6133::I,_
6143::I, 6151::I, 6163::I, 6173::I, 6197::I, 6199::I,_
6203::I, 6211::I, 6217::I, 6221::I, 6229::I, 6247::I,_
6257::I, 6263::I, 6269::I, 6271::I, 6277::I, 6287::I,_
6299::I, 6301::I, 6311::I, 6317::I, 6323::I, 6329::I,_
6337::I, 6343::I, 6353::I, 6359::I, 6361::I, 6367::I,_
6373::I, 6379::I, 6389::I, 6397::I, 6421::I, 6427::I,_
6449::I, 6451::I, 6469::I, 6473::I, 6481::I, 6491::I,_
6521::I, 6529::I, 6547::I, 6551::I, 6553::I, 6563::I,_
6569::I, 6571::I, 6577::I, 6581::I, 6599::I, 6607::I,_
6619::I, 6637::I, 6653::I, 6659::I, 6661::I, 6673::I,_
6679::I, 6689::I, 6691::I, 6701::I, 6703::I, 6709::I,_
6719::I, 6733::I, 6737::I, 6761::I, 6763::I, 6779::I,_
6781::I, 6791::I, 6793::I, 6803::I, 6823::I, 6827::I,_
6829::I, 6833::I, 6841::I, 6857::I, 6863::I, 6869::I,_
6871::I, 6883::I, 6899::I, 6907::I, 6911::I, 6917::I,_
6947::I, 6949::I, 6959::I, 6961::I, 6967::I, 6971::I,_
6977::I, 6983::I, 6991::I, 6997::I, 7001::I, 7013::I,_
7019::I, 7027::I, 7039::I, 7043::I, 7057::I, 7069::I,_
7079::I, 7103::I, 7109::I, 7121::I, 7127::I, 7129::I,_
7151::I, 7159::I, 7177::I, 7187::I, 7193::I, 7207::I,_
7211::I, 7213::I, 7219::I, 7229::I, 7237::I, 7243::I,_
7247::I, 7253::I, 7283::I, 7297::I, 7307::I, 7309::I,_
7321::I, 7331::I, 7333::I, 7349::I, 7351::I, 7369::I,_
7393::I, 7411::I, 7417::I, 7433::I, 7451::I, 7457::I,_
7459::I, 7477::I, 7481::I, 7487::I, 7489::I, 7499::I,_
7507::I, 7517::I, 7523::I, 7529::I, 7537::I, 7541::I,_
7547::I, 7549::I, 7559::I, 7561::I, 7573::I, 7577::I,_
7583::I, 7589::I, 7591::I, 7603::I, 7607::I, 7621::I,_
7639::I, 7643::I, 7649::I, 7669::I, 7673::I, 7681::I,_
7687::I, 7691::I, 7699::I, 7703::I, 7717::I, 7723::I,_
7727::I, 7741::I, 7753::I, 7757::I, 7759::I, 7789::I,_
7793::I, 7817::I, 7823::I, 7829::I, 7841::I, 7853::I,_
7867::I, 7873::I, 7877::I, 7879::I, 7883::I, 7901::I,_
7907::I, 7919::I, 7927::I, 7933::I, 7937::I, 7949::I,_
7951::I, 7963::I, 7993::I, 8009::I, 8011::I, 8017::I,_
8039::I, 8053::I, 8059::I, 8069::I, 8081::I, 8087::I,_
8089::I, 8093::I, 8101::I, 8111::I, 8117::I, 8123::I,_
8147::I, 8161::I, 8167::I, 8171::I, 8179::I, 8191::I,_
8209::I, 8219::I, 8221::I, 8231::I, 8233::I, 8237::I,_
8243::I, 8263::I, 8269::I, 8273::I, 8287::I, 8291::I,_
8293::I, 8297::I, 8311::I, 8317::I, 8329::I, 8353::I,_
8363::I, 8369::I, 8377::I, 8387::I, 8389::I, 8419::I,_
8423::I, 8429::I, 8431::I, 8443::I, 8447::I, 8461::I,_
8467::I, 8501::I, 8513::I, 8521::I, 8527::I, 8537::I,_
```

```
   8539::I, 8543::I, 8563::I, 8573::I, 8581::I, 8597::I,_
   8599::I, 8609::I, 8623::I, 8627::I, 8629::I, 8641::I,_
   8647::I, 8663::I, 8669::I, 8677::I, 8681::I, 8689::I,_
   8693::I, 8699::I, 8707::I, 8713::I, 8719::I, 8731::I,_
   8737::I, 8741::I, 8747::I, 8753::I, 8761::I, 8779::I,_
   8783::I, 8803::I, 8807::I, 8819::I, 8821::I, 8831::I,_
   8837::I, 8839::I, 8849::I, 8861::I, 8863::I, 8867::I,_
   8887::I, 8893::I, 8923::I, 8929::I, 8933::I, 8941::I,_
   8951::I, 8963::I, 8969::I, 8971::I, 8999::I, 9001::I,_
   9007::I, 9011::I, 9013::I, 9029::I, 9041::I, 9043::I,_
   9049::I, 9059::I, 9067::I, 9091::I, 9103::I, 9109::I,_
   9127::I, 9133::I, 9137::I, 9151::I, 9157::I, 9161::I,_
   9173::I, 9181::I, 9187::I, 9199::I, 9203::I, 9209::I,_
   9221::I, 9227::I, 9239::I, 9241::I, 9257::I, 9277::I,_
   9281::I, 9283::I, 9293::I, 9311::I, 9319::I, 9323::I,_
   9337::I, 9341::I, 9343::I, 9349::I, 9371::I, 9377::I,_
   9391::I, 9397::I, 9403::I, 9413::I, 9419::I, 9421::I,_
   9431::I, 9433::I, 9437::I, 9439::I, 9461::I, 9463::I,_
   9467::I, 9473::I, 9479::I, 9491::I, 9497::I, 9511::I,_
   9521::I, 9533::I, 9539::I, 9547::I, 9551::I, 9587::I,_
   9601::I, 9613::I, 9619::I, 9623::I, 9629::I, 9631::I,_
   9643::I, 9649::I, 9661::I, 9677::I, 9679::I, 9689::I,_
   9697::I, 9719::I, 9721::I, 9733::I, 9739::I, 9743::I,_
   9749::I, 9767::I, 9769::I, 9781::I, 9787::I, 9791::I,_
   9803::I, 9811::I, 9817::I, 9829::I, 9833::I, 9839::I,_
   9851::I, 9857::I, 9859::I, 9871::I, 9883::I, 9887::I,_
   9901::I, 9907::I, 9923::I, 9929::I, 9931::I, 9941::I,_
   9949::I, 9967::I, 9973::I]

productSmallPrimes    := */smallPrimes
nextSmallPrime        := 10007::I
nextSmallPrimeSquared := nextSmallPrime**2
two                   := 2::I
tenPowerTwenty:=(10::I)**20
PomeranceList:= [25326001::I, 161304001::I, 960946321::I, 1157839381::I,
                 -- 3215031751::I, -- has a factor of 151
                 3697278427::I, 5764643587::I, 6770862367::I,
                  14386156093::I, 15579919981::I, 18459366157::I,
                   19887974881::I, 21276028621::I ]::(List I)
PomeranceLimit:=27716349961::I  -- replaces (25*10**9) due to Pinch
PinchList:= _
  [3215031751::I, 118670087467::I, 128282461501::I, 354864744877::I,
   546348519181::I, 602248359169::I, 669094855201::I ]
PinchLimit:= (10**12)::I
PinchList2:= [2152302898747::I, 3474749660383::I]
PinchLimit2:= (10**13)::I
```

```
JaeschkeLimit:=341550071728321::I
rootsMinus1:Set I := empty()
-- used to check whether we detect too many roots of -1
count2Order:Vector NonNegativeInteger := new(1,0)
-- used to check whether we observe an element of maximal two-order
```

10.54.2 primes

⟨*package PRIMES IntegerPrimesPackage*⟩+≡

```
primes(m, n) ==
   -- computes primes from m to n inclusive using prime?
   l:List(I) :=
     m <= two => [two]
     empty()
   n < two or n < m => empty()
   if even? m then m := m + 1
   ll:List(I) := [k::I for k in
          convert(m)@Integer..convert(n)@Integer by 2 | prime?(k::I)]
   reverse_! concat_!(ll, l)

rabinProvesComposite : (I,I,I,I,NonNegativeInteger) -> Boolean
rabinProvesCompositeSmall : (I,I,I,I,NonNegativeInteger) -> Boolean
```

10.54.3 rabinProvesCompositeSmall

⟨*package PRIMES IntegerPrimesPackage*⟩+≡

```
rabinProvesCompositeSmall(p,n,nm1,q,k) ==
      -- probability n prime is > 3/4 for each iteration
      -- for most n this probability is much greater than 3/4
      t := powmod(p, q, n)
      -- neither of these cases tells us anything
      if not ((t = 1) or t = nm1) then
         for j in 1..k-1 repeat
            oldt := t
            t := mulmod(t, t, n)
            (t = 1) => return true
            -- we have squared someting not -1 and got 1
            t = nm1 =>
                leave
         not (t = nm1) => return true
      false
```

10.54.4 rabinProvesComposite

⟨*package PRIMES IntegerPrimesPackage*⟩+≡

```
rabinProvesComposite(p,n,nm1,q,k) ==
      -- probability n prime is > 3/4 for each iteration
      -- for most n this probability is much greater than 3/4
      t := powmod(p, q, n)
      -- neither of these cases tells us anything
      if t=nm1 then count2Order(1):=count2Order(1)+1
      if not ((t = 1) or t = nm1) then
         for j in 1..k-1 repeat
            oldt := t
            t := mulmod(t, t, n)
            (t = 1) => return true
            -- we have squared someting not -1 and got 1
            t = nm1 =>
                rootsMinus1:=union(rootsMinus1,oldt)
                count2Order(j+1):=count2Order(j+1)+1
                leave
         not (t = nm1) => return true
      # rootsMinus1 > 2 => true  -- Z/nZ can't be a field
      false
```

10.54.5 prime?

⟨*package PRIMES IntegerPrimesPackage*⟩+≡

```
prime? n ==
   n < two => false
   n < nextSmallPrime => member?(n, smallPrimes)
   not (gcd(n, productSmallPrimes) = 1) => false
   n < nextSmallPrimeSquared => true

   nm1 := n-1
   q := (nm1) quo two
   for k in 1.. while not odd? q repeat q := q quo two
   -- q = (n-1) quo 2**k for largest possible k

   n < JaeschkeLimit =>
       rabinProvesCompositeSmall(2::I,n,nm1,q,k) => return false
       rabinProvesCompositeSmall(3::I,n,nm1,q,k) => return false

       n < PomeranceLimit =>
           rabinProvesCompositeSmall(5::I,n,nm1,q,k) => return false
           member?(n,PomeranceList) => return false
           true

       rabinProvesCompositeSmall(7::I,n,nm1,q,k) => return false
       n < PinchLimit =>
           rabinProvesCompositeSmall(10::I,n,nm1,q,k) => return false
           member?(n,PinchList) => return false
           true

       rabinProvesCompositeSmall(5::I,n,nm1,q,k) => return false
       rabinProvesCompositeSmall(11::I,n,nm1,q,k) => return false
       n < PinchLimit2 =>
           member?(n,PinchList2) => return false
           true

       rabinProvesCompositeSmall(13::I,n,nm1,q,k) => return false
       rabinProvesCompositeSmall(17::I,n,nm1,q,k) => return false
       true

   rootsMinus1:= empty()
   count2Order := new(k,0) -- vector of k zeroes

   mn := minIndex smallPrimes
   for i in mn+1..mn+10 repeat
       rabinProvesComposite(smallPrimes i,n,nm1,q,k) => return false
   import IntegerRoots(I)
```

```
        q > 1 and perfectSquare?(3*n+1) => false
        ((n9:=n rem (9::I))=1 or n9 = -1) and perfectSquare?(8*n+1) => false
        -- Both previous tests from Damgard & Landrock
        currPrime:=smallPrimes(mn+10)
        probablySafe:=tenPowerTwenty
        while count2Order(k) = 0 or n > probablySafe repeat
            currPrime := nextPrime currPrime
            probablySafe:=probablySafe*(100::I)
            rabinProvesComposite(currPrime,n,nm1,q,k) => return false
        true
```

10.54.6 nextPrime

⟨*package PRIMES IntegerPrimesPackage*⟩+≡

```
    nextPrime n ==
       -- computes the first prime after n
       n < two => two
       if odd? n then n := n + two else n := n + 1
       while not prime? n repeat n := n + two
       n
```

10.54.7 prevPrime

⟨*package PRIMES IntegerPrimesPackage*⟩+≡

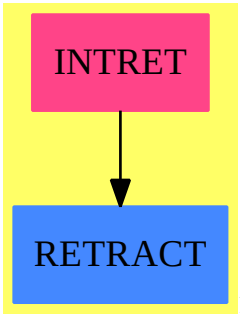
```
    prevPrime n ==
       -- computes the first prime before n
       n < 3::I => error "no primes less than 2"
       n = 3::I => two
       if odd? n then n := n - two else n := n - 1
       while not prime? n repeat n := n - two
       n
```

⟨*PRIMES.dotabb*⟩≡

```
"PRIMES" [color="#FF4488",href="bookvol10.4.pdf#nameddest=PRIMES"]
"FSAGG" [color="#4488FF",href="bookvol10.2.pdf#nameddest=FSAGG"]
"PRIMES" -> "FSAGG"
```

10.55 package INTRET IntegerRetractions

10.56 IntegerRetractions



1.00

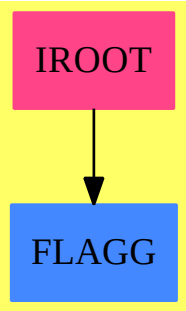
Exports:
integer integer? integerIfCan

⟨*package INTRET IntegerRetractions*⟩≡
```
)abbrev package INTRET IntegerRetractions
++ Author: Manuel Bronstein
++ Description: Provides integer testing and retraction functions.
++ Date Created: March 1990
++ Date Last Updated: 9 April 1991
IntegerRetractions(S:RetractableTo(Integer)): with
    integer     : S -> Integer
      ++ integer(x) returns x as an integer;
      ++ error if x is not an integer;
    integer?    : S -> Boolean
      ++ integer?(x) is true if x is an integer, false otherwise;
    integerIfCan: S -> Union(Integer, "failed")
      ++ integerIfCan(x) returns x as an integer,
      ++ "failed" if x is not an integer;
  == add
    integer s      == retract s
    integer? s     == retractIfCan(s) case Integer
    integerIfCan s == retractIfCan s
```

⟨*INTRET.dotabb*⟩≡
```
"INTRET" [color="#FF4488",href="bookvol10.4.pdf#nameddest=INTRET"]
"RETRACT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=RETRACT"]
"INTRET" -> "RETRACT"
```

10.57 package IROOT IntegerRoots

10.58 IntegerRoots



1.00

Exports:
approxSqrt approxNthRoot perfectNthPower? perfectNthRoot perfectSqrt perfectSquare

⟨*package IROOT IntegerRoots*⟩≡
```
)abbrev package IROOT IntegerRoots
++ Author: Michael Monagan
++ Date Created: November 1987
++ Date Last Updated:
++ Basic Operations:
++ Related Domains:
++ Also See:
++ AMS Classifications:
++ Keywords: integer roots
++ Examples:
++ References:
++ Description: The \spadtype{IntegerRoots} package computes square roots and
++   nth roots of integers efficiently.
IntegerRoots(I:IntegerNumberSystem): Exports == Implementation where
  NNI ==> NonNegativeInteger

  Exports ==> with
    perfectNthPower?: (I, NNI) -> Boolean
      ++ \spad{perfectNthPower?(n,r)} returns true if n is an \spad{r}th
      ++ power and false otherwise
    perfectNthRoot: (I,NNI) -> Union(I,"failed")
      ++ \spad{perfectNthRoot(n,r)} returns the \spad{r}th root of n if n
      ++ is an \spad{r}th power and returns "failed" otherwise
    perfectNthRoot: I -> Record(base:I, exponent:NNI)
      ++ \spad{perfectNthRoot(n)} returns \spad{[x,r]}, where \spad{n = x\^r}
      ++ and r is the largest integer such that n is a perfect \spad{r}th power
    approxNthRoot: (I,NNI) -> I
```

```
      ++ \spad{approxRoot(n,r)} returns an approximation x
      ++ to \spad{n**(1/r)} such that \spad{-1 < x - n**(1/r) < 1}
    perfectSquare?: I -> Boolean
      ++ \spad{perfectSquare?(n)} returns true if n is a perfect square
      ++ and false otherwise
    perfectSqrt: I -> Union(I,"failed")
      ++ \spad{perfectSqrt(n)} returns the square root of n if n is a
      ++ perfect square and returns "failed" otherwise
    approxSqrt: I -> I
      ++ \spad{approxSqrt(n)} returns an approximation x
      ++ to \spad{sqrt(n)} such that \spad{-1 < x - sqrt(n) < 1}.
      ++ Compute an approximation s to \spad{sqrt(n)} such that
      ++           \spad{-1 < s - sqrt(n) < 1}
      ++ A variable precision Newton iteration is used.
      ++ The running time is \spad{O( log(n)**2 )}.

  Implementation ==> add
    import IntegerPrimesPackage(I)

    resMod144: List I := [0::I,1::I,4::I,9::I,16::I,25::I,36::I,49::I,_
                    52::I,64::I,73::I,81::I,97::I,100::I,112::I,121::I]
    two := 2::I
```

10.58.1 perfectSquare?

⟨*package IROOT IntegerRoots*⟩+≡

```
    perfectSquare? a       == (perfectSqrt a) case I
```

10.58.2 perfectNthPower?

⟨*package IROOT IntegerRoots*⟩+≡

```
    perfectNthPower?(b, n) == perfectNthRoot(b, n) case I
```

10.58.3 perfectNthRoot

⟨*package IROOT IntegerRoots*⟩+≡

```
perfectNthRoot n ==  -- complexity (log log n)**2 (log n)**2
  m:NNI
  (n = 1) or zero? n or n = -1 => [n, 1]
  e:NNI := 1
  p:NNI := 2
  while p::I <= length(n) + 1 repeat
     for m in 0.. while (r := perfectNthRoot(n, p)) case I repeat
        n := r::I
     e := e * p ** m
     p := convert(nextPrime(p::I))@Integer :: NNI
  [n, e]
```

10.58.4 approxNthRoot

⟨*package IROOT IntegerRoots*⟩+≡

```
approxNthRoot(a, n) ==   -- complexity (log log n) (log n)**2
  zero? n => error "invalid arguments"
  (n = 1) => a
  n=2 => approxSqrt a
  negative? a =>
    odd? n => - approxNthRoot(-a, n)
    0
  zero? a => 0
  (a = 1) => 1
  -- quick check for case of large n
  ((3*n) quo 2)::I >= (l := length a) => two
  -- the initial approximation must be >= the root
  y := max(two, shift(1, (n::I+l-1) quo (n::I)))
  z:I := 1
  n1:= (n-1)::NNI
  while z > 0 repeat
    x := y
    xn:= x**n1
    y := (n1*x*xn+a) quo (n*xn)
    z := x-y
  x
```

10.58.5 perfectNthRoot

⟨*package IROOT IntegerRoots*⟩+≡
```
perfectNthRoot(b, n) ==
  (r := approxNthRoot(b, n)) ** n = b => r
  "failed"
```

10.58.6 perfectSqrt

⟨*package IROOT IntegerRoots*⟩+≡
```
perfectSqrt a ==
  a < 0 or not member?(a rem (144::I), resMod144) => "failed"
  (s := approxSqrt a) * s = a => s
  "failed"
```

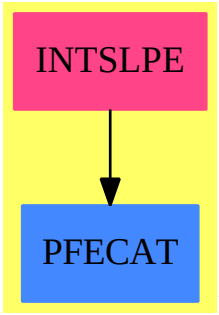
10.58.7 approxSqrt

⟨*package IROOT IntegerRoots*⟩+≡
```
approxSqrt a ==
  a < 1 => 0
  if (n := length a) > (100::I) then
     -- variable precision newton iteration
     n := n quo (4::I)
     s := approxSqrt shift(a, -2 * n)
     s := shift(s, n)
     return ((1 + s + a quo s) quo two)
  -- initial approximation for the root is within a factor of 2
  (new, old) := (shift(1, n quo two), 1)
  while new ^= old repeat
     (new, old) := ((1 + new + a quo new) quo two, new)
  new
```

⟨*IROOT.dotabb*⟩≡
```
"IROOT" [color="#FF4488",href="bookvol10.4.pdf#nameddest=IROOT"]
"FLAGG" [color="#4488FF",href="bookvol10.2.pdf#nameddest=FLAGG"]
"IROOT" -> "FLAGG"
```

10.59 package INTSLPE IntegerSolveLinear-PolynomialEquation

10.60 IntegerSolveLinearPolynomialEquation



1.00

Exports:
solveLinearPolynomialEquation

⟨*package INTSLPE IntegerSolveLinearPolynomialEquation*⟩≡

```
)abbrev package INTSLPE IntegerSolveLinearPolynomialEquation
++ Author: Davenport
++ Date Created: 1991
++ Date Last Updated:
++ Basic Functions:
++ Related Constructors:
++ Also See:
++ AMS Classifications:
++ Keywords:
++ References:
++ Description:
++ This package provides the implementation for the
++ \spadfun{solveLinearPolynomialEquation}
++ operation over the integers. It uses a lifting technique
++ from the package GenExEuclid
IntegerSolveLinearPolynomialEquation(): C ==T
 where
  ZP ==> SparseUnivariatePolynomial Integer
  C == with
      solveLinearPolynomialEquation: (List ZP,ZP) -> Union(List ZP,"failed")
           ++ solveLinearPolynomialEquation([f1, ..., fn], g)
           ++ (where the fi are relatively prime to each other)
           ++ returns a list of ai such that
           ++ \spad{g/prod fi = sum ai/fi}
           ++ or returns "failed" if no such list of ai's exists.
  T == add
```

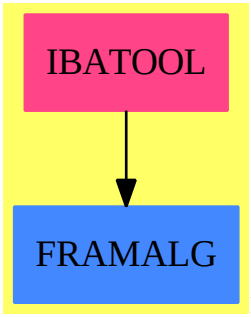
```
oldlp:List ZP := []
slpePrime:Integer:=(2::Integer)
oldtable:Vector List ZP := empty()
solveLinearPolynomialEquation(lp,p) ==
   if (oldlp ^= lp) then
      -- we have to generate a new table
      deg:= _+/[degree u for u in lp]
      ans:Union(Vector List ZP,"failed"):="failed"
      slpePrime:=2147483647::Integer   -- 2**31 -1 : a prime
           -- a good test case for this package is
           --  ([x**31-1,x-2],2)
      while (ans case "failed") repeat
        ans:=tablePow(deg,slpePrime,lp)$GenExEuclid(Integer,ZP)
        if (ans case "failed") then
           slpePrime:= prevPrime(slpePrime)$IntegerPrimesPackage(Integer)
      oldtable:=(ans:: Vector List ZP)
   answer:=solveid(p,slpePrime,oldtable)
   answer
```

⟨*INTSLPE.dotabb*⟩≡

```
"INTSLPE" [color="#FF4488",href="bookvol10.4.pdf#nameddest=INTSLPE"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"INTSLPE" -> "PFECAT"
```

10.61 package IBATOOL IntegralBasisTools

10.62 IntegralBasisTools



1.00

Exports:

diagonalProduct	divideIfCan!	idealiser	idealiserMatrix	leastPower
matrixGcd	moduleSum			

⟨*package IBATOOL IntegralBasisTools*⟩≡

```
)abbrev package IBATOOL IntegralBasisTools
++ Functions common to both integral basis packages
++ Author: Victor Miller, Barry Trager, Clifton Williamson
++ Date Created: 11 April 1990
++ Date Last Updated: 20 September 1994
++ Keywords: integral basis, function field, number field
++ Examples:
++ References:
++ Description:
++ This package contains functions used in the packages
++ FunctionFieldIntegralBasis and NumberFieldIntegralBasis.

IntegralBasisTools(R,UP,F): Exports == Implementation where
  R  : EuclideanDomain with
        squareFree: $ -> Factored $
                ++ squareFree(x) returns a square-free factorisation of x
  UP : UnivariatePolynomialCategory R
  F  : FramedAlgebra(R,UP)
  Mat ==> Matrix R
  NNI ==> NonNegativeInteger
  Ans ==> Record(basis: Mat, basisDen: R, basisInv:Mat)

  Exports ==> with

    diagonalProduct: Mat -> R
      ++ diagonalProduct(m) returns the product of the elements on the
```

```
    ++ diagonal of the matrix m
  matrixGcd: (Mat,R,NNI) -> R
    ++ matrixGcd(mat,sing,n) is \spad{gcd(sing,g)} where \spad{g} is the
    ++ gcd of the entries of the \spad{n}-by-\spad{n} upper-triangular
    ++ matrix \spad{mat}.
  divideIfCan_!: (Matrix R,Matrix R,R,Integer) -> R
    ++ divideIfCan!(matrix,matrixOut,prime,n) attempts to divide the
    ++ entries of \spad{matrix} by \spad{prime} and store the result in
    ++ \spad{matrixOut}.  If it is successful, 1 is returned and if not,
    ++ \spad{prime} is returned.  Here both \spad{matrix} and
    ++ \spad{matrixOut} are \spad{n}-by-\spad{n} upper triangular matrices.
  leastPower: (NNI,NNI) -> NNI
    ++ leastPower(p,n) returns e, where e is the smallest integer
    ++ such that \spad{p **e >= n}
  idealiser: (Mat,Mat) -> Mat
    ++ idealiser(m1,m2) computes the order of an ideal defined by m1 and m2
  idealiser: (Mat,Mat,R) -> Mat
    ++ idealiser(m1,m2,d) computes the order of an ideal defined by m1 and m2
    ++ where d is the known part of the denominator
  idealiserMatrix: (Mat, Mat) -> Mat
    ++ idealiserMatrix(m1, m2) returns the matrix representing the linear
    ++ conditions on the Ring associatied with an ideal defined by m1 and m2.
  moduleSum: (Ans,Ans) -> Ans
    ++ moduleSum(m1,m2) returns the sum of two modules in the framed
    ++ algebra \spad{F}.  Each module \spad{mi} is represented as follows:
    ++ F is a framed algebra with R-module basis \spad{w1,w2,...,wn} and
    ++ \spad{mi} is a record \spad{[basis,basisDen,basisInv]}.  If
    ++ \spad{basis} is the matrix \spad{(aij, i = 1..n, j = 1..n)}, then
    ++ a basis \spad{v1,...,vn} for \spad{mi} is given by
    ++ \spad{vi = (1/basisDen) * sum(aij * wj, j = 1..n)}, i.e. the
    ++ \spad{i}th row of 'basis' contains the coordinates of the
    ++ \spad{i}th basis vector.  Similarly, the \spad{i}th row of the
    ++ matrix \spad{basisInv} contains the coordinates of \spad{wi} with
    ++ respect to the basis \spad{v1,...,vn}: if \spad{basisInv} is the
    ++ matrix \spad{(bij, i = 1..n, j = 1..n)}, then
    ++ \spad{wi = sum(bij * vj, j = 1..n)}.

Implementation ==> add
  import ModularHermitianRowReduction(R)
  import TriangularMatrixOperations(R, Vector R, Vector R, Matrix R)

  diagonalProduct m ==
    ans : R := 1
    for i in minRowIndex m .. maxRowIndex m
      for j in minColIndex m .. maxColIndex m repeat
        ans := ans * qelt(m, i, j)
```

```
      ans

    matrixGcd(mat,sing,n) ==
      -- note: 'matrix' is upper triangular;
      -- no need to do anything below the diagonal
      d := sing
      for i in 1..n repeat
        for j in i..n repeat
          if not zero?(mij := qelt(mat,i,j)) then d := gcd(d,mij)
--          one? d => return d
          (d = 1) => return d
      d

    divideIfCan_!(matrix,matrixOut,prime,n) ==
    -- note: both 'matrix' and 'matrixOut' will be upper triangular;
    -- no need to do anything below the diagonal
      for i in 1..n repeat
        for j in i..n repeat
          (a := (qelt(matrix,i,j) exquo prime)) case "failed" => return prime
          qsetelt_!(matrixOut,i,j,a :: R)
      1

    leastPower(p,n) ==
      -- efficiency is not an issue here
      e : NNI := 1; q := p
      while q < n repeat (e := e + 1; q := q * p)
      e

    idealiserMatrix(ideal,idealinv) ==
      -- computes the Order of the ideal
      n  := rank()$F
      bigm := zero(n * n,n)$Mat
      mr := minRowIndex bigm; mc := minColIndex bigm
      v := basis()$F
      for i in 0..n-1 repeat
        r := regularRepresentation qelt(v,i + minIndex v)
        m := ideal * r * idealinv
        for j in 0..n-1 repeat
          for k in 0..n-1 repeat
            bigm(j * n + k + mr,i + mc) := qelt(m,j + mr,k + mc)
      bigm

    idealiser(ideal,idealinv) ==
      bigm := idealiserMatrix(ideal, idealinv)
      transpose squareTop rowEch bigm
```

```
idealiser(ideal,idealinv,denom) ==
  bigm := (idealiserMatrix(ideal, idealinv) exquo denom)::Mat
  transpose squareTop rowEchelon(bigm,denom)

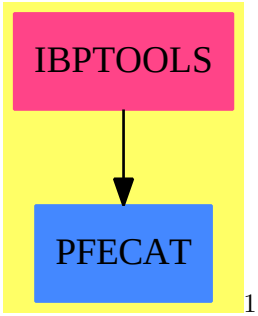
moduleSum(mod1,mod2) ==
  rb1 := mod1.basis; rbden1 := mod1.basisDen; rbinv1 := mod1.basisInv
  rb2 := mod2.basis; rbden2 := mod2.basisDen; rbinv2 := mod2.basisInv
  -- compatibility check: doesn't take much computation time
  (not square? rb1) or (not square? rbinv1) or (not square? rb2) _
    or (not square? rbinv2) =>
    error "moduleSum: matrices must be square"
  ((n := nrows rb1) ^= (nrows rbinv1)) or (n ^= (nrows rb2)) _
    or (n ^= (nrows rbinv2)) =>
    error "moduleSum: matrices of imcompatible dimensions"
  (zero? rbden1) or (zero? rbden2) =>
    error "moduleSum: denominator must be non-zero"
  den := lcm(rbden1,rbden2); c1 := den quo rbden1; c2 := den quo rbden2
  rb := squareTop rowEchelon(vertConcat(c1 * rb1,c2 * rb2),den)
  rbinv := UpTriBddDenomInv(rb,den)
  [rb,den,rbinv]
```

⟨*IBATOOL.dotabb*⟩≡

```
"IBATOOL" [color="#FF4488",href="bookvol10.4.pdf#nameddest=IBATOOL"]
"FRAMALG" [color="#4488FF",href="bookvol10.2.pdf#nameddest=FRAMALG"]
"IBATOOL" -> "FRAMALG"
```

10.63 package IBPTOOLS IntegralBasisPolynomialTools

10.64 IntegralBasisPolynomialTools



1.00

Exports:
mapBivariate mapMatrixIfCan mapUnivariate mapUnivariateIfCan

⟨*package IBPTOOLS IntegralBasisPolynomialTools*⟩≡

```
)abbrev package IBPTOOLS IntegralBasisPolynomialTools
++ Author: Clifton Williamson
++ Date Created: 13 August 1993
++ Date Last Updated: 17 August 1993
++ Basic Operations: mapUnivariate, mapBivariate
++ Related Domains: PAdicWildFunctionFieldIntegralBasis(K,R,UP,F)
++ Also See: WildFunctionFieldIntegralBasis, FunctionFieldIntegralBasis
++ AMS Classifications:
++ Keywords: function field, finite field, integral basis
++ Examples:
++ References:
++ Description:  IntegralBasisPolynomialTools provides functions for
++   mapping functions on the coefficients of univariate and bivariate
++   polynomials.

IntegralBasisPolynomialTools(K,R,UP,L): Exports == Implementation where
  K  : Ring
  R  : UnivariatePolynomialCategory K
  UP : UnivariatePolynomialCategory R
  L  : Ring

  MAT ==> Matrix
  SUP ==> SparseUnivariatePolynomial

  Exports ==> with
    mapUnivariate: (L -> K,SUP L) -> R
```

```
    ++ mapUnivariate(f,p(x)) applies the function \spad{f} to the
    ++ coefficients of \spad{p(x)}.

  mapUnivariate: (K -> L,R) -> SUP L
    ++ mapUnivariate(f,p(x)) applies the function \spad{f} to the
    ++ coefficients of \spad{p(x)}.

  mapUnivariateIfCan: (L -> Union(K,"failed"),SUP L) -> Union(R,"failed")
    ++ mapUnivariateIfCan(f,p(x)) applies the function \spad{f} to the
    ++ coefficients of \spad{p(x)}, if possible, and returns
    ++ \spad{"failed"} otherwise.

  mapMatrixIfCan: (L -> Union(K,"failed"),MAT SUP L) -> Union(MAT R,"failed")
    ++ mapMatrixIfCan(f,mat) applies the function \spad{f} to the
    ++ coefficients of the entries of \spad{mat} if possible, and returns
    ++ \spad{"failed"} otherwise.

  mapBivariate: (K -> L,UP) -> SUP SUP L
    ++ mapBivariate(f,p(x,y)) applies the function \spad{f} to the
    ++ coefficients of \spad{p(x,y)}.

Implementation ==> add

  mapUnivariate(f:L -> K,poly:SUP L) ==
    ans : R := 0
    while not zero? poly repeat
      ans := ans + monomial(f leadingCoefficient poly,degree poly)
      poly := reductum poly
    ans

  mapUnivariate(f:K -> L,poly:R) ==
    ans : SUP L := 0
    while not zero? poly repeat
      ans := ans + monomial(f leadingCoefficient poly,degree poly)
      poly := reductum poly
    ans

  mapUnivariateIfCan(f,poly) ==
    ans : R := 0
    while not zero? poly repeat
      (lc := f leadingCoefficient poly) case "failed" => return "failed"
      ans := ans + monomial(lc :: K,degree poly)
      poly := reductum poly
    ans

  mapMatrixIfCan(f,mat) ==
```

```
    m := nrows mat; n := ncols mat
    matOut : MAT R := new(m,n,0)
    for i in 1..m repeat for j in 1..n repeat
      (poly := mapUnivariateIfCan(f,qelt(mat,i,j))) case "failed" =>
        return "failed"
      qsetelt_!(matOut,i,j,poly :: R)
    matOut

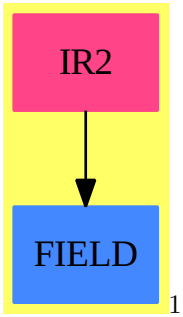
  mapBivariate(f,poly) ==
    ans : SUP SUP L := 0
    while not zero? poly repeat
      ans :=
        ans + monomial(mapUnivariate(f,leadingCoefficient poly),degree poly)
      poly := reductum poly
    ans
```

⟨*IBPTOOLS.dotabb*⟩≡

```
"IBPTOOLS" [color="#FF4488",href="bookvol10.4.pdf#nameddest=IBPTOOLS"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"IBPTOOLS" -> "PFECAT"
```

10.65 package IR2 IntegrationResultFunctions2

10.66 IntegrationResultFunctions2



1.00

Exports:
map

⟨*package IR2 IntegrationResultFunctions2*⟩≡

```
)abbrev package IR2 IntegrationResultFunctions2
++ Internally used by the integration packages
++ Author: Manuel Bronstein
++ Date Created: 1987
++ Date Last Updated: 12 August 1992
++ Keywords: integration.
IntegrationResultFunctions2(E, F): Exports == Implementation where
  E : Field
  F : Field

  SE  ==> Symbol
  Q   ==> Fraction Integer
  IRE ==> IntegrationResult E
  IRF ==> IntegrationResult F
  UPE ==> SparseUnivariatePolynomial E
  UPF ==> SparseUnivariatePolynomial F
  NEE ==> Record(integrand:E, intvar:E)
  NEF ==> Record(integrand:F, intvar:F)
  LGE ==> Record(scalar:Q, coeff:UPE, logand:UPE)
  LGF ==> Record(scalar:Q, coeff:UPF, logand:UPF)
  NLE ==> Record(coeff:E, logand:E)
  NLF ==> Record(coeff:F, logand:F)
  UFE ==> Union(Record(mainpart:E, limitedlogs:List NLE), "failed")
  URE ==> Union(Record(ratpart:E, coeff:E), "failed")
  UE  ==> Union(E, "failed")

  Exports ==> with
```

```
      map: (E -> F, IRE) -> IRF
          ++ map(f,ire) \undocumented
      map: (E -> F, URE) -> Union(Record(ratpart:F, coeff:F), "failed")
          ++ map(f,ure) \undocumented
      map: (E -> F,  UE) -> Union(F, "failed")
          ++ map(f,ue) \undocumented
      map: (E -> F, UFE) ->
                Union(Record(mainpart:F, limitedlogs:List NLF), "failed")
          ++ map(f,ufe) \undocumented

    Implementation ==> add
      import SparseUnivariatePolynomialFunctions2(E, F)

      NEE2F: (E -> F, NEE) -> NEF
      LGE2F: (E -> F, LGE) -> LGF
      NLE2F: (E -> F, NLE) -> NLF

      NLE2F(func, r)         == [func(r.coeff), func(r.logand)]
      NEE2F(func, n)         == [func(n.integrand), func(n.intvar)]
      map(func:E -> F, u:UE) == (u case "failed" => "failed"; func(u::E))

      map(func:E -> F, ir:IRE) ==
        mkAnswer(func ratpart ir, [LGE2F(func, f) for f in logpart ir],
                                    [NEE2F(func, g) for g in notelem ir])

      map(func:E -> F, u:URE) ==
        u case "failed" => "failed"
        [func(u.ratpart), func(u.coeff)]

      map(func:E -> F, u:UFE) ==
        u case "failed" => "failed"
        [func(u.mainpart), [NLE2F(func, f) for f in u.limitedlogs]]

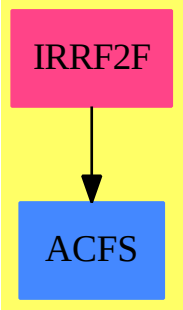
      LGE2F(func, lg) ==
        [lg.scalar, map(func, lg.coeff), map(func, lg.logand)]
```

⟨*IR2.dotabb*⟩≡

```
  "IR2" [color="#FF4488",href="bookvol10.4.pdf#nameddest=IR2"]
  "FIELD"  [color="#4488FF",href="bookvol10.2.pdf#nameddest=FIELD"]
  "IR2" -> "FIELD"
```

10.67 package IRRF2F IntegrationResultRFTo-Function

10.68 IntegrationResultRFToFunction



1.00

Exports:
complexExpand complexIntegrate expand integrate split

⟨*package IRRF2F IntegrationResultRFToFunction*⟩≡

```
)abbrev package IRRF2F IntegrationResultRFToFunction
++ Conversion of integration results to top-level expressions
++ Author: Manuel Bronstein
++ Description:
++   This package allows a sum of logs over the roots of a polynomial
++   to be expressed as explicit logarithms and arc tangents, provided
++   that the indexing polynomial can be factored into quadratics.
++ Date Created: 21 August 1988
++ Date Last Updated: 4 October 1993
IntegrationResultRFToFunction(R): Exports == Implementation where
  R: Join(GcdDomain, RetractableTo Integer, OrderedSet,
          LinearlyExplicitRingOver Integer)

  RF  ==> Fraction Polynomial R
  F   ==> Expression R
  IR  ==> IntegrationResult RF

  Exports ==> with
    split           : IR -> IR
       ++ split(u(x) + sum_{P(a)=0} Q(a,x)) returns
       ++ \spad{u(x) + sum_{P1(a)=0} Q(a,x) + ... + sum_{Pn(a)=0} Q(a,x)}
       ++ where P1,...,Pn are the factors of P.
    expand          : IR -> List F
       ++ expand(i) returns the list of possible real functions
       ++ corresponding to i.
    complexExpand   : IR -> F
```

```
        ++ complexExpand(i) returns the expanded complex function
        ++ corresponding to i.
      if R has CharacteristicZero then
        integrate       : (RF, Symbol) -> Union(F, List F)
          ++ integrate(f, x) returns the integral of \spad{f(x)dx}
          ++ where x is viewed as a real variable..
        complexIntegrate: (RF, Symbol) -> F
          ++ complexIntegrate(f, x) returns the integral of \spad{f(x)dx}
          ++ where x is viewed as a complex variable.

    Implementation ==> add
      import IntegrationTools(R, F)
      import TrigonometricManipulations(R, F)
      import IntegrationResultToFunction(R, F)

      toEF: IR -> IntegrationResult F

      toEF i          == map(z1+->z1::F, i)$IntegrationResultFunctions2(RF, F)
      expand i        == expand toEF i
      complexExpand i == complexExpand toEF i

      split i ==
        map(retract, split toEF i)$IntegrationResultFunctions2(F, RF)

      if R has CharacteristicZero then
        import RationalFunctionIntegration(R)

        complexIntegrate(f, x) == complexExpand internalIntegrate(f, x)

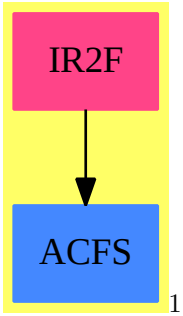
  -- do not use real integration if R is complex
        if R has imaginary: () -> R then integrate(f, x) == complexIntegrate(f, x)
        else
          integrate(f, x) ==
            l := [mkPrim(real g, x) for g in expand internalIntegrate(f, x)]
            empty? rest l => first l
            l
```

⟨*IRRF2F.dotabb*⟩≡

```
"IRRF2F" [color="#FF4488",href="bookvol10.4.pdf#nameddest=IRRF2F"]
"ACFS" [color="#4488FF",href="bookvol10.2.pdf#nameddest=ACFS"]
"IRRF2F" -> "ACFS"
```

10.69 package IR2F IntegrationResultToFunction

10.70 IntegrationResultToFunction



1.00

Exports:
complexExpand expand split

⟨*package IR2F IntegrationResultToFunction*⟩≡
```
)abbrev package IR2F IntegrationResultToFunction
++ Conversion of integration results to top-level expressions
++ Author: Manuel Bronstein
++ Date Created: 4 February 1988
++ Date Last Updated: 9 October 1991
++ Description:
++   This package allows a sum of logs over the roots of a polynomial
++   to be expressed as explicit logarithms and arc tangents, provided
++   that the indexing polynomial can be factored into quadratics.
++ Keywords: integration, expansion, function.
IntegrationResultToFunction(R, F): Exports == Implementation where
  R: Join(GcdDomain, RetractableTo Integer, OrderedSet,
          LinearlyExplicitRingOver Integer)
  F: Join(AlgebraicallyClosedFunctionSpace R,
          TranscendentalFunctionCategory)

  N   ==> NonNegativeInteger
  Z   ==> Integer
  Q   ==> Fraction Z
  K   ==> Kernel F
  P   ==> SparseMultivariatePolynomial(R, K)
  UP  ==> SparseUnivariatePolynomial F
  IR  ==> IntegrationResult F
  REC ==> Record(ans1:F, ans2:F)
  LOG ==> Record(scalar:Q, coeff:UP, logand:UP)
```

```
Exports ==> with
  split        : IR -> IR
     ++ split(u(x) + sum_{P(a)=0} Q(a,x)) returns
     ++ \spad{u(x) + sum_{P1(a)=0} Q(a,x) + ... + sum_{Pn(a)=0} Q(a,x)}
     ++ where P1,...,Pn are the factors of P.
  expand       : IR -> List F
     ++ expand(i) returns the list of possible real functions
     ++ corresponding to i.
  complexExpand: IR -> F
     ++ complexExpand(i) returns the expanded complex function
     ++ corresponding to i.

Implementation ==> add
  import AlgebraicManipulations(R, F)
  import ElementaryFunctionSign(R, F)

  IR2F       : IR -> F
  insqrt     : F  -> Record(sqrt:REC, sgn:Z)
  pairsum    : (List F, List F) -> List F
  pairprod   : (F, List F) -> List F
  quadeval   : (UP, F, F, F) -> REC
  linear     : (UP, UP) -> F
  tantrick   : (F, F) -> F
  ilog       : (F, F, List K) -> F
  ilog0      : (F, F, UP, UP, F) -> F
  nlogs      : LOG -> List LOG
  lg2func    : LOG -> List F
  quadratic  : (UP, UP) -> List F
  mkRealFunc : List LOG -> List F
  lg2cfunc   : LOG -> F
  loglist    : (Q, UP, UP) -> List LOG
  cmplex     : (F, UP) -> F
  evenRoots  : F -> List F
  compatible?: (List F, List F) -> Boolean

  cmplex(alpha, p) == alpha * log p alpha
  IR2F i           == retract mkAnswer(ratpart i, empty(), notelem i)
  pairprod(x, l)   == [x * y for y in l]

  evenRoots x ==
    [first argument k for k in tower x |
     is?(k,"nthRoot"::Symbol) and even?(retract(second argument k)@Z)
       and (not empty? variables first argument k)]

  expand i ==
    j := split i
```

```
      pairsum([IR2F j], mkRealFunc logpart j)

    split i ==
      mkAnswer(ratpart i,concat [nlogs l for l in logpart i],notelem i)

    complexExpand i ==
      j := split i
      IR2F j + +/[lg.scalar::F * lg2cfunc lg for lg in logpart j]

-- p = a t^2 + b t + c
-- Expands sum_{p(t) = 0} t log(lg(t))
    quadratic(p, lg) ==
      zero?(delta := (b := coefficient(p, 1))**2 - 4 *
       (a := coefficient(p,2)) * (p0 := coefficient(p, 0))) =>
         [linear(monomial(1, 1) + (b / a)::UP, lg)]
      e := (q := quadeval(lg, c := - b * (d := inv(2*a)),d, delta)).ans1
      lgp  := c * log(nrm := (e**2 - delta * (f := q.ans2)**2))
      s    := (sqr := insqrt delta).sqrt
      pp := nn := 0$F
      if sqr.sgn >= 0 then
        sqrp := s.ans1 * rootSimp sqrt(s.ans2)
        pp := lgp + d * sqrp * log(((2 * e * f) / nrm) * sqrp
                                       + (e**2 + delta * f**2) / nrm)
      if sqr.sgn <= 0 then
        sqrn := s.ans1 * rootSimp sqrt(-s.ans2)
        nn := lgp + d * sqrn * ilog(e, f * sqrn,
                    setUnion(setUnion(kernels a, kernels b), kernels p0))
      sqr.sgn > 0 => [pp]
      sqr.sgn < 0 => [nn]
      [pp, nn]

-- returns 2 atan(a/b) or 2 atan(-b/a) whichever looks better
-- they differ by a constant so it's ok to do it from an IR
    tantrick(a, b) ==
      retractIfCan(a)@Union(Q, "failed") case Q => 2 * atan(-b/a)
      2 * atan(a/b)

-- transforms i log((a + i b) / (a - i b)) into a sum of real
-- arc-tangents using Rioboo's algorithm
-- lk is a list of kernels which are parameters for the integral
    ilog(a, b, lk) ==
      l := setDifference(setUnion(variables numer a, variables numer b),
           setUnion(lk, setUnion(variables denom a, variables denom b)))
      empty? l => tantrick(a, b)
      k := "max"/l
      ilog0(a, b, numer univariate(a, k), numer univariate(b, k), k::F)
```

```
-- transforms i log((a + i b) / (a - i b)) into a sum of real
-- arc-tangents using Rioboo's algorithm
-- the arc-tangents will not have k in the denominator
-- we always keep upa(k) = a  and upb(k) = b
    ilog0(a, b, upa, upb, k) ==
      if degree(upa) < degree(upb) then
        (upa, upb) := (-upb, upa)
        (a, b) := (-b, a)
      zero? degree upb => tantrick(a, b)
      r := extendedEuclidean(upa, upb)
      (g:= retractIfCan(r.generator)@Union(F,"failed")) case "failed" =>
        tantrick(a, b)
      if degree(r.coef1) >= degree upb then
        qr := divide(r.coef1, upb)
        r.coef1 := qr.remainder
        r.coef2 := r.coef2 + qr.quotient * upa
      aa := (r.coef2) k
      bb := -(r.coef1) k
      tantrick(aa * a + bb * b, g::F) + ilog0(aa,bb,r.coef2,-r.coef1,k)

    lg2func lg ==
      zero?(d := degree(p := lg.coeff)) => error "poly has degree 0"
--      one? d => [linear(p, lg.logand)]
      (d = 1) => [linear(p, lg.logand)]
      d = 2  => quadratic(p, lg.logand)
      odd? d and
        ((r := retractIfCan(reductum p)@Union(F, "failed")) case F) =>
            pairsum([cmplex(alpha := rootSimp zeroOf p, lg.logand)],
                    lg2func [lg.scalar,
                     (p exquo (monomial(1, 1)$UP - alpha::UP))::UP,
                      lg.logand])
      [lg2cfunc lg]

    lg2cfunc lg ==
      +/[cmplex(alpha, lg.logand) for alpha in zerosOf(lg.coeff)]

    mkRealFunc l ==
      ans := empty()$List(F)
      for lg in l repeat
        ans := pairsum(ans, pairprod(lg.scalar::F, lg2func lg))
      ans

-- returns a log(b)
    linear(p, lg) ==
      alpha := - coefficient(p, 0) / coefficient(p, 1)
```

```
      alpha * log lg alpha

  -- returns (c, d) s.t. p(a + b t) = c + d t, where t^2 = delta
    quadeval(p, a, b, delta) ==
      zero? p => [0, 0]
      bi := c := d := 0$F
      ai := 1$F
      v  := vectorise(p, 1 + degree p)
      for i in minIndex v .. maxIndex v repeat
        c    := c + qelt(v, i) * ai
        d    := d + qelt(v, i) * bi
        temp := a * ai + b * bi * delta
        bi   := a * bi + b * ai
        ai   := temp
      [c, d]

    compatible?(lx, ly) ==
      empty? ly => true
      for x in lx repeat
        for y in ly repeat
          ((s := sign(x*y)) case Z) and (s::Z < 0) => return false
      true

    pairsum(lx, ly) ==
      empty? lx => ly
      empty? ly => lx
      l := empty()$List(F)
      for x in lx repeat
        ls := evenRoots x
        if not empty?(ln :=
          [x + y for y in ly | compatible?(ls, evenRoots y)]) then
            l := removeDuplicates concat(l, ln)
      l

  -- returns [[a, b], s] where sqrt(y) = a sqrt(b) and
  -- s = 1 if b > 0, -1 if b < 0, 0 if the sign of b cannot be determined
    insqrt y ==
      rec := froot(y, 2)$PolynomialRoots(IndexedExponents K, K, R, P, F)
  --      one?(rec.exponent) => [[rec.coef * rec.radicand, 1], 1]
      ((rec.exponent) = 1) => [[rec.coef * rec.radicand, 1], 1]
      rec.exponent ^=2 => error "Should not happen"
      [[rec.coef, rec.radicand],
          ((s := sign(rec.radicand)) case "failed" => 0; s::Z)]

    nlogs lg ==
      [[f.exponent * lg.scalar, f.factor, lg.logand] for f in factors
```

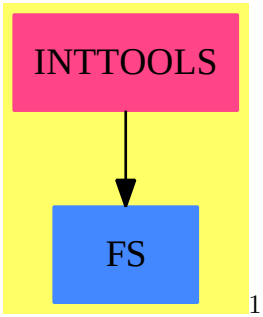
```
          ffactor(primitivePart(lg.coeff)
                    )$FunctionSpaceUnivariatePolynomialFactor(R, F, UP)]
```

⟨*IR2F.dotabb*⟩≡

```
  "IR2F" [color="#FF4488",href="bookvol10.4.pdf#nameddest=IR2F"]
  "ACFS" [color="#4488FF",href="bookvol10.2.pdf#nameddest=ACFS"]
  "IR2F" -> "ACFS"
```

10.71 package INTTOOLS IntegrationTools

10.72 IntegrationTools



1.00

Exports:

kmax	intPatternMatch	ksec	mkPrim	removeConstantTerm
union	vark	varselect		

⟨*package INTTOOLS IntegrationTools*⟩≡

```
)abbrev package INTTOOLS IntegrationTools
++ Tools for the integrator
++ Author: Manuel Bronstein
++ Date Created: 25 April 1990
++ Date Last Updated: 9 June 1993
++ Keywords: elementary, function, integration.
IntegrationTools(R:OrderedSet, F:FunctionSpace R): Exp == Impl where
  K   ==> Kernel F
  SE  ==> Symbol
  P   ==> SparseMultivariatePolynomial(R, K)
  UP  ==> SparseUnivariatePolynomial F
  IR  ==> IntegrationResult F
  ANS ==> Record(special:F, integrand:F)
  U   ==> Union(ANS, "failed")
  ALGOP ==> "%alg"

  Exp ==> with
    varselect: (List K, SE) -> List K
      ++ varselect([k1,...,kn], x) returns the ki which involve x.
    kmax     : List K -> K
      ++ kmax([k1,...,kn]) returns the top-level ki for integration.
    ksec     : (K, List K, SE) -> K
      ++ ksec(k, [k1,...,kn], x) returns the second top-level ki
      ++ after k involving x.
    union    : (List K, List K) -> List K
      ++ union(l1, l2) returns set-theoretic union of l1 and l2.
```

```
    vark     : (List F, SE) -> List K
      ++ vark([f1,...,fn],x) returns the set-theoretic union of
      ++ \spad{(varselect(f1,x),...,varselect(fn,x))}.
    if R has IntegralDomain then
      removeConstantTerm: (F, SE) -> F
        ++ removeConstantTerm(f, x) returns f minus any additive constant
        ++ with respect to x.
    if R has GcdDomain and F has ElementaryFunctionCategory then
      mkPrim: (F, SE) -> F
        ++ mkPrim(f, x) makes the logs in f which are linear in x
        ++ primitive with respect to x.
      if R has ConvertibleTo Pattern Integer and R has PatternMatchable Integer
        and F has LiouvillianFunctionCategory and F has RetractableTo SE then
          intPatternMatch: (F, SE, (F, SE) -> IR, (F, SE) -> U) -> IR
            ++ intPatternMatch(f, x, int, pmint) tries to integrate \spad{f}
            ++ first by using the integration function \spad{int}, and then
            ++ by using the pattern match intetgration function \spad{pmint}
            ++ on any remaining unintegrable part.

  Impl ==> add
    better?: (K, K) -> Boolean

    union(l1, l2)   == setUnion(l1, l2)
    varselect(l, x) == [k for k in l | member?(x, variables(k::F))]
    ksec(k, l, x)   == kmax setUnion(remove(k, l), vark(argument k, x))

    vark(l, x) ==
      varselect(reduce("setUnion",[kernels f for f in l],empty()$List(K)), x)

    kmax l ==
      ans := first l
      for k in rest l repeat
        if better?(k, ans) then ans := k
      ans

  -- true if x should be considered before y in the tower
    better?(x, y) ==
      height(y) ^= height(x) => height(y) < height(x)
      has?(operator y, ALGOP) or
              (is?(y, "exp"::SE) and not is?(x, "exp"::SE)
                                 and not has?(operator x, ALGOP))

    if R has IntegralDomain then
      removeConstantTerm(f, x) ==
        not freeOf?((den := denom f)::F, x) => f
        (u := isPlus(num := numer f)) case "failed" =>
```

```
            freeOf?(num::F, x) => 0
            f
          ans:P := 0
          for term in u::List(P) repeat
            if not freeOf?(term::F, x) then ans := ans + term
          ans / den

      if R has GcdDomain and F has ElementaryFunctionCategory then
        psimp     : (P, SE) -> Record(coef:Integer, logand:F)
        cont      : (P, List K) -> P
        logsimp   : (F, SE) -> F
        linearLog?: (K, F, SE)  -> Boolean

        logsimp(f, x) ==
          r1 := psimp(numer f, x)
          r2 := psimp(denom f, x)
          g := gcd(r1.coef, r2.coef)
          g * log(r1.logand ** (r1.coef quo g) / r2.logand ** (r2.coef quo g))

        cont(p, l) ==
          empty? l => p
          q := univariate(p, first l)
          cont(unitNormal(leadingCoefficient q).unit * content q, rest l)

        linearLog?(k, f, x) ==
          is?(k, "log"::SE) and
           ((u := retractIfCan(univariate(f,k))@Union(UP,"failed")) case UP)
--              and one?(degree(u::UP))
                and (degree(u::UP) = 1)
                   and not member?(x, variables leadingCoefficient(u::UP))

        mkPrim(f, x) ==
          lg := [k for k in kernels f | linearLog?(k, f, x)]
          eval(f, lg, [logsimp(first argument k, x) for k in lg])

        psimp(p, x) ==
          (u := isExpt(p := ((p exquo cont(p, varselect(variables p, x)))::P)))
            case "failed" => [1, p::F]
          [u.exponent, u.var::F]

        if R has Join(ConvertibleTo Pattern Integer, PatternMatchable Integer)
          and F has Join(LiouvillianFunctionCategory, RetractableTo SE) then
            intPatternMatch(f, x, int, pmint) ==
              ir := int(f, x)
              empty?(l := notelem ir) => ir
              ans := ratpart ir
```

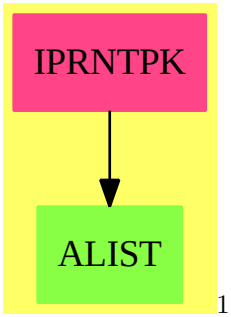
```
nl:List(Record(integrand:F, intvar:F)) := empty()
lg := logpart ir
for rec in l repeat
  u := pmint(rec.integrand, retract(rec.intvar))
  if u case ANS then
     rc := u::ANS
     ans := ans + rc.special
     if rc.integrand ^= 0 then
       ir0 := intPatternMatch(rc.integrand, x, int, pmint)
       ans := ans + ratpart ir0
       lg  := concat(logpart ir0, lg)
       nl  := concat(notelem ir0, nl)
  else nl := concat(rec, nl)
mkAnswer(ans, lg, nl)
```

⟨*INTTOOLS.dotabb*⟩≡

```
"INTTOOLS" [color="#FF4488",href="bookvol10.4.pdf#nameddest=INTTOOLS"]
"FS" [color="#4488FF",href="bookvol10.2.pdf#nameddest=FS"]
"INTTOOLS" -> "FS"
```

10.73 package IPRNTPK InternalPrintPackage

10.74 InternalPrintPackage



1.00

Exports:
iprint

⟨*package IPRNTPK InternalPrintPackage*⟩≡

```
)abbrev package IPRNTPK InternalPrintPackage
++ Author: Themos Tsikas
++ Date Created: 09/09/1998
++ Date Last Updated: 09/09/1998
++ Basic Functions:
++ Related Constructors:
++ Also See:
++ AMS Classifications:
++ Keywords:
++ References:
++ Description: A package to print strings without line-feed
++ nor carriage-return.

InternalPrintPackage(): Exports == Implementation where

  Exports ==  with
     iprint: String -> Void
       ++ \axiom{iprint(s)} prints \axiom{s} at the current position
       ++ of the cursor.

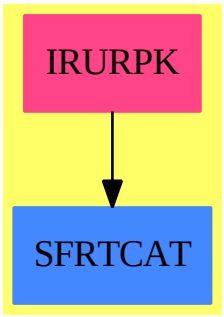
  Implementation == add
     iprint(s:String) ==
          PRINC(coerce(s)@Symbol)$Lisp
          FORCE_-OUTPUT()$Lisp
```

⟨*IPRNTPK.dotabb*⟩≡

```
"IPRNTPK" [color="#FF4488",href="bookvol10.4.pdf#nameddest=IPRNTPK"]
"ALIST" [color="#88FF44",href="bookvol10.3.pdf#nameddest=ALIST"]
"IPRNTPK" -> "ALIST"
```

10.75 package IRURPK InternalRationalUnivariateRepresentationPackage

10.76 InternalRationalUnivariateRepresentationPackage



1.00

Exports:
checkRur rur

⟨*package IRURPK InternalRationalUnivariateRepresentationPackage*⟩≡

```
)abbrev package IRURPK InternalRationalUnivariateRepresentationPackage
++ Author: Marc Moreno Maza
++ Date Created: 01/1999
++ Date Last Updated: 23/01/1999
++ Basic Functions:
++ Related Constructors:
++ Also See:
++ AMS Classifications:
++ Keywords:
++ Description:
++   An internal package for computing the rational univariate representation
++   of a zero-dimensional algebraic variety given by a square-free
++   triangular set.
++   The main operation is \axiomOpFrom{rur}{InternalRationalUnivariateRepresentationPackage
++   It is based on the {\em generic} algorithm description in [1]. \newline References:
++  [1] D. LAZARD "Solving Zero-dimensional Algebraic Systems"
++      Journal of Symbolic Computation, 1992, 13, 117-131
++ Version: 1.

InternalRationalUnivariateRepresentationPackage(R,E,V,P,TS): Exports == Implementation where
  R : Join(EuclideanDomain,CharacteristicZero)
  E : OrderedAbelianMonoidSup
  V : OrderedSet
  P : RecursivePolynomialCategory(R,E,V)
  TS : SquareFreeRegularTriangularSetCategory(R,E,V,P)
  N ==> NonNegativeInteger
```

```
Z ==> Integer
B ==> Boolean
LV ==> List V
LP ==> List P
PWT ==> Record(val: P, tower: TS)
LPWT ==> Record(val: LP, tower: TS)
WIP ==> Record(pol: P, gap: Z, tower: TS)
BWT ==> Record(val:Boolean, tower: TS)
polsetpack ==> PolynomialSetUtilitiesPackage(R,E,V,P)
normpack ==> NormalizationPackage(R,E,V,P,TS)

Exports ==  with

   rur: (TS,B) -> List TS
     ++ \spad{rur(ts,univ?)} returns a rational univariate representation
     ++ of \spad{ts}. This assumes that the lowest polynomial in \spad{ts}
     ++ is a variable \spad{v} which does not occur in the other polynomials
     ++ of \spad{ts}. This variable will be used to define the simple
     ++ algebraic extension over which these other polynomials will be
     ++ rewritten as univariate polynomials with degree one.
     ++ If \spad{univ?} is \spad{true} then these polynomials will have
     ++ a constant initial.
   checkRur: (TS, List TS) -> Boolean
     ++ \spad{checkRur(ts,lus)} returns \spad{true} if \spad{lus}
     ++ is a rational univariate representation of \spad{ts}.

Implementation == add

   checkRur(ts: TS, lts: List TS): Boolean ==
     f0 := last(ts)::P
     z := mvar(f0)
     ts := collectUpper(ts,z)
     dts: N := degree(ts)
     lp := parts(ts)
     dlts: N := 0
     for us in lts repeat
       dlts := dlts + degree(us)
       rems := [removeZero(p,us) for p in lp]
       not every?(zero?,rems) =>
         output(us::OutputForm)$OutputPackage
         return false
     (dts =$N dlts)@Boolean

   convert(p:P,sqfr?:B):TS ==
     -- if sqfr? ASSUME p is square-free
     newts: TS := empty()
```

```
        sqfr? => internalAugment(p,newts)
        p := squareFreePart(p)
        internalAugment(p,newts)

      prepareRur(ts: TS): List LPWT ==
        not purelyAlgebraic?(ts)$TS =>
          error "rur$IRURPK: #1 is not zero-dimensional"
        lp: LP := parts(ts)$TS
        lp := sort(infRittWu?,lp)
        empty? lp =>
          error "rur$IRURPK: #1 is empty"
        f0 := first lp; lp := rest lp
--        not (one?(init(f0)) and one?(mdeg(f0)) and zero?(tail(f0))) =>
        not ((init(f0) = 1) and (mdeg(f0) = 1) and zero?(tail(f0))) =>
          error "rur$IRURPK: #1 has no generating root."
        empty? lp =>
          error "rur$IRURPK: #1 has a generating root but no indeterminates"
        z: V :=  mvar(f0)
        f1 := first lp; lp := rest lp
        x1: V := mvar(f1)
        newf1 := x1::P - z::P
        toSave: List LPWT := []
        for ff1 in irreducibleFactors([f1])$polsetpack repeat
          newf0 := eval(ff1,mvar(f1),f0)
          ts := internalAugment(newf1,convert(newf0,true)@TS)
          toSave := cons([lp,ts],toSave)
        toSave

      makeMonic(z:V,c:P,r:P,ts:TS,s:P,univ?:B): TS ==
        --ASSUME r is a irreducible univariate polynomial in z
        --ASSUME c and s only depends on z and mvar(s)
        --ASSUME c and a have main degree 1
        --ASSUME c and s have a constant initial
        --ASSUME mvar(ts) < mvar(s)
        lp: LP := parts(ts)
        lp := sort(infRittWu?,lp)
        newts: TS := convert(r,true)@TS
        s := remainder(s,newts).polnum
        if univ?
          then
            s := normalizedAssociate(s,newts)$normpack
        for p in lp repeat
          p := lazyPrem(eval(p,z,c),s)
          p := remainder(p,newts).polnum
          newts := internalAugment(p,newts)
        internalAugment(s,newts)
```

```
      next(lambda:Z):Z ==
        if lambda < 0 then lambda := - lambda + 1 else lambda := - lambda

      makeLinearAndMonic(p: P, xi: V, ts: TS, univ?:B, check?: B, info?: B): List
        -- if check? THEN some VERIFICATIONS are performed
        -- if info? THEN some INFORMATION is displayed
        f0 := last(ts)::P
        z: V := mvar(f0)
        lambda: Z := 1
        ts := collectUpper(ts,z)
        toSee: List WIP := [[f0,lambda,ts]$WIP]
        toSave: List TS := []
        while not empty? toSee repeat
          wip := first toSee; toSee := rest toSee
          (f0, lambda, ts) := (wip.pol, wip.gap, wip.tower)
          if check? and ((not univariate?(f0)$polsetpack) or (mvar(f0) ~= z))
            then
                output("Bad f0: ")$OutputPackage
                output(f0::OutputForm)$OutputPackage
          c: P := lambda * xi::P + z::P
          f := eval(f0,z,c); q := eval(p,z,c)
          prs := subResultantChain(q,f)
          r := first prs; prs := rest prs
          check? and ((not zero? degree(r,xi)) or (empty? prs)) =>
            error "rur$IRURPK: should never happen !"
          s := first prs; prs := rest prs
          check? and (zero? degree(s,xi)) and (empty? prs) =>
            error "rur$IRURPK: should never happen !!"
          if zero? degree(s,xi) then s := first prs
--          not one? degree(s,xi) =>
          not (degree(s,xi) = 1) =>
            toSee := cons([f0,next(lambda),ts]$WIP,toSee)
          h := init(s)
          r := squareFreePart(r)
          ground?(h) or ground?(gcd(h,r)) =>
            for fr in irreducibleFactors([r])$polsetpack repeat
              ground? fr => "leave"
              toSave := cons(makeMonic(z,c,fr,ts,s,univ?),toSave)
          if info?
            then
              output("Unlucky lambda")$OutputPackage
              output(h::OutputForm)$OutputPackage
              output(r::OutputForm)$OutputPackage
          toSee := cons([f0,next(lambda),ts]$WIP,toSee)
        toSave
```

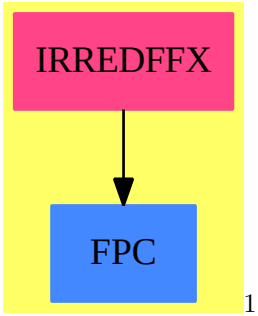
```
      rur (ts: TS,univ?:Boolean): List TS ==
        toSee: List LPWT := prepareRur(ts)
        toSave: List TS := []
        while not empty? toSee repeat
          wip := first toSee; toSee := rest toSee
          ts: TS := wip.tower
          lp: LP := wip.val
          empty? lp => toSave := cons(ts,toSave)
          p := first lp; lp := rest lp
          xi: V := mvar(p)
          p := remainder(p,ts).polnum
          if not univ?
            then
              p := primitivePart stronglyReduce(p,ts)
          ground?(p) or (mvar(p) < xi) =>
            error "rur$IRUROK: should never happen"
 --         (one? mdeg(p)) and (ground? init(p)) =>
          (mdeg(p) = 1) and (ground? init(p)) =>
            ts := internalAugment(p,ts)
            wip := [lp,ts]
            toSee := cons(wip,toSee)
          lts := makeLinearAndMonic(p,xi,ts,univ?,false,false)
          for ts in lts repeat
            wip := [lp,ts]
            toSee := cons(wip,toSee)
        toSave
```

⟨*IRURPK.dotabb*⟩≡

```
"IRURPK" [color="#FF4488",href="bookvol10.4.pdf#nameddest=IRURPK"]
"SFRTCAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=SFRTCAT"]
"IRURPK" -> "SFRTCAT"
```

10.77 package IRREDFFX IrredPolyOverFiniteField

10.78 IrredPolyOverFiniteField



1.00

Exports:
generateIrredPoly

⟨*package IRREDFFX IrredPolyOverFiniteField*⟩≡

```
)abbrev package IRREDFFX IrredPolyOverFiniteField
++ Author: Robert S. Sutor (original)
++ Date Created: ???
++ Date Last Updated: 29 May 1990
++ Description:
++ This package exports the function generateIrredPoly that computes
++ a monic irreducible polynomial of degree n over a finite field.

IrredPolyOverFiniteField(GF:FiniteFieldCategory): Exports == Impl where
  N    ==> PositiveInteger
  Z    ==> Integer
  SUP  ==> SparseUnivariatePolynomial GF
  QR   ==> Record(quotient: Z, remainder: Z)

  Exports ==> with
    generateIrredPoly: N -> SUP
      ++ generateIrredPoly(n) generates an irreducible univariate
      ++ polynomial of the given degree n over the finite field.

  Impl ==> add
    import DistinctDegreeFactorize(GF, SUP)

    getIrredPoly  : (Z, N) -> SUP
    qAdicExpansion: Z -> SUP

    p := characteristic()$GF :: N
```

```
    q := size()$GF :: N

    qAdicExpansion(z : Z): SUP ==
      -- expands z as a sum of powers of q, with coefficients in GF
      -- z = HornerEval(qAdicExpansion z,q)
      qr := divide(z, q)
      zero?(qr.remainder) => monomial(1, 1) * qAdicExpansion(qr.quotient)
      r := index(qr.remainder pretend N)$GF :: SUP
      zero?(qr.quotient) => r
      r + monomial(1, 1) * qAdicExpansion(qr.quotient)

    getIrredPoly(start : Z, n : N) : SUP ==
      -- idea is to iterate over possibly irreducible monic polynomials
      -- until we find an irreducible one. The obviously reducible ones
      -- are avoided.
      mon := monomial(1, n)$SUP
      pol: SUP := 0
      found: Boolean := false
      end: Z := q**n - 1
      while not ((end < start) or found) repeat
        if gcd(start, p) = 1 then
          if irreducible?(pol := mon + qAdicExpansion(start)) then
            found := true
        start := start + 1
      zero? pol => error "no irreducible poly found"
      pol

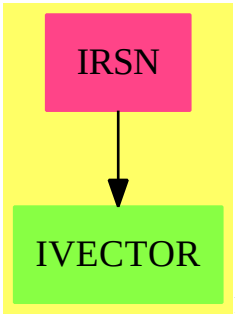
    generateIrredPoly(n : N) : SUP ==
      -- want same poly every time
--      one?(n) => monomial(1, 1)$SUP
      (n = 1) => monomial(1, 1)$SUP
--      one?(gcd(p, n)) or (n < q) =>
      (gcd(p, n) = 1) or (n < q) =>
        odd?(n) => getIrredPoly(2, n)
        getIrredPoly(1, n)
      getIrredPoly(q + 1, n)
```

⟨*IRREDFFX.dotabb*⟩≡

```
"IRREDFFX" [color="#FF4488",href="bookvol10.4.pdf#nameddest=IRREDFFX"]
"FPC" [color="#4488FF",href="bookvol10.2.pdf#nameddest=FPC"]
"IRREDFFX" -> "FPC"
```

10.79 package IRSN IrrRepSymNatPackage

10.80 IrrRepSymNatPackage



1.00

Exports:
dimensionOfIrreducibleRepresentation irreducibleRepresentation

⟨*package IRSN IrrRepSymNatPackage*⟩≡
```
)abbrev package IRSN IrrRepSymNatPackage
++ Authors: Johannes Grabmeier, Thorsten Werther
++ Date Created: 04 August 1988
++ Date Last Updated: 24 May 1991
++ Basic Operations: dimensionOfIrreducibleRepresentation
++   irreducibleRepresentation
++ Related Constructors: RepresentationTheoryPackage1
++   RepresentationTheoryPackage2
++ Also See: SymmetricGroupCombinatoricFunctions
++ AMS Classifications:
++ Keywords:
++ References:
++   G. James, A. Kerber: The Representation Theory of the Symmetric
++    Group. Encycl. of Math. and its Appl. Vol 16., Cambr. Univ Press 1981;
++   J. Grabmeier, A. Kerber: The Evaluation of Irreducible
++    Polynomial Representations of the General Linear Groups
++    and of the Unitary Groups over Fields of Characteristic 0,
++    Acta Appl. Math. 8 (1987), 271-291;
++   H. Gollan, J. Grabmeier: Algorithms in Representation Theory and
++    their Realization in the Computer Algebra System Scratchpad,
++    Bayreuther Mathematische Schriften, Heft 33, 1990, 1-23
++ Description:
++   IrrRepSymNatPackage contains functions for computing
++   the ordinary irreducible representations of symmetric groups on
++   n letters {\em {1,2,...,n}} in Young's natural form and their dimensions.
++   These representations can be labelled by number partitions of n,
++   i.e. a weakly decreasing sequence of integers summing up to n, e.g.
```

```
++   {\em [3,3,3,1]} labels an irreducible representation for n equals 10.
++   Note: whenever a \spadtype{List Integer} appears in a signature,
++   a partition required.
--   NOT TRUE in current system, but should:
--   also could be an element of \spadtype(Partition)

IrrRepSymNatPackage(): public == private where
  NNI   ==> NonNegativeInteger
  I     ==> Integer
  L     ==> List
  M     ==> Matrix
  V     ==> Vector
  B     ==> Boolean
  SGCF  ==> SymmetricGroupCombinatoricFunctions
  ICF   ==> IntegerCombinatoricFunctions Integer
  PP    ==> PartitionsAndPermutations
  PERM  ==> Permutation

  public ==> with

    dimensionOfIrreducibleRepresentation  : L I -> NNI
      ++ dimensionOfIrreducibleRepresentation(lambda) is the dimension
      ++ of the ordinary irreducible representation of the symmetric group
      ++ corresponding to {\em lambda}.
      ++ Note: the Robinson-Thrall hook formula is implemented.
    irreducibleRepresentation : (L I, PERM I) -> M I
      ++ irreducibleRepresentation(lambda,pi) is the irreducible representation
      ++ corresponding to partition {\em lambda} in Young's natural form of the
      ++ permutation {\em pi} in the symmetric group, whose elements permute
      ++ {\em {1,2,...,n}}.
    irreducibleRepresentation : L I -> L M I
      ++ irreducibleRepresentation(lambda) is the list of the two
      ++ irreducible representations corresponding to the partition {\em lambda}
      ++ in Young's natural form for the following two generators
      ++ of the symmetric group, whose elements permute
      ++ {\em {1,2,...,n}}, namely {\em (1 2)} (2-cycle) and
      ++ {\em (1 2 ... n)} (n-cycle).
    irreducibleRepresentation : (L I, L PERM I)  -> L M I
      ++ irreducibleRepresentation(lambda,listOfPerm) is the list of the
      ++ irreducible representations corresponding to {\em lambda}
      ++ in Young's natural form for the list of permutations
      ++ given by {\em listOfPerm}.

  private ==> add

    -- local variables
```

```
oldlambda : L I  := nil$(L I)
flambda   : NNI := 0           -- dimension of the irreducible repr.
younglist : L M I := nil$(L M I)     -- list of all standard tableaus
lprime    : L I  := nil$(L I)      -- conjugated partition of lambda
n         : NNI := 0           -- concerning symmetric group S_n
rows      : NNI := 0           -- # of rows of standard tableau
columns   : NNI := 0           -- # of columns of standard tableau
aId       : M I  := new(1,1,0)

-- declaration of local functions

aIdInverse : () -> Void
  -- computes aId, the inverse of the matrix
  -- (signum(k,l,id))_1 <= k,l <= flambda, where id
  -- denotes the identity permutation

alreadyComputed? : L I -> Void
  -- test if the last calling of an exported function concerns
  -- the same partition lambda as the previous call

listPermutation : PERM I -> L I   -- should be in Permutation
  -- converts a permutation pi into the list
  -- [pi(1),pi(2),..,pi(n)]

signum : (NNI, NNI, L I) -> I
  -- if there exists a vertical permutation v of the tableau
  -- tl := pi o younglist(l) (l-th standard tableau)
  -- and a horizontal permutation h of the tableau
  -- tk := younglist(k) (k-th standard tableau)  such that
  --                v o tl = h o tk,
  -- then
  --            signum(k,l,pi) = sign(v),
  -- otherwise
  --            signum(k,l,pi) = 0.

sumPartition : L I -> NNI
  -- checks if lambda is a proper partition and results in
  -- the sum of the entries

testPermutation : L I -> NNI
  -- testPermutation(pi) checks if pi is an element of S_n,
  -- the set of permutations of the set {1,2,...,n}.
  -- If not, an error message will occur, if yes it replies n.


-- definition of local functions
```

```
aIdInverse() ==

  aId := new(flambda,flambda,0)
  for k in 1..flambda repeat
    aId(k,k) := 1
  if n < 5 then return aId

  idperm      : L I  := nil$(L I)
  for k in n..1 by -1 repeat
    idperm := cons(k,idperm)
  for k in 1..(flambda-1) repeat
    for l in (k+1)..flambda repeat
      aId(k::NNI,l::NNI) := signum(k::NNI,l::NNI,idperm)

  -- invert the upper triangular matrix aId
  for j in flambda..2 by -1 repeat
    for i in (j-1)..1 by -1 repeat
      aId(i::NNI,j:NNI) := -aId(i::NNI,j::NNI)
    for k in (j+1)..flambda repeat
      for i in (j-1)..1 by -1 repeat
        aId(i::NNI,k:NNI) := aId(i::NNI,k::NNI) +
            aId(i::NNI,j:NNI) * aId(j::NNI,k::NNI)


alreadyComputed?(lambda) ==
  if not(lambda = oldlambda) then
    oldlambda := lambda
    lprime    := conjugate(lambda)$PP
    rows      := (first(lprime)$(L I))::NNI
    columns   := (first(lambda)$(L I))::NNI
    n         := (+/lambda)::NNI
    younglist := listYoungTableaus(lambda)$SGCF
    flambda   := #younglist
    aIdInverse()        -- side effect: creates actual aId

listPermutation(pi) ==
  li : L I := nil$(L I)
  for k in n..1 by -1 repeat
    li := cons(eval(pi,k)$(PERM I),li)
  li

signum(numberOfRowTableau, numberOfColumnTableau,pi) ==

  rowtab : M I  := copy younglist numberOfRowTableau
```

```
columntab : M I  := copy younglist numberOfColumnTableau
swap : I
sign : I   := 1
end  : B   := false
endk : B
ctrl : B

-- k-loop for all rows of tableau rowtab
k    : NNI := 1
while (k <= rows) and (not end) repeat
  -- l-loop along the k-th row of rowtab
  l : NNI := 1
  while (l <= oldlambda(k)) and (not end) repeat
    z : NNI := l
    endk := false
    -- z-loop for k-th row of rowtab beginning at column l.
    -- test wether the entry rowtab(k,z) occurs in the l-th column
    -- beginning at row k of pi o columntab
    while (z <= oldlambda(k)) and (not endk) repeat
      s : NNI := k
      ctrl := true
      while ctrl repeat
        if (s <= lprime(l))
          then
            if (1+rowtab(k,z) = pi(1+columntab(s,l)))
            -- if entries in the tableaus were from 1,..,n, then
            -- it should be ..columntab(s,l)... .
              then ctrl := false
              else s := s + 1
          else ctrl := false
      -- end of ctrl-loop
      endk := (s <= lprime(l)) -- same entry found ?
      if not endk
        then       -- try next entry
          z := z + 1
        else
          if k < s
            then     -- verticalpermutation
              sign := -sign
              swap := columntab(s,l)
              columntab(s,l) := columntab(k,l)
              columntab(k,l) := swap
          if l < z
            then     -- horizontalpermutation
              swap := rowtab(k,z)
              rowtab(k,z) := rowtab(k,l)
```

```
                rowtab(k,l) := swap
            -- end of else
        -- end of z-loop
        if (z > oldlambda(k))  -- no coresponding entry found
          then
            sign := 0
            end := true
        l := l + 1
      -- end of l-loop
      k := k + 1
    -- end of k-loop

    sign


  sumPartition(lambda) ==
    ok   : B := true
    prev : I := first lambda
    sum  : I := 0
    for x in lambda repeat
      sum := sum + x
      ok := ok and (prev >= x)
      prev := x
    if not ok then
      error("No proper partition ")
    sum::NNI


  testPermutation(pi : L I) : NNI ==
    ok : B := true
    n  : I := 0
    for i in pi repeat
      if i > n then n  := i     -- find the largest entry n in pi
      if i < 1 then ok := false -- check whether there are entries < 1
    -- now n should be the number of permuted objects
    if (not (n=#pi)) or (not ok) then
      error("No permutation of 1,2,..,n")
    -- now we know that pi has n Elements ranging from 1 to n
    test : Vector(B) := new((n)::NNI,false)
    for i in pi repeat
      test(i) := true   -- this means that i occurs in pi
    if member?(false,test) then error("No permutation")  -- pi not surjective
    n::NNI


  -- definitions of exported functions
```

```
dimensionOfIrreducibleRepresentation(lambda) ==
  nn : I :=  sumPartition(lambda)::I --also checks whether lambda
  dd : I := 1                        --is a partition
  lambdaprime : L I := conjugate(lambda)$PP
  -- run through all rows of the Youngtableau corr. to lambda
  for i in 1..lambdaprime.1 repeat
    -- run through all nodes in row i of the Youngtableau
    for j in 1..lambda.i repeat
        -- the hooklength of node (i,j) of the Youngtableau
        -- is the new factor, remember counting starts with 1
        dd := dd * (lambda.i + lambdaprime.j - i - j + 1)
  (factorial(nn)$ICF quo dd)::NNI


irreducibleRepresentation(lambda:(L I),pi:(PERM I)) ==
  nn : NNI := sumPartition(lambda)
  alreadyComputed?(lambda)
  piList : L I := listPermutation pi
  if not (nn = testPermutation(piList)) then
    error("Partition and permutation are not consistent")
  aPi : M I := new(flambda,flambda,0)
  for k in 1..flambda repeat
    for l in 1..flambda repeat
      aPi(k,l) := signum(k,l,piList)
  aId * aPi


irreducibleRepresentation(lambda) ==
  listperm : L PERM I := nil$(L PERM I)
  li       : L I  := nil$(L I)
  sumPartition(lambda)
  alreadyComputed?(lambda)
  listperm :=
    n = 1 =>  cons(1$(PERM I),listperm)
    n = 2 =>  cons(cycle([1,2])$(PERM I),listperm)
    -- the n-cycle (1,2,..,n) and the 2-cycle (1,2) generate S_n
    for k in n..1 by -1 repeat
      li := cons(k,li)  -- becomes n-cycle (1,2,..,n)
    listperm := cons(cycle(li)$(PERM I),listperm)
    -- 2-cycle (1,2)
    cons(cycle([1,2])$(PERM I),listperm)
  irreducibleRepresentation(lambda,listperm)
```

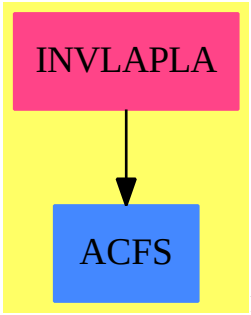
```
irreducibleRepresentation(lambda:(L I),listperm:(L PERM I)) ==
  sumPartition(lambda)
  alreadyComputed?(lambda)
  [irreducibleRepresentation(lambda, pi) for pi in listperm]
```

⟨*IRSN.dotabb*⟩≡

```
"IRSN" [color="#FF4488",href="bookvol10.4.pdf#nameddest=IRSN"]
"IVECTOR" [color="#88FF44",href="bookvol10.3.pdf#nameddest=IVECTOR"]
"IRSN" -> "IVECTOR"
```

10.81 package INVLAPLA InverseLaplaceTransform

10.82 InverseLaplaceTransform



1.00

Exports:
inverseLaplace

⟨*package INVLAPLA InverseLaplaceTransform*⟩≡

```
)abbrev package INVLAPLA InverseLaplaceTransform
++ Inverse Laplace transform
++ Author: Barry Trager
++ Date Created: 3 Sept 1991
++ Date Last Updated: 3 Sept 1991
++ Description: This package computes the inverse Laplace Transform.
InverseLaplaceTransform(R, F): Exports == Implementation where
  R : Join(EuclideanDomain, OrderedSet, CharacteristicZero,
           RetractableTo Integer, LinearlyExplicitRingOver Integer)
  F : Join(TranscendentalFunctionCategory, PrimitiveFunctionCategory,
           SpecialFunctionCategory, AlgebraicallyClosedFunctionSpace R)

  SE  ==> Symbol
  PI  ==> PositiveInteger
  N   ==> NonNegativeInteger
  K   ==> Kernel F
  UP  ==> SparseUnivariatePolynomial F
  RF  ==> Fraction UP

  Exports ==> with
    inverseLaplace: (F, SE, SE) -> Union(F,"failed")
      ++ inverseLaplace(f, s, t) returns the Inverse
      ++ Laplace transform of \spad{f(s)}
      ++ using t as the new variable or "failed" if unable to find
      ++ a closed form.
      ++ Handles only rational \spad{f(s)}.
```

```
Implementation ==> add

  -- local ops --
  ilt : (F,Symbol,Symbol) -> Union(F,"failed")
  ilt1 : (RF,F) -> F
  iltsqfr : (RF,F) -> F
  iltirred: (UP,UP,F) -> F
  freeOf?: (UP,Symbol) -> Boolean

  inverseLaplace(expr,ivar,ovar) == ilt(expr,ivar,ovar)

  freeOf?(p:UP,v:Symbol) ==
    "and"/[freeOf?(c,v) for c in coefficients p]

  ilt(expr,var,t) ==
    expr = 0 => 0
    r := univariate(expr,kernel(var))

    -- Check that r is a rational function such that degree of
    -- the numarator is lower that degree of denominator
    not(numer(r) quo denom(r) = 0) => "failed"
    not( freeOf?(numer r,var) and freeOf?(denom r,var)) => "failed"

    ilt1(r,t::F)

  hintpac := TranscendentalHermiteIntegration(F, UP)

  ilt1(r,t) ==
    r = 0 => 0
    rsplit := HermiteIntegrate(r, differentiate)$hintpac
    -t*ilt1(rsplit.answer,t) + iltsqfr(rsplit.logpart,t)

  iltsqfr(r,t) ==
     r = 0 => 0
     p:=numer r
     q:=denom r
   --  ql := [qq.factor for qq in factors factor q]
     ql := [qq.factor for qq in factors squareFree q]
     # ql = 1 => iltirred(p,q,t)
     nl := multiEuclidean(ql,p)::List(UP)
     +/[iltirred(a,b,t) for a in nl for b in ql]

  -- q is irreducible, monic, degree p < degree q
  iltirred(p,q,t) ==
    degree q = 1 =>
```

```
      cp := coefficient(p,0)
      (c:=coefficient(q,0))=0 => cp
      cp*exp(-c*t)
    degree q = 2 =>
      a := coefficient(p,1)
      b := coefficient(p,0)
      c:=(-1/2)*coefficient(q,1)
      d:= coefficient(q,0)
      e := exp(c*t)
      b := b+a*c
      d := d-c**2
      d > 0 =>
          alpha:F := sqrt d
          e*(a*cos(t*alpha) + b*sin(t*alpha)/alpha)
      alpha :F := sqrt(-d)
      e*(a*cosh(t*alpha) + b*sinh(t*alpha)/alpha)
    roots:List F := zerosOf q
    q1 := differentiate q
    +/[p(root)/q1(root)*exp(root*t) for root in roots]
```

⟨*INVLAPLA.dotabb*⟩≡

```
"INVLAPLA" [color="#FF4488",href="bookvol10.4.pdf#nameddest=INVLAPLA"]
"ACFS" [color="#4488FF",href="bookvol10.2.pdf#nameddest=ACFS"]
"INVLAPLA" -> "ACFS"
```

Chapter 11

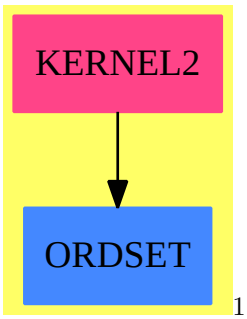
Chapter J

Chapter 12

Chapter K

12.1 package KERNEL2 KernelFunctions2

12.2 KernelFunctions2



1.00

Exports:
constantKernel constantIfCan

⟨*package KERNEL2 KernelFunctions2*⟩≡
```
)abbrev package KERNEL2 KernelFunctions2
++ Description:
++ This package exports some auxiliary functions on kernels
KernelFunctions2(R:OrderedSet, S:OrderedSet): with
  constantKernel: R -> Kernel S
        ++ constantKernel(r) \undocumented
  constantIfCan : Kernel S -> Union(R, "failed")
        ++ constantIfCan(k) \undocumented

 == add
  import BasicOperatorFunctions1(R)
```

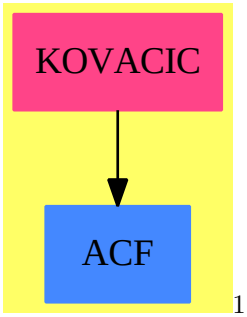
```
    constantKernel r == kernel(constantOperator r, nil(), 1)
    constantIfCan k  == constantOpIfCan operator k
```

⟨*KERNEL2.dotabb*⟩≡

```
  "KERNEL2" [color="#FF4488",href="bookvol10.4.pdf#nameddest=KERNEL2"]
  "ORDSET" [color="#4488FF",href="bookvol10.2.pdf#nameddest=ORDSET"]
  "KERNEL2" -> "ORDSET"
```

12.3 package KOVACIC Kovacic

12.4 Kovacic



1.00

Exports:
kovacic

⟨*package KOVACIC Kovacic*⟩≡
```
)abbrev package KOVACIC Kovacic
++ Author: Manuel Bronstein
++ Date Created: 14 January 1992
++ Date Last Updated: 3 February 1994
++ Description:
++ \spadtype{Kovacic} provides a modified Kovacic's algorithm for
++ solving explicitely irreducible 2nd order linear ordinary
++ differential equations.
++ Keywords: differential equation, ODE
Kovacic(F, UP): Exports == Impl where
  F  : Join(CharacteristicZero, AlgebraicallyClosedField,
            RetractableTo Integer, RetractableTo Fraction Integer)
  UP : UnivariatePolynomialCategory F

  RF  ==> Fraction UP
  SUP ==> SparseUnivariatePolynomial RF
  LF  ==> List Record(factor:UP, exponent:Integer)
  LODO==> LinearOrdinaryDifferentialOperator1 RF

  Exports ==> with
    kovacic: (RF, RF, RF) -> Union(SUP, "failed")
      ++ kovacic(a_0,a_1,a_2) returns either "failed" or P(u) such that
      ++ \spad{$e^{\int(-a_1/2a_2)} e^{\int u}$} is a solution of
      ++      \spad{a_2 y'' + a_1 y' + a0 y = 0}
      ++ whenever \spad{u} is a solution of \spad{P u = 0}.
      ++ The equation must be already irreducible over the rational functions.
    kovacic: (RF, RF, RF, UP -> Factored UP) -> Union(SUP, "failed")
```

```
      ++ kovacic(a_0,a_1,a_2,ezfactor) returns either "failed" or P(u) such
      ++ that \spad{$e^{\int(-a_1/2a_2)} e^{\int u}$} is a solution of
      ++      \spad{$a_2 y'' + a_1 y' + a0 y = 0$}
      ++ whenever \spad{u} is a solution of \spad{P u = 0}.
      ++ The equation must be already irreducible over the rational functions.
      ++ Argument \spad{ezfactor} is a factorisation in \spad{UP},
      ++ not necessarily into irreducibles.

    Impl ==> add
      import RationalRicDE(F, UP)

      case2       : (RF, LF, UP -> Factored UP) -> Union(SUP, "failed")
      cannotCase2?: LF -> Boolean

      kovacic(a0, a1, a2) == kovacic(a0, a1, a2, squareFree)

      -- it is assumed here that a2 y'' + a1 y' + a0 y is already irreducible
      -- over the rational functions, i.e. that the associated Riccati equation
      -- does NOT have rational solutions (so we don't check case 1 of Kovacic's
      -- algorithm)
      -- currently only check case 2, not 3
      kovacic(a0, a1, a2, ezfactor) ==
        -- transform first the equation to the form y'' = r y
        -- which makes the Galois group unimodular
        -- this does not change irreducibility over the rational functions
      -- the following is split into 5 lines in order to save a couple of
      -- hours of compile time.
        r:RF := a1**2
        r := r + 2 * a2 * differentiate a1
        r := r - 2 * a1 * differentiate a2
        r := r - 4 * a0 * a2
        r := r  / (4 * a2**2)
        lf := factors squareFree denom r
        case2(r, lf, ezfactor)

      -- this is case 2 of Kovacic's algorithm, i.e. look for a solution
      -- of the associated Riccati equation in a quadratic extension
      -- lf is the squarefree factorisation of denom(r) and is used to
      -- check the necessary condition
      case2(r, lf, ezfactor) ==
        cannotCase2? lf => "failed"
        -- build the symmetric square of the operator L = y'' - r y
        -- which is L2 = y''' - 4 r y' - 2 r' y
        l2:LODO := monomial(1, 3) - monomial(4*r, 1) - 2 * differentiate(r)::LODO
        -- no solution in this case if L2 has no rational solution
        empty?(sol := ricDsolve(l2, ezfactor)) => "failed"
```

```
      -- otherwise the defining polynomial for an algebraic solution
      -- of the Ricatti equation associated with L is
      -- u^2 - b u + (1/2 b' + 1/2 b^2 - r) = 0
      -- where b is a rational solution of the Ricatti of L2
      b := first sol
      monomial(1, 2)$SUP - monomial(b, 1)$SUP
                         + ((differentiate(b) + b**2 - 2 * r) / (2::RF))::SUP

    -- checks the necessary condition for case 2
    -- returns true if case 2 cannot have solutions
    -- the necessary condition is that there is either a factor with
    -- exponent 2 or odd exponent > 2
    cannotCase2? lf ==
      for rec in lf repeat
        rec.exponent = 2 or (odd?(rec.exponent) and rec.exponent > 2) =>
          return false
      true
```

⟨*KOVACIC.dotabb*⟩≡

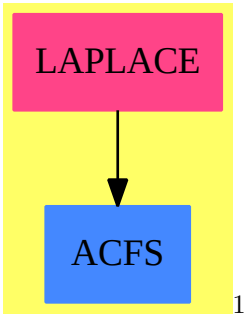
```
"KOVACIC" [color="#FF4488",href="bookvol10.4.pdf#nameddest=KOVACIC"]
"ACF" [color="#4488FF",href="bookvol10.2.pdf#nameddest=ACF"]
"KOVACIC" -> "ACF"
```

Chapter 13

Chapter L

13.1 package LAPLACE LaplaceTransform

13.2 LaplaceTransform



1.00

Exports:
laplace

⟨*package LAPLACE LaplaceTransform*⟩≡

```
)abbrev package LAPLACE LaplaceTransform
++ Laplace transform
++ Author: Manuel Bronstein
++ Date Created: 30 May 1990
++ Date Last Updated: 13 December 1995
++ Description: This package computes the forward Laplace Transform.
LaplaceTransform(R, F): Exports == Implementation where
  R : Join(EuclideanDomain, OrderedSet, CharacteristicZero,
           RetractableTo Integer, LinearlyExplicitRingOver Integer)
  F : Join(TranscendentalFunctionCategory, PrimitiveFunctionCategory,
           AlgebraicallyClosedFunctionSpace R)
```

```
    SE  ==> Symbol
    PI  ==> PositiveInteger
    N   ==> NonNegativeInteger
    K   ==> Kernel F
    OFE ==> OrderedCompletion F
    EQ  ==> Equation OFE

    ALGOP       ==> "%alg"
    SPECIALDIFF ==> "%specialDiff"

    Exports ==> with
      laplace: (F, SE, SE) -> F
        ++ laplace(f, t, s) returns the Laplace transform of \spad{f(t)}
        ++ using \spad{s} as the new variable.
        ++ This is \spad{integral(exp(-s*t)*f(t), t = 0..%plusInfinity)}.
        ++ Returns the formal object \spad{laplace(f, t, s)} if it cannot
        ++ compute the transform.

    Implementation ==> add
      import IntegrationTools(R, F)
      import ElementaryIntegration(R, F)
      import PatternMatchIntegration(R, F)
      import PowerSeriesLimitPackage(R, F)
      import FunctionSpaceIntegration(R, F)
      import TrigonometricManipulations(R, F)

      locallaplace : (F, SE, F, SE, F) -> F
      lapkernel    : (F, SE, F, F) -> Union(F, "failed")
      intlaplace   : (F, F, F, SE, F) -> Union(F, "failed")
      isLinear     : (F, SE) -> Union(Record(const:F, nconst:F), "failed")
      mkPlus       : F -> Union(List F, "failed")
      dvlap        : (List F, SE) -> F
      tdenom       : (F, F) -> Union(F, "failed")
      atn          : (F, SE) -> Union(Record(coef:F, deg:PI), "failed")
      aexp         : (F, SE) -> Union(Record(coef:F, coef1:F, coef0:F), "failed")
      algebraic?   : (F, SE) -> Boolean

      oplap := operator("laplace"::Symbol, 3)$BasicOperator

      laplace(f,t,s) == locallaplace(complexElementary(f,t),t,t::F,s,s::F)

  -- returns true if the highest kernel of f is algebraic over something
      algebraic?(f, t) ==
        l := varselect(kernels f, t)
        m:N := reduce(max, [height k for k in l], 0)$List(N)
```

```
      for k in l repeat
         height k = m and has?(operator k, ALGOP) => return true
      false

  -- differentiate a kernel of the form  laplace(l.1,l.2,l.3) w.r.t x.
  -- note that x is not necessarily l.3
  -- if x = l.3, then there is no use recomputing the laplace transform,
  -- it will remain formal anyways
    dvlap(l, x) ==
      l1 := first l
      l2 := second l
      x = (v := retract(l3 := third l)@SE) => - oplap(l2 * l1, l2, l3)
      e := exp(- l3 * l2)
      locallaplace(differentiate(e * l1, x) / e, retract(l2)@SE, l2, v, l3)

  -- returns [b, c] iff f = c * t + b
  -- and b and c do not involve t
    isLinear(f, t) ==
      ff := univariate(f, kernel(t)@K)
      ((d := retractIfCan(denom ff)@Union(F, "failed")) case "failed")
        or (degree(numer ff) > 1) => "failed"
      freeOf?(b := coefficient(numer ff, 0) / d, t) and
        freeOf?(c := coefficient(numer ff, 1) / d, t) => [b, c]
      "failed"

  -- returns [a, n] iff f = a * t**n
    atn(f, t) ==
      if ((v := isExpt f) case Record(var:K, exponent:Integer)) then
        w := v::Record(var:K, exponent:Integer)
        (w.exponent > 0) and
          ((vv := symbolIfCan(w.var)) case SE) and (vv::SE = t) =>
            return [1, w.exponent::PI]
      (u := isTimes f) case List(F) =>
        c:F  := 1
        d:N  := 0
        for g in u::List(F) repeat
          if (rec := atn(g, t)) case Record(coef:F, deg:PI) then
            r := rec::Record(coef:F, deg:PI)
            c := c * r.coef
            d := d + r.deg
          else c := c * g
        zero? d => "failed"
        [c, d::PI]
      "failed"

  -- returns [a, c, b] iff f = a * exp(c * t + b)
```

```
  -- and b and c do not involve t
      aexp(f, t) ==
        is?(f, "exp"::SE) =>
          (v := isLinear(first argument(retract(f)@K),t)) case "failed" =>
             "failed"
          [1, v.nconst, v.const]
        (u := isTimes f) case List(F) =>
          c:F := 1
          c1 := c0 := 0$F
          for g in u::List(F) repeat
            if (r := aexp(g,t)) case Record(coef:F,coef1:F,coef0:F) then
              rec := r::Record(coef:F, coef1:F, coef0:F)
              c   := c * rec.coef
              c0  := c0 + rec.coef0
              c1  := c1 + rec.coef1
            else c := c * g
          zero? c0 and zero? c1 => "failed"
          [c, c1, c0]
        if (v := isPower f) case Record(val:F, exponent:Integer) then
          w := v::Record(val:F, exponent:Integer)
          (w.exponent ^= 1) and
            ((r := aexp(w.val, t)) case Record(coef:F,coef1:F,coef0:F)) =>
              rec := r::Record(coef:F, coef1:F, coef0:F)
              return [rec.coef ** w.exponent, w.exponent * rec.coef1,
                                              w.exponent * rec.coef0]
        "failed"

      mkPlus f ==
        (u := isPlus numer f) case "failed" => "failed"
        d := denom f
        [p / d for p in u::List(SparseMultivariatePolynomial(R, K))]

  -- returns g if f = g/t
      tdenom(f, t) ==
        (denom f exquo numer t) case "failed" => "failed"
        t * f

      intlaplace(f, ss, g, v, vv) ==
        is?(g, oplap) or ((i := integrate(g, v)) case List(F)) => "failed"
        (u:=limit(i::F,equation(vv::OFE,plusInfinity()$OFE)$EQ)) case OFE =>
          (l := limit(i::F, equation(vv::OFE, ss::OFE)$EQ)) case OFE =>
            retractIfCan(u::OFE - l::OFE)@Union(F, "failed")
          "failed"
        "failed"

      lapkernel(f, t, tt, ss) ==
```

```
        (k := retractIfCan(f)@Union(K, "failed")) case "failed" => "failed"
        empty?(arg := argument(k::K)) => "failed"
        is?(op := operator k, "%diff"::SE) =>
          not( #arg = 3) => "failed"
          not(is?(arg.3, t)) => "failed"
          fint := eval(arg.1, arg.2, tt)
          s := name operator (kernels(ss).1)
          ss * locallaplace(fint, t, tt, s, ss) - eval(fint, tt = 0)
        not (empty?(rest arg)) => "failed"
        member?(t, variables(a := first(arg) / tt)) => "failed"
        is?(op := operator k, "Si"::SE) => atan(a / ss) / ss
        is?(op, "Ci"::SE) => log((ss**2 + a**2) / a**2) / (2 * ss)
        is?(op, "Ei"::SE) => log((ss + a) / a) / ss
        -- digamma (or Gamma) needs SpecialFunctionCategory
        -- which we do not have here
        -- is?(op, "log"::SE) => (digamma(1) - log(a) - log(ss)) / ss
        "failed"

      -- Below we try to apply one of the texbook rules for computing
      -- Laplace transforms, either reducing problem to simpler cases
      -- or using one of known base cases
      locallaplace(f, t, tt, s, ss) ==
        zero? f => 0
--      one? f  => inv ss
        (f = 1)  => inv ss

        -- laplace(f(t)/t,t,s)
        --              = integrate(laplace(f(t),t,v), v = s..%plusInfinity)
        (x := tdenom(f, tt)) case F =>
          g := locallaplace(x::F, t, tt, vv := new()$SE, vvv := vv::F)
          (x := intlaplace(f, ss, g, vv, vvv)) case F => x::F
          oplap(f, tt, ss)

        -- Use linearity
        (u := mkPlus f) case List(F) =>
          +/[locallaplace(g, t, tt, s, ss) for g in u::List(F)]
        (rec := splitConstant(f, t)).const ^= 1 =>
          rec.const * locallaplace(rec.nconst, t, tt, s, ss)

        -- laplace(t^n*f(t),t,s) = (-1)^n*D(laplace(f(t),t,s), s, n))
        (v := atn(f, t)) case Record(coef:F, deg:PI) =>
          vv := v::Record(coef:F, deg:PI)
          is?(la := locallaplace(vv.coef, t, tt, s, ss), oplap) => oplap(f,tt,ss)
          (-1$Integer)**(vv.deg) * differentiate(la, s, vv.deg)

        -- Complex shift rule
```

```
        (w := aexp(f, t)) case Record(coef:F, coef1:F, coef0:F) =>
          ww := w::Record(coef:F, coef1:F, coef0:F)
          exp(ww.coef0) * locallaplace(ww.coef,t,tt,s,ss - ww.coef1)

        -- Try base cases
        (x := lapkernel(f, t, tt, ss)) case F => x::F

  --    -- The following does not seem to help computing transforms, but
  --    -- quite frequently leads to loops, so I (wh) disabled it for now
  --    -- last chance option: try to use the fact that
  --    --    laplace(f(t),t,s) = s laplace(g(t),t,s) - g(0)  where dg/dt = f(t)
  --    elem?(int := lfintegrate(f, t)) and (rint := retractIfCan int) case F =>
  --        fint := rint :: F
  --        -- to avoid infinite loops, we don't call laplace recursively
  --        -- if the integral has no new logs and f is an algebraic function
  --        empty?(logpart int) and algebraic?(f, t) => oplap(fint, tt, ss)
  --        ss * locallaplace(fint, t, tt, s, ss) - eval(fint, tt = 0)
        oplap(f, tt, ss)

      setProperty(oplap,SPECIALDIFF,dvlap@((List F,SE)->F) pretend None)
```

⟨*LAPLACE.dotabb*⟩≡

```
"LAPLACE" [color="#FF4488",href="bookvol10.4.pdf#nameddest=LAPLACE"]
"ACFS" [color="#4488FF",href="bookvol10.2.pdf#nameddest=ACFS"]
"LAPLACE" -> "ACFS"
```

13.3 package LAZM3PK LazardSetSolving-Package

⟨*LazardSetSolvingPackage.input*⟩≡

```
)set break resume
)sys rm -f LazardSetSolvingPackage.output
)spool LazardSetSolvingPackage.output
)set message test on
)set message auto off
)clear all

--S 1 of 36
R := Integer
--R
--R   (1)  Integer
--R                                                              Type: Domain
--E 1

--S 2 of 36
ls : List Symbol := [b1,x,y,z,t,v,u,w]
--R
--R   (2)  [b1,x,y,z,t,v,u,w]
--R                                                        Type: List Symbol
--E 2

--S 3 of 36
V := OVAR(ls)
--R
--R   (3)  OrderedVariableList [b1,x,y,z,t,v,u,w]
--R                                                              Type: Domain
--E 3

--S 4 of 36
E := IndexedExponents V
--R
--R   (4)  IndexedExponents OrderedVariableList [b1,x,y,z,t,v,u,w]
--R                                                              Type: Domain
--E 4

--S 5 of 36
P := NSMP(R, V)
--R
--R   (5)
--R  NewSparseMultivariatePolynomial(Integer,OrderedVariableList [b1,x,y,z,t,v,u,w
--R  ])
```

```
--R                                                                    Type: Domain
--E 5

--S 6 of 36
b1: P := 'b1
--R
--R   (6)  b1
--RType: NewSparseMultivariatePolynomial(Integer,OrderedVariableList [b1,x,y,z,t,
--E 6

--S 7 of 36
x: P := 'x
--R
--R   (7)  x
--RType: NewSparseMultivariatePolynomial(Integer,OrderedVariableList [b1,x,y,z,t,
--E 7

--S 8 of 36
y: P := 'y
--R
--R   (8)  y
--RType: NewSparseMultivariatePolynomial(Integer,OrderedVariableList [b1,x,y,z,t,
--E 8

--S 9 of 36
z: P := 'z
--R
--R   (9)  z
--RType: NewSparseMultivariatePolynomial(Integer,OrderedVariableList [b1,x,y,z,t,
--E 9

--S 10 of 36
t: P := 't
--R
--R   (10)  t
--RType: NewSparseMultivariatePolynomial(Integer,OrderedVariableList [b1,x,y,z,t,
--E 10

--S 11 of 36
u: P := 'u
--R
--R   (11)  u
--RType: NewSparseMultivariatePolynomial(Integer,OrderedVariableList [b1,x,y,z,t,
--E 11

--S 12 of 36
```

```
v: P := 'v
--R
--R   (12)  v
--RType: NewSparseMultivariatePolynomial(Integer,OrderedVariableList [b1,x,y,z,t,v,u,w])
--E 12

--S 13 of 36
w: P := 'w
--R
--R   (13)  w
--RType: NewSparseMultivariatePolynomial(Integer,OrderedVariableList [b1,x,y,z,t,v,u,w])
--E 13

--S 14 of 36
T := REGSET(R,E,V,P)
--R
--R   (14)
--R  RegularTriangularSet(Integer,IndexedExponents OrderedVariableList [b1,x,y,z,t
--R  ,v,u,w],OrderedVariableList [b1,x,y,z,t,v,u,w],NewSparseMultivariatePolynomia
--R  l(Integer,OrderedVariableList [b1,x,y,z,t,v,u,w]))
--R                                                                  Type: Domain
--E 14

--S 15 of 36
p0 := b1 + y + z - t - w
--R
--R   (15)  b1 + y + z - t - w
--RType: NewSparseMultivariatePolynomial(Integer,OrderedVariableList [b1,x,y,z,t,v,u,w])
--E 15

--S 16 of 36
p1 := 2*z*u + 2*y*v + 2*t*w - 2*w**2 - w - 1
--R
--R                                  2
--R   (16)  2v y + 2u z + 2w t - 2w  - w - 1
--RType: NewSparseMultivariatePolynomial(Integer,OrderedVariableList [b1,x,y,z,t,v,u,w])
--E 16

--S 17 of 36
p2 := 3*z*u**2 + 3*y*v**2 - 3*t*w**2 + 3*w**3 + 3*w**2 - t + 4*w
--R
--R           2      2         2           3     2
--R   (17)  3v y + 3u z + (- 3w  - 1)t + 3w  + 3w  + 4w
--RType: NewSparseMultivariatePolynomial(Integer,OrderedVariableList [b1,x,y,z,t,v,u,w])
--E 17
```

```
--S 18 of 36
p3 := 6*x*z*v - 6*t*w**2 + 6*w**3 - 3*t*w + 6*w**2 - t + 4*w
--R
--R                        2                3     2
--R   (18)  6v z x + (- 6w  - 3w - 1)t + 6w  + 6w  + 4w
--RType: NewSparseMultivariatePolynomial(Integer,OrderedVariableList [b1,x,y,z,t,
--E 18

--S 19 of 36
p4 := 4*z*u**3+ 4*y*v**3+ 4*t*w**3- 4*w**4 - 6*w**3+ 4*t*w- 10*w**2- w- 1
--R
--R           3      3       3            4     3      2
--R   (19)  4v y + 4u z + (4w  + 4w)t - 4w  - 6w  - 10w  - w - 1
--RType: NewSparseMultivariatePolynomial(Integer,OrderedVariableList [b1,x,y,z,t,
--E 19

--S 20 of 36
p5 := 8*x*z*u*v +8*t*w**3 -8*w**4 +4*t*w**2 -12*w**3 +4*t*w -14*w**2 -3*w -1
--R
--R                        3     2            4      3      2
--R   (20)  8u v z x + (8w  + 4w  + 4w)t - 8w  - 12w  - 14w  - 3w - 1
--RType: NewSparseMultivariatePolynomial(Integer,OrderedVariableList [b1,x,y,z,t,
--E 20

--S 21 of 36
p6 := 12*x*z*v**2+12*t*w**3 -12*w**4 +12*t*w**2 -18*w**3 +8*t*w -14*w**2 -w -1
--R
--R            2          3      2             4      3      2
--R   (21)  12v z x + (12w  + 12w  + 8w)t - 12w  - 18w  - 14w  - w - 1
--RType: NewSparseMultivariatePolynomial(Integer,OrderedVariableList [b1,x,y,z,t,
--E 21

--S 22 of 36
p7 := -24*t*w**3 + 24*w**4 - 24*t*w**2 + 36*w**3 - 8*t*w + 26*w**2 + 7*w + 1
--R
--R               3      2             4      3      2
--R   (22)  (- 24w  - 24w  - 8w)t + 24w  + 36w  + 26w  + 7w + 1
--RType: NewSparseMultivariatePolynomial(Integer,OrderedVariableList [b1,x,y,z,t,
--E 22

--S 23 of 36
lp := [p0, p1, p2, p3, p4, p5, p6, p7]
--R
--R   (23)
--R                                                 2
--R   [b1 + y + z - t - w, 2v y + 2u z + 2w t - 2w  - w - 1,
```

```
--R      2      2         2           3     2
--R    3v y + 3u z + (- 3w  - 1)t + 3w  + 3w  + 4w,
--R                  2               3     2
--R    6v z x + (- 6w  - 3w - 1)t + 6w  + 6w  + 4w,
--R      3      3       3           4     3      2
--R    4v y + 4u z + (4w  + 4w)t - 4w  - 6w  - 10w  - w - 1,
--R                  3     2           4      3      2
--R    8u v z x + (8w  + 4w  + 4w)t - 8w  - 12w  - 14w  - 3w - 1,
--R       2          3      2             4      3      2
--R    12v z x + (12w  + 12w  + 8w)t - 12w  - 18w  - 14w  - w - 1,
--R          3      2             4      3      2
--R    (- 24w  - 24w  - 8w)t + 24w  + 36w  + 26w  + 7w + 1]
--RType: List NewSparseMultivariatePolynomial(Integer,OrderedVariableList [b1,x,y,z,t,v,u,w]
--E 23

--S 24 of 36
lts := zeroSetSplit(lp,false)$T
--R
--R   (24)
--R   [{w + 1,u,v,t + 1,b1 + y + z + 2}, {w + 1,v,t + 1,z,b1 + y + 2},
--R    {w + 1,t + 1,z,y,b1 + 2}, {w + 1,v - u,t + 1,y + z,x,b1 + 2},
--R    {w + 1,u,t + 1,y,x,b1 + z + 2},
--R
--R          5       4      3     2
--R     {144w  + 216w  + 96w  + 6w  - 11w - 1,
--R          2                 5       4      3     2
--R      (12w  + 9w + 1)u - 72w  - 108w  - 42w  - 9w  - 3w,
--R          2                 4      3      2
--R      (12w  + 9w + 1)v + 36w  + 54w  + 18w ,
--R          3      2             4      3      2
--R      (24w  + 24w  + 8w)t - 24w  - 36w  - 26w  - 7w - 1,
--R
--R                      2                 2                          4      3     2
--R         (12u v - 12u )z + (12w v + 12w  + 4)t + (3w - 5)v + 36w  + 42w  + 6w
--R       +
--R         - 16w
--R       ,
--R                             2
--R      2v y + 2u z + 2w t - 2w  - w - 1,
--R                     2               3     2
--R      6v z x + (- 6w  - 3w - 1)t + 6w  + 6w  + 4w, b1 + y + z - t - w}
--R     ]
--RType: List RegularTriangularSet(Integer,IndexedExponents OrderedVariableList [b1,x,y,z,t,
--E 24

--S 25 of 36
```

```
[coHeight(ts) for ts in lts]
--R
--R   (25)  [3,3,3,2,2,0]
--R                                                    Type: List NonNegativeInteger
--E 25

--S 26 of 36
ST := SREGSET(R,E,V,P)
--R
--R   (26)
--R  SquareFreeRegularTriangularSet(Integer,IndexedExponents OrderedVariableList
--R  b1,x,y,z,t,v,u,w],OrderedVariableList [b1,x,y,z,t,v,u,w],NewSparseMultivaria
--R  ePolynomial(Integer,OrderedVariableList [b1,x,y,z,t,v,u,w]))
--R                                                                  Type: Domain
--E 26

--S 27 of 36
pack := LAZM3PK(R,E,V,P,T,ST)
--R
--R   (27)
--R  LazardSetSolvingPackage(Integer,IndexedExponents OrderedVariableList [b1,x,y
--R  z,t,v,u,w],OrderedVariableList [b1,x,y,z,t,v,u,w],NewSparseMultivariatePolyn
--R  mial(Integer,OrderedVariableList [b1,x,y,z,t,v,u,w]),RegularTriangularSet(In
--R  eger,IndexedExponents OrderedVariableList [b1,x,y,z,t,v,u,w],OrderedVariable
--R  ist [b1,x,y,z,t,v,u,w],NewSparseMultivariatePolynomial(Integer,OrderedVariab
--R  eList [b1,x,y,z,t,v,u,w])),SquareFreeRegularTriangularSet(Integer,IndexedExp
--R  nents OrderedVariableList [b1,x,y,z,t,v,u,w],OrderedVariableList [b1,x,y,z,t
--R  v,u,w],NewSparseMultivariatePolynomial(Integer,OrderedVariableList [b1,x,y,z
--R  t,v,u,w])))
--R                                                                  Type: Domain
--E 27

--S 28 of 36
zeroSetSplit(lp,false)$pack
--R
--R   (28)
--R   [{w + 1,t + 1,z,y,b1 + 2}, {w + 1,v,t + 1,z,b1 + y + 2},
--R    {w + 1,u,v,t + 1,b1 + y + z + 2}, {w + 1,v - u,t + 1,y + z,x,b1 + 2},
--R    {w + 1,u,t + 1,y,x,b1 + z + 2},
--R
--R          5       4      3     2                    4      3      2
--R     {144w  + 216w  + 96w  + 6w  - 11w - 1, u - 24w  - 36w  - 14w  + w + 1,
--R              4      3      2                  4      3      2
--R      3v - 48w  - 60w  - 10w  + 8w + 2, t - 24w  - 36w  - 14w  - w + 1,
--R                  4        3        2
--R      486z - 2772w  - 4662w  - 2055w  + 30w + 127,
```

```
--R                    4         3        2
--R      2916y - 22752w  - 30312w  - 8220w  + 2064w + 1561,
--R                   4        3       2
--R      356x - 3696w  - 4536w  - 968w  + 822w + 371,
--R                    4         3         2
--R      2916b1 - 30600w  - 46692w  - 20274w  - 8076w + 593}
--R     ]
--RType: List SquareFreeRegularTriangularSet(Integer,IndexedExponents OrderedVariableList [b
--E 28

--S 29 of 36
f0 := (w - v) ** 2 + (u - t) ** 2 - 1
--R
--R          2           2           2    2
--R   (29)  t  - 2u t + v  - 2w v + u  + w  - 1
--RType: NewSparseMultivariatePolynomial(Integer,OrderedVariableList [b1,x,y,z,t,v,u,w])
--E 29

--S 30 of 36
f1 := t ** 2 - v ** 3
--R
--R          2    3
--R   (30)  t  - v
--RType: NewSparseMultivariatePolynomial(Integer,OrderedVariableList [b1,x,y,z,t,v,u,w])
--E 30

--S 31 of 36
f2 := 2 * t * (w - v) + 3 * v ** 2 * (u - t)
--R
--R              2                   2
--R   (31)  (- 3v  - 2v + 2w)t + 3u v
--RType: NewSparseMultivariatePolynomial(Integer,OrderedVariableList [b1,x,y,z,t,v,u,w])
--E 31

--S 32 of 36
f3 := (3 * z * v ** 2 - 1) * (2 * z * t - 1)
--R
--R           2   2             2
--R   (32)  6v t z  + (- 2t - 3v )z + 1
--RType: NewSparseMultivariatePolynomial(Integer,OrderedVariableList [b1,x,y,z,t,v,u,w])
--E 32

--S 33 of 36
lf := [f0, f1, f2, f3]
--R
--R   (33)
```

```
--R     2           2           2    2       2    3       2                   2
--R   [t  - 2u t + v  - 2w v + u  + w  - 1, t  - v , (- 3v  - 2v + 2w)t + 3u v ,
--R      2   2             2
--R    6v t z  + (- 2t - 3v )z + 1]
--RType: List NewSparseMultivariatePolynomial(Integer,OrderedVariableList [b1,x,y
--E 33

--S 34 of 36
zeroSetSplit(lf,true)$T
--R
--R   (34)
--R   [
--R     {
--R              6           3       2                 4
--R         729u  + (- 1458w  + 729w  - 4158w - 1685)u
--R       +
--R               6        5        4        3       2                2       8
--R         (729w  - 1458w  - 2619w  - 4892w  - 297w  + 5814w + 427)u  + 729w
--R       +
--R             7        6        5        4        3        2
--R         216w  - 2900w  - 2376w  + 3870w  + 4072w  - 1188w  - 1656w + 529
--R       ,
--R
--R                  4           3       2                  2        6        5
--R             2187u  + (- 4374w  - 972w  - 12474w - 2868)u  + 2187w  - 1944w
--R           +
--R                     4        3        2
--R             - 10125w  - 4800w  + 2501w  + 4968w - 1587
--R        *
--R           v
--R       +
--R                3       2  2       6        5        4       3        2
--R         (1944w  - 108w )u  + 972w  + 3024w  - 1080w  + 496w  + 1116w
--R       ,
--R         2                    2                     2  2           2
--R      (3v  + 2v - 2w)t - 3u v , ((4v - 4w)t - 6u v )z  + (2t + 3v )z - 1}
--R     ]
--RType: List RegularTriangularSet(Integer,IndexedExponents OrderedVariableList [
--E 34

--S 35 of 36
zeroSetSplit(lf,false)$T
--R
--R   (35)
--R   [
--R     {
```

```
--R             6           3       2                 4
--R          729u  + (- 1458w  + 729w  - 4158w - 1685)u
--R       +
--R              6        5        4        3       2                2       8
--R          (729w  - 1458w  - 2619w  - 4892w  - 297w  + 5814w + 427)u  + 729w
--R       +
--R             7        6        5        4        3        2
--R          216w  - 2900w  - 2376w  + 3870w  + 4072w  - 1188w  - 1656w + 529
--R       ,
--R
--R                  4           3       2                   2        6        5
--R             2187u  + (- 4374w  - 972w  - 12474w - 2868)u  + 2187w  - 1944w
--R           +
--R                     4        3        2
--R             - 10125w  - 4800w  + 2501w  + 4968w - 1587
--R        *
--R           v
--R       +
--R                3       2  2       6        5        4       3        2
--R          (1944w  - 108w )u  + 972w  + 3024w  - 1080w  + 496w  + 1116w
--R       ,
--R         2                   2                    2  2           2
--R      (3v  + 2v - 2w)t - 3u v , ((4v - 4w)t - 6u v )z  + (2t + 3v )z - 1}
--R     ,
--R
--R         4     3      2                                2
--R     {27w  + 4w  - 54w  - 36w + 23, u, (12w + 2)v - 9w  - 2w + 9,
--R        2          2
--R      6t  - 2v - 3w  + 2w + 3, 2t z - 1}
--R     ,
--R
--R            6         5         4          3         2
--R     {59049w  + 91854w  - 45198w  + 145152w  + 63549w  + 60922w + 21420,
--R
--R                              5                  4                   3
--R             31484448266904w  - 18316865522574w  + 23676995746098w
--R           +
--R                           2
--R             6657857188965w  + 8904703998546w + 3890631403260
--R        *
--R            2
--R           u
--R       +
--R                           5                  4                  3
--R          94262810316408w  - 82887296576616w  + 89801831438784w
--R       +
```

```
--R                          2
--R         28141734167208w  + 38070359425432w + 16003865949120
--R       ,
--R           2             2         2       3      2                    3     2
--R      (243w  + 36w + 85)v  + (- 81u  - 162w  + 36w  + 154w + 72)v - 72w  + 4w
--R         2                   2                    2  2          2
--R      (3v  + 2v - 2w)t - 3u v , ((4v - 4w)t - 6u v )z  + (2t + 3v )z - 1}
--R     ,
--R
--R         4     3      2                              2
--R     {27w  + 4w  - 54w  - 36w + 23, u, (12w + 2)v - 9w  - 2w + 9,
--R        2          2             2
--R      6t  - 2v - 3w  + 2w + 3, 3v z - 1}
--R     ]
--RType: List RegularTriangularSet(Integer,IndexedExponents OrderedVariableList [
--E 35

--S 36 of 36
zeroSetSplit(lf,false)$pack
--R
--R   (36)
--R   [
--R     {
--R              6           3       2                 4
--R         729u  + (- 1458w  + 729w  - 4158w - 1685)u
--R       +
--R               6        5        4        3       2                2       8
--R         (729w  - 1458w  - 2619w  - 4892w  - 297w  + 5814w + 427)u  + 729w
--R       +
--R              7        6        5        4        3        2
--R         216w  - 2900w  - 2376w  + 3870w  + 4072w  - 1188w  - 1656w + 529
--R       ,
--R
--R                  4           3       2                  2        6        5
--R            2187u  + (- 4374w  - 972w  - 12474w - 2868)u  + 2187w  - 1944w
--R           +
--R                     4        3        2
--R             - 10125w  - 4800w  + 2501w  + 4968w - 1587
--R        *
--R           v
--R       +
--R                3       2  2       6        5        4       3        2
--R         (1944w  - 108w )u  + 972w  + 3024w  - 1080w  + 496w  + 1116w
--R       ,
--R         2                   2                    2  2          2
--R      (3v  + 2v - 2w)t - 3u v , ((4v - 4w)t - 6u v )z  + (2t + 3v )z - 1}
```

```
--R     ,
--R
--R         2                 2                    2
--R     {81w  + 18w + 28, 729u  - 1890w - 533, 81v  + (- 162w + 27)v - 72w - 112,
--R      11881t + (972w + 2997)u v + (- 11448w - 11536)u,
--R
--R                          2
--R         641237934604288z
--R       +
--R                   (78614584763904w + 26785578742272)u + 236143618655616w
--R               +
--R                   70221988585728
--R            *
--R               v
--R           +
--R             (358520253138432w + 101922133759488)u + 142598803536000w
--R           +
--R             54166419595008
--R        *
--R           z
--R       +
--R         (32655103844499w - 44224572465882)u v
--R       +
--R         (43213900115457w - 32432039102070)u
--R       }
--R     ,
--R
--R         4     3      2                             3     2
--R     {27w  + 4w  - 54w  - 36w + 23, u, 218v - 162w  + 3w  + 160w + 153,
--R          2      3      2                               3      2
--R      109t  - 27w  - 54w  + 63w + 80, 1744z + (- 1458w  + 27w  + 1440w + 505)t}
--R     ,
--R
--R         4     3      2                             3     2
--R     {27w  + 4w  - 54w  - 36w + 23, u, 218v - 162w  + 3w  + 160w + 153,
--R          2      3      2                             3     2
--R      109t  - 27w  - 54w  + 63w + 80, 1308z + 162w  - 3w  - 814w - 153}
--R     ,
--R
--R          4       3        2                   2      2
--R     {729w  + 972w  - 1026w  + 1684w + 765, 81u  + 72w  + 16w - 72,
--R                 3       2
--R      702v - 162w  - 225w  + 40w - 99,
--R                    3       2
--R      11336t + (324w  - 603w  - 1718w - 1557)u,
--R
```

```
--R                   2
--R         595003968z
--R       +
--R                             3             2
--R             (- 963325386w  - 898607682w  + 1516286466w - 3239166186)u
--R           +
--R                             3              2
--R             - 1579048992w  - 1796454288w  + 2428328160w - 4368495024
--R        *
--R           z
--R       +
--R                       3              2
--R         (9713133306w  + 9678670317w  - 16726834476w + 28144233593)u
--R       }
--R     ]
--RType: List SquareFreeRegularTriangularSet(Integer,IndexedExponents OrderedVari
--E 36

)spool
)lisp (bye)
```

⟨*LazardSetSolvingPackage.help*⟩≡

```
==========================================================================
LazardSetSolvingPackage examples
==========================================================================

The LazardSetSolvingPackage package constructor solves polynomial
systems by means of Lazard triangular sets.  However one condition is
relaxed: Regular triangular sets whose saturated ideals have positive
dimension are not necessarily normalized.

The decompositions are computed in two steps.  First the algorithm of
Moreno Maza (implemented in the RegularTriangularSet domain
constructor) is called.  Then the resulting decompositions are
converted into lists of square-free regular triangular sets and the
redundant components are removed.  Moreover, zero-dimensional regular
triangular sets are normalized.

Note that the way of understanding triangular decompositions is
detailed in the example of the RegularTriangularSet constructor.

The LazardSetSolvingPackage constructor takes six arguments.  The
first one, R, is the coefficient ring of the polynomials; it must
belong to the category GcdDomain.  The second one, E, is the exponent
monoid of the polynomials; it must belong to the category
OrderedAbelianMonoidSup.  the third one, V, is the ordered set of
variables; it must belong to the category OrderedSet.  The fourth one
is the polynomial ring; it must belong to the category
RecursivePolynomialCategory(R,E,V).  The fifth one is a domain of the
category RegularTriangularSetCategory(R,E,V,P) and the last one is a
domain of the category SquareFreeRegularTriangularSetCategory(R,E,V,P).
The abbreviation for LazardSetSolvingPackage is LAZM3PK.

For the purpose of solving zero-dimensional algebraic systems, see
also LexTriangularPackage and ZeroDimensionalSolvePackage.  These
packages are easier to call than LAZM3PK.  Moreover, the
ZeroDimensionalSolvePackage package provides operations to compute
either the complex roots or the real roots.

We illustrate now the use of the LazardSetSolvingPackage package
constructor with two examples (Butcher and Vermeer).

Define the coefficient ring.

R := Integer
     Integer
```

```
Define the list of variables,

ls : List Symbol := [b1,x,y,z,t,v,u,w]
    [b1,x,y,z,t,v,u,w]

and make it an ordered set:

V := OVAR(ls)
    OrderedVariableList [b1,x,y,z,t,v,u,w]

then define the exponent monoid.

E := IndexedExponents V

Define the polynomial ring.

P := NSMP(R, V)

Let the variables be polynomial.

b1: P := 'b1

x: P := 'x

y: P := 'y

z: P := 'z

t: P := 't

u: P := 'u

v: P := 'v

w: P := 'w

Now call the {\tt RegularTriangularSet} domain constructor.

T := REGSET(R,E,V,P)

Define a polynomial system (the Butcher example).

p0 := b1 + y + z - t - w

p1 := 2*z*u + 2*y*v + 2*t*w - 2*w**2 - w - 1
```

```
p2 := 3*z*u**2 + 3*y*v**2 - 3*t*w**2 + 3*w**3 + 3*w**2 - t + 4*w

p3 := 6*x*z*v - 6*t*w**2 + 6*w**3 - 3*t*w + 6*w**2 - t + 4*w

p4 := 4*z*u**3+ 4*y*v**3+ 4*t*w**3- 4*w**4 - 6*w**3+ 4*t*w- 10*w**2- w- 1

p5 := 8*x*z*u*v +8*t*w**3 -8*w**4 +4*t*w**2 -12*w**3 +4*t*w -14*w**2 -3*w -1

p6 := 12*x*z*v**2+12*t*w**3 -12*w**4 +12*t*w**2 -18*w**3 +8*t*w -14*w**2 -w -1

p7 := -24*t*w**3 + 24*w**4 - 24*t*w**2 + 36*w**3 - 8*t*w + 26*w**2 + 7*w + 1

lp := [p0, p1, p2, p3, p4, p5, p6, p7]
   [b1 + y + z - t - w, 2v y + 2u z + 2w t - 2w  - w - 1,
     2      2         2           3     2
    3v y + 3u z + (- 3w  - 1)t + 3w  + 3w  + 4w,
                  2                3     2
    6v z x + (- 6w  - 3w - 1)t + 6w  + 6w  + 4w,
     3      3       3            4     3      2
    4v y + 4u z + (4w  + 4w)t - 4w  - 6w  - 10w  - w - 1,
                  3     2            4      3      2
    8u v z x + (8w  + 4w  + 4w)t - 8w  - 12w  - 14w  - 3w - 1,
       2          3      2             4      3      2
    12v z x + (12w  + 12w  + 8w)t - 12w  - 18w  - 14w  - w - 1,
          3      2             4      3      2
    (- 24w  - 24w  - 8w)t + 24w  + 36w  + 26w  + 7w + 1]

First of all, let us solve this system in the sense of Lazard by means
of the REGSET constructor:

lts := zeroSetSplit(lp,false)$T
   [{w + 1,u,v,t + 1,b1 + y + z + 2}, {w + 1,v,t + 1,z,b1 + y + 2},
    {w + 1,t + 1,z,y,b1 + 2}, {w + 1,v - u,t + 1,y + z,x,b1 + 2},
    {w + 1,u,t + 1,y,x,b1 + z + 2},

         5       4      3     2
     {144w  + 216w  + 96w  + 6w  - 11w - 1,
         2                 5       4      3     2
      (12w  + 9w + 1)u - 72w  - 108w  - 42w  - 9w  - 3w,
         2                 4      3      2
      (12w  + 9w + 1)v + 36w  + 54w  + 18w ,
         3      2             4      3      2
      (24w  + 24w  + 8w)t - 24w  - 36w  - 26w  - 7w - 1,

                     2                 2                          4      3     2
          (12u v - 12u )z + (12w v + 12w  + 4)t + (3w - 5)v + 36w  + 42w  + 6w
```

```
       +
         - 16w
       ,
                              2
        2v y + 2u z + 2w t - 2w  - w - 1,
                     2                3     2
        6v z x + (- 6w  - 3w - 1)t + 6w  + 6w  + 4w, b1 + y + z - t - w}
       ]
```

We can get the dimensions of each component
of a decomposition as follows.

```
[coHeight(ts) for ts in lts]
    [3,3,3,2,2,0]
```

The first five sets have a simple shape. However, the last one, which
has dimension zero, can be simplified by using Lazard triangular sets.

Thus we call the SquareFreeRegularTriangularSet domain constructor,

```
ST := SREGSET(R,E,V,P)
```

and set the LAZM3PK package constructor to our situation.

```
pack := LAZM3PK(R,E,V,P,T,ST)
```

We are ready to solve the system by means of Lazard triangular sets:

```
zeroSetSplit(lp,false)$pack
   [{w + 1,t + 1,z,y,b1 + 2}, {w + 1,v,t + 1,z,b1 + y + 2},
    {w + 1,u,v,t + 1,b1 + y + z + 2}, {w + 1,v - u,t + 1,y + z,x,b1 + 2},
    {w + 1,u,t + 1,y,x,b1 + z + 2},

         5       4      3     2                    4      3      2
     {144w  + 216w  + 96w  + 6w  - 11w - 1, u - 24w  - 36w  - 14w  + w + 1,
             4      3      2                  4      3      2
      3v - 48w  - 60w  - 10w  + 8w + 2, t - 24w  - 36w  - 14w  - w + 1,
                 4         3         2
      486z - 2772w  - 4662w  - 2055w  + 30w + 127,
                  4          3         2
      2916y - 22752w  - 30312w  - 8220w  + 2064w + 1561,
                 4         3        2
      356x - 3696w  - 4536w  - 968w  + 822w + 371,
                   4          3         2
      2916b1 - 30600w  - 46692w  - 20274w  - 8076w + 593}
     ]
```

```
We see the sixth triangular set is {\em nicer} now: each one of its
polynomials has a constant initial.

We follow with the Vermeer example. The ordering is the usual one
for this system.

Define the polynomial system.

f0 := (w - v) ** 2 + (u - t) ** 2 - 1
```

$$t^2 - 2ut + v^2 - 2wv + u^2 + w^2 - 1$$

```
f1 := t ** 2 - v ** 3
```

$$t^2 - v^3$$

```
f2 := 2 * t * (w - v) + 3 * v ** 2 * (u - t)
```

$$(-3v^2 - 2v + 2w)t + 3uv^2$$

```
f3 := (3 * z * v ** 2 - 1) * (2 * z * t - 1)
```

$$6vtz^2 + (-2t - 3v^2)z + 1$$

```
lf := [f0, f1, f2, f3]
```

$$[t^2 - 2ut + v^2 - 2wv + u^2 + w^2 - 1,\ t^2 - v^3,\ (-3v^2 - 2v + 2w)t + 3uv^2,\ 6vtz^2 + (-2t - 3v^2)z + 1]$$

```
First of all, let us solve this system in the sense of Kalkbrener by
means of the REGSET constructor:

zeroSetSplit(lf,true)$T
   [
     {
```

$$729u^6 + (-1458w^3 + 729w^2 - 4158w - 1685)u^4$$

```
        +
              6        5        4        3       2                2       8
          (729w  - 1458w  - 2619w  - 4892w  - 297w  + 5814w + 427)u  + 729w
        +
             7        6        5        4        3        2
          216w  - 2900w  - 2376w  + 3870w  + 4072w  - 1188w  - 1656w + 529
        ,

                   4           3       2                  2        6        5
              2187u  + (- 4374w  - 972w  - 12474w - 2868)u  + 2187w  - 1944w
            +
                      4        3        2
              - 10125w  - 4800w  + 2501w  + 4968w - 1587
         *
            v
        +
                3       2  2       6        5        4       3        2
          (1944w  - 108w )u  + 972w  + 3024w  - 1080w  + 496w  + 1116w
        ,
          2                   2                    2  2           2
       (3v  + 2v - 2w)t - 3u v , ((4v - 4w)t - 6u v )z  + (2t + 3v )z - 1}
      ]
```

We have obtained one regular chain (i.e. regular triangular set) with dimension 1. This set is in fact a characterist set of the (radical of) of the ideal generated by the input system lf. Thus we have only the generic points of the variety associated with lf (for the elimination ordering given by ls).

So let us get now a full description of this variety.

Hence, we solve this system in the sense of Lazard by means of the REGSET constructor:

```
zeroSetSplit(lf,false)$T
   [
     {
             6           3       2                 4
       729u  + (- 1458w  + 729w  - 4158w - 1685)u
       +
             6        5        4        3       2                2       8
       (729w  - 1458w  - 2619w  - 4892w  - 297w  + 5814w + 427)u  + 729w
       +
             7        6        5        4        3        2
       216w  - 2900w  - 2376w  + 3870w  + 4072w  - 1188w  - 1656w + 529
       ,
```

```
             4           3       2                   2        6        5
       2187u  + (- 4374w  - 972w  - 12474w - 2868)u  + 2187w  - 1944w
     +
               4        3        2
       - 10125w  - 4800w  + 2501w  + 4968w - 1587
   *
     v
  +
          3       2  2       6        5        4       3        2
    (1944w  - 108w )u  + 972w  + 3024w  - 1080w  + 496w  + 1116w
  ,
    2                   2                    2  2           2
 (3v  + 2v - 2w)t - 3u v , ((4v - 4w)t - 6u v )z  + (2t + 3v )z - 1}
,

    4     3      2                                2
{27w  + 4w  - 54w  - 36w + 23, u, (12w + 2)v - 9w  - 2w + 9,
   2          2
 6t  - 2v - 3w  + 2w + 3, 2t z - 1}
,

       6         5         4          3         2
{59049w  + 91854w  - 45198w  + 145152w  + 63549w  + 60922w + 21420,

                        5                  4                  3
       31484448266904w  - 18316865522574w  + 23676995746098w
     +
                      2
       6657857188965w  + 8904703998546w + 3890631403260
   *
      2
     u
  +
                  5                  4                  3
    94262810316408w  - 82887296576616w  + 89801831438784w
  +
                  2
    28141734167208w  + 38070359425432w + 16003865949120
  ,
     2             2         2       3      2                     3     2
 (243w  + 36w + 85)v  + (- 81u  - 162w  + 36w  + 154w + 72)v - 72w  + 4w ,
    2                   2                    2  2           2
 (3v  + 2v - 2w)t - 3u v , ((4v - 4w)t - 6u v )z  + (2t + 3v )z - 1}
,
```

```
        4     3      2                                  2
      {27w  + 4w  - 54w  - 36w + 23, u, (12w + 2)v - 9w  - 2w + 9,
         2          2             2
       6t  - 2v - 3w  + 2w + 3, 3v z - 1}
      ]
```

We retrieve our regular chain of dimension 1 and we get three regular chains of dimension 0 corresponding to the degenerated cases.
We want now to simplify these zero-dimensional regular chains by using Lazard triangular sets. Moreover, this will allow us to prove that the above decomposition has no redundant component.

Generally, decompositions computed by the REGSET constructor do not have redundant components. However, to be sure that no redundant component occurs one needs to use the SREGSET or LAZM3PK constructors.

So let us solve the input system in the sense of Lazard by means of the LAZM3PK constructor:

```
zeroSetSplit(lf,false)$pack
   [
     {
             6           3       2                 4
           729u  + (- 1458w  + 729w  - 4158w - 1685)u
       +
              6        5        4        3       2                2       8
           (729w  - 1458w  - 2619w  - 4892w  - 297w  + 5814w + 427)u  + 729w
       +
             7        6        5        4        3        2
           216w  - 2900w  - 2376w  + 3870w  + 4072w  - 1188w  - 1656w + 529
       ,

                  4           3       2                  2        6        5
              2187u  + (- 4374w  - 972w  - 12474w - 2868)u  + 2187w  - 1944w
          +
                    4        3        2
              - 10125w  - 4800w  + 2501w  + 4968w - 1587
        *
           v
       +
             3       2  2       6        5        4       3        2
           (1944w  - 108w )u  + 972w  + 3024w  - 1080w  + 496w  + 1116w
       ,
         2                   2                    2  2           2
       (3v  + 2v - 2w)t - 3u v , ((4v - 4w)t - 6u v )z  + (2t + 3v )z - 1}
```

```
,

   2                2                   2
{81w  + 18w + 28, 729u  - 1890w - 533, 81v  + (- 162w + 27)v - 72w - 112,
 11881t + (972w + 2997)u v + (- 11448w - 11536)u,

                   2
   641237934604288z
  +
            (78614584763904w + 26785578742272)u + 236143618655616w
          +
            70221988585728
       *
          v
      +
        (358520253138432w + 101922133759488)u + 142598803536000w
      +
        54166419595008
   *
      z
  +
    (32655103844499w - 44224572465882)u v
  +
    (43213900115457w - 32432039102070)u
  }
,

   4     3      2                            3     2
{27w  + 4w  - 54w  - 36w + 23, u, 218v - 162w  + 3w  + 160w + 153,
     2      3      2                              3      2
 109t  - 27w  - 54w  + 63w + 80, 1744z + (- 1458w  + 27w  + 1440w + 505)t}
,

   4     3      2                            3     2
{27w  + 4w  - 54w  - 36w + 23, u, 218v - 162w  + 3w  + 160w + 153,
     2      3      2                            3     2
 109t  - 27w  - 54w  + 63w + 80, 1308z + 162w  - 3w  - 814w - 153}
,

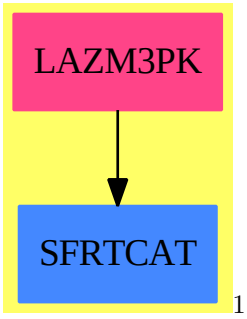
     4       3        2                   2      2
{729w  + 972w  - 1026w  + 1684w + 765, 81u  + 72w  + 16w - 72,
            3       2
 702v - 162w  - 225w  + 40w - 99,
               3       2
 11336t + (324w  - 603w  - 1718w - 1557)u,
```

```
                  2
          595003968z
        +
                            3              2
              (- 963325386w  - 898607682w  + 1516286466w - 3239166186)u
           +
                            3              2
              - 1579048992w  - 1796454288w  + 2428328160w - 4368495024
         *
           z
        +
                      3              2
          (9713133306w  + 9678670317w  - 16726834476w + 28144233593)u
        }
      ]
```

Due to square-free factorization, we obtained now four zero-dimensional regular chains. Moreover, each of them is normalized (the initials are constant). Note that these zero-dimensional components may be investigated further with the ZeroDimensionalSolvePackage package constructor.

See also:
o)show LazardSetSolvingPackage
o)show ZeroDimensionalSolvePackage

13.4 LazardSetSolvingPackage



1.00

Exports:
normalizeIfCan zeroSetSplit

⟨*package LAZM3PK LazardSetSolvingPackage*⟩≡
```
)abbrev package LAZM3PK LazardSetSolvingPackage
++ Author: Marc Moreno Maza
++ Date Created: 10/02/1998
++ Date Last Updated: 12/16/1998
++ Basic Functions:
++ Related Constructors:
++ Also See:
++ AMS Classifications:
++ Keywords:
++ Description:
++ A package for solving polynomial systems by means of Lazard triangular
++ sets [1].
++ This package provides two operations. One for solving in the sense
++ of the regular zeros, and the other for solving in the sense of
++ the Zariski closure. Both produce square-free regular sets.
++ Moreover, the decompositions do not contain any redundant component.
++ However, only zero-dimensional regular sets are normalized, since
++ normalization may be time consumming in positive dimension.
++ The decomposition process is that of [2].\newline
++ References :
++  [1] D. LAZARD "A new method for solving algebraic systems of
++      positive dimension" Discr. App. Math. 33:147-160,1991
++  [2] M. MORENO MAZA "A new algorithm for computing triangular
++      decomposition of algebraic varieties" NAG Tech. Rep. 4/98.
++ Version: 1.

LazardSetSolvingPackage(R,E,V,P,TS,ST): Exports == Implementation where

  R : GcdDomain
  E : OrderedAbelianMonoidSup
```

```
V : OrderedSet
P : RecursivePolynomialCategory(R,E,V)
TS: RegularTriangularSetCategory(R,E,V,P)
ST : SquareFreeRegularTriangularSetCategory(R,E,V,P)
N ==> NonNegativeInteger
Z ==> Integer
B ==> Boolean
S ==> String
K ==> Fraction R
LP ==> List P
PWT ==> Record(val : P, tower : TS)
BWT ==> Record(val : Boolean, tower : TS)
LpWT ==> Record(val : (List P), tower : TS)
Split ==> List TS
--KeyGcd ==> Record(arg1: P, arg2: P, arg3: TS, arg4: B)
--EntryGcd ==> List PWT
--HGcd ==> TabulatedComputationPackage(KeyGcd, EntryGcd)
--KeyInvSet ==> Record(arg1: P, arg3: TS)
--EntryInvSet ==> List TS
--HInvSet ==> TabulatedComputationPackage(KeyInvSet, EntryInvSet)
polsetpack ==> PolynomialSetUtilitiesPackage(R,E,V,P)
regsetgcdpack ==> SquareFreeRegularTriangularSetGcdPackage(R,E,V,P,ST)
quasicomppack ==> SquareFreeQuasiComponentPackage(R,E,V,P,ST)
normalizpack ==> NormalizationPackage(R,E,V,P,ST)

Exports ==  with

   normalizeIfCan: ST -> ST
     ++ \axiom{normalizeIfCan(ts)} returns \axiom{ts} in an normalized shape
     ++ if \axiom{ts} is zero-dimensional.
   zeroSetSplit: (LP, B) -> List ST
     ++ \axiom{zeroSetSplit(lp,clos?)} has the same specifications as
     ++ \axiomOpFrom{zeroSetSplit(lp,clos?)}{RegularTriangularSetCategory}.

Implementation == add

   convert(st: ST): TS ==
     ts: TS := empty()
     lp: LP := members(st)$ST
     lp := sort(infRittWu?,lp)
     for p in lp repeat
       ts := internalAugment(p,ts)$TS
     ts

   squareFree(ts: TS): List ST ==
     empty? ts => [empty()$ST]
```

```
    lp: LP := members(ts)$TS
    lp := sort(infRittWu?,lp)
    newts: ST := empty()$ST
    toSee: List ST := [newts]
    toSave: List ST
    for p in lp repeat
      toSave := []
      while (not empty? toSee) repeat
        us := first toSee; toSee := rest toSee
        lpwt := stoseSquareFreePart(p,us)$regsetgcdpack
        for pwt in lpwt repeat
          newus := internalAugment(pwt.val,pwt.tower)$ST
          toSave := cons(newus,toSave)
      toSee := toSave
    toSave

  normalizeIfCan(ts: ST): ST ==
    empty? ts => ts
    lp: LP := members(ts)$ST
    lp := sort(infRittWu?,lp)
    p: P := first lp
    not univariate?(p)$polsetpack => ts
    lp := rest lp
    newts: ST := empty()$ST
    newts := internalAugment(p,newts)$ST
    while (not empty? lp) repeat
      p := first lp
      lv := variables(p)
      for v in lv repeat
        v = mvar(p) => "leave"
        not algebraic?(v,newts) => return internalAugment(lp,newts)$ST
      lp := rest lp
      p := normalizedAssociate(p,newts)$normalizpack
      newts := internalAugment(p,newts)$ST
    newts

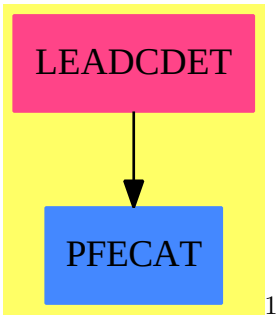
  zeroSetSplit(lp:List(P), clos?:B): List ST ==
    -- if clos? then SOLVE in the closure sense
    toSee: Split := zeroSetSplit(lp, clos?)$TS
    toSave: List ST := []
    for ts in toSee repeat
      toSave := concat(squareFree(ts),toSave)
    toSave := removeSuperfluousQuasiComponents(toSave)$quasicomppack
    [normalizeIfCan(ts) for ts in toSave]
```

⟨*LAZM3PK.dotabb*⟩≡

```
"LAZM3PK" [color="#FF4488",href="bookvol10.4.pdf#nameddest=LAZM3PK"]
"SFRTCAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=SFRTCAT"]
"LAZM3PK" -> "SFRTCAT"
```

13.5 package LEADCDET LeadingCoefDetermination

13.6 LeadingCoefDetermination



1.00

Exports:
distFact polCase

⟨*package LEADCDET LeadingCoefDetermination*⟩≡

```
)abbrev package LEADCDET LeadingCoefDetermination
++ Author : P.Gianni, May 1990
++ Description:
++ Package for leading coefficient determination in the lifting step.
++ Package working for every R euclidean with property "F".
LeadingCoefDetermination(OV,E,Z,P) : C == T
 where
  OV     :   OrderedSet
  E      :   OrderedAbelianMonoidSup
  Z      :   EuclideanDomain
  BP         ==> SparseUnivariatePolynomial Z
  P          :   PolynomialCategory(Z,E,OV)
  NNI        ==> NonNegativeInteger
  LeadFact   ==> Record(polfac:List(P),correct:Z,corrfact:List(BP))
  ParFact    ==> Record(irr:P,pow:Integer)
  FinalFact  ==> Record(contp:Z,factors:List(ParFact))

  C == with
   polCase  : (Z,NNI,List(Z)) -> Boolean
     ++ polCase(contprod, numFacts, evallcs), where contprod is the
     ++ product of the content of the leading coefficient of
     ++ the polynomial to be factored with the content of the
     ++ evaluated polynomial, numFacts is the number of factors
     ++ of the leadingCoefficient, and evallcs is the list of
     ++ the evaluated factors of the leadingCoefficient, returns
     ++ true if the factors of the leading Coefficient can be
```

```
    ++ distributed with this valuation.
  distFact : (Z,List(BP),FinalFact,List(Z),List(OV),List(Z)) ->
                                              Union(LeadFact,"failed")
    ++ distFact(contm,unilist,plead,vl,lvar,lval), where contm is
    ++ the content of the evaluated polynomial, unilist is the list
    ++ of factors of the evaluated polynomial, plead is the complete
    ++ factorization of the leading coefficient, vl is the list
    ++ of factors of the leading coefficient evaluated, lvar is the
    ++ list of variables, lval is the list of values, returns a record
    ++ giving the list of leading coefficients to impose on the univariate
    ++ factors,

 T == add
   distribute: (Z,List(BP),List(P),List(Z),List(OV),List(Z)) -> LeadFact
   checkpow  : (Z,Z) -> NNI

   polCase(d:Z,nk:NNI,lval:List(Z)):Boolean ==
     -- d is the product of the content lc m (case polynomial)
     -- and the cont of the polynomial evaluated
     q:Z
     distlist:List(Z) := [d]
     for i in 1..nk repeat
       q := unitNormal(lval.i).canonical
       for j in 0..(i-1)::NNI repeat
         y := distlist.((i-j)::NNI)
         while y^=1  repeat
           y := gcd(y,q)
           q := q quo y
         if q=1 then return false
       distlist := append(distlist,[q])
     true

   checkpow(a:Z,b:Z) : NonNegativeInteger ==
     qt: Union(Z,"failed")
     for i in 0.. repeat
       qt:= b exquo a
       if qt case "failed" then return i
       b:=qt::Z

   distribute(contm:Z,unilist:List(BP),pl:List(P),vl:List(Z),
                              lvar:List(OV),lval:List(Z)): LeadFact ==
     d,lcp : Z
     nf:NNI:=#unilist
     for i in 1..nf repeat
         lcp := leadingCoefficient (unilist.i)
         d:= gcd(lcp,vl.i)
```

```
        pl.i := (lcp quo d)*pl.i
        d := vl.i quo d
        unilist.i := d*unilist.i
        contm := contm quo d
    if contm ^=1 then for i in 1..nf repeat pl.i := contm*pl.i
    [pl,contm,unilist]$LeadFact

  distFact(contm:Z,unilist:List(BP),plead:FinalFact,
           vl:List(Z),lvar:List(OV),lval:List(Z)):Union(LeadFact,"failed") ==
    h:NonNegativeInteger
    c,d : Z
    lpol:List(P):=[]
    lexp:List(Integer):=[]
    nf:NNI := #unilist
    vl := reverse vl --lpol and vl reversed so test from right to left
    for fpl in plead.factors repeat
     lpol:=[fpl.irr,:lpol]
     lexp:=[fpl.pow,:lexp]
    vlp:List(Z):= [1$Z for i in 1..nf]
    aux : List(P) := [1$P for i in 1..nf]
    for i in 1..nf repeat
      c := contm*leadingCoefficient unilist.i
      c=1 or c=-1  => "next i"
      for k in 1..(# lpol) repeat
        lexp.k=0 => "next factor"
        h:= checkpow(vl.k,c)
        if h ^=0 then
         if h>lexp.k then return "failed"
         lexp.k:=lexp.k-h
         aux.i := aux.i*(lpol.k ** h)
         d:= vl.k**h
         vlp.i:= vlp.i*d
         c:= c quo d
      if contm=1 then vlp.i:=c
    for k in 1..(# lpol) repeat if lexp.k ^= 0 then return "failed"
    contm =1 => [[vlp.i*aux.i for i in 1..nf],1,unilist]$LeadFact
    distribute(contm,unilist,aux,vlp,lvar,lval)
```

⟨*LEADCDET.dotabb*⟩≡

```
"LEADCDET" [color="#FF4488",href="bookvol10.4.pdf#nameddest=LEADCDET"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"LEADCDET" -> "PFECAT"
```

13.7 package LEXTRIPK LexTriangularPackage

⟨*LexTriangularPackage.input*⟩≡

```
)set break resume
)spool LexTriangularPackage.output
)set message test on
)set message auto off
)clear all
--S 1 of 22
R := Integer
--R
--R
--R   (1)  Integer
--R                                                          Type: Domain
--E 1

--S 2 of 22
ls : List Symbol := [a,b,c,d,e,f]
--R
--R
--R   (2)  [a,b,c,d,e,f]
--R                                                   Type: List Symbol
--E 2

--S 3 of 22
V := OVAR(ls)
--R
--R
--R   (3)  OrderedVariableList [a,b,c,d,e,f]
--R                                                          Type: Domain
--E 3

--S 4 of 22
P := NSMP(R, V)
--R
--R
--R   (4)
--R   NewSparseMultivariatePolynomial(Integer,OrderedVariableList [a,b,c,d,e,f])
--R                                                          Type: Domain
--E 4

--S 5 of 22
p1: P :=  a*b*c*d*e*f - 1
--R
```

```
--R
--R   (5)  f e d c b a - 1
--RType: NewSparseMultivariatePolynomial(Integer,OrderedVariableList [a,b,c,d,e,f])
--E 5

--S 6 of 22
p2: P := a*b*c*d*e +a*b*c*d*f +a*b*c*e*f +a*b*d*e*f +a*c*d*e*f +b*c*d*e*f
--R
--R
--R   (6)  ((((e + f)d + f e)c + f e d)b + f e d c)a + f e d c b
--RType: NewSparseMultivariatePolynomial(Integer,OrderedVariableList [a,b,c,d,e,f])
--E 6

--S 7 of 22
p3: P :=  a*b*c*d + a*b*c*f + a*b*e*f + a*d*e*f + b*c*d*e + c*d*e*f
--R
--R
--R   (7)  (((d + f)c + f e)b + f e d)a + e d c b + f e d c
--RType: NewSparseMultivariatePolynomial(Integer,OrderedVariableList [a,b,c,d,e,f])
--E 7

--S 8 of 22
p4: P := a*b*c + a*b*f + a*e*f + b*c*d + c*d*e + d*e*f
--R
--R
--R   (8)  ((c + f)b + f e)a + d c b + e d c + f e d
--RType: NewSparseMultivariatePolynomial(Integer,OrderedVariableList [a,b,c,d,e,f])
--E 8

--S 9 of 22
p5: P := a*b + a*f + b*c + c*d + d*e + e*f
--R
--R
--R   (9)  (b + f)a + c b + d c + e d + f e
--RType: NewSparseMultivariatePolynomial(Integer,OrderedVariableList [a,b,c,d,e,f])
--E 9

--S 10 of 22
p6: P := a + b + c + d + e + f
--R
--R
--R   (10)  a + b + c + d + e + f
--RType: NewSparseMultivariatePolynomial(Integer,OrderedVariableList [a,b,c,d,e,f])
--E 10

--S 11 of 22
```

```
lp := [p1, p2, p3, p4, p5, p6]
--R
--R
--R   (11)
--R   [f e d c b a - 1, ((((e + f)d + f e)c + f e d)b + f e d c)a + f e d c b,
--R    (((d + f)c + f e)b + f e d)a + e d c b + f e d c,
--R    ((c + f)b + f e)a + d c b + e d c + f e d,
--R    (b + f)a + c b + d c + e d + f e, a + b + c + d + e + f]
--RType: List NewSparseMultivariatePolynomial(Integer,OrderedVariableList [a,b,c,
--E 11

--S 12 of 22
lextripack :=  LEXTRIPK(R,ls)
--R
--R
--R   (12)  LexTriangularPackage(Integer,[a,b,c,d,e,f])
--R                                                              Type: Domain
--E 12

--S 13 of 22
lg := groebner(lp)$lextripack
--R
--R
--R   (13)
--R   [a + b + c + d + e + f,
--R
--R                          2                                              2
--R       3968379498283200b  + 15873517993132800f b + 3968379498283200d
--R     +
--R                                                3 5                     4 4
--R       15873517993132800f d + 3968379498283200f e  - 15873517993132800f e
--R     +
--R                           5 3                          6                        2
--R       23810276989699200f e  + (206355733910726400f  + 230166010900425600)e
--R     +
--R                                43                         37
--R           - 7297059873166 87f   + 18636674968672054 21f
--R         +
--R                                   31                           25
--R           29167485377173110446 1f   + 36528599469110692174 5f
--R         +
--R                                19                            13
--R           549961185828911895f   - 365048404038768439269f
--R         +
--R                                   7
--R           - 2923828204315040276 69f  - 22718984676318654 97f
```

```
--R      *
--R         e
--R     +
--R                           44                          38
--R       - 3988812642545399f   + 10187423878429609997f
--R     +
--R                              32                            26
--R       1594377523424314053637f   + 1994739308439916238065f
--R     +
--R                             20                          14
--R       1596840088052642815f   - 1993494118301162145413f
--R     +
--R                                8                         2
--R       - 1596049742289689815053f  - 11488171330159667449f
--R     ,
--R                                                                  2
--R       (23810276989699200c - 2381027698969920f)b + 23810276989699200c
--R     +
--R                                                 2
--R       71430830969097600f c - 23810276989699200d  - 95241107958796800f d
--R     +
--R                             3 5                         4 4                       5 3
--R       - 55557312975964800f e  + 174608697924460800f e  - 174608697924460800f e
--R     +
--R                               6                          2
--R       (- 2428648252949318400f  - 2611193709870345600)e
--R     +
--R                              43                         37
--R           8305444561289527f   - 21212087151945459641f
--R         +
--R                                    31                            25
--R           - 3319815883093451385381f   - 41576916462616571364445f
--R         +
--R                                  19                             13
--R           - 6072721607510764095f   + 4154986709036460221649f
--R         +
--R                                   7
--R           3327761311138587096749f  + 25885340608290841637f
--R      *
--R         e
--R     +
--R                            44                          38
--R       45815897629010329f   - 1170137655821518912077f
--R     +
--R                                32                            26
```

```
--R       - 18313166848970865074187f   - 22909971239649297438915f
--R     +
--R                                20                              14
--R       - 16133250761305157265f   + 22897305857636178256623f
--R     +
--R                                 8                            2
--R       18329944781867242497923f  + 130258531002020420699f
--R     ,
--R
--R       (7936758996566400d - 7936758996566400f)b - 7936758996566400f d
--R     +
--R                            3 5                       4 4                       5 3
--R       - 7936758996566400f e  + 23810276989699200f e  - 23810276989699200f e
--R     +
--R                             6                         2
--R       (- 337312257354072000f  - 369059293340337600)e
--R     +
--R                            43                        37
--R           1176345388640471f   - 3004383582891473073f
--R         +
--R                                  31                          25
--R           - 470203502707246105653f   - 588858183402644348085f
--R         +
--R                                19                          13
--R           - 856939308623513535f   + 588472674242340526377f
--R         +
--R                                 7
--R           471313241958371103517f  + 3659742549078552381f
--R      *
--R         e
--R     +
--R                          44                          38
--R       6423170513956901f   - 164047721370364808O3f   - 256741916522752877446 3f
--R     +
--R                                26                          20
--R       - 3211938090825682172335f   - 2330490332697587485f
--R     +
--R                               14                           8
--R       3210100109444754864587f   + 2569858315395162617847f
--R     +
--R                             2
--R       183260894874277357 51f
--R     ,
--R
--R                                                                          3 5
--R       (11905138494849600e - 11905138494849600f)b - 3968379498283200f e
```

```
--R     +
--R                          4 4                       5 3
--R       15873517993132800f e  - 27778656487982400f e
--R     +
--R                             6                        2
--R       (- 208339923659868000f  - 240086959646133600)e
--R     +
--R                           43                       37
--R           786029984751110f   - 2007519008182245250f
--R         +
--R                                  31                          25
--R           - 3141880629080738070 90f   - 3934236675379295752 50f
--R         +
--R                                19                          13
--R           - 550329120654394950f   + 3931964087288896127 70f
--R         +
--R                                 7
--R           3148923727991764957 30f  + 2409386515146668530f
--R      *
--R         e
--R     +
--R                         44                          38                            32
--R       4177638546747827f   - 106696852946025763 81f   - 16698529804199495246 01f
--R     +
--R                                26                         20
--R       - 2089077057287904170745f   - 1569899763580278795f
--R     +
--R                              14                            8
--R       20878640268590155733 49f   + 16714960859451995779 69f
--R     +
--R                             2
--R       119402572262162801 77f
--R     ,
--R
--R                            6                                                2 5
--R       (11905138494849600f  - 11905138494849600)b - 15873517993132800f e
--R     +
--R                          3 4                       4 3
--R       39683794982832000f e  - 39683794982832000f e
--R     +
--R                              11                         5  2
--R       (- 686529653202993600f   - 607162063237329600f )e
--R     +
--R                            42                          36                          30
--R           65144531306704f   - 1663812809010886 52f   - 2603343450247028347 2f
--R         +
```

```
--R                                  24                      18
--R           - 31696259583860650140f   + 971492093167581360f
--R         +
--R                                12                      6
--R         32220085033691389548f   + 25526177666070529808f  + 138603268355749244
--R      *
--R         e
--R     +
--R                        43                      37                          31
--R       167620036074811f   - 428102417974791473f   - 669972438012316793l3f
--R     +
--R                               25                      19
--R       - 83426716722148750485f   + 203673895369980765f
--R     +
--R                             13                      7
--R       83523056326010432457f   + 669957896402380669 37f  + 478592855549587901f
--R     ,
--R
--R                    3                   2                 2                45
--R       801692827936c  + 2405078483808f c  - 2405078483808f c - 13752945467f
--R     +
--R                      39                    33                    27
--R       35125117815561f   + 5496946957826433f   + 6834659447749117f
--R     +
--R                        21                    15                    9
--R       - 44484880462461f   - 6873406230093057f   - 5450844938762633f
--R     +
--R                    3
--R       1216586044571f
--R     ,
--R
--R                                                                      2
--R       (23810276989699200d - 23810276989699200f)c + 23810276989699200d
--R     +
--R                                                3 5                    4 4
--R       71430830969097600f d + 7936758996566400f e  - 31747035986265600f e
--R     +
--R                          5 3                       6                       2
--R       31747035986265600f e  + (404774708824886400f  + 396837949828320000)e
--R     +
--R                               43                      37
--R           - 1247372229446701f   + 3185785654596621203f
--R         +
--R                                  31                       25
--R           4985948668499747514 63f   + 624542545845791047935f
--R         +
```

```
--R                              19                          13
--R           93108575576968288 5f   - 624150663582417063387f
--R         +
--R                                   7
--R           - 499881859388360475647f  - 39268853138195273 51f
--R      *
--R         e
--R     +
--R                           44                          38
--R       - 7026011547118141f   + 17944427051950691243f
--R     +
--R                               32                           26
--R       28083835225939866035 43f   + 3513624142354807530135f
--R     +
--R                             20                            14
--R       2860757006705537685f   - 35113567356421907372 67f
--R     +
--R                                 8                           2
--R       - 28113324946971038198 87f  - 203150116315228473 11f
--R     ,
--R
--R       (7936758996566400e - 7936758996566400f)c
--R     +
--R                             43                      37                          31
--R           - 4418748183673f   + 11285568707456559f   + 17659986172944510 19f
--R         +
--R                                25                       19
--R           2173749283622606155f   - 55788292195402895f
--R         +
--R                                13                         7
--R         - 2215291421788292951f   - 17181426653474308 51f  + 30256569458230237f
--R      *
--R         e
--R     +
--R                      44                       38                         32
--R       4418748183673f   - 11285568707456559f   - 17659986172944510 19f
--R     +
--R                              26                       20                          14
--R       - 2173749283622606155f   + 55788292195402895f   + 2215291421788292951f
--R     +
--R                             8                        2
--R       17181426653474308 51f  - 30256569458230237f
--R     ,
--R
--R                         6                                    43
--R       (72152354514240f  - 72152354514240)c + 40950859449f
```

```
--R     +
--R                          37                       31                       25
--R       - 104588980990367f   - 16367227395575307f   - 20268523416527355f
--R     +
--R                        19                      13                      7
--R       442205002259535f   + 20576059935789063f   + 15997133796970563f
--R     +
--R       - 275099892785581f
--R     ,
--R
--R                         3                       2                     2
--R       1984189749141600d  + 5952569247424800f d  - 5952569247424800f d
--R     +
--R                           4 5                      5 4                      3
--R       - 3968379498283200f e  + 15873517993132800f e  + 17857707742274400e
--R     +
--R                              7                        2
--R       (- 148814231185620000f  - 162703559429611200f)e
--R     +
--R                              44                      38
--R           - 390000914678878f   + 996062704593756434f
--R         +
--R                                 32                       26
--R           155886323972034823914f   + 194745956143985421330f
--R         +
--R                               20                         14
--R           6205077595574430f   - 1945965126532990687 86f
--R         +
--R                                  8                       2
--R           - 1557968979407569226 66f  - 10363757590773209 78f
--R      *
--R         e
--R     +
--R                           45                       39                          33
--R       - 3749986300359 91f   + 957747106595453993f   + 14988915556676489 1693f
--R     +
--R                              27                      21
--R       1871541714434946416 85f   - 127129015426348065f
--R     +
--R                              15                         9
--R     - 1872415332431150404 17f   - 14971998356797653403 7f  - 836654081239648061
--R     ,
--R
--R                                                                    3 5
--R       (5952569247424800e - 5952569247424800f)d - 3968379498283200f e
--R     +
```

```
--R                          4 4                     5 3
--R       9920948745708000f e  - 3968379498283200f e
--R     +
--R                             6                      2
--R       (- 148814231185620000f  - 150798420934761600)e
--R     +
--R                           43                      37
--R           492558110242553f   - 1257992359608074599f
--R         +
--R                                  31                        25
--R           - 196883094539368513959f   - 246562115745735428055f
--R         +
--R                                19                         13
--R           - 325698701993885505f   + 2464177698836518081l1f
--R         +
--R                                 7
--R           197327352068200652911f  + 1523373796389332143f
--R      *
--R         e
--R     +
--R                          44                       38                           32
--R       2679481081803026f   - 6843392695421906608f   - 1071020459642646913578f
--R     +
--R                                26                      20
--R       - 13397891696920412400l0f   - 852746750910750210f
--R     +
--R                               14                           8
--R       1339105101971878401312f   + 10719002897587129847l62f
--R     +
--R                             2
--R       7555239072072727756f
--R     ,
--R
--R                            6                                          2 5
--R       (11905138494849600f  - 11905138494849600)d - 7936758996566400f e
--R     +
--R                           3 4                     4 3
--R       31747035986265600f e  - 31747035986265600f e
--R     +
--R                              11                       5  2
--R       (- 420648226818019200f   - 404774708824886400f )e
--R     +
--R                            42                      36                         30
--R           15336187600889f   - 39169739565161107f   - 61271761274896908l27f
--R         +
--R                                   24                      18
```

```
--R           - 7217708742310509615f   + 538628483890722735f
--R         +
--R                                12                      6
--R           7506804353843507643f   + 5886160769782607203f  + 63576108396535879
--R      *
--R         e
--R     +
--R                       43                     37                        31
--R       71737781777066f   - 183218856207557938f   - 286728742711322760 78f
--R     +
--R                                25                     19
--R       - 356252236869398120 10f   + 164831339634084390f
--R     +
--R                               13                     7
--R       35724160423073052642f   + 28627022578664910622f  + 187459987029680506f
--R     ,
--R
--R                          6                     5                   2 4
--R       1322793166094400e  - 3968379498283200f e  + 3968379498283200f e
--R     +
--R                           3 3
--R       - 5291172664377600f e
--R     +
--R                              10                     4  2
--R       (- 230166010900425600f   - 226197631402142400f )e
--R     +
--R                                 47                        41
--R           - 152375364610443885f   + 3891666260648548 90415f
--R         +
--R                                    35                       29
--R           60906097841360558987335f   + 7616736793460879869727 5f
--R         +
--R                                 23                          17
--R           27855066785995181125f   - 7614495281705272314549 5f
--R         +
--R                                   11                        5
--R           - 60933629892463517546975f   - 4114150716820025477 95f
--R      *
--R         e
--R     +
--R                           42                      36                        30
--R       - 209493533143822f   + 535045979490560586f   + 8373794796497355314 6f
--R     +
--R                              24                     18
--R       1048895070842133715 70f   + 167117997269207870f
--R     +
```

```
--R                                12                          6
--R       - 104793725781390615514f   - 83842685189903180394f  - 569978796672974242
--R     ,
--R
--R                        6                   3
--R       (25438330117200f  + 25438330117200)e
--R     +
--R                        7                    2
--R       (76314990351600f  + 76314990351600f)e
--R     +
--R                             44                     38                         32
--R           - 1594966552735f   + 4073543370415745f   + 637527159231148925f
--R         +
--R                               26                    20                          14
--R           797521176113606525f   + 530440941097175f   - 797160527306433145f
--R         +
--R                                 8                    2
--R           - 638132320196044965f  - 4510507167940725f
--R      *
--R         e
--R     +
--R                          45                      39                          33
--R       - 6036376800443f   + 154169034214769O9f   + 2412807646192304449f
--R     +
--R                             27                     21                          15
--R       30176799230280137O5f   + 1422320037411955f   - 30165604024178439 41f
--R     +
--R                               9                     3
--R       - 2414249368183033161f  - 16561862361763873f
--R     ,
--R
--R                        12                  2
--R       (1387545279120f   - 1387545279120)e
--R     +
--R                        43                  37                    31
--R           4321823003f   - 11037922310209f   - 1727510711947989f
--R         +
--R                               25                  19                    13
--R           - 2165150991154425f   - 5114342560755f   + 2162682824948601f
--R         +
--R                              7
--R           1732620732685741f  + 13506088516033f
--R      *
--R         e
--R     +
--R                    44                  38                    32
```

```
--R       24177661775f   - 61749727185325f   - 9664106795754225f
--R     +
--R                           26                 20                       14
--R       - 12090487758628245f   - 8787672733575f   + 12083693383005045f
--R     +
--R                          8                  2
--R       9672870290826025f  + 68544102808525f
--R     ,
--R     48        42          36          30          18          12        6
--R    f   - 2554f   - 399710f   - 499722f   + 499722f   + 399710f   + 2554f  - 1
--RType: List NewSparseMultivariatePolynomial(Integer,OrderedVariableList [a,b,c,
--E 13

--S 14 of 22
lexTriangular(lg,false)$lextripack
--R
--R
--R   (14)
--R   [
--R       6       6       5     2 4     3 3     4 2     5
--R     {f  + 1, e  - 3f e  + 3f e  - 4f e  + 3f e  - 3f e - 1,
--R            2 5     3 4     4 3     5 2
--R      3d + f e  - 4f e  + 4f e  - 2f e  - 2e + 2f, c + f,
--R             2 5     3 4     4 3      5 2
--R      3b + 2f e  - 5f e  + 5f e  - 10f e  - 4e + 7f,
--R           2 5     3 4     4 3     5 2
--R      a - f e  + 3f e  - 3f e  + 4f e  + 3e - 3f}
--R     ,
--R      6                  2          2                   2
--R    {f  - 1,e - f,d - f,c  + 4f c + f ,(c - f)b - f c - 5f ,a + b + c + 3f},
--R      6                         2          2
--R    {f  - 1,e - f,d - f,c - f,b  + 4f b + f ,a + b + 4f},
--R      6            2          2                   2
--R    {f  - 1,e - f,d  + 4f d + f ,(d - f)c - f d - 5f ,b - f,a + c + d + 3f},
--R
--R       36        30          24          18          12        6
--R     {f   - 2554f   - 399709f   - 502276f   - 399709f   - 2554f  + 1,
--R
--R                    12             2
--R         (161718564f   - 161718564)e
--R       +
--R                       31              25                19                13
--R             - 504205f   + 1287737951f   + 201539391380f   + 253982817368f
--R           +
--R                            7
--R             201940704665f  + 1574134601f
```

```
--R        *
--R           e
--R       +
--R                    32              26                  20                  14
--R         - 2818405f   + 7198203911f   + 1126548149060f   + 1416530563364f
--R       +
--R                     8              2
--R         1127377589345f  + 7988820725f
--R       ,
--R
--R                     6                                  2 5                 3 4
--R         (693772639560f  - 693772639560)d - 462515093040f e  + 1850060372160f e
--R       +
--R                       4 3                      11                  5  2
--R         - 1850060372160f e  + (- 24513299931120f   - 23588269745040f )e
--R       +
--R                      30                 24                  18
--R             - 890810428f   + 2275181044754f   + 355937263869776f
--R           +
--R                         12                   6
--R             413736880104344f   + 342849304487996f  + 3704966481878
--R        *
--R           e
--R       +
--R                     31                  25                    19
--R         - 4163798003f   + 10634395752169f   + 1664161760192806f
--R       +
--R                       13                    7
--R         2079424391370694f   + 1668153650635921f  + 10924274392693f
--R       ,
--R
--R                     6                           31               25
--R         (12614047992f  - 12614047992)c - 7246825f   + 18508536599f
--R       +
--R                     19                  13                 7
--R         2896249516034f   + 3581539649666f   + 2796477571739f  - 48094301893f
--R       ,
--R
--R                     6                                  2 5                 3 4
--R         (693772639560f  - 693772639560)b - 925030186080f e  + 2312575465200f e
--R       +
--R                       4 3                      11                  5  2
--R         - 2312575465200f e  + (- 40007555547960f   - 35382404617560f )e
--R       +
--R                      30                 24                    18
--R             - 3781280823f   + 9657492291789f   + 1511158913397906f
```

```
--R           +
--R                                12                    6
--R             1837290892286154f   + 1487216006594361f  + 8077238712093
--R        *
--R           e
--R       +
--R                       31                  25                     19
--R         - 9736390478f   + 24866827916734f   + 3891495681905296f
--R       +
--R                            13                    7
--R         4872556418871424f   + 3904047887269606f  + 27890075838538f
--R       ,
--R      a + b + c + d + e + f}
--R     ,
--R      6      2           2                   2
--R    {f  - 1,e  + 4f e + f ,(e - f)d - f e - 5f ,c - f,b - f,a + d + e + 3f}]
--R                                Type: List RegularChain(Integer,[a,b,c,d,e,f])
--E 14

--S 15 of 22
lts := lexTriangular(lg,true)$lextripack
--R
--R
--R   (15)
--R   [
--R       6       6       5     2 4     3 3     4 2     5
--R     {f  + 1, e  - 3f e  + 3f e  - 4f e  + 3f e  - 3f e - 1,
--R            2 5     3 4     4 3     5 2
--R      3d + f e  - 4f e  + 4f e  - 2f e  - 2e + 2f, c + f,
--R             2 5     3 4     4 3      5 2
--R      3b + 2f e  - 5f e  + 5f e  - 10f e  - 4e + 7f,
--R           2 5     3 4     4 3     5 2
--R      a - f e  + 3f e  - 3f e  + 4f e  + 3e - 3f}
--R     ,
--R      6                  2           2
--R    {f  - 1,e - f,d - f,c  + 4f c + f ,b + c + 4f,a - f},
--R      6                        2           2
--R    {f  - 1,e - f,d - f,c - f,b  + 4f b + f ,a + b + 4f},
--R      6            2           2
--R    {f  - 1,e - f,d  + 4f d + f ,c + d + 4f,b - f,a - f},
--R
--R       36        30          24          18          12        6
--R     {f   - 2554f   - 399709f   - 502276f   - 399709f   - 2554f  + 1,
--R
--R                       2
--R         1387545279120e
```

```
--R       +
--R                         31                   25                     19
--R              4321823003f    - 11037922310209f    - 1727506390124986f
--R           +
--R                                13                    7
--R              - 2176188913464634f    - 1732620732685741f  - 13506088516033f
--R        *
--R           e
--R       +
--R                      32                   26                     20
--R         24177661775f    - 61749727185325f    - 9664082618092450f
--R       +
--R                              14                    8                   2
--R         - 12152237485813570f    - 9672870290826025f  - 68544102808525f
--R       ,
--R
--R         1387545279120d
--R       +
--R                           30                  24                    18
--R              - 1128983050f    + 2883434331830f    + 451234998755840f
--R           +
--R                              12                    6
--R              562426491685760f    + 447129055314890f  - 165557857270
--R        *
--R           e
--R       +
--R                        31                  25                   19
--R         - 1816935351f    + 4640452214013f    + 726247129626942f
--R       +
--R                          13                   7
--R         912871801716798f    + 726583262666877f  + 4909358645961f
--R       ,
--R
--R                                      31                  25                    19
--R         1387545279120c + 778171189f    - 1987468196267f    - 310993556954378f
--R       +
--R                            13                    7
--R         - 383262822316802f    - 300335488637543f  + 5289595037041f
--R       ,
--R
--R         1387545279120b
--R       +
--R                           30                  24                    18
--R              1128983050f    - 2883434331830f    - 451234998755840f
--R           +
--R                                 12                    6
```

```
--R             - 562426491685760f   - 447129055314890f  + 165557857270
--R        *
--R           e
--R       +
--R                      31                 25                     19
--R         - 3283058841f   + 8384938292463f   + 1312252817452422f
--R       +
--R                           13                    7
--R         1646579934064638f   + 1306372958656407f  + 4694680112151f
--R       ,
--R                                                          31                  25
--R         1387545279120a + 1387545279120e + 4321823003f   - 11037922310209f
--R       +
--R                               19                    13                    7
--R         - 1727506390124986f   - 2176188913464634f   - 1732620732685741f
--R       +
--R         - 13506088516033f
--R       }
--R     ,
--R      6      2           2
--R    {f  - 1,e  + 4f e + f ,d + e + 4f,c - f,b - f,a - f}]
--R                                Type: List RegularChain(Integer,[a,b,c,d,e,f])
--E 15

--S 16 of 22
[ [init(p) for p in (ts :: List(P))] for ts in lts]
--R
--R
--R   (16)
--R   [[1,3,1,3,1,1], [1,1,1,1,1,1], [1,1,1,1,1,1], [1,1,1,1,1,1],
--R    [1387545279120,1387545279120,1387545279120,1387545279120,1387545279120,1],
--R    [1,1,1,1,1,1]]
--RType: List List NewSparseMultivariatePolynomial(Integer,OrderedVariableList [a
--E 16

--S 17 of 22
squareFreeLexTriangular(lg,true)$lextripack
--R
--R
--R   (17)
--R   [
--R       6       6       5     2 4     3 3     4 2     5
--R     {f  + 1, e  - 3f e  + 3f e  - 4f e  + 3f e  - 3f e - 1,
--R            2 5     3 4     4 3     5 2
--R      3d + f e  - 4f e  + 4f e  - 2f e  - 2e + 2f, c + f,
```

```
--R             2 5     3 4     4 3      5 2
--R      3b + 2f e  - 5f e  + 5f e  - 10f e  - 4e + 7f,
--R           2 5     3 4     4 3     5 2
--R      a - f e  + 3f e  - 3f e  + 4f e  + 3e - 3f}
--R     ,
--R      6                  2           2
--R    {f  - 1,e - f,d - f,c  + 4f c + f ,b + c + 4f,a - f},
--R      6                        2           2
--R    {f  - 1,e - f,d - f,c - f,b  + 4f b + f ,a + b + 4f},
--R      6            2           2
--R    {f  - 1,e - f,d  + 4f d + f ,c + d + 4f,b - f,a - f},
--R
--R       36        30          24          18          12        6
--R     {f   - 2554f   - 399709f   - 502276f   - 399709f   - 2554f  + 1,
--R
--R                        2
--R         1387545279120e
--R       +
--R                         31                  25                     19
--R             4321823003f   - 11037922310209f   - 1727506390124986f
--R           +
--R                                13                    7
--R             - 2176188913464634f   - 1732620732685741f  - 13506088516033f
--R        *
--R           e
--R       +
--R                      32                  26                    20
--R         24177661775f   - 61749727185325f   - 9664082618092450f
--R       +
--R                              14                    8                  2
--R         - 12152237485813570f   - 9672870290826025f  - 68544102808525f
--R       ,
--R
--R         1387545279120d
--R       +
--R                           30                 24                    18
--R             - 1128983050f   + 2883434331830f   + 451234998755840f
--R           +
--R                              12                   6
--R             562426491685760f   + 447129055314890f  - 165557857270
--R        *
--R           e
--R       +
--R                         31                 25                   19
--R         - 1816935351f   + 4640452214013f   + 726247129626942f
--R       +
```

```
--R                          13                   7
--R         912871801716798f   + 726583262666877f  + 4909358645961f
--R       ,
--R
--R                                    31                 25                   19
--R         1387545279120c + 778171189f   - 1987468196267f   - 310993556954378f
--R       +
--R                           13                   7
--R         - 383262822316802f   - 300335488637543f  + 5289595037041f
--R       ,
--R
--R         1387545279120b
--R       +
--R                       30                 24                    18
--R             1128983050f   - 2883434331830f   - 451234998755840f
--R           +
--R                                12                   6
--R             - 562426491685760f   - 447129055314890f  + 165557857270
--R        *
--R           e
--R       +
--R                        31                 25                    19
--R         - 3283058841f   + 8384938292463f   + 1312252817452422f
--R       +
--R                            13                    7
--R         1646579934064638f   + 1306372958656407f  + 4694680112151f
--R       ,
--R
--R                                                          31                  25
--R         1387545279120a + 1387545279120e + 4321823003f   - 11037922310209f
--R       +
--R                               19                     13                    7
--R         - 1727506390124986f   - 2176188913464634f   - 1732620732685741f
--R       +
--R         - 13506088516033f
--R       }
--R     ,
--R      6      2           2
--R    {f  - 1,e  + 4f e + f ,d + e + 4f,c - f,b - f,a - f}]
--RType: List SquareFreeRegularTriangularSet(Integer,IndexedExponents OrderedVari
--E 17

--S 18 of 22
reduce(+,[degree(ts) for ts in lts])
--R
--R
```

```
--R   (18)  156
--R                                                        Type: PositiveInteger
--E 18

--S 19 of 22
ls2 : List Symbol := concat(ls,new()$Symbol)
--R
--R
--R   (19)  [a,b,c,d,e,f,%A]
--R                                                           Type: List Symbol
--E 19

--S 20 of 22
zdpack := ZDSOLVE(R,ls,ls2)
--R
--R
--R   (20)  ZeroDimensionalSolvePackage(Integer,[a,b,c,d,e,f],[a,b,c,d,e,f,%A])
--R                                                                 Type: Domain
--E 20

--S 21 of 22
concat [univariateSolve(ts)$zdpack for ts in lts]
--R
--R
--R   (21)
--R   [
--R                      4      2
--R     [complexRoots= ?  - 13?  + 49,
--R
--R       coordinates =
--R                  3                3                3                 3
--R         [7a + %A  - 6%A, 21b + %A  + %A, 21c - 2%A  + 19%A, 7d - %A  + 6%A,
--R                   3                3
--R          21e - %A  - %A, 21f + 2%A  - 19%A]
--R       ]
--R     ,
--R
--R                      4      2
--R     [complexRoots= ?  + 11?  + 49,
--R
--R       coordinates =
--R                    3                3                  3
--R         [35a + 3%A  + 19%A, 35b + %A  + 18%A, 35c - 2%A  - %A,
--R                    3                3                  3
--R          35d - 3%A  - 19%A, 35e - %A  - 18%A, 35f + 2%A  + %A]
--R       ]
```

```
--R     ,
--R
--R     [
--R       complexRoots =
--R          8      7      6       5       4       3       2
--R         ?  - 12?  + 58?  - 120?  + 207?  - 360?  + 802?  - 1332? + 1369
--R       ,
--R
--R       coordinates =
--R         [
--R                                  7           6            5            4
--R             43054532a + 33782%A  - 546673%A  + 3127348%A  - 6927123%A
--R           +
--R                       3             2
--R             4365212%A  - 25086957%A  + 39582814%A - 107313172
--R           ,
--R
--R                                  7           6            5            4
--R             43054532b - 33782%A  + 546673%A  - 3127348%A  + 6927123%A
--R           +
--R                       3             2
--R             - 4365212%A  + 25086957%A  - 39582814%A + 107313172
--R           ,
--R
--R                                  7           6            5            4
--R             21527266c - 22306%A  + 263139%A  - 1166076%A  + 1821805%A
--R           +
--R                       3             2
--R             - 2892788%A  + 10322663%A  - 9026596%A + 12950740
--R           ,
--R
--R                                  7           6            5            4
--R             43054532d + 22306%A  - 263139%A  + 1166076%A  - 1821805%A
--R           +
--R                       3             2
--R             2892788%A  - 10322663%A  + 30553862%A - 12950740
--R           ,
--R
--R                                  7           6            5            4
--R             43054532e - 22306%A  + 263139%A  - 1166076%A  + 1821805%A
--R           +
--R                       3             2
--R             - 2892788%A  + 10322663%A  - 30553862%A + 12950740
--R           ,
--R
--R                                  7           6            5            4
```

```
--R             21527266f + 22306%A  - 263139%A  + 1166076%A  - 1821805%A
--R           +
--R                       3             2
--R             2892788%A  - 10322663%A  + 9026596%A - 12950740
--R           ]
--R       ]
--R     ,
--R
--R     [
--R       complexRoots =
--R          8      7      6       5       4       3       2
--R         ?  + 12?  + 58?  + 120?  + 207?  + 360?  + 802?  + 1332? + 1369
--R       ,
--R
--R       coordinates =
--R         [
--R                                7           6           5           4
--R             43054532a + 33782%A  + 546673%A  + 3127348%A  + 6927123%A
--R           +
--R                       3             2
--R             4365212%A  + 25086957%A  + 39582814%A + 107313172
--R           ,
--R
--R                                7           6           5           4
--R             43054532b - 33782%A  - 546673%A  - 3127348%A  - 6927123%A
--R           +
--R                       3             2
--R             - 4365212%A  - 25086957%A  - 39582814%A - 107313172
--R           ,
--R
--R                                7           6           5           4
--R             21527266c - 22306%A  - 263139%A  - 1166076%A  - 1821805%A
--R           +
--R                       3             2
--R             - 2892788%A  - 10322663%A  - 9026596%A - 12950740
--R           ,
--R
--R                                7           6           5           4
--R             43054532d + 22306%A  + 263139%A  + 1166076%A  + 1821805%A
--R           +
--R                       3             2
--R             2892788%A  + 10322663%A  + 30553862%A + 12950740
--R           ,
--R
--R                                7           6           5           4
--R             43054532e - 22306%A  - 263139%A  - 1166076%A  - 1821805%A
```

```
--R           +
--R                        3             2
--R             - 2892788%A  - 10322663%A  - 30553862%A - 12950740
--R           ,
--R
--R                                  7           6            5            4
--R             21527266f + 22306%A  + 263139%A  + 1166076%A  + 1821805%A
--R           +
--R                      3             2
--R             2892788%A  + 10322663%A  + 9026596%A + 12950740
--R           ]
--R       ]
--R     ,
--R
--R                     4    2
--R     [complexRoots= ?  - ?  + 1,
--R                                  3            3             3            3
--R      coordinates= [a - %A,b + %A  - %A,c + %A ,d + %A,e - %A  + %A,f - %A ]]
--R     ,
--R
--R                     8     6      4      2
--R     [complexRoots= ?  + 4?  + 12?  + 16?  + 4,
--R
--R       coordinates =
--R                   7      5       3                  7      5       3
--R         [4a - 2%A  - 7%A  - 20%A  - 22%A, 4b + 2%A  + 7%A  + 20%A  + 22%A,
--R                  7      5       3                 7      5       3
--R          4c + %A  + 3%A  + 10%A  + 10%A, 4d + %A  + 3%A  + 10%A  + 6%A,
--R                 7      5       3                7      5       3
--R          4e - %A  - 3%A  - 10%A  - 6%A, 4f - %A  - 3%A  - 10%A  - 10%A]
--R       ]
--R     ,
--R
--R                     4     3      2
--R     [complexRoots= ?  + 6?  + 30?  + 36? + 36,
--R
--R       coordinates =
--R                   3      2                     3      2
--R         [30a - %A  - 5%A  - 30%A - 6, 6b + %A  + 5%A  + 24%A + 6,
--R                  3      2              3      2
--R          30c - %A  - 5%A  - 6, 30d - %A  - 5%A  - 30%A - 6,
--R                  3      2                     3      2
--R          30e - %A  - 5%A  - 30%A - 6, 30f - %A  - 5%A  - 30%A - 6]
--R       ]
--R     ,
--R
```

```
--R                     4     3      2
--R     [complexRoots= ?  - 6?  + 30?  - 36? + 36,
--R
--R       coordinates =
--R                  3      2                    3      2
--R         [30a - %A  + 5%A  - 30%A + 6, 6b + %A  - 5%A  + 24%A - 6,
--R                  3      2              3      2
--R          30c - %A  + 5%A  + 6, 30d - %A  + 5%A  - 30%A + 6,
--R                  3      2                     3      2
--R          30e - %A  + 5%A  - 30%A + 6, 30f - %A  + 5%A  - 30%A + 6]
--R       ]
--R     ,
--R
--R                     2
--R     [complexRoots= ?  + 6? + 6,
--R      coordinates= [a + 1,b - %A - 5,c + %A + 1,d + 1,e + 1,f + 1]]
--R     ,
--R
--R                     2
--R     [complexRoots= ?  - 6? + 6,
--R      coordinates= [a - 1,b - %A + 5,c + %A - 1,d - 1,e - 1,f - 1]]
--R     ,
--R
--R                     4     3      2
--R     [complexRoots= ?  + 6?  + 30?  + 36? + 36,
--R
--R       coordinates =
--R                 3      2                     3      2
--R         [6a + %A  + 5%A  + 24%A + 6, 30b - %A  - 5%A  - 6,
--R                  3      2                     3      2
--R          30c - %A  - 5%A  - 30%A - 6, 30d - %A  - 5%A  - 30%A - 6,
--R                  3      2                     3      2
--R          30e - %A  - 5%A  - 30%A - 6, 30f - %A  - 5%A  - 30%A - 6]
--R       ]
--R     ,
--R
--R                     4     3      2
--R     [complexRoots= ?  - 6?  + 30?  - 36? + 36,
--R
--R       coordinates =
--R                  3      2                    3      2
--R         [6a + %A  - 5%A  + 24%A - 6, 30b - %A  + 5%A  + 6,
--R                  3      2                     3      2
--R          30c - %A  + 5%A  - 30%A + 6, 30d - %A  + 5%A  - 30%A + 6,
--R                  3      2                     3      2
--R          30e - %A  + 5%A  - 30%A + 6, 30f - %A  + 5%A  - 30%A + 6]
```

```
--R       ]
--R     ,
--R
--R                     2
--R     [complexRoots= ?  + 6? + 6,
--R      coordinates= [a - %A - 5,b + %A + 1,c + 1,d + 1,e + 1,f + 1]]
--R     ,
--R
--R                     2
--R     [complexRoots= ?  - 6? + 6,
--R      coordinates= [a - %A + 5,b + %A - 1,c - 1,d - 1,e - 1,f - 1]]
--R     ,
--R
--R                     4     3      2
--R     [complexRoots= ?  + 6?  + 30?  + 36? + 36,
--R
--R       coordinates =
--R                   3      2                       3      2
--R         [30a - %A  - 5%A  - 30%A - 6, 30b - %A  - 5%A  - 30%A - 6,
--R                   3      2                       3      2
--R          6c + %A  + 5%A  + 24%A + 6, 30d - %A  - 5%A  - 6,
--R                   3      2                       3      2
--R          30e - %A  - 5%A  - 30%A - 6, 30f - %A  - 5%A  - 30%A - 6]
--R       ]
--R     ,
--R
--R                     4     3      2
--R     [complexRoots= ?  - 6?  + 30?  - 36? + 36,
--R
--R       coordinates =
--R                   3      2                       3      2
--R         [30a - %A  + 5%A  - 30%A + 6, 30b - %A  + 5%A  - 30%A + 6,
--R                   3      2                       3      2
--R          6c + %A  - 5%A  + 24%A - 6, 30d - %A  + 5%A  + 6,
--R                   3      2                       3      2
--R          30e - %A  + 5%A  - 30%A + 6, 30f - %A  + 5%A  - 30%A + 6]
--R       ]
--R     ,
--R
--R                     2
--R     [complexRoots= ?  + 6? + 6,
--R      coordinates= [a + 1,b + 1,c - %A - 5,d + %A + 1,e + 1,f + 1]]
--R     ,
--R
--R                     2
--R     [complexRoots= ?  - 6? + 6,
```

```
--R      coordinates= [a - 1,b - 1,c - %A + 5,d + %A - 1,e - 1,f - 1]]
--R     ,
--R
--R                     8     7      6      5      4     2
--R     [complexRoots= ?  + 6?  + 16?  + 24?  + 18?  - 8?  + 4,
--R
--R       coordinates =
--R                  7      6       5       4      3       2
--R         [2a + 2%A  + 9%A  + 18%A  + 19%A  + 4%A  - 10%A  - 2%A + 4,
--R                  7      6       5       4      3       2
--R          2b + 2%A  + 9%A  + 18%A  + 19%A  + 4%A  - 10%A  - 4%A + 4,
--R                  7      6      5      4      3
--R          2c - %A  - 4%A  - 8%A  - 9%A  - 4%A  - 2%A - 4,
--R                 7      6      5      4      3
--R          2d + %A  + 4%A  + 8%A  + 9%A  + 4%A  + 2%A + 4,
--R                  7      6       5       4      3       2
--R          2e - 2%A  - 9%A  - 18%A  - 19%A  - 4%A  + 10%A  + 4%A - 4,
--R                  7      6       5       4      3       2
--R          2f - 2%A  - 9%A  - 18%A  - 19%A  - 4%A  + 10%A  + 2%A - 4]
--R       ]
--R     ,
--R
--R     [
--R       complexRoots =
--R          8      7      6       5       4       3        2
--R         ?  + 12?  + 64?  + 192?  + 432?  + 768?  + 1024?  + 768? + 256
--R       ,
--R
--R       coordinates =
--R         [
--R                          7         6         5         4         3         2
--R             1408a - 19%A  - 200%A  - 912%A  - 2216%A  - 4544%A  - 6784%A
--R           +
--R             - 6976%A - 1792
--R           ,
--R
--R                          7         6         5         4          3          2
--R             1408b - 37%A  - 408%A  - 1952%A  - 5024%A  - 10368%A  - 16768%A
--R           +
--R             - 17920%A - 5120
--R           ,
--R
--R                          7         6         5         4          3          2
--R             1408c + 37%A  + 408%A  + 1952%A  + 5024%A  + 10368%A  + 16768%A
--R           +
--R             17920%A + 5120
```

```
--R           ,
--R
--R                           7        6        5         4         3         2
--R             1408d + 19%A  + 200%A  + 912%A  + 2216%A  + 4544%A  + 6784%A
--R           +
--R             6976%A + 1792
--R           ,
--R          2e + %A, 2f - %A]
--R       ]
--R     ,
--R
--R                      8     6      4      2
--R     [complexRoots= ?  + 4?  + 12?  + 16?  + 4,
--R
--R       coordinates =
--R                  7      5       3               7      5       3
--R         [4a - %A  - 3%A  - 10%A  - 6%A, 4b - %A  - 3%A  - 10%A  - 10%A,
--R                   7      5       3                  7      5       3
--R          4c - 2%A  - 7%A  - 20%A  - 22%A, 4d + 2%A  + 7%A  + 20%A  + 22%A,
--R                  7      5       3               7      5       3
--R          4e + %A  + 3%A  + 10%A  + 10%A, 4f + %A  + 3%A  + 10%A  + 6%A]
--R       ]
--R     ,
--R
--R                      8      6      4       2
--R     [complexRoots= ?  + 16?  - 96?  + 256?  + 256,
--R
--R       coordinates =
--R                  7       5        3
--R         [512a - %A  - 12%A  + 176%A  - 448%A,
--R                   7       5       3
--R          128b - %A  - 16%A  + 96%A  - 256%A,
--R                   7       5       3
--R          128c + %A  + 16%A  - 96%A  + 256%A,
--R                  7       5        3
--R          512d + %A  + 12%A  - 176%A  + 448%A, 2e + %A, 2f - %A]
--R       ]
--R     ,
--R
--R     [
--R       complexRoots =
--R          8      7      6       5       4       3        2
--R         ?  - 12?  + 64?  - 192?  + 432?  - 768?  + 1024?  - 768? + 256
--R       ,
--R
--R       coordinates =
```

```
--R         [
--R                            7        6        5         4         3         2
--R              1408a - 19%A  + 200%A  - 912%A  + 2216%A  - 4544%A  + 6784%A
--R            +
--R              - 6976%A + 1792
--R            ,
--R
--R                            7        6         5         4          3          2
--R              1408b - 37%A  + 408%A  - 1952%A  + 5024%A  - 10368%A  + 16768%A
--R            +
--R              - 17920%A + 5120
--R            ,
--R
--R                            7        6         5         4          3          2
--R              1408c + 37%A  - 408%A  + 1952%A  - 5024%A  + 10368%A  - 16768%A
--R            +
--R              17920%A - 5120
--R            ,
--R
--R                            7        6        5         4         3         2
--R              1408d + 19%A  - 200%A  + 912%A  - 2216%A  + 4544%A  - 6784%A
--R            +
--R              6976%A - 1792
--R            ,
--R           2e + %A, 2f - %A]
--R       ]
--R     ,
--R
--R                      8     7      6      5      4     2
--R     [complexRoots= ?  - 6?  + 16?  - 24?  + 18?  - 8?  + 4,
--R
--R       coordinates =
--R                   7      6       5       4      3       2
--R          [2a + 2%A  - 9%A  + 18%A  - 19%A  + 4%A  + 10%A  - 2%A - 4,
--R                    7      6       5       4      3       2
--R           2b + 2%A  - 9%A  + 18%A  - 19%A  + 4%A  + 10%A  - 4%A - 4,
--R                  7      6      5      4      3
--R           2c - %A  + 4%A  - 8%A  + 9%A  - 4%A  - 2%A + 4,
--R                  7      6      5      4      3
--R           2d + %A  - 4%A  + 8%A  - 9%A  + 4%A  + 2%A - 4,
--R                    7      6       5       4      3       2
--R           2e - 2%A  + 9%A  - 18%A  + 19%A  - 4%A  - 10%A  + 4%A + 4,
--R                  7      6       5       4      3       2
--R           2f - 2%A  + 9%A  - 18%A  + 19%A  - 4%A  - 10%A  + 2%A + 4]
--R       ]
--R     ,
```

```
--R
--R                    4      2
--R     [complexRoots= ?  + 12?  + 144,
--R
--R       coordinates =
--R                  2               2               2               2
--R         [12a - %A  - 12, 12b - %A  - 12, 12c - %A  - 12, 12d - %A  - 12,
--R                  2                    2
--R          6e + %A  + 3%A + 12, 6f + %A  - 3%A + 12]
--R       ]
--R     ,
--R
--R                    4     3      2
--R     [complexRoots= ?  + 6?  + 30?  + 36? + 36,
--R
--R       coordinates =
--R                  3      2                      3      2
--R         [6a - %A  - 5%A  - 24%A - 6, 30b + %A  + 5%A  + 30%A + 6,
--R                  3      2                      3      2
--R          30c + %A  + 5%A  + 30%A + 6, 30d + %A  + 5%A  + 30%A + 6,
--R                  3      2                      3      2
--R          30e + %A  + 5%A  + 30%A + 6, 30f + %A  + 5%A  + 6]
--R       ]
--R     ,
--R
--R                    4     3      2
--R     [complexRoots= ?  - 6?  + 30?  - 36? + 36,
--R
--R       coordinates =
--R                  3      2                      3      2
--R         [6a - %A  + 5%A  - 24%A + 6, 30b + %A  - 5%A  + 30%A - 6,
--R                  3      2                      3      2
--R          30c + %A  - 5%A  + 30%A - 6, 30d + %A  - 5%A  + 30%A - 6,
--R                  3      2                      3      2
--R          30e + %A  - 5%A  + 30%A - 6, 30f + %A  - 5%A  - 6]
--R       ]
--R     ,
--R
--R                    4      2
--R     [complexRoots= ?  + 12?  + 144,
--R
--R       coordinates =
--R                  2               2               2               2
--R         [12a + %A  + 12, 12b + %A  + 12, 12c + %A  + 12, 12d + %A  + 12,
--R                  2                    2
--R          6e - %A  + 3%A - 12, 6f - %A  - 3%A - 12]
```

```
--R       ]
--R     ,
--R
--R                     2
--R     [complexRoots= ?  - 12,
--R      coordinates= [a - 1,b - 1,c - 1,d - 1,2e + %A + 4,2f - %A + 4]]
--R     ,
--R
--R                     2
--R     [complexRoots= ?  + 6? + 6,
--R      coordinates= [a + %A + 5,b - 1,c - 1,d - 1,e - 1,f - %A - 1]]
--R     ,
--R
--R                     2
--R     [complexRoots= ?  - 6? + 6,
--R      coordinates= [a + %A - 5,b + 1,c + 1,d + 1,e + 1,f - %A + 1]]
--R     ,
--R
--R                     2
--R     [complexRoots= ?  - 12,
--R      coordinates= [a + 1,b + 1,c + 1,d + 1,2e + %A - 4,2f - %A - 4]]
--R     ,
--R
--R                     4     3      2
--R     [complexRoots= ?  + 6?  + 30?  + 36? + 36,
--R
--R       coordinates =
--R                  3      2                      3      2
--R         [30a - %A  - 5%A  - 30%A - 6, 30b - %A  - 5%A  - 30%A - 6,
--R                  3      2                      3      2
--R          30c - %A  - 5%A  - 30%A - 6, 6d + %A  + 5%A  + 24%A + 6,
--R                  3      2               3      2
--R          30e - %A  - 5%A  - 6, 30f - %A  - 5%A  - 30%A - 6]
--R       ]
--R     ,
--R
--R                     4     3      2
--R     [complexRoots= ?  - 6?  + 30?  - 36? + 36,
--R
--R       coordinates =
--R                  3      2                      3      2
--R         [30a - %A  + 5%A  - 30%A + 6, 30b - %A  + 5%A  - 30%A + 6,
--R                  3      2                      3      2
--R          30c - %A  + 5%A  - 30%A + 6, 6d + %A  - 5%A  + 24%A - 6,
--R                  3      2               3      2
--R          30e - %A  + 5%A  + 6, 30f - %A  + 5%A  - 30%A + 6]
```

```
--R       ]
--R     ,
--R
--R                      2
--R     [complexRoots= ?  + 6? + 6,
--R      coordinates= [a + 1,b + 1,c + 1,d - %A - 5,e + %A + 1,f + 1]]
--R     ,
--R
--R                      2
--R     [complexRoots= ?  - 6? + 6,
--R      coordinates= [a - 1,b - 1,c - 1,d - %A + 5,e + %A - 1,f - 1]]
--R     ]
--RType: List Record(complexRoots: SparseUnivariatePolynomial Integer,coordinates
--E 21

--S 22 of 22
concat [realSolve(ts)$zdpack for ts in lts]
--R
--R
--R   (22)
--R   [[%B1,%B1,%B1,%B5,- %B5 - 4%B1,%B1], [%B1,%B1,%B1,%B6,- %B6 - 4%B1,%B1],
--R    [%B2,%B2,%B2,%B3,- %B3 - 4%B2,%B2], [%B2,%B2,%B2,%B4,- %B4 - 4%B2,%B2],
--R    [%B7,%B7,%B7,%B7,%B11,- %B11 - 4%B7], [%B7,%B7,%B7,%B7,%B12,- %B12 - 4%B7]
--R    [%B8,%B8,%B8,%B8,%B9,- %B9 - 4%B8], [%B8,%B8,%B8,%B8,%B10,- %B10 - 4%B8],
--R    [%B13,%B13,%B17,- %B17 - 4%B13,%B13,%B13],
--R    [%B13,%B13,%B18,- %B18 - 4%B13,%B13,%B13],
--R    [%B14,%B14,%B15,- %B15 - 4%B14,%B14,%B14],
--R    [%B14,%B14,%B16,- %B16 - 4%B14,%B14,%B14],
--R
--R     [%B19, %B29,
--R
--R           7865521      31   6696179241     25   25769893181     19
--R         ---------- %B19   - ---------- %B19   - ----------- %B19
--R         6006689520          2002229840            49235160
--R       +
--R           1975912990729     13   1048460696489     7   21252634831
--R         - ------------- %B19   - ------------- %B19  - ----------- %B19
--R             3003344760             2002229840           6006689520
--R       ,
--R
--R             778171189       31   1987468196267     25   155496778477189     1
--R         - ------------- %B19   + ------------- %B19   + --------------- %B19
--R           1387545279120          1387545279120            693772639560
--R       +
--R         191631411158401     13   300335488637543     7   755656433863
--R         --------------- %B19   + --------------- %B19  - ------------ %B19
```

```
--R           693772639560            1387545279120          198220754160
--R       ,
--R
--R          1094352947      31   2794979430821     25   218708802908737     19
--R         ------------ %B19   - ------------- %B19   - --------------- %B19
--R         462515093040           462515093040            231257546520
--R       +
--R           91476663003591     13   145152550961823     7   1564893370717
--R         - -------------- %B19   - --------------- %B19  - ------------- %B19
--R             77085848840             154171697680           462515093040
--R       ,
--R
--R                    4321823003      31   180949546069     25
--R         - %B29 - ------------- %B19   + ------------ %B19
--R                  1387545279120           22746643920
--R       +
--R         863753195062493     19   1088094456732317     13
--R         --------------- %B19   + ---------------- %B19
--R           693772639560             693772639560
--R       +
--R         1732620732685741     7   13506088516033
--R         ---------------- %B19  + -------------- %B19
--R           1387545279120           1387545279120
--R       ]
--R     ,
--R
--R     [%B19, %B30,
--R
--R           7865521      31   6696179241     25   25769893181     19
--R         ---------- %B19   - ---------- %B19   - ----------- %B19
--R         6006689520          2002229840            49235160
--R       +
--R           1975912990729     13   1048460696489     7   21252634831
--R         - ------------- %B19   - ------------- %B19  - ----------- %B19
--R             3003344760             2002229840           6006689520
--R       ,
--R
--R             778171189       31   1987468196267     25   155496778477189     19
--R         - ------------- %B19   + ------------- %B19   + --------------- %B19
--R           1387545279120          1387545279120            693772639560
--R       +
--R         191631411158401     13   300335488637543     7   755656433863
--R         --------------- %B19   + --------------- %B19  - ------------ %B19
--R           693772639560            1387545279120          198220754160
--R       ,
--R
```

```
--R          1094352947      31   2794979430821     25   218708802908737     19
--R         ------------ %B19   - ------------- %B19   - ---------------- %B19
--R         462515093040           462515093040            231257546520
--R       +
--R           91476663003591     13   145152550961823     7   1564893370717
--R         - --------------- %B19   - ---------------- %B19  - ------------- %B19
--R             77085848840             154171697680           462515093040
--R       ,
--R
--R                    4321823003      31   180949546069     25
--R         - %B30 - ------------- %B19   + ------------ %B19
--R                 1387545279120           22746643920
--R       +
--R         863753195062493     19   1088094456732317     13
--R         ---------------- %B19   + ----------------- %B19
--R           693772639560             693772639560
--R       +
--R         1732620732685741     7   13506088516033
--R         ----------------- %B19  + --------------- %B19
--R           1387545279120           1387545279120
--R       ]
--R     ,
--R
--R     [%B20, %B27,
--R
--R           7865521      31   6696179241     25   25769893181     19
--R         ---------- %B20   - ---------- %B20   - ----------- %B20
--R         6006689520          2002229840            49235160
--R       +
--R           1975912990729     13   1048460696489     7   21252634831
--R         - ------------- %B20   - ------------- %B20  - ----------- %B20
--R             3003344760             2002229840           6006689520
--R       ,
--R
--R             778171189       31   1987468196267     25   155496778477189     1
--R         - ------------- %B20   + ------------- %B20   + ---------------- %B20
--R           1387545279120          1387545279120            693772639560
--R       +
--R         191631411158401     13   300335488637543     7   755656433863
--R         ---------------- %B20   + ---------------- %B20  - ------------ %B20
--R           693772639560           1387545279120          198220754160
--R       ,
--R
--R          1094352947      31   2794979430821     25   218708802908737     19
--R         ------------ %B20   - ------------- %B20   - ---------------- %B20
--R         462515093040           462515093040            231257546520
```

```
--R       +
--R           91476663003591     13   145152550961823     7   1564893370717
--R         - -------------- %B20   - --------------- %B20  - ------------- %B20
--R             77085848840             154171697680           462515093040
--R       ,
--R
--R                   4321823003      31   180949546069     25
--R         - %B27 - ------------- %B20   + ------------ %B20
--R                 1387545279120            22746643920
--R       +
--R         863753195062493     19   1088094456732317     13
--R         --------------- %B20   + ---------------- %B20
--R           693772639560             693772639560
--R       +
--R         1732620732685741     7   13506088516033
--R         ---------------- %B20  + -------------- %B20
--R           1387545279120           1387545279120
--R       ]
--R     ,
--R
--R     [%B20, %B28,
--R
--R           7865521      31   6696179241     25   25769893181     19
--R         ---------- %B20   - ---------- %B20   - ----------- %B20
--R         6006689520          2002229840            49235160
--R       +
--R           1975912990729     13   1048460696489     7   21252634831
--R         - ------------- %B20   - ------------- %B20  - ----------- %B20
--R             3003344760             2002229840           6006689520
--R       ,
--R
--R             778171189       31   1987468196267     25   155496778477189     19
--R         - ------------- %B20   + ------------- %B20   + --------------- %B20
--R           1387545279120          1387545279120            693772639560
--R       +
--R         191631411158401     13   300335488637543     7   755656433863
--R         --------------- %B20   + --------------- %B20  - ------------ %B20
--R           693772639560            1387545279120          198220754160
--R       ,
--R
--R          1094352947      31   2794979430821     25   218708802908737     19
--R         ------------ %B20   - ------------- %B20   - --------------- %B20
--R         462515093040           462515093040            231257546520
--R       +
--R           91476663003591     13   145152550961823     7   1564893370717
--R         - -------------- %B20   - --------------- %B20  - ------------- %B20
```

```
--R             77085848840              154171697680            462515093040
--R       ,
--R
--R                    4321823003      31   180949546069     25
--R         - %B28 - ------------- %B20   + ------------ %B20
--R                  1387545279120            22746643920
--R       +
--R         863753195062493     19   1088094456732317     13
--R         ---------------- %B20   + ----------------- %B20
--R           693772639560              693772639560
--R       +
--R         1732620732685741     7   13506088516033
--R         ----------------- %B20  + -------------- %B20
--R           1387545279120            1387545279120
--R       ]
--R     ,
--R
--R     [%B21, %B25,
--R
--R           7865521      31   6696179241     25   25769893181     19
--R         ---------- %B21   - ---------- %B21   - ----------- %B21
--R         6006689520           2002229840             49235160
--R       +
--R           1975912990729     13   1048460696489     7   21252634831
--R         - -------------- %B21   - -------------- %B21  - ------------ %B21
--R             3003344760              2002229840            6006689520
--R       ,
--R
--R             778171189       31   1987468196267     25   155496778477189     1
--R         - -------------- %B21   + -------------- %B21   + ---------------- %B21
--R           1387545279120           1387545279120             693772639560
--R       +
--R         191631411158401     13   300335488637543     7   755656433863
--R         ---------------- %B21   + ---------------- %B21  - ------------ %B21
--R           693772639560             1387545279120           198220754160
--R       ,
--R
--R          1094352947      31   2794979430821     25   218708802908737     19
--R         ------------ %B21   - -------------- %B21   - ---------------- %B21
--R         462515093040           462515093040             231257546520
--R       +
--R           91476663003591     13   145152550961823     7   1564893370717
--R         - --------------- %B21   - ---------------- %B21  - -------------- %B21
--R             77085848840              154171697680            462515093040
--R       ,
--R
```

```
--R                    4321823003      31   180949546069     25
--R         - %B25 - ------------- %B21   + ------------ %B21
--R                  1387545279120           22746643920
--R       +
--R         863753195062493     19   1088094456732317     13
--R         --------------- %B21   + ----------------- %B21
--R           693772639560             693772639560
--R       +
--R         1732620732685741     7   13506088516033
--R         ---------------- %B21  + -------------- %B21
--R           1387545279120           1387545279120
--R       ]
--R     ,
--R
--R     [%B21, %B26,
--R
--R           7865521      31   6696179241     25   25769893181     19
--R         ---------- %B21   - ---------- %B21   - ------------ %B21
--R         6006689520          2002229840            49235160
--R       +
--R           1975912990729     13   1048460696489     7   21252634831
--R         - ------------- %B21   - ------------- %B21  - ----------- %B21
--R             3003344760             2002229840           6006689520
--R       ,
--R
--R             778171189       31   1987468196267     25   155496778477189     19
--R         - ------------- %B21   + ------------- %B21   + --------------- %B21
--R           1387545279120          1387545279120            693772639560
--R       +
--R         191631411158401     13   300335488637543     7   755656433863
--R         --------------- %B21   + --------------- %B21  - ------------ %B21
--R           693772639560            1387545279120          198220754160
--R       ,
--R
--R          1094352947      31   2794979430821     25   218708802908737     19
--R         ------------ %B21   - ------------- %B21   - --------------- %B21
--R         462515093040           462515093040            231257546520
--R       +
--R           91476663003591     13   145152550961823     7   1564893370717
--R         - -------------- %B21   - --------------- %B21  - ------------- %B21
--R             77085848840            154171697680           462515093040
--R       ,
--R
--R                    4321823003      31   180949546069     25
--R         - %B26 - ------------- %B21   + ------------ %B21
--R                  1387545279120           22746643920
```

```
--R       +
--R         863753195062493     19   1088094456732317     13
--R         --------------- %B21   + ---------------- %B21
--R           693772639560             693772639560
--R       +
--R         1732620732685741     7   13506088516033
--R         ---------------- %B21  + -------------- %B21
--R           1387545279120           1387545279120
--R       ]
--R     ,
--R
--R     [%B22, %B23,
--R
--R           7865521      31   6696179241     25   25769893181     19
--R         ---------- %B22   - ---------- %B22   - ----------- %B22
--R         6006689520          2002229840            49235160
--R       +
--R           1975912990729     13   1048460696489     7   21252634831
--R         - ------------- %B22   - ------------- %B22  - ----------- %B22
--R             3003344760             2002229840           6006689520
--R       ,
--R
--R             778171189       31   1987468196267     25   155496778477189     1
--R         - ------------- %B22   + ------------- %B22   + --------------- %B22
--R           1387545279120          1387545279120            693772639560
--R       +
--R         191631411158401     13   300335488637543     7   755656433863
--R         --------------- %B22   + --------------- %B22  - ------------ %B22
--R           693772639560            1387545279120          198220754160
--R       ,
--R
--R          1094352947      31   2794979430821     25   218708802908737     19
--R         ------------ %B22   - ------------- %B22   - --------------- %B22
--R         462515093040           462515093040            231257546520
--R       +
--R           91476663003591     13   145152550961823     7   1564893370717
--R         - -------------- %B22   - --------------- %B22  - ------------- %B22
--R             77085848840             154171697680           462515093040
--R       ,
--R
--R                    4321823003      31   180949546069     25
--R         - %B23 - ------------- %B22   + ------------ %B22
--R                  1387545279120           22746643920
--R       +
--R         863753195062493     19   1088094456732317     13
--R         --------------- %B22   + ---------------- %B22
```

```
--R           693772639560              693772639560
--R       +
--R         1732620732685741     7   13506088516033
--R         ---------------- %B22  + -------------- %B22
--R           1387545279120           1387545279120
--R       ]
--R     ,
--R
--R     [%B22, %B24,
--R
--R           7865521      31   6696179241     25   25769893181     19
--R         ---------- %B22   - ---------- %B22   - ----------- %B22
--R         6006689520           2002229840            49235160
--R       +
--R           1975912990729     13   1048460696489     7   21252634831
--R         - ------------- %B22   - ------------- %B22  - ----------- %B22
--R             3003344760              2002229840           6006689520
--R       ,
--R
--R             778171189       31   1987468196267     25   155496778477189     19
--R         - ------------- %B22   + ------------- %B22   + --------------- %B22
--R           1387545279120           1387545279120            693772639560
--R       +
--R         191631411158401     13   300335488637543     7   755656433863
--R         ---------------- %B22   + ---------------- %B22  - ------------ %B22
--R           693772639560            1387545279120          198220754160
--R       ,
--R
--R          1094352947      31   2794979430821     25   218708802908737     19
--R         ------------ %B22   - ------------- %B22   - --------------- %B22
--R         462515093040           462515093040            231257546520
--R       +
--R           91476663003591     13   145152550961823     7   1564893370717
--R         - -------------- %B22   - ---------------- %B22  - ------------- %B22
--R             77085848840             154171697680           462515093040
--R       ,
--R
--R                    4321823003      31   180949546069     25
--R         - %B24 - ------------- %B22   + ------------ %B22
--R                  1387545279120           22746643920
--R       +
--R         863753195062493     19   1088094456732317     13
--R         ---------------- %B22   + ----------------- %B22
--R           693772639560            693772639560
--R       +
--R         1732620732685741     7   13506088516033
```

```
--R          ----------------- %B22  + -------------- %B22
--R            1387545279120            1387545279120
--R       ]
--R     ,
--R    [%B31,%B35,- %B35 - 4%B31,%B31,%B31,%B31],
--R    [%B31,%B36,- %B36 - 4%B31,%B31,%B31,%B31],
--R    [%B32,%B33,- %B33 - 4%B32,%B32,%B32,%B32],
--R    [%B32,%B34,- %B34 - 4%B32,%B32,%B32,%B32]]
--R                                    Type: List List RealClosure Fraction Integer
--E 22
)spool
)lisp (bye)
```

⟨*LexTriangularPackage.help*⟩≡

```
==========================================================================
LexTriangularPackage examples
==========================================================================

The LexTriangularPackage package constructor provides an
implementation of the lexTriangular algorithm (D. Lazard "Solving
Zero-dimensional Algebraic Systems", J. of Symbol. Comput., 1992).
This algorithm decomposes a zero-dimensional variety into zero-sets of
regular triangular sets.  Thus the input system must have a finite
number of complex solutions.  Moreover, this system needs to be a
lexicographical Groebner basis.

This package takes two arguments: the coefficient-ring R of the
polynomials, which must be a GcdDomain and their set of variables
given by ls a List Symbol.  The type of the input polynomials must be
NewSparseMultivariatePolynomial(R,V) where V is OrderedVariableList(ls).
The abbreviation for LexTriangularPackage is LEXTRIPK.  The main operations
are lexTriangular and squareFreeLexTriangular.  The later provide
decompositions by means of square-free regular triangular sets, built
with the SquareFreeRegularTriangularSet constructor, whereas the former
uses the RegularTriangularSet constructor.  Note that these constructors
also implement another algorithm for solving algebraic systems by
means of regular triangular sets; in that case no computations of
Groebner bases are needed and the input system may have any dimension
(i.e. it may have an infinite number of solutions).

The implementation of the lexTriangular algorithm provided in the
LexTriangularPackage constructor differs from that reported in
"Computations of gcd over algebraic towers of simple extensions" by
M. Moreno Maza and R. Rioboo (in proceedings of AAECC11, Paris, 1995).
Indeed, the squareFreeLexTriangular operation removes all multiplicities
of the solutions (i.e. the computed solutions are pairwise different)
and the lexTriangular operation may keep some multiplicities; this
later operation runs generally faster than the former.

The interest of the lexTriangular algorithm is due to the
following experimental remark.  For some examples, a triangular
decomposition of a zero-dimensional variety can be computed faster via
a lexicographical Groebner basis computation than by using a direct method
(like that of SquareFreeRegularTriangularSet and RegularTriangularSet).
This happens typically when the total degree of the system relies
essentially on its smallest variable (like in the Katsura systems).
When this is not the case, the direct method may give better timings
(like in the Rose system).
```

Of course, the direct method can also be applied to a lexicographical Groebner basis. However, the lexTriangular algorithm takes advantage of the structure of this basis and avoids many unnecessary computations which are performed by the direct method.

For this purpose of solving algebraic systems with a finite number of solutions, see also the ZeroDimensionalSolvePackage. It allows to use both strategies (the lexTriangular algorithm and the direct method) for computing either the complex or real roots of a system.

Note that the way of understanding triangular decompositions is detailed in the example of the RegularTriangularSet constructor.

Since the LexTriangularPackage package constructor is limited to zero-dimensional systems, it provides a zeroDimensional? operation to check whether this requirement holds. There is also a groebner operation to compute the lexicographical Groebner basis of a set of polynomials with type NewSparseMultivariatePolynomial(R,V). The elimination ordering is that given by ls (the greatest variable being the first element of ls). This basis is computed by the FLGM algorithm (Faugere et al. "Efficient Computation of Zero-Dimensional Groebner Bases by Change of Ordering" , J. of Symbol. Comput., 1993) implemented in the LinGroebnerPackage package constructor.

Once a lexicographical Groebner basis is computed, then one can call the operations lexTriangular and squareFreeLexTriangular. Note that these operations admit an optional argument to produce normalized triangular sets. There is also a zeroSetSplit operation which does all the job from the input system; an error is produced if this system is not zero-dimensional.

Let us illustrate the facilities of the LexTriangularPackage constructor by a famous example, the cyclic-6 root system.

Define the coefficient ring.

```
R := Integer
 Integer
                      Type: Domain
```

Define the list of variables,

```
ls : List Symbol := [a,b,c,d,e,f]
  [a,b,c,d,e,f]
                    Type: List Symbol
```

```
and make it an ordered set.

  V := OVAR(ls)
    OrderedVariableList [a,b,c,d,e,f]
                      Type: Domain

Define the polynomial ring.

  P := NSMP(R, V)
   NewSparseMultivariatePolynomial(Integer,OrderedVariableList [a,b,c,d,e,f])
                      Type: Domain

Define the polynomials.

  p1: P :=  a*b*c*d*e*f - 1
   f e d c b a - 1
    Type: NewSparseMultivariatePolynomial(Integer,
                                          OrderedVariableList [a,b,c,d,e,f])

  p2: P := a*b*c*d*e +a*b*c*d*f +a*b*c*e*f +a*b*d*e*f +a*c*d*e*f +b*c*d*e*f
   ((((e + f)d + f e)c + f e d)b + f e d c)a + f e d c b
    Type: NewSparseMultivariatePolynomial(Integer,
                                          OrderedVariableList [a,b,c,d,e,f])

  p3: P :=  a*b*c*d + a*b*c*f + a*b*e*f + a*d*e*f + b*c*d*e + c*d*e*f
   (((d + f)c + f e)b + f e d)a + e d c b + f e d c
    Type: NewSparseMultivariatePolynomial(Integer,
                                          OrderedVariableList [a,b,c,d,e,f])

  p4: P := a*b*c + a*b*f + a*e*f + b*c*d + c*d*e + d*e*f
   ((c + f)b + f e)a + d c b + e d c + f e d
    Type: NewSparseMultivariatePolynomial(Integer,
                                          OrderedVariableList [a,b,c,d,e,f])

  p5: P := a*b + a*f + b*c + c*d + d*e + e*f
   (b + f)a + c b + d c + e d + f e
    Type: NewSparseMultivariatePolynomial(Integer,
                                          OrderedVariableList [a,b,c,d,e,f])

  p6: P := a + b + c + d + e + f
   a + b + c + d + e + f
    Type: NewSparseMultivariatePolynomial(Integer,
                                          OrderedVariableList [a,b,c,d,e,f])

  lp := [p1, p2, p3, p4, p5, p6]
    [f e d c b a - 1, ((((e + f)d + f e)c + f e d)b + f e d c)a + f e d c b,
```

```
      (((d + f)c + f e)b + f e d)a + e d c b + f e d c,
      ((c + f)b + f e)a + d c b + e d c + f e d,
      (b + f)a + c b + d c + e d + f e, a + b + c + d + e + f]
   Type: List NewSparseMultivariatePolynomial(Integer,
                                        OrderedVariableList [a,b,c,d,e,f])
```

Now call LEXTRIPK.

```
  lextripack :=  LEXTRIPK(R,ls)
    LexTriangularPackage(Integer,[a,b,c,d,e,f])
                        Type: Domain
```

Compute the lexicographical Groebner basis of the system. This may take between 5 minutes and one hour, depending on your machine.

```
  lg := groebner(lp)$lextripack
   [a + b + c + d + e + f,
                          2                                              2
     3968379498283200b  + 15873517993132800f b + 3968379498283200d
    +
                                                3 5                     4 4
     15873517993132800f d + 3968379498283200f e  - 15873517993132800f e
    +
                          5 3                       6                       2
     23810276989699200f e  + (206355733910726400f  + 230166010900425600)e
    +
                              43                        37
          - 729705987316687f   + 18636674968672054 21f
       +
                                 31                           25
         291674853771731104461f   + 365285994691106921745f
       +
                               19                           13
         549961185828911895f   - 365048404038768439269f
       +
                                 7
         - 2923828204315040276 69f  - 2271898467631865497f
     *
       e
    +
                           44                         38
     - 3988812642545399f   + 10187423878429609997f
    +
                             32                            26
     1594377523424314053637f   + 19947393084399162380 65f
    +
```

```
                      20                            14
  1596840088052642815f   - 19934941183011621454l3f
+
                           8                         2
  - 1596049742289689815053f  - 114881713301596674491f
,
                                                                  2
  (23810276989699200c - 2381027698969920f)b + 23810276989699200c
+
                                        2
  71430830969097600f c - 23810276989699200d  - 952411079587968001 d
+
                      3 5                       4 4                       5 3
  - 55557312975964800f e  + 1746086979244608001 e  - 1746086979244608001 e
+
                         6                         2
  (- 2428648252949318400f  - 2611193709870345600)e
+
                       43                          37
      8305444561289527f   - 212120871519454596411
    +
                                31                           25
      - 33198158830934513853811   - 41576916462616571364451
    +
                             19                            13
      - 6072721607510764095f   + 4154986709036460221649f
    +
                              7
      33277613111385870967491  + 258853406082908416371
 *
    e
+
                     44                          38
  458158976290103291   - 1170137655821518912071
+
                              32                            26
  - 183131668489708650741871   - 229099712396492974389151
+
                           20                              14
  - 16133250761305157265f   + 228973058576361782566231
+
                            8                           2
  183299447818672424979231  + 1302585310020204206991
,
   (7936758996566400d - 793675899656640l)b - 7936758996566400l d
+
```

```
                      3 5                      4 4                      5 3
  - 7936758996566400f e  + 23810276989699200f e  - 23810276989699200f e
+
                        6                       2
  (- 337312257354072000f  - 369059293340337600)e
+
                       43                       37
      1176345388640471f   - 3004383582891473073f
    +
                            31                         25
      - 4702035027072461056 53f   - 588858183402644348085f
    +
                           19                          13
      - 856939308623513535f   + 588472674242340526377f
    +
                           7
      471313241958371103517f  + 3659742549078552381f
 *
    e
+
                    44                         38                            32
  6423170513956901f   - 164047721370364808 03f   - 2567419165227528774463f
+
                           26                        20
  - 32119380908256821723 35f   - 2330490332697587485f
+
                          14                          8
  3210100109444754864587f   + 2569858315395162617847f
+
                        2
  18326089487427735751f
,
                                                                        3 5
  (11905138494849600e - 11905138494849600f)b - 3968379498283200f e
+
                     4 4                      5 3
  15873517993132800f e  - 27778656487982400f e
+
                       6                       2
  (- 208339923659868000f  - 240086959646133600)e
+
                      43                       37
      786029984751110f   - 2007519008182245250f
    +
                           31                         25
      - 3141880629080738 07090f   - 3934236675379295 75250f
```

```
    +
                            19                            13
      - 550329120654394950f   + 393196408728889612770f
    +
                             7
      31489237279917649573 0f  + 2409386515146668530f
 *
    e
+
                     44                          38                              32
  4177638546747827f   - 10669685294602576381f   - 1669852980419949524601f
+
                              26                          20
  - 2089077057287904170745f   - 1569899763580278795f
+
                           14                             8
  2087864026859015573349f   + 16714960859451995779 69f
+
                         2
  11940257226216280177f
,
                        6                                                 2 5
  (11905138494849600f  - 11905138494849600)b - 158735179931328 00f e
+
                     3 4                         4 3
  39683794982832000f e  - 39683794982832000f e
+
                           11                          5  2
  (- 686529653202993600f   - 607162063237329600f )e
+
                       42                          36                            30
      65144531306704f   - 166381280901088652f   - 260334345024702834 72f
    +
                              24                          18
      - 316962595838606501 40f   + 971492093167581360f
    +
                           12                            6
    3222008503369138954 8f   + 255261776660705298 08f  + 138603268355749244
 *
    e
+
                   43                          37                            31
  167620036074811f   - 428102417974791473f   - 669972438012316793 13f
+
                           25                           19
  - 83426716722148750485f   + 203673895369980765f
```

```
+
                        13                          7
  835230563260104324 57f   + 669957896402380669 37f  + 478592855549587901f
,
                3                    2                 2                 45
  801692827936c  + 2405078483808f c  - 2405078483808f c - 13752945467f
+
                  39                     33                     27
  35125117815561f   + 5496946957826433f   + 6834659447749117f
+
                    21                     15                     9
  - 44484880462461f   - 6873406230093057f   - 5450844938762633f
+
                3
  1216586044571f
,
                                                                    2
  (23810276989699200d - 2381027698969920 0f)c + 23810276989699200d
+
                                           3 5                     4 4
  71430830969097600f d + 7936758996566400f e  - 31747035986265600f e
+
                     5 3                        6                       2
  31747035986265600f e  + (404774708824886400f  + 396837949828320000)e
+
                          43                         37
      - 1247372229446701f   + 3185785654596621203f
    +
                             31                            25
      4985948668499747514 63f   + 6245425458457910479 35f
    +
                          19                             13
      931085755769682885f   - 6241506635824170633 87f
    +
                              7
      - 4998818593883604756 47f  - 39268853138195273 51f
 *
    e
+
                       44                          38
  - 7026011547118141f   + 179444270519506912 43f
+
                          32                             26
  280838352259398660354 3f   + 351362414235480753013 5f
+
                        20                             14
```

```
  2860757006705537685f    - 3511356735642190737267f
+
                            8                        2
  - 2811332494697103819887f  - 203150116315228473 11f
,
   (7936758996566400e - 7936758996566400f)c
+
                       43                      37                          31
      - 4418748183673f   + 11285568707456559f   + 17659986172944510 19f
    +
                           25                     19
      2173749283622606155f   - 55788292195402895f
    +
                           13                       7
    - 2215291421788292951f   - 17181426653474308 51f  + 30256569458230237f
 *
    e
+
                44                      38                          32
  4418748183673f   - 11285568707456559f   - 17659986172944510 19f
+
                          26                      20                          14
  - 2173749283622606155f   + 55788292195402895f   + 22152914217882929 51f
+
                        8                      2
  17181426653474308 51f  - 30256569458230237f
,
                     6                                    43
  (72152354514240f  - 72152354514240)c + 40950859449f
+
                    37                      31                       25
  - 104588980990367f   - 16367227395575307f   - 20268523416527355f
+
                   19                      13                       7
  442205002259535f   + 20576059935789063f   + 15997133796970563f
+
  - 2750998927855 81f
,
                    3                       2                       2
  1984189749141600d  + 5952569247424800f d  - 5952569247424800f d
+
                     4 5                     5 4                      3
  - 3968379498283200f e  + 15873517993132800f e  + 17857707742274400e
+
                       7                        2
  (- 1488142311856200 00f  - 16270355942961120 0f)e
```

```
+
                        44                        38
      - 390000914678878f   + 996062704593756434f
    +
                             32                             26
      155886323972034823914f   + 194745956143985421330f
    +
                       20                           14
      6205077595574430f   - 1945965126532990687 86f
    +
                                8                          2
      - 1557968979407569226 66f  - 10363757590773209 78f
 *
    e
+
                     45                         39                              33
  - 374998630035991f   + 957747106595453993f   + 1498891555667648916 93f
+
                           27                          21
  1871541714434946416 85f   - 127129015426348065f
+
                           15                             9                          3
- 1872415332431150404 17f   - 1497199835679765340 37f  - 836654081239648061f
,
                                                                          3 5
  (5952569247424800e - 5952569247424800f)d - 3968379498283200f e
+
                     4 4                       5 3
  9920948745708000f e  - 3968379498283200f e
+
                           6                          2
  (- 148814231185620000f  - 150798420934761600)e
+
                         43                           37
      492558110242553f   - 1257992359608074599f
    +
                               31                            25
      - 1968830945393685139 59f   - 2465621157457354280 55f
    +
                             19                             13
      - 325698701993885505f   + 24641776988365180811 1f
    +
                              7
      1973273520682006529 11f  + 1523373796389332143f
 *
    e
```

```
+
                  44                        38                           32
  2679481081803026f   - 6843392695421906608f   - 1071020459642646913578f
+
                           26                       20
  - 13397891696920412400 60f   - 852746750910750210f
+
                         14                          8
  1339105101971878401312f   + 1071900289758712984762f
+
                       2
  7555239072072727756f
,
                       6                                          2 5
  (11905138494849600f  - 11905138494849600)d - 7936758996566400f e
+
                     3 4                      4 3
  31747035986265600f e  - 31747035986265600f e
+
                         11                       5  2
  (- 420648226818019200f   - 404774708824886400f )e
+
                      42                       36                         30
      15336187600889f   - 39169739565161107f   - 6127176127489690827f
    +
                           24                       18
      - 72177087423105096 15f   + 538628483890722735f
    +
                         12                         6
      75068043538435076 43f   + 588616076978260720 3f  + 63576108396535879
 *
    e
+
                  43                       37                           31
  71737781777066f   - 183218856207557938f   - 2867287427113227607 8f
+
                         25                        19
  - 356252236869398120 10f   + 164831339634084390f
+
                        13                          7
  35724160423073052642f   + 28627022578664910622f  + 1874599870296805 06f
,
                      6                        5                       2 4
  1322793166094400e  - 3968379498283200f e  + 3968379498283200f e
+
                       3 3
```

```
  - 5291172664377600f e
+
                          10                        4  2
  (- 230166010900425600f   - 226197631402142400f )e
+
                            47                           41
      - 152375364610443885f   + 3891666260648548904l5f
    +
                               35                             29
      6090609784136055898733 5f   + 7616736793460879869727 5f
    +
                             23                              17
      27855066785995181125f   - 7614495281705272314549 5f
    +
                               11                           5
      - 6093362989246351754697 5f   - 4114150716820025477 95f
 *
    e
+
                     42                        36                          30
  - 209493533143822f   + 535045979490560586f   + 837379479649735531 46f
+
                          24                         18
  1048895070842133715 70f   + 167117997269207870f
+
                            12                           6
  - 1047937257813906155 14f   - 83842685189903180394f  - 569978796672974242
,
                    6                    3
  (25438330117200f  + 25438330117200)e
+
                   7                    2
  (76314990351600f  + 76314990351600f)e
+
                       44                      38                        32
      - 1594966552735f   + 4073543370415745f   + 637527159231148925f
    +
                         26                    20                          14
      797521176113606525f   + 530440941097175f   - 797160527306433145f
    +
                           8                     2
      - 638132320196044965f  - 4510507167940725f
 *
    e
+
                   45                      39                          33
```

```
      - 6036376800443f    + 15416903421476909f    + 2412807646192304449f
    +
                          27                    21                        15
      3017679923028013705f    + 1422320037411955f    - 3016560402417843941f
    +
                            9                     3
      - 2414249368183033161f   - 16561862361763873f
    ,
                    12                   2
      (1387545279120f    - 1387545279120)e
    +
                      43                   37                     31
          4321823003f    - 11037922310209f    - 1727510711947989f
        +
                            25                   19                       13
          - 2165150991154425f    - 5114342560755f    + 2162682824948601f
        +
                          7
          1732620732685741f  + 13506088516033f
     *
        e
    +
                   44                   38                      32
      24177661775f    - 61749727185325f    - 9664106795754225f
    +
                          26                   20                       14
      - 12090487758628245f    - 8787672733575f    + 12083693383005045f
    +
                        8                   2
      9672870290826025f  + 68544102808525f
    ,
     48        42            36            30            18            12         6
    f    - 2554f    - 399710f    - 499722f    + 499722f    + 399710f    + 2554f   - 1]
  Type: List NewSparseMultivariatePolynomial(Integer,
    OrderedVariableList [a,b,c,d,e,f])
```

Apply lexTriangular to compute a decomposition into regular triangular sets. This should not take more than 5 seconds.

```
  lexTriangular(lg,false)$lextripack
  [
     6       6       5     2 4     3 3     4 2     5
   {f  + 1, e  - 3f e  + 3f e  - 4f e  + 3f e  - 3f e - 1,
          2 5     3 4     4 3     5 2
    3d + f e  - 4f e  + 4f e  - 2f e  - 2e + 2f, c + f,
           2 5     3 4     4 3      5 2
```

```
  3b + 2f e  - 5f e  + 5f e  - 10f e  - 4e + 7f,
       2 5     3 4     4 3     5 2
  a - f e  + 3f e  - 3f e  + 4f e  + 3e - 3f}
 ,
  6                  2           2                    2
{f  - 1,e - f,d - f,c  + 4f c + f ,(c - f)b - f c - 5f ,a + b + c + 3f},
  6                        2           2
{f  - 1,e - f,d - f,c - f,b  + 4f b + f ,a + b + 4f},
  6            2           2                    2
{f  - 1,e - f,d  + 4f d + f ,(d - f)c - f d - 5f ,b - f,a + c + d + 3f},
    36        30          24          18          12        6
 {f   - 2554f   - 399709f   - 502276f   - 399709f   - 2554f  + 1,
                 12              2
     (161718564f   - 161718564)e
   +
                  31              25                19                13
        - 504205f   + 1287737951f   + 201539391380f   + 253982817368f
      +
                     7
        201940704665f  + 1574134601f
    *
      e
   +
               32              26                  20                  14
     - 2818405f   + 7198203911f   + 1126548149060f   + 1416530563364f
   +
                  8              2
     1127377589345f  + 7988820725f
   ,
                    6                                  2 5                 3 4
     (693772639560f  - 693772639560)d - 462515093040f e  + 1850060372160f e
   +
                     4 3                       11                   5  2
     - 1850060372160f e  + (- 24513299931120f   - 23588269745040f )e
   +
                     30                  24                    18
        - 890810428f   + 2275181044754f   + 355937263869776f
      +
                       12                    6
        413736880104344f   + 342849304487996f  + 3704966481878
    *
      e
   +
                  31                  25                    19
     - 4163798003f   + 10634395752169f   + 1664161760192806f
   +
```

```
                      13                      7
        2079424391370694f   + 1668153650635921f  + 10924274392693f
      ,
                    6                            31               25
        (12614047992f  - 12614047992)c - 7246825f   + 18508536599f
      +
                    19                  13                 7
        2896249516034f   + 3581539649666f   + 2796477571739f  - 48094301893f
      ,
                     6                                  2 5                 3 4
        (693772639560f  - 693772639560)b - 925030186080f e  + 2312575465200f e
      +
                     4 3                       11                   5  2
        - 2312575465200f e  + (- 40007555547960f   - 35382404617560f )e
      +
                      30                 24                     18
          - 3781280823f   + 9657492291789f   + 1511158913397906f
        +
                           12                     6
          1837290892286154f   + 1487216006594361f  + 8077238712093
       *
          e
      +
                    31                  25                    19
        - 9736390478f   + 24866827916734f   + 3891495681905296f
      +
                      13                     7
        4872556418871424f   + 3904047887269606f  + 27890075838538f
      ,
     a + b + c + d + e + f}
    ,
     6      2           2                   2
   {f  - 1,e  + 4f e + f ,(e - f)d - f e - 5f ,c - f,b - f,a + d + e + 3f}]
                             Type: List RegularChain(Integer,[a,b,c,d,e,f])
```

Note that the first set of the decomposition is normalized (all initials are integer numbers) but not the second one (normalized triangular sets are defined in the description of the NormalizedTriangularSetCategory constructor).

So apply now lexTriangular to produce normalized triangular sets.

```
  lts := lexTriangular(lg,true)$lextripack
   [
      6       6       5     2 4     3 3     4 2     5
    {f  + 1, e  - 3f e  + 3f e  - 4f e  + 3f e  - 3f e - 1,
```

```
       2 5     3 4     4 3     5 2
  3d + f e  - 4f e  + 4f e  - 2f e  - 2e + 2f, c + f,
        2 5     3 4     4 3      5 2
  3b + 2f e  - 5f e  + 5f e  - 10f e  - 4e + 7f,
      2 5     3 4     4 3     5 2
  a - f e  + 3f e  - 3f e  + 4f e  + 3e - 3f}
 ,
  6                  2           2
{f  - 1,e - f,d - f,c  + 4f c + f ,b + c + 4f,a - f},
  6                        2           2
{f  - 1,e - f,d - f,c - f,b  + 4f b + f ,a + b + 4f},
  6            2           2
{f  - 1,e - f,d  + 4f d + f ,c + d + 4f,b - f,a - f},
    36        30          24          18          12        6
 {f   - 2554f   - 399709f   - 502276f   - 399709f   - 2554f  + 1,
                    2
     1387545279120e
   +
                   31                  25                     19
         4321823003f   - 11037922310209f   - 1727506390124986f
      +
                            13                    7
         - 2176188913464634f   - 1732620732685741f  - 13506088516033f
    *
      e
   +
                32                  26                    20
     24177661775f   - 61749727185325f   - 9664082618092450f
   +
                         14                    8                  2
     - 12152237485813570f   - 9672870290826025f  - 68544102808525f
   ,
      1387545279120d
   +
                     30                  24                    18
         - 1128983050f   + 2883434331830f   + 451234998755840f
      +
                         12                    6
         562426491685760f   + 447129055314890f  - 165557857270
    *
      e
   +
                  31                  25                    19
     - 1816935351f   + 4640452214013f   + 726247129626942f
   +
                    13                   7
```

```
          912871801716798f   + 726583262666877f  + 4909358645961f
        ,
                                    31                 25                    19
          1387545279120c + 778171189f   - 1987468196267f   - 310993556954378f
        +
                           13                   7
          - 383262822316802f   - 300335488637543f  + 5289595037041f
        ,
           1387545279120b
        +
                        30                 24                   18
              1128983050f   - 2883434331830f   - 451234998755840f
           +
                              12                   6
              - 562426491685760f   - 447129055314890f  + 165557857270
         *
           e
        +
                      31                 25                    19
          - 3283058841f   + 8384938292463f   + 1312252817452422f
        +
                         13                   7
          1646579934064638f   + 1306372958656407f  + 4694680112151f
        ,
                                                      31                   25
          1387545279120a + 1387545279120e + 4321823003f   - 11037922310209f
        +
                             19                    13                    7
          - 1727506390124986f   - 2176188913464634f   - 1732620732685741f
        +
          - 13506088516033f
        }
       ,
        6      2           2
      {f  - 1,e  + 4f e + f ,d + e + 4f,c - f,b - f,a - f}]
                                Type: List RegularChain(Integer,[a,b,c,d,e,f])
```

We check that all initials are constant.

```
  [ [init(p) for p in (ts :: List(P))] for ts in lts]
   [[1,3,1,3,1,1], [1,1,1,1,1,1], [1,1,1,1,1,1], [1,1,1,1,1,1],
    [1387545279120,1387545279120,1387545279120,1387545279120,1387545279120,1],
    [1,1,1,1,1,1]]
     Type: List List NewSparseMultivariatePolynomial(Integer,
                                       OrderedVariableList [a,b,c,d,e,f])
```

Note that each triangular set in lts is a lexicographical Groebner basis. Recall that a point belongs to the variety associated with lp if and only if it belongs to that associated with one triangular set ts in lts.

By running the squareFreeLexTriangular operation, we retrieve the above decomposition.

```
squareFreeLexTriangular(lg,true)$lextripack
[
    6       6       5     2 4     3 3     4 2     5
  {f  + 1, e  - 3f e  + 3f e  - 4f e  + 3f e  - 3f e - 1,
         2 5     3 4     4 3     5 2
   3d + f e  - 4f e  + 4f e  - 2f e  - 2e + 2f, c + f,
          2 5     3 4     4 3      5 2
   3b + 2f e  - 5f e  + 5f e  - 10f e  - 4e + 7f,
        2 5     3 4     4 3     5 2
   a - f e  + 3f e  - 3f e  + 4f e  + 3e - 3f}
  ,
   6                  2           2
 {f  - 1,e - f,d - f,c  + 4f c + f ,b + c + 4f,a - f},
   6                        2           2
 {f  - 1,e - f,d - f,c - f,b  + 4f b + f ,a + b + 4f},
   6            2           2
 {f  - 1,e - f,d  + 4f d + f ,c + d + 4f,b - f,a - f},
     36        30          24          18          12        6
  {f   - 2554f   - 399709f   - 502276f   - 399709f   - 2554f  + 1,
                    2
      1387545279120e
    +
                     31                   25                     19
          4321823003f   - 11037922310209f   - 1727506390124986f
        +
                              13                     7
          - 2176188913464634f   - 1732620732685741f  - 13506088516033f
     *
        e
    +
                 32                  26                    20
      24177661775f   - 61749727185325f   - 9664082618092450f
    +
                           14                    8                  2
      - 12152237485813570f   - 9672870290826025f  - 68544102808525f
    ,
       1387545279120d
    +
```

```
                      30                 24                   18
        - 1128983050f   + 2883434331830f   + 451234998755840f
      +
                         12                   6
        562426491685760f   + 447129055314890f  - 165557857270
     *
      e
    +
                  31                  25                    19
      - 1816935351f   + 4640452214013f   + 726247129626942f
    +
                     13                   7
      912871801716798f   + 726583262666877f  + 4909358645961f
    ,
                                  31                 25                    19
      1387545279120c + 778171189f   - 1987468196267f   - 310993556954378f
    +
                       13                   7
      - 383262822316802f   - 300335488637543f  + 5289595037041f
    ,
       1387545279120b
    +
                    30                 24                   18
        1128983050f   - 2883434331830f   - 451234998755840f
      +
                           12                   6
        - 562426491685760f   - 447129055314890f  + 165557857270
     *
      e
    +
                  31                  25                    19
      - 3283058841f   + 8384938292463f   + 1312252817452422f
    +
                     13                   7
      1646579934064638f   + 1306372958656407f  + 4694680112151f
    ,
                                                 31                   25
      1387545279120a + 1387545279120e + 4321823003f   - 11037922310209f
    +
                        19                   13                    7
      - 1727506390124986f   - 2176188913464634f   - 1732620732685741f
    +
      - 13506088516033f
    }
  ,
   6      2           2
```

```
   {f  - 1,e  + 4f e + f ,d + e + 4f,c - f,b - f,a - f}]
  Type: List SquareFreeRegularTriangularSet(Integer,
              IndexedExponents OrderedVariableList [a,b,c,d,e,f],
              OrderedVariableList [a,b,c,d,e,f],
              NewSparseMultivariatePolynomial(Integer,
                              OrderedVariableList [a,b,c,d,e,f]))
```

Thus the solutions given by lts are pairwise different.

We count them as follows.

```
  reduce(+,[degree(ts) for ts in lts])
    156
                      Type: PositiveInteger
```

We can investigate the triangular decomposition lts by using the ZeroDimensionalSolvePackage.

This requires to add an extra variable (smaller than the others) as follows.

```
  ls2 : List Symbol := concat(ls,new()$Symbol)
    [a,b,c,d,e,f,%A]
                      Type: List Symbol
```

Then we call the package.

```
  zdpack := ZDSOLVE(R,ls,ls2)
   (20)  ZeroDimensionalSolvePackage(Integer,[a,b,c,d,e,f],[a,b,c,d,e,f,%A])
                      Type: Domain
```

We compute a univariate representation of the variety associated with the input system as follows.

```
  concat [univariateSolve(ts)$zdpack for ts in lts]
  [
                    4      2
    [complexRoots= ?  - 13?  + 49,
       coordinates =
                3                3                3                3
        [7a + %A  - 6%A, 21b + %A  + %A, 21c - 2%A  + 19%A, 7d - %A  + 6%A,
                  3                3
         21e - %A  - %A, 21f + 2%A  - 19%A]
      ]
    ,
                    4      2
    [complexRoots= ?  + 11?  + 49,
```

```
   coordinates =
                3                   3                   3
    [35a + 3%A  + 19%A, 35b + %A  + 18%A, 35c - 2%A  - %A,
                3                   3                   3
     35d - 3%A  - 19%A, 35e - %A  - 18%A, 35f + 2%A  + %A]
  ]
,
  [
  complexRoots =
     8      7      6       5       4       3       2
    ?  - 12?  + 58?  - 120?  + 207?  - 360?  + 802?  - 1332? + 1369
  ,
   coordinates =
    [
                            7           6            5            4
      43054532a + 33782%A  - 546673%A  + 3127348%A  - 6927123%A
     +
                3             2
      4365212%A  - 25086957%A  + 39582814%A - 107313172
     ,
                            7           6            5            4
      43054532b - 33782%A  + 546673%A  - 3127348%A  + 6927123%A
     +
                3             2
      - 4365212%A  + 25086957%A  - 39582814%A + 107313172
     ,
                            7           6            5            4
      21527266c - 22306%A  + 263139%A  - 1166076%A  + 1821805%A
     +
                3             2
      - 2892788%A  + 10322663%A  - 9026596%A + 12950740
     ,
                            7           6            5            4
      43054532d + 22306%A  - 263139%A  + 1166076%A  - 1821805%A
     +
                3             2
      2892788%A  - 10322663%A  + 30553862%A - 12950740
     ,
                            7           6            5            4
      43054532e - 22306%A  + 263139%A  - 1166076%A  + 1821805%A
     +
                3             2
      - 2892788%A  + 10322663%A  - 30553862%A + 12950740
     ,
                            7           6            5            4
      21527266f + 22306%A  - 263139%A  + 1166076%A  - 1821805%A
```

```
      +
                3             2
        2892788%A  - 10322663%A  + 9026596%A - 12950740
      ]
    ]
  ,
   [
    complexRoots =
       8      7      6       5       4       3       2
      ?  + 12?  + 58?  + 120?  + 207?  + 360?  + 802?  + 1332? + 1369
    ,
     coordinates =
      [
                              7           6            5            4
        43054532a + 33782%A  + 546673%A  + 3127348%A  + 6927123%A
      +
                3             2
        4365212%A  + 25086957%A  + 39582814%A + 107313172
      ,
                              7           6            5            4
        43054532b - 33782%A  - 546673%A  - 3127348%A  - 6927123%A
      +
                3             2
        - 4365212%A  - 25086957%A  - 39582814%A - 107313172
      ,
                              7           6            5            4
        21527266c - 22306%A  - 263139%A  - 1166076%A  - 1821805%A
      +
                3             2
        - 2892788%A  - 10322663%A  - 9026596%A - 12950740
      ,
                              7           6            5            4
        43054532d + 22306%A  + 263139%A  + 1166076%A  + 1821805%A
      +
                3             2
        2892788%A  + 10322663%A  + 30553862%A + 12950740
      ,
                              7           6            5            4
        43054532e - 22306%A  - 263139%A  - 1166076%A  - 1821805%A
      +
                3             2
        - 2892788%A  - 10322663%A  - 30553862%A - 12950740
      ,
                              7           6            5            4
        21527266f + 22306%A  + 263139%A  + 1166076%A  + 1821805%A
      +
```

```
                3             2
       2892788%A  + 10322663%A  + 9026596%A + 12950740
     ]
  ]
,
                4    2
[complexRoots= ?  - ?  + 1,
                            3            3              3            3
 coordinates= [a - %A,b + %A  - %A,c + %A ,d + %A,e - %A  + %A,f - %A ]]
,
                8     6      4      2
[complexRoots= ?  + 4?  + 12?  + 16?  + 4,
   coordinates =
              7      5       3                 7      5       3
    [4a - 2%A  - 7%A  - 20%A  - 22%A, 4b + 2%A  + 7%A  + 20%A  + 22%A,
             7      5       3                 7      5       3
     4c + %A  + 3%A  + 10%A  + 10%A, 4d + %A  + 3%A  + 10%A  + 6%A,
            7      5       3                 7      5       3
     4e - %A  - 3%A  - 10%A  - 6%A, 4f - %A  - 3%A  - 10%A  - 10%A]
  ]
,
                4     3      2
[complexRoots= ?  + 6?  + 30?  + 36? + 36,
   coordinates =
              3      2                    3      2
    [30a - %A  - 5%A  - 30%A - 6, 6b + %A  + 5%A  + 24%A + 6,
             3      2              3      2
     30c - %A  - 5%A  - 6, 30d - %A  - 5%A  - 30%A - 6,
             3      2                    3      2
     30e - %A  - 5%A  - 30%A - 6, 30f - %A  - 5%A  - 30%A - 6]
  ]
,
                4     3      2
[complexRoots= ?  - 6?  + 30?  - 36? + 36,
   coordinates =
              3      2                    3      2
    [30a - %A  + 5%A  - 30%A + 6, 6b + %A  - 5%A  + 24%A - 6,
             3      2              3      2
     30c - %A  + 5%A  + 6, 30d - %A  + 5%A  - 30%A + 6,
             3      2                    3      2
     30e - %A  + 5%A  - 30%A + 6, 30f - %A  + 5%A  - 30%A + 6]
  ]
,
                2
[complexRoots= ?  + 6? + 6,
 coordinates= [a + 1,b - %A - 5,c + %A + 1,d + 1,e + 1,f + 1]]
```

```
,
                2
[complexRoots= ?  - 6? + 6,
 coordinates= [a - 1,b - %A + 5,c + %A - 1,d - 1,e - 1,f - 1]]
,
                4     3      2
[complexRoots= ?  + 6?  + 30?  + 36? + 36,
   coordinates =
             3      2                      3      2
    [6a + %A  + 5%A  + 24%A + 6, 30b - %A  - 5%A  - 6,
              3      2                      3      2
     30c - %A  - 5%A  - 30%A - 6, 30d - %A  - 5%A  - 30%A - 6,
              3      2                      3      2
     30e - %A  - 5%A  - 30%A - 6, 30f - %A  - 5%A  - 30%A - 6]
  ]
,
                4     3      2
[complexRoots= ?  - 6?  + 30?  - 36? + 36,
   coordinates =
             3      2                      3      2
    [6a + %A  - 5%A  + 24%A - 6, 30b - %A  + 5%A  + 6,
              3      2                      3      2
     30c - %A  + 5%A  - 30%A + 6, 30d - %A  + 5%A  - 30%A + 6,
              3      2                      3      2
     30e - %A  + 5%A  - 30%A + 6, 30f - %A  + 5%A  - 30%A + 6]
  ]
,
                2
[complexRoots= ?  + 6? + 6,
 coordinates= [a - %A - 5,b + %A + 1,c + 1,d + 1,e + 1,f + 1]]
,
                2
[complexRoots= ?  - 6? + 6,
 coordinates= [a - %A + 5,b + %A - 1,c - 1,d - 1,e - 1,f - 1]]
,
                4     3      2
[complexRoots= ?  + 6?  + 30?  + 36? + 36,
   coordinates =
             3      2                      3      2
    [30a - %A  - 5%A  - 30%A - 6, 30b - %A  - 5%A  - 30%A - 6,
              3      2                      3      2
     6c + %A  + 5%A  + 24%A + 6, 30d - %A  - 5%A  - 6,
              3      2                      3      2
     30e - %A  - 5%A  - 30%A - 6, 30f - %A  - 5%A  - 30%A - 6]
  ]
,
```

```
                4     3      2
[complexRoots= ?  - 6?  + 30?  - 36? + 36,
   coordinates =
             3      2                      3      2
    [30a - %A  + 5%A  - 30%A + 6, 30b - %A  + 5%A  - 30%A + 6,
             3      2                      3      2
     6c + %A  - 5%A  + 24%A - 6, 30d - %A  + 5%A  + 6,
             3      2                      3      2
     30e - %A  + 5%A  - 30%A + 6, 30f - %A  + 5%A  - 30%A + 6]
  ]
,
                2
[complexRoots= ?  + 6? + 6,
 coordinates= [a + 1,b + 1,c - %A - 5,d + %A + 1,e + 1,f + 1]]
,
                2
[complexRoots= ?  - 6? + 6,
 coordinates= [a - 1,b - 1,c - %A + 5,d + %A - 1,e - 1,f - 1]]
,
                8     7      6      5      4     2
[complexRoots= ?  + 6?  + 16?  + 24?  + 18?  - 8?  + 4,
   coordinates =
             7      6       5       4      3       2
    [2a + 2%A  + 9%A  + 18%A  + 19%A  + 4%A  - 10%A  - 2%A + 4,
             7      6       5       4      3       2
     2b + 2%A  + 9%A  + 18%A  + 19%A  + 4%A  - 10%A  - 4%A + 4,
             7      6      5      4      3
     2c - %A  - 4%A  - 8%A  - 9%A  - 4%A  - 2%A - 4,
             7      6      5      4      3
     2d + %A  + 4%A  + 8%A  + 9%A  + 4%A  + 2%A + 4,
             7      6       5       4      3       2
     2e - 2%A  - 9%A  - 18%A  - 19%A  - 4%A  + 10%A  + 4%A - 4,
             7      6       5       4      3       2
     2f - 2%A  - 9%A  - 18%A  - 19%A  - 4%A  + 10%A  + 2%A - 4]
  ]
,
 [
  complexRoots =
     8      7      6       5       4       3        2
    ?  + 12?  + 64?  + 192?  + 432?  + 768?  + 1024?  + 768? + 256
  ,
   coordinates =
    [
                    7        6        5         4         3         2
        1408a - 19%A  - 200%A  - 912%A  - 2216%A  - 4544%A  - 6784%A
      +
```

```
          - 6976%A - 1792
        ,
                       7        6         5         4          3          2
          1408b - 37%A  - 408%A  - 1952%A  - 5024%A  - 10368%A  - 16768%A
        +
          - 17920%A - 5120
        ,
                       7        6         5         4          3          2
          1408c + 37%A  + 408%A  + 1952%A  + 5024%A  + 10368%A  + 16768%A
        +
          17920%A + 5120
        ,
                       7        6        5         4         3         2
          1408d + 19%A  + 200%A  + 912%A  + 2216%A  + 4544%A  + 6784%A
        +
          6976%A + 1792
        ,
       2e + %A, 2f - %A]
  ]
,
                  8     6      4      2
[complexRoots= ?  + 4?  + 12?  + 16?  + 4,
   coordinates =
             7      5       3               7      5       3
    [4a - %A  - 3%A  - 10%A  - 6%A, 4b - %A  - 3%A  - 10%A  - 10%A,
              7      5       3                 7      5       3
     4c - 2%A  - 7%A  - 20%A  - 22%A, 4d + 2%A  + 7%A  + 20%A  + 22%A,
             7      5       3               7      5       3
     4e + %A  + 3%A  + 10%A  + 10%A, 4f + %A  + 3%A  + 10%A  + 6%A]
  ]
,
                  8      6      4       2
[complexRoots= ?  + 16?  - 96?  + 256?  + 256,
   coordinates =
              7       5        3
    [512a - %A  - 12%A  + 176%A  - 448%A,
              7       5       3
     128b - %A  - 16%A  + 96%A  - 256%A,
               7       5       3
     128c + %A  + 16%A  - 96%A  + 256%A,
              7       5        3
     512d + %A  + 12%A  - 176%A  + 448%A, 2e + %A, 2f - %A]
  ]
,
 [
  complexRoots =
```

```
      8      7      6       5       4       3        2
     ?  - 12?  + 64?  - 192?  + 432?  - 768?  + 1024?  - 768? + 256
  ,
    coordinates =
     [
                    7        6        5         4         3         2
        1408a - 19%A  + 200%A  - 912%A  + 2216%A  - 4544%A  + 6784%A
       +
        - 6976%A + 1792
       ,
                     7        6         5         4          3          2
        1408b - 37%A  + 408%A  - 1952%A  + 5024%A  - 10368%A  + 16768%A
       +
        - 17920%A + 5120
       ,
                     7        6         5         4          3          2
        1408c + 37%A  - 408%A  + 1952%A  - 5024%A  + 10368%A  - 16768%A
       +
        17920%A - 5120
       ,
                     7        6        5         4         3         2
        1408d + 19%A  - 200%A  + 912%A  - 2216%A  + 4544%A  - 6784%A
       +
        6976%A - 1792
       ,
      2e + %A, 2f - %A]
  ]
,
                  8     7      6      5      4     2
[complexRoots= ?  - 6?  + 16?  - 24?  + 18?  - 8?  + 4,
   coordinates =
              7      6       5       4      3       2
    [2a + 2%A  - 9%A  + 18%A  - 19%A  + 4%A  + 10%A  - 2%A - 4,
              7      6       5       4      3       2
     2b + 2%A  - 9%A  + 18%A  - 19%A  + 4%A  + 10%A  - 4%A - 4,
            7      6      5      4      3
     2c - %A  + 4%A  - 8%A  + 9%A  - 4%A  - 2%A + 4,
            7      6      5      4      3
     2d + %A  - 4%A  + 8%A  - 9%A  + 4%A  + 2%A - 4,
            7      6       5       4      3       2
     2e - 2%A  + 9%A  - 18%A  + 19%A  - 4%A  - 10%A  + 4%A + 4,
            7      6       5       4      3       2
     2f - 2%A  + 9%A  - 18%A  + 19%A  - 4%A  - 10%A  + 2%A + 4]
  ]
,
                  4      2
```

```
[complexRoots= ?  + 12?  + 144,
   coordinates =
                2                2                2                2
    [12a - %A  - 12, 12b - %A  - 12, 12c - %A  - 12, 12d - %A  - 12,
                2                2
     6e + %A  + 3%A + 12, 6f + %A  - 3%A + 12]
  ]
,
                  4     3      2
[complexRoots= ?  + 6?  + 30?  + 36? + 36,
   coordinates =
              3      2                     3      2
    [6a - %A  - 5%A  - 24%A - 6, 30b + %A  + 5%A  + 30%A + 6,
              3      2                     3      2
     30c + %A  + 5%A  + 30%A + 6, 30d + %A  + 5%A  + 30%A + 6,
              3      2                     3      2
     30e + %A  + 5%A  + 30%A + 6, 30f + %A  + 5%A  + 6]
  ]
,
                  4     3      2
[complexRoots= ?  - 6?  + 30?  - 36? + 36,
   coordinates =
              3      2                     3      2
    [6a - %A  + 5%A  - 24%A + 6, 30b + %A  - 5%A  + 30%A - 6,
              3      2                     3      2
     30c + %A  - 5%A  + 30%A - 6, 30d + %A  - 5%A  + 30%A - 6,
              3      2                     3      2
     30e + %A  - 5%A  + 30%A - 6, 30f + %A  - 5%A  - 6]
  ]
,
                  4      2
[complexRoots= ?  + 12?  + 144,
   coordinates =
                2                2                2                2
    [12a + %A  + 12, 12b + %A  + 12, 12c + %A  + 12, 12d + %A  + 12,
                2                2
     6e - %A  + 3%A - 12, 6f - %A  - 3%A - 12]
  ]
,
                  2
[complexRoots= ?  - 12,
 coordinates= [a - 1,b - 1,c - 1,d - 1,2e + %A + 4,2f - %A + 4]]
,
                  2
[complexRoots= ?  + 6? + 6,
 coordinates= [a + %A + 5,b - 1,c - 1,d - 1,e - 1,f - %A - 1]]
```

```
,
                  2
[complexRoots= ?  - 6? + 6,
 coordinates= [a + %A - 5,b + 1,c + 1,d + 1,e + 1,f - %A + 1]]
,
                  2
[complexRoots= ?  - 12,
 coordinates= [a + 1,b + 1,c + 1,d + 1,2e + %A - 4,2f - %A - 4]]
,
                  4     3      2
[complexRoots= ?  + 6?  + 30?  + 36? + 36,
   coordinates =
              3      2                      3      2
    [30a - %A  - 5%A  - 30%A - 6, 30b - %A  - 5%A  - 30%A - 6,
              3      2                     3      2
     30c - %A  - 5%A  - 30%A - 6, 6d + %A  + 5%A  + 24%A + 6,
              3      2              3      2
     30e - %A  - 5%A  - 6, 30f - %A  - 5%A  - 30%A - 6]
  ]
,
                  4     3      2
[complexRoots= ?  - 6?  + 30?  - 36? + 36,
   coordinates =
              3      2                      3      2
    [30a - %A  + 5%A  - 30%A + 6, 30b - %A  + 5%A  - 30%A + 6,
              3      2                     3      2
     30c - %A  + 5%A  - 30%A + 6, 6d + %A  - 5%A  + 24%A - 6,
              3      2              3      2
     30e - %A  + 5%A  + 6, 30f - %A  + 5%A  - 30%A + 6]
  ]
,
                  2
[complexRoots= ?  + 6? + 6,
 coordinates= [a + 1,b + 1,c + 1,d - %A - 5,e + %A + 1,f + 1]]
,
                  2
[complexRoots= ?  - 6? + 6,
 coordinates= [a - 1,b - 1,c - 1,d - %A + 5,e + %A - 1,f - 1]]
]
  Type: List Record(complexRoots: SparseUnivariatePolynomial Integer,
                    coordinates: List Polynomial Integer)
```

Since the univariateSolve operation may split a regular set, it returns a list. This explains the use of concat.

Look at the last item of the result. It consists of two parts. For

```
any complex root ? of the univariate polynomial in the first part, we
get a tuple of univariate polynomials (in a, ...,f respectively) by
replacing %A by ? in the second part.  Each of these tuples t
describes a point of the variety associated with lp by equaling to
zero the polynomials in t.

Note that the way of reading these univariate representations is
explained also in the example illustrating the ZeroDimensionalSolvePackage
constructor.

Now, we compute the points of the variety with real coordinates.

  concat [realSolve(ts)$zdpack for ts in lts]
  [[%B1,%B1,%B1,%B5,- %B5 - 4%B1,%B1], [%B1,%B1,%B1,%B6,- %B6 - 4%B1,%B1],
   [%B2,%B2,%B2,%B3,- %B3 - 4%B2,%B2], [%B2,%B2,%B2,%B4,- %B4 - 4%B2,%B2],
   [%B7,%B7,%B7,%B7,%B11,- %B11 - 4%B7], [%B7,%B7,%B7,%B7,%B12,- %B12 - 4%B7],
   [%B8,%B8,%B8,%B8,%B9,- %B9 - 4%B8], [%B8,%B8,%B8,%B8,%B10,- %B10 - 4%B8],
   [%B13,%B13,%B17,- %B17 - 4%B13,%B13,%B13],
   [%B13,%B13,%B18,- %B18 - 4%B13,%B13,%B13],
   [%B14,%B14,%B15,- %B15 - 4%B14,%B14,%B14],
   [%B14,%B14,%B16,- %B16 - 4%B14,%B14,%B14],
     [%B19, %B29,
           7865521      31   6696179241     25   25769893181     19
       ---------- %B19   - ---------- %B19   - ----------- %B19
       6006689520          2002229840            49235160
      +
          1975912990729     13   1048460696489     7   21252634831
       - ------------- %B19   - ------------- %B19  - ----------- %B19
           3003344760             2002229840           6006689520
      ,
            778171189       31   1987468196267     25   155496778477189     19
       - ------------- %B19   + ------------- %B19   + --------------- %B19
         1387545279120          1387545279120            693772639560
      +
         191631411158401     13   300335488637543     7   755656433863
         --------------- %B19   + --------------- %B19  - ------------ %B19
           693772639560            1387545279120          198220754160
      ,
           1094352947      31   2794979430821     25   218708802908737     19
         ------------ %B19   - ------------- %B19   - --------------- %B19
         462515093040          462515093040            231257546520
      +
          91476663003591     13   145152550961823     7   1564893370717
       - -------------- %B19   - --------------- %B19  - ------------- %B19
           77085848840             154171697680           462515093040
      ,
```

```
                4321823003      31   180949546069     25
    - %B29 - ------------- %B19   + ------------ %B19
             1387545279120           22746643920
  +
    863753195062493     19   1088094456732317     13
    --------------- %B19   + ---------------- %B19
      693772639560             693772639560
  +
    1732620732685741     7   13506088516033
    ---------------- %B19  + -------------- %B19
      1387545279120           1387545279120
  ]
,
  [%B19, %B30,
       7865521      31   6696179241     25   25769893181     19
    ---------- %B19   - ---------- %B19   - ----------- %B19
    6006689520          2002229840            49235160
  +
      1975912990729     13   1048460696489     7   21252634831
    - ------------- %B19   - ------------- %B19  - ----------- %B19
        3003344760             2002229840           6006689520
  ,
        778171189       31   1987468196267     25   155496778477189     19
    - ------------- %B19   + ------------- %B19   + --------------- %B19
      1387545279120          1387545279120            693772639560
  +
    191631411158401     13   300335488637543     7   755656433863
    --------------- %B19   + ---------------- %B19  - ------------ %B19
      693772639560            1387545279120          198220754160
  ,
      1094352947      31   2794979430821     25   218708802908737     19
    ------------ %B19   - ------------- %B19   - --------------- %B19
    462515093040          462515093040            231257546520
  +
      91476663003591     13   145152550961823     7   1564893370717
    - -------------- %B19   - --------------- %B19  - ------------- %B19
        77085848840             154171697680          462515093040
  ,
                4321823003      31   180949546069     25
    - %B30 - ------------- %B19   + ------------ %B19
             1387545279120           22746643920
  +
    863753195062493     19   1088094456732317     13
    --------------- %B19   + ---------------- %B19
      693772639560             693772639560
  +
```

```
    1732620732685741     7   13506088516033
    ---------------- %B19  + -------------- %B19
      1387545279120           1387545279120
  ]
,
  [%B20, %B27,
       7865521       31   6696179241     25   25769893181     19
    ---------- %B20   - ---------- %B20   - ----------- %B20
    6006689520          2002229840            49235160
  +
      1975912990729     13   1048460696489     7   21252634831
    - ------------- %B20   - ------------- %B20  - ----------- %B20
        3003344760             2002229840           6006689520
  ,
         778171189        31   1987468196267     25   155496778477189     19
    - ------------- %B20   + ------------- %B20   + --------------- %B20
      1387545279120          1387545279120            693772639560
  +
    191631411158401     13   300335488637543     7   755656433863
    --------------- %B20   + --------------- %B20  - ------------ %B20
      693772639560           1387545279120          198220754160
  ,
      1094352947      31   2794979430821     25   218708802908737     19
    ------------ %B20   - ------------- %B20   - --------------- %B20
    462515093040          462515093040            231257546520
  +
      91476663003591     13   145152550961823     7   1564893370717
    - -------------- %B20   - --------------- %B20  - ------------ %B20
        77085848840            154171697680           462515093040
  ,
                4321823003      31   180949546069     25
    - %B27 - ------------- %B20   + ------------ %B20
             1387545279120           22746643920
  +
    863753195062493     19   1088094456732317     13
    --------------- %B20   + ---------------- %B20
      693772639560           693772639560
  +
    1732620732685741     7   13506088516033
    ---------------- %B20  + -------------- %B20
      1387545279120           1387545279120
  ]
,
  [%B20, %B28,
       7865521       31   6696179241     25   25769893181     19
    ---------- %B20   - ---------- %B20   - ----------- %B20
```

```
      6006689520           2002229840             49235160
    +
        1975912990729      13   1048460696489      7   21252634831
      - ------------- %B20    - ------------- %B20  - ----------- %B20
          3003344760              2002229840            6006689520
    ,
           778171189        31   1987468196267      25   155496778477189      19
      - ------------- %B20    + ------------- %B20    + --------------- %B20
        1387545279120           1387545279120             693772639560
    +
      191631411158401      13   300335488637543      7   755656433863
      --------------- %B20    + --------------- %B20  - ------------ %B20
        693772639560             1387545279120           198220754160
    ,
        1094352947       31   2794979430821      25   218708802908737      19
      ------------ %B20    - ------------- %B20    - --------------- %B20
      462515093040             462515093040             231257546520
    +
        91476663003591      13   145152550961823      7   1564893370717
      - -------------- %B20    - --------------- %B20  - ------------- %B20
          77085848840              154171697680            462515093040
    ,
                  4321823003       31   180949546069      25
      - %B28 - ------------- %B20    + ------------ %B20
               1387545279120             22746643920
    +
      863753195062493      19   1088094456732317      13
      --------------- %B20    + ---------------- %B20
        693772639560              693772639560
    +
      1732620732685741      7   13506088516033
      ---------------- %B20  + -------------- %B20
        1387545279120            1387545279120
    ]
  ,
   [%B21, %B25,
        7865521       31   6696179241      25   25769893181      19
     ---------- %B21    - ---------- %B21    - ------------ %B21
     6006689520           2002229840             49235160
    +
        1975912990729      13   1048460696489      7   21252634831
      - ------------- %B21    - ------------- %B21  - ----------- %B21
          3003344760              2002229840            6006689520
    ,
           778171189        31   1987468196267      25   155496778477189      19
      - ------------- %B21    + ------------- %B21    + --------------- %B21
```

```
    1387545279120          1387545279120            693772639560
  +
    191631411158401     13   300335488637543     7   755656433863
    --------------- %B21   + --------------- %B21  - ------------ %B21
    693772639560            1387545279120          198220754160
  ,
    1094352947      31   2794979430821     25   218708802908737     19
    ------------ %B21   - ------------- %B21   - --------------- %B21
    462515093040           462515093040            231257546520
  +
    91476663003591     13   145152550961823     7   1564893370717
    - --------------- %B21   - --------------- %B21  - ------------- %B21
      77085848840             154171697680           462515093040
  ,
              4321823003      31   180949546069     25
    - %B25 - ------------- %B21   + ------------ %B21
             1387545279120           22746643920
  +
    863753195062493     19   1088094456732317     13
    --------------- %B21   + ---------------- %B21
    693772639560            693772639560
  +
    1732620732685741     7   13506088516033
    ---------------- %B21  + -------------- %B21
    1387545279120           1387545279120
  ]
,
  [%B21, %B26,
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    6006689520          2002229840            49235160
  +
    1975912990729     13   1048460696489     7   21252634831
    - ------------- %B21   - ------------- %B21  - ----------- %B21
      3003344760             2002229840           6006689520
  ,
      778171189       31   1987468196267     25   155496778477189     19
    - ------------- %B21   + ------------- %B21   + --------------- %B21
    1387545279120           1387545279120            693772639560
  +
    191631411158401     13   300335488637543     7   755656433863
    --------------- %B21   + --------------- %B21  - ------------ %B21
    693772639560            1387545279120          198220754160
  ,
    1094352947      31   2794979430821     25   218708802908737     19
    ------------ %B21   - ------------- %B21   - --------------- %B21
```

```
    462515093040            462515093040            231257546520
  +
      91476663003591     13   145152550961823     7   1564893370717
    - -------------- %B21   - --------------- %B21  - ------------- %B21
        77085848840             154171697680           462515093040
  ,
                4321823003      31   180949546069     25
    - %B26 - ------------- %B21   + ------------ %B21
             1387545279120            22746643920
  +
    863753195062493     19   1088094456732317     13
    --------------- %B21   + ---------------- %B21
      693772639560             693772639560
  +
    1732620732685741     7   13506088516033
    ---------------- %B21  + -------------- %B21
      1387545279120            1387545279120
  ]
,
 [%B22, %B23,
       7865521      31   6696179241     25   25769893181     19
    ---------- %B22   - ---------- %B22   - ----------- %B22
    6006689520           2002229840             49235160
  +
      1975912990729     13   1048460696489     7   21252634831
    - ------------- %B22   - ------------- %B22  - ----------- %B22
        3003344760             2002229840           6006689520
  ,
         778171189       31   1987468196267     25   155496778477189     19
    - ------------- %B22   + ------------- %B22   + --------------- %B22
      1387545279120           1387545279120             693772639560
  +
    191631411158401     13   300335488637543     7   755656433863
    --------------- %B22   + --------------- %B22  - ------------ %B22
      693772639560            1387545279120           198220754160
  ,
      1094352947      31   2794979430821     25   218708802908737     19
    ------------ %B22   - ------------- %B22   - --------------- %B22
    462515093040            462515093040            231257546520
  +
      91476663003591     13   145152550961823     7   1564893370717
    - -------------- %B22   - --------------- %B22  - ------------- %B22
        77085848840             154171697680           462515093040
  ,
                4321823003      31   180949546069     25
    - %B23 - ------------- %B22   + ------------ %B22
```

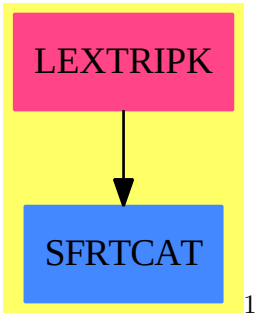
```
            1387545279120           22746643920
  +
    863753195062493     19   1088094456732317     13
    --------------- %B22   + ---------------- %B22
      693772639560             693772639560
  +
    1732620732685741     7   13506088516033
    ---------------- %B22  + -------------- %B22
      1387545279120           1387545279120
  ]
,
  [%B22, %B24,
       7865521      31   6696179241     25   25769893181     19
    ---------- %B22   - ---------- %B22   - ----------- %B22
    6006689520          2002229840            49235160
  +
      1975912990729     13   1048460696489     7   21252634831
    - ------------- %B22   - ------------- %B22  - ----------- %B22
        3003344760             2002229840           6006689520
  ,
         778171189       31   1987468196267     25   155496778477189     19
    - ------------- %B22   + ------------- %B22   + --------------- %B22
      1387545279120          1387545279120            693772639560
  +
    191631411158401     13   300335488637543     7   755656433863
    --------------- %B22   + --------------- %B22  - ------------ %B22
      693772639560           1387545279120          198220754160
  ,
      1094352947      31   2794979430821     25   218708802908737     19
    ------------ %B22   - ------------- %B22   - --------------- %B22
    462515093040           462515093040            231257546520
  +
      91476663003591     13   145152550961823     7   1564893370717
    - -------------- %B22   - --------------- %B22  - ------------- %B22
        77085848840            154171697680           462515093040
  ,
               4321823003      31   180949546069     25
    - %B24 - ------------- %B22   + ------------ %B22
             1387545279120           22746643920
  +
    863753195062493     19   1088094456732317     13
    --------------- %B22   + ---------------- %B22
      693772639560             693772639560
  +
    1732620732685741     7   13506088516033
    ---------------- %B22  + -------------- %B22
```

```
            1387545279120            1387545279120
       ]
     ,
    [%B31,%B35,- %B35 - 4%B31,%B31,%B31,%B31],
    [%B31,%B36,- %B36 - 4%B31,%B31,%B31,%B31],
    [%B32,%B33,- %B33 - 4%B32,%B32,%B32,%B32],
    [%B32,%B34,- %B34 - 4%B32,%B32,%B32,%B32]]
                                  Type: List List RealClosure Fraction Integer
```

We obtain 24 points given by lists of elements in the RealClosure
of Fraction of R. In each list, the first value corresponds
to the indeterminate f, the second to e and so on.

See Also:
o)help RegularChain
o)help RegularTriangularSet
o)help SquareFreeRegularTriangularSet
o)help ZeroDimensionalSolvePackage
o)help NewSparseMultivariatePolynomial
o)help LinGroebnerPackage
o)help NormalizedTriangularSetCategory
o)help RealClosure
o)help Fraction
o)show LexTriangularPackage

13.8 LexTriangularPackage



1.00

Exports:

fglmIfCan	groebner	lexTriangular
squareFreeLexTriangular	zeroDimensional?	zeroSetSplit

⟨*package LEXTRIPK LexTriangularPackage*⟩≡

```
)abbrev package LEXTRIPK LexTriangularPackage
++ Author: Marc Moreno Maza
++ Date Created: 08/02/1999
++ Date Last Updated: 08/02/1999
++ Basic Functions:
++ Related Constructors:
++ Also See:
++ AMS Classifications:
++ Keywords:
++ Description:
++ A package for solving polynomial systems with finitely many solutions.
++ The decompositions are given by means of regular triangular sets.
++ The computations use lexicographical Groebner bases.
++ The main operations are \axiomOpFrom{lexTriangular}{LexTriangularPackage}
++ and \axiomOpFrom{squareFreeLexTriangular}{LexTriangularPackage}.
++ The second one provide decompositions by means of square-free regular triangul
++ Both are based on the {\em lexTriangular} method described in [1].
++ They differ from the algorithm described in [2] by the fact that
++ multiciplities of the roots are not kept.
++ With the \axiomOpFrom{squareFreeLexTriangular}{LexTriangularPackage} operation
++ all multiciplities are removed. With the other operation some multiciplities m
++ Both operations admit an optional argument to produce normalized triangular se
++ References: \newline
++ [1] D. LAZARD "Solving Zero-dimensional Algebraic Systems"
++ published in the J. of Symbol. Comput. (1992) 13, 117-131.\newline
++ [2] M. MORENO MAZA and R. RIOBOO "Computations of gcd over
++ algebraic towers of simple extensions" In proceedings of AAECC11, Paris, 1995.
++ Version: 2.
```

```
LexTriangularPackage(R,ls): Exports == Implementation where

  R: GcdDomain
  ls: List Symbol
  V ==> OrderedVariableList ls
  E ==> IndexedExponents V
  P ==> NewSparseMultivariatePolynomial(R,V)
  TS  ==> RegularChain(R,ls)
  ST ==> SquareFreeRegularTriangularSet(R,E,V,P)
  Q1 ==> Polynomial R
  PS ==> GeneralPolynomialSet(R,E,V,P)
  N ==> NonNegativeInteger
  Z ==> Integer
  B ==> Boolean
  S ==> String
  K ==> Fraction R
  LP ==> List P
  BWTS ==> Record(val : Boolean, tower : TS)
  LpWTS ==> Record(val : (List P), tower : TS)
  BWST ==> Record(val : Boolean, tower : ST)
  LpWST ==> Record(val : (List P), tower : ST)
  polsetpack ==> PolynomialSetUtilitiesPackage(R,E,V,P)
  quasicomppackTS ==> QuasiComponentPackage(R,E,V,P,TS)
  regsetgcdpackTS ==> SquareFreeRegularTriangularSetGcdPackage(R,E,V,P,TS)
  normalizpackTS ==> NormalizationPackage(R,E,V,P,TS)
  quasicomppackST ==> QuasiComponentPackage(R,E,V,P,ST)
  regsetgcdpackST ==> SquareFreeRegularTriangularSetGcdPackage(R,E,V,P,ST)
  normalizpackST ==> NormalizationPackage(R,E,V,P,ST)

  Exports ==  with

     zeroDimensional?: LP -> B
         ++ \axiom{zeroDimensional?(lp)} returns true iff
         ++ \axiom{lp} generates a zero-dimensional ideal
         ++ w.r.t. the variables involved in \axiom{lp}.
     fglmIfCan:  LP -> Union(LP, "failed")
         ++ \axiom{fglmIfCan(lp)} returns the lexicographical Groebner
         ++ basis of \axiom{lp} by using the {\em FGLM} strategy,
         ++ if \axiom{zeroDimensional?(lp)} holds .
     groebner: LP -> LP
         ++ \axiom{groebner(lp)} returns the lexicographical Groebner
         ++ basis of \axiom{lp}. If \axiom{lp} generates a zero-dimensional
         ++ ideal then the {\em FGLM} strategy is used, otherwise
         ++ the {\em Sugar} strategy is used.
     lexTriangular: (LP, B) -> List TS
         ++ \axiom{lexTriangular(base, norm?)} decomposes the variety
```

```
        ++ associated with \axiom{base} into regular chains.
        ++ Thus a point belongs to this variety iff it is a regular
        ++ zero of a regular set in in the output.
        ++ Note that \axiom{base} needs to be a lexicographical Groebner basis
        ++ of a zero-dimensional ideal. If \axiom{norm?} is \axiom{true}
        ++ then the regular sets are normalized.
    squareFreeLexTriangular: (LP, B) -> List ST
        ++ \axiom{squareFreeLexTriangular(base, norm?)} decomposes the variety
        ++ associated with \axiom{base} into square-free regular chains.
        ++ Thus a point belongs to this variety iff it is a regular
        ++ zero of a regular set in in the output.
        ++ Note that \axiom{base} needs to be a lexicographical Groebner basis
        ++ of a zero-dimensional ideal. If \axiom{norm?} is \axiom{true}
        ++ then the regular sets are normalized.
    zeroSetSplit: (LP, B) -> List TS
        ++ \axiom{zeroSetSplit(lp, norm?)} decomposes the variety
        ++ associated with \axiom{lp} into regular chains.
        ++ Thus a point belongs to this variety iff it is a regular
        ++ zero of a regular set in in the output.
        ++ Note that \axiom{lp} needs to generate a zero-dimensional ideal.
        ++ If \axiom{norm?} is \axiom{true} then the regular sets are normalized
    zeroSetSplit: (LP, B) -> List ST
        ++ \axiom{zeroSetSplit(lp, norm?)} decomposes the variety
        ++ associated with \axiom{lp} into square-free regular chains.
        ++ Thus a point belongs to this variety iff it is a regular
        ++ zero of a regular set in in the output.
        ++ Note that \axiom{lp} needs to generate a zero-dimensional ideal.
        ++ If \axiom{norm?} is \axiom{true} then the regular sets are normalized

Implementation == add

    trueVariables(lp: List(P)): List Symbol ==
      lv: List V := variables([lp]$PS)
      truels: List Symbol := []
      for s in ls repeat
        if member?(variable(s)::V, lv) then truels := cons(s,truels)
      reverse truels

    zeroDimensional?(lp:List(P)): Boolean ==
      truels: List Symbol := trueVariables(lp)
      fglmpack := FGLMIfCanPackage(R,truels)
      lq1: List(Q1) := [p::Q1 for p in lp]
      zeroDimensional?(lq1)$fglmpack

    fglmIfCan(lp:List(P)): Union(List(P), "failed") ==
      truels: List Symbol := trueVariables(lp)
```

```
    fglmpack := FGLMIfCanPackage(R,truels)
    lq1: List(Q1) := [p::Q1 for p in lp]
    foo := fglmIfCan(lq1)$fglmpack
    foo case "failed" => return("failed" :: Union(List(P), "failed"))
    lp := [retract(q1)$P for q1 in (foo :: List(Q1))]
    lp::Union(List(P), "failed")

  groebner(lp:List(P)): List(P) ==
    truels: List Symbol := trueVariables(lp)
    fglmpack := FGLMIfCanPackage(R,truels)
    lq1: List(Q1) := [p::Q1 for p in lp]
    lq1 := groebner(lq1)$fglmpack
    lp := [retract(q1)$P for q1 in lq1]

  lexTriangular(base: List(P), norm?: Boolean): List(TS) ==
    base := sort(infRittWu?,base)
    base := remove(zero?, base)
    any?(ground?, base) => []
    ts: TS := empty()
    toSee: List LpWTS := [[base,ts]$LpWTS]
    toSave: List TS := []
    while not empty? toSee repeat
      lpwt := first toSee; toSee := rest toSee
      lp := lpwt.val; ts := lpwt.tower
      empty? lp => toSave := cons(ts, toSave)
      p := first lp; lp := rest lp; v := mvar(p)
      algebraic?(v,ts) =>
        error "lexTriangular$LEXTRIPK: should never happen !"
      norm? and zero? remainder(init(p),ts).polnum =>
        toSee := cons([lp, ts]$LpWTS, toSee)
      (not norm?) and zero? (initiallyReduce(init(p),ts)) =>
        toSee := cons([lp, ts]$LpWTS, toSee)
      lbwt: List BWTS := invertible?(init(p),ts)$TS
      while (not empty? lbwt) repeat
        bwt := first lbwt; lbwt := rest lbwt
        b := bwt.val; us := bwt.tower
        (not b) => toSee := cons([lp, us], toSee)
        lus: List TS
        if norm?
          then
            newp := normalizedAssociate(p,us)$normalizpackTS
            lus := [internalAugment(newp,us)$TS]
          else
            newp := p
            lus := augment(newp,us)$TS
        newlp := lp
```

```
      while (not empty? newlp) and (mvar(first newlp) = v) repeat
        newlp := rest newlp
      for us in lus repeat
        toSee := cons([newlp, us]$LpWTS, toSee)
  algebraicSort(toSave)$quasicomppackTS

zeroSetSplit(lp:List(P), norm?:B): List TS ==
  bar := fglmIfCan(lp)
  bar case "failed" =>
    error "zeroSetSplit$LEXTRIPK: #1 not zero-dimensional"
  lexTriangular(bar::(List P),norm?)

squareFreeLexTriangular(base: List(P), norm?: Boolean): List(ST) ==
  base := sort(infRittWu?,base)
  base := remove(zero?, base)
  any?(ground?, base) => []
  ts: ST := empty()
  toSee: List LpWST := [[base,ts]$LpWST]
  toSave: List ST := []
  while not empty? toSee repeat
    lpwt := first toSee; toSee := rest toSee
    lp := lpwt.val; ts := lpwt.tower
    empty? lp => toSave := cons(ts, toSave)
    p := first lp; lp := rest lp; v := mvar(p)
    algebraic?(v,ts) =>
      error "lexTriangular$LEXTRIPK: should never happen !"
    norm? and zero? remainder(init(p),ts).polnum =>
      toSee := cons([lp, ts]$LpWST, toSee)
    (not norm?) and zero? (initiallyReduce(init(p),ts)) =>
      toSee := cons([lp, ts]$LpWST, toSee)
    lbwt: List BWST := invertible?(init(p),ts)$ST
    while (not empty? lbwt) repeat
      bwt := first lbwt; lbwt := rest lbwt
      b := bwt.val; us := bwt.tower
      (not b) => toSee := cons([lp, us], toSee)
      lus: List ST
      if norm?
        then
          newp := normalizedAssociate(p,us)$normalizpackST
          lus := augment(newp,us)$ST
        else
          lus := augment(p,us)$ST
      newlp := lp
      while (not empty? newlp) and (mvar(first newlp) = v) repeat
        newlp := rest newlp
      for us in lus repeat
```

```
            toSee := cons([newlp, us]$LpWST, toSee)
        algebraicSort(toSave)$quasicomppackST

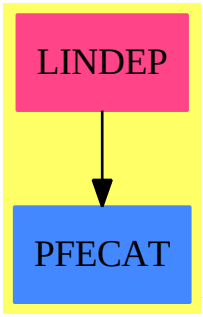
      zeroSetSplit(lp:List(P), norm?:B): List ST ==
        bar := fglmIfCan(lp)
        bar case "failed" =>
          error "zeroSetSplit$LEXTRIPK: #1 not zero-dimensional"
        squareFreeLexTriangular(bar::(List P),norm?)
```

⟨*LEXTRIPK.dotabb*⟩≡

```
"LEXTRIPK" [color="#FF4488",href="bookvol10.4.pdf#nameddest=LEXTRIPK"]
"SFRTCAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=SFRTCAT"]
"LEXTRIPK" -> "SFRTCAT"
```

13.9 package LINDEP LinearDependence

13.10 LinearDependence



1.00

Exports:
linearDependence linearlyDependent? solveLinear

⟨*package LINDEP LinearDependence*⟩≡
```
)abbrev package LINDEP LinearDependence
++ Test for linear dependence
++ Author: Manuel Bronstein
++ Date Created: ???
++ Date Last Updated: 14 May 1991
++ Description: Test for linear dependence.
LinearDependence(S, R): Exports == Implementation where
  S: IntegralDomain
  R: LinearlyExplicitRingOver S

  Q ==> Fraction S

  Exports ==> with
    linearlyDependent?: Vector R -> Boolean
      ++ \spad{linearlyDependent?([v1,...,vn])} returns true if
      ++ the vi's are linearly dependent over S, false otherwise.
    linearDependence  : Vector R -> Union(Vector S, "failed")
      ++ \spad{linearDependence([v1,...,vn])} returns \spad{[c1,...,cn]} if
      ++ \spad{c1*v1 + ... + cn*vn = 0} and not all the ci's are 0,
      ++ "failed" if the vi's are linearly independent over S.
    if S has Field then
      solveLinear: (Vector R, R) -> Union(Vector S, "failed")
        ++ \spad{solveLinear([v1,...,vn], u)} returns \spad{[c1,...,cn]}
        ++ such that \spad{c1*v1 + ... + cn*vn = u},
        ++ "failed" if no such ci's exist in S.
    else
      solveLinear: (Vector R, R) -> Union(Vector Q, "failed")
```

```
        ++ \spad{solveLinear([v1,...,vn], u)} returns \spad{[c1,...,cn]}
        ++ such that \spad{c1*v1 + ... + cn*vn = u},
        ++ "failed" if no such ci's exist in the quotient field of S.

  Implementation ==> add
    aNonZeroSolution: Matrix S -> Union(Vector S, "failed")

    aNonZeroSolution m ==
      every?(zero?, v := first nullSpace m) => "failed"
      v

    linearlyDependent? v ==
      zero?(n := #v) => true
--      one? n => zero?(v(minIndex v))
      (n = 1) => zero?(v(minIndex v))
      positive? nullity reducedSystem transpose v

    linearDependence v ==
      zero?(n := #v) => empty()
--      one? n =>
      (n = 1) =>
        zero?(v(minIndex v)) => new(1, 1)
        "failed"
      aNonZeroSolution reducedSystem transpose v

    if S has Field then
      solveLinear(v:Vector R, c:R):Union(Vector S, "failed") ==
        zero? c => new(#v, 0)
        empty? v => "failed"
        sys := reducedSystem(transpose v, new(1, c))
        particularSolution(sys.mat, sys.vec)$LinearSystemMatrixPackage(S,
                                          Vector S, Vector S, Matrix S)

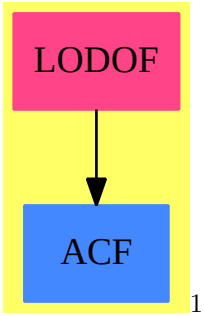
    else
      solveLinear(v:Vector R, c:R):Union(Vector Q, "failed") ==
        zero? c => new(#v, 0)
        empty? v => "failed"
        sys := reducedSystem(transpose v, new(1, c))
        particularSolution(map((z:S):Q+->z::Q, sys.mat)_
          $MatrixCategoryFunctions2(S,
               Vector S,Vector S,Matrix S,Q,Vector Q,Vector Q,Matrix Q),
                  map((z1:S):Q+->z1::Q, sys.vec)$VectorFunctions2(S, Q)
                                  )$LinearSystemMatrixPackage(Q,
                                          Vector Q, Vector Q, Matrix Q)
```

⟨*LINDEP.dotabb*⟩≡

```
"LINDEP" [color="#FF4488",href="bookvol10.4.pdf#nameddest=LINDEP"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"LINDEP" -> "PFECAT"
```

13.11 package LODOF LinearOrdinaryDifferentialOperatorFactorizer

13.12 LinearOrdinaryDifferentialOperatorFactorizer



1.00

Exports:
factor factor1

⟨*package LODOF LinearOrdinaryDifferentialOperatorFactorizer*⟩≡

```
)abbrev package LODOF LinearOrdinaryDifferentialOperatorFactorizer
++ Author: Fritz Schwarz, Manuel Bronstein
++ Date Created: 1988
++ Date Last Updated: 3 February 1994
++ Description:
++ \spadtype{LinearOrdinaryDifferentialOperatorFactorizer} provides a
++ factorizer for linear ordinary differential operators whose coefficients
++ are rational functions.
++ Keywords: differential equation, ODE, LODO, factoring
LinearOrdinaryDifferentialOperatorFactorizer(F, UP): Exports == Impl where
  F : Join(Field, CharacteristicZero,
           RetractableTo Integer, RetractableTo Fraction Integer)
  UP: UnivariatePolynomialCategory F

  RF ==> Fraction UP
  L  ==> LinearOrdinaryDifferentialOperator1 RF

  Exports ==> with
    factor: (L, UP -> List F) -> List L
      ++ factor(a, zeros) returns the factorisation of a.
      ++ \spad{zeros} is a zero finder in \spad{UP}.
    if F has AlgebraicallyClosedField then
      factor: L -> List L
        ++ factor(a) returns the factorisation of a.
      factor1: L -> List L
        ++ factor1(a) returns the factorisation of a,
```

```
        ++ assuming that a has no first-order right factor.

  Impl ==> add
    import RationalLODE(F, UP)
    import RationalRicDE(F, UP)
--  import AssociatedEquations RF

    dd := D()$L

    expsol     : (L, UP -> List F, UP -> Factored UP) -> Union(RF, "failed")
    expsols    : (L, UP -> List F, UP -> Factored UP, Boolean) -> List RF
    opeval     : (L, L) -> L
    recurfactor: (L, L, UP -> List F, UP -> Factored UP, Boolean) -> List L
    rfactor    : (L, L, UP -> List F, UP -> Factored UP, Boolean) -> List L
    rightFactor: (L, NonNegativeInteger, UP -> List F, UP -> Factored UP)
                                                        -> Union(L, "failed")
    innerFactor: (L, UP -> List F, UP -> Factored UP, Boolean) -> List L

    factor(l, zeros) == innerFactor(l, zeros, squareFree, true)

    expsol(l, zeros, ezfactor) ==
      empty?(sol := expsols(l, zeros, ezfactor, false)) => "failed"
      first sol

    expsols(l, zeros, ezfactor, all?) ==
      sol := [differentiate(f)/f for f in ratDsolve(l, 0).basis | f ^= 0]
      not(all? or empty? sol) => sol
      concat(sol, ricDsolve(l, zeros, ezfactor))

-- opeval(l1, l2) returns l1(l2)
    opeval(l1, l2) ==
      ans:L := 0
      l2n:L := 1
      for i in 0..degree l1 repeat
        ans := ans + coefficient(l1, i) * l2n
        l2n := l2 * l2n
      ans

    recurfactor(l, r, zeros, ezfactor, adj?) ==
      q := rightExactQuotient(l, r)::L
      if adj? then q := adjoint q
      innerFactor(q, zeros, ezfactor, true)

    rfactor(op, r, zeros, ezfactor, adj?) ==
--      degree r > 1 or not one? leadingCoefficient r =>
      degree r > 1 or not ((leadingCoefficient r) = 1) =>
```

```
        recurfactor(op, r, zeros, ezfactor, adj?)
      op1 := opeval(op, dd - coefficient(r, 0)::L)
      map_!((z:L):L+->opeval(z,r), recurfactor(op1, dd, zeros, ezfactor, adj?))

  -- r1? is true means look for 1st-order right-factor also
    innerFactor(l, zeros, ezfactor, r1?) ==
      (n := degree l) <= 1 => [l]
      ll := adjoint l
      for i in 1..(n quo 2) repeat
        (r1? or (i > 1)) and ((u := rightFactor(l,i,zeros,ezfactor)) case L) =>
           return concat_!(rfactor(l, u::L, zeros, ezfactor, false), u::L)
        (2 * i < n) and ((u := rightFactor(ll, i, zeros, ezfactor)) case L) =>
           return concat(adjoint(u::L), rfactor(ll, u::L, zeros,ezfactor,true))
      [l]

    rightFactor(l, n, zeros, ezfactor) ==
  --    one? n =>
      (n = 1) =>
        (u := expsol(l, zeros, ezfactor)) case "failed" => "failed"
        D() - u::RF::L
  --  rec := associatedEquations(l, n::PositiveInteger)
  --  empty?(sol := expsols(rec.eq, zeros, ezfactor, true)) => "failed"
      "failed"

    if F has AlgebraicallyClosedField then
      zro1: UP -> List F
      zro : (UP, UP -> Factored UP) -> List F

      zro(p, ezfactor) ==
        concat [zro1(r.factor) for r in factors ezfactor p]

      zro1 p ==
        [zeroOf(map((z1:F):F+->z1,p)_
          $UnivariatePolynomialCategoryFunctions2(F, UP,
                                          F, SparseUnivariatePolynomial F))]

      if F is AlgebraicNumber then
        import AlgFactor UP

        factor l  ==
          innerFactor(l,(p:UP):List(F)+->zro(p,factor),factor,true)
        factor1 l ==
          innerFactor(l,(p:UP):List(F)+->zro(p,factor),factor,false)

      else
        factor l  ==
```

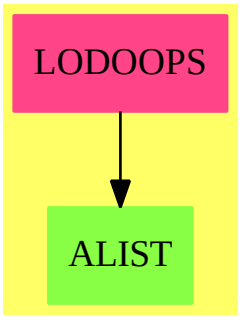
```
      innerFactor(l,(p:UP):List(F)+->zro(p,squareFree),squareFree,true)
    factor1 l ==
      innerFactor(l,(p:UP):List(F)+->zro(p,squareFree),squareFree,false)
```

⟨*LODOF.dotabb*⟩≡

```
"LODOF" [color="#FF4488",href="bookvol10.4.pdf#nameddest=LODOF"]
"ACF" [color="#4488FF",href="bookvol10.2.pdf#nameddest=ACF"]
"LODOF" -> "ACF"
```

13.13 package LODOOPS LinearOrdinaryDifferentialOperatorsOps

13.14 LinearOrdinaryDifferentialOperatorsOps



1.00

Exports:
directSum symmetricPower symmetricProduct

⟨*package LODOOPS LinearOrdinaryDifferentialOperatorsOps*⟩≡

```
)abbrev package LODOOPS LinearOrdinaryDifferentialOperatorsOps
++ Author: Manuel Bronstein
++ Date Created: 18 January 1994
++ Date Last Updated: 15 April 1994
++ Description:
++   \spad{LinearOrdinaryDifferentialOperatorsOps} provides symmetric
++   products and sums for linear ordinary differential operators.
-- Putting those operations here rather than defaults in LODOCAT allows
-- LODOCAT to be defined independently of the derivative used.
-- MB 1/94
LinearOrdinaryDifferentialOperatorsOps(A, L): Exports == Implementation where
    A: Field
    L: LinearOrdinaryDifferentialOperatorCategory A

    N  ==> NonNegativeInteger
    V  ==> OrderlyDifferentialVariable Symbol
    P  ==> DifferentialSparseMultivariatePolynomial(A, Symbol, V)

    Exports ==> with
          symmetricProduct: (L, L, A -> A) -> L
            ++ symmetricProduct(a,b,D) computes an operator \spad{c} of
            ++ minimal order such that the nullspace of \spad{c} is
            ++ generated by all the products of a solution of \spad{a} by
            ++ a solution of \spad{b}.
            ++ D is the derivation to use.
          symmetricPower: (L, N, A -> A) -> L
```

```
            ++ symmetricPower(a,n,D) computes an operator \spad{c} of
            ++ minimal order such that the nullspace of \spad{c} is
            ++ generated by all the products of \spad{n} solutions
            ++ of \spad{a}.
            ++ D is the derivation to use.
          directSum: (L, L, A -> A) -> L
            ++ directSum(a,b,D) computes an operator \spad{c} of
            ++ minimal order such that the nullspace of \spad{c} is
            ++ generated by all the sums of a solution of \spad{a} by
            ++ a solution of \spad{b}.
            ++ D is the derivation to use.

    Implementation ==> add
          import IntegerCombinatoricFunctions

          var1 := new()$Symbol
          var2 := new()$Symbol

          nonTrivial?: Vector A -> Boolean
          applyLODO  : (L, V) -> P
          killer     : (P, N, List V, List P, A -> A) -> L
          vec2LODO   : Vector A -> L

          nonTrivial? v == any?((x1:A):Boolean +-> x1 ^= 0, v)$Vector(A)
          vec2LODO v    == +/[monomial(v.i, (i-1)::N) for i in 1..#v]

          symmetricPower(l, m, diff) ==
            u := var1::V; n := degree l
            un := differentiate(u, n)
            a  := applyLODO(inv(- leadingCoefficient l) * reductum l, u)
            killer(u::P ** m, binomial(n + m - 1, n - 1)::N, [un], [a], diff)

-- returns an operator L such that L(u) = 0, for a given differential
-- polynomial u, given that the differential variables appearing in u
-- satisfy some linear ode's
-- m is a bound on the order of the operator searched.
-- lvar, lval describe the substitution(s) to perform when differentiating
--     the expression u (they encode the fact the the differential variables
--     satisfy some differential equations, which can be seen as the rewrite
--     rules   lvar --> lval)
-- diff is the derivation to use
          killer(u, m, lvar, lval, diff) ==
            lu:List P := [u]
            for q in 0..m repeat
              mat := reducedSystem(matrix([lu])@Matrix(P))@Matrix(A)
              (sol := find(nonTrivial?, l := nullSpace mat)) case Vector(A) =>
```

```
        return vec2LODO(sol::Vector(A))
      u := eval(differentiate(u, diff), lvar, lval)
      lu := concat_!(lu, [u])
    error "killer: no linear dependence found"

symmetricProduct(l1, l2, diff) ==
  u  := var1::V;   v  := var2::V
  n1 := degree l1; n2 := degree l2
  un := differentiate(u, n1); vn := differentiate(v, n2)
  a  := applyLODO(inv(- leadingCoefficient l1) * reductum l1, u)
  b  := applyLODO(inv(- leadingCoefficient l2) * reductum l2, v)
  killer(u::P * v::P, n1 * n2, [un, vn], [a, b], diff)

directSum(l1, l2, diff) ==
  u  := var1::V;   v  := var2::V
  n1 := degree l1; n2 := degree l2
  un := differentiate(u, n1); vn := differentiate(v, n2)
  a  := applyLODO(inv(- leadingCoefficient l1) * reductum l1, u)
  b  := applyLODO(inv(- leadingCoefficient l2) * reductum l2, v)
  killer(u::P + v::P, n1 + n2, [un, vn], [a, b], diff)

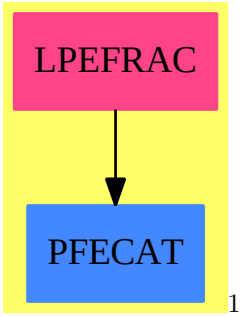
applyLODO(l, v) ==
  p:P := 0
  while l ^= 0 repeat
    p := p + monomial(leadingCoefficient(l)::P,
                      differentiate(v, degree l), 1)
    l := reductum l
  p
```

⟨*LODOOPS.dotabb*⟩≡

```
"LODOOPS" [color="#FF4488",href="bookvol10.4.pdf#nameddest=LODOOPS"]
"ALIST" [color="#88FF44",href="bookvol10.3.pdf#nameddest=ALIST"]
"LODOOPS" -> "ALIST"
```

13.15 package LPEFRAC LinearPolynomialEquationByFractions

13.16 LinearPolynomialEquationByFractions



1.00

Exports:
solveLinearPolynomialEquationByFractions

⟨*package LPEFRAC LinearPolynomialEquationByFractions*⟩≡

```
)abbrev package LPEFRAC LinearPolynomialEquationByFractions
++ Author: James Davenport
++ Date Created:
++ Date Last Updated:
++ Basic Functions:
++ Related Constructors:
++ Also See:
++ AMS Classifications:
++ Keywords:
++ References:
++ Description:
++ Given a PolynomialFactorizationExplicit ring, this package
++ provides a defaulting rule for the \spad{solveLinearPolynomialEquation}
++ operation, by moving into the field of fractions, and solving it there
++ via the \spad{multiEuclidean} operation.
LinearPolynomialEquationByFractions(R:PolynomialFactorizationExplicit): with
  solveLinearPolynomialEquationByFractions: ( _
           List SparseUnivariatePolynomial R, _
           SparseUnivariatePolynomial R) ->   _
           Union(List SparseUnivariatePolynomial R, "failed")
         ++ solveLinearPolynomialEquationByFractions([f1, ..., fn], g)
         ++ (where the fi are relatively prime to each other)
         ++ returns a list of ai such that
         ++ \spad{g/prod fi = sum ai/fi}
         ++ or returns "failed" if no such exists.
 == add
```

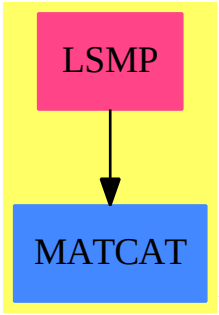
```
SupR ==> SparseUnivariatePolynomial R
F ==> Fraction R
SupF ==> SparseUnivariatePolynomial F
import UnivariatePolynomialCategoryFunctions2(R,SupR,F,SupF)
lp : List SupR
pp: SupR
pF: SupF
pullback : SupF -> Union(SupR,"failed")
pullback(pF) ==
  pF = 0 => 0
  c:=retractIfCan leadingCoefficient pF
  c case "failed" => "failed"
  r:=pullback reductum pF
  r case "failed" => "failed"
  monomial(c,degree pF) + r
solveLinearPolynomialEquationByFractions(lp,pp) ==
  lpF:List SupF:=[map(#1@R::F,u) for u in lp]
  pF:SupF:=map(#1@R::F,pp)
  ans:= solveLinearPolynomialEquation(lpF,pF)$F
  ans case "failed" => "failed"
  [(vv:= pullback v;
    vv case "failed" => return "failed";
     vv)
      for v in ans]
```

⟨*LPEFRAC.dotabb*⟩≡

```
"LPEFRAC" [color="#FF4488",href="bookvol10.4.pdf#nameddest=LPEFRAC"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"LPEFRAC" -> "PFECAT"
```

13.17 package LSMP LinearSystemMatrixPackage

13.18 LinearSystemMatrixPackage



1.00

Exports:
hasSolution? particularSolution rank solve

⟨*package LSMP LinearSystemMatrixPackage*⟩≡

```
)abbrev package LSMP LinearSystemMatrixPackage
++ Author: P.Gianni, S.Watt
++ Date Created: Summer 1985
++ Date Last Updated:Summer 1990
++ Basic Functions: solve, particularSolution, hasSolution?, rank
++ Related Constructors: LinearSystemMatrixPackage1
++ Also See:
++ AMS Classifications:
++ Keywords:
++ References:
++ Description:
++ This package solves linear system in the matrix form \spad{AX = B}.

LinearSystemMatrixPackage(F, Row, Col, M): Cat == Capsule where
    F: Field
    Row: FiniteLinearAggregate F with shallowlyMutable
    Col: FiniteLinearAggregate F with shallowlyMutable
    M  : MatrixCategory(F, Row, Col)

    N        ==> NonNegativeInteger
    PartialV ==> Union(Col, "failed")
    Both     ==> Record(particular: PartialV, basis: List Col)

    Cat ==> with
        solve       : (M, Col) -> Both
          ++  solve(A,B) finds a particular solution of the system \spad{AX = B}
```

```
      ++  and a basis of the associated homogeneous system \spad{AX = 0}.
    solve       : (M, List Col) -> List Both
      ++  solve(A,LB) finds a particular soln of the systems \spad{AX = B}
      ++  and a basis of the associated homogeneous systems \spad{AX = 0}
      ++  where B varies in the list of column vectors LB.

    particularSolution: (M, Col) -> PartialV
      ++ particularSolution(A,B) finds a particular solution of the linear
      ++ system \spad{AX = B}.
    hasSolution?: (M, Col) -> Boolean
      ++ hasSolution?(A,B) tests if the linear system \spad{AX = B}
      ++ has a solution.
    rank        : (M, Col) -> N
      ++ rank(A,B) computes the rank of the complete matrix \spad{(A|B)}
      ++ of the linear system \spad{AX = B}.

  Capsule ==> add
    systemMatrix      : (M, Col) -> M
    aSolution         :  M -> PartialV

    -- rank theorem
    hasSolution?(A, b) == rank A = rank systemMatrix(A, b)
    systemMatrix(m, v) == horizConcat(m, -(v::M))
    rank(A, b)         == rank systemMatrix(A, b)
    particularSolution(A, b) == aSolution rowEchelon systemMatrix(A,b)

    -- m should be in row-echelon form.
    -- last column of m is -(right-hand-side of system)
    aSolution m ==
       nvar := (ncols m - 1)::N
       rk := maxRowIndex m
       while (rk >= minRowIndex m) and every?(zero?, row(m, rk))
         repeat rk := dec rk
       rk < minRowIndex m => new(nvar, 0)
       ck := minColIndex m
       while (ck < maxColIndex m) and zero? qelt(m, rk, ck) repeat
         ck := inc ck
       ck = maxColIndex m => "failed"
       sol := new(nvar, 0)$Col
       -- find leading elements of diagonal
       v := new(nvar, minRowIndex m - 1)$PrimitiveArray(Integer)
       for i in minRowIndex m .. rk repeat
         for j in 0.. while zero? qelt(m, i, j+minColIndex m) repeat 0
         v.j := i
       for j in 0..nvar-1 repeat
         if v.j >= minRowIndex m then
```

```
            qsetelt_!(sol, j+minIndex sol, - qelt(m, v.j, maxColIndex m))
        sol

    solve(A:M, b:Col) ==
        -- Special case for homogeneous systems.
        every?(zero?, b) => [new(ncols A, 0), nullSpace A]
        -- General case.
        m   := rowEchelon systemMatrix(A, b)
        [aSolution m,
         nullSpace subMatrix(m, minRowIndex m, maxRowIndex m,
                                  minColIndex m, maxColIndex m - 1)]

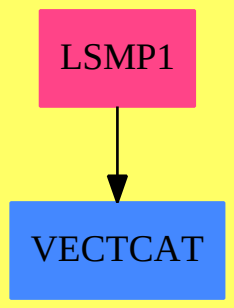
    solve(A:M, l:List Col) ==
        null l => [[new(ncols A, 0), nullSpace A]]
        nl := (sol0 := solve(A, first l)).basis
        cons(sol0,
               [[aSolution rowEchelon systemMatrix(A, b), nl]
                                                  for b in rest l])
```

⟨*LSMP.dotabb*⟩≡

```
"LSMP" [color="#FF4488",href="bookvol10.4.pdf#nameddest=LSMP"]
"MATCAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=MATCAT"]
"LSMP" -> "MATCAT"
```

13.19 package LSMP1 LinearSystemMatrixPackage1

13.20 LinearSystemMatrixPackage1



1.00

Exports:
hasSolution? particularSolution rank solve

⟨*package LSMP1 LinearSystemMatrixPackage1*⟩≡

```
)abbrev package LSMP1 LinearSystemMatrixPackage1
++ Author: R. Sutor
++ Date Created: June, 1994
++ Date Last Updated:
++ Basic Functions: solve, particularSolution, hasSolution?, rank
++ Related Constructors: LinearSystemMatrixPackage
++ Also See:
++ AMS Classifications:
++ Keywords: solve
++ References:
++ Description:
++ This package solves linear system in the matrix form \spad{AX = B}.
++ It is essentially a particular instantiation of the package
++ \spadtype{LinearSystemMatrixPackage} for Matrix and Vector. This
++ package's existence makes it easier to use \spadfun{solve} in the
++ AXIOM interpreter.

LinearSystemMatrixPackage1(F): Cat == Capsule where
    F: Field
    Row      ==> Vector F
    Col      ==> Vector F
    M        ==> Matrix(F)
    LL       ==> List List F

    N        ==> NonNegativeInteger
    PartialV ==> Union(Col, "failed")
```

```
    Both     ==> Record(particular: PartialV, basis: List Col)
    LSMP     ==> LinearSystemMatrixPackage(F, Row, Col, M)

    Cat ==> with
        solve       : (M, Col) -> Both
          ++  solve(A,B) finds a particular solution of the system \spad{AX = B}
          ++  and a basis of the associated homogeneous system \spad{AX = 0}.
        solve       : (LL, Col) -> Both
          ++  solve(A,B) finds a particular solution of the system \spad{AX = B}
          ++  and a basis of the associated homogeneous system \spad{AX = 0}.
        solve       : (M, List Col) -> List Both
          ++  solve(A,LB) finds a particular soln of the systems \spad{AX = B}
          ++  and a basis of the associated homogeneous systems \spad{AX = 0}
          ++  where B varies in the list of column vectors LB.
        solve       : (LL, List Col) -> List Both
          ++  solve(A,LB) finds a particular soln of the systems \spad{AX = B}
          ++  and a basis of the associated homogeneous systems \spad{AX = 0}
          ++  where B varies in the list of column vectors LB.

        particularSolution: (M, Col) -> PartialV
          ++ particularSolution(A,B) finds a particular solution of the linear
          ++ system \spad{AX = B}.
        hasSolution?: (M, Col) -> Boolean
          ++ hasSolution?(A,B) tests if the linear system \spad{AX = B}
          ++ has a solution.
        rank        : (M, Col) -> N
          ++ rank(A,B) computes the rank of the complete matrix \spad{(A|B)}
          ++ of the linear system \spad{AX = B}.

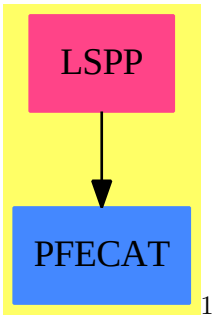
    Capsule ==> add
        solve(m : M, c: Col): Both == solve(m,c)$LSMP
        solve(ll : LL, c: Col): Both == solve(matrix(ll)$M,c)$LSMP
        solve(m : M, l : List Col): List Both == solve(m, l)$LSMP
        solve(ll : LL, l : List Col): List Both == solve(matrix(ll)$M, l)$LSMP
        particularSolution (m : M, c : Col): PartialV == particularSolution(m, c)
        hasSolution?(m :M, c : Col): Boolean == hasSolution?(m, c)$LSMP
        rank(m : M, c : Col): N == rank(m, c)$LSMP
```

⟨*LSMP1.dotabb*⟩≡

```
"LSMP1" [color="#FF4488",href="bookvol10.4.pdf#nameddest=LSMP1"]
"VECTCAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=VECTCAT"]
"LSMP1" -> "VECTCAT"
```

13.21 package LSPP LinearSystemPolynomial-Package

13.22 LinearSystemPolynomialPackage



1.00

Exports:
linSolve

⟨*package LSPP LinearSystemPolynomialPackage*⟩≡

```
)abbrev package LSPP LinearSystemPolynomialPackage
++ Author:  P.Gianni
++ Date Created: Summer 1985
++ Date Last Updated: Summer 1993
++ Basic Functions:
++ Related Constructors:
++ Also See:
++ AMS Classifications:
++ Keywords:
++ References: SystemSolvePackage
++ Description:
++ this package finds the solutions of linear systems presented as a
++ list of polynomials.

LinearSystemPolynomialPackage(R, E, OV, P): Cat == Capsule where
    R          :   IntegralDomain
    OV         :   OrderedSet
    E          :   OrderedAbelianMonoidSup
    P          :   PolynomialCategory(R,E,OV)

    F        ==> Fraction P
    NNI      ==> NonNegativeInteger
    V        ==> Vector
    M        ==> Matrix
    Soln     ==> Record(particular: Union(V F, "failed"), basis: List V F)
```

```
Cat == with
    linSolve:  (List P, List OV) -> Soln
      ++ linSolve(lp,lvar) finds the solutions of the linear system
      ++ of polynomials lp = 0 with respect to the list of symbols lvar.

Capsule == add

                    ---- Local Functions ----

    poly2vect:    (P,     List OV)    -> Record(coefvec: V F, reductum: F)
    intoMatrix:   (List P,   List OV) -> Record(mat: M F, vec: V F)


    poly2vect(p : P, vs : List OV) : Record(coefvec: V F, reductum: F) ==
        coefs := new(#vs, 0)$(V F)
        for v in vs for i in 1.. while p ^= 0 repeat
          u := univariate(p, v)
          degree u = 0 => "next v"
          coefs.i := (c := leadingCoefficient u)::F
          p := p - monomial(c,v, 1)
        [coefs, p :: F]

    intoMatrix(ps : List P, vs : List OV ) : Record(mat: M F, vec: V F) ==
        m := zero(#ps, #vs)$M(F)
        v := new(#ps, 0)$V(F)
        for p in ps for i in 1.. repeat
            totalDegree(p,vs) > 1 => error "The system is not linear"
            r   := poly2vect(p,vs)
            m:=setRow_!(m,i,r.coefvec)
            v.i := - r.reductum
        [m, v]

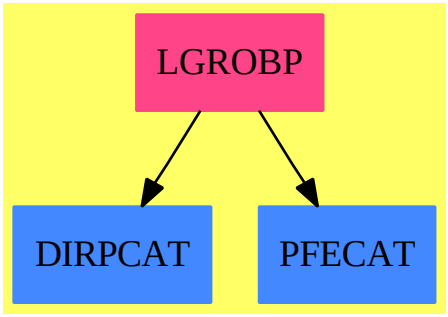
    linSolve(ps, vs) ==
        r := intoMatrix(ps, vs)
        solve(r.mat, r.vec)$LinearSystemMatrixPackage(F,V F,V F,M F)
```

⟨*LSPP.dotabb*⟩≡

```
"LSPP" [color="#FF4488",href="bookvol10.4.pdf#nameddest=LSPP"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"LSPP" -> "PFECAT"
```

13.23 package LGROBP LinGroebnerPackage

13.24 LinGroebnerPackage



1.00

Exports:

anticoord	choosemon	computeBasis	coord	groebgen
intcompBasis	linGenPos	minPol	totolex	transform

⟨*package LGROBP LinGroebnerPackage*⟩≡

```
)abbrev package LGROBP LinGroebnerPackage
++  Given a Groebner basis B with respect to the total degree ordering for
++  a zero-dimensional ideal I, compute
++  a Groebner basis with respect to the lexicographical ordering by using
++  linear algebra.
LinGroebnerPackage(lv,F) : C == T

 where
  Z      ==>  Integer
  lv     :    List Symbol
  F      :    GcdDomain

  DP     ==>  DirectProduct(#lv,NonNegativeInteger)
  DPoly  ==>  DistributedMultivariatePolynomial(lv,F)

  HDP    ==>  HomogeneousDirectProduct(#lv,NonNegativeInteger)
  HDPoly ==>  HomogeneousDistributedMultivariatePolynomial(lv,F)

  OV     ==>  OrderedVariableList(lv)
  NNI    ==>  NonNegativeInteger
  LVals  ==>  Record(gblist : List DPoly,gvlist : List Z)
  VF     ==>  Vector F
  VV     ==>  Vector NNI
  MF     ==>  Matrix F
  cLVars ==>  Record(glbase:List DPoly,glval:List Z)
```

```
C == with

   linGenPos    :           List HDPoly      -> LVals
      ++ linGenPos \undocumented
   groebgen     :           List DPoly       -> cLVars
      ++ groebgen \undocumented
   totolex      :           List HDPoly      -> List DPoly
      ++ totolex \undocumented
   minPol       : (List HDPoly,List HDPoly,OV) -> HDPoly
      ++ minPol \undocumented
   minPol       :        (List HDPoly,OV)     -> HDPoly
      ++ minPol \undocumented


   computeBasis :        List HDPoly           -> List HDPoly
      ++ computeBasis \undocumented
   coord        : (HDPoly,List HDPoly)         -> VF
      ++ coord \undocumented
   anticoord    : (List F,DPoly,List DPoly)    -> DPoly
      ++ anticoord \undocumented
   intcompBasis : (OV,List HDPoly,List HDPoly) -> List HDPoly
      ++ intcompBasis \undocumented
   choosemon    :     (DPoly,List  DPoly)      -> DPoly
      ++ choosemon \undocumented
   transform    :           DPoly              -> HDPoly
      ++ transform \undocumented


T == add

  import GroebnerPackage(F,DP,OV,DPoly)
  import GroebnerPackage(F,HDP,OV,HDPoly)
  import GroebnerInternalPackage(F,HDP,OV,HDPoly)
  import GroebnerInternalPackage(F,DP,OV,DPoly)

  lvar :=[variable(yx)::OV for yx in lv]

  reduceRow(M:MF, v : VF, lastRow: Integer, pivots: Vector(Integer)) : VF ==
    a1:F := 1
    b:F := 0
    dim := #v
    for j in 1..lastRow repeat -- scan over rows
       mj := row(M,j)
       k:=pivots(j)
       b:=mj.k
       vk := v.k
```

```
          for kk in 1..(k-1) repeat
             v(kk) := ((-b*v(kk)) exquo a1) :: F
          for kk in k..dim repeat
             v(kk) := ((vk*mj(kk)-b*v(kk)) exquo a1)::F
          a1 := b
        v

    rRedPol(f:HDPoly, B:List HDPoly):Record(poly:HDPoly, mult:F) ==
      gm := redPo(f,B)
      gm.poly = 0 => gm
      gg := reductum(gm.poly)
      ggm := rRedPol(gg,B)
      [ggm.mult*(gm.poly - gg) + ggm.poly, ggm.mult*gm.mult]

 ----- transform the total basis B in lex basis -----
    totolex(B : List HDPoly) : List DPoly ==
      result:List DPoly :=[]
      ltresult:List DPoly :=[]
      vBasis:= computeBasis B
      nBasis:List DPoly :=[1$DPoly]
      ndim:=(#vBasis)::PositiveInteger
      ndim1:NNI:=ndim+1
      lm:VF
      linmat:MF:=zero(ndim,2*ndim+1)
      linmat(1,1):=1$F
      linmat(1,ndim1):=1
      pivots:Vector Integer := new(ndim,0)
      pivots(1) := 1
      firstmon:DPoly:=1$DPoly
      ofirstmon:DPoly:=1$DPoly
      orecfmon:Record(poly:HDPoly, mult:F) := [1,1]
      i:NNI:=2
      while (firstmon:=choosemon(firstmon,ltresult))^=1 repeat
        if (v:=firstmon exquo ofirstmon) case "failed" then
          recfmon:=rRedPol(transform firstmon,B)
        else
          recfmon:=rRedPol(transform(v::DPoly) *orecfmon.poly,B)
          recfmon.mult := recfmon.mult * orecfmon.mult
        cc := gcd(content recfmon.poly, recfmon.mult)
        recfmon.poly := (recfmon.poly exquo cc)::HDPoly
        recfmon.mult := (recfmon.mult exquo cc)::F
        veccoef:VF:=coord(recfmon.poly,vBasis)
        ofirstmon:=firstmon
        orecfmon := recfmon
        lm:=zero(2*ndim+1)
        for j in 1..ndim repeat lm(j):=veccoef(j)
```

```
        lm(ndim+i):=recfmon.mult
        lm := reduceRow(linmat, lm, i-1, pivots)
        if i=ndim1 then j:=ndim1
        else
          j:=1
          while lm(j) = 0 and j< ndim1 repeat j:=j+1
        if j=ndim1 then
          cordlist:List F:=[lm(k) for k in ndim1..ndim1+(#nBasis)]
          antc:=+/[c*b for c in reverse cordlist
                       for b in concat(firstmon,nBasis)]
          antc:=primitivePart antc
          result:=concat(antc,result)
          ltresult:=concat(antc-reductum antc,ltresult)
        else
          pivots(i) := j
          setRow_!(linmat,i,lm)
          i:=i+1
          nBasis:=cons(firstmon,nBasis)
      result

---- Compute the univariate polynomial for x
----oldBasis is a total degree Groebner basis
    minPol(oldBasis:List HDPoly,x:OV) :HDPoly ==
      algBasis:= computeBasis oldBasis
      minPol(oldBasis,algBasis,x)

---- Compute the univariate polynomial for x
---- oldBasis is total Groebner, algBasis is the basis as algebra
    minPol(oldBasis:List HDPoly,algBasis:List HDPoly,x:OV) :HDPoly ==
      nvp:HDPoly:=x::HDPoly
      f:=1$HDPoly
      omult:F :=1
      ndim:=(#algBasis)::PositiveInteger
      ndim1:NNI:=ndim+1
      lm:VF
      linmat:MF:=zero(ndim,2*ndim+1)
      linmat(1,1):=1$F
      linmat(1,ndim1):=1
      pivots:Vector Integer := new(ndim,0)
      pivots(1) := 1
      for i in 2..ndim1 repeat
        recf:=rRedPol(f*nvp,oldBasis)
        omult := recf.mult * omult
        f := recf.poly
        cc := gcd(content f, omult)
        f := (f exquo cc)::HDPoly
```

```
          omult := (omult exquo cc)::F
          veccoef:VF:=coord(f,algBasis)
          lm:=zero(2*ndim+1)
          for j in 1..ndim repeat lm(j) := veccoef(j)
          lm(ndim+i):=omult
          lm := reduceRow(linmat, lm, i-1, pivots)
          j:=1
          while lm(j)=0 and j<ndim1 repeat j:=j+1
          if j=ndim1 then return
            g:HDPoly:=0
            for k in ndim1..2*ndim+1 repeat
              g:=g+lm(k) * nvp**((k-ndim1):NNI)
            primitivePart g
          pivots(i) := j
          setRow_!(linmat,i,lm)

  ----- transform a DPoly in a HDPoly -----
      transform(dpol:DPoly) : HDPoly ==
        dpol=0 => 0$HDPoly
        monomial(leadingCoefficient dpol,
                 directProduct(degree(dpol)::VV)$HDP)$HDPoly +
                                      transform(reductum dpol)

  ----- compute the basis for the vector space determined by B -----
      computeBasis(B:List HDPoly) : List HDPoly ==
        mB:List HDPoly:=[monomial(1$F,degree f)$HDPoly for f in B]
        result:List HDPoly := [1$HDPoly]
        for var in lvar repeat
          part:=intcompBasis(var,result,mB)
          result:=concat(result,part)
        result

  ----- internal function for computeBasis -----
      intcompBasis(x:OV,lr:List HDPoly,mB : List HDPoly):List HDPoly ==
        lr=[] => lr
        part:List HDPoly :=[]
        for f in lr repeat
          g:=x::HDPoly * f
          if redPo(g,mB).poly^=0 then part:=concat(g,part)
        concat(part,intcompBasis(x,part,mB))

  ----- coordinate of f with respect to the basis B -----
  ----- f is a reduced polynomial -----
      coord(f:HDPoly,B:List HDPoly) : VF ==
        ndim := #B
        vv:VF:=new(ndim,0$F)$VF
```

```
      while f^=0 repeat
        rf := reductum f
        lf := f-rf
        lcf := leadingCoefficient f
        i:Z:=position(monomial(1$F,degree lf),B)
        vv.i:=lcf
        f := rf
      vv

----- reconstruct the polynomial from its coordinate -----
    anticoord(vv:List F,mf:DPoly,B:List DPoly) : DPoly ==
      for f in B for c in vv repeat (mf:=mf-c*f)
      mf

----- choose the next monom -----
    choosemon(mf:DPoly,nB:List DPoly) : DPoly ==
      nB = [] => ((lvar.last)::DPoly)*mf
      for x in reverse lvar repeat
        xx:=x ::DPoly
        mf:=xx*mf
        if redPo(mf,nB).poly ^= 0 then return mf
        dx := degree(mf,x)
        mf := (mf exquo (xx ** dx))::DPoly
      mf

----- put B in general position, B is Groebner -----
    linGenPos(B : List HDPoly) : LVals ==
      result:List DPoly :=[]
      ltresult:List DPoly :=[]
      vBasis:= computeBasis B
      nBasis:List DPoly :=[1$DPoly]
      ndim:=#vBasis : PositiveInteger
      ndim1:NNI:=ndim+1
      lm:VF
      linmat:MF:=zero(ndim,2*ndim+1)
      linmat(1,1):=1$F
      linmat(1,ndim1):=1
      pivots:Vector Integer := new(ndim,0)
      pivots(1) := 1
      i:NNI:=2
      rval:List Z :=[]
      for ii in 1..(#lvar-1) repeat
        c:Z:=0
        while c=0 repeat c:=random()$Z rem 11
        rval:=concat(c,rval)
      nval:DPoly := (last.lvar)::DPoly -
```

```
              (+/[r*(vv)::DPoly for r in rval for vv in lvar])
        firstmon:DPoly:=1$DPoly
        ofirstmon:DPoly:=1$DPoly
        orecfmon:Record(poly:HDPoly, mult:F) := [1,1]
        lx:= lvar.last
        while (firstmon:=choosemon(firstmon,ltresult))^=1 repeat
          if (v:=firstmon exquo ofirstmon) case "failed" then
            recfmon:=rRedPol(transform(eval(firstmon,lx,nval)),B)
          else
            recfmon:=rRedPol(transform(eval(v,lx,nval))*orecfmon.poly,B)
            recfmon.mult := recfmon.mult * orecfmon.mult
          cc := gcd(content recfmon.poly, recfmon.mult)
          recfmon.poly := (recfmon.poly exquo cc)::HDPoly
          recfmon.mult := (recfmon.mult exquo cc)::F
          veccoef:VF:=coord(recfmon.poly,vBasis)
          ofirstmon:=firstmon
          orecfmon := recfmon
          lm:=zero(2*ndim+1)
          for j in 1..ndim repeat lm(j):=veccoef(j)
          lm(ndim+i):=recfmon.mult
          lm := reduceRow(linmat, lm, i-1, pivots)
          j:=1
          while lm(j) = 0 and j<ndim1 repeat j:=j+1
          if j=ndim1 then
            cordlist:List F:=[lm(j) for j in ndim1..ndim1+(#nBasis)]
            antc:=+/[c*b for c in reverse cordlist
                         for b in concat(firstmon,nBasis)]
            result:=concat(primitivePart antc,result)
            ltresult:=concat(antc-reductum antc,ltresult)
          else
            pivots(i) := j
            setRow_!(linmat,i,lm)
            i:=i+1
            nBasis:=concat(firstmon,nBasis)
        [result,rval]$LVals

  ----- given a basis of a zero-dimensional ideal,
  ----- performs a random change of coordinates
  ----- computes a Groebner basis for the lex ordering
      groebgen(L:List DPoly) : cLVars ==
        xn:=lvar.last
        val := xn::DPoly
        nvar1:NNI:=(#lvar-1):NNI
        ll: List Z :=[random()$Z rem 11 for i in 1..nvar1]
        val:=val+ +/[ll.i*(lvar.i)::DPoly for i in 1..nvar1]
        LL:=[elt(univariate(f,xn),val) for f in L]
```

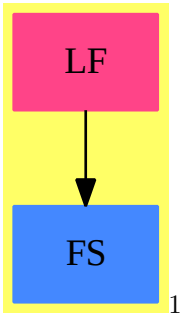
```
LL:=  groebner(LL)
[LL,ll]$cLVars
```

⟨*LGROBP.dotabb*⟩≡

```
"LGROBP" [color="#FF4488",href="bookvol10.4.pdf#nameddest=LGROBP"]
"DIRPCAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=DIRPCAT"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"LGROBP" -> "DIRPCAT"
"LGROBP" -> "PFECAT"
```

13.25 package LF LiouvillianFunction

13.26 LiouvillianFunction



1.00

Exports:

belong?	Ci	dilog	Ei	erf
integral	li	integral	operator	Si

⟨*package LF LiouvillianFunction*⟩≡

```
)abbrev package LF LiouvillianFunction
++ Author: Manuel Bronstein
++ Date Created: 1987
++ Date Last Updated: 10 August 1994
++ Keywords: liouvillian, function, primitive, exponential.
++ Examples:  )r LF INPUT
++ Description:
++   This package provides liouvillian functions over an integral domain.
LiouvillianFunction(R, F): Exports == Implementation where
  R:Join(OrderedSet, IntegralDomain)
  F:Join(FunctionSpace R,RadicalCategory,TranscendentalFunctionCategory)

  OP  ==> BasicOperator
  PR  ==> Polynomial R
  K   ==> Kernel F
  SE  ==> Symbol
  O   ==> OutputForm
  INP ==> InputForm
  INV ==> error "Invalid argument"

  SPECIALDIFF ==> "%specialDiff"
  SPECIALDISP ==> "%specialDisp"
  SPECIALINPUT ==> "%specialInput"
  SPECIALEQUAL ==> "%specialEqual"

  Exports ==> with
```

```
    belong? : OP -> Boolean
      ++ belong?(op) checks if op is Liouvillian
    operator: OP -> OP
      ++ operator(op) returns the Liouvillian operator based on op
    Ei      : F  -> F
      ++ Ei(f) denotes the exponential integral
    Si      : F  -> F
      ++ Si(f) denotes the sine integral
    Ci      : F  -> F
      ++ Ci(f) denotes the cosine integral
    li      : F  -> F
      ++ li(f) denotes the logarithmic integral
    erf     : F  -> F
      ++ erf(f) denotes the error function
    dilog   : F  -> F
      ++ dilog(f) denotes the dilogarithm
    integral: (F, SE) -> F
      ++ integral(f,x) indefinite integral of f with respect to x.
    integral: (F, SegmentBinding F) -> F
      ++ integral(f,x = a..b) denotes the definite integral of f with
      ++ respect to x from \spad{a} to b.

  Implementation ==> add
    iei        : F  -> F
    isi        : F  -> F
    ici        : F  -> F
    ierf       : F  -> F
    ili        : F  -> F
    ili2       : F  -> F
    iint       : List F -> F
    eqint      : (K,K) -> Boolean
    dvint      : (List F, SE) -> F
    dvdint     : (List F, SE) -> F
    ddint      : List F -> O
    integrand  : List F -> F

    dummy := new()$SE :: F

    opint  := operator("integral"::Symbol)$CommonOperators
    opdint := operator("%defint"::Symbol)$CommonOperators
    opei   := operator("Ei"::Symbol)$CommonOperators
    opli   := operator("li"::Symbol)$CommonOperators
    opsi   := operator("Si"::Symbol)$CommonOperators
    opci   := operator("Ci"::Symbol)$CommonOperators
    opli2  := operator("dilog"::Symbol)$CommonOperators
    operf  := operator("erf"::Symbol)$CommonOperators
```

```
    Si x                == opsi x
    Ci x                == opci x
    Ei x                == opei x
    erf x               == operf x
    li  x               == opli x
    dilog x             == opli2 x

    belong? op     == has?(op, "prim")
    isi x          == kernel(opsi, x)
    ici x          == kernel(opci, x)
    ierf x         == (zero? x => 0; kernel(operf, x))
--    ili2 x         == (one? x => INV; kernel(opli2, x))
    ili2 x         == ((x = 1) => INV; kernel(opli2, x))
    integrand l    == eval(first l, retract(second l)@K, third l)
    integral(f:F, x:SE) == opint [eval(f, k:=kernel(x)$K, dummy), dummy, k::F]

    iint l ==
      zero? first l => 0
      kernel(opint, l)

    ddint l ==
      int(integrand(l)::O * hconcat("d"::SE::O, third(l)::O),
                                    third(rest l)::O, third(rest rest l)::O)

    eqint(k1,k2) ==
      a1:=argument k1
      a2:=argument k2
      res:=operator k1 = operator k2
      if not res then return res
      res:= a1 = a2
      if res then return res
      res:= (a1.3 = a2.3) and (subst(a1.1,[retract(a1.2)@K],[a2.2]) = a2.1)

    dvint(l, x) ==
      k  := retract(second l)@K
      differentiate(third l, x) * integrand l
          + opint [differentiate(first l, x), second l, third l]


    dvdint(l, x) ==
      x = retract(y := third l)@SE => 0
      k := retract(d := second l)@K
      differentiate(h := third rest rest l,x) * eval(f := first l, k, h)
        - differentiate(g := third rest l, x) * eval(f, k, g)
             + opdint [differentiate(f, x), d, y, g, h]
```

```
integral(f:F, s: SegmentBinding F) ==
  x := kernel(variable s)$K
  opdint [eval(f,x,dummy), dummy, x::F, lo segment s, hi segment s]

ili x ==
  x = 1 => INV
  is?(x, "exp"::Symbol) => Ei first argument(retract(x)@K)
  kernel(opli, x)

iei x ==
  x = 0 => INV
  is?(x, "log"::Symbol) => li first argument(retract(x)@K)
  kernel(opei, x)

operator op ==
  is?(op, "integral"::Symbol)   => opint
  is?(op, "%defint"::Symbol)    => opdint
  is?(op, "Ei"::Symbol)         => opei
  is?(op, "Si"::Symbol)         => opsi
  is?(op, "Ci"::Symbol)         => opci
  is?(op, "li"::Symbol)         => opli
  is?(op, "erf"::Symbol)        => operf
  is?(op, "dilog"::Symbol)      => opli2
  error "Not a Liouvillian operator"

evaluate(opei,   iei)$BasicOperatorFunctions1(F)
evaluate(opli,   ili)
evaluate(opsi,   isi)
evaluate(opci,   ici)
evaluate(operf,  ierf)
evaluate(opli2,  ili2)
evaluate(opint,  iint)
derivative(opsi, sin(#1) / #1)
derivative(opci, cos(#1) / #1)
derivative(opei, exp(#1) / #1)
derivative(opli, inv log(#1))
derivative(operf, 2 * exp(-(#1**2)) / sqrt(pi()))
derivative(opli2, log(#1) / (1 - #1))
setProperty(opint,SPECIALEQUAL,eqint@((K,K) -> Boolean) pretend None)
setProperty(opint,SPECIALDIFF,dvint@((List F,SE) -> F) pretend None)
setProperty(opdint,SPECIALDIFF,dvdint@((List F,SE)->F) pretend None)
setProperty(opdint, SPECIALDISP, ddint@(List F -> O) pretend None)

if R has ConvertibleTo INP then
  inint : List F -> INP
```

```
indint: List F -> INP
pint  : List INP -> INP


pint l  == convert concat(convert("integral"::SE)@INP, l)
inint l ==
  r2:= convert([convert("::"::SE)@INP,convert(third l)@INP,convert("Symbol"::SE)@INP]@
  pint [convert(integrand l)@INP, r2]

indint l ==
  pint [convert(integrand l)@INP,
        convert concat(convert("="::SE)@INP,
                       [convert(third l)@INP,
                        convert concat(convert("SEGMENT"::SE)@INP,
                                       [convert(third rest l)@INP,
                                        convert(third rest rest l)@INP])])]

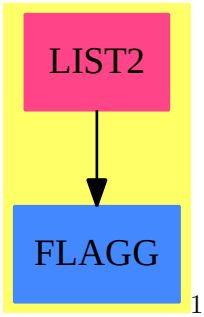
setProperty(opint, SPECIALINPUT, inint@(List F -> INP) pretend None)
setProperty(opdint, SPECIALINPUT, indint@(List F -> INP) pretend None)
```

⟨*LF.dotabb*⟩≡

```
"LF" [color="#FF4488",href="bookvol10.4.pdf#nameddest=LF"]
"FS" [color="#4488FF",href="bookvol10.2.pdf#nameddest=FS"]
"LF" -> "FS"
```

13.27 package LIST2 ListFunctions2

13.28 ListFunctions2



1.00

Exports:
map reduce scan

⟨*package LIST2 ListFunctions2*⟩≡
```
)abbrev package LIST2 ListFunctions2
++ Author:
++ Date Created:
++ Change History:
++ Basic Operations: map, reduce, scan
++ Related Constructors: List
++ Also See: ListFunctions3
++ AMS Classification:
++ Keywords: list, aggregate, map, reduce
++ Description:
++   \spadtype{ListFunctions2} implements utility functions that
++   operate on two kinds of lists, each with a possibly different
++   type of element.
ListFunctions2(A:Type, B:Type): public == private where
  LA     ==> List A
  LB     ==> List B
  O2     ==> FiniteLinearAggregateFunctions2(A, LA, B, LB)

  public ==> with
    scan:    ((A, B) -> B, LA, B) -> LB
      ++ scan(fn,u,ident) successively uses the binary function
      ++ \spad{fn} to reduce more and more of list \spad{u}.
      ++ \spad{ident} is returned if the \spad{u} is empty.
      ++ The result is a list of the reductions at each step. See
      ++ \spadfun{reduce} for more information. Examples:
      ++ \spad{scan(fn,[1,2],0) = [fn(2,fn(1,0)),fn(1,0)]} and
      ++ \spad{scan(*,[2,3],1) = [2 * 1, 3 * (2 * 1)]}.
```

```
    reduce:  ((A, B) -> B, LA, B) -> B
      ++ reduce(fn,u,ident) successively uses the binary function
      ++ \spad{fn} on the elements of list \spad{u} and the result
      ++ of previous applications. \spad{ident} is returned if the
      ++ \spad{u} is empty. Note the order of application in
      ++ the following examples:
      ++ \spad{reduce(fn,[1,2,3],0) = fn(3,fn(2,fn(1,0)))} and
      ++ \spad{reduce(*,[2,3],1) = 3 * (2 * 1)}.
    map:      (A -> B, LA) -> LB
      ++ map(fn,u) applies \spad{fn} to each element of
      ++ list \spad{u} and returns a new list with the results.
      ++ For example \spad{map(square,[1,2,3]) = [1,4,9]}.

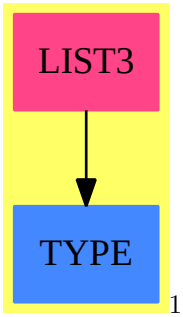
  private ==> add
    map(f, l)       == map(f, l)$O2
    scan(f, l, b)   == scan(f, l, b)$O2
    reduce(f, l, b) == reduce(f, l, b)$O2
```

⟨*LIST2.dotabb*⟩≡

```
"LIST2" [color="#FF4488",href="bookvol10.4.pdf#nameddest=LIST2"]
"FLAGG" [color="#4488FF",href="bookvol10.2.pdf#nameddest=FLAGG"]
"LIST2" -> "FLAGG"
```

13.29 package LIST3 ListFunctions3

13.30 ListFunctions3



1.00

Exports:
map

⟨*package LIST3 ListFunctions3*⟩≡
```
)abbrev package LIST3 ListFunctions3
++ Author:
++ Date Created:
++ Change History:
++ Basic Operations: map
++ Related Constructors: List
++ Also See: ListFunctions2
++ AMS Classification:
++ Keywords: list, aggregate, map
++ Description:
++   \spadtype{ListFunctions3} implements utility functions that
++   operate on three kinds of lists, each with a possibly different
++   type of element.
ListFunctions3(A:Type, B:Type, C:Type): public == private where
  LA     ==> List A
  LB     ==> List B
  LC     ==> List C

  public ==> with
    map: ( (A,B)->C, LA, LB) -> LC
      ++ map(fn,list1, u2) applies the binary function \spad{fn}
      ++ to corresponding elements of lists \spad{u1} and \spad{u2}
      ++ and returns a list of the results (in the same order). Thus
      ++ \spad{map(/,[1,2,3],[4,5,6]) = [1/4,2/4,1/2]}. The computation
      ++ terminates when the end of either list is reached. That is,
      ++ the length of the result list is equal to the minimum of the
      ++ lengths of \spad{u1} and \spad{u2}.
```

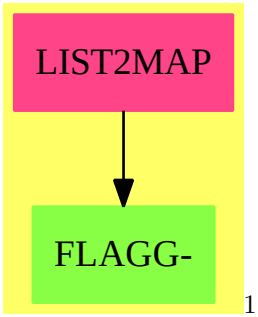
```
    private ==> add
      map(fn : (A,B) -> C, la : LA, lb : LB): LC ==
        empty?(la) or empty?(lb) => empty()$LC
        concat(fn(first la, first lb), map(fn, rest la, rest lb))
```

⟨*LIST3.dotabb*⟩≡

```
"LIST3" [color="#FF4488",href="bookvol10.4.pdf#nameddest=LIST3"]
"TYPE" [color="#4488FF",href="bookvol10.2.pdf#nameddest=TYPE"]
"LIST3" -> "TYPE"
```

13.31 package LIST2MAP ListToMap

13.32 ListToMap



1.00

Exports:
match

⟨*package LIST2MAP ListToMap*⟩≡
```
)abbrev package LIST2MAP ListToMap
++ Author: Manuel Bronstein
++ Date Created: 22 Mar 1988
++ Change History:
++   11 Oct 1989   MB   ?
++ Basic Operations: match
++ Related Constructors: List
++ Also See:
++ AMS Classification:
++ Keywords: mapping, list
++ Description:
++   \spadtype{ListToMap} allows mappings to be described by a pair of
++   lists of equal lengths.  The image of an element \spad{x},
++   which appears in position \spad{n} in the first list, is then
++   the \spad{n}th element of the second list.  A default value or
++   default function can be specified to be used when \spad{x}
++   does not appear in the first list.  In the absence of defaults,
++   an error will occur in that case.
ListToMap(A:SetCategory, B:Type): Exports == Implementation where
  LA  ==> List A
  LB  ==> List B
  AB  ==> (A -> B)

  Exports ==> with
    match: (LA, LB   ) -> AB
      ++ match(la, lb) creates a map with no default source or target values
      ++ defined by lists la and lb of equal length.
```

```
    ++ The target of a source value \spad{x} in la is the
    ++ value y with the same index lb.
    ++ Error: if la and lb are not of equal length.
    ++ Note: when this map is applied, an error occurs when
    ++ applied to a value missing from la.
  match: (LA, LB, A) -> B
    ++ match(la, lb, a) creates a map
    ++ defined by lists la and lb of equal length, where \spad{a} is used
    ++ as the default source value if the given one is not in \spad{la}.
    ++ The target of a source value \spad{x} in la is the
    ++ value y with the same index lb.
    ++ Error: if la and lb are not of equal length.
  match: (LA, LB, B)    -> AB
    ++ match(la, lb, b) creates a map
    ++ defined by lists la and lb of equal length, where \spad{b} is used
    ++ as the default target value if the given function argument is
    ++ not in \spad{la}.
    ++ The target of a source value \spad{x} in la is the
    ++ value y with the same index lb.
    ++ Error: if la and lb are not of equal length.
  match: (LA, LB, A, B) -> B
    ++ match(la, lb, a, b) creates a map
    ++ defined by lists la and lb of equal length.
    ++ and applies this map to a.
    ++ The target of a source value \spad{x} in la is the
    ++ value y with the same index lb.
    ++ Argument b is the default target value if a is not in la.
    ++ Error: if la and lb are not of equal length.
  match: (LA, LB, AB)    -> AB
    ++ match(la, lb, f) creates a map
    ++ defined by lists la and lb of equal length.
    ++ The target of a source value \spad{x} in la is the
    ++ value y with the same index lb.
    ++ Argument \spad{f} is used as the
    ++ function to call when the given function argument is not in
    ++ \spad{la}.
    ++ The value returned is f applied to that argument.
  match: (LA, LB, A, AB) -> B
    ++ match(la, lb, a, f) creates a map
    ++ defined by lists la and lb of equal length.
    ++ and applies this map to a.
    ++ The target of a source value \spad{x} in la is the
    ++ value y with the same index lb.
    ++ Argument \spad{f} is a default function to call if a is not in la.
    ++ The value returned is then obtained by applying f to argument a.
```

```
Implementation ==> add
  match(la, lb)              == match(la, lb, #1)
  match(la:LA, lb:LB, a:A)  == lb.position(a, la)
  match(la:LA, lb:LB, b:B)  == match(la, lb, #1, b)
  match(la:LA, lb:LB, f:AB) == match(la, lb, #1, f)

  match(la:LA, lb:LB, a:A, b:B) ==
    (p := position(a, la)) < minIndex(la) => b
    lb.p

  match(la:LA, lb:LB, a:A, f:AB) ==
    (p := position(a, la)) < minIndex(la) => f a
    lb.p
```

⟨*LIST2MAP.dotabb*⟩≡

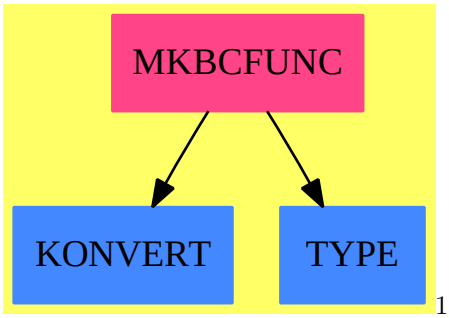
```
"LIST2MAP" [color="#FF4488",href="bookvol10.4.pdf#nameddest=LIST2MAP"]
"FLAGG-" [color="#88FF44",href="bookvol10.3.pdf#nameddest=FLAGG"]
"LIST2MAP" -> "FLAGG-"
```

Chapter 14

Chapter M

14.1 package MKBCFUNC MakeBinaryCompiledFunction

14.2 MakeBinaryCompiledFunction



1.00

Exports:
binaryFunction compiledFunction

⟨*package MKBCFUNC MakeBinaryCompiledFunction*⟩≡
```
)abbrev package MKBCFUNC MakeBinaryCompiledFunction
++ Tools for making compiled functions from top-level expressions
++ Author: Manuel Bronstein
++ Date Created: 1 Dec 1988
++ Date Last Updated: 5 Mar 1990
++ Description: transforms top-level objects into compiled functions.
MakeBinaryCompiledFunction(S, D1, D2, I):Exports == Implementation where
  S: ConvertibleTo InputForm
  D1, D2, I: Type
```

```
SY  ==> Symbol
DI  ==> devaluate((D1, D2) -> I)$Lisp

Exports ==> with
  binaryFunction  : SY -> ((D1, D2) -> I)
    ++ binaryFunction(s) is a local function
  compiledFunction: (S, SY, SY) -> ((D1, D2) -> I)
    ++ compiledFunction(expr,x,y) returns a function \spad{f: (D1, D2) -> I}
    ++ defined by \spad{f(x, y) == expr}.
    ++ Function f is compiled and directly
    ++ applicable to objects of type \spad{(D1, D2)}

Implementation ==> add
  import MakeFunction(S)

  func: (SY, D1, D2) -> I

  func(name, x, y)   == FUNCALL(name, x, y, NIL$Lisp)$Lisp
  binaryFunction name == func(name, #1, #2)

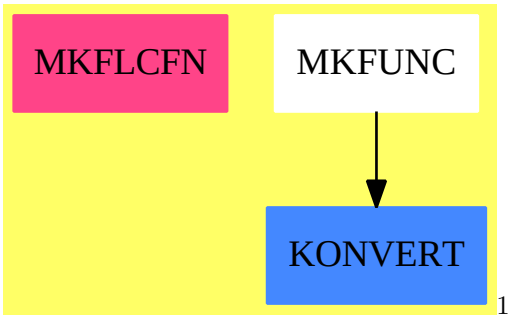
  compiledFunction(e, x, y) ==
    t := [devaluate(D1)$Lisp, devaluate(D2)$Lisp]$List(InputForm)
    binaryFunction compile(function(e, declare DI, x, y), t)
```

⟨*MKBCFUNC.dotabb*⟩≡

```
"MKBCFUNC" [color="#FF4488",href="bookvol10.4.pdf#nameddest=MKBCFUNC"]
"KONVERT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=KONVERT"]
"TYPE" [color="#4488FF",href="bookvol10.2.pdf#nameddest=TYPE"]
"MKBCFUNC" -> "KONVERT"
"MKBCFUNC" -> "TYPE"
```

14.3 package MKFLCFN MakeFloatCompiled-Function

14.4 MakeFloatCompiledFunction



1.00

Exports:

⟨*package MKFLCFN MakeFloatCompiledFunction*⟩≡

```
)abbrev package MKFLCFN MakeFloatCompiledFunction
++ Tools for making compiled functions from top-level expressions
++ Author: Manuel Bronstein
++ Date Created: 2 Mar 1990
++ Date Last Updated: 2 Dec 1996 (MCD)
++ Description:
++ MakeFloatCompiledFunction transforms top-level objects into
++ compiled Lisp functions whose arguments are Lisp floats.
++ This by-passes the \Language{} compiler and interpreter,
++ thereby gaining several orders of magnitude.
MakeFloatCompiledFunction(S): Exports == Implementation where
  S: ConvertibleTo InputForm

  INF ==> InputForm
  SF  ==> DoubleFloat
  DI1 ==> devaluate(SF -> SF)$Lisp
  DI2 ==> devaluate((SF, SF) -> SF)$Lisp

  Exports ==> with
    makeFloatFunction: (S, Symbol)         -> (SF -> SF)
      ++ makeFloatFunction(expr, x) returns a Lisp function
      ++ \spad{f: \axiomType{DoubleFloat} -> \axiomType{DoubleFloat}}
      ++ defined by \spad{f(x) == expr}.
      ++ Function f is compiled and directly
      ++ applicable to objects of type \axiomType{DoubleFloat}.
    makeFloatFunction: (S, Symbol, Symbol) -> ((SF, SF) -> SF)
```

```
      ++ makeFloatFunction(expr, x, y) returns a Lisp function
      ++ \spad{f: (\axiomType{DoubleFloat},
      ++ \axiomType{DoubleFloat}) -> \axiomType{DoubleFloat}}
      ++ defined by \spad{f(x, y) == expr}.
      ++ Function f is compiled and directly
      ++ applicable to objects of type \spad{(\axiomType{DoubleFloat},
      ++ \axiomType{DoubleFloat})}.

    Implementation ==> add
      import MakeUnaryCompiledFunction(S, SF, SF)
      import MakeBinaryCompiledFunction(S, SF, SF, SF)

      streq?    : (INF, String) -> Boolean
      streqlist?: (INF, List String) -> Boolean
      gencode   : (String, List INF) -> INF
      mkLisp    : INF -> Union(INF, "failed")
      mkLispList: List INF -> Union(List INF, "failed")
      mkDefun   : (INF, List INF) -> INF
      mkLispCall: INF -> INF
      mkPretend : INF -> INF
      mkCTOR : INF -> INF

      lsf := convert([convert("DoubleFloat"::Symbol)@INF]$List(INF))@INF

      streq?(s, st)    == s = convert(st::Symbol)@INF
      gencode(s, l)    == convert(concat(convert(s::Symbol)@INF, l))@INF
      streqlist?(s, l) == member?(string symbol s, l)

      mkPretend form ==
        convert([convert("pretend"::Symbol), form, lsf]$List(INF))@INF

      mkCTOR form ==
        convert([convert("C-TO-R"::Symbol), form]$List(INF))@INF


      mkLispCall name ==
        convert([convert("$elt"::Symbol),
                              convert("Lisp"::Symbol), name]$List(INF))@INF

      mkDefun(s, lv) ==
        name := convert(new()$Symbol)@INF
        fun  := convert([convert("DEFUN"::Symbol), name, convert lv,
                gencode("DECLARE",[gencode("FLOAT",lv)]),mkCTOR s]$List(INF))@INF
        EVAL(fun)$Lisp
        if _$compileDontDefineFunctions$Lisp then COMPILE(name)$Lisp
        name
```

```
makeFloatFunction(f, x, y) ==
  (u := mkLisp(convert(f)@INF)) case "failed" =>
    compiledFunction(f, x, y)
  name := mkDefun(u::INF, [ix := convert x, iy := convert y])
  t    := [lsf, lsf]$List(INF)
  spadname := declare DI2
  spadform:=mkPretend convert([mkLispCall name,ix,iy]$List(INF))@INF
  interpret function(spadform, [x, y], spadname)
  binaryFunction compile(spadname, t)

makeFloatFunction(f, var) ==
  (u := mkLisp(convert(f)@INF)) case "failed" =>
    compiledFunction(f, var)
  name := mkDefun(u::INF, [ivar := convert var])
  t    := [lsf]$List(INF)
  spadname := declare DI1
  spadform:= mkPretend convert([mkLispCall name,ivar]$List(INF))@INF
  interpret function(spadform, [var], spadname)
  unaryFunction compile(spadname, t)

mkLispList l ==
  ans := nil()$List(INF)
  for s in l repeat
    (u := mkLisp s) case "failed" => return "failed"
    ans := concat(u::INF, ans)
  reverse_! ans


mkLisp s ==
  atom? s => s
  op := first(l := destruct s)
  (u := mkLispList rest l) case "failed" => "failed"
  ll := u::List(INF)
  streqlist?(op, ["+","*","/","-"]) => convert(concat(op, ll))@INF
  streq?(op, "**") => gencode("EXPT", ll)
  streqlist?(op, ["exp","sin","cos","tan","atan",
     "log", "sinh","cosh","tanh","asinh","acosh","atanh","sqrt"]) =>
        gencode(upperCase string symbol op, ll)
  streq?(op, "nthRoot") =>
    second ll = convert(2::Integer)@INF =>gencode("SQRT",[first ll])
    gencode("EXPT", concat(first ll, [1$INF / second ll]))
  streq?(op, "float") =>
    a := ll.1
    e := ll.2
    b := ll.3
```

```
          _*(a, EXPT(b, e)$Lisp)$Lisp pretend INF
        "failed"
```

⟨*MKFLCFN.dotabb*⟩≡

```
  "MKFLCFN" [color="#FF4488",href="bookvol10.4.pdf#nameddest=MKFLCFN"]
  "KONVERT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=KONVERT"]
  "MKFUNC" -> "KONVERT"
```

14.5 package MKFUNC MakeFunction

⟨*MakeFunction.input*⟩≡

```
)set break resume
)spool MakeFunction.output
)set message test on
)set message auto off
)clear all
--S 1 of 9
expr := (x - exp x + 1)^2 * (sin(x^2) * x + 1)^3
--R
--R
--R   (1)
--R       3   x 2        4     3   x    5     4    3      2 3
--R     (x (%e )  + (- 2x  - 2x )%e  + x  + 2x  + x )sin(x )
--R   +
--R        2   x 2        3     2   x     4     3     2      2 2
--R     (3x (%e )  + (- 6x  - 6x )%e  + 3x  + 6x  + 3x )sin(x )
--R   +
--R            x 2        2        x     3     2           2       x 2
--R     (3x (%e )  + (- 6x  - 6x)%e  + 3x  + 6x  + 3x)sin(x ) + (%e )
--R   +
--R                  x    2
--R     (- 2x - 2)%e  + x  + 2x + 1
--R                                                    Type: Expression Integer
--E 1

--S 2 of 9
function(expr, f, x)
--R
--R
--R   (2)  f
--R                                                                 Type: Symbol
--E 2

--S 3 of 9
tbl := [f(0.1 * i - 1) for i in 0..20]
--R
--R   Compiling function f with type Float -> Float
--R
--R   (3)
--R   [0.0005391844 0362701574, 0.0039657551 1844206653,
--R    0.0088545187 4833983689 2, 0.0116524883 0907069695,
--R    0.0108618220 9245751364 5, 0.0076366823 2120869965 06,
--R    0.0040584985 7597822062 55, 0.0015349542 8910500836 48,
--R    0.0003424903 1549879905 716, 0.0000233304 8276098819 6001, 0.0,
```

```
--R    0.0000268186 8782862599 4229, 0.0004691571 3720051642 621,
--R    0.0026924576 5968519586 08, 0.0101486881 7369135148 8,
--R    0.0313833725 8543810564 3, 0.0876991144 5154615297 9,
--R    0.2313019789 3439968362, 0.5843743955 958098772, 1.4114930171 992819197,
--R    3.2216948276 75164252]
--R                                                          Type: List Float
--E 3

--S 4 of 9
e := (x - y + 1)^2 * (x^2 * y + 1)^2
--R
--R
--R   (4)
--R      4 4        5     4     2  3     6     5    4     3     2      2
--R     x y  + (- 2x  - 2x  + 2x )y  + (x  + 2x  + x  - 4x  - 4x  + 1)y
--R   +
--R        4     3     2              2
--R     (2x  + 4x  + 2x  - 2x - 2)y + x  + 2x + 1
--R                                                  Type: Polynomial Integer
--E 4

--S 5 of 9
function(e, g, [x, y])
--R
--R
--R   (5)  g
--R                                                              Type: Symbol
--E 5

--S 6 of 9
function(e, h, x, y)
--R
--R
--R   (6)  h
--R                                                              Type: Symbol
--E 6

--S 7 of 9
m1 := squareMatrix [ [1, 2], [3, 4] ]
--R
--R
--R        +1  2+
--R   (7)  |    |
--R        +3  4+
--R                                          Type: SquareMatrix(2,Integer)
--E 7
```

```
--S 8 of 9
m2 := squareMatrix [ [1, 0], [-1, 1] ]
--R
--R
--R        + 1   0+
--R   (8)  |      |
--R        +- 1  1+
--R                                                Type: SquareMatrix(2,Integer)
--E 8

--S 9 of 9
h(m1, m2)
--R
--R   Compiling function h with type (SquareMatrix(2,Integer),SquareMatrix
--R      (2,Integer)) -> SquareMatrix(2,Integer)
--R
--R        +- 7836   8960 +
--R   (9)  |              |
--R        +- 17132  19588+
--R                                                Type: SquareMatrix(2,Integer)
--E 9
)spool
)lisp (bye)
```

⟨*MakeFunction.help*⟩≡

```
==========================================================================
MakeFunction examples
==========================================================================

It is sometimes useful to be able to define a function given by
the result of a calculation.

Suppose that you have obtained the following expression after several
computations and that you now want to tabulate the numerical values of
f for x between -1 and +1 with increment 0.1.

  expr := (x - exp x + 1)^2 * (sin(x^2) * x + 1)^3
       3   x 2        4     3   x    5     4    3      2 3
    (x (%e )  + (- 2x  - 2x )%e  + x  + 2x  + x )sin(x )
   +
        2   x 2        3     2   x     4     3     2      2 2
    (3x (%e )  + (- 6x  - 6x )%e  + 3x  + 6x  + 3x )sin(x )
   +
            x 2        2        x     3     2           2       x 2
    (3x (%e )  + (- 6x  - 6x)%e  + 3x  + 6x  + 3x)sin(x ) + (%e )
   +
                 x    2
    (- 2x - 2)%e  + x  + 2x + 1
                                Type: Expression Integer

You could, of course, use the function eval within a loop and evaluate
expr twenty-one times, but this would be quite slow.  A better way is
to create a numerical function f such that f(x) is defined by the
expression expr above, but without retyping expr!  The package
MakeFunction provides the operation function which does exactly this.

Issue this to create the function f(x) given by expr.

  function(expr, f, x)
    f
                                Type: Symbol

To tabulate expr, we can now quickly evaluate f 21 times.

  tbl := [f(0.1 * i - 1) for i in 0..20];
                                Type: List Float

Use the list [x1,...,xn] as the third argument to function to create a
multivariate function f(x1,...,xn).
```

```
e := (x - y + 1)^2 * (x^2 * y + 1)^2
    4 4        5     4     2  3     6     5    4     3     2      2
   x y  + (- 2x  - 2x  + 2x )y  + (x  + 2x  + x  - 4x  - 4x  + 1)y
 +
      4     3     2               2
   (2x  + 4x  + 2x  - 2x - 2)y + x  + 2x + 1
                                Type: Polynomial Integer
```

```
function(e, g, [x, y])
  g
                               Type: Symbol
```

In the case of just two variables, they can be given as arguments without making them into a list.

```
function(e, h, x, y)
  h
                               Type: Symbol
```

Note that the functions created by function are not limited to floating point numbers, but can be applied to any type for which they are defined.

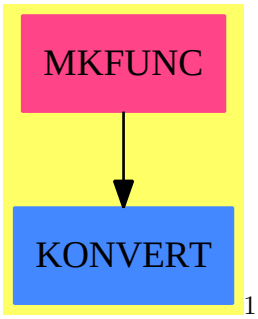
```
m1 := squareMatrix [ [1, 2], [3, 4] ]
  +1  2+
  |    |
  +3  4+
                               Type: SquareMatrix(2,Integer)
```

```
m2 := squareMatrix [ [1, 0], [-1, 1] ]
  + 1   0+
  |      |
  +- 1  1+
                               Type: SquareMatrix(2,Integer)
```

```
h(m1, m2)
  +- 7836   8960 +
  |              |
  +- 17132  19588+
                               Type: SquareMatrix(2,Integer)
```

See Also:
o)show MakeFunction

14.6 MakeFunction



1.00

Exports:
function

⟨*package MKFUNC MakeFunction*⟩≡

```
)abbrev package MKFUNC MakeFunction
++ Tools for making interpreter functions from top-level expressions
++ Author: Manuel Bronstein
++ Date Created: 22 Nov 1988
++ Date Last Updated: 8 Jan 1990
++ Description: transforms top-level objects into interpreter functions.
MakeFunction(S:ConvertibleTo InputForm): Exports == Implementation where
  SY ==> Symbol

  Exports ==> with
    function: (S, SY         ) -> SY
      ++ function(e, foo) creates a function \spad{foo() == e}.
    function: (S, SY,      SY) -> SY
      ++ function(e, foo, x) creates a function \spad{foo(x) == e}.
    function: (S, SY, SY,  SY) -> SY
      ++ function(e, foo, x, y) creates a function \spad{foo(x, y) = e}.
    function: (S, SY, List SY) -> SY
      ++ \spad{function(e, foo, [x1,...,xn])} creates a function
      ++ \spad{foo(x1,...,xn) == e}.

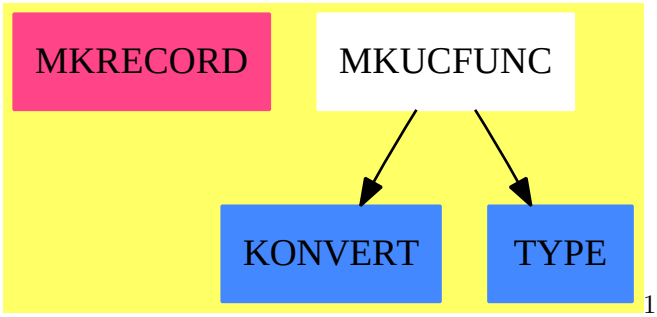
  Implementation ==> add
    function(s, name)            == function(s, name, nil())
    function(s:S, name:SY, x:SY) == function(s, name, [x])
    function(s, name, x, y)      == function(s, name, [x, y])

    function(s:S, name:SY, args:List SY) ==
      interpret function(convert s, args, name)$InputForm
      name
```

⟨*MKFUNC.dotabb*⟩≡
```
"MKFUNC" [color="#FF4488",href="bookvol10.4.pdf#nameddest=MKFUNC"]
"KONVERT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=KONVERT"]
"MKFUNC" -> "KONVERT"
```

14.7 package MKRECORD MakeRecord

14.8 MakeRecord



1.00

Exports:
makeRecord

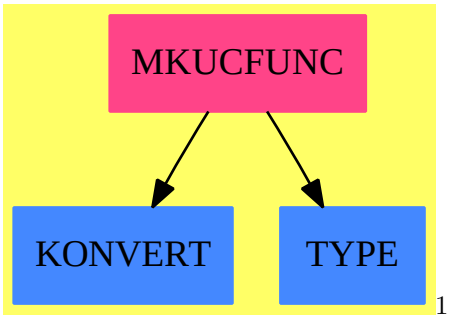
⟨*package MKRECORD MakeRecord*⟩≡
```
)abbrev package MKRECORD MakeRecord
++ Description:
++  MakeRecord is used internally by the interpreter to create record
++  types which are used for doing parallel iterations on streams.
MakeRecord(S: Type, T: Type): public == private where
  public == with
    makeRecord: (S,T) -> Record(part1: S, part2: T)
      ++ makeRecord(a,b) creates a record object with type
      ++ Record(part1:S, part2:R),
      ++ where part1 is \spad{a} and part2 is \spad{b}.
  private == add
    makeRecord(s: S, t: T)  ==
      [s,t]$Record(part1: S, part2: T)
```

⟨*MKRECORD.dotabb*⟩≡

```
"MKRECORD" [color="#FF4488",href="bookvol10.4.pdf#nameddest=MKRECORD"]
"KONVERT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=KONVERT"]
"TYPE" [color="#4488FF",href="bookvol10.2.pdf#nameddest=TYPE"]
"MKUCFUNC" -> "KONVERT"
"MKUCFUNC" -> "TYPE"
```

14.9 package MKUCFUNC MakeUnaryCompiledFunction

14.10 MakeUnaryCompiledFunction



1.00

Exports:
unaryFunction compiledFunction

⟨*package MKUCFUNC MakeUnaryCompiledFunction*⟩≡

```
)abbrev package MKUCFUNC MakeUnaryCompiledFunction
++ Tools for making compiled functions from top-level expressions
++ Author: Manuel Bronstein
++ Date Created: 1 Dec 1988
++ Date Last Updated: 5 Mar 1990
++ Description: transforms top-level objects into compiled functions.
MakeUnaryCompiledFunction(S, D, I): Exports == Implementation where
  S: ConvertibleTo InputForm
  D, I: Type

  SY  ==> Symbol
  DI  ==> devaluate(D -> I)$Lisp

  Exports ==> with
    unaryFunction   : SY -> (D -> I)
      ++ unaryFunction(a) is a local function
    compiledFunction: (S, SY) -> (D -> I)
      ++ compiledFunction(expr, x) returns a function \spad{f: D -> I}
      ++ defined by \spad{f(x) == expr}.
      ++ Function f is compiled and directly
      ++ applicable to objects of type D.

  Implementation ==> add
    import MakeFunction(S)

    func: (SY, D) -> I
```

```
func(name, x)       == FUNCALL(name, x, NIL$Lisp)$Lisp
unaryFunction name  == func(name, #1)

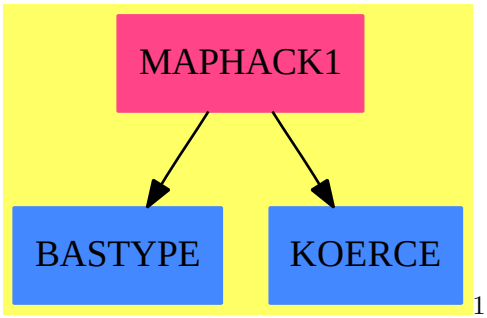
compiledFunction(e:S, x:SY) ==
  t := [convert([devaluate(D)$Lisp]$List(InputForm))
       ]$List(InputForm)
  unaryFunction compile(function(e, declare DI, x), t)
```

⟨*MKUCFUNC.dotabb*⟩≡

```
"MKUCFUNC" [color="#FF4488",href="bookvol10.4.pdf#nameddest=MKUCFUNC"]
"KONVERT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=KONVERT"]
"TYPE" [color="#4488FF",href="bookvol10.2.pdf#nameddest=TYPE"]
"MKUCFUNC" -> "KONVERT"
"MKUCFUNC" -> "TYPE"
```

14.11 package MAPHACK1 MappingPackageInternalHacks1

14.12 MappingPackageInternalHacks1



1.00

Exports:
iter recur

⟨*package MAPHACK1 MappingPackageInternalHacks1*⟩≡

```
)abbrev package MAPHACK1 MappingPackageInternalHacks1
++ Author: S.M.Watt and W.H.Burge
++ Date Created:Jan 87
++ Date Last Updated:Feb 92
++ Basic Operations:
++ Related Domains:
++ Also See:
++ AMS Classifications:
++ Keywords:
++ Examples:
++ References:
++ Description: various Currying operations.
MappingPackageInternalHacks1(A: SetCategory): MPcat == MPdef where
    NNI ==> NonNegativeInteger

    MPcat == with
        iter:  ((A -> A), NNI, A) -> A
          ++\spad{iter(f,n,x)} applies \spad{f n} times to \spad{x}.
        recur: ((NNI, A)->A, NNI, A) -> A
          ++\spad{recur(n,g,x)} is \spad{g(n,g(n-1,..g(1,x)..))}.

    MPdef == add
        iter(g,n,x)  ==
            for i in 1..n repeat x := g x     -- g(g(..(x)..))
            x
        recur(g,n,x) ==
```

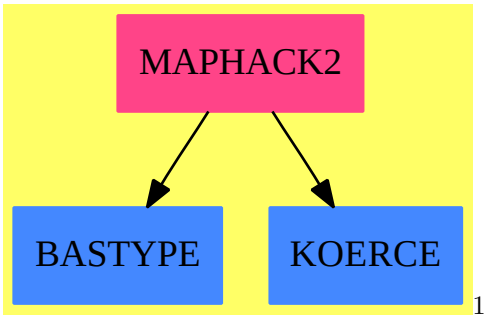
```
            for i in 1..n repeat x := g(i,x)  -- g(n,g(n-1,..g(1,x)..))
            x
```

⟨*MAPHACK1.dotabb*⟩≡

```
"MAPHACK1" [color="#FF4488",href="bookvol10.4.pdf#nameddest=MAPHACK1"]
"BASTYPE" [color="#4488FF",href="bookvol10.2.pdf#nameddest=BASTYPE"]
"KOERCE" [color="#4488FF",href="bookvol10.2.pdf#nameddest=KOERCE"]
"MAPHACK1" -> "BASTYPE"
"MAPHACK1" -> "KOERCE"
```

14.13 package MAPHACK2 MappingPackageInternalHacks2

14.14 MappingPackageInternalHacks2



1.00

Exports:
arg1 arg2

⟨*package MAPHACK2 MappingPackageInternalHacks2*⟩≡
```
)abbrev package MAPHACK2 MappingPackageInternalHacks2
++ Description: various Currying operations.
MappingPackageInternalHacks2(A: SetCategory, C: SetCategory):_
  MPcat == MPdef where
    NNI ==> NonNegativeInteger

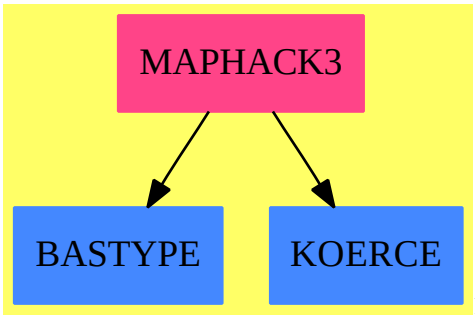
    MPcat == with
        arg1:  (A, C) -> A
          ++\spad{arg1(a,c)} selects its first argument.
        arg2:  (A, C) -> C
          ++\spad{arg2(a,c)} selects its second argument.

    MPdef == add
        arg1(a, c)   == a
        arg2(a, c)   == c
```

⟨*MAPHACK2.dotabb*⟩≡
```
"MAPHACK2" [color="#FF4488",href="bookvol10.4.pdf#nameddest=MAPHACK2"]
"BASTYPE" [color="#4488FF",href="bookvol10.2.pdf#nameddest=BASTYPE"]
"KOERCE" [color="#4488FF",href="bookvol10.2.pdf#nameddest=KOERCE"]
"MAPHACK2" -> "BASTYPE"
"MAPHACK2" -> "KOERCE"
```

14.15 package MAPHACK3 MappingPackageInternalHacks3

14.16 MappingPackageInternalHacks3



1.00

Exports:
comp

⟨*package MAPHACK3 MappingPackageInternalHacks3*⟩≡
```
)abbrev package MAPHACK3 MappingPackageInternalHacks3
++ Description: various Currying operations.
MappingPackageInternalHacks3(A: SetCategory, B: SetCategory, C: SetCategory):_
  MPcat == MPdef where
    NNI ==> NonNegativeInteger

    MPcat == with
        comp:  (B->C, A->B, A) -> C
          ++\spad{comp(f,g,x)} is \spad{f(g x)}.

    MPdef == add
        comp(g,h,x)  == g h x
```

⟨*MAPHACK3.dotabb*⟩≡
```
"MAPHACK3" [color="#FF4488",href="bookvol10.4.pdf#nameddest=MAPHACK3"]
"BASTYPE" [color="#4488FF",href="bookvol10.2.pdf#nameddest=BASTYPE"]
"KOERCE" [color="#4488FF",href="bookvol10.2.pdf#nameddest=KOERCE"]
"MAPHACK3" -> "BASTYPE"
"MAPHACK3" -> "KOERCE"
```

14.17 package MAPPKG1 MappingPackage1

⟨*MappingPackage1.input*⟩≡

```
)set break resume
)spool MappingPackage1.output
)set message test on
)set message auto off
)clear all

--S 1 of 26
power(q: FRAC INT, n: INT): FRAC INT == q**n
--R
--R   Function declaration power : (Fraction Integer,Integer) -> Fraction
--R      Integer has been added to workspace.
--R                                                                   Type: Void
--E 1

--S 2 of 26
power(2,3)
--R
--R   Compiling function power with type (Fraction Integer,Integer) ->
--R      Fraction Integer
--R
--R   (2)  8
--R                                                    Type: Fraction Integer
--E 2

--S 3 of 26
rewop := twist power
--R
--R
--I   (3)  theMap(MAPPKG3;twist;MM;5!0)
--R                       Type: ((Integer,Fraction Integer) -> Fraction Integer)
--E 3

--S 4 of 26
rewop(3, 2)
--R
--R
--R   (4)  8
--R                                                    Type: Fraction Integer
--E 4

--S 5 of 26
square: FRAC INT -> FRAC INT
--R
```

```
--R                                                                    Type: Void
--E 5

--S 6 of 26
square:= curryRight(power, 2)
--R
--R
--I   (6)  theMap(MAPPKG3;curryRight;MBM;1!0,0)
--R                                  Type: (Fraction Integer -> Fraction Integer)
--E 6

--S 7 of 26
square 4
--R
--R
--R   (7)  16
--R                                                        Type: Fraction Integer
--E 7

--S 8 of 26
squirrel:= constantRight(square)$MAPPKG3(FRAC INT,FRAC INT,FRAC INT)
--R
--R
--I   (8)  theMap(MAPPKG3;constantRight;MM;3!0)
--R              Type: ((Fraction Integer,Fraction Integer) -> Fraction Integer)
--E 8

--S 9 of 26
squirrel(1/2, 1/3)
--R
--R
--R        1
--R   (9)  -
--R        4
--R                                                        Type: Fraction Integer
--E 9

--S 10 of 26
sixteen := curry(square, 4/1)
--R
--R
--I   (10)  theMap(MAPPKG2;curry;MAM;2!0,0)
--R                                                 Type: (() -> Fraction Integer)
--E 10

--S 11 of 26
```

```
sixteen()
--R
--R
--R   (11)  16
--R                                                    Type: Fraction Integer
--E 11

--S 12 of 26
square2:=square*square
--R
--R
--I   (12)  theMap(MAPPKG3;*;MMM;6!0,0)
--R                                 Type: (Fraction Integer -> Fraction Integer)
--E 12

--S 13 of 26
square2 3
--R
--R
--R   (13)  81
--R                                                    Type: Fraction Integer
--E 13

--S 14 of 26
sc(x: FRAC INT): FRAC INT == x + 1
--R
--R   Function declaration sc : Fraction Integer -> Fraction Integer has
--R      been added to workspace.
--R                                                                 Type: Void
--E 14

--S 15 of 26
incfns := [sc**i for i in 0..10]
--R
--R   Compiling function sc with type Fraction Integer -> Fraction Integer
--R
--R
--R   (15)
--I   [theMap(MAPPKG1;**;MNniM;6!0,0), theMap(MAPPKG1;**;MNniM;6!0,0),
--I    theMap(MAPPKG1;**;MNniM;6!0,0), theMap(MAPPKG1;**;MNniM;6!0,0),
--I    theMap(MAPPKG1;**;MNniM;6!0,0), theMap(MAPPKG1;**;MNniM;6!0,0),
--I    theMap(MAPPKG1;**;MNniM;6!0,0), theMap(MAPPKG1;**;MNniM;6!0,0),
--I    theMap(MAPPKG1;**;MNniM;6!0,0), theMap(MAPPKG1;**;MNniM;6!0,0),
--I    theMap(MAPPKG1;**;MNniM;6!0,0)]
--R                            Type: List (Fraction Integer -> Fraction Integer)
--E 15
```

```
--S 16 of 26
[f 4 for f in incfns]
--R
--R
--R   (16)  [4,5,6,7,8,9,10,11,12,13,14]
--R                                                    Type: List Fraction Integer
--E 16

--S 17 of 26
times(n:NNI, i:INT):INT == n*i
--R
--R   Function declaration times : (NonNegativeInteger,Integer) -> Integer
--R      has been added to workspace.
--R                                                                        Type: Void
--E 17

--S 18 of 26
r := recur(times)
--R
--R   Compiling function times with type (NonNegativeInteger,Integer) ->
--R      Integer
--R
--I   (18)  theMap(MAPPKG1;recur;2M;7!0,0)
--R                                Type: ((NonNegativeInteger,Integer) -> Integer)
--E 18

--S 19 of 26
fact := curryRight(r, 1)
--R
--R
--I   (19)  theMap(MAPPKG3;curryRight;MBM;1!0,0)
--R                                        Type: (NonNegativeInteger -> Integer)
--E 19

--S 20 of 26
fact 4
--R
--R
--R   (20)  24
--R                                                        Type: PositiveInteger
--E 20

--S 21 of 26
mto2ton(m, n) ==
  raiser := square^n
```

```
  raiser m
--R
--R                                                                  Type: Void
--E 21

--S 22 of 26
mto2ton(3, 3)
--R
--R   Compiling function mto2ton with type (PositiveInteger,
--R      PositiveInteger) -> Fraction Integer
--R
--R   (22)  6561
--R                                                    Type: Fraction Integer
--E 22

--S 23 of 26
shiftfib(r: List INT) : INT ==
  t := r.1
  r.1 := r.2
  r.2 := r.2 + t
  t
--R
--R   Function declaration shiftfib : List Integer -> Integer has been
--R      added to workspace.
--R                                                                  Type: Void
--E 23

--S 24 of 26
fibinit: List INT := [0, 1]
--R
--R
--R   (24)  [0,1]
--R                                                      Type: List Integer
--E 24

--S 25 of 26
fibs := curry(shiftfib, fibinit)
--R
--R   Compiling function shiftfib with type List Integer -> Integer
--R
--I   (25)  theMap(MAPPKG2;curry;MAM;2!0,0)
--R                                                   Type: (() -> Integer)
--E 25

--S 26 of 26
[fibs() for i in 0..30]
```

```
--R
--R
--R   (26)
--R   [0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597,
--R    2584, 4181, 6765, 10946, 17711, 28657, 46368, 75025, 121393, 196418,
--R    317811, 514229, 832040]
--R                                                        Type: List Integer
--E 26

)spool
)lisp (bye)
```

⟨*MappingPackage1.help*⟩≡

```
==========================================================================
MappingPackage examples
==========================================================================

Function are objects of type Mapping.  In this section we demonstrate
some library operations from the packages MappingPackage1, MappingPackage2,
and MappingPackage3 that manipulate and create functions. Some terminology:
a nullary function takes no arguments, a unary function takes one argument,
and a binary function takes two arguments.

We begin by creating an example function that raises a rational number
to an integer exponent.

  power(q: FRAC INT, n: INT): FRAC INT == q**n
                       Type: Void

  power(2,3)
    8
                        Type: Fraction Integer

The twist operation transposes the arguments of a binary function.
Here rewop(a, b) is power(b, a).

  rewop := twist power
    theMap(MAPPKG3;twist;MM;5!0)
                        Type: ((Integer,Fraction Integer) -> Fraction Integer)

This is 2^3.

  rewop(3, 2)
    8
                        Type: Fraction Integer

Now we define square in terms of power.

  square: FRAC INT -> FRAC INT
                        Type: Void
The curryRight operation creates a unary function from a binary one by
providing a constant argument on the right.

  square:= curryRight(power, 2)
    theMap(MAPPKG3;curryRight;MBM;1!0,0)
                        Type: (Fraction Integer -> Fraction Integer)

Likewise, the curryLeft operation provides a constant argument on the
```

```
left.

  square 4
    16
                        Type: Fraction Integer

The constantRight operation creates (in a trivial way) a binary
function from a unary one: constantRight(f) is the function g such
that g(a,b)= f(a).

  squirrel:= constantRight(square)$MAPPKG3(FRAC INT,FRAC INT,FRAC INT)
    theMap(MAPPKG3;constantRight;MM;3!0)
              Type: ((Fraction Integer,Fraction Integer) -> Fraction Integer)

Likewise, constantLeft(f) is the function g such that g(a,b)= f(b).

  squirrel(1/2, 1/3)
    1
    -
    4
                        Type: Fraction Integer

The curry operation makes a unary function nullary.

  sixteen := curry(square, 4/1)
    theMap(MAPPKG2;curry;MAM;2!0,0)
                        Type: (() -> Fraction Integer)

  sixteen()
    16
                        Type: Fraction Integer

The * operation constructs composed functions.

  square2:=square*square
    theMap(MAPPKG3;*;MMM;6!0,0)
                        Type: (Fraction Integer -> Fraction Integer)

  square2 3
    81
                        Type: Fraction Integer

Use the ** operation to create functions that are n-fold iterations of
other functions.

  sc(x: FRAC INT): FRAC INT == x + 1
```

```
                        Type: Void

This is a list of Mapping objects.

  incfns := [sc**i for i in 0..10]
   [theMap(MAPPKG1;**;MNniM;6!0,0), theMap(MAPPKG1;**;MNniM;6!0,0),
    theMap(MAPPKG1;**;MNniM;6!0,0), theMap(MAPPKG1;**;MNniM;6!0,0),
    theMap(MAPPKG1;**;MNniM;6!0,0), theMap(MAPPKG1;**;MNniM;6!0,0),
    theMap(MAPPKG1;**;MNniM;6!0,0), theMap(MAPPKG1;**;MNniM;6!0,0),
    theMap(MAPPKG1;**;MNniM;6!0,0), theMap(MAPPKG1;**;MNniM;6!0,0),
    theMap(MAPPKG1;**;MNniM;6!0,0)]
                        Type: List (Fraction Integer -> Fraction Integer)

This is a list of applications of those functions.

  [f 4 for f in incfns]
    [4,5,6,7,8,9,10,11,12,13,14]
                        Type: List Fraction Integer

Use the recur operation for recursion:

  g := recur f means g(n,x) == f(n,f(n-1,...f(1,x))).

  times(n:NNI, i:INT):INT == n*i
                        Type: Void

  r := recur(times)
    theMap(MAPPKG1;recur;2M;7!0,0)
                        Type: ((NonNegativeInteger,Integer) -> Integer)

This is a factorial function.

  fact := curryRight(r, 1)
    theMap(MAPPKG3;curryRight;MBM;1!0,0)
                        Type: (NonNegativeInteger -> Integer)

  fact 4
    24
                        Type: PositiveInteger

Constructed functions can be used within other functions.

  mto2ton(m, n) ==
    raiser := square^n
    raiser m
                        Type: Void
```

```
This is 3^(2^3).

  mto2ton(3, 3)
    6561
                        Type: Fraction Integer

Here shiftfib is a unary function that modifies its argument.

  shiftfib(r: List INT) : INT ==
    t := r.1
    r.1 := r.2
    r.2 := r.2 + t
    t
                        Type: Void

By currying over the argument we get a function with private state.

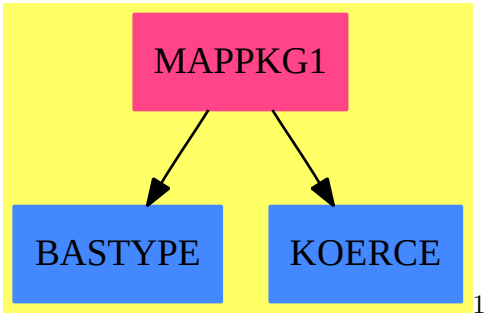
  fibinit: List INT := [0, 1]
    [0,1]
                        Type: List Integer

  fibs := curry(shiftfib, fibinit)
    theMap(MAPPKG2;curry;MAM;2!0,0)
                        Type: (() -> Integer)

  [fibs() for i in 0..30]
   [0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597,
    2584, 4181, 6765, 10946, 17711, 28657, 46368, 75025, 121393, 196418,
    317811, 514229, 832040]
                        Type: List Integer

See Also:
o )show MappingPackage1
o )help MappingPackage2
o )help MappingPackage3
o )help MappingPackage4
```

14.18 MappingPackage1



1.00

Exports:
coerce fixedPoint id nullary recur ?**?

⟨*package MAPPKG1 MappingPackage1*⟩≡
```
)abbrev package MAPPKG1 MappingPackage1
++ Author: S.M.Watt and W.H.Burge
++ Date Created:Jan 87
++ Date Last Updated:Feb 92
++ Basic Operations:
++ Related Domains:
++ Also See:
++ AMS Classifications:
++ Keywords:
++ Examples:
++ References:
++ Description: various Currying operations.
MappingPackage1(A:SetCategory): MPcat == MPdef where
    NNI   ==>  NonNegativeInteger

    MPcat ==  with
        nullary: A           -> (()->A)
          ++\spad{nullary A} changes its argument into a
          ++ nullary function.
        coerce:  A           -> (()->A)
          ++\spad{coerce A} changes its argument into a
          ++ nullary function.

        fixedPoint: (A->A) -> A
          ++\spad{fixedPoint f} is the fixed point of function \spad{f}.
          ++ i.e. such that \spad{fixedPoint f = f(fixedPoint f)}.
        fixedPoint: (List A->List A, Integer) -> List A
          ++\spad{fixedPoint(f,n)} is the fixed point of function
          ++ \spad{f} which is assumed to transform a list of length
          ++ \spad{n}.
```

```
    id:    A -> A
      ++\spad{id x} is \spad{x}.
    "**":  (A->A, NNI)  -> (A->A)
      ++\spad{f**n} is the  function which is the n-fold application
      ++ of \spad{f}.

    recur: ((NNI, A)->A) -> ((NNI, A)->A)
      ++\spad{recur(g)} is the function \spad{h} such that
      ++ \spad{h(n,x)= g(n,g(n-1,..g(1,x)..))}.


MPdef == add

    MappingPackageInternalHacks1(A)

    a: A
    faa:  A -> A
    f0a:  ()-> A

    nullary a   == a
    coerce  a   == nullary a
    fixedPoint faa ==
        g0 := GENSYM()$Lisp
        g1 := faa g0
        EQ(g0, g1)$Lisp => error "All points are fixed points"
        GEQNSUBSTLIST([g0]$Lisp, [g1]$Lisp, g1)$Lisp

    fixedPoint(fll, n) ==
        g0 := [(GENSYM()$Lisp):A for i in 1..n]
        g1 := fll g0
        or/[EQ(e0,e1)$Lisp for e0 in g0 for e1 in g1] =>
            error "All points are fixed points"
        GEQNSUBSTLIST(g0, g1, g1)$Lisp

    -- Composition and recursion.
    id a        == a
    g**n        == iter(g, n, #1)

    recur fnaa  == recur(fnaa, #1, #2)
```

⟨*MAPPKG1.dotabb*⟩≡

```
"MAPPKG1" [color="#FF4488",href="bookvol10.4.pdf#nameddest=MAPPKG1"]
"BASTYPE" [color="#4488FF",href="bookvol10.2.pdf#nameddest=BASTYPE"]
"KOERCE" [color="#4488FF",href="bookvol10.2.pdf#nameddest=KOERCE"]
"MAPPKG1" -> "BASTYPE"
"MAPPKG1" -> "KOERCE"
```

14.19 package MAPPKG2 MappingPackage2

⟨*MappingPackage2.input*⟩≡

```
)set break resume
)spool MappingPackage2.output
)set message test on
)set message auto off
)clear all
--S 1 of 26
power(q: FRAC INT, n: INT): FRAC INT == q**n
--R
--R   Function declaration power : (Fraction Integer,Integer) -> Fraction
--R      Integer has been added to workspace.
--R                                                                   Type: Void
--E 1

--S 2 of 26
power(2,3)
--R
--R   Compiling function power with type (Fraction Integer,Integer) ->
--R      Fraction Integer
--R
--R   (2)  8
--R                                                    Type: Fraction Integer
--E 2

--S 3 of 26
rewop := twist power
--R
--R
--I   (3)  theMap(MAPPKG3;twist;MM;5!0)
--R                        Type: ((Integer,Fraction Integer) -> Fraction Integer)
--E 3

--S 4 of 26
rewop(3, 2)
--R
--R
--R   (4)  8
--R                                                    Type: Fraction Integer
--E 4

--S 5 of 26
square: FRAC INT -> FRAC INT
--R
--R                                                                   Type: Void
```

```
--E 5

--S 6 of 26
square:= curryRight(power, 2)
--R
--R
--I   (6)  theMap(MAPPKG3;curryRight;MBM;1!0,0)
--R                                  Type: (Fraction Integer -> Fraction Integer)
--E 6

--S 7 of 26
square 4
--R
--R
--R   (7)  16
--R                                                        Type: Fraction Integer
--E 7

--S 8 of 26
squirrel:= constantRight(square)$MAPPKG3(FRAC INT,FRAC INT,FRAC INT)
--R
--R
--I   (8)  theMap(MAPPKG3;constantRight;MM;3!0)
--R              Type: ((Fraction Integer,Fraction Integer) -> Fraction Integer)
--E 8

--S 9 of 26
squirrel(1/2, 1/3)
--R
--R
--R        1
--R   (9)  -
--R        4
--R                                                        Type: Fraction Integer
--E 9

--S 10 of 26
sixteen := curry(square, 4/1)
--R
--R
--I   (10)  theMap(MAPPKG2;curry;MAM;2!0,0)
--R                                                Type: (() -> Fraction Integer)
--E 10

--S 11 of 26
sixteen()
```

```
--R
--R
--R   (11)  16
--R                                                    Type: Fraction Integer
--E 11

--S 12 of 26
square2:=square*square
--R
--R
--I   (12)  theMap(MAPPKG3;*;MMM;6!0,0)
--R                                  Type: (Fraction Integer -> Fraction Integer)
--E 12

--S 13 of 26
square2 3
--R
--R
--R   (13)  81
--R                                                    Type: Fraction Integer
--E 13

--S 14 of 26
sc(x: FRAC INT): FRAC INT == x + 1
--R
--R   Function declaration sc : Fraction Integer -> Fraction Integer has
--R      been added to workspace.
--R                                                                 Type: Void
--E 14

--S 15 of 26
incfns := [sc**i for i in 0..10]
--R
--R   Compiling function sc with type Fraction Integer -> Fraction Integer
--R
--R
--R   (15)
--I   [theMap(MAPPKG1;**;MNniM;6!0,0), theMap(MAPPKG1;**;MNniM;6!0,0),
--I    theMap(MAPPKG1;**;MNniM;6!0,0), theMap(MAPPKG1;**;MNniM;6!0,0),
--I    theMap(MAPPKG1;**;MNniM;6!0,0), theMap(MAPPKG1;**;MNniM;6!0,0),
--I    theMap(MAPPKG1;**;MNniM;6!0,0), theMap(MAPPKG1;**;MNniM;6!0,0),
--I    theMap(MAPPKG1;**;MNniM;6!0,0), theMap(MAPPKG1;**;MNniM;6!0,0),
--I    theMap(MAPPKG1;**;MNniM;6!0,0)]
--R                            Type: List (Fraction Integer -> Fraction Integer)
--E 15
```

```
--S 16 of 26
[f 4 for f in incfns]
--R
--R
--R   (16)  [4,5,6,7,8,9,10,11,12,13,14]
--R                                                    Type: List Fraction Integer
--E 16

--S 17 of 26
times(n:NNI, i:INT):INT == n*i
--R
--R   Function declaration times : (NonNegativeInteger,Integer) -> Integer
--R      has been added to workspace.
--R                                                                        Type: Void
--E 17

--S 18 of 26
r := recur(times)
--R
--R   Compiling function times with type (NonNegativeInteger,Integer) ->
--R      Integer
--R
--I   (18)  theMap(MAPPKG1;recur;2M;7!0,0)
--R                                Type: ((NonNegativeInteger,Integer) -> Integer)
--E 18

--S 19 of 26
fact := curryRight(r, 1)
--R
--R
--I   (19)  theMap(MAPPKG3;curryRight;MBM;1!0,0)
--R                                          Type: (NonNegativeInteger -> Integer)
--E 19

--S 20 of 26
fact 4
--R
--R
--R   (20)  24
--R                                                          Type: PositiveInteger
--E 20

--S 21 of 26
mto2ton(m, n) ==
  raiser := square^n
  raiser m
```

```
--R
--R                                                                    Type: Void
--E 21

--S 22 of 26
mto2ton(3, 3)
--R
--R   Compiling function mto2ton with type (PositiveInteger,
--R      PositiveInteger) -> Fraction Integer
--R
--R   (22)  6561
--R                                                  Type: Fraction Integer
--E 22

--S 23 of 26
shiftfib(r: List INT) : INT ==
  t := r.1
  r.1 := r.2
  r.2 := r.2 + t
  t
--R
--R   Function declaration shiftfib : List Integer -> Integer has been
--R      added to workspace.
--R                                                                    Type: Void
--E 23

--S 24 of 26
fibinit: List INT := [0, 1]
--R
--R
--R   (24)  [0,1]
--R                                                     Type: List Integer
--E 24

--S 25 of 26
fibs := curry(shiftfib, fibinit)
--R
--R   Compiling function shiftfib with type List Integer -> Integer
--R
--I   (25)  theMap(MAPPKG2;curry;MAM;2!0,0)
--R                                                   Type: (() -> Integer)
--E 25

--S 26 of 26
[fibs() for i in 0..30]
--R
```

```
--R
--R   (26)
--R   [0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597,
--R    2584, 4181, 6765, 10946, 17711, 28657, 46368, 75025, 121393, 196418,
--R    317811, 514229, 832040]
--R                                                        Type: List Integer
--E 26
)spool
)lisp (bye)
```

⟨*MappingPackage2.help*⟩≡

```
==========================================================================
MappingPackage examples
==========================================================================

Function are objects of type Mapping.  In this section we demonstrate
some library operations from the packages MappingPackage1, MappingPackage2,
and MappingPackage3 that manipulate and create functions. Some terminology:
a nullary function takes no arguments, a unary function takes one argument,
and a binary function takes two arguments.

We begin by creating an example function that raises a rational number
to an integer exponent.

  power(q: FRAC INT, n: INT): FRAC INT == q**n
                       Type: Void

  power(2,3)
    8
                        Type: Fraction Integer

The twist operation transposes the arguments of a binary function.
Here rewop(a, b) is power(b, a).

  rewop := twist power
    theMap(MAPPKG3;twist;MM;5!0)
                        Type: ((Integer,Fraction Integer) -> Fraction Integer)

This is 2^3.

  rewop(3, 2)
    8
                        Type: Fraction Integer

Now we define square in terms of power.

  square: FRAC INT -> FRAC INT
                        Type: Void
The curryRight operation creates a unary function from a binary one by
providing a constant argument on the right.

  square:= curryRight(power, 2)
    theMap(MAPPKG3;curryRight;MBM;1!0,0)
                        Type: (Fraction Integer -> Fraction Integer)

Likewise, the curryLeft operation provides a constant argument on the
```

```
left.

  square 4
    16
                        Type: Fraction Integer

The constantRight operation creates (in a trivial way) a binary
function from a unary one: constantRight(f) is the function g such
that g(a,b)= f(a).

  squirrel:= constantRight(square)$MAPPKG3(FRAC INT,FRAC INT,FRAC INT)
    theMap(MAPPKG3;constantRight;MM;3!0)
              Type: ((Fraction Integer,Fraction Integer) -> Fraction Integer)

Likewise, constantLeft(f) is the function g such that g(a,b)= f(b).

  squirrel(1/2, 1/3)
    1
    -
    4
                        Type: Fraction Integer

The curry operation makes a unary function nullary.

  sixteen := curry(square, 4/1)
    theMap(MAPPKG2;curry;MAM;2!0,0)
                        Type: (() -> Fraction Integer)

  sixteen()
    16
                        Type: Fraction Integer

The * operation constructs composed functions.

  square2:=square*square
    theMap(MAPPKG3;*;MMM;6!0,0)
                        Type: (Fraction Integer -> Fraction Integer)

  square2 3
    81
                        Type: Fraction Integer

Use the ** operation to create functions that are n-fold iterations of
other functions.

  sc(x: FRAC INT): FRAC INT == x + 1
```

```
                        Type: Void
```

This is a list of Mapping objects.

```
  incfns := [sc**i for i in 0..10]
   [theMap(MAPPKG1;**;MNniM;6!0,0), theMap(MAPPKG1;**;MNniM;6!0,0),
    theMap(MAPPKG1;**;MNniM;6!0,0), theMap(MAPPKG1;**;MNniM;6!0,0),
    theMap(MAPPKG1;**;MNniM;6!0,0), theMap(MAPPKG1;**;MNniM;6!0,0),
    theMap(MAPPKG1;**;MNniM;6!0,0), theMap(MAPPKG1;**;MNniM;6!0,0),
    theMap(MAPPKG1;**;MNniM;6!0,0), theMap(MAPPKG1;**;MNniM;6!0,0),
    theMap(MAPPKG1;**;MNniM;6!0,0)]
                        Type: List (Fraction Integer -> Fraction Integer)
```

This is a list of applications of those functions.

```
  [f 4 for f in incfns]
    [4,5,6,7,8,9,10,11,12,13,14]
                        Type: List Fraction Integer
```

Use the recur operation for recursion:

```
  g := recur f means g(n,x) == f(n,f(n-1,...f(1,x))).

  times(n:NNI, i:INT):INT == n*i
                        Type: Void

  r := recur(times)
    theMap(MAPPKG1;recur;2M;7!0,0)
                        Type: ((NonNegativeInteger,Integer) -> Integer)
```

This is a factorial function.

```
  fact := curryRight(r, 1)
    theMap(MAPPKG3;curryRight;MBM;1!0,0)
                        Type: (NonNegativeInteger -> Integer)

  fact 4
    24
                        Type: PositiveInteger
```

Constructed functions can be used within other functions.

```
  mto2ton(m, n) ==
    raiser := square^n
    raiser m
                        Type: Void
```

```
This is 3^(2^3).

  mto2ton(3, 3)
    6561
                        Type: Fraction Integer

Here shiftfib is a unary function that modifies its argument.

  shiftfib(r: List INT) : INT ==
    t := r.1
    r.1 := r.2
    r.2 := r.2 + t
    t
                        Type: Void

By currying over the argument we get a function with private state.

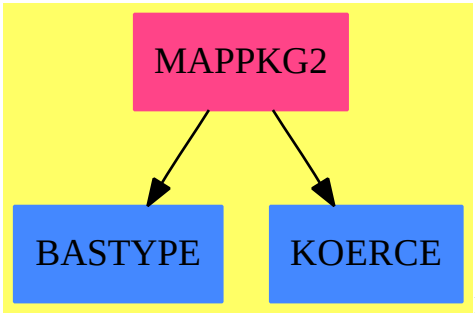
  fibinit: List INT := [0, 1]
    [0,1]
                        Type: List Integer

  fibs := curry(shiftfib, fibinit)
    theMap(MAPPKG2;curry;MAM;2!0,0)
                        Type: (() -> Integer)

  [fibs() for i in 0..30]
   [0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597,
    2584, 4181, 6765, 10946, 17711, 28657, 46368, 75025, 121393, 196418,
    317811, 514229, 832040]
                        Type: List Integer

See Also:
o )help MappingPackage1
o )show MappingPackage2
o )help MappingPackage3
o )help MappingPackage4
```

14.20 MappingPackage2



1.00

Exports:
const constant curry diag

⟨*package MAPPKG2 MappingPackage2*⟩≡
```
)abbrev package MAPPKG2 MappingPackage2
++ Description: various Currying operations.
MappingPackage2(A:SetCategory, C:SetCategory): MPcat == MPdef where
    NNI   ==>  NonNegativeInteger

    MPcat ==  with
        const:   C           -> (A ->C)
          ++\spad{const c} is a function which produces \spad{c} when
          ++ applied to its argument.

        curry:    (A ->C, A)    -> (()->C)
          ++\spad{cu(f,a)} is the function \spad{g}
          ++ such that \spad{g ()= f a}.
        constant:    (()->C)       -> (A ->C)
          ++\spad{vu(f)} is the function \spad{g}
          ++ such that \spad{g a= f ()}.

        diag:  ((A,A)->C)    -> (A->C)
          ++\spad{diag(f)} is the function \spad{g}
          ++ such that \spad{g a = f(a,a)}.


    MPdef == add

        MappingPackageInternalHacks2(A, C)

        a: A
        c: C
        faa:  A -> A
        f0c:  ()-> C
```

```
fac:  A -> C
faac: (A,A)->C

const c     == arg2(#1, c)
curry(fac, a)  == fac a
constant f0c      == arg2(#1, f0c())

diag  faac  == faac(#1, #1)
```

⟨*MAPPKG2.dotabb*⟩≡

```
"MAPPKG2" [color="#FF4488",href="bookvol10.4.pdf#nameddest=MAPPKG2"]
"BASTYPE" [color="#4488FF",href="bookvol10.2.pdf#nameddest=BASTYPE"]
"KOERCE" [color="#4488FF",href="bookvol10.2.pdf#nameddest=KOERCE"]
"MAPPKG2" -> "BASTYPE"
"MAPPKG2" -> "KOERCE"
```

14.21 package MAPPKG3 MappingPackage3

⟨*MappingPackage3.input*⟩≡

```
)set break resume
)spool MappingPackage3.output
)set message test on
)set message auto off
)clear all
--S 1 of 26
power(q: FRAC INT, n: INT): FRAC INT == q**n
--R
--R   Function declaration power : (Fraction Integer,Integer) -> Fraction
--R      Integer has been added to workspace.
--R                                                                   Type: Void
--E 1

--S 2 of 26
power(2,3)
--R
--R   Compiling function power with type (Fraction Integer,Integer) ->
--R      Fraction Integer
--R
--R   (2)  8
--R                                                    Type: Fraction Integer
--E 2

--S 3 of 26
rewop := twist power
--R
--R
--I   (3)  theMap(MAPPKG3;twist;MM;5!0)
--R                        Type: ((Integer,Fraction Integer) -> Fraction Integer)
--E 3

--S 4 of 26
rewop(3, 2)
--R
--R
--R   (4)  8
--R                                                    Type: Fraction Integer
--E 4

--S 5 of 26
square: FRAC INT -> FRAC INT
--R
--R                                                                   Type: Void
```

```
--E 5

--S 6 of 26
square:= curryRight(power, 2)
--R
--R
--I   (6)  theMap(MAPPKG3;curryRight;MBM;1!0,0)
--R                                  Type: (Fraction Integer -> Fraction Integer)
--E 6

--S 7 of 26
square 4
--R
--R
--R   (7)  16
--R                                                        Type: Fraction Integer
--E 7

--S 8 of 26
squirrel:= constantRight(square)$MAPPKG3(FRAC INT,FRAC INT,FRAC INT)
--R
--R
--I   (8)  theMap(MAPPKG3;constantRight;MM;3!0)
--R              Type: ((Fraction Integer,Fraction Integer) -> Fraction Integer)
--E 8

--S 9 of 26
squirrel(1/2, 1/3)
--R
--R
--R        1
--R   (9)  -
--R        4
--R                                                        Type: Fraction Integer
--E 9

--S 10 of 26
sixteen := curry(square, 4/1)
--R
--R
--I   (10)  theMap(MAPPKG2;curry;MAM;2!0,0)
--R                                              Type: (() -> Fraction Integer)
--E 10

--S 11 of 26
sixteen()
```

```
--R
--R
--R   (11)  16
--R                                                    Type: Fraction Integer
--E 11

--S 12 of 26
square2:=square*square
--R
--R
--I   (12)  theMap(MAPPKG3;*;MMM;6!0,0)
--R                                  Type: (Fraction Integer -> Fraction Integer)
--E 12

--S 13 of 26
square2 3
--R
--R
--R   (13)  81
--R                                                    Type: Fraction Integer
--E 13

--S 14 of 26
sc(x: FRAC INT): FRAC INT == x + 1
--R
--R   Function declaration sc : Fraction Integer -> Fraction Integer has
--R      been added to workspace.
--R                                                                 Type: Void
--E 14

--S 15 of 26
incfns := [sc**i for i in 0..10]
--R
--R   Compiling function sc with type Fraction Integer -> Fraction Integer
--R
--R
--R   (15)
--I   [theMap(MAPPKG1;**;MNniM;6!0,0), theMap(MAPPKG1;**;MNniM;6!0,0),
--I    theMap(MAPPKG1;**;MNniM;6!0,0), theMap(MAPPKG1;**;MNniM;6!0,0),
--I    theMap(MAPPKG1;**;MNniM;6!0,0), theMap(MAPPKG1;**;MNniM;6!0,0),
--I    theMap(MAPPKG1;**;MNniM;6!0,0), theMap(MAPPKG1;**;MNniM;6!0,0),
--I    theMap(MAPPKG1;**;MNniM;6!0,0), theMap(MAPPKG1;**;MNniM;6!0,0),
--I    theMap(MAPPKG1;**;MNniM;6!0,0)]
--R                             Type: List (Fraction Integer -> Fraction Integer)
--E 15
```

```
--S 16 of 26
[f 4 for f in incfns]
--R
--R
--R   (16)  [4,5,6,7,8,9,10,11,12,13,14]
--R                                                    Type: List Fraction Integer
--E 16

--S 17 of 26
times(n:NNI, i:INT):INT == n*i
--R
--R   Function declaration times : (NonNegativeInteger,Integer) -> Integer
--R      has been added to workspace.
--R                                                                        Type: Void
--E 17

--S 18 of 26
r := recur(times)
--R
--R   Compiling function times with type (NonNegativeInteger,Integer) ->
--R      Integer
--R
--I   (18)  theMap(MAPPKG1;recur;2M;7!0,0)
--R                                Type: ((NonNegativeInteger,Integer) -> Integer)
--E 18

--S 19 of 26
fact := curryRight(r, 1)
--R
--R
--I   (19)  theMap(MAPPKG3;curryRight;MBM;1!0,0)
--R                                          Type: (NonNegativeInteger -> Integer)
--E 19

--S 20 of 26
fact 4
--R
--R
--R   (20)  24
--R                                                          Type: PositiveInteger
--E 20

--S 21 of 26
mto2ton(m, n) ==
  raiser := square^n
  raiser m
```

```
--R
--R                                                                    Type: Void
--E 21

--S 22 of 26
mto2ton(3, 3)
--R
--R   Compiling function mto2ton with type (PositiveInteger,
--R      PositiveInteger) -> Fraction Integer
--R
--R   (22)  6561
--R                                                  Type: Fraction Integer
--E 22

--S 23 of 26
shiftfib(r: List INT) : INT ==
  t := r.1
  r.1 := r.2
  r.2 := r.2 + t
  t
--R
--R   Function declaration shiftfib : List Integer -> Integer has been
--R      added to workspace.
--R                                                                    Type: Void
--E 23

--S 24 of 26
fibinit: List INT := [0, 1]
--R
--R
--R   (24)  [0,1]
--R                                                     Type: List Integer
--E 24

--S 25 of 26
fibs := curry(shiftfib, fibinit)
--R
--R   Compiling function shiftfib with type List Integer -> Integer
--R
--I   (25)  theMap(MAPPKG2;curry;MAM;2!0,0)
--R                                                   Type: (() -> Integer)
--E 25

--S 26 of 26
[fibs() for i in 0..30]
--R
```

```
--R
--R   (26)
--R   [0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597,
--R    2584, 4181, 6765, 10946, 17711, 28657, 46368, 75025, 121393, 196418,
--R    317811, 514229, 832040]
--R                                                        Type: List Integer
--E 26
)spool
)lisp (bye)
```

⟨*MappingPackage3.help*⟩≡

```
==========================================================================
MappingPackage examples
==========================================================================

Function are objects of type Mapping.  In this section we demonstrate
some library operations from the packages MappingPackage1, MappingPackage2,
and MappingPackage3 that manipulate and create functions. Some terminology:
a nullary function takes no arguments, a unary function takes one argument,
and a binary function takes two arguments.

We begin by creating an example function that raises a rational number
to an integer exponent.

  power(q: FRAC INT, n: INT): FRAC INT == q**n
                       Type: Void

  power(2,3)
    8
                        Type: Fraction Integer

The twist operation transposes the arguments of a binary function.
Here rewop(a, b) is power(b, a).

  rewop := twist power
    theMap(MAPPKG3;twist;MM;5!0)
                        Type: ((Integer,Fraction Integer) -> Fraction Integer)

This is 2^3.

  rewop(3, 2)
    8
                        Type: Fraction Integer

Now we define square in terms of power.

  square: FRAC INT -> FRAC INT
                        Type: Void
The curryRight operation creates a unary function from a binary one by
providing a constant argument on the right.

  square:= curryRight(power, 2)
    theMap(MAPPKG3;curryRight;MBM;1!0,0)
                        Type: (Fraction Integer -> Fraction Integer)

Likewise, the curryLeft operation provides a constant argument on the
```

```
left.

  square 4
    16
                        Type: Fraction Integer

The constantRight operation creates (in a trivial way) a binary
function from a unary one: constantRight(f) is the function g such
that g(a,b)= f(a).

  squirrel:= constantRight(square)$MAPPKG3(FRAC INT,FRAC INT,FRAC INT)
    theMap(MAPPKG3;constantRight;MM;3!0)
              Type: ((Fraction Integer,Fraction Integer) -> Fraction Integer)

Likewise, constantLeft(f) is the function g such that g(a,b)= f(b).

  squirrel(1/2, 1/3)
    1
    -
    4
                        Type: Fraction Integer

The curry operation makes a unary function nullary.

  sixteen := curry(square, 4/1)
    theMap(MAPPKG2;curry;MAM;2!0,0)
                        Type: (() -> Fraction Integer)

  sixteen()
    16
                        Type: Fraction Integer

The * operation constructs composed functions.

  square2:=square*square
    theMap(MAPPKG3;*;MMM;6!0,0)
                        Type: (Fraction Integer -> Fraction Integer)

  square2 3
    81
                        Type: Fraction Integer

Use the ** operation to create functions that are n-fold iterations of
other functions.

  sc(x: FRAC INT): FRAC INT == x + 1
```

```
                        Type: Void
```

This is a list of Mapping objects.

```
  incfns := [sc**i for i in 0..10]
   [theMap(MAPPKG1;**;MNniM;6!0,0), theMap(MAPPKG1;**;MNniM;6!0,0),
    theMap(MAPPKG1;**;MNniM;6!0,0), theMap(MAPPKG1;**;MNniM;6!0,0),
    theMap(MAPPKG1;**;MNniM;6!0,0), theMap(MAPPKG1;**;MNniM;6!0,0),
    theMap(MAPPKG1;**;MNniM;6!0,0), theMap(MAPPKG1;**;MNniM;6!0,0),
    theMap(MAPPKG1;**;MNniM;6!0,0), theMap(MAPPKG1;**;MNniM;6!0,0),
    theMap(MAPPKG1;**;MNniM;6!0,0)]
                        Type: List (Fraction Integer -> Fraction Integer)
```

This is a list of applications of those functions.

```
  [f 4 for f in incfns]
    [4,5,6,7,8,9,10,11,12,13,14]
                        Type: List Fraction Integer
```

Use the recur operation for recursion:

```
  g := recur f means g(n,x) == f(n,f(n-1,...f(1,x))).

  times(n:NNI, i:INT):INT == n*i
                        Type: Void

  r := recur(times)
    theMap(MAPPKG1;recur;2M;7!0,0)
                        Type: ((NonNegativeInteger,Integer) -> Integer)
```

This is a factorial function.

```
  fact := curryRight(r, 1)
    theMap(MAPPKG3;curryRight;MBM;1!0,0)
                        Type: (NonNegativeInteger -> Integer)

  fact 4
    24
                        Type: PositiveInteger
```

Constructed functions can be used within other functions.

```
  mto2ton(m, n) ==
    raiser := square^n
    raiser m
                        Type: Void
```

```
This is 3^(2^3).

  mto2ton(3, 3)
    6561
                        Type: Fraction Integer

Here shiftfib is a unary function that modifies its argument.

  shiftfib(r: List INT) : INT ==
    t := r.1
    r.1 := r.2
    r.2 := r.2 + t
    t
                        Type: Void

By currying over the argument we get a function with private state.

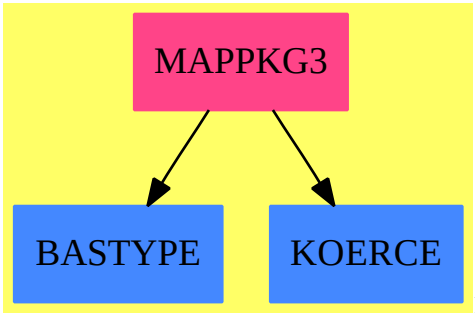
  fibinit: List INT := [0, 1]
    [0,1]
                        Type: List Integer

  fibs := curry(shiftfib, fibinit)
    theMap(MAPPKG2;curry;MAM;2!0,0)
                        Type: (() -> Integer)

  [fibs() for i in 0..30]
   [0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597,
    2584, 4181, 6765, 10946, 17711, 28657, 46368, 75025, 121393, 196418,
    317811, 514229, 832040]
                        Type: List Integer

See Also:
o )help MappingPackage1
o )help MappingPackage2
o )show MappingPackage3
o )help MappingPackage4
```

14.22 MappingPackage3



1.00

Exports:
constantLeft constantRight curryLeft curryRight twist ?*?

⟨*package MAPPKG3 MappingPackage3*⟩≡

```
)abbrev package MAPPKG3 MappingPackage3
++ Description: various Currying operations.
MappingPackage3(A:SetCategory, B:SetCategory, C:SetCategory):_
  MPcat == MPdef where
    NNI   ==>  NonNegativeInteger

    MPcat ==  with
        curryRight:   ((A,B)->C, B) -> (A ->C)
          ++\spad{curryRight(f,b)} is the function \spad{g} such that
          ++ \spad{g a = f(a,b)}.
        curryLeft:   ((A,B)->C, A) -> (B ->C)
          ++\spad{curryLeft(f,a)} is the function \spad{g}
          ++ such that \spad{g b = f(a,b)}.

        constantRight:   (A -> C)      -> ((A,B)->C)
          ++\spad{constantRight(f)} is the function \spad{g}
          ++ such that \spad{g (a,b)= f a}.
        constantLeft:   (B -> C)      -> ((A,B)->C)
          ++\spad{constantLeft(f)} is the function \spad{g}
          ++ such that \spad{g (a,b)= f b}.

        twist: ((A,B)->C)    -> ((B,A)->C)
          ++\spad{twist(f)} is the function \spad{g}
          ++ such that \spad{g (a,b)= f(b,a)}.

        "*":   (B->C, A->B) -> (A->C)
          ++\spad{f*g} is the function \spad{h}
          ++ such that \spad{h x= f(g x)}.
```

```
MPdef == add

    MappingPackageInternalHacks3(A, B, C)

    a: A
    b: B
    c: C
    faa:  A -> A
    f0c:  ()-> C
    fac:  A -> C
    fbc:  B -> C
    fab:  A -> B
    fabc: (A,B)->C
    faac: (A,A)->C

    -- Fix left and right arguments as constants.
    curryRight(fabc,b) == fabc(#1,b)
    curryLeft(fabc,a) == fabc(a, #1)

    -- Add left and right arguments which are ignored.
    constantRight fac     == fac #1
    constantLeft fbc     == fbc #2

    -- Combinators to rearrange arguments.
    twist fabc  == fabc(#2, #1)
    -- Functional composition
    fbc*fab == comp(fbc,fab,#1)
```

⟨*MAPPKG3.dotabb*⟩≡

```
"MAPPKG3" [color="#FF4488",href="bookvol10.4.pdf#nameddest=MAPPKG3"]
"BASTYPE" [color="#4488FF",href="bookvol10.2.pdf#nameddest=BASTYPE"]
"KOERCE" [color="#4488FF",href="bookvol10.2.pdf#nameddest=KOERCE"]
"MAPPKG3" -> "BASTYPE"
"MAPPKG3" -> "KOERCE"
```

14.23 package MAPPKG4 MappingPackage4

⟨*MappingPackage4.input*⟩≡

```
)set break resume
)spool MappingPackage4.output
)set message test on
)set message auto off
)clear all

--S 1 of 21
p:=(x:EXPR(INT)):EXPR(INT)+->3*x
--R
--R
--R   (1)  theMap(Closure)
--R                              Type: (Expression Integer -> Expression Integer)
--E 1

--S 2 of 21
q:=(x:EXPR(INT)):EXPR(INT)+->2*x+3
--R
--R
--R   (2)  theMap(Closure)
--R                              Type: (Expression Integer -> Expression Integer)
--E 2

--S 3 of 21
(p+q)(4)-(p(4)+q(4))
--R
--R
--R   (3)  0
--R                                                      Type: Expression Integer
--E 3

--S 4 of 21
(p+q)(x)-(p(x)+q(x))
--R
--R
--R   (4)  0
--R                                                      Type: Expression Integer
--E 4

--S 5 of 21
(p-q)(4)-(p(4)-q(4))
--R
--R
--R   (5)  0
```

```
--R                                                    Type: Expression Integer
--E 5

--S 6 of 21
(p-q)(x)-(p(x)-q(x))
--R
--R
--R   (6)  0
--R                                                    Type: Expression Integer
--E 6

--S 7 of 21
(p*q)(4)-(p(4)*q(4))
--R
--R
--R   (7)  0
--R                                                    Type: Expression Integer
--E 7

--S 8 of 21
(p*q)(x)-(p(x)*q(x))
--R
--R
--R   (8)  0
--R                                                    Type: Expression Integer
--E 8

--S 9 of 21
(p/q)(4)-(p(4)/q(4))
--R
--R
--R   (9)  0
--R                                                    Type: Expression Integer
--E 9

--S 10 of 21
(p/q)(x)-(p(x)/q(x))
--R
--R
--R   (10)  0
--R                                                    Type: Expression Integer
--E 10

--S 11 of 21
r:=(x:INT):INT+-> (x*x*x)
--R
```

```
--R
--R   (11)  theMap(Closure)
--R                                                Type: (Integer -> Integer)
--E 11

--S 12 of 21
s:=(y:INT):INT+-> (y*y+3)
--R
--R
--R   (12)  theMap(Closure)
--R                                                Type: (Integer -> Integer)
--E 12

--S 13 of 21
(r+s)(4)-(r(4)+s(4))
--R
--R
--R   (13)  0
--R                                                  Type: NonNegativeInteger
--E 13

--S 14 of 21
(r-s)(4)-(r(4)-s(4))
--R
--R
--R   (14)  0
--R                                                  Type: NonNegativeInteger
--E 14

--S 15 of 21
(r*s)(4)-(r(4)*s(4))
--R
--R
--R   (15)  0
--R                                                  Type: NonNegativeInteger
--E 15

--S 16 of 21
t:=(x:INT):EXPR(INT)+-> (x*x*x)
--R
--R
--R   (16)  theMap(Closure)
--R                                       Type: (Integer -> Expression Integer)
--E 16

--S 17 of 21
```

```
u:=(y:INT):EXPR(INT)+-> (y*y+3)
--R
--R
--R   (17)  theMap(Closure)
--R                                          Type: (Integer -> Expression Integer)
--E 17

--S 18 of 21
(t/u)(4)-(t(4)/u(4))
--R
--R
--R   (18)  0
--R                                                    Type: Expression Integer
--E 18

--S 19 of 21
h:=(x:EXPR(INT)):EXPR(INT)+->1
--R
--R   (19)  theMap(Closure)
--R                               Type: (Expression Integer -> Expression Integer)
--E 19

--S 20 of 21
(p/h)(x)
--R
--R   (20)  3x
--R                                                    Type: Expression Integer
--E 20

--S 21 of 21
(q/h)(x)
--R
--R   (21)  2x + 3
--R                                                    Type: Expression Integer
--E 21

)spool
)lisp (bye)
```

⟨*MappingPackage4.help*⟩≡

```
==========================================================================
MappingPackage examples
==========================================================================

We can construct some simple maps that take a variable x
into an equation:

  p:=(x:EXPR(INT)):EXPR(INT)+->3*x
  q:=(x:EXPR(INT)):EXPR(INT)+->2*x+3

Now we can do the four arithmetic operations, +, -, *, / on these
newly constructed mappings. Since the maps are from the domain
Expression Integer to the same domain we can also use symbolic
values for the argument. All of the following will return 0,
showing that function composition is equivalent to the result
of doing the operations individually.

  (p+q)(4)-(p(4)+q(4))
  (p+q)(x)-(p(x)+q(x))

  (p-q)(4)-(p(4)-q(4))
  (p-q)(x)-(p(x)-q(x))

  (p*q)(4)-(p(4)*q(4))
  (p*q)(x)-(p(x)*q(x))

  (p/q)(4)-(p(4)/q(4))
  (p/q)(x)-(p(x)/q(x))

We can construct simple maps from Integer to Integer but this
limits our ability to do division.

  r:=(x:INT):INT+-> (x*x*x)
  s:=(y:INT):INT+-> (y*y+3)

Again, all of these will return 0:

  (r+s)(4)-(r(4)+s(4))
  (r-s)(4)-(r(4)-s(4))
  (r*s)(4)-(r(4)*s(4))

If we want to do division with Integer inputs we create the
appropriate map:

  t:=(x:INT):EXPR(INT)+-> (x*x*x)
```

```
  u:=(y:INT):EXPR(INT)+-> (y*y+3)

  (t/u)(4)-(t(4)/u(4))

We can even recover the original functions if we make a map
that always returns the constant 1:

  h:=(x:EXPR(INT)):EXPR(INT)+->1

    theMap(Closure)
             Type: (Expression Integer -> Expression Integer)

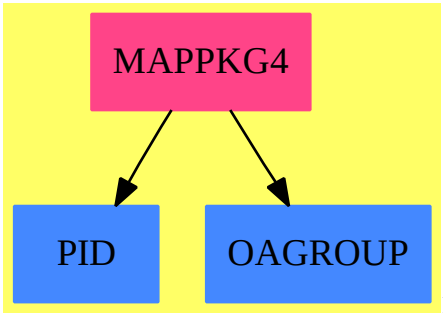
  (p/h)(x)

     3x
                                       Type: Expression Integer
  (q/h)(x)

    2x + 3
                                       Type: Expression Integer

See Also:
o )show MappingPackage1
o )help MappingPackage2
o )help MappingPackage3
o )help MappingPackage4
```

14.24 MappingPackage4



1.00

Exports:
?*? ?+? ?-? ?/?

⟨*package MAPPKG4 MappingPackage4*⟩≡
```
)abbrev package MAPPKG4 MappingPackage4
++ Author: Timothy Daly
++ Description: Functional Composition
++ Given functions f and g, returns the applicable closure
MappingPackage4(A:SetCategory, B:Ring):
 with
   "+": (A->B, A->B) -> (A->B)
        ++ \spad(+) does functional addition
        ++
        ++X f:=(x:INT):INT +-> 3*x
        ++X g:=(x:INT):INT +-> 2*x+3
        ++X (f+g)(4)
   "-": (A->B, A->B) -> (A->B)
        ++ \spad(+) does functional addition
        ++
        ++X f:=(x:INT):INT +-> 3*x
        ++X g:=(x:INT):INT +-> 2*x+3
        ++X (f-g)(4)
   "*": (A->B, A->B) -> (A->B)
        ++ \spad(+) does functional addition
        ++
        ++X f:=(x:INT):INT +-> 3*x
        ++X g:=(x:INT):INT +-> 2*x+3
        ++X (f*g)(4)
   "/": (A->Expression(Integer), A->Expression(Integer)) -> (A->Expression(Intege
        ++ \spad(+) does functional addition
        ++
        ++X p:=(x:EXPR(INT)):EXPR(INT)+->3*x
        ++X q:=(x:EXPR(INT)):EXPR(INT)+->2*x+3
        ++X (p/q)(4)
```

```
        ++X (p/q)(x)
    == add
      fab ==> (A -> B)
      faei ==> (A -> Expression(Integer))

      funcAdd(g:fab,h:fab,x:A):B == ((g x) + (h x))$B

      (a:fab)+(b:fab) == funcAdd(a,b,#1)

      funcSub(g:fab,h:fab,x:A):B == ((g x) - (h x))$B

      (a:fab)-(b:fab) == funcSub(a,b,#1)

      funcMul(g:fab,h:fab,x:A):B == ((g x) * (h x))$B

      (a:fab)*(b:fab) == funcMul(a,b,#1)

      funcDiv(g:faei,h:faei,x:A):Expression(Integer)
             == ((g x) / (h x))$Expression(Integer)

      (a:faei)/(b:faei) == funcDiv(a,b,#1)
```

⟨*MAPPKG4.dotabb*⟩≡

```
"MAPPKG4" [color="#FF4488",href="bookvol10.4.pdf#nameddest=MAPPKG4"]
"PID" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PID"]
"OAGROUP" [color="#4488FF",href="bookvol10.2.pdf#nameddest=OAGROUP"]
"MAPPKG4" -> "PID"
"MAPPKG4" -> "OAGROUP"
```

14.25 package MMLFORM MathMLFormat

Both this code and documentation are still under development and I don't pretend they are anywhere close to perfect or even finished. However the code does work and I hope it might be useful to somebody both for it's ability to output MathML from Axiom and as an example of how to write a new output form.

14.25.1 Introduction to Mathematical Markup Language

MathML exists in two forms: presentation and content. At this time (2007-02-11) the package only has a presentation package. A content package is in the works however it is more difficult. Unfortunately Axiom does not make its semantics easily available. The **OutputForm** domain mediates between the individual Axiom domains and the user visible output but **OutputForm** does not provide full semantic information. From my currently incomplete understanding of Axiom it appears that remedying this would entail going back to the individual domains and rewriting a lot of code. However some semantics are conveyed directly by **OutputForm** and other things can be deduced from **OutputForm** or from the original user command.

14.25.2 Displaying MathML

The MathML string produced by ")set output mathml on" can be pasted directly into an appropriate xhtml page and then viewed in Firefox or some other MathML aware browser. The boiler plate code needed for a test page, testmathml.xml, is:

```
<?xml version="1.0" ?>

<!DOCTYPE html PUBLIC "-//W3C//DTD XHTML 1.1 plus MathML 2.0//EN"
                      "http://www.w3.org/Math/DTD/mathml2/xhtml-math11-f.dtd" [
<!ENTITY mathml "http://www.w3.org/1998/Math/MathML">
]>

<html xmlns="http://www.w3.org/1999/xhtml"
      xmlns:xlink="http://www.w3.org/1999/xlink" >


  <head>
    <title>MathML Test </title>
  </head>

  <body>

  </body>
```

```
</html>
```

Paste the MathML string into the body element and it should display nicely in Firefox.

14.25.3 Test Cases

Here's a list of test cases that currently format correctly:

1. (x+y)**2
2. integrate(x**x,x)
3. integral(x**x,x)
4. (5 + sqrt 63 + sqrt 847)**(1/3)
5. set [1,2,3]
6. multiset [x rem 5 for x in primes(2,1000)]
7. series(sin(a*x),x=0)
8. matrix [[x**i + y**j for i in 1..10] for j in 1..10]
9. y := operator 'y a. D(y(x,z),[x,x,z,x]) b. D(y x,x,2)
10. x := series 'x a. sin(1+x)
11. series(1/log(y),y=1)
12. y:UTS(FLOAT,'z,0) := exp(z)
13. a. c := continuedFraction(314159/100000) b. c := continuedFraction(314159/100000)

The **TexFormat** domain has the capability to format an object with subscripts, superscripts, presubscripts and presuperscripts however I don't know of any Axiom command that produces such an object. In fact at present I see the case of "SUPERSUB" being used for putting primes in the superscript position to denote ordinary differentiation. I also only see the "SUB" case being used to denote partial derivatives.

14.25.4)set output mathml on

Making mathml appear as output during a normal Axiom session by invoking ")set output mathml on" proved to be a bit tedious and seems to be undocumented. I document my experience here in case it proves useful to somebody else trying to get a new output format from Axiom.

In **MathMLFormat** the functions *coerce(expr : OutputForm) : String* and *display(s : String) : Void* provide the desired mathml output. Note that this package was constructed by close examination of Robert Sutor's **TexFormat** domain and much remains from that source. To have mathml displayed as

output we need to get Axiom to call display(coerce(expr)) at the appropriate place. Here's what I did to get that to happen. Note that my starting point here was an attempt by Andrey Grozin to do the same. To figure things out I searched through files for "tex" to see what was done for the **TexFormat** domain, and used grep to find which files had mention of **TexFormat**.

14.25.5 File src/interp/setvars.boot.pamphlet

Create an output mathml section by analogy to the tex section. Remember to add the code chunk "outputmathmlCode" at the end.

setvars.boot is a bootstrap file which means that it has to be precompiled into lisp code and then that code has to be inserted back into setvars.boot. To do this extract the boot code by running "notangle" on it. I did this from the "tmp" directory. From inside axiom run ")lisp (boottran::boottocl "tmp/setvars.boot") which put "setvars.clisp" into "int/interp/setvars.clisp". Then replace the lisp in "setvars.boot.pamphlet" with that in the newly generated "setvars.clisp".

The relevant code chunks appearing in "setvars.boot.pamphlet" are:

```
outputmathmlCode
setOutputMathml
describeSetOutputMathml
```

and the relevant variables are:

```
setOutputMathml
$mathmlOutputStream
$mathmlOutputFile
$mathmlFormat
describeSetOutputMathml
```

14.25.6 File setvart.boot.pamphlet

Create an output mathml section in "setvart.boot.pamphlet" again patterned after the tex section. I changed the default file extension from ".stex" to ".smml".

To the "sectionoutput" table I added the line

```
mathml                  created output in MathML style       Off:CONSOLE
```

Added the code chunk "outputmathml" to the code chunk "output" in "sectionoutput".

Relevant code chunks:

```
outputmathml
```

Relevant variables:

```
setOutputMathml
$mathmlFormat
$mathmlOutputFile
```

Note when copying the tex stuff I changed occurrences of "tex" to "mathml", "Tex" to "Mathml" and "TeX" to "MathML".

14.25.7 File src/algebra/Makefile.pamphlet

The file "src/algebra/tex.spad.pamphlet" contains the domain **TexFormat** (TEX) and the package **TexFormat1** (TEX1). However the sole function of **TexFormat1** is to *coerce* objects from a domain into **OutputForm** and then apply **TexFormat** to them. It is to save programmers the trouble of doing the coercion themselves from inside spad code. It does not appear to be used for the main purpose of delivering Axiom output in TeX format. In order to keep the mathml package as simple as possible, and because I didn't see much use for this, I didn't copy the **TexFormat1** package. So no analog of the TEX1 entries in "Makefile.pamphlet" were needed. One curiosity I don't understand is why TEX1 appears in layer 4 when it seems to depend on TEX which appears in layer 14.

Initially I added "$OUT/MMLFORM.o" to layer 14 and "mathml.spad.pamphlet" to completed spad files in layer 14. When trying to compile the build failed at MMLFORM. It left "MMLFORM.erlib" in "int/algebra" instead of "MMLFORM.NRLIB" which confused me at first because mathml.spad compiled under a running axiom. By examining the file "obj/tmp/trace" I saw that a new dependency had been introduced, compared to TexFormat, with the function eltName depending on the domain FSAGG in layer 16. So the lines had to be moved from layer 14 to layer 17.

Added appropriate lines to "SPADFILES" and "DOCFILES".

14.25.8 File src/algebra/exposed.lsp.pamphlet

Add the line "(|MathMLFormat| . MMLFORM)"

14.25.9 File src/algebra/Lattice.pamphlet

I don't see that this file is used anywhere but I made the appropriate changes anyway by searching for "TEX" and mimicing everything for MMLFORM.

14.25.10 File src/doc/axiom.bib.pamphlet

Added mathml.spad subsection to "src/doc/axiom.bib.pamphlet".

14.25.11 File interp/i-output.boot.pamphlet

This is where the *coerce* and *display* functions from MathMLFormat actually get called. The following was added:

```
mathmlFormat expr ==
  mml := '(MathMLFormat)
  mmlrep := '(String)
  formatFn := getFunctionFromDomain("coerce",mml,[$OutputForm])
  displayFn := getFunctionFromDomain("display",mml,[mmlrep])
  SPADCALL(SPADCALL(expr,formatFn),displayFn)
  TERPRI $mathmlOutputStream
  FORCE_-OUTPUT $mathmlOutputStream
  NIL
```

Note that compared to the texFormat function there are a couple of differences. Since **MathMLFormat** is currently a package rather than a domain there is the "mmlrep" variable whereas in texFormat the argument of the "display" function is an instance of the domain. Also the *coerce* function here only has one argument, namely "$OutputForm".

Also for the function "output(expr,domain)" add lines for mathml, e.g. "if $mathmlFormat then mathmlFormat expr".

After these changes Axiom compiled with mathml enabled under)set output.

14.25.12 Public Declarations

The declarations

```
  E      ==> OutputForm
  I      ==> Integer
  L      ==> List
  S      ==> String
  US     ==> UniversalSegment(Integer)
```

provide abbreviations for domains used heavily in the code. The publicly exposed functions are:

coerce: E -> S This function is the main one for converting an expression in domain OutputForm into a MathML string.

coerceS: E -> S This function is for use from the command line. It converts an OutputForm expression into a MathML string and does some formatting so that the output is not one long line. If you take the output from this function, stick it in an emacs buffer in nxml-mode and then indent according to mode, you'll get something that's nicer to look at than what comes from coerce. Note that coerceS returns the same value as coerce but invokes a display function as well so that the result will be printed twice in different formats. The need for

this is that the output from coerce is automatically formatted with line breaks by Axiom's output routine that are not in the right place.

coerceL: E -> S Similar to coerceS except that the displayed result is the MathML string in one long line. These functions can be used, for instance, to get the MathML for the previous result by typing coerceL(

exprex: E -> S Converts **OutputForm** to **String** with the structure preserved with braces. This is useful in developing this package. Actually this is not quite accurate. The function *precondition* is first applied to the **OutputForm** expression before *exprex*. Raw **OutputForm** and the nature of the *precondition* function is still obscure to me at the time of this writing (2007-02-14), however I probably need to understand it to make sure I'm not missing any semantics. The spad function *precondition* is just a wrapper for the lisp function outputTran$Lisp, which I guess is compiled from boot.

display: S -> Void This one prints the string returned by coerce as one long line, adding "math" tags: $...$. Thus the output from this can be stuck directly into an appropriate html/xhtml page and will be displayed nicely by a MathML aware browser.

displayF: S -> Void This function doesn't exist yet but it would be nice to have a humanly readable formatted output as well. The basics do exist in the coerceS function however the formatting still needs some work to be really good.

⟨*public declarations*⟩≡

```
)abbrev domain MMLFORM MathMLFormat
++ Author: Arthur C. Ralfs
++ Date: January 2007
++ This package is based on the TeXFormat domain by Robert S. Sutor
++ without which I wouldn't have known where to start.
++ Basic Operations: coerce, coerceS, coerceL, exprex, display
++ Description:
++    \spadtype{MathMLFormat} provides a coercion from \spadtype{OutputForm}
++    to MathML format.

MathMLFormat(): public == private where
  E      ==> OutputForm
  I      ==> Integer
  L      ==> List
  S      ==> String
  US     ==> UniversalSegment(Integer)

  public == SetCategory with
    coerce:    E -> S
      ++ coerceS(o) changes o in the standard output format to MathML
      ++ format.
    coerceS:   E -> S
      ++ coerceS(o) changes o in the standard output format to MathML
```

```
  ++ format and displays formatted result.
coerceL:   E -> S
  ++ coerceS(o) changes o in the standard output format to MathML
  ++ format and displays result as one long string.
exprex:    E -> S
  ++ coverts \spadtype{OutputForm} to \spadtype{String} with the
  ++ structure preserved with braces.  Actually this is not quite
  ++ accurate.  The function \spadfun{precondition} is first
  ++ applied to the
  ++ \spadtype{OutputForm} expression before \spadfun{exprex}.
  ++ The raw \spadtype{OutputForm} and
  ++ the nature of the \spadfun{precondition} function is
  ++ still obscure to me
  ++ at the time of this writing (2007-02-14).
display:   S -> Void
  ++ prints the string returned by coerce, adding <math ...> tags.
```

14.25.13 Private Constant Declarations

⟨*private constant declarations*⟩≡

```
    private == add
      import OutputForm
      import Character
      import Integer
      import List OutputForm
      import List String

      -- local variable declarations and definitions

      expr: E
      prec,opPrec: I
      str:  S
      blank         : S := " \  "

      maxPrec       : I   := 1000000
      minPrec       : I   := 0

      unaryOps      : L S := ["-","^"]$(L S)
      unaryPrecs    : L I := [700,260]$(L I)

      -- the precedence of / in the following is relatively low because
      -- the bar obviates the need for parentheses.
      binaryOps     : L S := ["+->","|","**","/","<",">","=","OVER"]$(L S)
      binaryPrecs   : L I := [0,0,900, 700,400,400,400,   700]$(L I)

      naryOps       : L S := ["-","+","*",blank,",",";"," ","ROW","",
         " \cr ","&","</mtd></mtr><mtr><mtd>"]$(L S)
      naryPrecs     : L I := [700,700,800,  800,110,110,  0,    0, 0,
               0,  0,   0]$(L I)
      naryNGOps     : L S := ["ROW","&"]$(L S)

      plexOps       : L S := ["SIGMA","SIGMA2","PI","PI2","INTSIGN","INDEFINTEGRAL"]$(L S)
      plexPrecs     : L I := [    700, 800,     700, 800 , 700,      700]$(L I)

      specialOps    : L S := ["MATRIX","BRACKET","BRACE","CONCATB","VCONCAT",  _
                              "AGGLST","CONCAT","OVERBAR","ROOT","SUB","TAG", _
                              "SUPERSUB","ZAG","AGGSET","SC","PAREN", _
                              "SEGMENT","QUOTE","theMap" ]

      -- the next two lists provide translations for some strings for
      -- which MML provides special macros.

      specialStrings : L S :=
```

```
  ["cos", "cot", "csc", "log", "sec", "sin", "tan",
    "cosh", "coth", "csch", "sech", "sinh", "tanh",
      "acos","asin","atan","erf","...","$","infinity"]
specialStringsInMML : L S :=
  ["<mo>cos</mo>","<mo>cot</mo>","<mo>csc</mo>","<mo>log</mo>","<mo>sec</mo>"
    "<mo>cosh</mo>","<mo>coth</mo>","<mo>csch</mo>","<mo>sech</mo>","<mo>sinh
      "<mo>arccos</mo>","<mo>arcsin</mo>","<mo>arctan</mo>","<mo>erf</mo>","<
```

14.25.14 Private Function Declarations

These are the local functions:

addBraces:S -> S

addBrackets:S -> S

atomize:E -> L E

displayElt:S -> Void function for recursively displaying mathml nicely formatted

eltLimit:(S,I,S) -> I demarcates end postion of mathml element with name:S starting at position i:I in mathml string s:S and returns end of end tag as i:I position in mathml string, i.e. find start and end of substring: <name ...>...</name>

eltName:(I,S) -> S find name of mathml element starting at position i:I in string s:S

group:S -> S

formatBinary:(S,L E, I) -> S

formatFunction:(S,L E, I) -> S

formatMatrix:L E -> S

formatNary:(S,L E, I) -> S

formatNaryNoGroup:(S,L E, I) -> S

formatNullary:S -> S

formatPlex:(S,L E, I) -> S

formatSpecial:(S,L E, I) -> S

formatUnary:(S, E, I) -> S

formatMml:(E,I) -> S

newWithNum:I -> $ this is a relic from tex.spad and is not used here so far. I'll probably remove it.

parenthesize:S -> S

precondition:E -> E this function is applied to the OutputForm expression before doing anything else.

postcondition:S -> S this function is applied after all other OutputForm -> MathML transformations. In the TexFormat domain the ungroup function first peels off the outermost set of braces however I have replaced braces with <mrow>s here and sometimes the outermost set of <mrow>s is necessary to get proper display in Firefox. For instance with getting the correct size of brackets on a matrix the whole expression needs to be enclosed in a mrow element. It also checks for +− and removes the +.

stringify:E -> S

tagEnd:(S,I,S) -> I finds closing ">" of start or end tag for mathML element for formatting MathML string for human readability. No analog in TexFormat.

ungroup:S -> S

⟨*private function declarations*⟩≡

```
-- local function signatures

addBraces:      S -> S
addBrackets:    S -> S
atomize:        E -> L E
displayElt:     S -> Void
  ++ function for recursively displaying mathml nicely formatted
eltLimit:       (S,I,S) -> I
  ++ demarcates end postion of mathml element with name:S starting at
  ++ position i:I in mathml string s:S and returns end of end tag as
  ++  i:I position in mathml string, i.e. find start and end of
  ++  substring:  <name ...>...</name>
eltName:        (I,S) -> S
  ++ find name of mathml element starting at position i:I in string s:S
group:          S -> S
formatBinary:   (S,L E, I) -> S
formatFunction: (S,L E, I) -> S
formatIntSign:  (L E, I) -> S
formatMatrix:   L E -> S
formatNary:     (S,L E, I) -> S
formatNaryNoGroup: (S,L E, I) -> S
formatNullary:  S -> S
formatPlex:     (S,L E, I) -> S
formatSpecial:  (S,L E, I) -> S
formatSub:      (E, L E, I) -> S
formatSuperSub: (E, L E, I) -> S
formatSuperSub1: (E, L E, I) -> S
formatUnary:    (S,  E, I) -> S
formatMml:      (E,I) -> S
formatZag:      L E -> S
formatZag1:     L E -> S
newWithNum:     I -> $
parenthesize:   S -> S
precondition:   E -> E
postcondition:  S -> S
stringify:      E -> S
tagEnd:         (S,I,S) -> I
  ++  finds closing ">" of start or end tag for mathML element
ungroup:        S -> S
```

14.25.15 Public Function Definitions

Note that I use the function sayTeX$Lisp much as I would printf in a C program. I've noticed in grepping the code that there are other "say" functions, sayBrightly and sayMessage for instance, but I have no idea what the difference is between them at this point. sayTeX$Lisp does the job so for the time being I'll use that until I learn more.

The functions coerceS and coerceL should probably be changed to display functions, *i.e. displayS* and *display L*, returning Void. I really only need the one coerce function.

⟨*public function definitions*⟩≡

```
-- public function definitions

coerce(expr : E): S ==
  s : S := postcondition formatMml(precondition expr, minPrec)
  s

coerceS(expr : E): S ==
  s : S := postcondition formatMml(precondition expr, minPrec)
  sayTeX$Lisp "<math xmlns=_"http://www.w3.org/1998/Math/MathML_" mathsize=_"big_" displ
  displayElt(s)
  sayTeX$Lisp "</math>"
  s

coerceL(expr : E): S ==
  s : S := postcondition formatMml(precondition expr, minPrec)
  sayTeX$Lisp "<math xmlns=_"http://www.w3.org/1998/Math/MathML_" mathsize=_"big_" displ
  sayTeX$Lisp s
  sayTeX$Lisp "</math>"
  s

display(mathml : S): Void ==
  sayTeX$Lisp "<math xmlns=_"http://www.w3.org/1998/Math/MathML_" mathsize=_"big_" displ
  sayTeX$Lisp mathml
  sayTeX$Lisp "</math>"
  void()$Void



exprex(expr : E): S ==
  -- This breaks down an expression into atoms and returns it as
  -- a string.  It's for developmental purposes to help understand
  -- the expressions.
  a : E
  expr := precondition expr
```

```
--      sayTeX$Lisp "0: "stringify expr
      (ATOM(expr)$Lisp@Boolean) or (stringify expr = "NOTHING") =>
        concat ["{",stringify expr,"}"]
      le : L E := (expr pretend L E)
      op := first le
      sop : S := exprex op
      args : L E := rest le
      nargs : I := #args
--      sayTeX$Lisp concat ["1: ",stringify first le," : ",string(nargs)$S]
      s : S := concat ["{",sop]
      if nargs > 0  then
        for a in args repeat
--        sayTeX$Lisp concat ["2: ",stringify a]
          s1 : S := exprex a
          s := concat [s,s1]
      s := concat [s,"}"]
```

14.25.16 Private Function Definitions

Display Functions

displayElt(mathml:S):Void

eltName(pos:I,mathml:S):S

eltLimit(name:S,pos:I,mathml:S):I

tagEnd(name:S,pos:I,mathml:S):I

⟨*display functions*⟩≡

```
      displayElt(mathML:S): Void ==
        -- Takes a string of syntactically complete mathML
        -- and formats it for display.
  --      sayTeX$Lisp "****displayElt1****"
  --      sayTeX$Lisp mathML
        enT:I -- marks end of tag, e.g. "<name>"
        enE:I -- marks end of element, e.g. "<name> ... </name>"
        end:I -- marks end of mathML string
        u:US
        end := #mathML
        length:I := 60
  --      sayTeX$Lisp "****displayElt1.1****"
        name:S := eltName(1,mathML)
  --      sayTeX$Lisp name
  --      sayTeX$Lisp concat("****displayElt1.2****",name)
        enE := eltLimit(name,2+#name,mathML)
  --      sayTeX$Lisp "****displayElt2****"
        if enE < length then
  --        sayTeX$Lisp "****displayElt3****"
          u := segment(1,enE)$US
          sayTeX$Lisp mathML.u
        else
  --        sayTeX$Lisp "****displayElt4****"
          enT := tagEnd(name,1,mathML)
          u := segment(1,enT)$US
          sayTeX$Lisp mathML.u
          u := segment(enT+1,enE-#name-3)$US
          displayElt(mathML.u)
          u := segment(enE-#name-2,enE)$US
          sayTeX$Lisp mathML.u
        if end > enE then
  --        sayTeX$Lisp "****displayElt5****"
          u := segment(enE+1,end)$US
          displayElt(mathML.u)
```

```
  void()$Void

eltName(pos:I,mathML:S): S ==
  -- Assuming pos is the position of "<" for a start tag of a mathML
  -- element finds and returns the element's name.
  i:I := pos+1
  --sayTeX$Lisp "eltName:mathmML string: "mathML
  while member?(mathML.i,lowerCase()$CharacterClass)$CharacterClass repeat
     i := i+1
  u:US := segment(pos+1,i-1)
  name:S := mathML.u

eltLimit(name:S,pos:I,mathML:S): I ==
  -- Finds the end of a mathML element like "<name ...> ... </name>"
  -- where pos is the position of the space after name in the start tag
  -- although it could point to the closing ">".  Returns the position
  -- of the ">" in the end tag.
  pI:I := pos
  startI:I
  endI:I
  startS:S := concat ["<",name]
  endS:S := concat ["</",name,">"]
  level:I := 1
  --sayTeX$Lisp "eltLimit: element name: "name
  while (level > 0) repeat
    startI := position(startS,mathML,pI)$String

    endI := position(endS,mathML,pI)$String

    if (startI = 0) then
      level := level-1
      --sayTeX$Lisp "****eltLimit 1******"
      pI := tagEnd(name,endI,mathML)
    else
      if (startI < endI) then
        level := level+1
        pI := tagEnd(name,startI,mathML)
      else
        level := level-1
        pI := tagEnd(name,endI,mathML)
  pI


tagEnd(name:S,pos:I,mathML:S):I ==
  -- Finds the closing ">" for either a start or end tag of a mathML
```

```
-- element, so the return value is the position of ">" in mathML.
pI:I := pos
while  (mathML.pI ^= char ">") repeat
  pI := pI+1
u:US := segment(pos,pI)$US
--sayTeX$Lisp "tagEnd: "mathML.u
pI
```

Formatting Functions

Still need to format \zag in formatSpecial!

In formatPlex the case op = "INTSIGN" is now passed off to formatIntSign which is a change from the TexFormat domain. This is done here for presentation mark up to replace the ugly bound variable that Axiom delivers. For content mark up this has to be done anyway.

The formatPlex function also allows for op = "INDEFINTEGRAL". However I don't know what Axiom command gives rise to this case. The INTSIGN case already allows for both definite and indefinite integrals.

In the function formatSpecial various cases are handled including SUB and SUPERSUB. These cases are now caught in formatMml and so the code in formatSpecial doesn't get executed. The only cases I know of using these are partial derivatives for SUB and ordinary derivatives or SUPERSUB however in TexFormat the capability is there to handle multiscripts, i.e. an object with subscripts, superscripts, pre-subscripts and pre-superscripts but I am so far unaware of any Axiom command that produces such a multiscripted object.

Another question is how to represent derivatives. At present I have differential notation for partials and prime notation for ordinary derivatives, but it would be nice to allow for different derivative notations in different circumstances, maybe some options to)set output mathml on.

Ordinary derivatives are formatted in formatSuperSub and there are 2 versions, formatSuperSub and formatSuperSub1, which at this point have to be switched by swapping names.

⟨*formatting functions*⟩≡

```
atomize(expr : E): L E ==
  -- This breaks down an expression into a flat list of atomic expressions.
  -- expr should be preconditioned.
  le : L E := nil()
  a : E
  letmp : L E
  (ATOM(expr)$Lisp@Boolean) or (stringify expr = "NOTHING") =>
    le := append(le,list(expr))
  letmp := expr pretend L E
  for a in letmp repeat
    le := append(le,atomize a)
  le


ungroup(str: S): S ==
  len : I := #str
  len < 14 => str
  lrow : S :=  "<mrow>"
```

```
      rrow : S :=  "</mrow>"
      -- drop leading and trailing mrows
      u1 : US := segment(1,6)$US
      u2 : US := segment(len-6,len)$US
      if (str.u1 =$S lrow) and (str.u2 =$S rrow) then
        u : US := segment(7,len-7)$US
        str := str.u
      str

    postcondition(str: S): S ==
--      str := ungroup str
      len : I := #str
      plusminus : S := "<mo>+</mo><mo>-</mo>"
      pos : I := position(plusminus,str,1)
      if pos > 0 then
        ustart:US := segment(1,pos-1)$US
        uend:US := segment(pos+20,len)$US
        str := concat [str.ustart,"<mo>-</mo>",str.uend]
        if pos < len-18 then
          str := postcondition(str)
      str

    stringify expr == (mathObject2String$Lisp expr)@S

    group str ==
      concat ["<mrow>",str,"</mrow>"]

    addBraces str ==
      concat ["<mo>{</mo>",str,"<mo>}</mo>"]

    addBrackets str ==
      concat ["<mo>[</mo>",str,"<mo>]</mo>"]

    parenthesize str ==
      concat ["<mo>(</mo>",str,"<mo>)</mo>"]

    precondition expr ==
      outputTran$Lisp expr

    formatSpecial(op : S, args : L E, prec : I) : S ==
      arg : E
      prescript : Boolean := false
      op = "theMap" => "<mtext>theMap(...)</mtext>"
      op = "AGGLST" =>
        formatNary(",",args,prec)
      op = "AGGSET" =>
```

```
  formatNary(";",args,prec)
op = "TAG" =>
  group concat [formatMml(first args,prec),
                "<mo>&#x02192;</mo>",
                  formatMml(second args,prec)]
                   --RightArrow
op = "VCONCAT" =>
  group concat("<mtable><mtr>",
               concat(concat([concat("<mtd>",concat(formatMml(u, minPrec),"
                              for u in args]::L S),
                     "</mtr></mtable>"))
op = "CONCATB" =>
  formatNary(" ",args,prec)
op = "CONCAT" =>
  formatNary("",args,minPrec)
op = "QUOTE" =>
  group concat("<mo>'</mo>",formatMml(first args, minPrec))
op = "BRACKET" =>
  group addBrackets ungroup formatMml(first args, minPrec)
op = "BRACE" =>
  group addBraces ungroup formatMml(first args, minPrec)
op = "PAREN" =>
  group parenthesize ungroup formatMml(first args, minPrec)
op = "OVERBAR" =>
  null args => ""
  group concat ["<mover accent='true'><mrow>",formatMml(first args,minPrec)
   --OverBar
op = "ROOT" =>
  null args => ""
  tmp : S := group formatMml(first args, minPrec)
  null rest args => concat ["<msqrt>",tmp,"</msqrt>"]
  group concat
    ["<mroot><mrow>",tmp,"</mrow>",formatMml(first rest args, minPrec),"</m
op = "SEGMENT" =>
  tmp : S := concat [formatMml(first args, minPrec),"<mo>..</mo>"]
  group
    null rest args =>  tmp
    concat [tmp,formatMml(first rest args, minPrec)]
-- SUB should now be diverted in formatMml although I'll leave
-- the code here for now.
op = "SUB" =>
  group concat ["<msub>",formatMml(first args, minPrec),
    formatSpecial("AGGLST",rest args,minPrec),"</msub>"]
-- SUPERSUB should now be diverted in formatMml although I'll leave
-- the code here for now.
op = "SUPERSUB" =>
```

```
          base:S := formatMml(first args, minPrec)
          args := rest args
          if #args = 1 then
            "<msub><mrow>"base"</mrow><mrow>"formatMml(first args, minPrec)"</mrow></msub>"
          else if #args = 2 then
          -- it would be nice to substitue &#x2032; for , in the case of
          -- an ordinary derivative, it looks a lot better.
            "<msubsup><mrow>"base"</mrow><mrow>"formatMml(first args,minPrec)"</mrow><mrow>"fo
          else if #args = 3 then
            "<mmultiscripts><mrow>"base"</mrow><mrow>"formatMml(first args,minPrec)"</mrow><mr
          else if #args = 4 then
            "<mmultiscripts><mrow>"base"</mrow><mrow>"formatMml(first args,minPrec)"</mrow><mr
          else
            "<mtext>Problem with multiscript object</mtext>"
        op = "SC" =>
          -- need to handle indentation someday
          null args => ""
          tmp := formatNaryNoGroup("</mtd></mtr><mtr><mtd>", args, minPrec)
          group concat ["<mtable><mtr><mtd>",tmp,"</mtd></mtr></mtable>"]
        op = "MATRIX" => formatMatrix rest args
        op = "ZAG" =>
  -- {{+}{3}{{ZAG}{1}{7}}{{ZAG}{1}{15}}{{ZAG}{1}{1}}{{ZAG}{1}{25}}{{ZAG}{1}{1}}{{ZAG}{1}{7}}{+
  -- to format continued fraction traditionally need to intercept it at the
  -- formatNary of the "+"
          concat [" \zag{",formatMml(first args, minPrec),"}{",
            formatMml(first rest args,minPrec),"}"]
        concat ["<mtext>not done yet for: ",op,"</mtext>"]

      formatSub(expr : E, args : L E, opPrec : I) : S ==
        -- This one produces differential notation partial derivatives.
        -- It doesn't work in all cases and may not be workable, use
        -- formatSub1 below for now.
        -- At this time this is only to handle partial derivatives.
        -- If the SUB case handles anything else I'm not aware of it.
        -- This an example of the 4th partial of y(x,z) w.r.t. x,x,z,x
        -- {{{SUB}{y}{{CONCAT}{{CONCAT}{{CONCAT}{{CONCAT}{,}{1}}{{CONCAT}{,}{1}}}{{CONCAT}{,}+
        atomE : L E := atomize(expr)
        op : S := stringify first atomE
        op ^= "SUB" => "<mtext>Mistake in formatSub: no SUB</mtext>"
        stringify first rest rest atomE ^= "CONCAT" => "<mtext>Mistake in formatSub: no CONCAT
        -- expecting form for atomE like
        --[{SUB}{func}{CONCAT}...{CONCAT}{,}{n}{CONCAT}{,}{n}...{CONCAT}{,}{n}],
        --counting the first CONCATs before the comma gives the number of
        --derivatives
        ndiffs : I := 0
        tmpLE : L E := rest rest atomE
```

```
      while stringify first tmpLE = "CONCAT" repeat
        ndiffs := ndiffs+1
        tmpLE := rest tmpLE
      numLS : L S := nil
      i : I := 1
      while i < ndiffs repeat
        numLS := append(numLS,list(stringify first rest tmpLE))
        tmpLE := rest rest rest tmpLE
        i := i+1
      numLS := append(numLS,list(stringify first rest tmpLE))
      -- numLS contains the numbers of the bound variables as strings
      -- for the differentiations, thus for the differentiation [x,x,z,x]
      -- for y(x,z) numLS = ["1","1","2","1"]
      posLS : L S := nil
      i := 0
--    sayTeX$Lisp "formatSub: nargs = "string(#args)
      while i < #args repeat
        posLS := append(posLS,list(string(i+1)))
        i := i+1
      -- posLS contains the positions of the bound variables in args
      -- as a list of strings, e.g. for the above example ["1","2"]
      tmpS: S := stringify atomE.2
      if ndiffs = 1 then
        s : S := "<mfrac><mo>&#x02202;</mo><mi>"tmpS"</mi><mrow>"
      else
        s : S := "<mfrac><mrow><msup><mo>&#x02202;</mo><mn>"string(ndiffs)"</mn><
      -- need to find the order of the differentiation w.r.t. the i-th
      -- variable
      i := 1
      j : I
      k : I
      tmpS: S
      while i < #posLS+1 repeat
        j := 0
        k := 1
        while k < #numLS + 1 repeat
          if numLS.k = string i then j := j + 1
          k := k+1
        if j > 0 then
          tmpS := stringify args.i
          if j = 1 then
            s := s"<mo>&#x02202;</mo><mi>"tmpS"</mi>"
          else
            s := s"<mo>&#x02202;</mo><msup><mi>"tmpS"</mi><mn>"string(j)"</mn></m
        i := i + 1
      s := s"</mrow></mfrac><mo>(</mo>"
```

```
      i := 1
      while i < #posLS+1 repeat
        tmpS := stringify args.i
        s := s"<mi>"tmpS"</mi>"
        if i < #posLS then s := s"<mo>,</mo>"
        i := i+1
      s := s"<mo>)</mo>"

    formatSub1(expr : E, args : L E, opPrec : I) : S ==
      -- This one produces partial derivatives notated by ",n" as
      -- subscripts.
      -- At this time this is only to handle partial derivatives.
      -- If the SUB case handles anything else I'm not aware of it.
      -- This an example of the 4th partial of y(x,z) w.r.t. x,x,z,x
      -- {{{SUB}{y}{{CONCAT}{{CONCAT}{{CONCAT}{{CONCAT}{,}{1}}
      -- {{CONCAT}{,}{1}}}{{CONCAT}{,}{2}}}{{CONCAT}{,}{1}}}}{x}{z}},
      -- here expr is everything in the first set of braces and
      -- args is {{x}{z}}
      atomE : L E := atomize(expr)
      op : S := stringify first atomE
      op ^= "SUB" => "<mtext>Mistake in formatSub: no SUB</mtext>"
      stringify first rest rest atomE ^= "CONCAT" => "<mtext>Mistake in formatSub: no CONCAT
      -- expecting form for atomE like
      --[{SUB}{func}{CONCAT}...{CONCAT}{,}{n}{CONCAT}{,}{n}...{CONCAT}{,}{n}],
      --counting the first CONCATs before the comma gives the number of
      --derivatives
      ndiffs : I := 0
      tmpLE : L E := rest rest atomE
      while stringify first tmpLE = "CONCAT" repeat
        ndiffs := ndiffs+1
        tmpLE := rest tmpLE
      numLS : L S := nil
      i : I := 1
      while i < ndiffs repeat
        numLS := append(numLS,list(stringify first rest tmpLE))
        tmpLE := rest rest rest tmpLE
        i := i+1
      numLS := append(numLS,list(stringify first rest tmpLE))
      -- numLS contains the numbers of the bound variables as strings
      -- for the differentiations, thus for the differentiation [x,x,z,x]
      -- for y(x,z) numLS = ["1","1","2","1"]
      posLS : L S := nil
      i := 0
--    sayTeX$Lisp "formatSub: nargs = "string(#args)
      while i < #args repeat
        posLS := append(posLS,list(string(i+1)))
```

```
      i := i+1
    -- posLS contains the positions of the bound variables in args
    -- as a list of strings, e.g. for the above example ["1","2"]
    funcS: S := stringify atomE.2
    s : S := "<msub><mi>"funcS"</mi><mrow>"
    i := 1
    while i < #numLS+1 repeat
      s := s"<mo>,</mo><mn>"numLS.i"</mn>"
      i := i + 1
    s := s"</mrow></msub><mo>(</mo>"
    i := 1
    while i < #posLS+1 repeat
--      tmpS := stringify args.i
      tmpS := formatMml(first args,minPrec)
      args := rest args
      s := s"<mi>"tmpS"</mi>"
      if i < #posLS then s := s"<mo>,</mo>"
      i := i+1
    s := s"<mo>)</mo>"

  formatSuperSub(expr : E, args : L E, opPrec : I) : S ==
    -- this produces prime notation ordinary derivatives.
    -- first have to divine the semantics, add cases as needed
--    WriteLine$Lisp "SuperSub1 begin"
    atomE : L E := atomize(expr)
    op : S := stringify first atomE
--    WriteLine$Lisp "op: "op
    op ^= "SUPERSUB" => _
        "<mtext>Mistake in formatSuperSub: no SUPERSUB1</mtext>"
    #args ^= 1 => "<mtext>Mistake in SuperSub1: #args <> 1</mtext>"
    var : E := first args
    -- should be looking at something like {{SUPERSUB}{var}{ }{,,...,}} for
    -- example here's the second derivative of y w.r.t. x
    -- {{{SUPERSUB}{y}{ }{,,}}{x}}, expr is the first {} and args is the
    -- {x}
    funcS : S := stringify first rest atomE
--    WriteLine$Lisp "funcS: "funcS
    bvarS : S := stringify first args
--    WriteLine$Lisp "bvarS: "bvarS
    -- count the number of commas
    commaS : S := stringify first rest rest rest atomE
    commaTest : S := ","
    i : I := 0
    while position(commaTest,commaS,1) > 0 repeat
      i := i+1
      commaTest := commaTest","
```

```
      s : S := "<msup><mi>"funcS"</mi><mrow>"
--      WriteLine$Lisp "s: "s
      j : I := 0
      while j < i repeat
        s := s"<mo>&#x02032;</mo>"
        j := j + 1
      s := s"</mrow></msup><mo>&#x02061;</mo><mo>(</mo>"formatMml(first args,minPrec)"<mo>)<

    formatSuperSub1(expr : E, args : L E, opPrec : I) : S ==
      -- This one produces ordinary derivatives with differential notation,
      -- it needs a little more work yet.
      -- first have to divine the semantics, add cases as needed
--      WriteLine$Lisp "SuperSub begin"
      atomE : L E := atomize(expr)
      op : S := stringify first atomE
      op ^= "SUPERSUB" => _
         "<mtext>Mistake in formatSuperSub: no SUPERSUB</mtext>"
      #args ^= 1 => "<mtext>Mistake in SuperSub: #args <> 1</mtext>"
      var : E := first args
      -- should be looking at something like {{SUPERSUB}{var}{ }{,,...,}} for
      -- example here's the second derivative of y w.r.t. x
      -- {{{SUPERSUB}{y}{ }{,,}}{x}}, expr is the first {} and args is the
      -- {x}
      funcS : S := stringify first rest atomE
      bvarS : S := stringify first args
      -- count the number of commas
      commaS : S := stringify first rest rest rest atomE
      commaTest : S := ","
      ndiffs : I := 0
      while position(commaTest,commaS,1) > 0 repeat
        ndiffs := ndiffs+1
        commaTest := commaTest","
      s : S := "<mfrac><mrow><msup><mo>&#x02146;</mo><mn>"string(ndiffs)"</mn></msup><mi>"fu

    formatPlex(op : S, args : L E, prec : I) : S ==
      checkarg:Boolean := false
      hold : S
      p : I := position(op,plexOps)
      p < 1 => error "unknown plex op"
      op = "INTSIGN" => formatIntSign(args,minPrec)
      opPrec := plexPrecs.p
      n : I := #args
      (n ^= 2) and (n ^= 3) => error "wrong number of arguments for plex"
      s : S :=
        op = "SIGMA"   =>
         checkarg := true
```

```
          "<mo>&#x02211;</mo>"
        -- Sum
        op = "SIGMA2"   =>
         checkarg := true
         "<mo>&#x02211;</mo>"
        -- Sum
        op = "PI"      =>
         checkarg := true
         "<mo>&#x0220F;</mo>"
        -- Product
        op = "PI2"     =>
         checkarg := true
         "<mo>&#x0220F;</mo>"
        -- Product
--        op = "INTSIGN" => "<mo>&#x0222B;</mo>"
        -- Integral, int
        op = "INDEFINTEGRAL" => "<mo>&#x0222B;</mo>"
        -- Integral, int
        "????"
      hold := formatMml(first args,minPrec)
      args := rest args
      if op ^= "INDEFINTEGRAL" then
        if hold ^= "" then
          s := concat ["<munderover>",s,group hold]
        else
          s := concat ["<munderover>",s,group " "]
        if not null rest args then
          hold := formatMml(first args,minPrec)
          if hold ^= "" then
            s := concat [s,group hold,"</munderover>"]
          else
            s := concat [s,group " ","</munderover>"]
          args := rest args
        -- if checkarg true need to test op arg for "+" at least
        -- and wrap parentheses if so
        if checkarg then
          la : L E := (first args pretend L E)
          opa : S := stringify first la
          if opa = "+" then
            s := concat [s,"<mo>(</mo>",formatMml(first args,minPrec),"<mo>)</mo>
          else s := concat [s,formatMml(first args,minPrec)]
        else s := concat [s,formatMml(first args,minPrec)]
      else
        hold := group concat [hold,formatMml(first args,minPrec)]
        s := concat [s,hold]
--      if opPrec < prec then s := parenthesize s
```

```
-- getting ugly parentheses on fractions
      group s

    formatIntSign(args : L E, opPrec : I) : S ==
      -- the original OutputForm expression looks something like this:
      -- {{INTSIGN}{NOTHING or lower limit?}
      -- {bvar or upper limit?}{{*}{integrand}{{CONCAT}{d}{axiom var}}}}
      -- the args list passed here consists of the rest of this list, i.e.
      -- starting at the NOTHING or ...
      (stringify first args) = "NOTHING" =>
        -- the bound variable is the second one in the argument list
        bvar : E := first rest args
        bvarS : S := stringify bvar
        tmpS : S
        i : I := 0
        u1 : US
        u2 : US
        -- this next one atomizes the integrand plus differential
        atomE : L E := atomize(first rest rest args)
        -- pick out the bound variable used by axiom
        varRS : S := stringify last(atomE)
        tmpLE : L E := ((first rest rest args) pretend L E)
        integrand : S := formatMml(first rest tmpLE,minPrec)
        -- replace the bound variable, i.e. axiom uses someting of the form
        -- %A for the bound variable and puts the original variable used
        -- in the input command as a superscript on the integral sign.
        -- I'm assuming that the axiom variable is 2 characters.
        while (i := position(varRS,integrand,i+1)) > 0 repeat
          u1 := segment(1,i-1)$US
          u2 := segment(i+2,#integrand)$US
          integrand := concat [integrand.u1,bvarS,integrand.u2]
        concat ["<mrow><mo>&#x0222B;</mo>" integrand "<mo>&#x02146;</mo><mi>" bvarS "</mi><

      lowlim : S := stringify first args
      highlim : S := stringify first rest args
      bvar : E := last atomize(first rest rest args)
      bvarS : S := stringify bvar
      tmpLE : L E := ((first rest rest args) pretend L E)
      integrand : S := formatMml(first rest tmpLE,minPrec)
      concat ["<mrow><munderover><mo>&#x0222B;</mo><mi>" lowlim "</mi><mi>" highlim "</mi><


    formatMatrix(args : L E) : S ==
      -- format for args is [[ROW ...],[ROW ...],[ROW ...]]
      -- generate string for formatting columns (centered)
      group addBrackets concat
```

```
        ["<mtable><mtr><mtd>",formatNaryNoGroup("</mtd></mtr><mtr><mtd>",args,min
          "</mtd></mtr></mtable>"]

    formatFunction(op : S, args : L E, prec : I) : S ==
      group concat ["<mo>",op,"</mo>",parenthesize formatNary(",",args,minPrec)]

    formatNullary(op : S) ==
      op = "NOTHING" => ""
      group concat ["<mo>",op,"</mo><mo>(</mo><mo>)</mo>"]

    formatUnary(op : S, arg : E, prec : I) ==
      p : I := position(op,unaryOps)
      p < 1 => error "unknown unary op"
      opPrec := unaryPrecs.p
      s : S := concat ["<mo>",op,"</mo>",formatMml(arg,opPrec)]
      opPrec < prec => group parenthesize s
      op = "-" => s
      group s

    formatBinary(op : S, args : L E, prec : I) : S ==
      p : I := position(op,binaryOps)
      p < 1 => error "unknown binary op"
      opPrec := binaryPrecs.p
      -- if base op is product or sum need to add parentheses
      if ATOM(first args)$Lisp@Boolean then
        opa:S := stringify first args
      else
        la : L E := (first args pretend L E)
        opa : S := stringify first la
      if (opa = "SIGMA" or opa = "SIGMA2" or opa = "PI" or opa = "PI2") _
         and op = "**" then
        s1:S:=concat ["<mo>(</mo>",formatMml(first args, opPrec),"<mo>)</mo>"]
      else
       s1 : S := formatMml(first args, opPrec)
      s2 : S := formatMml(first rest args, opPrec)
      op :=
        op = "|"     =>  s := concat ["<mrow>",s1,"</mrow><mo>",op,"</mo><mrow>",
        op = "**"    =>  s := concat ["<msup><mrow>",s1,"</mrow><mrow>",s2,"</mro
        op = "/"     =>  s := concat ["<mfrac><mrow>",s1,"</mrow><mrow>",s2,"</mr
        op = "OVER"  =>  s := concat ["<mfrac><mrow>",s1,"</mrow><mrow>",s2,"</mr
        op = "+->"   =>  s := concat ["<mrow>",s1,"</mrow><mo>",op,"</mo><mrow>",
        s := concat ["<mrow>",s1,"</mrow><mo>",op,"</mo><mrow>",s2,"</mrow>"]
      group
        op = "OVER" => s
--        opPrec < prec => parenthesize s
-- ugly parentheses?
```

```
        s

    formatNary(op : S, args : L E, prec : I) : S ==
      group formatNaryNoGroup(op, args, prec)

    formatNaryNoGroup(op : S, args : L E, prec : I) : S ==
      checkargs:Boolean := false
      null args => ""
      p : I := position(op,naryOps)
      p < 1 => error "unknown nary op"
      -- need to test for "ZAG" case and divert it here
      -- ex 1. continuedFraction(314159/100000)
      -- {{+}{3}{{ZAG}{1}{7}}{{ZAG}{1}{15}}{{ZAG}{1}{1}}{{ZAG}{1}{25}}
      -- {{ZAG}{1}{1}}{{ZAG}{1}{7}}{{ZAG}{1}{4}}}
      -- this is the preconditioned output form
      -- including "op", the args list would be the rest of this
      -- i.e op = '+' and args = {{3}{{ZAG}{1}{7}}{{ZAG}{1}{15}}
      -- {{ZAG}{1}{1}}{{ZAG}{1}{25}}{{ZAG}{1}{1}}{{ZAG}{1}{7}}{{ZAG}{1}{4}}}
      -- ex 2. continuedFraction(14159/100000)
      -- this one doesn't have the leading integer
      -- {{+}{{ZAG}{1}{7}}{{ZAG}{1}{15}}{{ZAG}{1}{1}}{{ZAG}{1}{25}}
      -- {{ZAG}{1}{1}}{{ZAG}{1}{7}}{{ZAG}{1}{4}}}
      --
      -- ex 3. continuedFraction(3,repeating [1], repeating [3,6])
      -- {{+}{3}{{ZAG}{1}{3}}{{ZAG}{1}{6}}{{ZAG}{1}{3}}{{ZAG}{1}{6}}
      -- {{ZAG}{1}{3}}{{ZAG}{1}{6}}{{ZAG}{1}{3}}{{ZAG}{1}{6}}
      -- {{ZAG}{1}{3}}{{ZAG}{1}{6}}{...}}
      -- In each of these examples the args list consists of the terms
      -- following the '+' op
      -- so the first arg could be a "ZAG" or something
      -- else, but the second arg looks like it has to be "ZAG", so maybe
      -- test for #args > 1 and args.2 contains "ZAG".
      -- Note that since the resulting MathML <mfrac>s are nested we need
      -- to handle the whole continued fraction at once, i.e. we can't
      -- just look for, e.g., {{ZAG}{1}{6}}
      (#args > 1) and (position("ZAG",stringify first rest args,1) > 0) =>
           tmpS : S := stringify first args
           position("ZAG",tmpS,1) > 0 => formatZag(args)
--         position("ZAG",tmpS,1) > 0 => formatZag1(args)
           concat [formatMml(first args,minPrec) "<mo>+</mo>" formatZag(rest args)]
      -- At least for the ops "*","+","-" we need to test to see if a sigma
      -- or pi is one of their arguments because we might need parentheses
      -- as indicated by the problem with
      -- summation(operator(f)(i),i=1..n)+1 versus
      -- summation(operator(f)(i)+1,i=1..n) having identical displays as
      -- of 2007-21-21
```

```
      op :=
        op = ","     => "<mo>,</mo>" --originally , \:
        op = ";"     => "<mo>;</mo>" --originally ; \: should figure these out
        op = "*"     => "<mspace width='0.3em'/>"
        -- InvisibleTimes
        op = " "     => "<mspace width='0.5em'/>"
        op = "ROW"   => "</mtd><mtd>"
        op = "+"     =>
            checkargs := true
            "<mo>+</mo>"
        op = "-"     =>
            checkargs := true
            "<mo>-</mo>"
        op
      l : L S := nil
      opPrec := naryPrecs.p
      -- if checkargs is true check each arg except last one to see if it's
      -- a sigma or pi and if so add parentheses. Other op's may have to be
      -- checked for in future
      count:I := 1
      for a in args repeat
--        WriteLine$Lisp "checking args"
        if checkargs then
          if count < #args then
            -- check here for sum or product
            if ATOM(a)$Lisp@Boolean then
              opa:S := stringify a
            else
              la : L E := (a pretend L E)
              opa : S := stringify first la
            if opa = "SIGMA" or opa = "SIGMA2" or _
               opa = "PI" or opa = "PI2" then
              l := concat(op,concat(_
                    concat ["<mo>(</mo>",formatMml(a,opPrec),_
                            "<mo>)</mo>"],l)$L(S))$L(S)
            else l := concat(op,concat(formatMml(a,opPrec),l)$L(S))$L(S)
          else l := concat(op,concat(formatMml(a,opPrec),l)$L(S))$L(S)
        else l := concat(op,concat(formatMml(a,opPrec),l)$L(S))$L(S)
        count := count + 1
      s : S := concat reverse rest l
      opPrec < prec => parenthesize s
      s

    formatZag(args : L E) : S ==
    -- args will be a list of things like this {{ZAG}{1}{7}}, the ZAG
    -- must be there, the '1' and '7' could conceivably be more complex
```

```
-- expressions
  tmpZag : L E := first args pretend L E
  -- may want to test that tmpZag contains 'ZAG'
  #args > 1 => "<mfrac>"formatMml(first rest tmpZag,minPrec)"<mrow><mn>"formatMml(first
  -- EQUAL(tmpZag, "...")$Lisp => "<mo>&#x2026;</mo>"
  (first args = "..."::E)@Boolean => "<mo>&#x2026;</mo>"
  position("ZAG",stringify first args,1) > 0 =>
      "<mfrac>"formatMml(first rest tmpZag,minPrec)formatMml(first rest rest tmpZag,minP
  "<mtext>formatZag: Unexpected kind of ZAG</mtext>"


formatZag1(args : L E) : S ==
-- make alternative ZAG format without diminishing fonts, maybe
-- use a table
-- {{ZAG}{1}{7}}
  tmpZag : L E := first args pretend L E
  #args > 1 => "<mfrac>"formatMml(first rest tmpZag,minPrec)"<mrow><mn>"formatMml(first
  (first args = "..."::E)@Boolean => "<mo>&#x2026;</mo>"
  error "formatZag1: Unexpected kind of ZAG"


formatMml(expr : E,prec : I) ==
  i,len : Integer
  intSplitLen : Integer := 20
  ATOM(expr)$Lisp@Boolean =>
    str := stringify expr
    len := #str
    -- this bit seems to deal with integers
    FIXP$Lisp expr =>
      i := expr pretend Integer
      if (i < 0) or (i > 9)
        then
          group
             nstr : String := ""
             -- insert some blanks into the string, if too long
             while ((len := #str) > intSplitLen) repeat
               nstr := concat [nstr," ",
                 elt(str,segment(1,intSplitLen)$US)]
               str := elt(str,segment(intSplitLen+1)$US)
             empty? nstr => concat ["<mn>",str,"</mn>"]
             nstr :=
               empty? str => nstr
               concat [nstr," ",str]
             concat ["<mn>",elt(nstr,segment(2)$US),"</mn>"]
        else str := concat ["<mn>",str,"</mn>"]
    str = "%pi" => "<mi>&#x003C0;</mi>"
```

```
  -- pi
  str = "%e"  => "<mi>&#x02147;</mi>"
  -- ExponentialE
  str = "%i"  => "<mi>&#x02148;</mi>"
  -- ImaginaryI
  len > 0 and str.1 = char "%" => concat(concat("<mi>",str),"</mi>")
  len > 1 and digit? str.1 => concat ["<mn>",str,"</mn>"] -- should handle
  -- presumably this is a literal string
  len > 0 and str.1 = char "_"" =>
    concat(concat("<mtext>",str),"</mtext>")
  len = 1 and str.1 = char " " => " "
  (i := position(str,specialStrings)) > 0 =>
    specialStringsInMML.i
  (i := position(char " ",str)) > 0 =>
    -- We want to preserve spacing, so use a roman font.
    -- What's this for?  Leave the \rm in for now so I can see
    -- where it arises.  Removed 2007-02-14
    concat(concat("<mtext>",str),"</mtext>")
  -- if we get to here does that mean it's a variable?
  concat ["<mi>",str,"</mi>"]
l : L E := (expr pretend L E)
null l => blank
op : S := stringify first l
args : L E := rest l
nargs : I := #args
-- need to test here in case first l is SUPERSUB case and then
-- pass first l and args to formatSuperSub.
position("SUPERSUB",op,1) > 0 =>
  formatSuperSub(first l,args,minPrec)
-- now test for SUB
position("SUB",op,1) > 0 =>
  formatSub1(first l,args,minPrec)

-- special cases
member?(op, specialOps) => formatSpecial(op,args,prec)
member?(op, plexOps)    => formatPlex(op,args,prec)

-- nullary case
0 = nargs => formatNullary op

-- unary case
(1 = nargs) and member?(op, unaryOps) =>
  formatUnary(op, first args, prec)

-- binary case
(2 = nargs) and member?(op, binaryOps) =>
```

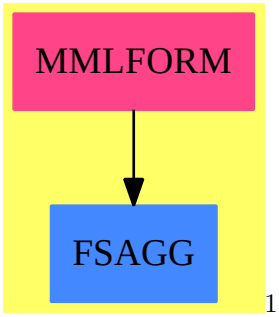
```
      formatBinary(op, args, prec)

    -- nary case
    member?(op,naryNGOps) => formatNaryNoGroup(op,args, prec)
    member?(op,naryOps) => formatNary(op,args, prec)

    op := formatMml(first l,minPrec)
    formatFunction(op,args,prec)
```

14.25.17 Mathematical Markup Language Form

14.26 MathMLForm



1.00

Exports:

coerce	coerceL	coerceS	display	exprex
hash	latex	?=?	?~=?	

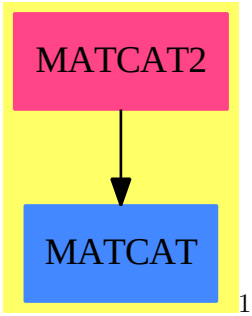
⟨*package MMLFORM MathMLForm*⟩≡
⟨*public declarations*⟩
⟨*private constant declarations*⟩
⟨*private function declarations*⟩
⟨*public function definitions*⟩
⟨*display functions*⟩
⟨*formatting functions*⟩

⟨*MMLFORM.dotabb*⟩≡

```
"MMLFORM" [color="#FF4488",href="bookvol10.4.pdf#nameddest=MMLFORM"]
"FSAGG" [color="#4488FF",href="bookvol10.2.pdf#nameddest=FSAGG"]
"MMLFORM" -> "FSAGG"
```

14.27 package MATCAT2 MatrixCategoryFunctions2

14.28 MatrixCategoryFunctions2



1.00

Exports:
map reduce

⟨*package MATCAT2 MatrixCategoryFunctions2*⟩≡

```
)abbrev package MATCAT2 MatrixCategoryFunctions2
++ Author: Clifton J. Williamson
++ Date Created: 21 November 1989
++ Date Last Updated: 21 March 1994
++ Basic Operations:
++ Related Domains: IndexedMatrix(R,minRow,minCol), Matrix(R),
++    RectangularMatrix(n,m,R), SquareMatrix(n,R)
++ Also See:
++ AMS Classifications:
++ Keywords:
++ Keywords: matrix, map, reduce
++ Examples:
++ References:
++ Description:
++   \spadtype{MatrixCategoryFunctions2} provides functions between two matrix
++   domains.  The functions provided are \spadfun{map} and \spadfun{reduce}.
MatrixCategoryFunctions2(R1,Row1,Col1,M1,R2,Row2,Col2,M2):_
         Exports == Implementation where
  R1   : Ring
  Row1 : FiniteLinearAggregate R1
  Col1 : FiniteLinearAggregate R1
  M1   : MatrixCategory(R1,Row1,Col1)
  R2   : Ring
  Row2 : FiniteLinearAggregate R2
  Col2 : FiniteLinearAggregate R2
  M2   : MatrixCategory(R2,Row2,Col2)
```

```
    Exports ==> with
      map: (R1 -> R2,M1) -> M2
       ++ \spad{map(f,m)} applies the function f to the elements of the matrix m.
      map: (R1 -> Union(R2,"failed"),M1) -> Union(M2,"failed")
       ++ \spad{map(f,m)} applies the function f to the elements of the matrix m.
      reduce: ((R1,R2) -> R2,M1,R2) -> R2
       ++ \spad{reduce(f,m,r)} returns a matrix n where
       ++ \spad{n[i,j] = f(m[i,j],r)} for all indices i and j.

    Implementation ==> add
      minr ==> minRowIndex
      maxr ==> maxRowIndex
      minc ==> minColIndex
      maxc ==> maxColIndex

      map(f:(R1->R2),m:M1):M2 ==
        ans : M2 := new(nrows m,ncols m,0)
        for i in minr(m)..maxr(m) for k in minr(ans)..maxr(ans) repeat
          for j in minc(m)..maxc(m) for l in minc(ans)..maxc(ans) repeat
            qsetelt_!(ans,k,l,f qelt(m,i,j))
        ans

      map(f:(R1 -> (Union(R2,"failed"))),m:M1):Union(M2,"failed") ==
        ans : M2 := new(nrows m,ncols m,0)
        for i in minr(m)..maxr(m) for k in minr(ans)..maxr(ans) repeat
          for j in minc(m)..maxc(m) for l in minc(ans)..maxc(ans) repeat
            (r := f qelt(m,i,j)) = "failed" => return "failed"
            qsetelt_!(ans,k,l,r::R2)
        ans

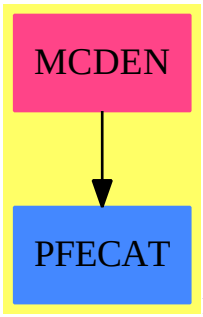
      reduce(f,m,ident) ==
        s := ident
        for i in minr(m)..maxr(m) repeat
         for j in minc(m)..maxc(m) repeat
           s := f(qelt(m,i,j),s)
        s
```

⟨*MATCAT2.dotabb*⟩≡

```
"MATCAT2" [color="#FF4488",href="bookvol10.4.pdf#nameddest=MATCAT2"]
"MATCAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=MATCAT"]
"MATCAT2" -> "MATCAT"
```

14.29 package MCDEN MatrixCommonDenominator

14.30 MatrixCommonDenominator



1.00

Exports:
commonDenominator clearDenominator splitDenominator

⟨*package MCDEN MatrixCommonDenominator*⟩≡

```
)abbrev package MCDEN MatrixCommonDenominator
--% MatrixCommonDenominator
++ Author: Manuel Bronstein
++ Date Created: 2 May 1988
++ Date Last Updated: 20 Jul 1990
++ Description: MatrixCommonDenominator provides functions to
++ compute the common denominator of a matrix of elements of the
++ quotient field of an integral domain.
++ Keywords: gcd, quotient, matrix, common, denominator.
MatrixCommonDenominator(R, Q): Exports == Implementation where
  R: IntegralDomain
  Q: QuotientFieldCategory R

  VR ==> Vector R
  VQ ==> Vector Q

  Exports ==> with
    commonDenominator: Matrix Q -> R
      ++ commonDenominator(q) returns a common denominator d for
      ++ the elements of q.
    clearDenominator : Matrix Q -> Matrix R
      ++ clearDenominator(q) returns p such that \spad{q = p/d} where d is
      ++ a common denominator for the elements of q.
    splitDenominator : Matrix Q -> Record(num: Matrix R, den: R)
      ++ splitDenominator(q) returns \spad{[p, d]} such that \spad{q = p/d} and d
      ++ is a common denominator for the elements of q.
```

```
Implementation ==> add
  import ListFunctions2(Q, R)
  import MatrixCategoryFunctions2(Q,VQ,VQ,Matrix Q,R,VR,VR,Matrix R)

  clearDenominator m ==
    d := commonDenominator m
    map(numer(d * #1), m)

  splitDenominator m ==
    d := commonDenominator m
    [map(numer(d * #1), m), d]

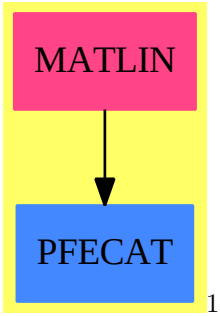
  if R has GcdDomain then
    commonDenominator m == lcm map(denom, parts m)
  else
    commonDenominator m == reduce("*",map(denom, parts m),1)$List(R)
```

⟨*MCDEN.dotabb*⟩≡

```
"MCDEN" [color="#FF4488",href="bookvol10.4.pdf#nameddest=MCDEN"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"MCDEN" -> "PFECAT"
```

14.31 package MATLIN MatrixLinearAlgebraFunctions

14.32 MatrixLinearAlgebraFunctions



1.00

Exports:

adjoint	determinant	elColumn2!	elRow1!	elRow2!
fractionFreeGauss!	inverse	invertIfCan	minordet	normalizedDivide
nullSpace	nullity	rank	rowEchelon	

⟨*package MATLIN MatrixLinearAlgebraFunctions*⟩≡

```
)abbrev package MATLIN MatrixLinearAlgebraFunctions
++ Author: Clifton J. Williamson, P.Gianni
++ Date Created: 13 November 1989
++ Date Last Updated: December 1992
++ Basic Operations:
++ Related Domains: IndexedMatrix(R,minRow,minCol), Matrix(R),
++    RectangularMatrix(n,m,R), SquareMatrix(n,R)
++ Also See:
++ AMS Classifications:
++ Keywords: matrix, canonical forms, linear algebra
++ Examples:
++ References:
++ Description:
++   \spadtype{MatrixLinearAlgebraFunctions} provides functions to compute
++   inverses and canonical forms.
MatrixLinearAlgebraFunctions(R,Row,Col,M):Exports == Implementation where
  R   : CommutativeRing
  Row : FiniteLinearAggregate R
  Col : FiniteLinearAggregate R
  M   : MatrixCategory(R,Row,Col)
  I ==> Integer

  Exports ==> with
```

```
determinant: M -> R
  ++ \spad{determinant(m)} returns the determinant of the matrix m.
  ++ an error message is returned if the matrix is not square.
minordet: M -> R
  ++ \spad{minordet(m)} computes the determinant of the matrix m using
  ++ minors. Error: if the matrix is not square.
elRow1!       :   (M,I,I)         ->  M
  ++ elRow1!(m,i,j) swaps rows i and j of matrix m : elementary operation
  ++ of first kind
elRow2!       :  (M,R,I,I)        ->  M
  ++ elRow2!(m,a,i,j)  adds to row i a*row(m,j) : elementary operation of
  ++ second kind. (i ^=j)
elColumn2!    :  (M,R,I,I)        ->  M
  ++ elColumn2!(m,a,i,j)  adds to column i a*column(m,j) : elementary
  ++ operation of second kind. (i ^=j)

if R has IntegralDomain then
  rank: M -> NonNegativeInteger
    ++ \spad{rank(m)} returns the rank of the matrix m.
  nullity: M -> NonNegativeInteger
    ++ \spad{nullity(m)} returns the mullity of the matrix m. This is
    ++ the dimension of the null space of the matrix m.
  nullSpace: M -> List Col
    ++ \spad{nullSpace(m)} returns a basis for the null space of the
    ++ matrix m.
  fractionFreeGauss! : M -> M
    ++ \spad{fractionFreeGauss(m)} performs the fraction
    ++ free gaussian  elimination on the matrix m.
  invertIfCan : M -> Union(M,"failed")
    ++ \spad{invertIfCan(m)} returns the inverse of m over R
  adjoint : M -> Record(adjMat:M, detMat:R)
    ++ \spad{adjoint(m)} returns the ajoint matrix of m (i.e. the matrix
    ++ n such that m*n = determinant(m)*id) and the detrminant of m.

if R has EuclideanDomain then
  rowEchelon: M -> M
    ++ \spad{rowEchelon(m)} returns the row echelon form of the matrix m.

  normalizedDivide: (R, R) -> Record(quotient:R, remainder:R)
    ++ normalizedDivide(n,d) returns a normalized quotient and
    ++ remainder such that consistently unique representatives
    ++ for the residue class are chosen, e.g. positive remainders

if R has Field then
  inverse: M -> Union(M,"failed")
    ++ \spad{inverse(m)} returns the inverse of the matrix.
```

```
        ++ If the matrix is not invertible, "failed" is returned.
        ++ Error: if the matrix is not square.

    Implementation ==> add

      rowAllZeroes?: (M,I) -> Boolean
      rowAllZeroes?(x,i) ==
        -- determines if the ith row of x consists only of zeroes
        -- internal function: no check on index i
        for j in minColIndex(x)..maxColIndex(x) repeat
          qelt(x,i,j) ^= 0 => return false
        true

      colAllZeroes?: (M,I) -> Boolean
      colAllZeroes?(x,j) ==
        -- determines if the ith column of x consists only of zeroes
        -- internal function: no check on index j
        for i in minRowIndex(x)..maxRowIndex(x) repeat
          qelt(x,i,j) ^= 0 => return false
        true

      minorDet:(M,I,List I,I,PrimitiveArray(Union(R,"uncomputed")))-> R
      minorDet(x,m,l,i,v) ==
        z := v.m
        z case R => z
        ans : R := 0; rl : List I := nil()
        j := first l; l := rest l; pos := true
        minR := minRowIndex x; minC := minColIndex x;
        repeat
          if qelt(x,j + minR,i + minC) ^= 0 then
            ans :=
              md := minorDet(x,m - 2**(j :: NonNegativeInteger),_
                             concat_!(reverse rl,l),i + 1,v) *_
                             qelt(x,j + minR,i + minC)
              pos => ans + md
              ans - md
          null l =>
            v.m := ans
            return ans
          pos := not pos; rl := cons(j,rl); j := first l; l := rest l

      minordet x ==
        (ndim := nrows x) ^= (ncols x) =>
          error "determinant: matrix must be square"
        -- minor expansion with (s---loads of) memory
        n1 : I := ndim - 1
```

```
    v : PrimitiveArray(Union(R,"uncomputed")) :=
         new((2**ndim - 1) :: NonNegativeInteger,"uncomputed")
    minR := minRowIndex x; maxC := maxColIndex x
    for i in 0..n1 repeat
      qsetelt_!(v,(2**i - 1),qelt(x,i + minR,maxC))
    minorDet(x, 2**ndim - 2, [i for i in 0..n1], 0, v)

     -- elementary operation of first kind: exchange two rows --
  elRow1!(m:M,i:I,j:I) : M ==
    vec:=row(m,i)
    setRow!(m,i,row(m,j))
    setRow!(m,j,vec)
    m

           -- elementary operation of second kind: add to row i--
                       -- a*row j  (i^=j) --
  elRow2!(m : M,a:R,i:I,j:I) : M ==
    vec:= map(a*#1,row(m,j))
    vec:=map("+",row(m,i),vec)
    setRow!(m,i,vec)
    m
           -- elementary operation of second kind: add to column i --
                        -- a*column j (i^=j) --
  elColumn2!(m : M,a:R,i:I,j:I) : M ==
    vec:= map(a*#1,column(m,j))
    vec:=map("+",column(m,i),vec)
    setColumn!(m,i,vec)
    m

  if R has IntegralDomain then
    -- Fraction-Free Gaussian Elimination
    fractionFreeGauss! x  ==
      (ndim := nrows x) = 1 => x
      ans := b := 1$R
      minR := minRowIndex x; maxR := maxRowIndex x
      minC := minColIndex x; maxC := maxColIndex x
      i := minR
      for j in minC..maxC repeat
        if qelt(x,i,j) = 0 then -- candidate for pivot = 0
          rown := minR - 1
          for k in (i+1)..maxR repeat
            if qelt(x,k,j) ^= 0 then
               rown := k -- found a pivot
               leave
          if rown > minR - 1 then
             swapRows_!(x,i,rown)
```

```
          ans := -ans
      (c := qelt(x,i,j)) = 0 =>  "next j" -- try next column
      for k in (i+1)..maxR repeat
        if qelt(x,k,j) = 0 then
          for l in (j+1)..maxC repeat
            qsetelt_!(x,k,l,(c * qelt(x,k,l) exquo b) :: R)
        else
          pv := qelt(x,k,j)
          qsetelt_!(x,k,j,0)
          for l in (j+1)..maxC repeat
            val := c * qelt(x,k,l) - pv * qelt(x,i,l)
            qsetelt_!(x,k,l,(val exquo b) :: R)
      b := c
      (i := i+1)>maxR => leave
    if ans=-1 then
      lasti := i-1
      for j in 1..maxC repeat x(lasti, j) := -x(lasti,j)
    x

  --
  lastStep(x:M)  : M ==
    ndim := nrows x
    minR := minRowIndex x; maxR := maxRowIndex x
    minC := minColIndex x; maxC := minC+ndim -1
    exCol:=maxColIndex x
    det:=x(maxR,maxC)
    maxR1:=maxR-1
    maxC1:=maxC+1
    minC1:=minC+1
    iRow:=maxR
    iCol:=maxC-1
    for i in maxR1..1 by -1 repeat
      for j in maxC1..exCol repeat
        ss:=+/[x(i,iCol+k)*x(i+k,j) for k in 1..(maxR-i)]
        x(i,j) := _exquo((det * x(i,j) - ss),x(i,iCol))::R
      iCol:=iCol-1
    subMatrix(x,minR,maxR,maxC1,exCol)

  invertIfCan(y) ==
    (nr:=nrows y) ^= (ncols y) =>
        error "invertIfCan: matrix must be square"
    adjRec := adjoint y
    (den:=recip(adjRec.detMat)) case "failed" => "failed"
    den::R * adjRec.adjMat

  adjoint(y) ==
```

```
    (nr:=nrows y) ^= (ncols y) => error "adjoint: matrix must be square"
    maxR := maxRowIndex y
    maxC := maxColIndex y
    x := horizConcat(copy y,scalarMatrix(nr,1$R))
    ffr:= fractionFreeGauss!(x)
    det:=ffr(maxR,maxC)
    [lastStep(ffr),det]


if R has Field then

  VR      ==> Vector R
  IMATLIN ==> InnerMatrixLinearAlgebraFunctions(R,Row,Col,M)
  MMATLIN ==> InnerMatrixLinearAlgebraFunctions(R,VR,VR,Matrix R)
  FLA2    ==> FiniteLinearAggregateFunctions2(R, VR, R, Col)
  MAT2    ==> MatrixCategoryFunctions2(R,Row,Col,M,R,VR,VR,Matrix R)

  rowEchelon  y == rowEchelon(y)$IMATLIN
  rank        y == rank(y)$IMATLIN
  nullity     y == nullity(y)$IMATLIN
  determinant y == determinant(y)$IMATLIN
  inverse     y == inverse(y)$IMATLIN
  if Col has shallowlyMutable then
    nullSpace y == nullSpace(y)$IMATLIN
  else
    nullSpace y ==
      [map(#1, v)$FLA2 for v in nullSpace(map(#1, y)$MAT2)$MMATLIN]

else if R has IntegralDomain then
  QF      ==> Fraction R
  Row2    ==> Vector QF
  Col2    ==> Vector QF
  M2      ==> Matrix QF
  IMATQF  ==> InnerMatrixQuotientFieldFunctions(R,Row,Col,M,QF,Row2,Col2,M2)

  nullSpace m == nullSpace(m)$IMATQF

  determinant y ==
    (nrows y) ^= (ncols y) => error "determinant: matrix must be square"
    fm:=fractionFreeGauss!(copy y)
    fm(maxRowIndex fm,maxColIndex fm)

  rank x ==
    y :=
      (rk := nrows x) > (rh := ncols x) =>
          rk := rh
```

```
        transpose x
    copy x
  y := fractionFreeGauss! y
  i := maxRowIndex y
  while rk > 0 and rowAllZeroes?(y,i) repeat
    i := i - 1
    rk := (rk - 1) :: NonNegativeInteger
  rk :: NonNegativeInteger

nullity x == (ncols x - rank x) :: NonNegativeInteger

if R has EuclideanDomain then

  if R has IntegerNumberSystem then
      normalizedDivide(n:R, d:R):Record(quotient:R, remainder:R) ==
         qr := divide(n, d)
         qr.remainder >= 0 => qr
         d > 0 =>
            qr.remainder := qr.remainder + d
            qr.quotient := qr.quotient - 1
            qr
         qr.remainder := qr.remainder - d
         qr.quotient := qr.quotient + 1
         qr
  else
      normalizedDivide(n:R, d:R):Record(quotient:R, remainder:R) ==
         divide(n, d)

  rowEchelon y ==
    x := copy y
    minR := minRowIndex x; maxR := maxRowIndex x
    minC := minColIndex x; maxC := maxColIndex x
    n := minR - 1
    i := minR
    for j in minC..maxC repeat
      if i > maxR then leave x
      n := minR - 1
      xnj: R
      for k in i..maxR repeat
        if not zero?(xkj:=qelt(x,k,j)) and ((n = minR - 1) _
               or sizeLess?(xkj,xnj)) then
          n := k
          xnj := xkj
      n = minR - 1 => "next j"
      swapRows_!(x,i,n)
      for k in (i+1)..maxR repeat
```

```
              qelt(x,k,j) = 0 => "next k"
              aa := extendedEuclidean(qelt(x,i,j),qelt(x,k,j))
              (a,b,d) := (aa.coef1,aa.coef2,aa.generator)
              b1 := (qelt(x,i,j) exquo d) :: R
              a1 := (qelt(x,k,j) exquo d) :: R
              -- a*b1+a1*b = 1
              for k1 in (j+1)..maxC repeat
                val1 := a * qelt(x,i,k1) + b * qelt(x,k,k1)
                val2 := -a1 * qelt(x,i,k1) + b1 * qelt(x,k,k1)
                qsetelt_!(x,i,k1,val1); qsetelt_!(x,k,k1,val2)
              qsetelt_!(x,i,j,d); qsetelt_!(x,k,j,0)

            un := unitNormal qelt(x,i,j)
            qsetelt_!(x,i,j,un.canonical)
            if un.associate ^= 1 then for jj in (j+1)..maxC repeat
                qsetelt_!(x,i,jj,un.associate * qelt(x,i,jj))

            xij := qelt(x,i,j)
            for k in minR..(i-1) repeat
              qelt(x,k,j) = 0 => "next k"
              qr := normalizedDivide(qelt(x,k,j), xij)
              qsetelt_!(x,k,j,qr.remainder)
              for k1 in (j+1)..maxC repeat
                qsetelt_!(x,k,k1,qelt(x,k,k1) - qr.quotient * qelt(x,i,k1))
            i := i + 1
          x

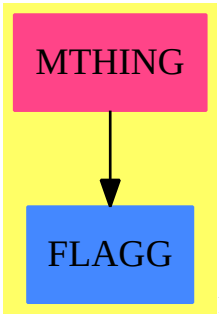
    else determinant x == minordet x
```

⟨*MATLIN.dotabb*⟩≡

```
"MATLIN" [color="#FF4488",href="bookvol10.4.pdf#nameddest=MATLIN"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"MATLIN" -> "PFECAT"
```

14.33 package MTHING MergeThing

14.34 MergeThing



1.00

Exports:
mergeDifference

⟨*package MTHING MergeThing*⟩≡
```
)abbrev package MTHING MergeThing
++ This package exports tools for merging lists
MergeThing(S:OrderedSet): Exports == Implementation where
  Exports == with
    mergeDifference: (List(S),List(S)) -> List(S)
        ++ mergeDifference(l1,l2) returns a list of elements in l1 not present
        ++ in l2. Assumes lists are ordered and all x in l2 are also in l1.
  Implementation == add
    mergeDifference1: (List S,S,List S) -> List S
    mergeDifference(x,y) ==
      null x or null y => x
      mergeDifference1(x,y.first,y.rest)
      x.first=y.first => x.rest
      x
    mergeDifference1(x,fy,ry) ==
      rx := x
      while not null rx repeat
        rx := rx.rest
        frx := rx.first
        while fy < frx repeat
          null ry => return x
          fy := first ry
          ry := rest ry
        frx = fy =>
          x.rest := rx.rest
          null ry => return x
          fy := ry.first
```

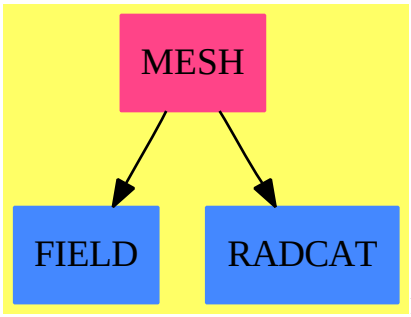
```
            ry := ry.rest
          x := rx
```

⟨*MTHING.dotabb*⟩≡

```
"MTHING" [color="#FF4488",href="bookvol10.4.pdf#nameddest=MTHING"]
"FLAGG" [color="#4488FF",href="bookvol10.2.pdf#nameddest=FLAGG"]
"MTHING" -> "FLAGG"
```

14.35 package MESH MeshCreationRoutinesForThreeDimensions

14.36 MeshCreationRoutinesForThreeDimensions



1.00

Exports:
meshFun2Var meshPar1Var meshPar2Var ptFunc

⟨*package MESH MeshCreationRoutinesForThreeDimensions*⟩≡

```
)abbrev package MESH MeshCreationRoutinesForThreeDimensions
++ <description of package>
++ Author: Jim Wen
++ Date Created: ??
++ Date Last Updated: October 1991 by Jon Steinbach
++ Keywords:
++ Examples:
++ References:
MeshCreationRoutinesForThreeDimensions():Exports == Implementation where

  I       ==> Integer
  PI      ==> PositiveInteger
  SF      ==> DoubleFloat
  L       ==> List
  SEG     ==> Segment
  S       ==> String
  Fn1     ==> SF -> SF
  Fn2     ==> (SF,SF) -> SF
  Fn3     ==> (SF,SF,SF) -> SF
  FnPt    ==> (SF,SF) -> Point(SF)
  FnU     ==> Union(Fn3,"undefined")
  EX      ==> Expression
  DROP    ==> DrawOption
  POINT   ==> Point(SF)
  SPACE3  ==> ThreeSpace(SF)
  COMPPROP ==> SubSpaceComponentProperty
```

```
TUBE    ==> TubePlot

Exports ==> with
  meshPar2Var: (Fn2,Fn2,Fn2,FnU,SEG SF,SEG SF,L DROP) -> SPACE3
      ++ meshPar2Var(f,g,h,j,s1,s2,l) \undocumented
  meshPar2Var: (FnPt,SEG SF,SEG SF,L DROP) -> SPACE3
      ++ meshPar2Var(f,s1,s2,l) \undocumented
  meshPar2Var: (SPACE3,FnPt,SEG SF,SEG SF,L DROP) -> SPACE3
      ++ meshPar2Var(sp,f,s1,s2,l) \undocumented
  meshFun2Var: (Fn2,FnU,SEG SF,SEG SF,L DROP) -> SPACE3
      ++ meshFun2Var(f,g,s1,s2,l) \undocumented
  meshPar1Var: (EX I,EX I,EX I,Fn1,SEG SF,L DROP) -> SPACE3
      ++ meshPar1Var(s,t,u,f,s1,l) \undocumented
  ptFunc: (Fn2,Fn2,Fn2,Fn3) -> ((SF,SF) -> POINT)
    ++ ptFunc(a,b,c,d) is an internal function exported in
    ++ order to compile packages.

Implementation ==> add
  import ViewDefaultsPackage()
  import SubSpaceComponentProperty()
  import DrawOptionFunctions0
  import SPACE3
  --import TUBE()

  -- local functions
  numberCheck(nums:Point SF):Void ==
    -- this function checks to see that the small floats are
    -- actually just that - rather than complex numbers or
    -- whatever (the whatever includes nothing presently
    -- since NaN, Not a Number, is not necessarily supported
    -- by common lisp). note that this function is dependent
    -- upon the fact that Common Lisp supports complex numbers.
    for i in minIndex(nums)..maxIndex(nums) repeat
      COMPLEXP(nums.(i::PositiveInteger))$Lisp =>
        error "An unexpected complex number was encountered in the calculations."

  makePt:(SF,SF,SF,SF) -> POINT
  makePt(x,y,z,c) == point(l : List SF := [x,y,z,c])
  ptFunc(f,g,h,c) ==
    x := f(#1,#2); y := g(#1,#2); z := h(#1,#2)
    makePt(x,y,z,c(x,y,z))

  -- parameterized equations of two variables
  meshPar2Var(sp,ptFun,uSeg,vSeg,opts) ==
    -- the issue of open and closed needs to be addressed, here, we are
    -- defaulting to open (which is probably the correct default)
```

```
      -- the user should be able to override that (optional argument?)
      llp : L L POINT := nil()
      uNum : PI  := var1Steps(opts,var1StepsDefault())
      vNum : PI  := var2Steps(opts,var2StepsDefault())
      ustep := (lo uSeg - hi uSeg)/uNum
      vstep := (lo vSeg - hi vSeg)/vNum
      someV := hi vSeg
      for iv in vNum..0 by -1 repeat
        if zero? iv then someV := lo vSeg
        -- hack: get last number in segment within segment
        lp : L POINT := nil()
        someU := hi uSeg
        for iu in uNum..0 by -1 repeat
          if zero? iu then someU := lo uSeg
          -- hack: get last number in segment within segment
          pt := ptFun(someU,someV)
          numberCheck pt
          lp := concat(pt,lp)
          someU := someU + ustep
        llp := concat(lp,llp)
        someV := someV + vstep
      -- now llp contains a list of lists of points
      -- for a surface that is a result of a function of 2 variables,
      -- the main component is open and each sublist is open as well
      lProp : L COMPPROP := [ new() for l in llp ]
      for aProp in lProp repeat
        close(aProp,false)
        solid(aProp,false)
      aProp : COMPPROP:= new()
      close(aProp,false)
      solid(aProp,false)
      space := sp
--      space := create3Space()
      mesh(space,llp,lProp,aProp)
      space

    meshPar2Var(ptFun,uSeg,vSeg,opts) ==
      sp := create3Space()
      meshPar2Var(sp,ptFun,uSeg,vSeg,opts)

    zCoord: (SF,SF,SF) -> SF
    zCoord(x,y,z) == z

    meshPar2Var(xFun,yFun,zFun,colorFun,uSeg,vSeg,opts) ==
      -- the color function should be parameterized by (u,v) as well,
      -- not (x,y,z) but we also want some sort of consistency and so
```

```
      -- changing this over would mean possibly changing the explicit
      -- stuff over and there, we probably do want the color function
      -- to be parameterized by (x,y,z) - not just (x,y) (this being
      -- for convinience only since z is also defined in terms of (x,y)).
      (colorFun case Fn3) =>
        meshPar2Var(ptFunc(xFun,yFun,zFun,colorFun :: Fn3),uSeg,vSeg,opts)
      meshPar2Var(ptFunc(xFun,yFun,zFun,zCoord),uSeg,vSeg,opts)

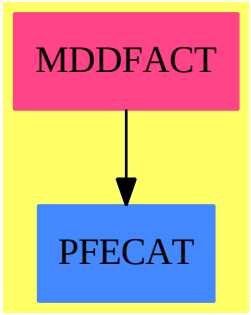
    -- explicit equations of two variables
    meshFun2Var(zFun,colorFun,xSeg,ySeg,opts) ==
      -- here, we construct the data for a function of two variables
      meshPar2Var(#1,#2,zFun,colorFun,xSeg,ySeg,opts)
```

⟨*MESH.dotabb*⟩≡

```
"MESH" [color="#FF4488",href="bookvol10.4.pdf#nameddest=MESH"]
"FIELD"  [color="#4488FF",href="bookvol10.2.pdf#nameddest=FIELD"]
"RADCAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=RADCAT"]
"MESH" -> "FIELD"
"MESH" -> "RADCAT"
```

14.37 package MDDFACT ModularDistinctDegreeFactorizer

14.38 ModularDistinctDegreeFactorizer



1.00

Exports:
factor gcd linears ddFact exptMod separateFactors

⟨*package MDDFACT ModularDistinctDegreeFactorizer*⟩≡

```
)abbrev package MDDFACT ModularDistinctDegreeFactorizer
++ Author: Barry Trager
++ Date Created:
++ Date Last Updated: 20.9.95 (JHD)
++ Basic Functions:
++ Related Constructors:
++ Also See:
++ AMS Classifications:
++ Keywords:
++ References:
++ Description:
++ This package supports factorization and gcds
++ of univariate polynomials over the integers modulo different
++ primes. The inputs are given as polynomials over the integers
++ with the prime passed explicitly as an extra argument.

ModularDistinctDegreeFactorizer(U):C == T where
  U : UnivariatePolynomialCategory(Integer)
  I ==> Integer
  NNI ==> NonNegativeInteger
  PI ==> PositiveInteger
  V ==> Vector
  L ==> List
  DDRecord ==> Record(factor:EMR,degree:I)
  UDDRecord ==> Record(factor:U,degree:I)
  DDList ==> L DDRecord
```

```
UDDList ==> L UDDRecord


C == with
  gcd:(U,U,I) -> U
    ++ gcd(f1,f2,p) computes the gcd of the univariate polynomials
    ++ f1 and f2 modulo the integer prime p.
  linears: (U,I) -> U
    ++ linears(f,p) returns the product of all the linear factors
    ++ of f modulo p. Potentially incorrect result if f is not
    ++ square-free modulo p.
  factor:(U,I) -> L U
    ++ factor(f1,p) returns the list of factors of the univariate
    ++ polynomial f1 modulo the integer prime p.
    ++ Error: if f1 is not square-free modulo p.
  ddFact:(U,I) -> UDDList
    ++ ddFact(f,p) computes a distinct degree factorization of the
    ++ polynomial f modulo the prime p, i.e. such that each factor
    ++ is a product of irreducibles of the same degrees. The input
    ++ polynomial f is assumed to be square-free modulo p.
  separateFactors:(UDDList,I) -> L U
    ++ separateFactors(ddl, p) refines the distinct degree factorization
    ++ produced by \spadfunFrom{ddFact}{ModularDistinctDegreeFactorizer}
    ++ to give a complete list of factors.
  exptMod:(U,I,U,I) -> U
    ++ exptMod(f,n,g,p) raises the univariate polynomial f to the nth
    ++ power modulo the polynomial g and the prime p.

T == add
  reduction(u:U,p:I):U ==
     zero? p => u
     map(positiveRemainder(#1,p),u)
  merge(p:I,q:I):Union(I,"failed") ==
     p = q => p
     p = 0 => q
     q = 0 => p
     "failed"
  modInverse(c:I,p:I):I ==
      (extendedEuclidean(c,p,1)::Record(coef1:I,coef2:I)).coef1
  exactquo(u:U,v:U,p:I):Union(U,"failed") ==
      invlcv:=modInverse(leadingCoefficient v,p)
      r:=monicDivide(u,reduction(invlcv*v,p))
      reduction(r.remainder,p) ^=0 => "failed"
      reduction(invlcv*r.quotient,p)
  EMR := EuclideanModularRing(Integer,U,Integer,
                              reduction,merge,exactquo)
```

```
probSplit2:(EMR,EMR,I) -> Union(List EMR,"failed")
trace:(EMR,I,EMR) -> EMR
ddfactor:EMR -> L EMR
ddfact:EMR -> DDList
sepFact1:DDRecord -> L EMR
sepfact:DDList -> L EMR
probSplit:(EMR,EMR,I) -> Union(L EMR,"failed")
makeMonic:EMR -> EMR
exptmod:(EMR,I,EMR) -> EMR

lc(u:EMR):I == leadingCoefficient(u::U)
degree(u:EMR):I == degree(u::U)
makeMonic(u) == modInverse(lc(u),modulus(u)) * u

i:I

exptmod(u1,i,u2) ==
  i < 0 => error("negative exponentiation not allowed for exptMod")
  ans:= 1$EMR
  while i > 0 repeat
    if odd?(i) then ans:= (ans * u1) rem u2
    i:= i quo 2
    u1:= (u1 * u1) rem u2
  ans

exptMod(a,i,b,q) ==
  ans:= exptmod(reduce(a,q),i,reduce(b,q))
  ans::U

ddfactor(u) ==
  if (c:= lc(u)) ^= 1$I then u:= makeMonic(u)
  ans:= sepfact(ddfact(u))
  cons(c::EMR,[makeMonic(f) for f in ans | degree(f) > 0])

gcd(u,v,q) == gcd(reduce(u,q),reduce(v,q))::U

factor(u,q) ==
  v:= reduce(u,q)
  dv:= reduce(differentiate(u),q)
  degree gcd(v,dv) > 0 =>
    error("Modular factor: polynomial must be squarefree")
  ans:= ddfactor v
  [f::U for f in ans]

ddfact(u) ==
```

```
    p:=modulus u
    w:= reduce(monomial(1,1)$U,p)
    m:= w
    d:I:= 1
    if (c:= lc(u)) ^= 1$I then u:= makeMonic u
    ans:DDList:= []
    repeat
      w:= exptmod(w,p,u)
      g:= gcd(w - m,u)
      if degree g > 0 then
        g:= makeMonic(g)
        ans:= [[g,d],:ans]
        u:= (u quo g)
      degree(u) = 0 => return [[c::EMR,0$I],:ans]
      d:= d+1
      d > (degree(u):I quo 2) =>
             return [[c::EMR,0$I],[u,degree(u)],:ans]

  ddFact(u,q) ==
    ans:= ddfact(reduce(u,q))
    [[(dd.factor)::U,dd.degree]$UDDRecord for dd in ans]$UDDList

  linears(u,q) ==
     uu:=reduce(u,q)
     m:= reduce(monomial(1,1)$U,q)
     gcd(exptmod(m,q,uu)-m,uu)::U

  sepfact(factList) ==
    "append"/[sepFact1(f) for f in factList]

  separateFactors(uddList,q) ==
    ans:= sepfact [[reduce(udd.factor,q),udd.degree]$DDRecord for
      udd in uddList]$DDList
    [f::U for f in ans]

  decode(s:Integer, p:Integer, x:U):U ==
    s<p => s::U
    qr := divide(s,p)
    qr.remainder :: U + x*decode(qr.quotient, p, x)

  sepFact1(f) ==
    u:= f.factor
    p:=modulus u
    (d := f.degree) = 0 => [u]
    if (c:= lc(u)) ^= 1$I then u:= makeMonic(u)
    d = (du := degree(u)) => [u]
```

```
      ans:L EMR:= []
      x:U:= monomial(1,1)
      -- for small primes find linear factors by exhaustion
      d=1 and p < 1000  =>
        for i in 0.. while du > 0 repeat
          if u(i::U) = 0 then
            ans := cons(reduce(x-(i::U),p),ans)
            du := du-1
        ans
      y:= x
      s:I:= 0
      ss:I := 1
      stack:L EMR:= [u]
      until null stack repeat
        t:= reduce(((s::U)+x),p)
        if not ((flist:= probSplit(first stack,t,d)) case "failed") then
          stack:= rest stack
          for fact in flist repeat
            f1:= makeMonic(fact)
            (df1:= degree(f1)) = 0 => nil
            df1 > d => stack:= [f1,:stack]
            ans:= [f1,:ans]
        p = 2 =>
          ss:= ss + 1
          x := y * decode(ss, p, y)
        s:= s+1
        s = p =>
          s:= 0
          ss := ss + 1
          x:= y * decode(ss, p, y)
--          not one? leadingCoefficient(x) =>
          not (leadingCoefficient(x) = 1) =>
              ss := p ** degree x
              x:= y ** (degree(x) + 1)
      [c * first(ans),:rest(ans)]

    probSplit(u,t,d) ==
      (p:=modulus(u)) = 2 => probSplit2(u,t,d)
      f1:= gcd(u,t)
      r:= ((p**(d:NNI)-1) quo 2):NNI
      n:= exptmod(t,r,u)
      f2:= gcd(u,n + 1)
      (g:= f1 * f2) = 1 => "failed"
      g = u => "failed"
      [f1,f2,(u quo g)]
```

```
probSplit2(u,t,d) ==
  f:= gcd(u,trace(t,d,u))
  f = 1 => "failed"
  degree u = degree f => "failed"
  [1,f,u quo f]

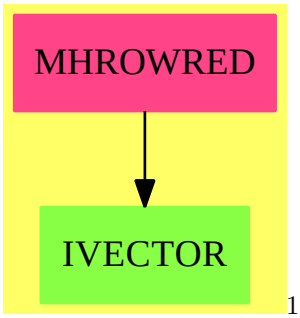
trace(t,d,u) ==
  p:=modulus(t)
  d:= d - 1
  tt:=t
  while d > 0 repeat
    tt:= (tt + (t:=exptmod(t,p,u))) rem u
    d:= d - 1
  tt
```

⟨*MDDFACT.dotabb*⟩≡

```
"MDDFACT" [color="#FF4488",href="bookvol10.4.pdf#nameddest=MDDFACT"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"MDDFACT" -> "PFECAT"
```

14.39 package MHROWRED ModularHermitianRowReduction

14.40 ModularHermitianRowReduction



1.00

Exports:
rowEch normalizedDivide rowEchLocal rowEchelon rowEchelonLocal

⟨*package MHROWRED ModularHermitianRowReduction*⟩≡

```
)abbrev package MHROWRED ModularHermitianRowReduction
++ Modular hermitian row reduction.
++ Author: Manuel Bronstein
++ Date Created: 22 February 1989
++ Date Last Updated: 24 November 1993
++ Keywords: matrix, reduction.
-- should be moved into matrix whenever possible
ModularHermitianRowReduction(R): Exports == Implementation where
  R: EuclideanDomain

  Z   ==> Integer
  V   ==> Vector R
  M   ==> Matrix R
  REC ==> Record(val:R, cl:Z, rw:Z)

  Exports ==> with
    rowEch       : M -> M
      ++ rowEch(m) computes a modular row-echelon form of m, finding
      ++ an appropriate modulus.
    rowEchelon   : (M, R) -> M
      ++ rowEchelon(m, d) computes a modular row-echelon form mod d of
      ++    [d     ]
      ++    [  d   ]
      ++    [    . ]
      ++    [     d]
      ++    [   M  ]
```

```
      ++ where \spad{M = m mod d}.
    rowEchLocal    : (M, R) -> M
      ++ rowEchLocal(m,p) computes a modular row-echelon form of m, finding
      ++ an appropriate modulus over a local ring where p is the only prime.
    rowEchelonLocal: (M, R, R) -> M
      ++ rowEchelonLocal(m, d, p) computes the row-echelon form of m
      ++ concatenated with d times the identity matrix
      ++ over a local ring where p is the only prime.
    normalizedDivide: (R, R) -> Record(quotient:R, remainder:R)
      ++ normalizedDivide(n,d) returns a normalized quotient and
      ++ remainder such that consistently unique representatives
      ++ for the residue class are chosen, e.g. positive remainders



  Implementation ==> add
    order   : (R, R) -> Z
    vconc   : (M, R) -> M
    non0    : (V, Z) -> Union(REC, "failed")
    nonzero?: V -> Boolean
    mkMat   : (M, List Z) -> M
    diagSubMatrix: M -> Union(Record(val:R, mat:M), "failed")
    determinantOfMinor: M -> R
    enumerateBinomial: (List Z, Z, Z) -> List Z

    nonzero? v == any?(#1 ^= 0, v)

-- returns [a, i, rown] if v = [0,...,0,a,0,...,0]
-- where a <> 0 and i is the index of a, "failed" otherwise.
    non0(v, rown) ==
      ans:REC
      allZero:Boolean := true
      for i in minIndex v .. maxIndex v repeat
        if qelt(v, i) ^= 0 then
          if allZero then
            allZero := false
            ans := [qelt(v, i), i, rown]
          else return "failed"
      allZero => "failed"
      ans

-- returns a matrix made from the non-zero rows of x whose row number
-- is not in l
    mkMat(x, l) ==
      empty?(ll := [parts row(x, i)
         for i in minRowIndex x .. maxRowIndex x |
```

```
            (not member?(i, l)) and nonzero? row(x, i)]$List(List R)) =>
               zero(1, ncols x)
        matrix ll

  -- returns [m, d] where m = x with the zero rows and the rows of
  -- the diagonal of d removed, if x has a diagonal submatrix of d's,
  -- "failed" otherwise.
      diagSubMatrix x ==
        l  := [u::REC for i in minRowIndex x .. maxRowIndex x |
                                      (u := non0(row(x, i), i)) case REC]
        for a in removeDuplicates([r.val for r in l]$List(R)) repeat
          {[r.cl for r in l | r.val = a]$List(Z)}$Set(Z) =
            {[z for z in minColIndex x .. maxColIndex x]$List(Z)}$Set(Z)
              => return [a, mkMat(x, [r.rw for r in l | a = r.val])]
        "failed"

  -- returns a non-zero determinant of a minor of x of rank equal to
  -- the number of columns of x, if there is one, 0 otherwise
      determinantOfMinor x ==
  -- do not compute a modulus for square matrices, since this is as expensive
  -- as the Hermite reduction itself
        (nr := nrows x) <= (nc := ncols x) => 0
        lc := [i for i in minColIndex x .. maxColIndex x]$List(Integer)
        lr := [i for i in minRowIndex x .. maxRowIndex x]$List(Integer)
        for i in 1..(n := binomial(nr, nc)) repeat
          (d := determinant x(enumerateBinomial(lr, nc, i), lc)) ^= 0 =>
            j := i + 1 + (random()$Z rem (n - i))
            return gcd(d, determinant x(enumerateBinomial(lr, nc, j), lc))
        0

  -- returns the i-th selection of m elements of l = (a1,...,an),
  --                 /n\
  -- where 1 <= i <= | |
  --                 \m/
      enumerateBinomial(l, m, i) ==
        m1 := minIndex l - 1
        zero?(m := m - 1) => [l(m1 + i)]
        for j in 1..(n := #l) repeat
          i <= (b := binomial(n - j, m)) =>
            return concat(l(m1 + j), enumerateBinomial(rest(l, j), m, i))
          i := i - b
        error "Should not happen"

      rowEch x ==
        (u := diagSubMatrix x) case "failed" =>
          zero?(d := determinantOfMinor x) => rowEchelon x
```

```
    rowEchelon(x, d)
  rowEchelon(u.mat, u.val)

vconc(y, m) ==
  vertConcat(diagonalMatrix new(ncols y, m)$V, map(#1 rem m, y))

order(m, p) ==
  zero? m => -1
  for i in 0.. repeat
    (mm := m exquo p) case "failed" => return i
    m := mm::R

if R has IntegerNumberSystem then
    normalizedDivide(n:R, d:R):Record(quotient:R, remainder:R) ==
        qr := divide(n, d)
        qr.remainder >= 0 => qr
        d > 0 =>
            qr.remainder := qr.remainder + d
            qr.quotient := qr.quotient - 1
            qr
        qr.remainder := qr.remainder - d
        qr.quotient := qr.quotient + 1
        qr
else
    normalizedDivide(n:R, d:R):Record(quotient:R, remainder:R) ==
        divide(n, d)

rowEchLocal(x,p) ==
  (u := diagSubMatrix x) case "failed" =>
    zero?(d := determinantOfMinor x) => rowEchelon x
    rowEchelonLocal(x, d, p)
  rowEchelonLocal(u.mat, u.val, p)

rowEchelonLocal(y, m, p) ==
    m := p**(order(m,p)::NonNegativeInteger)
    x     := vconc(y, m)
    nrows := maxRowIndex x
    ncols := maxColIndex x
    minr  := i := minRowIndex x
    for j in minColIndex x .. ncols repeat
      if i > nrows then leave x
      rown := minr - 1
      pivord : Integer
      npivord : Integer
      for k in i .. nrows repeat
        qelt(x,k,j) = 0 => "next k"
```

```
            npivord := order(qelt(x,k,j),p)
            (rown = minr - 1) or (npivord  <  pivord) =>
                    rown := k
                    pivord := npivord
          rown = minr - 1 => "enuf"
          x := swapRows_!(x, i, rown)
          (a, b, d) := extendedEuclidean(qelt(x,i,j), m)
          qsetelt_!(x,i,j,d)
          pivot := d
          for k in j+1 .. ncols repeat
            qsetelt_!(x,i,k, a * qelt(x,i,k) rem m)
          for k in i+1 .. nrows repeat
            zero? qelt(x,k,j) => "next k"
            q := (qelt(x,k,j) exquo pivot) :: R
            for k1 in j+1 .. ncols repeat
              v2 := (qelt(x,k,k1) - q * qelt(x,i,k1)) rem m
              qsetelt_!(x, k, k1, v2)
            qsetelt_!(x, k, j, 0)
          for k in minr .. i-1 repeat
            zero? qelt(x,k,j) => "enuf"
            qr    := normalizedDivide(qelt(x,k,j), pivot)
            qsetelt_!(x,k,j, qr.remainder)
            for k1 in j+1 .. ncols x repeat
              qsetelt_!(x,k,k1,
                      (qelt(x,k,k1) - qr.quotient * qelt(x,i,k1)) rem m)
          i := i+1
        x

      if R has Field then
        rowEchelon(y, m) == rowEchelon vconc(y, m)

      else

        rowEchelon(y, m) ==
          x     := vconc(y, m)
          nrows := maxRowIndex x
          ncols := maxColIndex x
          minr  := i := minRowIndex x
          for j in minColIndex x .. ncols repeat
            if i > nrows then leave
            rown := minr - 1
            for k in i .. nrows repeat
              if (qelt(x,k,j) ^= 0) and ((rown = minr - 1) or
                    sizeLess?(qelt(x,k,j), qelt(x,rown,j))) then rown := k
            rown = minr - 1 => "next j"
            x := swapRows_!(x, i, rown)
```

```
            for k in i+1 .. nrows repeat
              zero? qelt(x,k,j) => "next k"
              (a, b, d) := extendedEuclidean(qelt(x,i,j), qelt(x,k,j))
              (b1, a1) :=
                 ((qelt(x,i,j) exquo d)::R, (qelt(x,k,j) exquo d)::R)
              -- a*b1+a1*b = 1
              for k1 in j+1 .. ncols repeat
                v1 := (a  * qelt(x,i,k1) +  b * qelt(x,k,k1)) rem m
                v2 := (b1 * qelt(x,k,k1) - a1 * qelt(x,i,k1)) rem m
                qsetelt_!(x, i, k1, v1)
                qsetelt_!(x, k, k1, v2)
              qsetelt_!(x, i, j, d)
              qsetelt_!(x, k, j, 0)
            un := unitNormal qelt(x,i,j)
            qsetelt_!(x,i,j,un.canonical)
            if un.associate ^= 1 then for jj in (j+1)..ncols repeat
                qsetelt_!(x,i,jj,un.associate * qelt(x,i,jj))

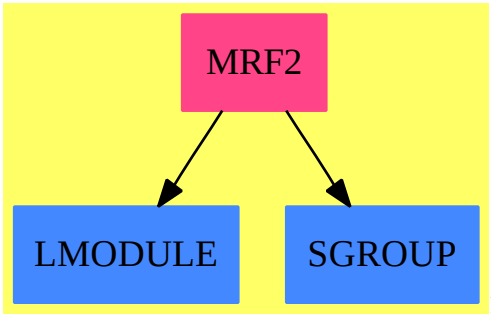
            xij := qelt(x,i,j)
            for k in minr .. i-1 repeat
              zero? qelt(x,k,j) => "next k"
              qr    := normalizedDivide(qelt(x,k,j), xij)
              qsetelt_!(x,k,j, qr.remainder)
              for k1 in j+1 .. ncols x repeat
                qsetelt_!(x,k,k1,
                       (qelt(x,k,k1) - qr.quotient * qelt(x,i,k1)) rem m)
            i := i+1
          x
```

⟨*MHROWRED.dotabb*⟩≡

```
"MHROWRED" [color="#FF4488",href="bookvol10.4.pdf#nameddest=MHROWRED"]
"IVECTOR" [color="#88FF44",href="bookvol10.3.pdf#nameddest=IVECTOR"]
"MHROWRED" -> "IVECTOR"
```

14.41 package MRF2 MonoidRingFunctions2

14.42 MonoidRingFunctions2



1.00

Exports:
map

⟨*package MRF2 MonoidRingFunctions2*⟩≡
```
)abbrev package MRF2 MonoidRingFunctions2
++ Author: Johannes Grabmeier
++ Date Created: 14 May 1991
++ Date Last Updated: 14 May 1991
++ Basic Operations: map
++ Related Constructors:
++ Also See:
++ AMS Classifications:
++ Keywords: monoid ring, group ring, change of coefficient domain
++ References:
++ Description:
++  MonoidRingFunctions2 implements functions between
++  two monoid rings defined with the same monoid over different rings.
MonoidRingFunctions2(R,S,M) : Exports == Implementation where
    R  : Ring
    S  : Ring
    M  : Monoid
    Exports ==> with
      map: (R -> S, MonoidRing(R,M)) -> MonoidRing(S,M)
        ++ map(f,u) maps f onto the coefficients f the element
        ++ u of the monoid ring to create an element of a monoid
        ++ ring with the same monoid b.
    Implementation ==> add
      map(fn, u) ==
        res : MonoidRing(S,M) := 0
        for te in terms u repeat
          res := res + monomial(fn(te.coef), te.monom)
```

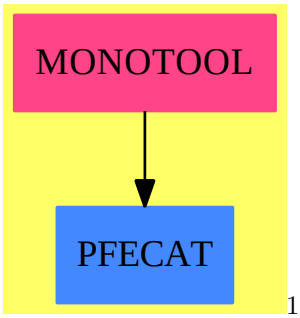
```
          res
```

⟨*MRF2.dotabb*⟩≡

```
"MRF2" [color="#FF4488",href="bookvol10.4.pdf#nameddest=MRF2"]
"LMODULE" [color="#4488FF",href="bookvol10.2.pdf#nameddest=LMODULE"]
"SGROUP" [color="#4488FF",href="bookvol10.2.pdf#nameddest=SGROUP"]
"MRF2" -> "LMODULE"
"MRF2" -> "SGROUP"
```

14.43 package MONOTOOL MonomialExtensionTools

14.44 MonomialExtensionTools



1.00

Exports:
decompose normalDenom split splitSquarefree

⟨*package MONOTOOL MonomialExtensionTools*⟩≡
```
)abbrev package MONOTOOL MonomialExtensionTools
++ Tools for handling monomial extensions
++ Author: Manuel Bronstein
++ Date Created: 18 August 1992
++ Date Last Updated: 3 June 1993
++ Description: Tools for handling monomial extensions.
MonomialExtensionTools(F, UP): Exports == Implementation where
  F : Field
  UP: UnivariatePolynomialCategory F

  RF ==> Fraction UP
  FR ==> Factored UP

  Exports ==> with
    split      : (UP, UP -> UP) -> Record(normal:UP, special:UP)
      ++ split(p, D) returns \spad{[n,s]} such that \spad{p = n s},
      ++ all the squarefree factors of n are normal w.r.t. D,
      ++ and s is special w.r.t. D.
      ++ D is the derivation to use.
    splitSquarefree: (UP, UP -> UP) -> Record(normal:FR, special:FR)
      ++ splitSquarefree(p, D) returns
      ++ \spad{[n_1 n_2\^2 ... n_m\^m, s_1 s_2\^2 ... s_q\^q]} such that
      ++ \spad{p = n_1 n_2\^2 ... n_m\^m s_1 s_2\^2 ... s_q\^q}, each
      ++ \spad{n_i} is normal w.r.t. D and each \spad{s_i} is special
      ++ w.r.t D.
      ++ D is the derivation to use.
```

```
      normalDenom: (RF, UP -> UP) -> UP
        ++ normalDenom(f, D) returns the product of all the normal factors
        ++ of \spad{denom(f)}.
        ++ D is the derivation to use.
      decompose  : (RF, UP -> UP) -> Record(poly:UP, normal:RF, special:RF)
        ++ decompose(f, D) returns \spad{[p,n,s]} such that \spad{f = p+n+s},
        ++ all the squarefree factors of \spad{denom(n)} are normal w.r.t. D,
        ++ \spad{denom(s)} is special w.r.t. D,
        ++ and n and s are proper fractions (no pole at infinity).
        ++ D is the derivation to use.

    Implementation ==> add
      normalDenom(f, derivation) == split(denom f, derivation).normal

      split(p, derivation) ==
        pbar := (gcd(p, derivation p) exquo gcd(p, differentiate p))::UP
        zero? degree pbar => [p, 1]
        rec := split((p exquo pbar)::UP, derivation)
        [rec.normal, pbar * rec.special]

      splitSquarefree(p, derivation) ==
        s:Factored(UP) := 1
        n := s
        q := squareFree p
        for rec in factors q repeat
          r := rec.factor
          g := gcd(r, derivation r)
          if not ground? g then s := s * sqfrFactor(g, rec.exponent)
          h := (r exquo g)::UP
          if not ground? h then n := n * sqfrFactor(h, rec.exponent)
        [n, unit(q) * s]

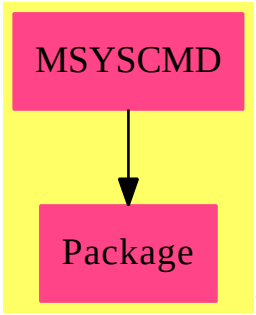
      decompose(f, derivation) ==
        qr := divide(numer f, denom f)
    -- rec.normal * rec.special = denom f
        rec := split(denom f, derivation)
    -- eeu.coef1 * rec.normal + eeu.coef2 * rec.special = qr.remainder
    -- and degree(eeu.coef1) < degree(rec.special)
    -- and degree(eeu.coef2) < degree(rec.normal)
    -- qr.remainder/denom(f) = eeu.coef1 / rec.special + eeu.coef2 / rec.normal
        eeu := extendedEuclidean(rec.normal, rec.special,
                                 qr.remainder)::Record(coef1:UP, coef2:UP)
        [qr.quotient, eeu.coef2 / rec.normal, eeu.coef1 / rec.special]
```

⟨*MONOTOOL.dotabb*⟩≡

```
"MONOTOOL" [color="#FF4488",href="bookvol10.4.pdf#nameddest=MONOTOOL"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"MONOTOOL" -> "PFECAT"
```

14.45 package MSYSCMD MoreSystemCommands

14.46 MoreSystemCommands



1.00

Exports:
systemCommand

⟨*package MSYSCMD MoreSystemCommands*⟩≡

```
)abbrev package MSYSCMD MoreSystemCommands
++ Author:
++ Date Created:
++ Change History:
++ Basic Operations: systemCommand
++ Related Constructors:
++ Also See:
++ AMS Classification:
++ Keywords: command
++ Description:
++   \spadtype{MoreSystemCommands} implements an interface with the
++   system command facility. These are the commands that are issued
++   from source files or the system interpreter and they start with
++   a close parenthesis, e.g., \spadsyscom{what} commands.

MoreSystemCommands: public == private where

  public == with

    systemCommand: String -> Void
      ++ systemCommand(cmd) takes the string \spadvar{cmd} and passes
      ++ it to the runtime environment for execution as a system
      ++ command. Although various things may be printed, no usable
      ++ value is returned.

  private == add
```

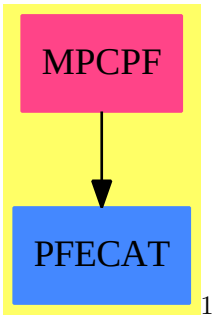
```
systemCommand cmd == doSystemCommand(cmd)$Lisp
```

⟨*MSYSCMD.dotabb*⟩≡

```
"MSYSCMD" [color="#FF4488",href="bookvol10.4.pdf#nameddest=MSYSCMD"]
"Package" [color="#FF4488"]
"MSYSCMD" -> "Package"
```

14.47 package MPCPF MPolyCatPolyFactorizer

14.48 MPolyCatPolyFactorizer



1.00

Exports:
factor

⟨*package MPCPF MPolyCatPolyFactorizer*⟩≡
```
)abbrev package MPCPF MPolyCatPolyFactorizer
++ Author: P. Gianni
++ Date Created:
++ Date Last Updated: March 1995
++ Basic Functions:
++ Related Constructors:
++ Also See:
++ AMS Classifications:
++ Keywords:
++ References:
++ Description:
++    This package exports a factor operation for multivariate polynomials
++ with coefficients which are polynomials over
++ some ring R over which we can factor. It is used internally by packages
++ such as the solve package which need to work with polynomials in a specific
++ set of variables with coefficients which are polynomials in all the other
++ variables.

MPolyCatPolyFactorizer(E,OV,R,PPR) : C == T
 where
  R     :   EuclideanDomain
  E     :   OrderedAbelianMonoidSup
    -- following type is required by PushVariables
  OV    :   OrderedSet  with
                convert : % -> Symbol
                  ++ convert(x) converts x to a symbol
```

```
                variable: Symbol -> Union(%, "failed")
                  ++ variable(s) makes an element from symbol s or fails.
  PR    ==> Polynomial R
  PPR   :   PolynomialCategory(PR,E,OV)
  NNI   ==> NonNegativeInteger
  ISY   ==> IndexedExponents Symbol
  SE    ==> Symbol
  UP    ==> SparseUnivariatePolynomial PR
  UPPR  ==> SparseUnivariatePolynomial PPR

  C  == with
     factor      :        PPR             ->   Factored PPR
       ++ factor(p) factors a polynomial with polynomial
       ++ coefficients.

           ---  Local Functions  ----
  T  == add

     import PushVariables(R,E,OV,PPR)

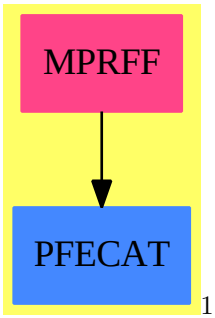
        ----  factorization of p ----
     factor(p:PPR) : Factored PPR ==
       ground? p => nilFactor(p,1)
       c := content p
       p := (p exquo c)::PPR
       vars:List OV :=variables p
       g:PR:=retract pushdown(p, vars)
       flist := factor(g)$GeneralizedMultivariateFactorize(Symbol,ISY,R,R,PR)
       ffact : List(Record(irr:PPR,pow:Integer))
       ffact:=[[pushup(u.factor::PPR,vars),u.exponent] for u in factors flist]
       fcont:=(unit flist)::PPR
       nilFactor(c*fcont,1)*(_*/[primeFactor(ff.irr,ff.pow) for ff in ffact])
```

⟨*MPCPF.dotabb*⟩≡

```
"MPCPF" [color="#FF4488",href="bookvol10.4.pdf#nameddest=MPCPF"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"MPCPF" -> "PFECAT"
```

14.49 package MPRFF MPolyCatRational-FunctionFactorizer

14.50 MPolyCatRationalFunctionFactorizer



1.00

Exports:

factor	pushdown	pushup	pushdterm	pushucoef
pushuconst	totalfract			

⟨*package MPRFF MPolyCatRationalFunctionFactorizer*⟩≡

```
)abbrev package MPRFF MPolyCatRationalFunctionFactorizer
++ Author: P. Gianni
++ Date Created:
++ Date Last Updated:
++ Basic Functions:
++ Related Constructors:
++ Also See:
++ AMS Classifications:
++ Keywords:
++ References:
++ Description:
++    This package exports a factor operation for multivariate polynomials
++ with coefficients which are rational functions over
++ some ring R over which we can factor. It is used internally by packages
++ such as primary decomposition which need to work with polynomials
++ with rational function coefficients, i.e. themselves fractions of
++ polynomials.

MPolyCatRationalFunctionFactorizer(E,OV,R,PRF) : C == T
 where
  R     :   IntegralDomain
  F     ==> Fraction Polynomial R
  RN    ==> Fraction Integer
  E     :   OrderedAbelianMonoidSup
  OV    :   OrderedSet  with
```

```
              convert : % -> Symbol
                ++ convert(x) converts x to a symbol
PRF   :   PolynomialCategory(F,E,OV)
NNI   ==> NonNegativeInteger
P     ==> Polynomial R
ISE   ==> IndexedExponents  SE
SE    ==> Symbol
UP    ==> SparseUnivariatePolynomial P
UF    ==> SparseUnivariatePolynomial F
UPRF  ==> SparseUnivariatePolynomial PRF
QuoForm   ==> Record(sup:P,inf:P)

C  == with
   totalfract  :        PRF             ->   QuoForm
     ++ totalfract(prf) takes a polynomial whose coefficients are
     ++ themselves fractions of polynomials and returns a record
     ++ containing the numerator and denominator resulting from
     ++ putting prf over a common denominator.
   pushdown    :      (PRF,OV)          ->   PRF
     ++ pushdown(prf,var) pushes all top level occurences of the
     ++ variable var into the coefficient domain for the polynomial prf.
   pushdterm   :      (UPRF,OV)         ->   PRF
     ++ pushdterm(monom,var) pushes all top level occurences of the
     ++ variable var into the coefficient domain for the monomial monom.
   pushup      :      (PRF,OV)          ->   PRF
     ++ pushup(prf,var) raises all occurences of the
     ++ variable var in the coefficients of the polynomial prf
     ++ back to the polynomial level.
   pushucoef   :       (UP,OV)          ->   PRF
     ++ pushucoef(upoly,var) converts the anonymous univariate
     ++ polynomial upoly to a polynomial in var over rational functions.
   pushuconst  :        (F,OV)          ->   PRF
     ++ pushuconst(r,var) takes a rational function and raises
     ++ all occurances of the variable var to the polynomial level.
   factor      :        PRF             ->   Factored PRF
     ++ factor(prf) factors a polynomial with rational function
     ++ coefficients.

           ---  Local Functions  ----
T  == add

      ----  factorization of p ----
   factor(p:PRF) : Factored PRF ==
     truelist:List OV :=variables p
     tp:=totalfract(p)
     nump:P:= tp.sup
```

```
        denp:F:=inv(tp.inf ::F)
        ffact : List(Record(irr:PRF,pow:Integer))
        flist:Factored P
        if R is Fraction Integer then
          flist:=
            ((factor nump)$MRationalFactorize(ISE,SE,Integer,P))
                          pretend (Factored P)
        else
          if R has FiniteFieldCategory  then
             flist:= ((factor nump)$MultFiniteFactorize(SE,ISE,R,P))
                     pretend (Factored P)

          else
             if R has Field then error "not done yet"

             else
               if R has CharacteristicZero then
                 flist:= ((factor nump)$MultivariateFactorize(SE,ISE,R,P))
                                              pretend (Factored P)
               else error "can't happen"
        ffact:=[[u.factor::F::PRF,u.exponent] for u in factors flist]
        fcont:=(unit flist)::F::PRF
        for x in truelist repeat
          fcont:=pushup(fcont,x)
          ffact:=[[pushup(ff.irr,x),ff.pow] for ff in ffact]
        (denp*fcont)*(_*/[primeFactor(ff.irr,ff.pow) for ff in ffact])


    -- the following functions are used to "push" x in the coefficient ring -

           ----  push x in the coefficient domain for a polynomial ----
        pushdown(g:PRF,x:OV) : PRF ==
          ground? g => g
          rf:PRF:=0$PRF
          ug:=univariate(g,x)
          while ug^=0 repeat
            rf:=rf+pushdterm(ug,x)
            ug := reductum ug
          rf

         ----  push x in the coefficient domain for a term ----
        pushdterm(t:UPRF,x:OV):PRF ==
          n:=degree(t)
          cf:=monomial(1,convert x,n)$P :: F
          cf * leadingCoefficient t
```

```
          ----  push back the variable  ----
pushup(f:PRF,x:OV) :PRF ==
  ground? f => pushuconst(retract f,x)
  v:=mainVariable(f)::OV
  g:=univariate(f,v)
  multivariate(map(pushup(#1,x),g),v)

 ----  push x back from the coefficient domain ----
pushuconst(r:F,x:OV):PRF ==
  xs:SE:=convert x
  degree(denom r,xs)>0 => error "bad polynomial form"
  inv((denom r)::F)*pushucoef(univariate(numer r,xs),x)


pushucoef(c:UP,x:OV):PRF ==
  c = 0 => 0
  monomial((leadingCoefficient c)::F::PRF,x,degree c) +
            pushucoef(reductum c,x)


       ----  write p with a common denominator  ----

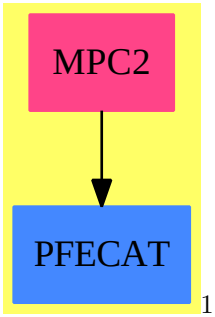
totalfract(p:PRF) : QuoForm ==
  p=0 => [0$P,1$P]$QuoForm
  for x in variables p repeat p:=pushdown(p,x)
  g:F:=retract p
  [numer g,denom g]$QuoForm
```

⟨*MPRFF.dotabb*⟩≡

```
"MPRFF" [color="#FF4488",href="bookvol10.4.pdf#nameddest=MPRFF"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"MPRFF" -> "PFECAT"
```

14.51 package MPC2 MPolyCatFunctions2

14.52 MPolyCatFunctions2



1.00

Exports:
map reshape

⟨*package MPC2 MPolyCatFunctions2*⟩≡
```
)abbrev package MPC2 MPolyCatFunctions2
++ Utilities for MPolyCat
++ Author: Manuel Bronstein
++ Date Created: 1987
++ Date Last Updated: 28 March 1990  (PG)
MPolyCatFunctions2(VarSet,E1,E2,R,S,PR,PS) : public == private where

  VarSet : OrderedSet
  E1     : OrderedAbelianMonoidSup
  E2     : OrderedAbelianMonoidSup
  R      : Ring
  S      : Ring
  PR     : PolynomialCategory(R,E1,VarSet)
  PS     : PolynomialCategory(S,E2,VarSet)
  SUPR   ==> SparseUnivariatePolynomial PR
  SUPS   ==> SparseUnivariatePolynomial PS

  public == with
    map:     (R -> S,PR) -> PS
        ++ map(f,p) \undocumented
    reshape: (List S, PR) -> PS
        ++ reshape(l,p) \undocumented

  private == add

    supMap:  (R -> S, SUPR) -> SUPS
```

```
supMap(fn : R -> S, supr : SUPR): SUPS ==
  supr = 0 => monomial(fn(0$R) :: PS,0)$SUPS
  c : PS := map(fn,leadingCoefficient supr)$%
  monomial(c,degree supr)$SUPS + supMap(fn, reductum supr)

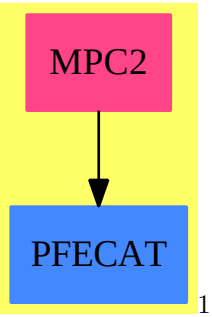
map(fn : R -> S, pr : PR): PS ==
  varu : Union(VarSet,"failed") := mainVariable pr
  varu case "failed" =>  -- have a constant
    fn(retract pr) :: PS
  var : VarSet := varu :: VarSet
  supr : SUPR := univariate(pr,var)$PR
  multivariate(supMap(fn,supr),var)$PS
```

⟨*MPC2.dotabb*⟩≡

```
"MPC2" [color="#FF4488",href="bookvol10.4.pdf#nameddest=MPC2"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"MPC2" -> "PFECAT"
```

14.53 package MPC3 MPolyCatFunctions3

14.54 MPolyCatFunctions3



1.00

Exports:
map

⟨*package MPC3 MPolyCatFunctions3*⟩≡
```
)abbrev package MPC3 MPolyCatFunctions3
++ Description:
++ This package \undocumented
MPolyCatFunctions3(Vars1,Vars2,E1,E2,R,PR1,PR2): C == T where
  E1   : OrderedAbelianMonoidSup
  E2   : OrderedAbelianMonoidSup
  Vars1: OrderedSet
  Vars2: OrderedSet
  R    : Ring
  PR1  : PolynomialCategory(R,E1,Vars1)
  PR2  : PolynomialCategory(R,E2,Vars2)

  C ==> with
    map: (Vars1 -> Vars2, PR1) -> PR2
        ++ map(f,x) \undocumented

  T ==> add

    map(f:Vars1 -> Vars2, p:PR1):PR2 ==
      (x1 := mainVariable p) case "failed" =>
        c:R:=(retract p)
        c::PR2
      up := univariate(p, x1::Vars1)
      x2 := f(x1::Vars1)
      ans:PR2 := 0
      while up ^= 0 repeat
        ans := ans + monomial(map(f,leadingCoefficient up),x2,degree up)
```

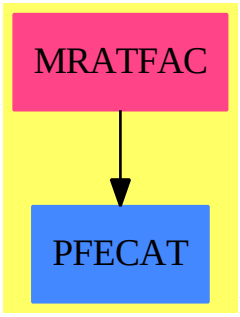
```
        up  := reductum up
      ans
```

⟨*MPC3.dotabb*⟩≡

```
"MPC3" [color="#FF4488",href="bookvol10.4.pdf#nameddest=MPC3"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"MPC3" -> "PFECAT"
```

14.55 package MRATFAC MRationalFactorize

14.56 MRationalFactorize



1.00

Exports:
factor

⟨*package MRATFAC MRationalFactorize*⟩≡

```
)abbrev package MRATFAC MRationalFactorize
++ Author: P. Gianni
++ Date Created:
++ Date Last Updated:
++ Basic Functions:
++ Related Constructors: MultivariateFactorize
++ Also See:
++ AMS Classifications:
++ Keywords:
++ References:
++ Description:  MRationalFactorize contains the factor function for multivariate
++ polynomials over the quotient field of a ring R such that the package
++ MultivariateFactorize can factor multivariate polynomials over R.


MRationalFactorize(E,OV,R,P) : C == T
 where
  E   :   OrderedAbelianMonoidSup
  OV  :   OrderedSet
  R   :   Join(EuclideanDomain, CharacteristicZero)  -- with factor over R[x]
  FR  ==> Fraction R
  P   :    PolynomialCategory(FR,E,OV)
  MPR ==> SparseMultivariatePolynomial(R,OV)
  SUP ==> SparseUnivariatePolynomial

  C  == with
     factor      : P   ->  Factored P
```

```
      ++ factor(p) factors the multivariate polynomial p with coefficients
      ++ which are fractions of elements of R.

  T  == add
     IE     ==> IndexedExponents OV
     PCLFRR ==> PolynomialCategoryLifting(E,OV,FR,P,MPR)
     PCLRFR ==> PolynomialCategoryLifting(IE,OV,R,MPR,P)
     MFACT  ==> MultivariateFactorize(OV,IE,R,MPR)
     UPCF2  ==> UnivariatePolynomialCategoryFunctions2

     numer1(c:FR): MPR   == (numer c) :: MPR
     numer2(pol:P) : MPR == map(coerce,numer1,pol)$PCLFRR
     coerce1(d:R) : P == (d::FR)::P
     coerce2(pp:MPR) :P == map(coerce,coerce1,pp)$PCLRFR

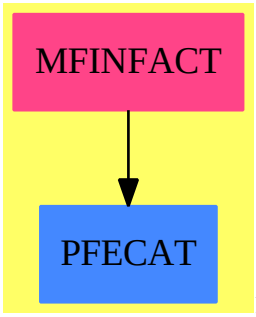
     factor(p:P) : Factored P ==
       pden:R:=lcm([denom c for c in coefficients p])
       pol :P:= (pden::FR)*p
       ipol:MPR:= map(coerce,numer1,pol)$PCLFRR
       ffact:=(factor ipol)$MFACT
       (1/pden)*map(coerce,coerce1,(unit ffact))$PCLRFR *
           _*/[primeFactor(map(coerce,coerce1,u.factor)$PCLRFR,
                           u.exponent) for u in factors ffact]
```

⟨*MRATFAC.dotabb*⟩≡

```
"MRATFAC" [color="#FF4488",href="bookvol10.4.pdf#nameddest=MRATFAC"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"MRATFAC" -> "PFECAT"
```

14.57 package MFINFACT MultFiniteFactorize

14.58 MultFiniteFactorize



1.00

Exports:
factor

⟨*package MFINFACT MultFiniteFactorize*⟩≡

```
)abbrev package MFINFACT MultFiniteFactorize
++ Author: P. Gianni
++ Date Created: Summer 1990
++ Date Last Updated: 19 March 1992
++ Basic Functions:
++ Related Constructors: PrimeField, FiniteField, Polynomial
++ Also See:
++ AMS Classifications:
++ Keywords:
++ References:
++ Description: Package for factorization of multivariate polynomials
++ over finite fields.


MultFiniteFactorize(OV,E,F,PG) : C == T
 where
  F          :   FiniteFieldCategory
  OV         :   OrderedSet
  E          :   OrderedAbelianMonoidSup
  PG         :   PolynomialCategory(F,E,OV)
  SUP        ==> SparseUnivariatePolynomial
  R          ==> SUP F
  P          ==> SparseMultivariatePolynomial(R,OV)
  Z          ==> Integer
  FFPOLY     ==> FiniteFieldPolynomialPackage(F)
  MParFact   ==> Record(irr:P,pow:Z)
  MFinalFact ==> Record(contp:R,factors:List MParFact)
```

```
SUParFact    ==> Record(irr:SUP P,pow:Z)
SUPFinalFact ==> Record(contp:R,factors:List SUParFact)

               --  contp   =  content,
               --  factors =  List of irreducible factors with exponent

C == with

  factor      : PG     ->  Factored PG
    ++ factor(p) produces the complete factorization of the multivariate
    ++ polynomial p over a finite field.
  factor      : SUP PG     ->  Factored SUP PG
    ++ factor(p) produces the complete factorization of the multivariate
    ++ polynomial p over a finite field. p is represented as a univariate
    ++ polynomial with multivariate coefficients over a finite field.

T == add

  import LeadingCoefDetermination(OV,IndexedExponents OV,R,P)
  import MultivariateLifting(IndexedExponents OV,OV,R,P)
  import FactoringUtilities(IndexedExponents OV,OV,R,P)
  import FactoringUtilities(E,OV,F,PG)
  import GenExEuclid(R,SUP R)

  NNI       ==> NonNegativeInteger
  L         ==> List
  UPCF2     ==> UnivariatePolynomialCategoryFunctions2
  LeadFact  ==> Record(polfac:L P,correct:R,corrfact:L SUP R)
  ContPrim  ==> Record(cont:P,prim:P)
  ParFact   ==> Record(irr:SUP R,pow:Z)
  FinalFact ==> Record(contp:R,factors:L ParFact)
  NewOrd    ==> Record(npol:SUP P,nvar:L OV,newdeg:L NNI)
  Valuf     ==> Record(inval:L L R,unvfact:L SUP R,lu:R,complead:L R)

                ----  Local Functions  ----
  ran       :                  Z             -> R
  mFactor   :               (P,Z)            -> MFinalFact
  supFactor :             (SUP P,Z)          -> SUPFinalFact
  mfconst   :        (SUP P,Z,L OV,L NNI)    -> L SUP P
  mfpol     :        (SUP P,Z,L OV,L NNI)    -> L SUP P
  varChoose :           (P,L OV,L NNI)       -> NewOrd
  simplify  :         (P,Z,L OV,L NNI)       -> MFinalFact
  intChoose :        (SUP P,L OV,R,L P,L L R) -> Valuf
  pretest   :         (P,NNI,L OV,L R)       -> FinalFact
  checkzero :            (SUP P,SUP R)       -> Boolean
  pushdcoef :                 PG             -> P
```

```
pushdown  :                (PG,OV)           -> P
pushupconst :               (R,OV)           -> PG
pushup    :                 (P,OV)           -> PG
norm      :               L SUP R            -> Integer
constantCase :        (P,L MParFact)         -> MFinalFact
pM          :             L SUP R            -> R
intfact     :     (SUP P,L OV,L NNI,MFinalFact,L L R) -> L SUP P

basicVar:OV:=NIL$Lisp pretend OV  -- variable for the basic step


convertPUP(lfg:MFinalFact): SUPFinalFact ==
  [lfg.contp,[[lff.irr ::SUP P,lff.pow]$SUParFact
               for lff in lfg.factors]]$SUPFinalFact

supFactor(um:SUP P,dx:Z) : SUPFinalFact ==
    degree(um)=0 => convertPUP(mFactor(ground um,dx))
    lvar:L OV:= "setUnion"/[variables cf for cf in coefficients um]
    lcont:SUP P
    lf:L SUP P

    flead : SUPFinalFact:=[0,empty()]$SUPFinalFact
    factorlist:L SUParFact :=empty()

    mdeg :=minimumDegree um     ---- is the Mindeg > 0? ----
    if mdeg>0 then
      f1:SUP P:=monomial(1,mdeg)
      um:=(um exquo f1)::SUP P
      factorlist:=cons([monomial(1,1),mdeg],factorlist)
      if degree um=0 then return
        lfg:=convertPUP mFactor(ground um, dx)
        [lfg.contp,append(factorlist,lfg.factors)]


    om:=map(pushup(#1,basicVar),um)$UPCF2(P,SUP P,PG,SUP PG)
    sqfacs:=squareFree(om)
    lcont:=map(pushdown(#1,basicVar),unit sqfacs)$UPCF2(PG,SUP PG,P,SUP P)

                               ----   Factorize the content  ----
    if ground? lcont then
      flead:=convertPUP constantCase(ground lcont,empty())
    else
      flead:=supFactor(lcont,dx)

    factorlist:=flead.factors
```

```
                              ----  Make the polynomial square-free  ----
    sqqfact:=[[map(pushdown(#1,basicVar),ff.factor),ff.exponent]
                 for ff in factors sqfacs]

                   ---  Factorize the primitive square-free terms ---
    for fact in sqqfact repeat
      ffactor:SUP P:=fact.irr
      ffexp:=fact.pow
      ffcont:=content ffactor
      coefs := coefficients ffactor
      ldeg:= ["max"/[degree(fc,xx) for fc in coefs] for xx in lvar]
      if ground?(leadingCoefficient ffactor) then
         lf:= mfconst(ffactor,dx,lvar,ldeg)
      else lf:=mfpol(ffactor,dx,lvar,ldeg)
      auxfl:=[[lfp,ffexp]$SUParFact  for lfp in lf]
      factorlist:=append(factorlist,auxfl)
    lcfacs := */[leadingCoefficient leadingCoefficient(f.irr)**((f.pow)::NNI)
                          for f in factorlist]
    [(leadingCoefficient leadingCoefficient(um) exquo lcfacs)::R,
                  factorlist]$SUPFinalFact

factor(um:SUP PG):Factored SUP PG ==
    lv:List OV:=variables um
    ld:=degree(um,lv)
    dx:="min"/ld
    basicVar:=lv.position(dx,ld)
    cm:=map(pushdown(#1,basicVar),um)$UPCF2(PG,SUP PG,P,SUP P)
    flist := supFactor(cm,dx)
    pushupconst(flist.contp,basicVar)::SUP(PG) *
      (*/[primeFactor(map(pushup(#1,basicVar),u.irr)$UPCF2(P,SUP P,PG,SUP PG)
             u.pow) for u in flist.factors])



mFactor(m:P,dx:Z) : MFinalFact ==
  ground?(m) => constantCase(m,empty())
  lvar:L OV:= variables m
  lcont:P
  lf:L SUP P
  flead : MFinalFact:=[1,empty()]$MFinalFact
  factorlist:L MParFact :=empty()
                              ---- is the Mindeg > 0? ----
  lmdeg :=minimumDegree(m,lvar)
  or/[n>0 for n in lmdeg] => simplify(m,dx,lvar,lmdeg)
                             ----  Make the polynomial square-free  ----
  om:=pushup(m,basicVar)
```

```
        sqfacs:=squareFree(om)
        lcont := pushdown(unit sqfacs,basicVar)

                                  ----  Factorize the content  ----
        if ground? lcont then
          flead:=constantCase(lcont,empty())
        else
          flead:=mFactor(lcont,dx)
        factorlist:=flead.factors
        sqqfact:List Record(factor:P,exponent:Integer)
        sqqfact:=[[pushdown(ff.factor,basicVar),ff.exponent]
                                              for ff in factors sqfacs]
                        ---  Factorize the primitive square-free terms ---
        for fact in sqqfact repeat
          ffactor:P:=fact.factor
          ffexp := fact.exponent
          ground? ffactor =>
            for lterm in constantCase(ffactor,empty()).factors repeat
              factorlist:=cons([lterm.irr,lterm.pow * ffexp], factorlist)
          lvar := variables ffactor
          x:OV:=lvar.1
          ldeg:=degree(ffactor,lvar)
               ---  Is the polynomial linear in one of the variables ? ---
          member?(1,ldeg) =>
            x:OV:=lvar.position(1,ldeg)
            lcont:= gcd coefficients(univariate(ffactor,x))
            ffactor:=(ffactor exquo lcont)::P
            factorlist:=cons([ffactor,ffexp]$MParFact,factorlist)
            for lcterm in mFactor(lcont,dx).factors repeat
             factorlist:=cons([lcterm.irr,lcterm.pow * ffexp], factorlist)

          varch:=varChoose(ffactor,lvar,ldeg)
          um:=varch.npol


          ldeg:=ldeg.rest
          lvar:=lvar.rest
          if varch.nvar.1 ^= x then
            lvar:= varch.nvar
            x := lvar.1
            lvar:=lvar.rest
            pc:= gcd coefficients um
            if pc^=1 then
              um:=(um exquo pc)::SUP P
              ffactor:=multivariate(um,x)
              for lcterm in mFactor(pc,dx).factors repeat
```

```
            factorlist:=cons([lcterm.irr,lcterm.pow*ffexp],factorlist)
          ldeg:= degree(ffactor,lvar)

        -- should be unitNormal if unified, but for now it is easier
        lcum:F:= leadingCoefficient leadingCoefficient
                leadingCoefficient um
        if lcum ^=1  then
          um:=((inv lcum)::R::P) * um
          flead.contp := (lcum::R) *flead.contp

        if ground?(leadingCoefficient um)
        then lf:= mfconst(um,dx,lvar,ldeg)
        else lf:=mfpol(um,dx,lvar,ldeg)
        auxfl:=[[multivariate(lfp,x),ffexp]$MParFact  for lfp in lf]
        factorlist:=append(factorlist,auxfl)
      flead.factors:= factorlist
      flead


    pM(lum:L SUP R) : R ==
      x := monomial(1,1)$R
      for i in 1..size()$F repeat
         p := x + (index(i::PositiveInteger)$F) ::R
         testModulus(p,lum) => return p
      for e in 2.. repeat
          p :=  (createIrreduciblePoly(e::PositiveInteger))$FFPOLY
          testModulus(p,lum) => return p
          while not((q := nextIrreduciblePoly(p)$FFPOLY) case "failed") repeat
             p := q::SUP F
             if testModulus(p, lum)$GenExEuclid(R, SUP R) then return p

      ----  push x in the coefficient domain for a term ----
    pushdcoef(t:PG):P ==
       map(coerce(#1)$R,t)$MPolyCatFunctions2(OV,E,
                                           IndexedExponents OV,F,R,PG,P)


             ----  internal function, for testing bad cases  ----
    intfact(um:SUP P,lvar: L OV,ldeg:L NNI,
            tleadpol:MFinalFact,ltry:L L R):  L SUP P ==
      polcase:Boolean:=(not empty? tleadpol.factors )
      vfchoo:Valuf:=
        polcase =>
          leadpol:L P:=[ff.irr for ff in tleadpol.factors]
          intChoose(um,lvar,tleadpol.contp,leadpol,ltry)
        intChoose(um,lvar,1,empty(),empty())
```

```
        unifact:List SUP R := vfchoo.unvfact
        nfact:NNI := #unifact
        nfact=1 => [um]
        ltry:L L R:= vfchoo.inval
        lval:L R:=first ltry
        dd:= vfchoo.lu
        lpol:List P:=empty()
        leadval:List R:=empty()
        if polcase then
          leadval := vfchoo.complead
          distf := distFact(vfchoo.lu,unifact,tleadpol,leadval,lvar,lval)
          distf case "failed" =>
               return intfact(um,lvar,ldeg,tleadpol,ltry)
          dist := distf :: LeadFact
            -- check the factorization of leading coefficient
          lpol:= dist.polfac
          dd := dist.correct
          unifact:=dist.corrfact
        if dd^=1 then
          unifact := [dd*unifact.i for i in 1..nfact]
          um := ((dd**(nfact-1)::NNI)::P)*um
        (ffin:= lifting(um,lvar,unifact,lval,lpol,ldeg,pM(unifact)))
            case "failed" => intfact(um,lvar,ldeg,tleadpol,ltry)
        factfin: L SUP P:=ffin :: L SUP P
        if dd^=1 then
          factfin:=[primitivePart ff  for ff in  factfin]
        factfin

  -- the following functions are used to "push" x in the coefficient ring -
                  ----  push back the variable  ----
      pushup(f:P,x:OV) :PG ==
         ground? f => pushupconst((retract f)@R,x)
         rr:PG:=0
         while f^=0 repeat
           lf:=leadingMonomial f
           cf:=pushupconst(leadingCoefficient f,x)
           lvf:=variables lf
           rr:=rr+monomial(cf,lvf, degree(lf,lvf))$PG
           f:=reductum f
         rr

          ----  push x in the coefficient domain for a polynomial ----
      pushdown(g:PG,x:OV) : P ==
         ground? g => ((retract g)@F)::R::P
         rf:P:=0$P
         ug:=univariate(g,x)
```

```
    while ug^=0 repeat
      cf:=monomial(1,degree ug)$R
      rf:=rf+cf*pushdcoef(leadingCoefficient ug)
      ug := reductum ug
    rf

  ----  push x back from the coefficient domain ----
pushupconst(r:R,x:OV):PG ==
    ground? r => (retract r)@F ::PG
    rr:PG:=0
    while r^=0 repeat
      rr:=rr+monomial((leadingCoefficient r)::PG,x,degree r)$PG
      r:=reductum r
    rr

-- This function has to be added to Eucliden domain
ran(k1:Z) : R ==
  --if R case Integer then random()$R rem (2*k1)-k1
  --else
  +/[monomial(random()$F,i)$R for i in 0..k1]

checkzero(u:SUP P,um:SUP R) : Boolean ==
  u=0 => um =0
  um = 0 => false
  degree u = degree um => checkzero(reductum u, reductum um)
  false

          ---  Choose the variable of least degree  ---
varChoose(m:P,lvar:L OV,ldeg:L NNI) : NewOrd ==
  k:="min"/[d for d in ldeg]
  k=degree(m,first lvar) =>
                          [univariate(m,first lvar),lvar,ldeg]$NewOrd
  i:=position(k,ldeg)
  x:OV:=lvar.i
  ldeg:=cons(k,delete(ldeg,i))
  lvar:=cons(x,delete(lvar,i))
  [univariate(m,x),lvar,ldeg]$NewOrd


norm(lum: L SUP R): Integer == "max"/[degree lup for lup in lum]

      ---  Choose the values to reduce to the univariate case  ---
intChoose(um:SUP P,lvar:L OV,clc:R,plist:L P,ltry:L L R) : Valuf ==
  -- declarations
  degum:NNI := degree um
  nvar1:=#lvar
```

```
      range:NNI:=0
      unifact:L SUP R
      ctf1 : R := 1
      testp:Boolean :=             -- polynomial leading coefficient
        plist = empty() => false
        true
      leadcomp,leadcomp1 : L R
      leadcomp:=leadcomp1:=empty()
      nfatt:NNI := degum+1
      lffc:R:=1
      lffc1:=lffc
      newunifact : L SUP R:=empty()
      leadtest:=true --- the lc test with polCase has to be performed
      int:L R:=empty()

   --  New sets of values are chosen until we find twice the
   --  same number of "univariate" factors:the set smaller in modulo is
   --  is chosen.
      while true repeat
       lval := [ ran(range) for i in 1..nvar1]
       member?(lval,ltry) => range:=1+range
       ltry := cons(lval,ltry)
       leadcomp1:=[retract eval(pol,lvar,lval) for pol in plist]
       testp and or/[unit? epl for epl in leadcomp1] => range:=range+1
       newm:SUP R:=completeEval(um,lvar,lval)
       degum ^= degree newm or minimumDegree newm ^=0 => range:=range+1
       lffc1:=content newm
       newm:=(newm exquo lffc1)::SUP R
       testp and leadtest and ^ polCase(lffc1*clc,#plist,leadcomp1)
                           => range:=range+1
       Dnewm := differentiate newm
       D2newm := map(differentiate, newm)
       degree(gcd [newm,Dnewm,D2newm])^=0 => range:=range+1
      -- if R has Integer then luniv:=henselFact(newm,false)$
      -- else
       lcnm:F:=1
        -- should be unitNormal if unified, but for now it is easier
       if (lcnm:=leadingCoefficient leadingCoefficient newm)^=1 then
         newm:=((inv lcnm)::R)*newm
       dx:="max"/[degree uc  for uc in coefficients newm]
       luniv:=generalTwoFactor(newm)$TwoFactorize(F)
       lunivf:= factors luniv
       nf:= #lunivf

       nf=0 or nf>nfatt => "next values"      ---  pretest failed ---
```

```
                    --- the univariate polynomial is irreducible ---
    if nf=1 then leave (unifact:=[newm])

    lffc1:=lcnm * retract(unit luniv)@R * lffc1

--  the new integer give the same number of factors
    nfatt = nf =>
    -- if this is the first univariate factorization with polCase=true
    -- or if the last factorization has smaller norm and satisfies
    -- polCase
      if leadtest or
        ((norm unifact > norm [ff.factor for ff in lunivf]) and
          (^testp or polCase(lffc1*clc,#plist,leadcomp1))) then
             unifact:=[uf.factor for uf in lunivf]
             int:=lval
             lffc:=lffc1
             if testp then leadcomp:=leadcomp1
      leave "foundit"

--  the first univariate factorization, inizialize
    nfatt > degum =>
      unifact:=[uf.factor for uf in lunivf]
      lffc:=lffc1
      if testp then leadcomp:=leadcomp1
      int:=lval
      leadtest := false
      nfatt := nf

    nfatt>nf =>  -- for the previous values there were more factors
      if testp then leadtest:=^polCase(lffc*clc,#plist,leadcomp)
      else leadtest:= false
      -- if polCase=true we can consider the univariate decomposition
      if ^leadtest then
        unifact:=[uf.factor for uf in lunivf]
        lffc:=lffc1
        if testp then leadcomp:=leadcomp1
        int:=lval
      nfatt := nf
   [cons(int,ltry),unifact,lffc,leadcomp]$Valuf


 constantCase(m:P,factorlist:List MParFact) : MFinalFact ==
 --if R case Integer then [const m,factorlist]$MFinalFact
 --else
   lunm:=distdfact((retract m)@R,false)$DistinctDegreeFactorize(F,R)
   [(lunm.cont)::R, append(factorlist,
```

```
        [[(pp.irr)::P,pp.pow] for pp in lunm.factors])]$MFinalFact

             ----  The polynomial has mindeg>0   ----

simplify(m:P,dm:Z,lvar:L OV,lmdeg:L NNI):MFinalFact ==
  factorlist:L MParFact:=empty()
  pol1:P:= 1$P
  for x in lvar repeat
    i := lmdeg.(position(x,lvar))
    i=0 => "next value"
    pol1:=pol1*monomial(1$P,x,i)
    factorlist:=cons([x::P,i]$MParFact,factorlist)
  m := (m exquo pol1)::P
  ground? m => constantCase(m,factorlist)
  flead:=mFactor(m,dm)
  flead.factors:=append(factorlist,flead.factors)
  flead

             ----  m square-free,primitive,lc constant  ----
mfconst(um:SUP P,dm:Z,lvar:L OV,ldeg:L NNI):L SUP P ==
  nsign:Boolean
  factfin:L SUP P:=empty()
  empty? lvar =>
      um1:SUP R:=map(ground,
          um)$UPCF2(P,SUP P,R,SUP R)
      lum:= generalTwoFactor(um1)$TwoFactorize(F)
      [map(coerce,lumf.factor)$UPCF2(R,SUP R,P,SUP P)
            for lumf in factors lum]
  intfact(um,lvar,ldeg,[0,empty()]$MFinalFact,empty())

           --- m is square-free,primitive,lc is a polynomial  ---
mfpol(um:SUP P,dm:Z,lvar:L OV,ldeg:L NNI):L SUP P ==
  dist : LeadFact
  tleadpol:=mFactor(leadingCoefficient um,dm)
  intfact(um,lvar,ldeg,tleadpol,empty())

factor(m:PG):Factored PG ==
   lv:=variables m
   lv=empty() => makeFR(m,empty() )
-- reduce to multivariate over SUP
   ld:=[degree(m,x) for x in lv]
   dx:="min"/ld
   basicVar:=lv(position(dx,ld))
   cm:=pushdown(m,basicVar)
   flist := mFactor(cm,dx)
   pushupconst(flist.contp,basicVar) *
```

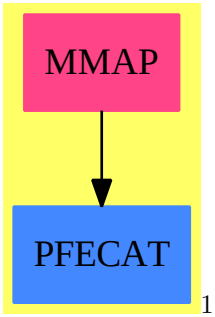
```
          (*/[primeFactor(pushup(u.irr,basicVar),u.pow)
                                                 for u in flist.factors])
```

⟨*MFINFACT.dotabb*⟩≡

```
"MFINFACT" [color="#FF4488",href="bookvol10.4.pdf#nameddest=MFINFACT"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"MFINFACT" -> "PFECAT"
```

14.59 package MMAP MultipleMap

14.60 MultipleMap



1.00

Exports:
map

⟨*package MMAP MultipleMap*⟩≡

```
)abbrev package MMAP MultipleMap
++ Lifting a map through 2 levels of polynomials
++ Author: Manuel Bronstein
++ Date Created: May 1988
++ Date Last Updated: 11 Jul 1990
++ Description: Lifting of a map through 2 levels of polynomials;
MultipleMap(R1,UP1,UPUP1,R2,UP2,UPUP2): Exports == Implementation where
  R1   : IntegralDomain
  UP1  : UnivariatePolynomialCategory R1
  UPUP1: UnivariatePolynomialCategory Fraction UP1
  R2   : IntegralDomain
  UP2  : UnivariatePolynomialCategory R2
  UPUP2: UnivariatePolynomialCategory Fraction UP2

  Q1 ==> Fraction UP1
  Q2 ==> Fraction UP2

  Exports ==> with
    map: (R1 -> R2, UPUP1) -> UPUP2
      ++ map(f, p) lifts f to the domain of p then applies it to p.

  Implementation ==> add
    import UnivariatePolynomialCategoryFunctions2(R1, UP1, R2, UP2)

    rfmap: (R1 -> R2, Q1) -> Q2

    rfmap(f, q) == map(f, numer q) / map(f, denom q)
```

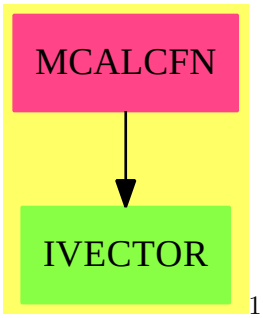
```
      map(f, p) ==
        map(rfmap(f, #1),
            p)$UnivariatePolynomialCategoryFunctions2(Q1, UPUP1, Q2, UPUP2)
```

⟨*MMAP.dotabb*⟩≡

```
  "MMAP" [color="#FF4488",href="bookvol10.4.pdf#nameddest=MMAP"]
  "PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
  "MMAP" -> "PFECAT"
```

14.61 package MCALCFN MultiVariableCalculusFunctions

14.62 MultiVariableCalculusFunctions



1.00

Exports:

bandedHessian	bandedJacobian	divergence	gradient	hessian
jacobian	laplacian			

⟨*package MCALCFN MultiVariableCalculusFunctions*⟩≡

```
)abbrev package MCALCFN MultiVariableCalculusFunctions
++ Author: Themos Tsikas, Grant Keady
++ Date Created: December 1992
++ Date Last Updated: June 1993
++ Basic Operations:
++ Related Constructors:
++ Also See:
++ AMS Classifications:
++ Keywords:
++ References:
++ Description:
++  \spadtype{MultiVariableCalculusFunctions} Package provides several
++  functions for multivariable calculus.
++ These include gradient, hessian and jacobian,
++ divergence and laplacian.
++ Various forms for banded and sparse storage of matrices are
++ included.
MultiVariableCalculusFunctions(S,F,FLAF,FLAS) : Exports == Implementation where
  PI ==> PositiveInteger
  NNI ==> NonNegativeInteger

  S: SetCategory
  F: PartialDifferentialRing(S)
  FLAS: FiniteLinearAggregate(S)
    with finiteAggregate
```

```
FLAF: FiniteLinearAggregate(F)

Exports ==> with
  gradient: (F,FLAS) -> Vector F
   ++ \spad{gradient(v,xlist)}
   ++ computes the gradient, the vector of first partial derivatives,
   ++ of the scalar field v,
   ++ v a function of the variables listed in xlist.
  divergence: (FLAF,FLAS) ->  F
   ++ \spad{divergence(vf,xlist)}
   ++ computes the divergence of the vector field vf,
   ++ vf a vector function of the variables listed in xlist.
  laplacian: (F,FLAS) -> F
   ++ \spad{laplacian(v,xlist)}
   ++ computes the laplacian of the scalar field v,
   ++ v a function of the variables listed in xlist.
  hessian: (F,FLAS) -> Matrix F
   ++ \spad{hessian(v,xlist)}
   ++ computes the hessian, the matrix of second partial derivatives,
   ++ of the scalar field v,
   ++ v a function of the variables listed in xlist.
  bandedHessian: (F,FLAS,NNI) -> Matrix F
   ++ \spad{bandedHessian(v,xlist,k)}
   ++ computes the hessian, the matrix of second partial derivatives,
   ++ of the scalar field v,
   ++ v a function of the variables listed in xlist,
   ++ k is the semi-bandwidth, the number of nonzero subdiagonals,
   ++ 2*k+1 being actual bandwidth.
   ++ Stores the nonzero band in lower triangle in a matrix,
   ++ dimensions k+1 by #xlist,
   ++ whose rows are the vectors formed by diagonal, subdiagonal, etc.
   ++ of the real, full-matrix, hessian.
   ++ (The notation conforms to LAPACK/NAG-F07 conventions.)
  -- At one stage it seemed a good idea to help the ASP<n> domains
  -- with the types of their input arguments and this led to the
  -- standard Gradient|Hessian|Jacobian functions.
  --standardJacobian: (Vector(F),List(S)) -> Matrix F
  -- ++ \spad{jacobian(vf,xlist)}
  -- ++ computes the jacobian, the matrix of first partial derivatives,
  -- ++ of the vector field vf,
  -- ++ vf a vector function of the variables listed in xlist.
  jacobian: (FLAF,FLAS) -> Matrix F
   ++ \spad{jacobian(vf,xlist)}
   ++ computes the jacobian, the matrix of first partial derivatives,
   ++ of the vector field vf,
   ++ vf a vector function of the variables listed in xlist.
```

```
    bandedJacobian: (FLAF,FLAS,NNI,NNI) -> Matrix F
     ++ \spad{bandedJacobian(vf,xlist,kl,ku)}
     ++ computes the jacobian, the matrix of first partial derivatives,
     ++ of the vector field vf,
     ++ vf a vector function of the variables listed in xlist,
     ++ kl is the number of nonzero subdiagonals,
     ++ ku is the number of nonzero superdiagonals,
     ++ kl+ku+1 being actual bandwidth.
     ++ Stores the nonzero band in a matrix,
     ++ dimensions kl+ku+1 by #xlist.
     ++ The upper triangle is in the top ku rows,
     ++ the diagonal is in row ku+1,
     ++ the lower triangle in the last kl rows.
     ++ Entries in a column in the band store correspond to entries
     ++ in same column of full store.
     ++ (The notation conforms to LAPACK/NAG-F07 conventions.)

  Implementation ==> add
    localGradient(v:F,xlist:List(S)):Vector(F) ==
       vector([D(v,x) for x in xlist])
    gradient(v,xflas) ==
       --xlist:List(S) := [xflas(i) for i in 1 .. maxIndex(xflas)]
       xlist:List(S) := parts(xflas)
       localGradient(v,xlist)
    localDivergence(vf:Vector(F),xlist:List(S)):F ==
       i: PI
       n: NNI
       ans: F
       -- Perhaps should report error if two args of min different
       n:= min(#(xlist),((maxIndex(vf))::NNI))$NNI
       ans:= 0
       for i in 1 .. n repeat ans := ans + D(vf(i),xlist(i))
       ans
    divergence(vf,xflas) ==
       xlist:List(S) := parts(xflas)
       i: PI
       n: NNI
       ans: F
       -- Perhaps should report error if two args of min different
       n:= min(#(xlist),((maxIndex(vf))::NNI))$NNI
       ans:= 0
       for i in 1 .. n repeat ans := ans + D(vf(i),xlist(i))
       ans
    laplacian(v,xflas) ==
       xlist:List(S) := parts(xflas)
       gv:Vector(F) := localGradient(v,xlist)
```

```
   localDivergence(gv,xlist)
hessian(v,xflas) ==
   xlist:List(S) := parts(xflas)
   matrix([[D(v,[x,y]) for x in xlist] for y in xlist])
--standardJacobian(vf,xlist) ==
--   i: PI
--   matrix([[D(vf(i),x) for x in xlist] for i in 1 .. maxIndex(vf)])
jacobian(vf,xflas) ==
   xlist:List(S) := parts(xflas)
   i: PI
   matrix([[D(vf(i),x) for x in xlist] for i in 1 .. maxIndex(vf)])
bandedHessian(v,xflas,k) ==
   xlist:List(S) := parts(xflas)
   j,iw: PI
   n: NNI
   bandM: Matrix F
   n:= #(xlist)
   bandM:= new(k+1,n,0)
   for j in 1 .. n repeat setelt(bandM,1,j,D(v,xlist(j),2))
   for iw in 2 .. (k+1) repeat (_
     for j in 1 .. (n-iw+1) repeat (_
       setelt(bandM,iw,j,D(v,[xlist(j),xlist(j+iw-1)])) ) )
   bandM
jacobian(vf,xflas) ==
   xlist:List(S) := parts(xflas)
   i: PI
   matrix([[D(vf(i),x) for x in xlist] for i in 1 .. maxIndex(vf)])
bandedJacobian(vf,xflas,kl,ku) ==
   xlist:List(S) := parts(xflas)
   j,iw: PI
   n: NNI
   bandM: Matrix F
   n:= #(xlist)
   bandM:= new(kl+ku+1,n,0)
   for j in 1 .. n repeat setelt(bandM,ku+1,j,D(vf(j),xlist(j)))
   for iw in (ku+2) .. (ku+kl+1) repeat (_
     for j in 1 .. (n-iw+ku+1) repeat (_
       setelt(bandM,iw,j,D(vf(j+iw-1-ku),xlist(j))) ) )
   for iw in 1 .. ku repeat (_
     for j in (ku+2-iw) .. n repeat (_
       setelt(bandM,iw,j,D(vf(j+iw-1-ku),xlist(j))) ) )
   bandM
```

⟨*MCALCFN.dotabb*⟩≡

```
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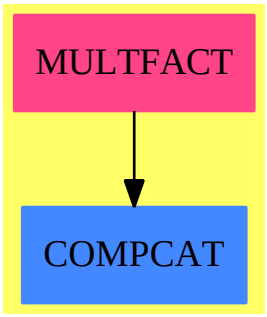
"IVECTOR" [color="#88FF44",href="bookvol10.3.pdf#nameddest=IVECTOR"]
"MCALCFN" -> "IVECTOR"
```

14.63 package MULTFACT MultivariateFactorize

14.64 MultivariateFactorize



1.00

Exports:
factor

⟨*package MULTFACT MultivariateFactorize*⟩≡

```
)abbrev package MULTFACT MultivariateFactorize
++ Author: P. Gianni
++ Date Created: 1983
++ Date Last Updated: Sept. 1990
++ Basic Functions:
++ Related Constructors: MultFiniteFactorize, AlgebraicMultFact, UnivariateFactor
++ Also See:
++ AMS Classifications:
++ Keywords:
++ References:
++ Description:
++   This is the top level package for doing multivariate factorization
++ over basic domains like \spadtype{Integer} or \spadtype{Fraction Integer}.

MultivariateFactorize(OV,E,R,P) : C == T
 where
  R          :   Join(EuclideanDomain, CharacteristicZero)
                    -- with factor on R[x]
  OV         :   OrderedSet
  E          :   OrderedAbelianMonoidSup
  P          :   PolynomialCategory(R,E,OV)
  Z            ==> Integer
  MParFact     ==> Record(irr:P,pow:Z)
  USP          ==> SparseUnivariatePolynomial P
  SUParFact    ==> Record(irr:USP,pow:Z)
  SUPFinalFact ==> Record(contp:R,factors:List SUParFact)
```

```
    MFinalFact   ==> Record(contp:R,factors:List MParFact)

                  --  contp   =  content,
                  --  factors =  List of irreducible factors with exponent
    L          ==> List

    C == with
      factor      :      P  ->  Factored P
        ++ factor(p) factors the multivariate polynomial p over its coefficient
        ++ domain
      factor      :      USP  ->  Factored USP
        ++ factor(p) factors the multivariate polynomial p over its coefficient
        ++ domain where p is represented as a univariate polynomial with
        ++ multivariate coefficients
    T == add
      factor(p:P) : Factored P ==
        R is Fraction Integer =>
           factor(p)$MRationalFactorize(E,OV,Integer,P)
        R is Fraction Complex Integer =>
           factor(p)$MRationalFactorize(E,OV,Complex Integer,P)
        R is Fraction Polynomial Integer and OV has convert: % -> Symbol =>
           factor(p)$MPolyCatRationalFunctionFactorizer(E,OV,Integer,P)
        factor(p,factor$GenUFactorize(R))$InnerMultFact(OV,E,R,P)

      factor(up:USP) : Factored USP ==
        factor(up,factor$GenUFactorize(R))$InnerMultFact(OV,E,R,P)
```

⟨*MULTFACT.dotabb*⟩≡

```
"MULTFACT" [color="#FF4488",href="bookvol10.4.pdf#nameddest=MULTFACT"]
"COMPCAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=COMPCAT"]
"MULTFACT" -> "COMPCAT"
```

14.65 package MLIFT MultivariateLifting

⟨*package MLIFT MultivariateLifting*⟩≡

```
)abbrev package MLIFT MultivariateLifting
++ Author : P.Gianni.
++ Description:
++ This package provides the functions for the multivariate "lifting", using
++ an algorithm of Paul Wang.
++ This package will work for every euclidean domain R which has property
++ F, i.e. there exists a factor operation in \spad{R[x]}.

MultivariateLifting(E,OV,R,P) : C == T
 where
  OV        :   OrderedSet
  E         :   OrderedAbelianMonoidSup
  R         :   EuclideanDomain  -- with property "F"
  Z         ==> Integer
  BP        ==> SparseUnivariatePolynomial R
  P         :   PolynomialCategory(R,E,OV)
  SUP       ==> SparseUnivariatePolynomial P
  NNI       ==> NonNegativeInteger
  Term      ==> Record(expt:NNI,pcoef:P)
  VTerm     ==> List Term
  Table     ==> Vector List BP
  L         ==> List

  C == with
    corrPoly:      (SUP,L OV,L R,L NNI,L SUP,Table,R) -> Union(L SUP,"failed")
        ++ corrPoly(u,lv,lr,ln,lu,t,r) \undocumented
    lifting:       (SUP,L OV,L BP,L R,L P,L NNI,R) -> Union(L SUP,"failed")
        ++ lifting(u,lv,lu,lr,lp,ln,r) \undocumented
    lifting1:      (SUP,L OV,L SUP,L R,L P,L VTerm,L NNI,Table,R) ->
                                                  Union(L SUP,"failed")
        ++ lifting1(u,lv,lu,lr,lp,lt,ln,t,r) \undocumented

  T == add
    GenExEuclid(R,BP)
    NPCoef(BP,E,OV,R,P)
    IntegerCombinatoricFunctions(Z)

    SUPF2   ==> SparseUnivariatePolynomialFunctions2

    DetCoef ==> Record(deter:L SUP,dterm:L VTerm,nfacts:L BP,
                       nlead:L P)

              ---   local functions   ---
```

```
normalDerivM    :    (P,Z,OV)     ->  P
normalDeriv     :     (SUP,Z)     ->  SUP
subslead        :     (SUP,P)     ->  SUP
subscoef        :   (SUP,L Term)  ->  SUP
maxDegree       :   (SUP,OV)      ->  NonNegativeInteger


corrPoly(m:SUP,lvar:L OV,fval:L R,ld:L NNI,flist:L SUP,
         table:Table,pmod:R):Union(L SUP,"failed") ==
  --  The correction coefficients are evaluated recursively.
  --   Extended Euclidean algorithm for the multivariate case.

  -- the polynomial is univariate  --
  #lvar=0 =>
    lp:=solveid(map(ground,m)$SUPF2(P,R),pmod,table)
    if lp case "failed" then return "failed"
    lcoef:= [map(coerce,mp)$SUPF2(R,P) for mp in lp::L BP]


  diff,ddiff,pol,polc:SUP
  listpolv,listcong:L SUP
  deg1:NNI:= ld.first
  np:NNI:= #flist
  a:P:= fval.first ::P
  y:OV:=lvar.first
  lvar:=lvar.rest
  listpolv:L SUP := [map(eval(#1,y,a),f1) for f1 in flist]
  um:=map(eval(#1,y,a),m)
  flcoef:=corrPoly(um,lvar,fval.rest,ld.rest,listpolv,table,pmod)
  if flcoef case "failed" then return "failed"
  else lcoef:=flcoef :: L SUP
  listcong:=[*/[flist.i for i in 1..np | i^=l] for l in 1..np]
  polc:SUP:= (monomial(1,y,1) - a)::SUP
  pol := 1$SUP
  diff:=m- +/[lcoef.i*listcong.i for i in 1..np]
  for l in 1..deg1 repeat
    if diff=0 then return lcoef
    pol := pol*polc
    (ddiff:= map(eval(normalDerivM(#1,l,y),y,a),diff)) = 0 => "next l"
    fbeta := corrPoly(ddiff,lvar,fval.rest,ld.rest,listpolv,table,pmod)
    if fbeta case "failed" then return "failed"
    else beta:=fbeta :: L SUP
    lcoef := [lcoef.i+beta.i*pol  for i in 1..np]
    diff:=diff- +/[listcong.i*beta.i for i in 1..np]*pol
  lcoef
```

```
lifting1(m:SUP,lvar:L OV,plist:L SUP,vlist:L R,tlist:L P,_
  coeflist:L VTerm,listdeg:L NNI,table:Table,pmod:R) :Union(L SUP,"failed") =
-- The factors of m (multivariate) are determined ,
-- We suppose to know the true univariate factors
-- some coefficients are determined
  conglist:L SUP:=empty()
  nvar : NNI:= #lvar
  pol,polc:P
  mc,mj:SUP
  testp:Boolean:= (not empty?(tlist))
  lalpha : L SUP := empty()
  tlv:L P:=empty()
  subsvar:L OV:=empty()
  subsval:L R:=empty()
  li:L OV := lvar
  ldeg:L NNI:=empty()
  clv:L VTerm:=empty()
  --j =#variables, i=#factors
  for j in 1..nvar repeat
    x  := li.first
    li := rest li
    conglist:= plist
    v := vlist.first
    vlist := rest vlist
    degj := listdeg.j
    ldeg := cons(degj,ldeg)
    subsvar:=cons(x,subsvar)
    subsval:=cons(v,subsval)

  --substitute the determined coefficients
    if testp then
      if j<nvar then
        tlv:=[eval(p,li,vlist) for p in tlist]
        clv:=[[[term.expt,eval(term.pcoef,li,vlist)]$Term
               for term in clist] for clist  in coeflist]
      else (tlv,clv):=(tlist,coeflist)
      plist :=[subslead(p,lcp) for p in plist for lcp in tlv]
      if not(empty? coeflist) then
        plist:=[subscoef(tpol,clist)
               for tpol in plist for clist in clv]
    mj := map(eval(#1,li,vlist),m)  --m(x1,..,xj,aj+1,..,an
    polc := x::P - v::P  --(xj-aj)
    pol:= 1$P
  --Construction of Rik, k in 1..right degree for xj+1
```

```
    for k in 1..degj repeat  --I can exit before
     pol := pol*polc
     mc := */[term for term in plist]-mj
     if mc=0 then leave "next var"
     --Modulus Dk
     mc:=map(normalDerivM(#1,k,x),mc)
     (mc := map(eval(#1,[x],[v]),mc))=0 => "next k"
     flalpha:=corrPoly(mc,subsvar.rest,subsval.rest,
                     ldeg.rest,conglist,table,pmod)
     if flalpha case "failed" then return "failed"
     else lalpha:=flalpha :: L SUP
     plist:=[term-alpha*pol for term in plist for alpha in lalpha]
    -- PGCD may call with a smaller valure of degj
    idegj:Integer:=maxDegree(m,x)
    for term in plist repeat idegj:=idegj -maxDegree(term,x)
    idegj < 0 => return "failed"
  plist
    --There are not extraneous factors

maxDegree(um:SUP,x:OV):NonNegativeInteger ==
   ans:NonNegativeInteger:=0
   while um ^= 0 repeat
      ans:=max(ans,degree(leadingCoefficient um,x))
      um:=reductum um
   ans

lifting(um:SUP,lvar:L OV,plist:L BP,vlist:L R,
        tlist:L P,listdeg:L NNI,pmod:R):Union(L SUP,"failed") ==
-- The factors of m (multivariate) are determined, when the
--  univariate true factors are known and some coefficient determined
  nplist:List SUP:=[map(coerce,pp)$SUPF2(R,P) for pp in plist]
  empty? tlist =>
    table:=tablePow(degree um,pmod,plist)
    table case "failed" => error "Table construction failed in MLIFT"
    lifting1(um,lvar,nplist,vlist,tlist,empty(),listdeg,table,pmod)
  ldcoef:DetCoef:=npcoef(um,plist,tlist)
  if not empty?(listdet:=ldcoef.deter) then
    if #listdet = #plist  then return listdet
    plist:=ldcoef.nfacts
    nplist:=[map(coerce,pp)$SUPF2(R,P) for pp in plist]
    um:=(um exquo */[pol for pol in listdet])::SUP
    tlist:=ldcoef.nlead
    tab:=tablePow(degree um,pmod,plist.rest)
  else tab:=tablePow(degree um,pmod,plist)
  tab case "failed" => error "Table construction failed in MLIFT"
  table:Table:=tab
```

```
      ffl:=lifting1(um,lvar,nplist,vlist,tlist,ldcoef.dterm,listdeg,table,pmod)
      if ffl case "failed" then return "failed"
      append(listdet,ffl:: L SUP)

    -- normalDerivM(f,m,x) = the normalized (divided by m!) m-th
    -- derivative with respect to x of the multivariate polynomial f
    normalDerivM(g:P,m:Z,x:OV) : P ==
     multivariate(normalDeriv(univariate(g,x),m),x)

    normalDeriv(f:SUP,m:Z) : SUP ==
     (n1:Z:=degree f) < m => 0$SUP
     n1=m => leadingCoefficient f :: SUP
     k:=binomial(n1,m)
     ris:SUP:=0$SUP
     n:Z:=n1
     while n>= m repeat
       while n1>n repeat
         k:=(k*(n1-m)) quo n1
         n1:=n1-1
       ris:=ris+monomial(k*leadingCoefficient f,(n-m)::NNI)
       f:=reductum f
       n:=degree f
     ris

    subslead(m:SUP,pol:P):SUP ==
      dm:NNI:=degree m
      monomial(pol,dm)+reductum m

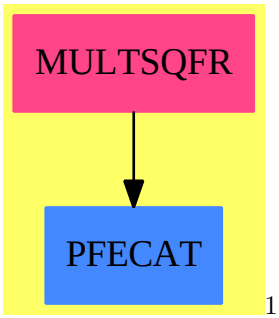
    subscoef(um:SUP,lterm:L Term):SUP ==
      dm:NNI:=degree um
      new:=monomial(leadingCoefficient um,dm)
      for k in dm-1..0 by -1 repeat
        i:NNI:=k::NNI
        empty?(lterm) or lterm.first.expt^=i =>
                                  new:=new+monomial(coefficient(um,i),i)
        new:=new+monomial(lterm.first.pcoef,i)
        lterm:=lterm.rest
      new
```

⟨*MLIFT.dotabb*⟩≡

```
"MLIFT" [color="#FF4488",href="bookvol10.4.pdf#nameddest=MLIFT"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"MLIFT" -> "PFECAT"
```

14.66 package MULTSQFR MultivariateSquareFree

14.67 MultivariateSquareFree



1.00

Exports:

check	coefChoose	compdegd	consnewpol	intChoose
lift	myDegree	normDeriv2	nsqfree	squareFree
squareFreePrim	univcase			

⟨*package MULTSQFR MultivariateSquareFree*⟩≡

```
)abbrev package MULTSQFR MultivariateSquareFree
++ Author : P.Gianni
++ This package provides the functions for the computation of the square
++ free decomposition of a multivariate polynomial.
++ It uses the package GenExEuclid for the resolution of
++ the equation \spad{Af + Bg = h} and its generalization to n polynomials
++ over an integral domain and the package \spad{MultivariateLifting}
++ for the "multivariate" lifting.

MultivariateSquareFree (E,OV,R,P) : C == T where
 Z   ==> Integer
 NNI ==> NonNegativeInteger
 R   : EuclideanDomain
 OV  : OrderedSet
 E   : OrderedAbelianMonoidSup
 P   : PolynomialCategory(R,E,OV)
 SUP ==> SparseUnivariatePolynomial P

 BP          ==> SparseUnivariatePolynomial(R)
 fUnion      ==> Union("nil","sqfr","irred","prime")
 ffSUP       ==> Record(flg:fUnion,fctr:SUP,xpnt:Integer)
 ffP         ==> Record(flg:fUnion,fctr:P,xpnt:Integer)
 FFE         ==> Record(factor:BP,exponent:Z)
 FFEP        ==> Record(factor:P,exponent:Z)
```

```
FFES        ==> Record(factor:SUP,exponent:Z)
Choice      ==> Record(upol:BP,Lval:List(R),Lfact:List FFE,ctpol:R)
squareForm  ==> Record(unitPart:P,suPart:List FFES)
Twopol      ==> Record(pol:SUP,polval:BP)
UPCF2       ==> UnivariatePolynomialCategoryFunctions2


C == with
  squareFree      :      P     -> Factored P
    ++ squareFree(p) computes the square free
    ++ decomposition of a multivariate polynomial p.
  squareFree      :      SUP     -> Factored SUP
    ++ squareFree(p) computes the square free
    ++ decomposition of a multivariate polynomial p presented as
    ++ a univariate polynomial with multivariate coefficients.
  squareFreePrim :      P     -> Factored P
    ++ squareFreePrim(p) compute the square free decomposition
    ++ of a primitive multivariate polynomial p.



                  ----  local functions  ----
  compdegd   :              List FFE                    -> Z
       ++ compdegd should be local
  univcase   :               (P,OV)                     -> Factored(P)
       ++ univcase should be local
  consnewpol :             (SUP,BP,Z)                   -> Twopol
       ++ consnewpol should be local
  nsqfree    :           (SUP,List(OV), List List R)    -> squareForm
       ++ nsqfree should be local
  intChoose  :           (SUP,List(OV),List List R)     -> Choice
       ++ intChoose should be local
  coefChoose :           (Z,Factored P)                 -> P
       ++ coefChoose should be local
  check      :     (List(FFE),List(FFE))                -> Boolean
       ++ check should be local
  lift       : (SUP,BP,BP,P,List(OV),List(NNI),List(R)) -> Union(List(SUP),"fail
       ++ lift should be local
  myDegree   :       (SUP,List OV,NNI)                  -> List NNI
       ++ myDegree should be local
  normDeriv2 :             (BP,Z)                       ->  BP
       ++ normDeriv2 should be local



T == add
```

```
    pmod:R   :=  (prevPrime(2**26)$IntegerPrimesPackage(Integer))::R


    import GenExEuclid()
    import MultivariateLifting(E,OV,R,P)
    import PolynomialGcdPackage(E,OV,R,P)
    import FactoringUtilities(E,OV,R,P)
    import IntegerCombinatoricFunctions(Z)


     ----  Are the univariate square-free decompositions consistent?  ----

      ----  new square-free algorithm for primitive polynomial  ----
    nsqfree(oldf:SUP,lvar:List(OV),ltry:List List R) : squareForm ==
      f:=oldf
      univPol := intChoose(f,lvar,ltry)
--      debug msg
--      if not empty? ltry then output("ltry =", (ltry::OutputForm))$OutputPackage
      f0:=univPol.upol
      --the polynomial is square-free
      f0=1$BP => [1$P,[[f,1]$FFES]]$squareForm
      lfact:List(FFE):=univPol.Lfact
      lval:=univPol.Lval
      ctf:=univPol.ctpol
      leadpol:Boolean:=false
      sqdec:List FFES := empty()
      exp0:Z:=0
      unitsq:P:=1
      lcf:P:=leadingCoefficient f
      if ctf^=1 then
        f0:=ctf*f0
        f:=(ctf::P)*f
        lcf:=ctf*lcf
      sqlead:List FFEP:= empty()
      sqlc:Factored P:=1
      if lcf^=1$P then
        leadpol:=true
        sqlc:=squareFree lcf
        unitsq:=unitsq*(unit sqlc)
        sqlead:= factors sqlc
      lfact:=sort(#1.exponent > #2.exponent,lfact)
      while lfact^=[] repeat
        pfact:=lfact.first
        (g0,exp0):=(pfact.factor,pfact.exponent)
        lfact:=lfact.rest
```

```
      lfact=[] and exp0 =1 =>
        f := (f exquo (ctf::P))::SUP
        gg := unitNormal leadingCoefficient f
        sqdec:=cons([gg.associate*f,exp0],sqdec)
        return  [gg.unit, sqdec]$squareForm
      if ctf^=1 then g0:=ctf*g0
      npol:=consnewpol(f,f0,exp0)
      (d,d0):=(npol.pol,npol.polval)
      if leadpol then lcoef:=coefChoose(exp0,sqlc)
      else lcoef:=1$P
      ldeg:=myDegree(f,lvar,exp0::NNI)
      result:=lift(d,g0,(d0 exquo g0)::BP,lcoef,lvar,ldeg,lval)
      result case "failed" => return nsqfree(oldf,lvar,ltry)
      result0:SUP:= (result::List SUP).1
      r1:SUP:=result0**(exp0:NNI)
      if (h:=f exquo r1) case "failed" then return nsqfree(oldf,lvar,empty())
      sqdec:=cons([result0,exp0],sqdec)
      f:=h::SUP
      f0:=completeEval(h,lvar,lval)
      lcr:P:=leadingCoefficient result0
      if leadpol and lcr^=1$P then
        for lpfact in sqlead  while lcr^=1 repeat
          ground? lcr =>
            unitsq:=(unitsq exquo lcr)::P
            lcr:=1$P
          (h1:=lcr exquo lpfact.factor) case "failed" => "next"
          lcr:=h1::P
          lpfact.exponent:=(lpfact.exponent)-exp0
    [((retract f) exquo ctf)::P,sqdec]$squareForm


  squareFree(f:SUP) : Factored SUP ==
    degree f =0 =>
      fu:=squareFree retract f
      makeFR((unit fu)::SUP,[["sqfr",ff.fctr::SUP,ff.xpnt]
              for ff in factorList fu])
    lvar:= "setUnion"/[variables cf for cf in coefficients f]
    empty? lvar =>  -- the polynomial is univariate
      upol:=map(ground,f)$UPCF2(P,SUP,R,BP)
      usqfr:=squareFree upol
      makeFR(map(coerce,unit usqfr)$UPCF2(R,BP,P,SUP),
             [["sqfr",map(coerce,ff.fctr)$UPCF2(R,BP,P,SUP),ff.xpnt]
                for ff in factorList usqfr])

    lcf:=content f
    f:=(f exquo lcf) ::SUP
```

```
    lcSq:=squareFree lcf
    lfs:List ffSUP:=[["sqfr",ff.fctr ::SUP,ff.xpnt]
                      for ff in factorList lcSq]
    partSq:=nsqfree(f,lvar,empty())

    lfs:=append([["sqfr",fu.factor,fu.exponent]$ffSUP
                  for fu in partSq.suPart],lfs)
    makeFR((unit lcSq * partSq.unitPart) ::SUP,lfs)

  squareFree(f:P) : Factored P ==
    ground? f => makeFR(f,[])      ---   the polynomial is constant  ---

    lvar:List(OV):=variables(f)
    result1:List ffP:= empty()

    lmdeg :=minimumDegree(f,lvar)     ---       is the mindeg > 0 ? ---
    p:P:=1$P
    for im in 1..#lvar repeat
      (n:=lmdeg.im)=0 => "next im"
      y:=lvar.im
      p:=p*monomial(1$P,y,n)
      result1:=cons(["sqfr",y::P,n],result1)
    if p^=1$P then
      f := (f exquo p)::P
      if ground? f then return makeFR(f, result1)
      lvar:=variables(f)


    #lvar=1 =>                     ---  the polynomial is univariate ---
      result:=univcase(f,lvar.first)
      makeFR(unit result,append(result1,factorList result))

    ldeg:=degree(f,lvar)          ---  general case ---
    m:="min"/[j for j in ldeg|j^=0]
    i:Z:=1
    for j in ldeg while j>m repeat i:=i+1
    x:=lvar.i
    lvar:=delete(lvar,i)
    f0:=univariate (f,x)
    lcont:P:= content f0
    nsqfftot:=nsqfree((f0 exquo lcont)::SUP,lvar,empty())
    nsqff:List ffP:=[["sqfr",multivariate(fu.factor,x),fu.exponent]$ffP
                       for fu in nsqfftot.suPart]
    result1:=append(result1,nsqff)
    ground? lcont => makeFR(lcont*nsqfftot.unitPart,result1)
    sqlead:=squareFree(lcont)
```

```
      makeFR(unit sqlead*nsqfftot.unitPart,append(result1,factorList sqlead))

    -- Choose the integer for the evaluation.                          --
    -- If the polynomial is square-free the function returns upol=1. --

     intChoose(f:SUP,lvar:List(OV),ltry:List List R):Choice ==
       degf:= degree f
       try:NNI:=0
       nvr:=#lvar
       range:Z:=10
       lfact1:List(FFE):=[]
       lval1:List R := []
       lfact:List(FFE)
       ctf1:R:=1
       f1:BP:=1$BP
       d1:Z
       while range<10000000000 repeat
         range:=2*range
         lval:= [ran(range) for i in 1..nvr]
         member?(lval,ltry) => "new integer"
         ltry:=cons(lval,ltry)
         f0:=completeEval(f,lvar,lval)
         degree f0 ^=degf  => "new integer"
         ctf:=content f0
         lfact:List(FFE):=factors(squareFree((f0 exquo (ctf:R)::BP)::BP))

              ----  the univariate polynomial is square-free  ----
         if #lfact=1 and (lfact.1).exponent=1 then
           return [1$BP,lval,lfact,1$R]$Choice

         d0:=compdegd lfact
                   ----      inizialize lfact1      ----
         try=0 =>
           f1:=f0
           lfact1:=lfact
           ctf1:=ctf
           lval1:=lval
           d1:=d0
           try:=1
         d0=d1 =>
           return [f1,lval1,lfact1,ctf1]$Choice
         d0 < d1 =>
           try:=1
           f1:=f0
           lfact1:=lfact
           ctf1:=ctf
```

```
      lval1:=lval
      d1:=d0


     ----  Choose the leading coefficient for the lifting  ----
coefChoose(exp:Z,sqlead:Factored(P)) : P ==
  lcoef:P:=unit(sqlead)
  for term in factors(sqlead) repeat
    texp:=term.exponent
    texp<exp => "next term"
    texp=exp => lcoef:=lcoef*term.factor
    lcoef:=lcoef*(term.factor)**((texp quo exp)::NNI)
  lcoef

     ----  Construction of the polynomials for the lifting  ----
consnewpol(g:SUP,g0:BP,deg:Z):Twopol ==
  deg=1 => [g,g0]$Twopol
  deg:=deg-1
  [normalDeriv(g,deg),normDeriv2(g0,deg)]$Twopol

      ----  lift the univariate square-free factor  ----
lift(ud:SUP,g0:BP,g1:BP,lcoef:P,lvar:List(OV),
                     ldeg:List(NNI),lval:List(R)) : Union(List SUP,"failed") ==
  leadpol:Boolean:=false
  lcd:P:=leadingCoefficient ud
  leadlist:List(P):=empty()

  if ^ground?(leadingCoefficient ud) then
    leadpol:=true
    ud:=lcoef*ud
    lcg0:R:=leadingCoefficient g0
    if ground? lcoef then g0:=retract(lcoef) quo lcg0 *g0
    else g0:=(retract(eval(lcoef,lvar,lval)) quo lcg0) * g0
    g1:=lcg0*g1
    leadlist:=[lcoef,lcd]
  plist:=lifting(ud,lvar,[g0,g1],lval,leadlist,ldeg,pmod)
  plist case "failed" => "failed"
  (p0:SUP,p1:SUP):=((plist::List SUP).1,(plist::List SUP).2)
  if completeEval(p0,lvar,lval) ^= g0 then (p0,p1):=(p1,p0)
  [primitivePart p0,primitivePart p1]

             ----  the polynomial is univariate  ----
univcase(f:P,x:OV) : Factored(P) ==
  uf := univariate f
  cf:=content uf
  uf :=(uf exquo cf)::BP
```

```
      result:Factored BP:=squareFree uf
      makeFR(multivariate(cf*unit result,x),
          [["sqfr",multivariate(term.factor,x),term.exponent]
            for term in factors result])

  --   squareFreePrim(p:P) : Factored P ==
  --     -- p is content free
  --     ground? p => makeFR(p,[])      ---   the polynomial is constant  ---
  --
  --     lvar:List(OV):=variables p
  --     #lvar=1 =>                     ---  the polynomial is univariate ---
  --       univcase(p,lvar.first)
  --     nsqfree(p,lvar,1)

    compdegd(lfact:List(FFE)) : Z ==
      ris:Z:=0
      for pfact in lfact repeat
        ris:=ris+(pfact.exponent -1)*degree pfact.factor
      ris

    normDeriv2(f:BP,m:Z) : BP ==
      (n1:Z:=degree f) < m => 0$BP
      n1=m => (leadingCoefficient f)::BP
      k:=binomial(n1,m)
      ris:BP:=0$BP
      n:Z:=n1
      while n>= m repeat
        while n1>n repeat
          k:=(k*(n1-m)) quo n1
          n1:=n1-1
        ris:=ris+monomial(k*leadingCoefficient f,(n-m)::NNI)
        f:=reductum f
        n:=degree f
      ris

    myDegree(f:SUP,lvar:List OV,exp:NNI) : List NNI==
      [n quo exp for n in degree(f,lvar)]
```

⟨*MULTSQFR.dotabb*⟩≡

```
"MULTSQFR" [color="#FF4488",href="bookvol10.4.pdf#nameddest=MULTSQFR"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"MULTSQFR" -> "PFECAT"
```

Chapter 15

Chapter N

15.1 package NAGF02 NagEigenPackage

⟨*NagEigenPackage.help*⟩≡

```
F02 -- Eigenvalues and Eigenvectors          Introduction -- F02
                          Chapter F02
                Eigenvalues and Eigenvectors

1. Scope of the Chapter

This chapter is concerned with computing

--    eigenvalues and eigenvectors of a matrix

--    eigenvalues and eigenvectors of generalized matrix
      eigenvalue problems

--    singular values and singular vectors of a matrix.

2. Background to the Problems

2.1. Eigenvalue Problems

In the most usual form of eigenvalue problem we are given a
square n by n matrix A and wish to compute (lambda) (an
eigenvalue) and x/=0 (an eigenvector) which satisfy the equation

                          Ax=(lambda)x
```

Such problems are called 'standard' eigenvalue problems in contrast to 'generalized' eigenvalue problems where we wish to satisfy the equation

Ax=(lambda)Bx

B also being a square n by n matrix.

Section 2.1.1 and Section 2.1.2 discuss, respectively, standard and generalized eigenvalue problems where the matrices involved are dense; Section 2.1.3 discusses both types of problem in the case where A and B are sparse (and symmetric).

2.1.1. Standard eigenvalue problems

Some of the routines in this chapter find all the n eigenvalues, some find all the n eigensolutions (eigenvalues and corresponding eigenvectors), and some find a selected group of eigenvalues and/or eigenvectors. The matrix A may be:

(i) general (real or complex)

(ii) real symmetric, or

(iii) complex Hermitian (so that if a_{ij} =(alpha)+i(beta) then a_{ji} =(alpha)-i(beta)).

In all cases the computation starts with a similarity transformation $S^{-1}AS=T$, where S is non-singular and is the product of fairly simple matrices, and T has an 'easier form' than A so that its eigensolutions are easily determined. The matrices A and T, of course, have the same eigenvalues, and if y is an eigenvector of T then Sy is the corresponding eigenvector of A.

In case (i) (general real or complex A), the selected form of T is an upper Hessenberg matrix (t_{ij} =0 if i-j>1) and S is the product of n-2 stabilised elementary transformation matrices. There is no easy method of computing selected eigenvalues of a Hessenberg matrix, so that all eigenvalues are always calculated. In the real case this computation is performed via the Francis QR algorithm with double shifts, and in the complex case by means of

the LR algorithm. If the eigenvectors are required they are computed by back-substitution following the QR and LR algorithm.

In case (ii) (real and symmetric A) the selected simple form of T is a tridiagonal matrix ($t_{ij}=0$ if $|i-j|>1$), and S is the product of n-2 orthogonal Householder transformation matrices. If only selected eigenvalues are required, they are obtained by the method of bisection using the Sturm sequence property, and the corresponding eigenvectors of T are computed by inverse iteration. If all eigenvalues are required, they are computed from T via the QL algorithm (an adaptation of the QR algorithm), and the corresponding eigenvectors of T are the product of the transformations for the QL reduction. In all cases the corresponding eigenvectors of A are recovered from the computation of x=Sy.

In case (iii) (complex Hermitian A) analogous transformations as in case (ii) are used. T has complex elements in off-diagonal positions, but a simple diagonal similarity transformation is then used to produce a real tridiagonal form, after which the QL algorithm and succeeding methods described in the previous paragraph are used to complete the solution.

2.1.2. Generalized eigenvalue problems

Here we distinguish as a special case those problems in which both A and B are symmetric and B is positive-definite and well-conditioned with respect to inversion (i.e., all the eigenvalues of B are significantly greater than zero). Such problems can be satisfactorily treated by first reducing them to case (ii) of Section 2.1.1 and then using the methods described there to compute the eigensolutions. If B is factorized as LL^T (L lower triangular), then $Ax=(lambda)Bx$ is equivalent to the standard symmetric problem $Ry=(lambda)y$, where $R=L^{-1}A(L^T)^{-1}$ and $y=L^T x$. After finding an eigenvector y of R, the required x is computed by back-substitution in $y=L^T x$.

For generalized problems of the form $Ax=(lambda)Bx$ which do not fall into the special case, the QZ algorithm is provided.

In order to appreciate the domain in which this algorithm is appropriate we remark first that when B is non-singular the

problem Ax=(lambda)Bx is fully equivalent to the problem
  -1
(B A)x=(lambda)x; both the eigenvalues and eigenvectors being
the same. When A is non-singular Ax=(lambda)Bx is equivalent to
              -1
the problem (A B)x=(mu)x; the eigenvalues (mu) being the
reciprocals of the required eigenvalues and the eigenvectors
remaining the same. In theory then, provided at least one of the
matrices A and B is non-singular, the generalized problem
Ax=(lambda)Bx could be solved via the standard problem
Cx=(lambda)x with an appropriate matrix C, and as far as economy
of effort is concerned this is quite satisfactory. However, in
practice, for this reduction to be satisfactory from the
standpoint of numerical stability, one requires more than the
                                                        -1
mere non-singularity of A or B. It is necessary that B A (or
 -1
A B) should not only exist but that B (or A) should be well-
conditioned with respect to inversion. The nearer B (or A) is to
                                      -1   -1
singularity the more unsatisfactory B A (or A B) will be as a
vehicle for determining the required eigenvalues. Unfortunately
                                                             -1
one cannot counter ill-conditioning in B (or A) by computing B A
     -1
(or A B) accurately to single precision using iterative
refinement. Well-determined eigenvalues of the original
Ax=(lambda)Bx may be poorly determined even by the correctly
                    -1   -1
rounded version of B A (or A B). The situation may in some
instances be saved by the observation that if Ax=(lambda)Bx then
(A-kB)x=((lambda)-k)Bx. Hence if A-kB is non-singular we may
                                  -1
solve the standard problem [(A-kB) B]x=(mu)x and for numerical
stability we require only that (A-kB) be well-conditioned with
respect to inversion.

In practice one may well be in a situation where no k is known
for which (A-kB) is well-conditioned with respect to inversion
and indeed (A-kB) may be singular for all k. The QZ algorithm is
designed to deal directly with the problem Ax=(lambda)Bx itself
and its performance is unaffected by singularity or near-
singularity of A, B or A-kB.

2.1.3. Sparse symmetric problems

If the matrices A and B are large and sparse (i.e., only a small proportion of the elements are non-zero), then the methods described in the previous Section are unsuitable, because in reducing the problem to a simpler form, much of the sparsity of the problem would be lost; hence the computing time and the storage required would be very large. Instead, for symmetric problems, the method of simultaneous iteration may be used to determine selected eigenvalues and the corresponding eigenvectors. The routine provided has been designed to handle both symmetric and generalized symmetric problems.

2.2. Singular Value Problems

The singular value decomposition of an m by n real matrix A is given by

$$A=QDP^{T},$$

where Q is an m by m orthogonal matrix, P is an n by n orthogonal matrix and D is an m by n diagonal matrix with non-negative diagonal elements. The first k==min(m,n) columns of Q and P are the left- and right-hand singular vectors of A and the k diagonal elements of D are the singular values.

When A is complex then the singular value decomposition is given by

$$A=QDP^{H},$$

where Q and P are unitary, P^{H} denotes the complex conjugate of P^{T} and D is as above for the real case.

If the matrix A has column means of zero, then AP is the matrix of principal components of A and the singular values are the square roots of the sample variances of the observations with respect to the principal components. (See also Chapter G03.)

Routines are provided to return the singular values and vectors of a general real or complex matrix.

3. Recommendations on Choice and Use of Routines

3.1. General Discussion

There is one routine, F02FJF, which is designed for sparse symmetric eigenvalue problems, either standard or generalized. The remainder of the routines are designed for dense matrices.

3.2. Eigenvalue and Eigenvector Routines

These reduce the matrix A to a simpler form by a similarity transformation $S^{-1}AS=T$ where T is an upper Hessenberg or tridiagonal matrix, compute the eigensolutions of T, and then recover the eigenvectors of A via the matrix S. The eigenvectors are normalised so that

$$\sum_{r=1}^{n} |x_r|^2 = 1$$

x_r being the rth component of the eigenvector x, and so that the element of largest modulus is real if x is complex. For problems of the type $Ax=(lambda)Bx$ with A and B symmetric and B positive-definite, the eigenvectors are normalised so that $x^T Bx=1$, x always being real for such problems.

3.3. Singular Value and Singular Vector Routines

These reduce the matrix A to real bidiagonal form, B say, by orthogonal transformations $Q^T AP=B$ in the real case, and by unitary transformations $Q^H AP=B$ in the complex case, and the singular values and vectors are computed via this bidiagonal form. The singular values are returned in descending order.

3.4. Decision Trees

(i) Eigenvalues and Eigenvectors

Please see figure in printed Reference Manual

(ii) Singular Values and Singular Vectors

Please see figure in printed Reference Manual

F02 -- Eigenvalues and Eigenvectors Contents -- F02
Chapter F02

Eigenvalues and Eigenvectors

F02AAF All eigenvalues of real symmetric matrix

F02ABF All eigenvalues and eigenvectors of real symmetric matrix

F02ADF All eigenvalues of generalized real symmetric-definite eigenproblem

F02AEF All eigenvalues and eigenvectors of generalized real symmetric-definite eigenproblem

F02AFF All eigenvalues of real matrix

F02AGF All eigenvalues and eigenvectors of real matrix

F02AJF All eigenvalues of complex matrix

F02AKF All eigenvalues and eigenvectors of complex matrix

F02AWF All eigenvalues of complex Hermitian matrix

F02AXF All eigenvalues and eigenvectors of complex Hermitian matrix

F02BBF Selected eigenvalues and eigenvectors of real symmetric matrix

F02BJF All eigenvalues and optionally eigenvectors of generalized eigenproblem by QZ algorithm, real matrices

F02FJF Selected eigenvalues and eigenvectors of sparse symmetric eigenproblem

F02WEF SVD of real matrix

F02XEF SVD of complex matrix

%%%

F02 -- Eigenvalue and Eigenvectors F02AAF
F02AAF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for your implementation to check implementation-dependent details. The symbol (*) after a NAG routine name denotes a routine that is not included in the Foundation Library.

1. Purpose

F02AAF calculates all the eigenvalues of a real symmetric matrix.

2. Specification

```
SUBROUTINE F02AAF (A, IA, N, R, E, IFAIL)
INTEGER          IA, N, IFAIL
DOUBLE PRECISION A(IA,N), R(N), E(N)
```

3. Description

This routine reduces the real symmetric matrix A to a real symmetric tridiagonal matrix using Householder's method. The eigenvalues of the tridiagonal matrix are then determined using the QL algorithm.

4. References

[1] Wilkinson J H and Reinsch C (1971) Handbook for Automatic Computation II, Linear Algebra. Springer-Verlag.

5. Parameters

1: A(IA,N) -- DOUBLE PRECISION array Input/Output
On entry: the lower triangle of the n by n symmetric matrix A. The elements of the array above the diagonal need not be set. On exit: the elements of A below the diagonal are overwritten, and the rest of the array is unchanged.

2: IA -- INTEGER Input
On entry:
the first dimension of the array A as declared in the (sub)program from which F02AAF is called.
Constraint: IA >= N.

```
 3:  N -- INTEGER                                              Input
     On entry: n, the order of the matrix A.

 4:  R(N) -- DOUBLE PRECISION array                            Output
     On exit: the eigenvalues in ascending order.

 5:  E(N) -- DOUBLE PRECISION array                          Workspace

 6:  IFAIL -- INTEGER                                     Input/Output
     On entry: IFAIL must be set to 0, -1 or 1. For users not
     familiar with this parameter (described in the Essential
     Introduction) the recommended value is 0.

     On exit: IFAIL = 0 unless the routine detects an error (see
     Section 6).

6. Error Indicators and Warnings

Errors detected by the routine:

IFAIL= 1
     Failure in F02AVF(*) indicating that more than 30*N
     iterations are required to isolate all the eigenvalues.

7. Accuracy

The accuracy of the eigenvalues depends on the sensitivity of the
matrix to rounding errors produced in tridiagonalisation. For a
detailed error analysis see Wilkinson and Reinsch [1] pp 222 and
235.

8. Further Comments

                                                                  3
The time taken by the routine is approximately proportional to n

9. Example

To calculate all the eigenvalues of the real symmetric matrix:

                    ( 0.5  0.0  2.3 -2.6)
                    ( 0.0  0.5 -1.4 -0.7)
                    ( 2.3 -1.4  0.5  0.0).
                    (-2.6 -0.7  0.0  0.5)

The example program is not reproduced here. The source code for
```

```
all example programs is distributed with the NAG Foundation
Library software and should be available on-line.

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

F02 -- Eigenvalue and Eigenvectors                          F02ABF
        F02ABF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for
your implementation to check implementation-dependent details.
The symbol (*) after a NAG routine name denotes a routine that is
not included in the Foundation Library.

1. Purpose

F02ABF calculates all the eigenvalues and eigenvectors of a real
symmetric matrix.

2. Specification

       SUBROUTINE F02ABF (A, IA, N, R, V, IV, E, IFAIL)
       INTEGER          IA, N, IV, IFAIL
       DOUBLE PRECISION A(IA,N), R(N), V(IV,N), E(N)

3. Description

This routine reduces the real symmetric matrix A to a real
symmetric tridiagonal matrix by Householder's method. The
eigenvalues and eigenvectors are calculated using the QL
algorithm.

4. References

[1]   Wilkinson J H and Reinsch C (1971) Handbook for Automatic
      Computation II, Linear Algebra. Springer-Verlag.

5. Parameters

 1:  A(IA,N) -- DOUBLE PRECISION array                       Input
     On entry: the lower triangle of the n by n symmetric matrix
     A. The elements of the array above the diagonal need not be
     set. See also Section 8.

 2:  IA -- INTEGER                                           Input
     On entry:
     the first dimension of the array A as declared in the
```

```
     (sub)program from which F02ABF is called.
     Constraint: IA >= N.

 3:  N -- INTEGER                                              Input
     On entry: n, the order of the matrix A.

 4:  R(N) -- DOUBLE PRECISION array                            Output
     On exit: the eigenvalues in ascending order.

 5:  V(IV,N) -- DOUBLE PRECISION array                         Output
     On exit: the normalised eigenvectors, stored by columns;
     the ith column corresponds to the ith eigenvalue. The
     eigenvectors are normalised so that the sum of squares of
     the elements is equal to 1.

 6:  IV -- INTEGER                                             Input
     On entry:
     the first dimension of the array V as declared in the
     (sub)program from which F02ABF is called.
     Constraint: IV >= N.

 7:  E(N) -- DOUBLE PRECISION array                          Workspace

 8:  IFAIL -- INTEGER                                     Input/Output
     On entry: IFAIL must be set to 0, -1 or 1. For users not
     familiar with this parameter (described in the Essential
     Introduction) the recommended value is 0.

     On exit: IFAIL = 0 unless the routine detects an error (see
     Section 6).

6. Error Indicators and Warnings

Errors detected by the routine:

IFAIL= 1
     Failure in F02AMF(*) indicating that more than 30*N
     iterations are required to isolate all the eigenvalues.

7. Accuracy

The eigenvectors are always accurately orthogonal but the
accuracy of the individual eigenvectors is dependent on their
inherent sensitivity to changes in the original matrix. For a
detailed error analysis see Wilkinson and Reinsch [1] pp 222 and
235.
```

```
8. Further Comments

                                                                      3
The time taken by the routine is approximately proportional to n

Unless otherwise stated in the Users' Note for your
implementation, the routine may be called with the same actual
array supplied for parameters A and V, in which case the
eigenvectors will overwrite the original matrix. However this is
not standard Fortran 77, and may not work on all systems.

9. Example

To calculate all the eigenvalues and eigenvectors of the real
symmetric matrix:

                    ( 0.5  0.0  2.3 -2.6)
                    ( 0.0  0.5 -1.4 -0.7)
                    ( 2.3 -1.4  0.5  0.0).
                    (-2.6 -0.7  0.0  0.5)

The example program is not reproduced here. The source code for
all example programs is distributed with the NAG Foundation
Library software and should be available on-line.

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

F02 -- Eigenvalue and Eigenvectors                          F02ADF
        F02ADF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for
your implementation to check implementation-dependent details.
The symbol (*) after a NAG routine name denotes a routine that is
not included in the Foundation Library.

1. Purpose

F02ADF calculates all the eigenvalues of  Ax=(lambda)Bx,  where A
is a real symmetric matrix and B is a real symmetric positive-
definite matrix.

2. Specification

       SUBROUTINE F02ADF (A, IA, B, IB, N, R, DE, IFAIL)
       INTEGER          IA, IB, N, IFAIL
```

```
      DOUBLE PRECISION A(IA,N), B(IB,N), R(N), DE(N)

3. Description

The problem is reduced to the standard symmetric eigenproblem
using Cholesky's method to decompose B into triangular matrices,
    T
B=LL , where L is lower triangular. Then Ax=(lambda)Bx implies
  -1  -T   T             T
(L  AL  )(L x)=(lambda)(L x); hence the eigenvalues of
Ax=(lambda)Bx are those of Py=(lambda)y where P is the symmetric
       -1  -T
matrix L  AL  . Householder's method is used to tridiagonalise
the matrix P and the eigenvalues are then found using the QL
algorithm.

4. References

[1]   Wilkinson J H and Reinsch C (1971) Handbook for Automatic
      Computation II, Linear Algebra. Springer-Verlag.

5. Parameters

 1:  A(IA,N) -- DOUBLE PRECISION array              Input/Output
     On entry: the upper triangle of the n by n symmetric matrix
     A. The elements of the array below the diagonal need not be
     set. On exit: the lower triangle of the array is
     overwritten. The rest of the array is unchanged.

 2:  IA -- INTEGER                                        Input
     On entry:
     the first dimension of the array A as declared in the
     (sub)program from which F02ADF is called.
     Constraint: IA >= N.

 3:  B(IB,N) -- DOUBLE PRECISION array              Input/Output
     On entry: the upper triangle of the n by n symmetric
     positive-definite matrix B. The elements of the array below
     the diagonal need not be set. On exit: the elements below
     the diagonal are overwritten. The rest of the array is
     unchanged.

 4:  IB -- INTEGER                                        Input
     On entry:
     the first dimension of the array B as declared in the
     (sub)program from which F02ADF is called.
```

```
     Constraint: IB >= N.

 5:  N -- INTEGER                                              Input
     On entry: n, the order of the matrices A and B.

 6:  R(N) -- DOUBLE PRECISION array                            Output
     On exit: the eigenvalues in ascending order.

 7:  DE(N) -- DOUBLE PRECISION array                        Workspace

 8:  IFAIL -- INTEGER                                     Input/Output
     On entry: IFAIL must be set to 0, -1 or 1. For users not
     familiar with this parameter (described in the Essential
     Introduction) the recommended value is 0.

     On exit: IFAIL = 0 unless the routine detects an error (see
     Section 6).

6. Error Indicators and Warnings

Errors detected by the routine:

IFAIL= 1
     Failure in F01AEF(*); the matrix B is not positive-definite
     possibly due to rounding errors.

IFAIL= 2
     Failure in F02AVF(*), more than 30*N iterations are required
     to isolate all the eigenvalues.

7. Accuracy

In general this routine is very accurate. However, if B is ill-
conditioned with respect to inversion, the eigenvalues could be
inaccurately determined. For a detailed error analysis see
Wilkinson and Reinsch [1] pp 310, 222 and 235.

8. Further Comments

                                                                  3
The time taken by the routine is approximately proportional to n

9. Example

To calculate all the eigenvalues of the general symmetric
eigenproblem Ax=(lambda) Bx where A is the symmetric matrix:
```

```
                          (0.5  1.5  6.6   4.8)
                          (1.5  6.5 16.2   8.6)
                          (6.6 16.2 37.6   9.8)
                          (4.8  8.6  9.8 -17.1)
```

and B is the symmetric positive-definite matrix:

```
                             (1  3  4  1)
                             (3 13 16 11)
                             (4 16 24 18).
                             (1 11 18 27)
```

The example program is not reproduced here. The source code for all example programs is distributed with the NAG Foundation Library software and should be available on-line.

%%%

F02 -- Eigenvalue and Eigenvectors F02AEF
F02AEF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for your implementation to check implementation-dependent details. The symbol (*) after a NAG routine name denotes a routine that is not included in the Foundation Library.

1. Purpose

F02AEF calculates all the eigenvalues and eigenvectors of Ax=(lambda)Bx, where A is a real symmetric matrix and B is a real symmetric positive-definite matrix.

2. Specification

```
      SUBROUTINE F02AEF (A, IA, B, IB, N, R, V, IV, DL, E, IFAIL)
      INTEGER          IA, IB, N, IV, IFAIL
      DOUBLE PRECISION A(IA,N), B(IB,N), R(N), V(IV,N), DL(N), E
     1                 (N)
```

3. Description

The problem is reduced to the standard symmetric eigenproblem using Cholesky's method to decompose B into triangular matrices $B=LL^T$, where L is lower triangular. Then Ax=(lambda)Bx implies

-1 -T T T
(L AL)(L x)=(lambda)(L x); hence the eigenvalues of
Ax=(lambda)Bx are those of Py=(lambda)y, where P is the symmetric
-1 -T
matrix L AL . Householder's method is used to tridiagonalise
the matrix P and the eigenvalues are found using the QL
algorithm. An eigenvector z of the derived problem is related to
T
an eigenvector x of the original problem by z=L x. The
eigenvectors z are determined using the QL algorithm and are
T
normalised so that z z=1; the eigenvectors of the original
T
problem are then determined by solving L x=z, and are normalised
T
so that x Bx=1.

4. References

[1] Wilkinson J H and Reinsch C (1971) Handbook for Automatic Computation II, Linear Algebra. Springer-Verlag.

5. Parameters

1: A(IA,N) -- DOUBLE PRECISION array Input/Output

On entry: the upper triangle of the n by n symmetric matrix A. The elements of the array below the diagonal need not be set. On exit: the lower triangle of the array is overwritten. The rest of the array is unchanged. See also Section 8.

2: IA -- INTEGER Input
On entry:
the first dimension of the array A as declared in the (sub)program from which F02AEF is called.
Constraint: IA >= N.

3: B(IB,N) -- DOUBLE PRECISION array Input/Output
On entry: the upper triangle of the n by n symmetric positive-definite matrix B. The elements of the array below the diagonal need not be set. On exit: the elements below the diagonal are overwritten. The rest of the array is unchanged.

4: IB -- INTEGER Input

```
     On entry:
     the first dimension of the array B as declared in the
     (sub)program from which F02AEF is called.
     Constraint: IB >= N.

 5:  N -- INTEGER                                              Input
     On entry: n, the order of the matrices A and B.

 6:  R(N) -- DOUBLE PRECISION array                           Output
     On exit: the eigenvalues in ascending order.

 7:  V(IV,N) -- DOUBLE PRECISION array                        Output
     On exit: the normalised eigenvectors, stored by columns;
     the ith column corresponds to the ith eigenvalue. The
                                          T
     eigenvectors x are normalised so that x Bx=1. See also
     Section 8.

 8:  IV -- INTEGER                                             Input
     On entry:
     the first dimension of the array V as declared in the
     (sub)program from which F02AEF is called.
     Constraint: IV >= N.

 9:  DL(N) -- DOUBLE PRECISION array                        Workspace

10:  E(N) -- DOUBLE PRECISION array                         Workspace

11:  IFAIL -- INTEGER                                     Input/Output
     On entry: IFAIL must be set to 0, -1 or 1. For users not
     familiar with this parameter (described in the Essential
     Introduction) the recommended value is 0.
     On exit: IFAIL = 0 unless the routine detects an error (see
     Section 6).

6. Error Indicators and Warnings

Errors detected by the routine:

IFAIL= 1
     Failure in F01AEF(*); the matrix B is not positive-definite,
     possibly due to rounding errors.

IFAIL= 2
     Failure in F02AMF(*); more than 30*N iterations are required
     to isolate all the eigenvalues.
```

7. Accuracy

In general this routine is very accurate. However, if B is ill-conditioned with respect to inversion, the eigenvectors could be inaccurately determined. For a detailed error analysis see Wilkinson and Reinsch [1] pp 310, 222 and 235.

8. Further Comments

The time taken by the routine is approximately proportional to n^3

Unless otherwise stated in the Users' Note for your implementation, the routine may be called with the same actual array supplied for parameters A and V, in which case the eigenvectors will overwrite the original matrix A. However this is not standard Fortran 77, and may not work on all systems.

9. Example

To calculate all the eigenvalues and eigenvectors of the general symmetric eigenproblem Ax=(lambda) Bx where A is the symmetric matrix:

```
(0.5  1.5  6.6   4.8)
(1.5  6.5 16.2   8.6)
(6.6 16.2 37.6   9.8)
(4.8  8.6  9.8 -17.1)
```

and B is the symmetric positive-definite matrix:

```
(1  3  4  1)
(3 13 16 11)
(4 16 24 18).
(1 11 18 27)
```

The example program is not reproduced here. The source code for all example programs is distributed with the NAG Foundation Library software and should be available on-line.

%%%

F02 -- Eigenvalue and Eigenvectors F02AFF

F02AFF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for your implementation to check implementation-dependent details. The symbol (*) after a NAG routine name denotes a routine that is not included in the Foundation Library.

1. Purpose

F02AFF calculates all the eigenvalues of a real unsymmetric matrix.

2. Specification

```
SUBROUTINE F02AFF (A, IA, N, RR, RI, INTGER, IFAIL)
INTEGER          IA, N, INTGER(N), IFAIL
DOUBLE PRECISION A(IA,N), RR(N), RI(N)
```

3. Description

The matrix A is first balanced and then reduced to upper Hessenberg form using stabilised elementary similarity transformations. The eigenvalues are then found using the QR algorithm for real Hessenberg matrices.

4. References

[1] Wilkinson J H and Reinsch C (1971) Handbook for Automatic Computation II, Linear Algebra. Springer-Verlag.

5. Parameters

1: A(IA,N) -- DOUBLE PRECISION array Input/Output
On entry: the n by n matrix A. On exit: the array is overwritten.

2: IA -- INTEGER Input
On entry:
the dimension of the array A as declared in the (sub)program from which F02AFF is called.
Constraint: IA >= N.

3: N -- INTEGER Input
On entry: n, the order of the matrix A.

4: RR(N) -- DOUBLE PRECISION array Output
On exit: the real parts of the eigenvalues.

5: RI(N) -- DOUBLE PRECISION array Output
On exit: the imaginary parts of the eigenvalues.

6: INTGER(N) -- INTEGER array Output
On exit: INTGER(i) contains the number of iterations used to find the ith eigenvalue. If INTGER(i) is negative, the i th eigenvalue is the second of a pair found simultaneously.

Note that the eigenvalues are found in reverse order, starting with the nth.

7: IFAIL -- INTEGER Input/Output
On entry: IFAIL must be set to 0, -1 or 1. For users not familiar with this parameter (described in the Essential Introduction) the recommended value is 0.

On exit: IFAIL = 0 unless the routine detects an error (see Section 6).

6. Error Indicators and Warnings

Errors detected by the routine:

IFAIL= 1
More than 30*N iterations are required to isolate all the eigenvalues.

7. Accuracy

The accuracy of the results depends on the original matrix and the multiplicity of the roots. For a detailed error analysis see Wilkinson and Reinsch [1] pp 352 and 367.

8. Further Comments

The time taken by the routine is approximately proportional to n^3

9. Example

To calculate all the eigenvalues of the real matrix:

$$\begin{pmatrix} 1.5 & 0.1 & 4.5 & -1.5 \\ -22.5 & 3.5 & 12.5 & -2.5 \\ -2.5 & 0.3 & 4.5 & -2.5 \\ -2.5 & 0.1 & 4.5 & 2.5 \end{pmatrix}.$$

```
The example program is not reproduced here. The source code for
all example programs is distributed with the NAG Foundation
Library software and should be available on-line.
```

%%%

```
F02 -- Eigenvalue and Eigenvectors                          F02AGF
        F02AGF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for
your implementation to check implementation-dependent details.
The symbol (*) after a NAG routine name denotes a routine that is
not included in the Foundation Library.

1. Purpose

F02AGF calculates all the eigenvalues and eigenvectors of a real
unsymmetric matrix.

2. Specification

       SUBROUTINE F02AGF (A, IA, N, RR, RI, VR, IVR, VI, IVI,
      1                   INTGER, IFAIL)
       INTEGER          IA, N, IVR, IVI, INTGER(N), IFAIL
       DOUBLE PRECISION A(IA,N), RR(N), RI(N), VR(IVR,N), VI
      1                 (IVI,N)

3. Description

The matrix A is first balanced and then reduced to upper
Hessenberg form using real stabilised elementary similarity
transformations. The eigenvalues and eigenvectors of the
Hessenberg matrix are calculated using the QR algorithm. The
eigenvectors of the Hessenberg matrix are back-transformed to
give the eigenvectors of the original matrix A.

4. References

[1]   Wilkinson J H and Reinsch C (1971) Handbook for Automatic
      Computation II, Linear Algebra. Springer-Verlag.

5. Parameters

 1:  A(IA,N) -- DOUBLE PRECISION array              Input/Output
     On entry: the n by n matrix A. On exit: the array is
```

overwritten.

2: IA -- INTEGER Input
On entry:
the first dimension of the array A as declared in the (sub)program from which F02AGF is called.
Constraint: IA >= N.

3: N -- INTEGER Input
On entry: n, the order of the matrix A.

4: RR(N) -- DOUBLE PRECISION array Output
On exit: the real parts of the eigenvalues.

5: RI(N) -- DOUBLE PRECISION array Output
On exit: the imaginary parts of the eigenvalues.

6: VR(IVR,N) -- DOUBLE PRECISION array Output
On exit: the real parts of the eigenvectors, stored by columns. The ith column corresponds to the ith eigenvalue. The eigenvectors are normalised so that the sum of the squares of the moduli of the elements is equal to 1 and the element of largest modulus is real. This ensures that real eigenvalues have real eigenvectors.

7: IVR -- INTEGER Input
On entry:
the first dimension of the array VR as declared in the (sub)program from which F02AGF is called.
Constraint: IVR >= N.

8: VI(IVI,N) -- DOUBLE PRECISION array Output
On exit: the imaginary parts of the eigenvectors, stored by columns. The ith column corresponds to the ith eigenvalue.

9: IVI -- INTEGER Input
On entry:
the first dimension of the array VI as declared in the (sub)program from which F02AGF is called.
Constraint: IVI >= N.

10: INTGER(N) -- INTEGER array Output
On exit: INTGER(i) contains the number of iterations used to find the ith eigenvalue. If INTGER(i) is negative, the i th eigenvalue is the second of a pair found simultaneously.

```
     Note that the eigenvalues are found in reverse order,
     starting with the nth.

11:  IFAIL -- INTEGER                                  Input/Output
     On entry: IFAIL must be set to 0, -1 or 1. For users not
     familiar with this parameter (described in the Essential
     Introduction) the recommended value is 0.

     On exit: IFAIL = 0 unless the routine detects an error (see
     Section 6).

6. Error Indicators and Warnings

Errors detected by the routine:

IFAIL= 1
     More than 30*N iterations are required to isolate all the
     eigenvalues.

7. Accuracy

The accuracy of the results depends on the original matrix and
the multiplicity of the roots. For a detailed error analysis see
Wilkinson and Reinsch [1] pp 352 and 390.

8. Further Comments

                                                                  3
The time taken by the routine is approximately proportional to n

9. Example

To calculate all the eigenvalues and eigenvectors of the real
matrix:

                    (  1.5 0.1  4.5 -1.5)
                    (-22.5 3.5 12.5 -2.5)
                    ( -2.5 0.3  4.5 -2.5).
                    ( -2.5 0.1  4.5  2.5)

The example program is not reproduced here. The source code for
all example programs is distributed with the NAG Foundation
Library software and should be available on-line.
```

%%%

F02 -- Eigenvalue and Eigenvectors F02AJF
F02AJF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for your implementation to check implementation-dependent details. The symbol (*) after a NAG routine name denotes a routine that is not included in the Foundation Library.

1. Purpose

F02AJF calculates all the eigenvalues of a complex matrix.

2. Specification

```
      SUBROUTINE F02AJF (AR, IAR, AI, IAI, N, RR, RI, INTGER,
     1                   IFAIL)
      INTEGER           IAR, IAI, N, INTGER(N), IFAIL
      DOUBLE PRECISION AR(IAR,N), AI(IAI,N), RR(N), RI(N)
```

3. Description

The complex matrix A is first balanced and then reduced to upper Hessenberg form using stabilised elementary similarity transformations. The eigenvalues are then found using the modified LR algorithm for complex Hessenberg matrices.

4. References

[1] Wilkinson J H and Reinsch C (1971) Handbook for Automatic Computation II, Linear Algebra. Springer-Verlag.

5. Parameters

1: AR(IAR,N) -- DOUBLE PRECISION array Input/Output
On entry: the real parts of the elements of the n by n complex matrix A. On exit: the array is overwritten.

2: IAR -- INTEGER Input
On entry:
the first dimension of the array AR as declared in the (sub)program from which F02AJF is called.
Constraint: IAR >= N.

3: AI(IAI,N) -- DOUBLE PRECISION array Input/Output
On entry: the imaginary parts of the elements of the n by n complex matrix A. On exit: the array is overwritten.

```
 4:  IAI -- INTEGER                                              Input
     On entry:
     the first dimension of the array AI as declared in the
     (sub)program from which F02AJF is called.
     Constraint: IAI >= N.

 5:  N -- INTEGER                                                Input
     On entry: n, the order of the matrix A.

 6:  RR(N) -- DOUBLE PRECISION array                             Output
     On exit: the real parts of the eigenvalues.

 7:  RI(N) -- DOUBLE PRECISION array                             Output
     On exit: the imaginary parts of the eigenvalues.

 8:  INTGER(N) -- INTEGER array                                  Workspace

 9:  IFAIL -- INTEGER                                        Input/Output
     On entry: IFAIL must be set to 0, -1 or 1. For users not
     familiar with this parameter (described in the Essential
     Introduction) the recommended value is 0.

     On exit: IFAIL = 0 unless the routine detects an error (see
     Section 6).

6. Error Indicators and Warnings

Errors detected by the routine:

IFAIL= 1
     More than 30*N iterations are required to isolate all the
     eigenvalues.

7. Accuracy

The accuracy of the results depends on the original matrix and
the multiplicity of the roots. For a detailed error analysis see
Wilkinson and Reinsch [1] pp 352 and 401.

8. Further Comments

                                                                  3
The time taken by the routine is approximately proportional to n

9. Example
```

To calculate all the eigenvalues of the complex matrix:

```
(-21.0-5.0i      24.60i 13.6+10.2i     4.0i)
(     22.5i 26.00-5.00i  7.5-10.0i 2.5     )
( -2.0+1.5i  1.68+2.24i  4.5-5.0i  1.5+2.0i).
(     -2.5i -2.60       -2.7+3.6i  2.5-5.0i)
```

The example program is not reproduced here. The source code for all example programs is distributed with the NAG Foundation Library software and should be available on-line.

%%%

F02 -- Eigenvalue and Eigenvectors F02AKF
F02AKF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for your implementation to check implementation-dependent details. The symbol (*) after a NAG routine name denotes a routine that is not included in the Foundation Library.

1. Purpose

F02AKF calculates all the eigenvalues and eigenvectors of a complex matrix.

2. Specification

```
 SUBROUTINE F02AKF (AR, IAR, AI, IAI, N, RR, RI, VR, IVR,
1                   VI, IVI, INTGER, IFAIL)
 INTEGER          IAR, IAI, N, IVR, IVI, INTGER(N), IFAIL
 DOUBLE PRECISION AR(IAR,N), AI(IAI,N), RR(N), RI(N), VR
1                 (IVR,N), VI(IVI,N)
```

3. Description

The complex matrix A is first balanced and then reduced to upper Hessenberg form by stabilised elementary similarity transformations. The eigenvalues and eigenvectors of the Hessenberg matrix are calculated using the LR algorithm. The eigenvectors of the Hessenberg matrix are back-transformed to give the eigenvectors of the original matrix.

4. References

[1] Wilkinson J H and Reinsch C (1971) Handbook for Automatic Computation II, Linear Algebra. Springer-Verlag.

5. Parameters

1: AR(IAR,N) -- DOUBLE PRECISION array Input/Output
On entry: the real parts of the elements of the n by n complex matrix A. On exit: the array is overwritten.

2: IAR -- INTEGER Input
On entry:
the first dimension of the array AR as declared in the (sub)program from which F02AKF is called.
Constraint: IAR >= N.

3: AI(IAI,N) -- DOUBLE PRECISION array Input/Output
On entry: the imaginary parts of the elements of the n by n complex matrix A. On exit: the array is overwritten.

4: IAI -- INTEGER Input
On entry:
the first dimension of the array AI as declared in the (sub)program from which F02AKF is called.
Constraint: IAI >= N.

5: N -- INTEGER Input
On entry: n, the order of the matrix A.

6: RR(N) -- DOUBLE PRECISION array Output
On exit: the real parts of the eigenvalues.

7: RI(N) -- DOUBLE PRECISION array Output
On exit: the imaginary parts of the eigenvalues.

8: VR(IVR,N) -- DOUBLE PRECISION array Output
On exit: the real parts of the eigenvectors, stored by columns. The ith column corresponds to the ith eigenvalue. The eigenvectors are normalised so that the sum of squares of the moduli of the elements is equal to 1 and the element of largest modulus is real.

9: IVR -- INTEGER Input
On entry:
the first dimension of the array VR as declared in the (sub)program from which F02AKF is called.
Constraint: IVR >= N.

10: VI(IVI,N) -- DOUBLE PRECISION array Output
On exit: the imaginary parts of the eigenvectors, stored by columns. The ith column corresponds to the ith eigenvalue.

11: IVI -- INTEGER Input
On entry:
the first dimension of the array VI as declared in the (sub)program from which F02AKF is called.
Constraint: IVI >= N.

12: INTGER(N) -- INTEGER array Workspace

13: IFAIL -- INTEGER Input/Output
On entry: IFAIL must be set to 0, -1 or 1. For users not familiar with this parameter (described in the Essential Introduction) the recommended value is 0.

On exit: IFAIL = 0 unless the routine detects an error (see Section 6).

6. Error Indicators and Warnings

Errors detected by the routine:

IFAIL= 1
More than 30*N iterations are required to isolate all the eigenvalues.

7. Accuracy

The accuracy of the results depends on the conditioning of the original matrix and the multiplicity of the roots. For a detailed error analysis see Wilkinson and Reinsch [1] pp 352 and 390.

8. Further Comments

The time taken by the routine is approximately proportional to n^3

9. Example

To calculate all the eigenvalues and eigenvectors of the complex matrix:

(-21.0-5.0i 24.60i 13.6+10.2i 4.0i)

```
(     22.5i 26.00-5.00i  7.5-10.0i 2.5     )
( -2.0+1.5i  1.68+2.24i  4.5-5.0i  1.5+2.0i).
(     -2.5i -2.60       -2.7+3.6i  2.5-5.0i)
```

The example program is not reproduced here. The source code for all example programs is distributed with the NAG Foundation Library software and should be available on-line.

%%%

F02 -- Eigenvalue and Eigenvectors F02AWF
F02AWF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for your implementation to check implementation-dependent details. The symbol (*) after a NAG routine name denotes a routine that is not included in the Foundation Library.

1. Purpose

F02AWF calculates all the eigenvalues of a complex Hermitian matrix.

2. Specification

```
 SUBROUTINE F02AWF (AR, IAR, AI, IAI, N, R, WK1, WK2, WK3,
1                   IFAIL)
 INTEGER          IAR, IAI, N, IFAIL
 DOUBLE PRECISION AR(IAR,N), AI(IAI,N), R(N), WK1(N),
1                 WK2(N), WK3(N)
```

3. Description

The complex Hermitian matrix A is first reduced to a real tridiagonal matrix by n-2 unitary transformations, and a subsequent diagonal transformation. The eigenvalues are then derived using the QL algorithm, an adaptation of the QR algorithm.

4. References

[1] Peters G (1967) NPL Algorithms Library. Document No. F1/04/A.

[2] Wilkinson J H and Reinsch C (1971) Handbook for Automatic Computation II, Linear Algebra. Springer-Verlag.

5. Parameters

1: AR(IAR,N) -- DOUBLE PRECISION array Input/Output
On entry: the real parts of the elements of the lower triangle of the n by n complex Hermitian matrix A. Elements of the array above the diagonal need not be set. On exit: the array is overwritten.

2: IAR -- INTEGER Input
On entry:
the first dimension of the array AR as declared in the (sub)program from which F02AWF is called.
Constraint: IAR >= N.

3: AI(IAI,N) -- DOUBLE PRECISION array Input/Output
On entry: the imaginary parts of the elements of the lower triangle of the n by n complex Hermitian matrix A. Elements of the array above the diagonal need not be set. On exit: the array is overwritten.

4: IAI -- INTEGER Input
On entry:
the first dimension of the array AI as declared in the (sub)program from which F02AWF is called.
Constraint: IAI >= N.

5: N -- INTEGER Input
On entry: n, the order of the complex Hermitian matrix, A.

6: R(N) -- DOUBLE PRECISION array Output
On exit: the eigenvalues in ascending order.

7: WK1(N) -- DOUBLE PRECISION array Workspace

8: WK2(N) -- DOUBLE PRECISION array Workspace

9: WK3(N) -- DOUBLE PRECISION array Workspace

10: IFAIL -- INTEGER Input/Output
On entry: IFAIL must be set to 0, -1 or 1. For users not familiar with this parameter (described in the Essential Introduction) the recommended value is 0.

On exit: IFAIL = 0 unless the routine detects an error (see Section 6).

```
6. Error Indicators and Warnings

Errors detected by the routine:

IFAIL= 1
     More than 30*N iterations are required to isolate all the
     eigenvalues.

7. Accuracy

For a detailed error analysis see Peters [1] page 3 and Wilkinson
and Reinsch [2] page 235.

8. Further Comments

                                                                  3
The time taken by the routine is approximately proportional to n

9. Example

To calculate all the eigenvalues of the complex Hermitian matrix:

            (0.50        0.00       1.84+1.38i  2.08-1.56i)
            (0.00        0.50       1.12+0.84i -0.56+0.42i)
            (1.84-1.38i  1.12-0.84i 0.50        0.00      ).
            (2.08+1.56i -0.56-0.42i 0.00        0.50      )

The example program is not reproduced here. The source code for
all example programs is distributed with the NAG Foundation
Library software and should be available on-line.
```

%%%

```
F02 -- Eigenvalue and Eigenvectors                        F02AXF
        F02AXF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for
your implementation to check implementation-dependent details.
The symbol (*) after a NAG routine name denotes a routine that is
not included in the Foundation Library.

1. Purpose

F02AXF calculates all the eigenvalues and eigenvectors of a
complex Hermitian matrix.
```

2. Specification

```
      SUBROUTINE F02AXF (AR, IAR, AI, IAI, N, R, VR, IVR, VI,
     1                   IVI, WK1, WK2, WK3, IFAIL)
      INTEGER           IAR, IAI, N, IVR, IVI, IFAIL
      DOUBLE PRECISION AR(IAR,N), AI(IAI,N), R(N), VR(IVR,N), VI
     1                 (IVI,N), WK1(N), WK2(N), WK3(N)
```

3. Description

The complex Hermitian matrix is first reduced to a real tridiagonal matrix by n-2 unitary transformations and a subsequent diagonal transformation. The eigenvalues and eigenvectors are then derived using the QL algorithm, an adaptation of the QR algorithm.

4. References

[1] Peters G (1967) NPL Algorithms Library. Document No. F2/03/A.

[2] Peters G (1967) NPL Algorithms Library. Document No. F1/04/A.

5. Parameters

1: AR(IAR,N) -- DOUBLE PRECISION array Input
On entry: the real parts of the elements of the lower triangle of the n by n complex Hermitian matrix A. Elements of the array above the diagonal need not be set. See also Section 8.

2: IAR -- INTEGER Input
On entry:
the first dimension of the array AR as declared in the (sub)program from which F02AXF is called.
Constraint: IAR >= N.

3: AI(IAI,N) -- DOUBLE PRECISION array Input
On entry: the imaginary parts of the elements of the lower triangle of the n by n complex Hermitian matrix A. Elements of the array above the diagonal need not be set. See also Section 8.

4: IAI -- INTEGER Input

```
        On entry:
        the first dimension of the array AI as declared in the
        (sub)program from which F02AXF is called.
        Constraint: IAI >= N.

 5:  N -- INTEGER                                              Input
        On entry: n, the order of the matrix, A.

 6:  R(N) -- DOUBLE PRECISION array                            Output
        On exit: the eigenvalues in ascending order.

 7:  VR(IVR,N) -- DOUBLE PRECISION array                       Output
        On exit: the real parts of the eigenvectors, stored by
        columns. The ith column corresponds to the ith eigenvector.
        The eigenvectors are normalised so that the sum of the
        squares of the moduli of the elements is equal to 1 and the
        element of largest modulus is real. See also Section 8.

 8:  IVR -- INTEGER                                            Input
        On entry:
        the first dimension of the array VR as declared in the
        (sub)program from which F02AXF is called.
        Constraint: IVR >= N.

 9:  VI(IVI,N) -- DOUBLE PRECISION array                       Output
        On exit: the imaginary parts of the eigenvectors, stored by
        columns. The ith column corresponds to the ith eigenvector.
        See also Section 8.

10:  IVI -- INTEGER                                            Input
        On entry:
        the first dimension of the array VI as declared in the
        (sub)program from which F02AXF is called.
        Constraint: IVI >= N.

11:  WK1(N) -- DOUBLE PRECISION array                       Workspace

12:  WK2(N) -- DOUBLE PRECISION array                       Workspace

13:  WK3(N) -- DOUBLE PRECISION array                       Workspace

14:  IFAIL -- INTEGER                                    Input/Output
        On entry: IFAIL must be set to 0, -1 or 1. For users not
        familiar with this parameter (described in the Essential
        Introduction) the recommended value is 0.
```

```
     On exit: IFAIL = 0 unless the routine detects an error (see
     Section 6).

6. Error Indicators and Warnings

Errors detected by the routine:

IFAIL= 1
     More than 30*N iterations are required to isolate all the
     eigenvalues.

IFAIL= 2
     The diagonal elements of AI are not all zero, i.e., the
     complex matrix is not Hermitian.

7. Accuracy

The eigenvectors are always accurately orthogonal but the
accuracy of the individual eigenvalues and eigenvectors is
dependent on their inherent sensitivity to small changes in the
original matrix. For a detailed error analysis see Peters [1]
page 3 and [2] page 3.

8. Further Comments

                                                                  3
The time taken by the routine is approximately proportional to n

Unless otherwise stated in the implementation document, the
routine may be called with the same actual array supplied for
parameters AR and VR, and for AI and VI, in which case the
eigenvectors will overwrite the original matrix A. However this
is not standard Fortran 77, and may not work on all systems.

9. Example

To calculate the eigenvalues and eigenvectors of the complex
Hermitian matrix:

            (0.50        0.00       1.84+1.38i  2.08-1.56i)
            (0.00        0.50       1.12+0.84i -0.56+0.42i)
            (1.84-1.38i  1.12-0.84i 0.50        0.00      ).
            (2.08+1.56i -0.56-0.42i 0.00        0.50      )

The example program is not reproduced here. The source code for
all example programs is distributed with the NAG Foundation
```

Library software and should be available on-line.

%%%

F02 -- Eigenvalue and Eigenvectors F02BBF
F02BBF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for your implementation to check implementation-dependent details. The symbol (*) after a NAG routine name denotes a routine that is not included in the Foundation Library.

1. Purpose

F02BBF calculates selected eigenvalues and eigenvectors of a real symmetric matrix by reduction to tridiagonal form, bisection and inverse iteration, where the selected eigenvalues lie within a given interval.

2. Specification

```
 SUBROUTINE F02BBF (A, IA, N, ALB, UB, M, MM, R, V, IV, D,
1                   E, E2, X, G, C, ICOUNT, IFAIL)
 INTEGER          IA, N, M, MM, IV, ICOUNT(M), IFAIL
 DOUBLE PRECISION A(IA,N), ALB, UB, R(M), V(IV,M), D(N), E
1                 (N), E2(N), X(N,7), G(N)
 LOGICAL          C(N)
```

3. Description

The real symmetric matrix A is reduced to a symmetric tridiagonal matrix T by Householder's method. The eigenvalues which lie within a given interval [l,u], are calculated by the method of bisection. The corresponding eigenvectors of T are calculated by inverse iteration. A back-transformation is then performed to obtain the eigenvectors of the original matrix A.

4. References

[1] Wilkinson J H and Reinsch C (1971) Handbook for Automatic Computation II, Linear Algebra. Springer-Verlag.

5. Parameters

1: A(IA,N) -- DOUBLE PRECISION array Input/Output
On entry: the lower triangle of the n by n symmetric matrix

```
     A. The elements of the array above the diagonal need not be
     set. On exit: the elements of A below the diagonal are
     overwritten, and the rest of the array is unchanged.

 2:  IA -- INTEGER                                              Input
     On entry:
     the first dimension of the array A as declared in the
     (sub)program from which F02BBF is called.
     Constraint: IA >= N.

 3:  N -- INTEGER                                               Input
     On entry: n, the order of the matrix A.

 4:  ALB -- DOUBLE PRECISION                                    Input

 5:  UB -- DOUBLE PRECISION                                     Input
     On entry: l and u, the lower and upper end-points of the
     interval within which eigenvalues are to be calculated.

 6:  M -- INTEGER                                               Input
     On entry: an upper bound for the number of eigenvalues
     within the interval.

 7:  MM -- INTEGER                                             Output
     On exit: the actual number of eigenvalues within the
     interval.

 8:  R(M) -- DOUBLE PRECISION array                            Output
     On exit: the eigenvalues, not necessarily in ascending
     order.

 9:  V(IV,M) -- DOUBLE PRECISION array                         Output
     On exit: the eigenvectors, stored by columns. The ith
     column corresponds to the ith eigenvalue. The eigenvectors
     are normalised so that the sum of the squares of the
     elements are equal to 1.

10:  IV -- INTEGER                                             Input
     On entry:
     the first dimension of the array V as declared in the
     (sub)program from which F02BBF is called.
     Constraint: IV >= N.

11:  D(N) -- DOUBLE PRECISION array                         Workspace

12:  E(N) -- DOUBLE PRECISION array                         Workspace
```

13: E2(N) -- DOUBLE PRECISION array Workspace

14: X(N,7) -- DOUBLE PRECISION array Workspace

15: G(N) -- DOUBLE PRECISION array Workspace

16: C(N) -- LOGICAL array Workspace

17: ICOUNT(M) -- INTEGER array Output
On exit: ICOUNT(i) contains the number of iterations for the ith eigenvalue.

18: IFAIL -- INTEGER Input/Output
On entry: IFAIL must be set to 0, -1 or 1. For users not familiar with this parameter (described in the Essential Introduction) the recommended value is 0.

On exit: IFAIL = 0 unless the routine detects an error (see Section 6).

6. Error Indicators and Warnings

Errors detected by the routine:

IFAIL= 1
M is less than the number of eigenvalues in the given interval. On exit MM contains the number of eigenvalues in the interval. Rerun with this value for M.

IFAIL= 2
More than 5 iterations are required to determine any one eigenvector.

7. Accuracy

There is no guarantee of the accuracy of the eigenvectors as the results depend on the original matrix and the multiplicity of the roots. For a detailed error analysis see Wilkinson and Reinsch [1] pp 222 and 436.

8. Further Comments

The time taken by the routine is approximately proportional to n^3

This subroutine should only be used when less than 25% of the eigenvalues and the corresponding eigenvectors are required. Also this subroutine is less efficient with matrices which have multiple eigenvalues.

9. Example

To calculate the eigenvalues lying between -2.0 and 3.0, and the corresponding eigenvectors of the real symmetric matrix:

```
( 0.5  0.0  2.3 -2.6)
( 0.0  0.5 -1.4 -0.7)
( 2.3 -1.4  0.5  0.0).
(-2.6 -0.7  0.0  0.5)
```

The example program is not reproduced here. The source code for all example programs is distributed with the NAG Foundation Library software and should be available on-line.

%%%

F02 -- Eigenvalue and Eigenvectors F02BJF
F02BJF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for your implementation to check implementation-dependent details. The symbol (*) after a NAG routine name denotes a routine that is not included in the Foundation Library.

1. Purpose

F02BJF calculates all the eigenvalues and, if required, all the eigenvectors of the generalized eigenproblem Ax=(lambda)Bx where A and B are real, square matrices, using the QZ algorithm.

2. Specification

```
      SUBROUTINE F02BJF (N, A, IA, B, IB, EPS1, ALFR, ALFI,
     1                   BETA, MATV, V, IV, ITER, IFAIL)
      INTEGER           N, IA, IB, IV, ITER(N), IFAIL
      DOUBLE PRECISION A(IA,N), B(IB,N), EPS1, ALFR(N), ALFI(N),
     1                 BETA(N), V(IV,N)
      LOGICAL           MATV
```

3. Description

All the eigenvalues and, if required, all the eigenvectors of the generalized eigenproblem Ax=(lambda)Bx where A and B are real, square matrices, are determined using the QZ algorithm. The QZ algorithm consists of 4 stages:

(a) A is reduced to upper Hessenberg form and at the same time B is reduced to upper triangular form.

(b) A is further reduced to quasi-triangular form while the triangular form of B is maintained.

(c) The quasi-triangular form of A is reduced to triangular form and the eigenvalues extracted. This routine does not actually produce the eigenvalues (lambda) , but instead
j
returns (alpha) and (beta) such that

j j
(lambda) =(alpha) /(beta) , j=1,2,...,.n
j j j
The division by (beta) becomes the responsibility of the
j
user's program, since (beta) may be zero indicating an
j
infinite eigenvalue. Pairs of complex eigenvalues occur with (alpha) /(beta) and (alpha) /(beta) complex
j j j+1 j+1

conjugates, even though (alpha) and (alpha) are not
j j+1
conjugate.

(d) If the eigenvectors are required (MATV = .TRUE.), they are obtained from the triangular matrices and then transformed back into the original co-ordinate system.

4. References

[1] Moler C B and Stewart G W (1973) An Algorithm for Generalized Matrix Eigenproblems. SIAM J. Numer. Anal. 10 241--256.

[2] Ward R C (1975) The Combination Shift QZ Algorithm. SIAM J. Numer. Anal. 12 835--853.

[3] Wilkinson J H (1979) Kronecker's Canonical Form and the QZ

```
     Algorithm. Linear Algebra and Appl. 28 285--303.

5. Parameters

 1:  N -- INTEGER                                                Input
     On entry: n, the order of the matrices A and B.

 2:  A(IA,N) -- DOUBLE PRECISION array                 Input/Output
     On entry: the n by n matrix A. On exit: the array is
     overwritten.

 3:  IA -- INTEGER                                               Input
     On entry:
     the first dimension of the array A as declared in the
     (sub)program from which F02BJF is called.
     Constraint: IA >= N.

 4:  B(IB,N) -- DOUBLE PRECISION array                 Input/Output
     On entry: the n by n matrix B. On exit: the array is
     overwritten.

 5:  IB -- INTEGER                                               Input
     On entry:
     the first dimension of the array B as declared in the
     (sub)program from which F02BJF is called.
     Constraint: IB >= N.

 6:  EPS1 -- DOUBLE PRECISION                                    Input
     On entry: the tolerance used to determine negligible
     elements. If EPS1 > 0.0, an element will be considered
     negligible if it is less than EPS1 times the norm of its
     matrix. If EPS1 <= 0.0, machine precision is used in place
     of EPS1. A positive value of EPS1 may result in faster
     execution but less accurate results.

 7:  ALFR(N) -- DOUBLE PRECISION array                          Output

 8:  ALFI(N) -- DOUBLE PRECISION array                          Output
     On exit: the real and imaginary parts of (alpha) , for
                                                             j
     j=1,2,...,n.

 9:  BETA(N) -- DOUBLE PRECISION array                          Output
     On exit: (beta) , for j=1,2,...,n.
                    j
```

```
10:  MATV -- LOGICAL                                          Input
     On entry: MATV must be set .TRUE. if the eigenvectors are
     required, otherwise .FALSE..

11:  V(IV,N) -- DOUBLE PRECISION array                        Output
     On exit: if MATV = .TRUE., then
          (i)if the jth eigenvalue is real, the jth column of V
          contains its eigenvector;

          (ii) if the jth and (j+1)th eigenvalues form a complex
          pair, the jth and (j+1)th columns of V contain the
          real and imaginary parts of the eigenvector associated
          with the first eigenvalue of the pair. The conjugate
          of this vector is the eigenvector for the conjugate
          eigenvalue.
     Each eigenvector is normalised so that the component of
     largest modulus is real and the sum of squares of the moduli
     equal one.

     If MATV = .FALSE., V is not used.

12:  IV -- INTEGER                                            Input
     On entry:
     the first dimension of the array V as declared in the
     (sub)program from which F02BJF is called.
     Constraint: IV >= N.

13:  ITER(N) -- INTEGER array                                 Output
     On exit: ITER(j) contains the number of iterations needed
     to obtain the jth eigenvalue. Note that the eigenvalues are
     obtained in reverse order, starting with the nth.

14:  IFAIL -- INTEGER                                   Input/Output
     On entry: IFAIL must be set to 0, -1 or 1. For users not
     familiar with this parameter (described in the Essential
     Introduction) the recommended value is 0.

     On exit: IFAIL = 0 unless the routine detects an error (see
     Section 6).

6. Error Indicators and Warnings

Errors detected by the routine:

IFAIL= i
     More than 30*N iterations are required to determine all the
```

diagonal 1 by 1 or 2 by 2 blocks of the quasi-triangular
form in the second step of the QZ algorithm. IFAIL is set to
the index i of the eigenvalue at which this failure occurs.
If the soft failure option is used, (alpha) and (beta) are
j j
correct for j=i+1,i+2,...,n, but V does not contain any
correct eigenvectors.

7. Accuracy

The computed eigenvalues are always exact for a problem
(A+E)x=(lambda)(B+F)x where ||E||/||A|| and ||F||/||B||
are both of the order of max(EPS1,(epsilon)), EPS1 being defined
as in Section 5 and (epsilon) being the machine precision.

Note: interpretation of results obtained with the QZ algorithm
often requires a clear understanding of the effects of small
changes in the original data. These effects are reviewed in
Wilkinson [3], in relation to the significance of small values of
(alpha) and (beta) . It should be noted that if (alpha) and
j j j
(beta) are both small for any j, it may be that no reliance can
j
be placed on any of the computed eigenvalues
(lambda) =(alpha) /(beta) . The user is recommended to study [3]
i i i
and, if in difficulty, to seek expert advice on determining the
sensitivity of the eigenvalues to perturbations in the data.

8. Further Comments

3
The time taken by the routine is approximately proportional to n
and also depends on the value chosen for parameter EPS1.

9. Example

To find all the eigenvalues and eigenvectors of Ax=(lambda) Bx
where

$$A=\begin{pmatrix} 3.9 & 12.5 & -34.5 & -0.5 \\ 4.3 & 21.5 & -47.5 & 7.5 \\ 4.3 & 21.5 & -43.5 & 3.5 \\ 4.4 & 26.0 & -46.0 & 6.0 \end{pmatrix}$$

and

```
                                  (1 2 -3 1)
                                  (1 3 -5 4)
                               B=(1 3 -4 3).
                                  (1 3 -4 4)
```

The example program is not reproduced here. The source code for all example programs is distributed with the NAG Foundation Library software and should be available on-line.

%%%

F02 -- Eigenvalue and Eigenvectors F02FJF
F02FJF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for your implementation to check implementation-dependent details. The symbol (*) after a NAG routine name denotes a routine that is not included in the Foundation Library.

1. Purpose

To find eigenvalues and eigenvectors of a real sparse symmetric or generalized symmetric eigenvalue problem.

2. Specification

```
 SUBROUTINE F02FJF (N, M, K, NOITS, TOL, DOT, IMAGE, MONIT,
1                   NOVECS, X, NRX, D, WORK, LWORK, RWORK,
2                   LRWORK, IWORK, LIWORK, IFAIL)
 INTEGER          N, M, K, NOITS, NOVECS, NRX, LWORK,
1                 LRWORK, IWORK(LIWORK), LIWORK, IFAIL
 DOUBLE PRECISION TOL, DOT, X(NRX,K), D(K), WORK(LWORK),
1                 RWORK(LRWORK)
 EXTERNAL         DOT, IMAGE, MONIT
```

3. Description

F02FJF finds the m eigenvalues of largest absolute value and the corresponding eigenvectors for the real eigenvalue problem

```
                          Cx=(lambda)x                          (1)
```

where C is an n by n matrix such that

```
                                T
```

$$BC=C\ B \tag{2}$$

for a given positive-definite matrix B. C is said to be B-symmetric. Different specifications of C allow for the solution of a variety of eigenvalue problems. For example, when

$$C=A \text{ and } B=I \text{ where } A=A^{T}$$

the routine finds the m eigenvalues of largest absolute magnitude for the standard symmetric eigenvalue problem

$$Ax=(lambda)x. \tag{3}$$

The routine is intended for the case where A is sparse.

As a second example, when

$$C=B^{-1}A$$

where

$$A=A^{T}$$

the routine finds the m eigenvalues of largest absolute magnitude for the generalized symmetric eigenvalue problem

$$Ax=(lambda)Bx. \tag{4}$$

The routine is intended for the case where A and B are sparse.

The routine does not require C explicitly, but C is specified via a user-supplied routine IMAGE which, given an n element vector z, computes the image w given by

$$w=Cz.$$

For instance, in the above example, where $C=B^{-1}A$, routine IMAGE will need to solve the positive-definite system of equations $Bw=Az$ for w.

To find the m eigenvalues of smallest absolute magnitude of (3)

$$-1$$

we can choose C=A and hence find the reciprocals of the required eigenvalues, so that IMAGE will need to solve Aw=z for A^{-1} w, and correspondingly for (4) we can choose C=A B and solve Aw=Bz for w.

A table of examples of choice of IMAGE is given in Table 3.1. It should be remembered that the routine also returns the corresponding eigenvectors and that B is positive-definite. Throughout A is assumed to be symmetric and, where necessary, non-singularity is also assumed.

Eigenvalues Required	Problem		
	Ax=(lambda)x	(B=I)Ax=(lambda)Bx	ABx=(lambda)x
Largest	Compute w=Az	Solve Bw=Az	Compute w=ABz
Smallest (Find 1/(lambda))	Solve Aw=z	Solve Aw=Bz	Solve Av=z, Bw=(nu)
Furthest from (sigma) (Find (lambda)-(sigma))	Compute w=(A-(sigma)I)z	Solve Bw=(A-(sigma)B)z	Compute w=(AB-(sigma)I)z
Closest to (sigma) (Find 1/((lambda)-(sigma)))	Solve (A-(sigma)I)w=z	Solve (A-(sigma)B)w=Bz	Solve (AB-(sigma)I)w=z

Table 3.1
The Requirement of IMAGE for Various Problems

The matrix B also need not be supplied explicitly, but is specified via a user-supplied routine DOT which, given n element vectors z and w, computes the generalized dot product $w^T Bz$.

F02FJF is based upon routine SIMITZ (see Nikolai [1]), which is itself a derivative of the Algol procedure ritzit (see Rutishauser [4]), and uses the method of simultaneous (subspace) iteration. (See Parlett [2] for description, analysis and advice on the use of the method.)

The routine performs simultaneous iteration on k>m vectors. Initial estimates to p<=k eigenvectors, corresponding to the p eigenvalues of C of largest absolute value, may be supplied by the user to F02FJF. When possible k should be chosen so that the kth eigenvalue is not too close to the m required eigenvalues, but if k is initially chosen too small then F02FJF may be re-entered, supplying approximations to the k eigenvectors found so far and with k then increased.

At each major iteration F02FJF solves an r by r (r<=k) eigenvalue sub-problem in order to obtain an approximation to the eigenvalues for which convergence has not yet occurred. This approximation is refined by Chebyshev acceleration.

4. References

[1] Nikolai P J (1979) Algorithm 538: Eigenvectors and eigenvalues of real generalized symmetric matrices by simultaneous iteration. ACM Trans. Math. Softw. 5 118--125.

[2] Parlett B N (1980) The Symmetric Eigenvalue Problem. Prentice-Hall.

[3] Rutishauser H (1969) Computational aspects of F L Bauer's simultaneous iteration method. Num. Math. 13 4--13.

[4] Rutishauser H (1970) Simultaneous iteration method for symmetric matrices. Num. Math. 16 205--223.

5. Parameters

1: N -- INTEGER Input
On entry: n, the order of the matrix C. Constraint: N >= 1.

2: M -- INTEGER Input/Output
On entry: m, the number of eigenvalues required.
,
Constraint: M >= 1. On exit: m, the number of eigenvalues actually found. It is equal to m if IFAIL = 0 on exit, and is less than m if IFAIL = 2, 3 or 4. See Section 6 and

Section 8 for further information.

3: K -- INTEGER Input
On entry: the number of simultaneous iteration vectors to be used. Too small a value of K may inhibit convergence, while a larger value of K incurs additional storage and additional work per iteration. Suggested value: K = M + 4 will often be a reasonable choice in the absence of better information. Constraint: M < K <= N.

4: NOITS -- INTEGER Input/Output
On entry: the maximum number of major iterations (eigenvalue sub-problems) to be performed. If NOITS <= 0, then the value 100 is used in place of NOITS. On exit: the number of iterations actually performed.

5: TOL -- DOUBLE PRECISION Input
On entry: a relative tolerance to be used in accepting eigenvalues and eigenvectors. If the eigenvalues are required to about t significant figures, then TOL should be set to about 10^{-t}. d_i is accepted as an eigenvalue as soon as two successive approximations to d_i differ by less than $(|\tilde{d}_i|*TOL)/10$, where $\tilde{d}_i$ is the latest aproximation to d_i. Once an eigenvalue has been accepted, then an eigenvector is accepted as soon as $(d_i f_i)/(d_i - d_k)<TOL$, where f_i is the normalised residual of the current approximation to the eigenvector (see Section 8 for further information). The values of the f_i and d_i can be printed from routine MONIT.
If TOL is supplied outside the range ((epsilon), 1.0), where (epsilon) is the machine precision, then the value (epsilon) is used in place of TOL.

6: DOT -- DOUBLE PRECISION FUNCTION, supplied by the user.
External Procedure
DOT must return the value $w^T Bz$ for given vectors w and z. For the standard eigenvalue problem, where B=I, DOT must return the dot product $w^T z$.

Its specification is:

```
 DOUBLE PRECISION FUNCTION DOT (IFLAG, N, Z, W,
1                              RWORK, LRWORK,
2                              IWORK, LIWORK)
 INTEGER          IFLAG, N, LRWORK, IWORK(LIWORK),
1                 LIWORK
 DOUBLE PRECISION Z(N), W(N), RWORK(LRWORK)
```

1: IFLAG -- INTEGER Input/Output
On entry: IFLAG is always non-negative. On exit: IFLAG may be used as a flag to indicate a failure in the computation of $w^T Bz$. If IFLAG is negative on exit from DOT, then F02FJF will exit immediately with IFAIL set to IFLAG. Note that in this case DOT must still be assigned a value.

2: N -- INTEGER Input
On entry: the number of elements in the vectors z and w and the order of the matrix B.

3: Z(N) -- DOUBLE PRECISION array Input
On entry: the vector z for which $w^T Bz$ is required.

4: W(N) -- DOUBLE PRECISION array Input
On entry: the vector w for which $w^T Bz$ is required.

5: RWORK(LRWORK) -- DOUBLE PRECISION array User Workspace

6: LRWORK -- INTEGER Input

7: IWORK(LIWORK) -- INTEGER array User Workspace

8: LIWORK -- INTEGER Input
DOT is called from F02FJF with the parameters RWORK, LRWORK, IWORK and LIWORK as supplied to F02FJF. The user is free to use the arrays RWORK and IWORK to supply information to DOT and to IMAGE as an alternative to using COMMON.

```
    DOT must be declared as EXTERNAL in the (sub)program
    from which F02FJF is called. Parameters denoted as
    Input must not be changed by this procedure.

7:  IMAGE -- SUBROUTINE, supplied by the user.
                                              External Procedure
    IMAGE must return the vector w=Cz for a given vector z.

    Its specification is:

          SUBROUTINE IMAGE (IFLAG, N, Z, W, RWORK, LRWORK,
         1                  IWORK, LIWORK)
          INTEGER           IFLAG, N, LRWORK, IWORK(LIWORK),
         1                  LIWORK
          DOUBLE PRECISION Z(N), W(N), RWORK(LRWORK)

     1: IFLAG -- INTEGER                            Input/Output
        On entry: IFLAG is always non-negative. On exit: IFLAG
        may be used as a flag to indicate a failure in the
        computation of w. If IFLAG is negative on exit from
        IMAGE, then F02FJF will exit immediately with IFAIL set
        to IFLAG.

     2: N -- INTEGER                                    Input
        On entry: n, the number of elements in the vectors w
        and z, and the order of the matrix C.

     3: Z(N) -- DOUBLE PRECISION array                  Input
        On entry: the vector z for which Cz is required.

     4: W(N) -- DOUBLE PRECISION array                 Output
        On exit: the vector w=Cz.

     5: RWORK(LRWORK) -- DOUBLE PRECISION array  User Workspace

     6: LRWORK -- INTEGER                               Input

     7: IWORK(LIWORK) -- INTEGER array           User Workspace

     8: LIWORK -- INTEGER                               Input
        IMAGE is called from F02FJF with the parameters RWORK,
        LRWORK, IWORK and LIWORK as supplied to F02FJF. The
        user is free to use the arrays RWORK and IWORK to
        supply information to IMAGE and DOT as an alternative
        to using COMMON.
    IMAGE must be declared as EXTERNAL in the (sub)program
```

from which F02FJF is called. Parameters denoted as Input must not be changed by this procedure.

8: MONIT -- SUBROUTINE, supplied by the user. External Procedure

MONIT is used to monitor the progress of F02FJF. MONIT may be the dummy subroutine F02FJZ if no monitoring is actually required. (F02FJZ is included in the NAG Foundation Library and so need not be supplied by the user. The routine name F02FJZ may be implementation dependent: see the Users' Note for your implementation for details.) MONIT is called after the solution of each eigenvalue sub-problem and also just prior to return from F02FJF. The parameters ISTATE and NEXTIT allow selective printing by MONIT.

Its specification is:

```
      SUBROUTINE MONIT (ISTATE, NEXTIT, NEVALS,
     1                  NEVECS, K, F, D)
      INTEGER           ISTATE, NEXTIT, NEVALS, NEVECS,
     1                  K
      DOUBLE PRECISION F(K), D(K)
```

1: ISTATE -- INTEGER Input

On entry: ISTATE specifies the state of F02FJF and will have values as follows:

ISTATE = 0
: No eigenvalue or eigenvector has just been accepted.

ISTATE = 1
: One or more eigenvalues have been accepted since the last call to MONIT.

ISTATE = 2
: One or more eigenvectors have been accepted since the last call to MONIT.

ISTATE = 3
: One or more eigenvalues and eigenvectors have been accepted since the last call to MONIT.

ISTATE = 4
: Return from F02FJF is about to occur.

2: NEXTIT -- INTEGER Input

On entry: the number of the next iteration.

3: NEVALS -- INTEGER Input
On entry: the number of eigenvalues accepted so far.

4: NEVECS -- INTEGER Input
On entry: the number of eigenvectors accepted so far.

5: K -- INTEGER Input
On entry: k, the number of simultaneous iteration vectors.

6: F(K) -- DOUBLE PRECISION array Input
On entry: a vector of error quantities measuring the state of convergence of the simultaneous iteration vectors. See the parameter TOL of F02FJF above and Section 8 for further details. Each element of F is initially set to the value 4.0 and an element remains at 4.0 until the corresponding vector is tested.

7: D(K) -- DOUBLE PRECISION array Input
On entry: D(i) contains the latest approximation to the absolute value of the ith eigenvalue of C.

MONIT must be declared as EXTERNAL in the (sub)program from which F02FJF is called. Parameters denoted as Input must not be changed by this procedure.

9: NOVECS -- INTEGER Input
On entry: the number of approximate vectors that are being supplied in X. If NOVECS is outside the range (0,K), then the value 0 is used in place of NOVECS.

10: X(NRX,K) -- DOUBLE PRECISION array Input/Output
On entry: if 0 < NOVECS <= K, the first NOVECS columns of X must contain approximations to the eigenvectors corresponding to the NOVECS eigenvalues of largest absolute value of C. Supplying approximate eigenvectors can be useful when reasonable approximations are known, or when the routine is being restarted with a larger value of K. Otherwise it is not necessary to supply approximate vectors, as simultaneous iteration vectors will be generated randomly by the routine. On exit: if IFAIL = 0, 2, 3 or 4, the first m' columns contain the eigenvectors corresponding to the eigenvalues returned in the first m' elements of D (see below); and the next k-m'-1 columns contain approximations to the eigenvectors corresponding to the approximate

eigenvalues returned in the next k-m'-1 elements of D. Here m' is the value returned in M (see above), the number of eigenvalues actually found. The kth column is used as workspace.

11: NRX -- INTEGER Input
On entry:
the first dimension of the array X as declared in the (sub)program from which F02FJF is called.
Constraint: NRX >= N.

12: D(K) -- DOUBLE PRECISION array Output
On exit: if IFAIL = 0, 2, 3 or 4, the first m' elements contain the first m' eigenvalues in decreasing order of magnitude; and the next k-m'-1 elements contain approximations to the next k-m'-1 eigenvalues. Here m' is the value returned in M (see above), the number of eigenvalues actually found. D(k) contains the value e where (-e,e) is the latest interval over which Chebyshev acceleration is performed.

13: WORK(LWORK) -- DOUBLE PRECISION array Workspace

14: LWORK -- INTEGER Input
On entry: the length of the array WORK, as declared in the (sub)program from which F02FJF is called. Constraint: LWORK>=3*K+max(K*K,2*N).

15: RWORK(LRWORK) -- DOUBLE PRECISION array User Workspace
RWORK is not used by F02FJF, but is passed directly to routines DOT and IMAGE and may be used to supply information to these routines.

16: LRWORK -- INTEGER Input
On entry: the length of the array RWORK, as declared in the (sub)program from which F02FJF is called. Constraint: LRWORK >= 1.

17: IWORK(LIWORK) -- INTEGER array User Workspace
IWORK is not used by F02FJF, but is passed directly to routines DOT and IMAGE and may be used to supply information to these routines.

18: LIWORK -- INTEGER Input
On entry: the length of the array IWORK, as declared in the (sub)program from which F02FJF is called. Constraint: LIWORK

```
     >= 1.

19:  IFAIL -- INTEGER                                    Input/Output
     On entry: IFAIL must be set to 0, -1 or 1. Users who are
     unfamiliar with this parameter should refer to the Essential
     Introduction for details.

     On exit: IFAIL = 0 unless the routine detects an error or
     gives a warning (see Section 6).

     For this routine, because the values of output parameters
     may be useful even if IFAIL /=0 on exit, users are
     recommended to set IFAIL to -1 before entry. It is then
     essential to test the value of IFAIL on exit. To suppress
     the output of an error message when soft failure occurs, set
     IFAIL to 1.

6. Error Indicators and Warnings

Errors or warnings specified by the routine:


IFAIL< 0
     A negative value of IFAIL indicates an exit from F02FJF
     because the user has set IFLAG negative in DOT or IMAGE. The
     value of IFAIL will be the same as the user's setting of
     IFLAG.

IFAIL= 1
     On entry N < 1,

     or       M < 1,

     or       M >= K,

     or       K > N,

     or       NRX < N,

     or       LWORK <3*K+max(K*K*N),

     or       LRWORK < 1,

     or       LIWORK < 1.

IFAIL= 2
```

Not all the requested eigenvalues and vectors have been obtained. Approximations to the rth eigenvalue are oscillating rapidly indicating that severe cancellation is occurring in the rth eigenvector and so M is returned as (r-1). A restart with a larger value of K may permit convergence.

IFAIL= 3
Not all the requested eigenvalues and vectors have been obtained. The rate of convergence of the remaining eigenvectors suggests that more than NOITS iterations would be required and so the input value of M has been reduced. A restart with a larger value of K may permit convergence.

IFAIL= 4
Not all the requested eigenvalues and vectors have been obtained. NOITS iterations have been performed. A restart, possibly with a larger value of K, may permit convergence.

IFAIL= 5
This error is very unlikely to occur, but indicates that convergence of the eigenvalue sub-problem has not taken place. Restarting with a different set of approximate vectors may allow convergence. If this error occurs the user should check carefully that F02FJF is being called correctly.

7. Accuracy

Eigenvalues and eigenvectors will normally be computed to the accuracy requested by the parameter TOL, but eigenvectors corresponding to small or to close eigenvalues may not always be computed to the accuracy requested by the parameter TOL. Use of the routine MONIT to monitor acceptance of eigenvalues and eigenvectors is recommended.

8. Further Comments

The time taken by the routine will be principally determined by the time taken to solve the eigenvalue sub-problem and the time taken by the routines DOT and IMAGE. The time taken to solve an eigenvalue sub-problem is approximately proportional to nk^2. It is important to be aware that several calls to DOT and IMAGE may occur on each major iteration.

As can be seen from Table 3.1, many applications of F02FJF will require routine IMAGE to solve a system of linear equations. For example, to find the smallest eigenvalues of Ax=(lambda)Bx, IMAGE needs to solve equations of the form Aw=Bz for w and routines from Chapters F01 and F04 of the NAG Foundation Library will frequently be useful in this context. In particular, if A is a positive-definite variable band matrix, F04MCF may be used after A has been factorized by F01MCF. Thus factorization need be performed only once prior to calling F02FJF. An illustration of this type of use is given in the example program in Section 9.

An approximation $\tilde{d}_h$, to the ith eigenvalue, is accepted as soon as $\tilde{d}_h$ and the previous approximation differ by less than $|\tilde{d}_h|*TOL/10$. Eigenvectors are accepted in groups corresponding to clusters of eigenvalues that are equal, or nearly equal, in absolute value and that have already been accepted. If d_r is the last eigenvalue in such a group and we define the residual r_j as

$$r_j = Cx_j - y_r$$

where y_r is the projection of Cx_j, with respect to B, onto the space spanned by $x_1, x_2, \ldots, x_r$ and x_j is the current approximation to the jth eigenvector, then the value f_i returned in MONIT is given by

$$f_i = \max\left(\|r_j\|_B / \|Cx_j\|_B, \|x_j\|_B^2 = x_j^T Bx_j\right)$$

and each vector in the group is accepted as an eigenvector if

$$(|d_r| f_r)/(|d_r| - e) < TOL$$

where e is the current approximation to $|\tilde{d}_k|$. The values of the f_i are systematically increased if the convergence criteria appear to be too strict. See Rutishauser [4] for further details.

The algorithm implemented by F02FJF differs slightly from SIMITZ (Nikolai [1]) in that the eigenvalue sub-problem is solved using the singular value decomposition of the upper triangular matrix R_T of the Gram-Schmidt factorization of Cx_r, rather than forming $R_r R_r^T$

9. Example

To find the four eigenvalues of smallest absolute value and corresponding eigenvectors for the generalized symmetric eigenvalue problem Ax=(lambda)Bx, where A and B are the 16 by 16 matrices

$$A=\begin{pmatrix}
1 & a & & a & & & & & & & & & & & & \\
a & 1 & a & & a & & & & & & & & & & & \\
 & a & 1 & a & & a & & & & & & & & & & \\
 & & a & 1 & a & & a & & & & & & & & & \\
a & & & a & 1 & a & & a & & & & & & & & \\
 & a & & & a & 1 & a & & a & & & & & & & \\
 & & a & & & a & 1 & a & & a & & & & & & \\
 & & & a & & & a & 1 & a & & a & & & & & \\
 & & & & a & & & a & 1 & a & & a & & & & \\
 & & & & & a & & & a & 1 & a & & a & & & \\
 & & & & & & a & & & a & 1 & a & & a & & \\
 & & & & & & & a & & & a & 1 & a & & a & \\
 & & & & & & & & a & & & a & 1 & a & & \\
 & & & & & & & & & a & & & a & 1 & a & \\
 & & & & & & & & & & a & & & a & 1 &
\end{pmatrix}$$

where $a=-\frac{1}{4}$

$$\begin{pmatrix}
1 & b & & & \\
b & 1 & b & & \\
 & b & 1 & b & \\
 & & b & 1 &
\end{pmatrix}$$

```
                  (      b 1 b                    )
                  (        b 1 b                  )
                  (          b 1 b                )
                B=(            b 1 b              )
                  (              b 1 b            )
                  (                b 1 b          )
                  (                  b 1 b        )
                  (                    b 1 b      )
                  (                      b 1 b    )
                  (                        b 1 b  )
                  (                          b 1 b)

           1
where b=- -
           2
```

TOL is taken as 0.0001 and 6 iteration vectors are used. F01MAF is used to factorize the matrix A, prior to calling F02FJF, and F04MAF is used within IMAGE to solve the equations Aw=Bz for w. Details of the factorization of A are passed from F01MAF to F04MAF by means of the COMMON block BLOCK1.

Output from MONIT occurs each time ISTATE is non-zero. Note that the required eigenvalues are the reciprocals of the eigenvalues returned by F02FJF.

The example program is not reproduced here. The source code for all example programs is distributed with the NAG Foundation Library software and should be available on-line.

%%%

F02 -- Eigenvalue and Eigenvectors F02WEF
F02WEF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for your implementation to check implementation-dependent details. The symbol (*) after a NAG routine name denotes a routine that is not included in the Foundation Library.

1. Purpose

F02WEF returns all, or part, of the singular value decomposition of a general real matrix.

2. Specification

```
      SUBROUTINE F02WEF (M, N, A, LDA, NCOLB, B, LDB, WANTQ, Q,
     1                   LDQ, SV, WANTP, PT, LDPT, WORK, IFAIL)
      INTEGER          M, N, LDA, NCOLB, LDB, LDQ, LDPT, IFAIL
      DOUBLE PRECISION A(LDA,*), B(LDB,*), Q(LDQ,*), SV(*), PT
     1                 (LDPT,*), WORK(*)
      LOGICAL          WANTQ, WANTP
```

3. Description

The m by n matrix A is factorized as

$$A=QDP^{T},$$

where

$$D=\begin{pmatrix}S\\0\end{pmatrix}, \quad m>n,$$

$$D=S, \quad m=n,$$

$$D=\begin{pmatrix}S & 0\end{pmatrix}, \quad m<n,$$

Q is an m by m orthogonal matrix, P is an n by n orthogonal matrix and S is a min(m,n) by min(m,n) diagonal matrix with non-negative diagonal elements, $sv_1, sv_2, \ldots, sv_{min(m,n)}$, ordered such that

$$sv_1 \geq sv_2 \geq \ldots \geq sv_{min(m,n)} \geq 0.$$

The first min(m,n) columns of Q are the left-hand singular vectors of A, the diagonal elements of S are the singular values of A and the first min(m,n) columns of P are the right-hand singular vectors of A.

Either or both of the left-hand and right-hand singular vectors of A may be requested and the matrix C given by

$$C=Q^{T}B,$$

where B is an m by ncolb given matrix, may also be requested.

The routine obtains the singular value decomposition by first reducing A to upper triangular form by means of Householder transformations, from the left when m>=n and from the right when m<n. The upper triangular form is then reduced to bidiagonal form by Givens plane rotations and finally the QR algorithm is used to obtain the singular value decomposition of the bidiagonal form.

Good background descriptions to the singular value decomposition are given in Dongarra et al [1], Hammarling [2] and Wilkinson [3] DSVDC.

Note that if K is any orthogonal diagonal matrix so that

$$KK^T = I,$$

(so that K has elements +1 or -1 on the diagonal)

then

$$A=(QK)D(PK)^T$$

is also a singular value decomposition of A.

4. References

[1] Dongarra J J, Moler C B, Bunch J R and Stewart G W (1979) LINPACK Users' Guide. SIAM, Philadelphia.

[2] Hammarling S (1985) The Singular Value Decomposition in Multivariate Statistics. ACM Signum Newsletter. 20, 3 2--25.

[3] Wilkinson J H (1978) Singular Value Decomposition -- Basic Aspects. Numerical Software -- Needs and Availability. (ed D A H Jacobs) Academic Press.

5. Parameters

1: M -- INTEGER Input
On entry: the number of rows, m, of the matrix A.
Constraint: M >= 0.

When M = 0 then an immediate return is effected.

2: N -- INTEGER Input

On entry: the number of columns, n, of the matrix A.
Constraint: N >= 0.

When N = 0 then an immediate return is effected.

3: A(LDA,*) -- DOUBLE PRECISION array Input/Output
Note: the second dimension of the array A must be at least max(1,N).
On entry: the leading m by n part of the array A must contain the matrix A whose singular value decomposition is required. On exit: if M >= N and WANTQ = .TRUE., then the leading m by n part of A will contain the first n columns of the orthogonal matrix Q.

If M < N and WANTP = .TRUE., then the leading m by n part of
T
A will contain the first m rows of the orthogonal matrix P .

If M >= N and WANTQ = .FALSE. and WANTP = .TRUE., then the leading n by n part of A will contain the first n rows of
T
the orthogonal matrix P .

Otherwise the leading m by n part of A is used as internal workspace.

4: LDA -- INTEGER Input
On entry:
the first dimension of the array A as declared in the (sub)program from which F02WEF is called.
Constraint: LDA >= max(1,M).

5: NCOLB -- INTEGER Input
On entry: ncolb, the number of columns of the matrix B.
When NCOLB = 0 the array B is not referenced. Constraint: NCOLB >= 0.

6: B(LDB,*) -- DOUBLE PRECISION array Input/Output
Note: the second dimension of the array B must be at least max(1,ncolb) On entry: if NCOLB > 0, the leading m by ncolb part of the array B must contain the matrix to be transformed. On exit: B is overwritten by the m by ncolb
T
matrix Q B.

7: LDB -- INTEGER Input

```
     On entry:
     the first dimension of the array B as declared in the
     (sub)program from which F02WEF is called.
     Constraint: if NCOLB > 0 then LDB >= max(1,M).

 8:  WANTQ -- LOGICAL                                          Input
     On entry: WANTQ must be .TRUE., if the left-hand singular
     vectors are required. If WANTQ = .FALSE., then the array Q
     is not referenced.

 9:  Q(LDQ,*) -- DOUBLE PRECISION array                        Output
     Note: the second dimension of the array Q must be at least
     max(1,M).
     On exit: if M < N and WANTQ = .TRUE., the leading m by m
     part of the array Q will contain the orthogonal matrix Q.
     Otherwise the array Q is not referenced.

10:  LDQ -- INTEGER                                           Input
     On entry:
     the first dimension of the array Q as declared in the
     (sub)program from which F02WEF is called.
     Constraint: if M < N and WANTQ = .TRUE., LDQ >= max(1,M).

11:  SV(*) -- DOUBLE PRECISION array                          Output
     Note: the length of SV must be at least min(M,N). On exit:
     the min(M,N) diagonal elements of the matrix S.

12:  WANTP -- LOGICAL                                         Input
     On entry: WANTP must be .TRUE. if the right-hand singular
     vectors are required. If WANTP = .FALSE., then the array PT
     is not referenced.

13:  PT(LDPT,*) -- DOUBLE PRECISION array                     Output
     Note: the second dimension of the array PT must be at least
     max(1,N).
     On exit: if M >= N and WANTQ and WANTP are .TRUE., the
     leading n by n part of the array PT will contain the
                          T
     orthogonal matrix P  . Otherwise the array PT is not
     referenced.

14:  LDPT -- INTEGER                                          Input
     On entry:
     the first dimension of the array PT as declared in the
     (sub)program from which F02WEF is called.
     Constraint: if M >= N and WANTQ and WANTP are .TRUE., LDPT
```

```
     >= max(1,N).

15:  WORK(*) -- DOUBLE PRECISION array                        Output
     Note: the length of WORK must be at least max(1,lwork),
     where lwork must be as given in the following table:

     M >= N
     WANTQ is .TRUE. and WANTP = .TRUE.
                      2
           lwork=max(N +5*(N-1),N+NCOLB,4)

     WANTQ = .TRUE. and WANTP = .FALSE.
                      2
           lwork=max(N +4*(N-1),N+NCOLB,4)

     WANTQ = .FALSE. and WANTP = .TRUE.
           lwork=max(3*(N-1),2) when NCOLB = 0

           lwork=max(5*(N-1),2) when NCOLB > 0

     WANTQ = .FALSE. and WANTP = .FALSE.
           lwork=max(2*(N-1),2) when NCOLB = 0

           lwork=max(3*(N-1),2) when NCOLB > 0

     M < N
     WANTQ = .TRUE. and WANTP = .TRUE.
                      2
           lwork=max(M +5*(M-1),2)

     WANTQ = .TRUE. and WANTP = .FALSE.
           lwork=max(3*(M-1),1)

     WANTQ = .FALSE. and WANTP = .TRUE.
                      2
           lwork=max(M +3*(M-1),2) when NCOLB = 0

                      2
           lwork=max(M +5*(M-1),2) when NCOLB > 0

     WANTQ = .FALSE. and WANTP = .FALSE.
           lwork=max(2*(M-1),1) when NCOLB = 0

           lwork=max(3*(M-1),1) when NCOLB > 0
     On exit: WORK(min(M,N)) contains the total number of
     iterations taken by the R algorithm.
```

```
     The rest of the array is used as workspace.

16:  IFAIL -- INTEGER                                    Input/Output
     On entry: IFAIL must be set to 0, -1 or 1. For users not
     familiar with this parameter (described in the Essential
     Introduction) the recommended value is 0.

     On exit: IFAIL = 0 unless the routine detects an error (see
     Section 6).

6. Error Indicators and Warnings

Errors detected by the routine:

If on entry IFAIL = 0 or -1, explanatory error messages are
output on the current error message unit (as defined by X04AAF).

IFAIL=-1
     One or more of the following conditions holds:
          M < 0,

          N < 0,

          LDA < M,

          NCOLB < 0,

          LDB < M and NCOLB > 0,

          LDQ < M and M < N and WANTQ = .TRUE.,

          LDPT < N and M >= N and WANTQ = .TRUE., and WANTP = .
          TRUE..

IFAIL> 0
     The QR algorithm has failed to converge in 50*min(m,n)
     iterations. In this case SV(1), SV(2),..., SV(IFAIL) may not
     have been found correctly and the remaining singular values
     may not be the smallest. The matrix A will nevertheless have
                                T
     been factorized as  A=QEP ,  where the leading min(m,n) by
     min(m,n) part of E is a bidiagonal matrix with SV(1), SV(2),
     ..., SV(min(m,n)) as the diagonal elements and WORK(1), WORK
     (2),..., WORK(min(m,n)-1) as the super-diagonal elements.
```

This failure is not likely to occur.

7. Accuracy

The computed factors Q, D and P satisfy the relation

$$QDP^T = A+E,$$

where

$$||E|| <= c(epsilon)||A||,$$

(epsilon) being the machine precision, c is a modest function of m and n and ||.|| denotes the spectral (two) norm. Note that $||A|| = sv_1$.

8. Further Comments

Following the use of this routine the rank of A may be estimated by a call to the INTEGER FUNCTION F06KLF(*). The statement:

```
IRANK = F06KLF(MIN(M, N), SV, 1, TOL)
```

returns the value (k-1) in IRANK, where k is the smallest integer for which SV(k)<tol*SV(1), where tol is the tolerance supplied in TOL, so that IRANK is an estimate of the rank of S and thus also of A. If TOL is supplied as negative then the machine precision is used in place of TOL.

9. Example

9.1. Example 1

To find the singular value decomposition of the 5 by 3 matrix

$$A = \begin{pmatrix} 2.0 & 2.5 & 2.5 \\ 2.0 & 2.5 & 2.5 \\ 1.6 & -0.4 & 2.8 \\ 2.0 & -0.5 & 0.5 \\ 1.2 & -0.3 & -2.9 \end{pmatrix}$$

together with the vector $Q^T b$ for the vector

```
                                    ( 1.1)
                                    ( 0.9)
                                b=( 0.6)
                                    ( 0.0)
                                    (-0.8)
```

The example program is not reproduced here. The source code for all example programs is distributed with the NAG Foundation Library software and should be available on-line.

9.2. Example 2

To find the singular value decomposition of the 3 by 5 matrix

```
                      (2.0 2.0  1.6  2.0  1.2)
                  A=(2.5 2.5 -0.4 -0.5 -0.3)
                      (2.5 2.5  2.8  0.5 -2.9)
```

The example program is not reproduced here. The source code for all example programs is distributed with the NAG Foundation Library software and should be available on-line.

%%%

F02 -- Eigenvalue and Eigenvectors F02XEF
F02XEF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for your implementation to check implementation-dependent details. The symbol (*) after a NAG routine name denotes a routine that is not included in the Foundation Library.

1. Purpose

F02XEF returns all, or part, of the singular value decomposition of a general complex matrix.

2. Specification

```
      SUBROUTINE F02XEF (M, N, A, LDA, NCOLB, B, LDB, WANTQ, Q,
     1                   LDQ, SV, WANTP, PH, LDPH, RWORK, CWORK,
     2                   IFAIL)
      INTEGER                   M, N, LDA, NCOLB, LDB, LDQ, LDPH,
     1                          IFAIL
      DOUBLE PRECISION          SV(*), RWORK(*)
      COMPLEX(KIND=KIND(1.0D0)) A(LDA,*), B(LDB,*), Q(LDQ,*),
```

```
      1                          PH(LDPH,*), CWORK(*)
       LOGICAL                   WANTQ, WANTP
```

3. Description

The m by n matrix A is factorized as

$$A=QDP^{H},$$

where

$$D=\begin{cases}(S) & m>n,\\ S, & m=n,\\ (S\ 0), & m<n,\end{cases}$$

Q is an m by m unitary matrix, P is an n by n unitary matrix and S is a min(m,n) by min(m,n) diagonal matrix with real non-negative diagonal elements, $sv_1, sv_2, \ldots, sv_{min(m,n)}$, ordered such that

$$sv_1 \geq sv_2 \geq \ldots \geq sv_{min(m,n)} \geq 0.$$

The first min(m,n) columns of Q are the left-hand singular vectors of A, the diagonal elements of S are the singular values of A and the first min(m,n) columns of P are the right-hand singular vectors of A.

Either or both of the left-hand and right-hand singular vectors of A may be requested and the matrix C given by

$$C=Q^{H}B,$$

where B is an m by ncolb given matrix, may also be requested.

The routine obtains the singular value decomposition by first reducing A to upper triangular form by means of Householder transformations, from the left when m>=n and from the right when m<n. The upper triangular form is then reduced to bidiagonal form by Givens plane rotations and finally the QR algorithm is used to

obtain the singular value decomposition of the bidiagonal form.

Good background descriptions to the singular value decomposition are given in Dongarra et al [1], Hammarling [2] and Wilkinson [3] ZSVDC.

Note that if K is any unitary diagonal matrix so that

$$KK^H = I,$$

then

$$A = (QK)D(PK)^H$$

is also a singular value decomposition of A.

4. References

[1] Dongarra J J, Moler C B, Bunch J R and Stewart G W (1979) LINPACK Users' Guide. SIAM, Philadelphia.

[2] Hammarling S (1985) The Singular Value Decomposition in Multivariate Statistics. ACM Signum Newsletter. 20, 3 2--25.

[3] Wilkinson J H (1978) Singular Value Decomposition -- Basic Aspects. Numerical Software -- Needs and Availability. (ed D A H Jacobs) Academic Press.

5. Parameters

1: M -- INTEGER Input
On entry: the number of rows, m, of the matrix A.
Constraint: M >= 0.

When M = 0 then an immediate return is effected.

2: N -- INTEGER Input
On entry: the number of columns, n, of the matrix A.
Constraint: N >= 0.

When N = 0 then an immediate return is effected.

3: A(LDA,*) -- COMPLEX(KIND(1.0D)) array Input/Output
Note: the second dimension of the array A must be at least

max(1,N).
On entry: the leading m by n part of the array A must contain the matrix A whose singular value decomposition is required. On exit: if M >= N and WANTQ = .TRUE., then the leading m by n part of A will contain the first n columns of the unitary matrix Q.
If M < N and WANTP = .TRUE., then the leading m by n part of
H
A will contain the first m rows of the unitary matrix P .
will contain the first m rows of the unitary matrix P If M >= N and WANTQ = .FALSE. and WANTP = .TRUE., then the leading n by n part of A will contain the first n
H
rows of the unitary matrix P . Otherwise the leading m by n part of A is used as internal workspace.

4: LDA -- INTEGER Input
On entry:
the first dimension of the array A as declared in the (sub)program from which F02XEF is called.
Constraint: LDA >= max(1,M).

5: NCOLB -- INTEGER Input
On entry: ncolb, the number of columns of the matrix B. When NCOLB = 0 the array B is not referenced. Constraint: NCOLB >= 0.

6: B(LDB,*) -- COMPLEX(KIND(1.0D)) array Input/Output
Note: the second dimension of the array B must be at least max(1,NCOLB).
On entry: if NCOLB > 0, the leading m by ncolb part of the array B must contain the matrix to be transformed. On exit:
H
B is overwritten by the m by ncolb matrix Q B.

7: LDB -- INTEGER Input
On entry:
the first dimension of the array B as declared in the (sub)program from which F02XEF is called.
Constraint: if NCOLB > 0, then LDB >= max(1,M).

8: WANTQ -- LOGICAL Input
On entry: WANTQ must be .TRUE. if the left-hand singular vectors are required. If WANTQ = .FALSE. then the array Q is not referenced.

```
 9:  Q(LDQ,*) -- COMPLEX(KIND(1.0D)) array                  Output
     Note: the second dimension of the array Q must be at least
     max(1,M).
     On exit: if M < N and WANTQ = .TRUE., the leading m by m
     part of the array Q will contain the unitary matrix Q.
     Otherwise the array Q is not referenced.

10:  LDQ -- INTEGER                                         Input
     On entry:
     the first dimension of the array Q as declared in the
     (sub)program from which F02XEF is called.
     Constraint: if M < N and WANTQ = .TRUE., LDQ >= max(1,M).

11:  SV(*) -- DOUBLE PRECISION array                        Output
     Note: the length of SV must be at least min(M,N). On exit:
     the min(m,n) diagonal elements of the matrix S.

12:  WANTP -- LOGICAL                                       Input
     On entry: WANTP must be .TRUE. if the right-hand singular
     vectors are required. If WANTP = .FALSE. then the array PH
     is not referenced.

13:  PH(LDPH,*) -- DOUBLE PRECISION array                   Output
     Note: the second dimension of the array PH must be at least
     max(1,N).
     On exit: if M >= N and WANTQ and WANTP are .TRUE., the
     leading n by n part of the array PH will contain the unitary
             H
     matrix P . Otherwise the array PH is not referenced.

14:  LDPH -- INTEGER                                        Input
     On entry:
     the first dimension of the array PH as declared in the
     (sub)program from which F02XEF is called.
     Constraint: if M >= N and WANTQ and WANTP are .TRUE.,  LDPH
     >= max(1,N).

15:  RWORK(*) -- DOUBLE PRECISION array                     Output
     Note: the length of RWORK must be at least max(1,lrwork),
     where lrwork must satisfy:
     lrwork=2*(min(M,N)-1) when
           NCOLB = 0 and WANTQ and WANTP are .FALSE.,

     lrwork=3*(min(M,N)-1) when
           either NCOLB = 0 and WANTQ = .FALSE. and WANTP = .
           TRUE., or WANTP = .FALSE. and one or both of NCOLB > 0
```

```
      and WANTQ = .TRUE.

lrwork=5*(min(M,N)-1)
      otherwise.
On exit: RWORK(min(M,N)) contains the total number of
iterations taken by the QR algorithm.

The rest of the array is used as workspace.

16:  CWORK(*) -- COMPLEX(KIND(1.0D)) array              Workspace
     Note: the length of CWORK must be at least max(1,lcwork),
     where lcwork must satisfy:
                   2
     lcwork=N+max(N ,NCOLB) when
           M >= N and WANTQ and WANTP are both .TRUE.

                   2
     lcwork=N+max(N +N,NCOLB) when
           M >= N and WANTQ = .TRUE., but WANTP = .FALSE.

     lcwork=N+max(N,NCOLB) when
           M >= N and WANTQ = .FALSE.

             2
     lcwork=M +M when
           M < N and WANTP = .TRUE.

     lcwork = M when
           M < N and WANTP = .FALSE.

17:  IFAIL -- INTEGER                                   Input/Output
     On entry: IFAIL must be set to 0, -1 or 1. For users not
     familiar with this parameter (described in the Essential
     Introduction) the recommended value is 0.

     On exit: IFAIL = 0 unless the routine detects an error (see
     Section 6).

6. Error Indicators and Warnings

Errors detected by the routine:

If on entry IFAIL = 0 or -1, explanatory error messages are
output on the current error message unit (as defined by X04AAF).

IFAIL=-1
```

```
     One or more of the following conditions holds:
          M < 0,

          N < 0,

          LDA < M,

          NCOLB < 0,

          LDB < M and NCOLB > 0,

          LDQ < M and M < N and WANTQ = .TRUE.,

          LDPH < N and M >= N and WANTQ = .TRUE. and WANTP = .
          TRUE..

IFAIL> 0
     The QR algorithm has failed to converge in 50*min(m,n)
     iterations. In this case SV(1), SV(2),..., SV(IFAIL) may not
     have been found correctly and the remaining singular values
     may not be the smallest. The matrix A will nevertheless have
                              H
     been factorized as  A=QEP   where the leading min(m,n) by
     min(m,n) part of E is a bidiagonal matrix with SV(1), SV(2),
     ..., SV(min(m,n)) as the diagonal elements and RWORK(1),
     RWORK(2),..., RWORK(min(m,n)-1) as the super-diagonal
     elements.

     This failure is not likely to occur.

7. Accuracy

The computed factors Q, D and P satisfy the relation

                                 H
                              QDP =A+E,

where

                        ||E||<=c(epsilon)||A||,

(epsilon) being the machine precision, c is a modest function of
m and n and ||.|| denotes the spectral (two) norm. Note that
||A||=sv .
        1
```

8. Further Comments

Following the use of this routine the rank of A may be estimated by a call to the INTEGER FUNCTION F06KLF(*). The statement:

```
IRANK = F06KLF(MIN(M, N), SV, 1, TOL)
```

returns the value (k-1) in IRANK, where k is the smallest integer for which SV(k)<tol*SV(1), where tol is the tolerance supplied in TOL, so that IRANK is an estimate of the rank of S and thus also of A. If TOL is supplied as negative then the machine precision is used in place of TOL.

9. Example

9.1. Example 1

To find the singular value decomposition of the 5 by 3 matrix

$$A = \begin{pmatrix} 0.5i & -0.5+1.5i & -1.0+1.0i \\ 0.4+0.3i & 0.9+1.3i & 0.2+1.4i \\ 0.4 & -0.4+0.4i & 1.8 \\ 0.3-0.4i & 0.1+0.7i & 0.0 \\ -0.3i & 0.3+0.3i & 2.4i \end{pmatrix}$$

together with the vector $Q^H b$ for the vector

$$b = \begin{pmatrix} -0.55+1.05i \\ 0.49+0.93i \\ 0.56-0.16i \\ 0.39+0.23i \\ 1.13+0.83i \end{pmatrix}$$

The example program is not reproduced here. The source code for all example programs is distributed with the NAG Foundation Library software and should be available on-line.

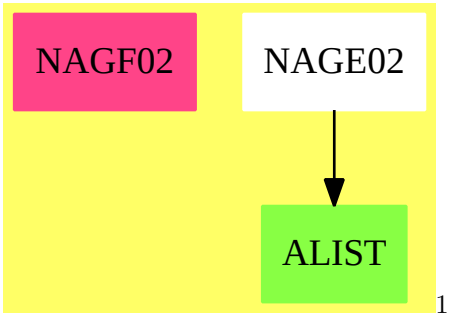
9.2. Example 2

To find the singular value decompostition of the 3 by 5 matrix

$$A = \begin{pmatrix} 0.5i & 0.4-0.3i & 0.4 & 0.3+0.4i & 0.3i \\ -0.5-1.5i & 0.9-1.3i & -0.4-0.4i & 0.1-0.7i & 0.3-0.3i \\ -1.0-1.0i & 0.2-1.4i & 1.8 & 0.0 & -2.4i \end{pmatrix}$$

```
The example program is not reproduced here. The source code for
all example programs is distributed with the NAG Foundation
Library software and should be available on-line.
```

15.2 NagEigenPackage



1.00

Exports:

f02aaf	f02abf	f02adf	f02aef	f02aff
f02agf	f02ajf	f02akf	f02awf	f02axf
f02bbf	f02bjf	f02fjf	f02fjf	f02wef
f02xef				

⟨*package NAGF02 NagEigenPackage*⟩≡

```
)abbrev package NAGF02 NagEigenPackage
++ Author: Godfrey Nolan and Mike Dewar
++ Date Created: Jan 1994
++ Date Last Updated: Thu May 12 17:45:20 1994
++ Description:
++ This package uses the NAG Library to compute
++ \begin{items}
++ \item eigenvalues and eigenvectors of a matrix
++ \item eigenvalues and eigenvectors of generalized matrix
++ eigenvalue problems
++ \item singular values and singular vectors of a matrix.
++ \end{items}
++ See \downlink{Manual Page}{manpageXXf02}.

NagEigenPackage(): Exports == Implementation where
  S ==> Symbol
  FOP ==> FortranOutputStackPackage

  Exports ==> with
    f02aaf : (Integer,Integer,Matrix DoubleFloat,Integer) -> Result
     ++ f02aaf(ia,n,a,ifail)
     ++ calculates all the eigenvalue.
     ++ See \downlink{Manual Page}{manpageXXf02aaf}.
    f02abf : (Matrix DoubleFloat,Integer,Integer,Integer,_
        Integer) -> Result
     ++ f02abf(a,ia,n,iv,ifail)
     ++ calculates all the eigenvalues of a real
```

```
 ++ symmetric matrix.
 ++ See \downlink{Manual Page}{manpageXXf02abf}.
f02adf : (Integer,Integer,Integer,Matrix DoubleFloat,_
    Matrix DoubleFloat,Integer) -> Result
 ++ f02adf(ia,ib,n,a,b,ifail)
 ++ calculates all the eigenvalues of  Ax=(lambda)Bx,  where A
 ++ is a real symmetric matrix and B is a real symmetric positive-
 ++ definite matrix.
 ++ See \downlink{Manual Page}{manpageXXf02adf}.
f02aef : (Integer,Integer,Integer,Integer,_
    Matrix DoubleFloat,Matrix DoubleFloat,Integer) -> Result
 ++ f02aef(ia,ib,n,iv,a,b,ifail)
 ++ calculates all the eigenvalues of
 ++ Ax=(lambda)Bx,  where A is a real symmetric matrix and B is a
 ++ real symmetric positive-definite matrix.
 ++ See \downlink{Manual Page}{manpageXXf02aef}.
f02aff : (Integer,Integer,Matrix DoubleFloat,Integer) -> Result
 ++ f02aff(ia,n,a,ifail)
 ++ calculates all the eigenvalues of a real unsymmetric
 ++ matrix.
 ++ See \downlink{Manual Page}{manpageXXf02aff}.
f02agf : (Integer,Integer,Integer,Integer,_
    Matrix DoubleFloat,Integer) -> Result
 ++ f02agf(ia,n,ivr,ivi,a,ifail)
 ++ calculates all the eigenvalues of a real
 ++ unsymmetric matrix.
 ++ See \downlink{Manual Page}{manpageXXf02agf}.
f02ajf : (Integer,Integer,Integer,Matrix DoubleFloat,_
    Matrix DoubleFloat,Integer) -> Result
 ++ f02ajf(iar,iai,n,ar,ai,ifail)
 ++ calculates all the eigenvalue.
 ++ See \downlink{Manual Page}{manpageXXf02ajf}.
f02akf : (Integer,Integer,Integer,Integer,_
    Integer,Matrix DoubleFloat,Matrix DoubleFloat,Integer) -> Result
 ++ f02akf(iar,iai,n,ivr,ivi,ar,ai,ifail)
 ++ calculates all the eigenvalues of a
 ++ complex matrix.
 ++ See \downlink{Manual Page}{manpageXXf02akf}.
f02awf : (Integer,Integer,Integer,Matrix DoubleFloat,_
    Matrix DoubleFloat,Integer) -> Result
 ++ f02awf(iar,iai,n,ar,ai,ifail)
 ++ calculates all the eigenvalues of a complex Hermitian
 ++ matrix.
 ++ See \downlink{Manual Page}{manpageXXf02awf}.
f02axf : (Matrix DoubleFloat,Integer,Matrix DoubleFloat,Integer,_
    Integer,Integer,Integer,Integer) -> Result
```

```
    ++ f02axf(ar,iar,ai,iai,n,ivr,ivi,ifail)
    ++ calculates all the eigenvalues of a
    ++ complex Hermitian matrix.
    ++ See \downlink{Manual Page}{manpageXXf02axf}.
  f02bbf : (Integer,Integer,DoubleFloat,DoubleFloat,_
      Integer,Integer,Matrix DoubleFloat,Integer) -> Result
    ++ f02bbf(ia,n,alb,ub,m,iv,a,ifail)
    ++ calculates selected eigenvalues of a real
    ++ symmetric matrix by reduction to tridiagonal form, bisection and
    ++ inverse iteration, where the selected eigenvalues lie within a
    ++ given interval.
    ++ See \downlink{Manual Page}{manpageXXf02bbf}.
  f02bjf : (Integer,Integer,Integer,DoubleFloat,_
      Boolean,Integer,Matrix DoubleFloat,Matrix DoubleFloat,Integer) -> Result
    ++ f02bjf(n,ia,ib,eps1,matv,iv,a,b,ifail)
    ++ calculates all the eigenvalues and, if required, all the
    ++ eigenvectors of the generalized eigenproblem  Ax=(lambda)Bx
    ++ where A and B are real, square matrices, using the QZ algorithm.
    ++ See \downlink{Manual Page}{manpageXXf02bjf}.
  f02fjf : (Integer,Integer,DoubleFloat,Integer,_
      Integer,Integer,Integer,Integer,Integer,Integer,Matrix DoubleFloat,Intege
    ++ f02fjf(n,k,tol,novecs,nrx,lwork,lrwork,liwork,m,noits,x,ifail,dot,image)
    ++ finds eigenvalues of a real sparse symmetric
    ++ or generalized symmetric eigenvalue problem.
    ++ See \downlink{Manual Page}{manpageXXf02fjf}.
  f02fjf : (Integer,Integer,DoubleFloat,Integer,_
      Integer,Integer,Integer,Integer,Integer,Integer,Matrix DoubleFloat,Intege
    ++ f02fjf(n,k,tol,novecs,nrx,lwork,lrwork,liwork,m,noits,x,ifail,dot,image,m
    ++ finds eigenvalues of a real sparse symmetric
    ++ or generalized symmetric eigenvalue problem.
    ++ See \downlink{Manual Page}{manpageXXf02fjf}.
  f02wef : (Integer,Integer,Integer,Integer,_
      Integer,Boolean,Integer,Boolean,Integer,Matrix DoubleFloat,Matrix DoubleF
    ++ f02wef(m,n,lda,ncolb,ldb,wantq,ldq,wantp,ldpt,a,b,ifail)
    ++ returns all, or part, of the singular value decomposition
    ++ of a general real matrix.
    ++ See \downlink{Manual Page}{manpageXXf02wef}.
  f02xef : (Integer,Integer,Integer,Integer,_
      Integer,Boolean,Integer,Boolean,Integer,Matrix Complex DoubleFloat,Matrix
    ++ f02xef(m,n,lda,ncolb,ldb,wantq,ldq,wantp,ldph,a,b,ifail)
    ++ returns all, or part, of the singular value decomposition
    ++ of a general complex matrix.
    ++ See \downlink{Manual Page}{manpageXXf02xef}.
Implementation ==> add

  import Lisp
```

```
import DoubleFloat
import Any
import Record
import Integer
import Matrix DoubleFloat
import Boolean
import NAGLinkSupportPackage
import FortranPackage
import AnyFunctions1(Integer)
import AnyFunctions1(Boolean)
import AnyFunctions1(Matrix DoubleFloat)
import AnyFunctions1(Matrix Complex DoubleFloat)
import AnyFunctions1(DoubleFloat)


f02aaf(iaArg:Integer,nArg:Integer,aArg:Matrix DoubleFloat,_
    ifailArg:Integer): Result ==
    [(invokeNagman(NIL$Lisp,_
    "f02aaf",_
    ["ia"::S,"n"::S,"ifail"::S,"r"::S,"a"::S,"e"::S]$Lisp,_
    ["r"::S,"e"::S]$Lisp,_
    [["double"::S,["r"::S,"n"::S]$Lisp,["a"::S,"ia"::S,"n"::S]$Lisp_
    ,["e"::S,"n"::S]$Lisp]$Lisp_
    ,["integer"::S,"ia"::S,"n"::S,"ifail"::S]$Lisp_
    ]$Lisp,_
    ["r"::S,"a"::S,"ifail"::S]$Lisp,_
    [([iaArg::Any,nArg::Any,ifailArg::Any,aArg::Any ])_
    @List Any]$Lisp)$Lisp)_
    pretend List (Record(key:Symbol,entry:Any))]$Result

f02abf(aArg:Matrix DoubleFloat,iaArg:Integer,nArg:Integer,_
    ivArg:Integer,ifailArg:Integer): Result ==
    [(invokeNagman(NIL$Lisp,_
    "f02abf",_
    ["ia"::S,"n"::S,"iv"::S,"ifail"::S,"a"::S,"r"::S,"v"::S,"e"::S]$Lisp,_
    ["r"::S,"v"::S,"e"::S]$Lisp,_
    [["double"::S,["a"::S,"ia"::S,"n"::S]$Lisp_
    ,["r"::S,"n"::S]$Lisp,["v"::S,"iv"::S,"n"::S]$Lisp,["e"::S,"n"::S]$Lisp]$Lisp_
    ,["integer"::S,"ia"::S,"n"::S,"iv"::S,"ifail"::S_
    ]$Lisp_
    ]$Lisp,_
    ["r"::S,"v"::S,"ifail"::S]$Lisp,_
    [([iaArg::Any,nArg::Any,ivArg::Any,ifailArg::Any,aArg::Any ])_
    @List Any]$Lisp)$Lisp)_
    pretend List (Record(key:Symbol,entry:Any))]$Result
```

```
f02adf(iaArg:Integer,ibArg:Integer,nArg:Integer,_
    aArg:Matrix DoubleFloat,bArg:Matrix DoubleFloat,ifailArg:Integer): Result
    [(invokeNagman(NIL$Lisp,_
    "f02adf",_
    ["ia"::S,"ib"::S,"n"::S,"ifail"::S,"r"::S,"a"::S,"b"::S,"de"::S]$Lisp,_
    ["r"::S,"de"::S]$Lisp,_
    [["double"::S,["r"::S,"n"::S]$Lisp,["a"::S,"ia"::S,"n"::S]$Lisp_
    ,["b"::S,"ib"::S,"n"::S]$Lisp,["de"::S,"n"::S]$Lisp]$Lisp_
    ,["integer"::S,"ia"::S,"ib"::S,"n"::S,"ifail"::S_
    ]$Lisp_
    ]$Lisp,_
    ["r"::S,"a"::S,"b"::S,"ifail"::S]$Lisp,_
    [([iaArg::Any,ibArg::Any,nArg::Any,ifailArg::Any,aArg::Any,bArg::Any ])_
    @List Any]$Lisp)$Lisp)_
    pretend List (Record(key:Symbol,entry:Any))]$Result

f02aef(iaArg:Integer,ibArg:Integer,nArg:Integer,_
    ivArg:Integer,aArg:Matrix DoubleFloat,bArg:Matrix DoubleFloat,_
    ifailArg:Integer): Result ==
    [(invokeNagman(NIL$Lisp,_
    "f02aef",_
    ["ia"::S,"ib"::S,"n"::S,"iv"::S,"ifail"::S_
    ,"r"::S,"v"::S,"a"::S,"b"::S,"dl"::S_
    ,"e"::S]$Lisp,_
    ["r"::S,"v"::S,"dl"::S,"e"::S]$Lisp,_
    [["double"::S,["r"::S,"n"::S]$Lisp,["v"::S,"iv"::S,"n"::S]$Lisp_
    ,["a"::S,"ia"::S,"n"::S]$Lisp,["b"::S,"ib"::S,"n"::S]$Lisp,["dl"::S,"n"::
    ]$Lisp_
    ,["integer"::S,"ia"::S,"ib"::S,"n"::S,"iv"::S_
    ,"ifail"::S]$Lisp_
    ]$Lisp,_
    ["r"::S,"v"::S,"a"::S,"b"::S,"ifail"::S]$Lisp,_
    [([iaArg::Any,ibArg::Any,nArg::Any,ivArg::Any,ifailArg::Any,aArg::Any,bAr
    @List Any]$Lisp)$Lisp)_
    pretend List (Record(key:Symbol,entry:Any))]$Result

f02aff(iaArg:Integer,nArg:Integer,aArg:Matrix DoubleFloat,_
    ifailArg:Integer): Result ==
    [(invokeNagman(NIL$Lisp,_
    "f02aff",_
    ["ia"::S,"n"::S,"ifail"::S,"rr"::S,"ri"::S,"intger"::S,"a"::S]$Lisp,_
    ["rr"::S,"ri"::S,"intger"::S]$Lisp,_
    [["double"::S,["rr"::S,"n"::S]$Lisp,["ri"::S,"n"::S]$Lisp_
    ,["a"::S,"ia"::S,"n"::S]$Lisp]$Lisp_
    ,["integer"::S,"ia"::S,"n"::S,["intger"::S,"n"::S]$Lisp_
    ,"ifail"::S]$Lisp_
```

```
    ]$Lisp,_
    ["rr"::S,"ri"::S,"intger"::S,"a"::S,"ifail"::S]$Lisp,_
    [([iaArg::Any,nArg::Any,ifailArg::Any,aArg::Any ])_
    @List Any]$Lisp)$Lisp)_
    pretend List (Record(key:Symbol,entry:Any))]$Result

f02agf(iaArg:Integer,nArg:Integer,ivrArg:Integer,_
    iviArg:Integer,aArg:Matrix DoubleFloat,ifailArg:Integer): Result ==
    [(invokeNagman(NIL$Lisp,_
    "f02agf",_
    ["ia"::S,"n"::S,"ivr"::S,"ivi"::S,"ifail"::S_
    ,"rr"::S,"ri"::S,"vr"::S,"vi"::S,"intger"::S_
    ,"a"::S]$Lisp,_
    ["rr"::S,"ri"::S,"vr"::S,"vi"::S,"intger"::S]$Lisp,_
    [["double"::S,["rr"::S,"n"::S]$Lisp,["ri"::S,"n"::S]$Lisp_
    ,["vr"::S,"ivr"::S,"n"::S]$Lisp,["vi"::S,"ivi"::S,"n"::S]$Lisp,["a"::S,"ia"::S,"n":
    ,["integer"::S,"ia"::S,"n"::S,"ivr"::S,"ivi"::S_
    ,["intger"::S,"n"::S]$Lisp,"ifail"::S]$Lisp_
    ]$Lisp,_
    ["rr"::S,"ri"::S,"vr"::S,"vi"::S,"intger"::S,"a"::S,"ifail"::S]$Lisp,_
    [([iaArg::Any,nArg::Any,ivrArg::Any,iviArg::Any,ifailArg::Any,aArg::Any ])_
    @List Any]$Lisp)$Lisp)_
    pretend List (Record(key:Symbol,entry:Any))]$Result

f02ajf(iarArg:Integer,iaiArg:Integer,nArg:Integer,_
    arArg:Matrix DoubleFloat,aiArg:Matrix DoubleFloat,ifailArg:Integer): Result ==
    [(invokeNagman(NIL$Lisp,_
    "f02ajf",_
    ["iar"::S,"iai"::S,"n"::S,"ifail"::S,"rr"::S,"ri"::S,"ar"::S,"ai"::S,"intger"::S_
    ]$Lisp,_
    ["rr"::S,"ri"::S,"intger"::S]$Lisp,_
    [["double"::S,["rr"::S,"n"::S]$Lisp,["ri"::S,"n"::S]$Lisp_
    ,["ar"::S,"iar"::S,"n"::S]$Lisp,["ai"::S,"iai"::S,"n"::S]$Lisp]$Lisp_
    ,["integer"::S,"iar"::S,"iai"::S,"n"::S,"ifail"::S_
    ,["intger"::S,"n"::S]$Lisp]$Lisp_
    ]$Lisp,_
    ["rr"::S,"ri"::S,"ar"::S,"ai"::S,"ifail"::S]$Lisp,_
    [([iarArg::Any,iaiArg::Any,nArg::Any,ifailArg::Any,arArg::Any,aiArg::Any ])_
    @List Any]$Lisp)$Lisp)_
    pretend List (Record(key:Symbol,entry:Any))]$Result

f02akf(iarArg:Integer,iaiArg:Integer,nArg:Integer,_
    ivrArg:Integer,iviArg:Integer,arArg:Matrix DoubleFloat,_
    aiArg:Matrix DoubleFloat,ifailArg:Integer): Result ==
    [(invokeNagman(NIL$Lisp,_
    "f02akf",_
```

```
    ["iar"::S,"iai"::S,"n"::S,"ivr"::S,"ivi"::S_
    ,"ifail"::S,"rr"::S,"ri"::S,"vr"::S,"vi"::S,"ar"::S_
    ,"ai"::S,"intger"::S]$Lisp,_
    ["rr"::S,"ri"::S,"vr"::S,"vi"::S,"intger"::S]$Lisp,_
    [["double"::S,["rr"::S,"n"::S]$Lisp,["ri"::S,"n"::S]$Lisp_
    ,["vr"::S,"ivr"::S,"n"::S]$Lisp,["vi"::S,"ivi"::S,"n"::S]$Lisp,["ar"::S,"
    ]$Lisp_
    ,["integer"::S,"iar"::S,"iai"::S,"n"::S,"ivr"::S_
    ,"ivi"::S,"ifail"::S,["intger"::S,"n"::S]$Lisp]$Lisp_
    ]$Lisp,_
    ["rr"::S,"ri"::S,"vr"::S,"vi"::S,"ar"::S,"ai"::S,"ifail"::S]$Lisp,_
    [([iarArg::Any,iaiArg::Any,nArg::Any,ivrArg::Any,iviArg::Any,ifailArg::An
    @List Any]$Lisp)$Lisp)_
    pretend List (Record(key:Symbol,entry:Any))]$Result

f02awf(iarArg:Integer,iaiArg:Integer,nArg:Integer,_
    arArg:Matrix DoubleFloat,aiArg:Matrix DoubleFloat,ifailArg:Integer): Resu
    [(invokeNagman(NIL$Lisp,_
    "f02awf",_
    ["iar"::S,"iai"::S,"n"::S,"ifail"::S,"r"::S,"ar"::S,"ai"::S,"wk1"::S,"wk2
    ,"wk3"::S]$Lisp,_
    ["r"::S,"wk1"::S,"wk2"::S,"wk3"::S]$Lisp,_
    [["double"::S,["r"::S,"n"::S]$Lisp,["ar"::S,"iar"::S,"n"::S]$Lisp_
    ,["ai"::S,"iai"::S,"n"::S]$Lisp,["wk1"::S,"n"::S]$Lisp,["wk2"::S,"n"::S]$
    ]$Lisp_
    ,["integer"::S,"iar"::S,"iai"::S,"n"::S,"ifail"::S_
    ]$Lisp_
    ]$Lisp,_
    ["r"::S,"ar"::S,"ai"::S,"ifail"::S]$Lisp,_
    [([iarArg::Any,iaiArg::Any,nArg::Any,ifailArg::Any,arArg::Any,aiArg::Any
    @List Any]$Lisp)$Lisp)_
    pretend List (Record(key:Symbol,entry:Any))]$Result

f02axf(arArg:Matrix DoubleFloat,iarArg:Integer,aiArg:Matrix DoubleFloat,_
    iaiArg:Integer,nArg:Integer,ivrArg:Integer,_
    iviArg:Integer,ifailArg:Integer): Result ==
    [(invokeNagman(NIL$Lisp,_
    "f02axf",_
    ["iar"::S,"iai"::S,"n"::S,"ivr"::S,"ivi"::S_
    ,"ifail"::S,"ar"::S,"ai"::S,"r"::S,"vr"::S,"vi"::S_
    ,"wk1"::S,"wk2"::S,"wk3"::S]$Lisp,_
    ["r"::S,"vr"::S,"vi"::S,"wk1"::S,"wk2"::S,"wk3"::S]$Lisp,_
    [["double"::S,["ar"::S,"iar"::S,"n"::S]$Lisp_
    ,["ai"::S,"iai"::S,"n"::S]$Lisp,["r"::S,"n"::S]$Lisp,["vr"::S,"ivr"::S,"n
    ,["wk2"::S,"n"::S]$Lisp,["wk3"::S,"n"::S]$Lisp]$Lisp_
    ,["integer"::S,"iar"::S,"iai"::S,"n"::S,"ivr"::S_
```

```
    ,"ivi"::S,"ifail"::S]$Lisp_
    ]$Lisp,_
    ["r"::S,"vr"::S,"vi"::S,"ifail"::S]$Lisp,_
    [([iarArg::Any,iaiArg::Any,nArg::Any,ivrArg::Any,iviArg::Any,ifailArg::Any,arArg::Ar
    @List Any]$Lisp)$Lisp)_
    pretend List (Record(key:Symbol,entry:Any))]$Result

f02bbf(iaArg:Integer,nArg:Integer,albArg:DoubleFloat,_
    ubArg:DoubleFloat,mArg:Integer,ivArg:Integer,_
    aArg:Matrix DoubleFloat,ifailArg:Integer): Result ==
    [(invokeNagman(NIL$Lisp,_
    "f02bbf",_
    ["ia"::S,"n"::S,"alb"::S,"ub"::S,"m"::S_
    ,"iv"::S,"mm"::S,"ifail"::S,"r"::S,"v"::S,"icount"::S,"a"::S,"d"::S_
    ,"e"::S,"e2"::S,"x"::S,"g"::S,"c"::S_
    ]$Lisp,_
    ["mm"::S,"r"::S,"v"::S,"icount"::S,"d"::S,"e"::S,"e2"::S,"x"::S,"g"::S,"c"::S]$Lisp
    [["double"::S,"alb"::S,"ub"::S,["r"::S,"m"::S]$Lisp_
    ,["v"::S,"iv"::S,"m"::S]$Lisp,["a"::S,"ia"::S,"n"::S]$Lisp,["d"::S,"n"::S]$Lisp,["e
    ,["x"::S,"n"::S,7$Lisp]$Lisp,["g"::S,"n"::S]$Lisp]$Lisp_
    ,["integer"::S,"ia"::S,"n"::S,"m"::S,"iv"::S_
    ,"mm"::S,["icount"::S,"m"::S]$Lisp,"ifail"::S]$Lisp_
    ,["logical"::S,["c"::S,"n"::S]$Lisp]$Lisp_
    ]$Lisp,_
    ["mm"::S,"r"::S,"v"::S,"icount"::S,"a"::S,"ifail"::S]$Lisp,_
    [([iaArg::Any,nArg::Any,albArg::Any,ubArg::Any,mArg::Any,ivArg::Any,ifailArg::Any,aA
    @List Any]$Lisp)$Lisp)_
    pretend List (Record(key:Symbol,entry:Any))]$Result

f02bjf(nArg:Integer,iaArg:Integer,ibArg:Integer,_
    eps1Arg:DoubleFloat,matvArg:Boolean,ivArg:Integer,_
    aArg:Matrix DoubleFloat,bArg:Matrix DoubleFloat,ifailArg:Integer): Result ==
    [(invokeNagman(NIL$Lisp,_
    "f02bjf",_
    ["n"::S,"ia"::S,"ib"::S,"eps1"::S,"matv"::S_
    ,"iv"::S,"ifail"::S,"alfr"::S,"alfi"::S,"beta"::S,"v"::S,"iter"::S_
    ,"a"::S,"b"::S]$Lisp,_
    ["alfr"::S,"alfi"::S,"beta"::S,"v"::S,"iter"::S]$Lisp,_
    [["double"::S,"eps1"::S,["alfr"::S,"n"::S]$Lisp_
    ,["alfi"::S,"n"::S]$Lisp,["beta"::S,"n"::S]$Lisp,["v"::S,"iv"::S,"n"::S]$Lisp,["a":
    ]$Lisp_
    ,["integer"::S,"n"::S,"ia"::S,"ib"::S,"iv"::S_
    ,["iter"::S,"n"::S]$Lisp,"ifail"::S]$Lisp_
    ,["logical"::S,"matv"::S]$Lisp_
    ]$Lisp,_
    ["alfr"::S,"alfi"::S,"beta"::S,"v"::S,"iter"::S,"a"::S,"b"::S,"ifail"::S]$Lisp,_
```

```
    [([nArg::Any,iaArg::Any,ibArg::Any,eps1Arg::Any,matvArg::Any,ivArg::Any,i
    @List Any]$Lisp)$Lisp)_
    pretend List (Record(key:Symbol,entry:Any))]$Result

f02fjf(nArg:Integer,kArg:Integer,tolArg:DoubleFloat,_
    novecsArg:Integer,nrxArg:Integer,lworkArg:Integer,_
    lrworkArg:Integer,liworkArg:Integer,mArg:Integer,_
    noitsArg:Integer,xArg:Matrix DoubleFloat,ifailArg:Integer,_
    dotArg:Union(fn:FileName,fp:Asp27(DOT)),imageArg:Union(fn:FileName,fp:Asp
    pushFortranOutputStack(dotFilename := aspFilename "dot")$FOP
    if dotArg case fn
              then outputAsFortran(dotArg.fn)
              else outputAsFortran(dotArg.fp)
    popFortranOutputStack()$FOP
    pushFortranOutputStack(imageFilename := aspFilename "image")$FOP
    if imageArg case fn
              then outputAsFortran(imageArg.fn)
              else outputAsFortran(imageArg.fp)
    popFortranOutputStack()$FOP
    pushFortranOutputStack(monitFilename := aspFilename "monit")$FOP
    outputAsFortran()$Asp29(MONIT)
    popFortranOutputStack()$FOP
    [(invokeNagman([dotFilename,imageFilename,monitFilename]$Lisp,_
    "f02fjf",_
    ["n"::S,"k"::S,"tol"::S,"novecs"::S,"nrx"::S_
    ,"lwork"::S,"lrwork"::S,"liwork"::S,"m"::S,"noits"::S_
    ,"ifail"::S,"dot"::S,"image"::S,"monit"::S,"d"::S,"x"::S,"work"::S,"rwork
    ]$Lisp,_
    ["d"::S,"work"::S,"rwork"::S,"iwork"::S,"dot"::S,"image"::S,"monit"::S]$L
    [["double"::S,"tol"::S,["d"::S,"k"::S]$Lisp_
    ,["x"::S,"nrx"::S,"k"::S]$Lisp,["work"::S,"lwork"::S]$Lisp,["rwork"::S,"l
    ]$Lisp_
    ,["integer"::S,"n"::S,"k"::S,"novecs"::S,"nrx"::S_
    ,"lwork"::S,"lrwork"::S,"liwork"::S,"m"::S,"noits"::S,"ifail"::S,["iwork"
    ]$Lisp,_
    ["d"::S,"m"::S,"noits"::S,"x"::S,"ifail"::S]$Lisp,_
    [([nArg::Any,kArg::Any,tolArg::Any,novecsArg::Any,nrxArg::Any,lworkArg::A
    @List Any]$Lisp)$Lisp)_
    pretend List (Record(key:Symbol,entry:Any))]$Result

f02fjf(nArg:Integer,kArg:Integer,tolArg:DoubleFloat,_
    novecsArg:Integer,nrxArg:Integer,lworkArg:Integer,_
    lrworkArg:Integer,liworkArg:Integer,mArg:Integer,_
    noitsArg:Integer,xArg:Matrix DoubleFloat,ifailArg:Integer,_
    dotArg:Union(fn:FileName,fp:Asp27(DOT)),imageArg:Union(fn:FileName,fp:Asp
    pushFortranOutputStack(dotFilename := aspFilename "dot")$FOP
```

```
    if dotArg case fn
              then outputAsFortran(dotArg.fn)
              else outputAsFortran(dotArg.fp)
    popFortranOutputStack()$FOP
    pushFortranOutputStack(imageFilename := aspFilename "image")$FOP
    if imageArg case fn
              then outputAsFortran(imageArg.fn)
              else outputAsFortran(imageArg.fp)
    popFortranOutputStack()$FOP
    pushFortranOutputStack(monitFilename := aspFilename "monit")$FOP
    outputAsFortran(monitArg)
    [(invokeNagman([dotFilename,imageFilename,monitFilename]$Lisp,_
    "f02fjf",_
    ["n"::S,"k"::S,"tol"::S,"novecs"::S,"nrx"::S_
    ,"lwork"::S,"lrwork"::S,"liwork"::S,"m"::S,"noits"::S_
    ,"ifail"::S,"dot"::S,"image"::S,"monit"::S,"d"::S,"x"::S,"work"::S,"rwork"::S,"iwork
    ]$Lisp,_
    ["d"::S,"work"::S,"rwork"::S,"iwork"::S,"dot"::S,"image"::S,"monit"::S]$Lisp,_
    [["double"::S,"tol"::S,["d"::S,"k"::S]$Lisp_
    ,["x"::S,"nrx"::S,"k"::S]$Lisp,["work"::S,"lwork"::S]$Lisp,["rwork"::S,"lrwork"::S]$
    ]$Lisp_
    ,["integer"::S,"n"::S,"k"::S,"novecs"::S,"nrx"::S_
    ,"lwork"::S,"lrwork"::S,"liwork"::S,"m"::S,"noits"::S,"ifail"::S,["iwork"::S,"liwork
    ]$Lisp,_
    ["d"::S,"m"::S,"noits"::S,"x"::S,"ifail"::S]$Lisp,_
    [([nArg::Any,kArg::Any,tolArg::Any,novecsArg::Any,nrxArg::Any,lworkArg::Any,lrworkAr
    @List Any]$Lisp)$Lisp)_
    pretend List (Record(key:Symbol,entry:Any))]$Result

f02wef(mArg:Integer,nArg:Integer,ldaArg:Integer,_
    ncolbArg:Integer,ldbArg:Integer,wantqArg:Boolean,_
    ldqArg:Integer,wantpArg:Boolean,ldptArg:Integer,_
    aArg:Matrix DoubleFloat,bArg:Matrix DoubleFloat,ifailArg:Integer): Result ==
    workLength : Integer :=
      mArg >= nArg =>
        wantqArg and wantpArg =>
          max(max(nArg**2 + 5*(nArg - 1),nArg + ncolbArg),4)
        wantqArg =>
          max(max(nArg**2 + 4*(nArg - 1),nArg + ncolbArg),4)
        wantpArg =>
          zero? ncolbArg => max(3*(nArg - 1),2)
          max(5*(nArg - 1),2)
        zero? ncolbArg => max(2*(nArg - 1),2)
        max(3*(nArg - 1),2)
      wantqArg and wantpArg =>
        max(mArg**2 + 5*(mArg - 1),2)
```

```
      wantqArg =>
        max(3*(mArg - 1),1)
      wantpArg =>
        zero? ncolbArg => max(mArg**2+3*(mArg - 1),2)
        max(mArg**2+5*(mArg - 1),2)
      zero? ncolbArg => max(2*(mArg - 1),1)
      max(3*(mArg - 1),1)

    [(invokeNagman(NIL$Lisp,_
    "f02wef",_
    ["m"::S,"n"::S,"lda"::S,"ncolb"::S,"ldb"::S_
    ,"wantq"::S,"ldq"::S,"wantp"::S,"ldpt"::S,"ifail"::S_
    ,"q"::S,"sv"::S,"pt"::S,"work"::S,"a"::S_
    ,"b"::S]$Lisp,_
    ["q"::S,"sv"::S,"pt"::S,"work"::S]$Lisp,_
    [["double"::S,["q"::S,"ldq"::S,"m"::S]$Lisp_
    ,["sv"::S,"m"::S]$Lisp,["pt"::S,"ldpt"::S,"n"::S]$Lisp,["work"::S,workLen
    ]$Lisp_
    ,["integer"::S,"m"::S,"n"::S,"lda"::S,"ncolb"::S_
    ,"ldb"::S,"ldq"::S,"ldpt"::S,"ifail"::S]$Lisp_
    ,["logical"::S,"wantq"::S,"wantp"::S]$Lisp_
    ]$Lisp,_
    ["q"::S,"sv"::S,"pt"::S,"work"::S,"a"::S,"b"::S,"ifail"::S]$Lisp,_
    [([mArg::Any,nArg::Any,ldaArg::Any,ncolbArg::Any,ldbArg::Any,wantqArg::An
    @List Any]$Lisp)$Lisp)_
    pretend List (Record(key:Symbol,entry:Any))]$Result

f02xef(mArg:Integer,nArg:Integer,ldaArg:Integer,_
    ncolbArg:Integer,ldbArg:Integer,wantqArg:Boolean,_
    ldqArg:Integer,wantpArg:Boolean,ldphArg:Integer,_
    aArg:Matrix Complex DoubleFloat,bArg:Matrix Complex DoubleFloat,ifailArg:
    -- This segment added by hand, to deal with an assumed size array GDN
    tem : Integer := (min(mArg,nArg)-1)
    rLen : Integer :=
      zero? ncolbArg and not wantqArg and not wantpArg => 2*tem
      zero? ncolbArg and wantpArg and not wantqArg => 3*tem
      not wantpArg =>
        ncolbArg >0 or wantqArg => 3*tem
      5*tem
    cLen : Integer :=
      mArg >= nArg =>
        wantqArg and wantpArg => 2*(nArg + max(nArg**2,ncolbArg))
        wantqArg and not wantpArg => 2*(nArg + max(nArg**2+nArg,ncolbArg))
        2*(nArg + max(nArg,ncolbArg))
      wantpArg => 2*(mArg**2 + mArg)
      2*mArg
```

```
svLength : Integer :=
  min(mArg,nArg)
[(invokeNagman(NIL$Lisp,_
"f02xef",_
["m"::S,"n"::S,"lda"::S,"ncolb"::S,"ldb"::S_
,"wantq"::S,"ldq"::S,"wantp"::S,"ldph"::S,"ifail"::S_
,"q"::S,"sv"::S,"ph"::S,"rwork"::S,"a"::S_
,"b"::S,"cwork"::S]$Lisp,_
["q"::S,"sv"::S,"ph"::S,"rwork"::S,"cwork"::S]$Lisp,_
[["double"::S,["sv"::S,svLength]$Lisp,["rwork"::S,rLen]$Lisp_
]$Lisp_
,["integer"::S,"m"::S,"n"::S,"lda"::S,"ncolb"::S_
,"ldb"::S,"ldq"::S,"ldph"::S,"ifail"::S]$Lisp_
,["logical"::S,"wantq"::S,"wantp"::S]$Lisp_
,["double complex"::S,["q"::S,"ldq"::S,"m"::S]$Lisp,["ph"::S,"ldph"::S,"n"::S]$Lisp
]$Lisp,_
["q"::S,"sv"::S,"ph"::S,"rwork"::S,"a"::S,"b"::S,"ifail"::S]$Lisp,_
[([mArg::Any,nArg::Any,ldaArg::Any,ncolbArg::Any,ldbArg::Any,wantqArg::Any,ldqArg::A
@List Any]$Lisp)$Lisp)_
pretend List (Record(key:Symbol,entry:Any))]$Result
```

⟨*NAGF02.dotabb*⟩≡

```
"NAGF02" [color="#FF4488",href="bookvol10.4.pdf#nameddest=NAGF02"]
"ALIST" [color="#88FF44",href="bookvol10.3.pdf#nameddest=ALIST"]
"NAGE02" -> "ALIST"
```

15.3 package NAGE02 NagFittingPackage

⟨*NagFittingPackage.help*⟩≡

```
E02 -- Curve and Surface Fitting              Introduction -- E02
                          Chapter E02
                  Curve and Surface Fitting

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```

1. Scope of the Chapter

The main aim of this chapter is to assist the user in finding a function which approximates a set of data points. Typically the data contain random errors, as of experimental measurement, which need to be smoothed out. To seek an approximation to the data, it is first necessary to specify for the approximating function a mathematical form (a polynomial, for example) which contains a number of unspecified coefficients: the appropriate fitting routine then derives for the coefficients the values which provide the best fit of that particular form. The chapter deals mainly with curve and surface fitting (i.e., fitting with functions of one and of two variables) when a polynomial or a cubic spline is used as the fitting function, since these cover the most common needs. However, fitting with other functions and/or more variables can be undertaken by means of general

linear or nonlinear routines (some of which are contained in other chapters) depending on whether the coefficients in the function occur linearly or nonlinearly. Cases where a graph rather than a set of data points is given can be treated simply by first reading a suitable set of points from the graph.

The chapter also contains routines for evaluating, differentiating and integrating polynomial and spline curves and surfaces, once the numerical values of their coefficients have been determined.

2. Background to the Problems

2.1. Preliminary Considerations

In the curve-fitting problems considered in this chapter, we have a dependent variable y and an independent variable x, and we are given a set of data points (x_r, y_r), for r=1,2,...,m. The preliminary matters to be considered in this section will, for simplicity, be discussed in this context of curve-fitting problems. In fact, however, these considerations apply equally well to surface and higher-dimensional problems. Indeed, the discussion presented carries over essentially as it stands if, for these cases, we interpret x as a vector of several independent variables and correspondingly each x_r as a vector containing the rth data value of each independent variable.

We wish, then, to approximate the set of data points as closely as possible with a specified function, f(x) say, which is as smooth as possible -- f(x) may, for example, be a polynomial. The requirements of smoothness and closeness conflict, however, and a balance has to be struck between them. Most often, the smoothness requirement is met simply by limiting the number of coefficients allowed in the fitting function -- for example, by restricting the degree in the case of a polynomial. Given a particular number of coefficients in the function in question, the fitting routines of this chapter determine the values of the coefficients such that the 'distance' of the function from the data points is as small as possible. The necessary balance is struck by the user comparing a selection of such fits having different numbers of coefficients. If the number of coefficients is too low, the approximation to the data will be poor. If the number is too high, the fit will be too close to the data, essentially following the random errors and tending to have unwanted

fluctuations between the data points. Between these extremes, there is often a group of fits all similarly close to the data points and then, particularly when least-squares polynomials are used, the choice is clear: it is the fit from this group having the smallest number of coefficients.

The above process can be seen as the user minimizing the smoothness measure (i.e., the number of coefficients) subject to the distance from the data points being acceptably small. Some of the routines, however, do this task themselves. They use a different measure of smoothness (in each case one that is continuous) and minimize it subject to the distance being less than a threshold specified by the user. This is a much more automatic process, requiring only some experimentation with the threshold.

2.1.1. Fitting criteria: norms

A measure of the above 'distance' between the set of data points and the function f(x) is needed. The distance from a single data point (x_r, y_r) to the function can simply be taken as

$$\epsilon_r = y_r - f(x_r), \tag{1}$$

and is called the residual of the point. (With this definition, the residual is regarded as a function of the coefficients contained in f(x); however, the term is also used to mean the particular value of ϵ_r which corresponds to the fitted values of the coefficients.) However, we need a measure of distance for the set of data points as a whole. Three different measures are used in the different routines (which measure to select, according to circumstances, is discussed later in this sub-section). With ϵ_r defined in (1), these measures, or norms, are

$$\sum_{r=1}^{m} |\epsilon_r|, \tag{2}$$

```
                    / m
                   /  --          2
                  /   >  (epsilon) ,  and                    (3)
                 /    --          r
                \/    r=1

                    max |(epsilon) |,                        (4)
                       r          r
```

respectively the l norm, the l norm and the l norm.
1 2 infty

Minimization of one or other of these norms usually provides the fitting criterion, the minimization being carried out with respect to the coefficients in the mathematical form used for f(x): with respect to the b for example if the mathematical form
i
is the power series in (8) below. The fit which results from minimizing (2) is known as the l fit, or the fit in the l norm:
1 1
that which results from minimizing (3) is the l fit, the well-
2
known least-squares fit (minimizing (3) is equivalent to minimizing the square of (3), i.e., the sum of squares of residuals, and it is the latter which is used in practice), and that from minimizing (4) is the l , or minimax, fit.
infty

Strictly speaking, implicit in the use of the above norms are the statistical assumptions that the random errors in the y are
r
independent of one another and that any errors in the x are
r
negligible by comparison. From this point of view, the use of the l norm is appropriate when the random errors in the y have a
2 r
normal distribution, and the l norm is appropriate when they
infty
have a rectangular distribution, as when fitting a table of values rounded to a fixed number of decimal places. The l norm
1
is appropriate when the error distribution has its frequency function proportional to the negative exponential of the modulus of the normalised error -- not a common situation.

However, the user is often indifferent to these statistical considerations, and simply seeks a fit which he can assess by inspection, perhaps visually from a graph of the results. In this event, the l_1 norm is particularly appropriate when the data are thought to contain some 'wild' points (since fitting in this norm tends to be unaffected by the presence of a small number of such points), though of course in simple situations the user may prefer to identify and reject these points. The l_∞ norm should be used only when the maximum residual is of particular concern, as may be the case for example when the data values have been obtained by accurate computation, as of a mathematical function. Generally, however, a routine based on least-squares should be preferred, as being computationally faster and usually providing more information on which to assess the results. In many problems the three fits will not differ significantly for practical purposes.

Some of the routines based on the l_2 norm do not minimize the norm itself but instead minimize some (intuitively acceptable) measure of smoothness subject to the norm being less than a user-specified threshold. These routines fit with cubic or bicubic splines (see (10) and (14) below) and the smoothing measures relate to the size of the discontinuities in their third derivatives. A much more automatic fitting procedure follows from this approach.

2.1.2. Weighting of data points

The use of the above norms also assumes that the data values y_r are of equal (absolute) accuracy. Some of the routines enable an allowance to be made to take account of differing accuracies. The allowance takes the form of 'weights' applied to the y-values so that those values known to be more accurate have a greater influence on the fit than others. These weights, to be supplied by the user, should be calculated from estimates of the absolute accuracies of the y-values, these estimates being expressed as standard deviations, probable errors or some other measure which has the same dimensions as y. Specifically, for each y_r the corresponding weight w_r should be inversely proportional to the accuracy estimate of y_r. For example, if the percentage accuracy

                    r
is the same for all y , then the absolute accuracy of y is
                     r   r
proportional to y (assuming y to be positive, as it usually is
                r   r
in such cases) and so w =K/y , for r=1,2,...,m, for an arbitrary
                      r   r
positive constant K. (This definition of weight is stressed
because often weight is defined as the square of that used here.)
The norms (2), (3) and (4) above are then replaced respectively
by

$$\sum_{r=1}^{m} |w_r(\epsilon)_r|, \tag{5}$$

$$\left(\sum_{r=1}^{m} w_r^2(\epsilon)_r^2\right)^{1/2}, \text{ and} \tag{6}$$

$$\max_r |w_r(\epsilon)_r|. \tag{7}$$

Again it is the square of (6) which is used in practice rather
than (6) itself.

2.2. Curve Fitting

When, as is commonly the case, the mathematical form of the
fitting function is immaterial to the problem, polynomials and
cubic splines are to be preferred because their simplicity and
ease of handling confer substantial benefits. The cubic spline is
the more versatile of the two. It consists of a number of cubic
polynomial segments joined end to end with continuity in first
and second derivatives at the joins. The third derivative at the
joins is in general discontinuous. The x-values of the joins are
called knots, or, more precisely, interior knots. Their number
determines the number of coefficients in the spline, just as the
degree determines the number of coefficients in a polynomial.

2.2.1. Representation of polynomials

Rather than using the power-series form

$$f(x)==b_0+b_1x+b_2x^2+\ldots+b_kx^k \tag{8}$$

to represent a polynomial, the routines in this chapter use the Chebyshev series form

$$f(x)==\tfrac{1}{2}a_0T_0(x)+a_1T_1(x)+a_2T_2(x)+\ldots+a_kT_k(x), \tag{9}$$

where $T_i(x)$ is the Chebyshev polynomial of the first kind of degree i (see Cox and Hayes [1], page 9), and where the range of x has been normalised to run from -1 to +1. The use of either form leads theoretically to the same fitted polynomial, but in practice results may differ substantially because of the effects of rounding error. The Chebyshev form is to be preferred, since it leads to much better accuracy in general, both in the computation of the coefficients and in the subsequent evaluation of the fitted polynomial at specified points. This form also has other advantages: for example, since the later terms in (9) generally decrease much more rapidly from left to right than do those in (8), the situation is more often encountered where the last terms are negligible and it is obvious that the degree of the polynomial can be reduced (note that on the interval -1<=x<=1 for all i, $T_i(x)$ attains the value unity but never exceeds it, so that the coefficient a_i gives directly the maximum value of the term containing it).

2.2.2. Representation of cubic splines

A cubic spline is represented in the form

$$f(x)==c_1N_1(x)+c_2N_2(x)+\ldots+c_pN_p(x), \tag{10}$$

where $N_i(x)$, for i=1,2,...,p, is a normalised cubic B-spline (see Hayes [2]). This form, also, has advantages of computational

speed and accuracy over alternative representations.

2.3. Surface Fitting

There are now two independent variables, and we shall denote these by x and y. The dependent variable, which was denoted by y in the curve-fitting case, will now be denoted by f. (This is a rather different notation from that indicated for the general-dimensional problem in the first paragraph of Section 2.1 , but it has some advantages in presentation.)

Again, in the absence of contrary indications in the particular application being considered, polynomials and splines are the approximating functions most commonly used. Only splines are used by the surface-fitting routines in this chapter.

2.3.1. Bicubic splines: definition and representation

The bicubic spline is defined over a rectangle R in the (x,y) plane, the sides of R being parallel to the x- and y-axes. R is divided into rectangular panels, again by lines parallel to the axes. Over each panel the bicubic spline is a bicubic polynomial, that is it takes the form

$$\sum_{i=0}^{3}\sum_{j=0}^{3} a_{ij}\, x^i y^j . \qquad (13)$$

Each of these polynomials joins the polynomials in adjacent panels with continuity up to the second derivative. The constant x-values of the dividing lines parallel to the y-axis form the set of interior knots for the variable x, corresponding precisely to the set of interior knots of a cubic spline. Similarly, the constant y-values of dividing lines parallel to the x-axis form the set of interior knots for the variable y. Instead of representing the bicubic spline in terms of the above set of bicubic polynomials, however, it is represented, for the sake of computational speed and accuracy, in the form

$$f(x,y) = \sum_{i=1}^{p}\sum_{j=1}^{q} c_{ij} M_i(x) N_j(y), \qquad (14)$$

where M (x), for i=1,2,...,p, and N (y), for j=1,2,...,q, are
       i   j
normalised B-splines (see Hayes and Halliday [4] for further details of bicubic splines and Hayes [2] for normalised B-splines).

2.4. General Linear and Nonlinear Fitting Functions

We have indicated earlier that, unless the data-fitting application under consideration specifically requires some other type of fitting function, a polynomial or a spline is usually to be preferred. Special routines for these functions, in one and in two variables, are provided in this chapter. When the application does specify some other fitting function, however, it may be treated by a routine which deals with a general linear function, or by one for a general nonlinear function, depending on whether the coefficients in the given function occur linearly or nonlinearly.

The general linear fitting function can be written in the form

$$f(x)==c_1\,\phi_1(x)+c_2\,\phi_2(x)+\ldots+c_p\,\phi_p(x), \qquad (15)$$

where x is a vector of one or more independent variables, and the ϕ_i are any given functions of these variables (though they must be linearly independent of one another if there is to be the possibility of a unique solution to the fitting problem). This is not intended to imply that each ϕ_i is necessarily a function of all the variables: we may have, for example, that each ϕ_i is a function of a different single variable, and even that one of the ϕ_i is a constant. All that is required is that a value of each $\phi_i(x)$ can be computed when a value of each independent variable is given.

When the fitting function f(x) is not linear in its coefficients, no more specific representation is available in general than f(x) itself. However, we shall find it helpful later on to indicate the fact that f(x) contains a number of coefficients (to be determined by the fitting process) by using instead the notation

f(x;c), where c denotes the vector of coefficients. An example of a nonlinear fitting function is

$$f(x;c) = c_1 + c_2 \exp(-c_4 x) + c_3 \exp(-c_5 x), \qquad (16)$$

which is in one variable and contains five coefficients. Note that here, as elsewhere in this Chapter Introduction, we use the term 'coefficients' to include all the quantities whose values are to be determined by the fitting process, not just those which occur linearly. We may observe that it is only the presence of the coefficients c_4 and c_5 which makes the form (16) nonlinear. If the values of these two coefficients were known beforehand, (16) would instead be a linear function which, in terms of the general linear form (15), has p=3 and

$$\phi_1(x) = 1, \quad \phi_2(x) = \exp(-c_4 x), \text{ and } \phi_3(x) = \exp(-c_5 x).$$

We may note also that polynomials and splines, such as (9) and (14), are themselves linear in their coefficients. Thus if, when fitting with these functions, a suitable special routine is not available (as when more than two independent variables are involved or when fitting in the l_1 norm), it is appropriate to use a routine designed for a general linear function.

2.5. Constrained Problems

So far, we have considered only fitting processes in which the values of the coefficients in the fitting function are determined by an unconstrained minimization of a particular norm. Some fitting problems, however, require that further restrictions be placed on the determination of the coefficient values. Sometimes these restrictions are contained explicitly in the formulation of the problem in the form of equalities or inequalities which the coefficients, or some function of them, must satisfy. For example, if the fitting function contains a term Aexp(-kx), it may be required that k>=0. Often, however, the equality or inequality constraints relate to the value of the fitting function or its derivatives at specified values of the independent variable(s), but these too can be expressed in terms of the coefficients of the fitting function, and it is appropriate to do this if a general linear or nonlinear routine

is being used. For example, if the fitting function is that given in (10), the requirement that the first derivative of the function at x=x be non-negative can be expressed as
0

c N ’(x)+c N ’(x)+...+c N ’(x)>=0, (17)
1 1 0 2 2 0 p p 0

where the prime denotes differentiation with respect to x and each derivative is evaluated at x=x . On the other hand, if the
0
requirement had been that the derivative at x=x be exactly zero,
0
the inequality sign in (17) would be replaced by an equality.

Routines which provide a facility for minimizing the appropriate norm subject to such constraints are discussed in Section 3.6.

2.6. References

[1] Cox M G and Hayes J G (1973) Curve fitting: a guide and suite of algorithms for the non-specialist user. Report NAC26. National Physical Laboratory.

[2] Hayes J G (1974) Numerical Methods for Curve and Surface Fitting. Bull Inst Math Appl. 10 144--152.

(For definition of normalised B-splines and details of numerical methods.)

[3] Hayes J G (1970) Curve Fitting by Polynomials in One Variable. Numerical Approximation to Functions and Data. (ed J G Hayes) Athlone Press, London.

[4] Hayes J G and Halliday J (1974) The Least-squares Fitting of Cubic Spline Surfaces to General Data Sets. J. Inst. Math. Appl. 14 89--103.

3. Recommendations on Choice and Use of Routines

3.1. General

The choice of a routine to treat a particular fitting problem will depend first of all on the fitting function and the norm to be used. Unless there is good reason to the contrary, the fitting function should be a polynomial or a cubic spline (in the

appropriate number of variables) and the norm should be the l_2 norm (leading to the least-squares fit). If some other function is to be used, the choice of routine will depend on whether the function is nonlinear (in which case see Section 3.5.2) or linear in its coefficients (see Section 3.5.1), and, in the latter case, on whether the l_1 or l_2 norm is to be used. The latter section is appropriate for polynomials and splines, too, if the l_1 norm is preferred.

In the case of a polynomial or cubic spline, if there is only one independent variable, the user should choose a spline (Section 3.3) when the curve represented by the data is of complicated form, perhaps with several peaks and troughs. When the curve is of simple form, first try a polynomial (see Section 3.2) of low degree, say up to degree 5 or 6, and then a spline if the polynomial fails to provide a satisfactory fit. (Of course, if third-derivative discontinuities are unacceptable to the user, a polynomial is the only choice.) If the problem is one of surface fitting, one of the spline routines should be used (Section 3.4). If the problem has more than two independent variables, it may be treated by the general linear routine in Section 3.5.1, again using a polynomial in the first instance.

Another factor which affects the choice of routine is the presence of constraints, as previously discussed in Section 2.5. Indeed this factor is likely to be overriding at present, because of the limited number of routines which have the necessary facility. See Section 3.6.

3.1.1. Data considerations

A satisfactory fit cannot be expected by any means if the number and arrangement of the data points do not adequately represent the character of the underlying relationship: sharp changes in behaviour, in particular, such as sharp peaks, should be well covered. Data points should extend over the whole range of interest of the independent variable(s): extrapolation outside the data ranges is most unwise. Then, with polynomials, it is advantageous to have additional points near the ends of the ranges, to counteract the tendency of polynomials to develop fluctuations in these regions. When, with polynomial curves, the user can precisely choose the x-values of the data, the special points defined in Section 3.2.2 should be selected. With splines

the choice is less critical as long as the character of the relationship is adequately represented. All fits should be tested graphically before accepting them as satisfactory.

For this purpose it should be noted that it is not sufficient to plot the values of the fitted function only at the data values of the independent variable(s); at the least, its values at a similar number of intermediate points should also be plotted, as unwanted fluctuations may otherwise go undetected. Such fluctuations are the less likely to occur the lower the number of coefficients chosen in the fitting function. No firm guide can be given, but as a rough rule, at least initially, the number of coefficients should not exceed half the number of data points (points with equal or nearly equal values of the independent variable, or both independent variables in surface fitting, counting as a single point for this purpose). However, the situation may be such, particularly with a small number of data points, that a satisfactorily close fit to the data cannot be achieved without unwanted fluctuations occurring. In such cases, it is often possible to improve the situation by a transformation of one or more of the variables, as discussed in the next paragraph: otherwise it will be necessary to provide extra data points. Further advice on curve fitting is given in Cox and Hayes [1] and, for polynomials only, in Hayes [3] of Section 2.7. Much of the advice applies also to surface fitting; see also the Routine Documents.

3.1.2. Transformation of variables

Before starting the fitting, consideration should be given to the choice of a good form in which to deal with each of the variables: often it will be satisfactory to use the variables as they stand, but sometimes the use of the logarithm, square root, or some other function of a variable will lead to a better-behaved relationship. This question is customarily taken into account in preparing graphs and tables of a relationship and the same considerations apply when curve or surface fitting. The practical context will often give a guide. In general, it is best to avoid having to deal with a relationship whose behaviour in one region is radically different from that in another. A steep rise at the left-hand end of a curve, for example, can often best be treated by curve fitting in terms of log(x+c) with some suitable value of the constant c. A case when such a transformation gave substantial benefit is discussed in Hayes [3] page 60. According to the features exhibited in any particular case, transformation of either dependent variable or independent

variable(s) or both may be beneficial. When there is a choice it is usually better to transform the independent variable(s): if the dependent variable is transformed, the weights attached to the data points must be adjusted. Thus (denoting the dependent variable by y, as in the notation for curves) if the y_r to be fitted have been obtained by a transformation y=g(Y) from original data values Y_r, with weights W_r, for r=1,2,...,m, we must take

$$w_r = W_r / (dy/dY), \qquad (18)$$

where the derivative is evaluated at Y_r. Strictly, the transformation of Y and the adjustment of weights are valid only when the data errors in the Y_r are small compared with the range spanned by the Y_r, but this is usually the case.

3.2. Polynomial Curves

3.2.1. Least-squares polynomials: arbitrary data points

E02ADF fits to arbitrary data points, with arbitrary weights, polynomials of all degrees up to a maximum degree k, which is at choice. If the user is seeking only a low degree polynomial, up to degree 5 or 6 say, k=10 is an appropriate value, providing there are about 20 data points or more. To assist in deciding the degree of polynomial which satisfactorily fits the data, the routine provides the root-mean-square-residual s_i for all degrees i=1,2,...,k. In a satisfactory case, these s_i will decrease steadily as i increases and then settle down to a fairly constant value, as shown in the example

i	s_i
0	3.5215
1	0.7708

```
        2    0.1861

        3    0.0820

        4    0.0554

        5    0.0251

        6    0.0264

        7    0.0280

        8    0.0277

        9    0.0297

        10   0.0271

If the s  values settle down in this way, it indicates that the
        i
closest polynomial approximation justified by the data has been
achieved. The degree which first gives the approximately constant
value of s  (degree 5 in the example) is the appropriate degree
          i
to select. (Users who are prepared to accept a fit higher than
sixth degree, should simply find a high enough value of k to
enable the type of behaviour indicated by the example to be
detected: thus they should seek values of k for which at least 4
or 5 consecutive values of s  are approximately the same.) If the
                            i
degree were allowed to go high enough, s  would, in most cases,
                                        i
eventually start to decrease again, indicating that the data
points are being fitted too closely and that undesirable
fluctuations are developing between the points. In some cases,
particularly with a small number of data points, this final
decrease is not distinguishable from the initial decrease in s .
                                                              i
In such cases, users may seek an acceptable fit by examining the
graphs of several of the polynomials obtained. Failing this, they
may (a) seek a transformation of variables which improves the
behaviour, (b) try fitting a spline, or (c) provide more data
points. If data can be provided simply by drawing an
approximating curve by hand and reading points from it, use the
points discussed in Section 3.2.2.
```

3.2.2. Least-squares polynomials: selected data points

When users are at liberty to choose the x-values of data points, such as when the points are taken from a graph, it is most advantageous when fitting with polynomials to use the values $x_r = \cos((\pi r)/n)$, for r=0,1,...,n for some value of n, a suitable value for which is discussed at the end of this section. Note that these x_r relate to the variable x after it has been normalised so that its range of interest is -1 to +1. E02ADF may then be used as in Section 3.2.1 to seek a satisfactory fit.

3.3. Cubic Spline Curves

3.3.1. Least-squares cubic splines

E02BAF fits to arbitrary data points, with arbitrary weights, a cubic spline with interior knots specified by the user. The choice of these knots so as to give an acceptable fit must largely be a matter of trial and error, though with a little experience a satisfactory choice can often be made after one or two trials. It is usually best to start with a small number of knots (too many will result in unwanted fluctuations in the fit, or even in there being no unique solution) and, examining the fit graphically at each stage, to add a few knots at a time at places where the fit is particularly poor. Moving the existing knots towards these places will also often improve the fit. In regions where the behaviour of the curve underlying the data is changing rapidly, closer knots will be needed than elsewhere. Otherwise, positioning is not usually very critical and equally-spaced knots are often satisfactory. See also the next section, however.

A useful feature of the routine is that it can be used in applications which require the continuity to be less than the normal continuity of the cubic spline. For example, the fit may be required to have a discontinuous slope at some point in the range. This can be achieved by placing three coincident knots at the given point. Similarly a discontinuity in the second derivative at a point can be achieved by placing two knots there. Analogy with these discontinuous cases can provide guidance in more usual cases: for example, just as three coincident knots can produce a discontinuity in slope, so three close knots can produce a rapid change in slope. The closer the knots are, the more rapid can the change be.

Figure 1
Please see figure in printed Reference Manual

An example set of data is given in Figure 1. It is a rather tricky set, because of the scarcity of data on the right, but it will serve to illustrate some of the above points and to show some of the dangers to be avoided. Three interior knots (indicated by the vertical lines at the top of the diagram) are chosen as a start. We see that the resulting curve is not steep enough in the middle and fluctuates at both ends, severely on the right. The spline is unable to cope with the shape and more knots are needed.

In Figure 2, three knots have been added in the centre, where the data shows a rapid change in behaviour, and one further out at each end, where the fit is poor. The fit is still poor, so a further knot is added in this region and, in Figure 3, disaster ensues in rather spectacular fashion.

Figure 2
Please see figure in printed Reference Manual

Figure 3
Please see figure in printed Reference Manual

The reason is that, at the right-hand end, the fits in Figure 1 and Figure 2 have been interpreted as poor simply because of the fluctuations about the curve underlying the data (or what it is naturally assumed to be). But the fitting process knows only about the data and nothing else about the underlying curve, so it is important to consider only closeness to the data when deciding goodness of fit.

Thus, in Figure 1, the curve fits the last two data points quite well compared with the fit elsewhere, so no knot should have been added in this region. In Figure 2, the curve goes exactly through the last two points, so a further knot is certainly not needed here.

Figure 4
Please see figure in printed Reference Manual

Figure 4 shows what can be achieved without the extra knot on each of the flat regions. Remembering that within each knot interval the spline is a cubic polynomial, there is really no

need to have more than one knot interval covering each flat region.

What we have, in fact, in Figure 2 and Figure 3 is a case of too many knots (so too many coefficients in the spline equation) for the number of data points. The warning in the second paragraph of Section 2.1 was that the fit will then be too close to the data, tending to have unwanted fluctuations between the data points. The warning applies locally for splines, in the sense that, in localities where there are plenty of data points, there can be a lot of knots, as long as there are few knots where there are few points, especially near the ends of the interval. In the present example, with so few data points on the right, just the one extra knot in Figure 2 is too many! The signs are clearly present, with the last two points fitted exactly (at least to the graphical accuracy and actually much closer than that) and fluctuations within the last two knot-intervals (cf. Figure 1, where only the final point is fitted exactly and one of the wobbles spans several data points).

The situation in Figure 3 is different. The fit, if computed exactly, would still pass through the last two data points, with even more violent fluctuations. However, the problem has become so ill-conditioned that all accuracy has been lost. Indeed, if the last interior knot were moved a tiny amount to the right, there would be no unique solution and an error message would have been caused. Near-singularity is, sadly, not picked up by the routine, but can be spotted readily in a graph, as Figure 3. B-spline coefficients becoming large, with alternating signs, is another indication. However, it is better to avoid such situations, firstly by providing, whenever possible, data adequately covering the range of interest, and secondly by placing knots only where there is a reasonable amount of data.

The example here could, in fact, have utilised from the start the observation made in the second paragraph of this section, that three close knots can produce a rapid change in slope. The example has two such rapid changes and so requires two sets of three close knots (in fact, the two sets can be so close that one knot can serve in both sets, so only five knots prove sufficient in Figure 4). It should be noted, however, that the rapid turn occurs within the range spanned by the three knots. This is the reason that the six knots in Figure 2 are not satisfactory as they do not quite span the two turns.

Some more examples to illustrate the choice of knots are given in

Cox and Hayes [1].

3.3.2. Automatic fitting with cubic splines

E02BEF also fits cubic splines to arbitrary data points with arbitrary weights but itself chooses the number and positions of the knots. The user has to supply only a threshold for the sum of squares of residuals. The routine first builds up a knot set by a series of trial fits in the l_2 norm. Then, with the knot set decided, the final spline is computed to minimize a certain smoothing measure subject to satisfaction of the chosen threshold. Thus it is easier to use than E02BAF (see previous section), requiring only some experimentation with this threshold. It should therefore be first choice unless the user has a preference for the ordinary least-squares fit or, for example, wishes to experiment with knot positions, trying to keep their number down (E02BEF aims only to be reasonably frugal with knots).

3.4. Spline Surfaces

3.4.1. Least-squares bicubic splines

E02DAF fits to arbitrary data points, with arbitrary weights, a bicubic spline with its two sets of interior knots specified by the user. For choosing these knots, the advice given for cubic splines, in Section 3.3.1 above, applies here too. (See also the next section, however.) If changes in the behaviour of the surface underlying the data are more marked in the direction of one variable than of the other, more knots will be needed for the former variable than the latter. Note also that, in the surface case, the reduction in continuity caused by coincident knots will extend across the whole spline surface: for example, if three knots associated with the variable x are chosen to coincide at a value L, the spline surface will have a discontinuous slope across the whole extent of the line x=L.

With some sets of data and some choices of knots, the least-squares bicubic spline will not be unique. This will not occur, with a reasonable choice of knots, if the rectangle R is well covered with data points: here R is defined as the smallest rectangle in the (x,y) plane, with sides parallel to the axes, which contains all the data points. Where the least-squares solution is not unique, the minimal least-squares solution is computed, namely that least-squares solution which has the

smallest value of the sum of squares of the B-spline coefficients
c   (see the end of Section 2.3.2 above). This choice of least-
 ij
squares solution tends to minimize the risk of unwanted
fluctuations in the fit. The fit will not be reliable, however,
in regions where there are few or no data points.

3.4.2. Automatic fitting with bicubic splines

E02DDF also fits bicubic splines to arbitrary data points with
arbitrary weights but chooses the knot sets itself. The user has
to supply only a threshold for the sum of squares of residuals.
Just like the automatic curve E02BEF (Section 3.3.2), E02DDF then
builds up the knot sets and finally fits a spline minimizing a
smoothing measure subject to satisfaction of the threshold.
Again, this easier to use routine is normally to be preferred, at
least in the first instance.

E02DCF is a very similar routine to E02DDF but deals with data
points of equal weight which lie on a rectangular mesh in the
(x,y) plane. This kind of data allows a very much faster
computation and so is to be preferred when applicable.
Substantial departures from equal weighting can be ignored if the
user is not concerned with statistical questions, though the
quality of the fit will suffer if this is taken too far. In such
cases, the user should revert to E02DDF.

3.5. General Linear and Nonlinear Fitting Functions

3.5.1. General linear functions

For the general linear function (15), routines are available for
fitting in the l and l norms. The least-squares routines (which
              1   2
are to be preferred unless there is good reason to use another
norm -- see Section 2.1.1) are in Chapter F04. The l routine is
                                                    1
E02GAF.

All the above routines are essentially linear algebra routines,
and in considering their use we need to view the fitting process
in a slightly different way from hitherto. Taking y to be the
dependent variable and x the vector of independent variables, we
have, as for equation (1) but with each x now a vector,
                                        r

$$(\epsilon)_r = y_r - f(x_r) \quad r=1,2,\ldots,m.$$

Substituting for f(x) the general linear form (15), we can write this as

$$c_1(\phi_1)(x_r) + c_2(\phi_2)(x_r) + \ldots + c_p(\phi_p)(x_r) = y_r - (\epsilon)_r, \quad r=1,2,\ldots,m \tag{19}$$

Thus we have a system of linear equations in the coefficients c_j. Usually, in writing these equations, the $(\epsilon)_r$ are omitted and simply taken as implied. The system of equations is then described as an overdetermined system (since we must have m>=p if there is to be the possibility of a unique solution to our fitting problem), and the fitting process of computing the c_j to minimize one or other of the norms (2), (3) and (4) can be described, in relation to the system of equations, as solving the overdetermined system in that particular norm. In matrix notation, the system can be written as

$$(\Phi)c = y, \tag{20}$$

where (Φ) is the m by p matrix whose element in row r and column j is $(\phi_j)(x_r)$, for r=1,2,...,m; j=1,2,...,p. The vectors c and y respectively contain the coefficients c_j and the data values y_r.

The routines, however, use the standard notation of linear algebra, the overdetermined system of equations being denoted by

$$Ax = b \tag{21}$$

The correspondence between this notation and that which we have used for the data-fitting problem (equation (20)) is therefore given by

$$A == (\Phi), \quad x == c \quad b == y \tag{22}$$

Note that the norms used by these routines are the unweighted norms (2) and (3). If the user wishes to apply weights to the data points, that is to use the norms (5) or (6), the equivalences (22) should be replaced by

$$A==D(Phi), \quad x==c \quad b==Dy$$

where D is a diagonal matrix with w_r as the rth diagonal element. Here w_r, for r=1,2,...,m, is the weight of the rth data point as defined in Section 2.1.2.

3.5.2. Nonlinear functions

Routines for fitting with a nonlinear function in the l_2 norm are provided in Chapter E04, and that chapter's Introduction should be consulted for the appropriate choice of routine. Again, however, the notation adopted is different from that we have used for data fitting. In the latter, we denote the fitting function by f(x;c), where x is the vector of independent variables and c is the vector of coefficients, whose values are to be determined. The squared l_2 norm, to be minimized with respect to the elements of c, is then

$$\sum_{r=1}^{m} w_r^2 [y_r - f(x_r;c)]^2 \qquad (23)$$

where y_r is the rth data value of the dependent variable, x_r is the vector containing the rth values of the independent variables, and w_r is the corresponding weight as defined in Section 2.1.2.

On the other hand, in the nonlinear least-squares routines of Chapter E04, the function to be minimized is denoted by

$$\sum_{}^{m} f_r^2(x), \qquad (24)$$

$$--\ i$$
$$i=1$$

the minimization being carried out with respect to the elements of the vector x. The correspondence between the two notations is given by

x==c and

$$f_i(x)==w_r\,[y_r - f(x;c)],\quad i=r=1,2,\ldots,m.$$

Note especially that the vector x of variables of the nonlinear least-squares routines is the vector c of coefficients of the data-fitting problem, and in particular that, if the selected routine requires derivatives of the $f_i(x)$ to be provided, these are derivatives of $w_r\,[y_r - f(x;c)]$ with respect to the coefficients of the data-fitting problem.

3.6. Constraints

At present, there are only a limited number of routines which fit subject to constraints. Chapter E04 contains a routine, E04UCF, which can be used for fitting with a nonlinear function in the l_2 norm subject to equality or inequality constraints. This routine, unlike those in that chapter suited to the unconstrained case, is not designed specifically for minimizing functions which are sums of squares, and so the function (23) has to be treated as a general nonlinear function. The E04 Chapter Introduction should be consulted.

The remaining constraint routine relates to fitting with polynomials in the l_2 norm. E02AGF deals with polynomial curves and allows precise values of the fitting function and (if required) all its derivatives up to a given order to be prescribed at one or more values of the independent variable.

3.7. Evaluation, Differentiation and Integration

Routines are available to evaluate, differentiate and integrate polynomials in Chebyshev-series form and cubic or bicubic splines in B-spline form. These polynomials and splines may have been

produced by the various fitting routines or, in the case of polynomials, from prior calls of the differentiation and integration routines themselves.

E02AEF and E02AKF evaluate polynomial curves: the latter has a longer parameter list but does not require the user to normalise the values of the independent variable and can accept coefficients which are not stored in contiguous locations. E02BBF evaluates cubic spline curves, and E02DEF and E02DFF bicubic spline surfaces.

Differentiation and integration of polynomial curves are carried out by E02AHF and E02AJF respectively. The results are provided in Chebyshev-series form and so repeated differentiation and integration are catered for. Values of the derivative or integral can then be computed using the appropriate evaluation routine.

For splines the differentiation and integration routines provided are of a different nature from those for polynomials. E02BCF provides values of a cubic spline curve and its first three derivatives (the rest, of course, are zero) at a given value of x spline over its whole range. These routines can also be applied to surfaces of the form (14). For example, if, for each value of j in turn, the coefficients c_{ij}, for i=1,2,...,p are supplied to E02BCF with $x=x_0$ and on each occasion we select from the output the value of the second derivative, d_j say, and if the whole set of d_j are then supplied to the same routine with $x=y_0$, the output will contain all the values at (x_0,y_0) of

$$\frac{\partial^2 f}{\partial x^2} \quad \text{and} \quad \frac{\partial^{r+2} f}{\partial x^2 \partial y^r}, \quad r=1,2,3.$$

Equally, if after each of the first p calls of E02BCF we had selected the function value (E02BBF would also provide this) instead of the second derivative and we had supplied these values to E02BDF, the result obtained would have been the value of

```
                                  /
                                  |f(x ,y)dy,
                                  /   0
                                  A
```

where A and B are the end-points of the y interval over which the spline was defined.

3.8. Index

Chapter E02

Curve and Surface Fitting

E02ADF Least-squares curve fit, by polynomials, arbitrary data points

E02AEF Evaluation of fitted polynomial in one variable from Chebyshev series form (simplified parameter list)

E02AGF Least-squares polynomial fit, values and derivatives may be constrained, arbitrary data points,

E02AHF Derivative of fitted polynomial in Chebyshev series form

E02AJF Integral of fitted polynomial in Chebyshev series form

E02AKF Evaluation of fitted polynomial in one variable, from Chebyshev series form

E02BAF Least-squares curve cubic spline fit (including interpolation)

E02BBF Evaluation of fitted cubic spline, function only

E02BCF Evaluation of fitted cubic spline, function and derivatives

E02BDF Evaluation of fitted cubic spline, definite integral

E02BEF Least-squares cubic spline curve fit, automatic knot placement

E02DAF Least-squares surface fit, bicubic splines

E02DCF Least-squares surface fit by bicubic splines with automatic knot placement, data on rectangular grid

E02DDF Least-squares surface fit by bicubic splines with automatic knot placement, scattered data

E02DEF Evaluation of a fitted bicubic spline at a vector of points

E02DFF Evaluation of a fitted bicubic spline at a mesh of points

E02GAF L -approximation by general linear function
1

E02ZAF Sort 2-D data into panels for fitting bicubic splines

%%

E02 -- Curve and Surface Fitting E02ADF
E02ADF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for your implementation to check implementation-dependent details. The symbol (*) after a NAG routine name denotes a routine that is not included in the Foundation Library.

1. Purpose

E02ADF computes weighted least-squares polynomial approximations to an arbitrary set of data points.

2. Specification

```
 SUBROUTINE E02ADF (M, KPLUS1, NROWS, X, Y, W, WORK1,
1                   WORK2, A, S, IFAIL)
 INTEGER          M, KPLUS1, NROWS, IFAIL
 DOUBLE PRECISION X(M), Y(M), W(M), WORK1(3*M), WORK2
1                 (2*KPLUS1), A(NROWS,KPLUS1), S(KPLUS1)
```

3. Description

This routine determines least-squares polynomial approximations of degrees 0,1,...,k to the set of data points (x_r, y_r) with weights w_r, for r=1,2,...,m.

The approximation of degree i has the property that it minimizes σ_i, the sum of squares of the weighted residuals ϵ_r, where

$$\epsilon_r = w_r (y_r - f_r)$$

and f_r is the value of the polynomial of degree i at the rth data point.

Each polynomial is represented in Chebyshev-series form with

normalised argument x. This argument lies in the range -1 to +1 and is related to the original variable x by the linear transformation

$$x = \frac{(2x - x_{max} - x_{min})}{(x_{max} - x_{min})}.$$

Here x_{max} and x_{min} are respectively the largest and smallest values of x_r. The polynomial approximation of degree i is represented as

$$\frac{1}{2}a_{i+1,1}T_0(x) + a_{i+1,2}T_1(x) + a_{i+1,3}T_2(x) + \ldots + a_{i+1,i+1}T_i(x),$$

where $T_j(x)$ is the Chebyshev polynomial of the first kind of degree j with argument (x).

For i=0,1,...,k, the routine produces the values of $a_{i+1,j+1}$, for j=0,1,...,i, together with the value of the root mean square residual $s_i = \sqrt{\frac{\sigma_i}{m-i-1}}$. In the case m=i+1 the routine sets the value of s_i to zero.

The method employed is due to Forsythe [4] and is based upon the generation of a set of polynomials orthogonal with respect to summation over the normalised data set. The extensions due to Clenshaw [1] to represent these polynomials as well as the approximating polynomials in their Chebyshev-series forms are

incorporated. The modifications suggested by Reinsch and Gentleman (see [5]) to the method originally employed by Clenshaw for evaluating the orthogonal polynomials from their Chebyshev-series representations are used to give greater numerical stability.

For further details of the algorithm and its use see Cox [2] and [3].

Subsequent evaluation of the Chebyshev-series representations of the polynomial approximations should be carried out using E02AEF.

4. References

[1] Clenshaw C W (1960) Curve Fitting with a Digital Computer. Comput. J. 2 170--173.

[2] Cox M G (1974) A Data-fitting Package for the Non-specialist User. Software for Numerical Mathematics. (ed D J Evans) Academic Press.

[3] Cox M G and Hayes J G (1973) Curve fitting: a guide and suite of algorithms for the non-specialist user. Report NAC26. National Physical Laboratory.

[4] Forsythe G E (1957) Generation and use of orthogonal polynomials for data fitting with a digital computer. J. Soc. Indust. Appl. Math. 5 74--88.

[5] Gentlemen W M (1969) An Error Analysis of Goertzel's (Watt's) Method for Computing Fourier Coefficients. Comput. J. 12 160--165.

[6] Hayes J G (1970) Curve Fitting by Polynomials in One Variable. Numerical Approximation to Functions and Data. (ed J G Hayes) Athlone Press, London.

5. Parameters

1: M -- INTEGER Input
On entry: the number m of data points. Constraint: M >= MDIST >= 2, where MDIST is the number of distinct x values in the data.

2: KPLUS1 -- INTEGER Input
On entry: k+1, where k is the maximum degree required.

Constraint: 0 < KPLUS1 <= MDIST, where MDIST is the number of distinct x values in the data.

3: NROWS -- INTEGER Input
On entry:
the first dimension of the array A as declared in the (sub)program from which E02ADF is called.
Constraint: NROWS >= KPLUS1.

4: X(M) -- DOUBLE PRECISION array Input
On entry: the values x_r of the independent variable, for r=1,2,...,m. Constraint: the values must be supplied in non-decreasing order with X(M) > X(1).

5: Y(M) -- DOUBLE PRECISION array Input
On entry: the values y_r of the dependent variable, for r=1,2,...,m.

6: W(M) -- DOUBLE PRECISION array Input
On entry: the set of weights, w_r, for r=1,2,...,m. For advice on the choice of weights, see Section 2.1.2 of the Chapter Introduction. Constraint: W(r) > 0.0, for r=1,2,...m.

7: WORK1(3*M) -- DOUBLE PRECISION array Workspace

8: WORK2(2*KPLUS1) -- DOUBLE PRECISION array Workspace

9: A(NROWS,KPLUS1) -- DOUBLE PRECISION array Output

On exit: the coefficients of $T_j(x)$ in the approximating polynomial of degree i. A(i+1,j+1) contains the coefficient $a_{i+1,j+1}$, for i=0,1,...,k; j=0,1,...,i.

10: S(KPLUS1) -- DOUBLE PRECISION array Output
On exit: S(i+1) contains the root mean square residual s_i, for i=0,1,...,k, as described in Section 3. For the interpretation of the values of the s_i and their use in selecting an appropriate degree, see Section 3.1 of the

```
     Chapter Introduction.

11:  IFAIL -- INTEGER                                    Input/Output
     On entry: IFAIL must be set to 0, -1 or 1. For users not
     familiar with this parameter (described in the Essential
     Introduction) the recommended value is 0.

     On exit: IFAIL = 0 unless the routine detects an error (see
     Section 6).

6. Error Indicators and Warnings

Errors detected by the routine:

IFAIL= 1
     The weights are not all strictly positive.

IFAIL= 2
     The values of X(r), for r=1,2,...,M are not in non-
     decreasing order.

IFAIL= 3
     All X(r) have the same value: thus the normalisation of X is
     not possible.

IFAIL= 4
     On entry KPLUS1 < 1 (so the maximum degree required is
               negative)

     or        KPLUS1 > MDIST, where MDIST is the number of
               distinct x values in the data (so there cannot be a
               unique solution for degree k=KPLUS1-1).

IFAIL= 5
     NROWS < KPLUS1.

7. Accuracy

No error analysis for the method has been published. Practical
experience with the method, however, is generally extremely
satisfactory.

8. Further Comments

The time taken by the routine is approximately proportional to
m(k+1)(k+11).
```

```
The approximating polynomials may exhibit undesirable
oscillations (particularly near the ends of the range) if the
maximum degree k exceeds a critical value which depends on the
number of data points m and their relative positions. As a rough
guide, for equally-spaced data, this critical value is about


2*\/m. For further details see Hayes [6] page 60.

9. Example

Determine weighted least-squares polynomial approximations of
degrees 0, 1, 2 and 3 to a set of 11 prescribed data points. For
the approximation of degree 3, tabulate the data and the
corresponding values of the approximating polynomial, together
with the residual errors, and also the values of the
approximating polynomial at points half-way between each pair of
adjacent data points.

The example program supplied is written in a general form that
will enable polynomial approximations of degrees 0,1,...,k to be
obtained to m data points, with arbitrary positive weights, and
the approximation of degree k to be tabulated. E02AEF is used to
evaluate the approximating polynomial. The program is self-
starting in that any number of data sets can be supplied.

The example program is not reproduced here. The source code for
all example programs is distributed with the NAG Foundation
Library software and should be available on-line.

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

E02 -- Curve and Surface Fitting                              E02AEF
        E02AEF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for
your implementation to check implementation-dependent details.
The symbol (*) after a NAG routine name denotes a routine that is
not included in the Foundation Library.

1. Purpose

E02AEF evaluates a polynomial from its Chebyshev-series
representation.
```

2. Specification

```
SUBROUTINE E02AEF (NPLUS1, A, XCAP, P, IFAIL)
INTEGER          NPLUS1, IFAIL
DOUBLE PRECISION A(NPLUS1), XCAP, P
```

3. Description

This routine evaluates the polynomial

$$\frac{1}{2}a_1 T_0(x) + a_2 T_1(x) + a_3 T_2(x) + \ldots + a_{n+1} T_n(x)$$

for any value of x satisfying -1<=x<=1. Here $T_j(x)$ denotes the Chebyshev polynomial of the first kind of degree j with argument x. The value of n is prescribed by the user.

In practice, the variable x will usually have been obtained from an original variable x, where $x_{min} <= x <= x_{max}$ and

$$x = \frac{(x - x_{min}) - (x_{max} - x)}{(x_{max} - x_{min})}$$

Note that this form of the transformation should be used computationally rather than the mathematical equivalent

$$x = \frac{(2x - x_{min} - x_{max})}{(x_{max} - x_{min})}$$

since the former guarantees that the computed value of x differs from its true value by at most 4(epsilon), where (epsilon) is the machine precision, whereas the latter has no such guarantee.

The method employed is based upon the three-term recurrence relation due to Clenshaw [1], with modifications to give greater numerical stability due to Reinsch and Gentleman (see [4]).

For further details of the algorithm and its use see Cox [2] and [3].

4. References

[1] Clenshaw C W (1955) A Note on the Summation of Chebyshev Series. Math. Tables Aids Comput. 9 118--120.

[2] Cox M G (1974) A Data-fitting Package for the Non-specialist User. Software for Numerical Mathematics. (ed D J Evans) Academic Press.

[3] Cox M G and Hayes J G (1973) Curve fitting: a guide and suite of algorithms for the non-specialist user. Report NAC26. National Physical Laboratory.

[4] Gentlemen W M (1969) An Error Analysis of Goertzel's (Watt's) Method for Computing Fourier Coefficients. Comput. J. 12 160--165.

5. Parameters

1: NPLUS1 -- INTEGER Input
On entry: the number n+1 of terms in the series (i.e., one greater than the degree of the polynomial). Constraint: NPLUS1 >= 1.

2: A(NPLUS1) -- DOUBLE PRECISION array Input
On entry: A(i) must be set to the value of the ith coefficient in the series, for i=1,2,...,n+1.

3: XCAP -- DOUBLE PRECISION Input

On entry: x, the argument at which the polynomial is to be evaluated. It should lie in the range -1 to +1, but a value just outside this range is permitted (see Section 6) to allow for possible rounding errors committed in the

transformation from x to x discussed in Section 3. Provided the recommended form of the transformation is used, a successful exit is thus assured whenever the value of x lies in the range x to x .
min max

4: P -- DOUBLE PRECISION Output
On exit: the value of the polynomial.

5: IFAIL -- INTEGER Input/Output
On entry: IFAIL must be set to 0, -1 or 1. For users not familiar with this parameter (described in the Essential Introduction) the recommended value is 0.

On exit: IFAIL = 0 unless the routine detects an error (see Section 6).

6. Error Indicators and Warnings

Errors detected by the routine:

IFAIL= 1
ABS(XCAP) > 1.0 + 4(epsilon), where (epsilon) is the machine precision. In this case the value of P is set arbitrarily to zero.

IFAIL= 2
On entry NPLUS1 < 1.

7. Accuracy

The rounding errors committed are such that the computed value of the polynomial is exact for a slightly perturbed set of coefficients $a_i + (\delta)a_i$. The ratio of the sum of the absolute values of the $(\delta)a_i$ to the sum of the absolute values of the a_i is less than a small multiple of (n+1) times machine precision.

8. Further Comments

The time taken by the routine is approximately proportional to

```
n+1.

It is expected that a common use of E02AEF will be the evaluation
of the polynomial approximations produced by E02ADF and E02AFF(*)

9. Example



Evaluate at 11 equally-spaced points in the interval -1<=x<=1 the
polynomial of degree 4 with Chebyshev coefficients, 2.0, 0.5, 0.
25, 0.125, 0.0625.

The example program is written in a general form that will enable
a polynomial of degree n in its Chebyshev-series form to be


evaluated at m equally-spaced points in the interval -1<=x<=1.
The program is self-starting in that any number of data sets can
be supplied.

The example program is not reproduced here. The source code for
all example programs is distributed with the NAG Foundation
Library software and should be available on-line.
```

%%

```
E02 -- Curve and Surface Fitting                              E02AGF
        E02AGF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for
your implementation to check implementation-dependent details.
The symbol (*) after a NAG routine name denotes a routine that is
not included in the Foundation Library.

1. Purpose

E02AGF computes constrained weighted least-squares polynomial
approximations in Chebyshev-series form to an arbitrary set of
data points. The values of the approximations and any number of
their derivatives can be specified at selected points.

2. Specification

       SUBROUTINE E02AGF (M, KPLUS1, NROWS, XMIN, XMAX, X, Y, W,
      1                   MF, XF, YF, LYF, IP, A, S, NP1, WRK,
```

```
2                   LWRK, IWRK, LIWRK, IFAIL)
 INTEGER           M, KPLUS1, NROWS, MF, LYF, IP(MF), NP1,
1                  LWRK, IWRK(LIWRK), LIWRK, IFAIL
 DOUBLE PRECISION XMIN, XMAX, X(M), Y(M), W(M), XF(MF), YF
1                  (LYF), A(NROWS,KPLUS1), S(KPLUS1), WRK
2                  (LWRK)
```

3. Description

This routine determines least-squares polynomial approximations of degrees up to k to the set of data points (x_r, y_r) with weights w_r, for r=1,2,...,m. The value of k, the maximum degree required, is prescribed by the user. At each of the values XF_r, for r = 1,2,...,MF, of the independent variable x, the approximations and their derivatives up to order p_r are constrained to have one of the user-specified values YF_s, for s=1,2,...,n, where $n=MF+\sum_{r=1}^{MF} p_r$.

The approximation of degree i has the property that, subject to the imposed contraints, it minimizes $(Sigma)_i$, the sum of the squares of the weighted residuals $(epsilon)_r$ for r=1,2,...,m

where

$$(epsilon)_r = w_r (y_r - f_i(x_r))$$

and $f_i(x_r)$ is the value of the polynomial approximation of degree i at the rth data point.

Each polynomial is represented in Chebyshev-series form with

normalised argument x. This argument lies in the range -1 to +1 and is related to the original variable x by the linear transformation

$$x = \frac{2x - (x_{max} + x_{min})}{(x_{max} - x_{min})}$$

where x_{min} and x_{max}, specified by the user, are respectively the lower and upper end-points of the interval of x over which the polynomials are to be defined.

The polynomial approximation of degree i can be written as

$$\frac{1}{2}a_{i,0} + a_{i,1}T_1(x) + \ldots + a_{ij}T_j(x) + \ldots + a_{ii}T_i(x)$$

where $T_j(x)$ is the Chebyshev polynomial of the first kind of degree j with argument x. For $i=n,n+1,\ldots,k$, the routine produces the values of the coefficients a_{ij}, for $j=0,1,\ldots,i$, together with the value of the root mean square residual, S_i, defined as

$$S_i = \sqrt{\frac{\sum_i}{(m'+n-i-1)}}$$

, where m' is the number of data points with non-zero weight.

Values of the approximations may subsequently be computed using E02AEF or E02AKF.

First E02AGF determines a polynomial (mu)(x), of degree n-1, which satisfies the given constraints, and a polynomial (nu)(x), of degree n, which has value (or derivative) zero wherever a

constrained value (or derivative) is specified. It then fits
y -(mu)(x), for r=1,2,...,m with polynomials of the required
 r r

degree in x each with factor (nu)(x). Finally the coefficients of

(mu)(x) are added to the coefficients of these fits to give the
coefficients of the constrained polynomial approximations to the
data points (x ,y), for r=1,2,...,m. The method employed is
 r r
given in Hayes [3]: it is an extension of Forsythe's orthogonal
polynomials method [2] as modified by Clenshaw [1].

4. References

[1] Clenshaw C W (1960) Curve Fitting with a Digital Computer. Comput. J. 2 170--173.

[2] Forsythe G E (1957) Generation and use of orthogonal polynomials for data fitting with a digital computer. J. Soc. Indust. Appl. Math. 5 74--88.

[3] Hayes J G (1970) Curve Fitting by Polynomials in One Variable. Numerical Approximation to Functions and Data. (ed J G Hayes) Athlone Press, London.

5. Parameters

1: M -- INTEGER Input
On entry: the number m of data points to be fitted.
Constraint: M >= 1.

2: KPLUS1 -- INTEGER Input
On entry: k+1, where k is the maximum degree required.
Constraint: n+1<=KPLUS1<=m''+n, where n is the total number of constraints and m'' is the number of data points with non-zero weights and distinct abscissae which do not coincide with any of the XF(r).

3: NROWS -- INTEGER Input
On entry:
the first dimension of the array A as declared in the (sub)program from which E02AGF is called.
Constraint: NROWS >= KPLUS1.

4: XMIN -- DOUBLE PRECISION Input

5: XMAX -- DOUBLE PRECISION Input
On entry: the lower and upper end-points, respectively, of the interval [x_{min},x_{max}]. Unless there are specific reasons to the contrary, it is recommended that XMIN and XMAX be set respectively to the lowest and highest value among the x_r and XF(r). This avoids the danger of extrapolation provided there is a constraint point or data point with non-zero weight at each end-point. Constraint: XMAX > XMIN.

6: X(M) -- DOUBLE PRECISION array Input
On entry: the value x_r of the independent variable at the rth data point, for r=1,2,...,m. Constraint: the X(r) must be in non-decreasing order and satisfy XMIN <= X(r) <= XMAX.

7: Y(M) -- DOUBLE PRECISION array Input
On entry: Y(r) must contain y_r, the value of the dependent variable at the rth data point, for r=1,2,...,m.

8: W(M) -- DOUBLE PRECISION array Input
On entry: the weights w_r to be applied to the data points x_r, for r=1,2...,m. For advice on the choice of weights see the Chapter Introduction. Negative weights are treated as positive. A zero weight causes the corresponding data point to be ignored. Zero weight should be given to any data point whose x and y values both coincide with those of a constraint (otherwise the denominators involved in the root-mean-square residuals s_i will be slightly in error).

9: MF -- INTEGER Input
On entry: the number of values of the independent variable at which a constraint is specified. Constraint: MF >= 1.

10: XF(MF) -- DOUBLE PRECISION array Input
On entry: the rth value of the independent variable at which a constraint is specified, for r = 1,2,...,MF. Constraint: these values need not be ordered but must be

distinct and satisfy XMIN <= XF(r) <= XMAX.

11: YF(LYF) -- DOUBLE PRECISION array Input
On entry: the values which the approximating polynomials and their derivatives are required to take at the points specified in XF. For each value of XF(r), YF contains in successive elements the required value of the approximation, its first derivative, second derivative,..., p th
r
derivative, for r = 1,2,...,MF. Thus the value which the kth derivative of each approximation (k=0 referring to the approximation itself) is required to take at the point XF(r) must be contained in YF(s), where
s=r+k+p +p +...+p ,
1 2 r-1
for k=0,1,...,p and r = 1,2,...,MF. The derivatives are
r
with respect to the user's variable x.

12: LYF -- INTEGER Input
On entry:
the dimension of the array YF as declared in the (sub)program from which E02AGF is called.
Constraint: LYF>=n, where n=MF+p +p +...+p .
1 2 MF

13: IP(MF) -- INTEGER array Input
On entry: IP(r) must contain p , the order of the highest-
r
order derivative specified at XF(r), for r = 1,2,...,MF.
p =0 implies that the value of the approximation at XF(r) is
r
specified, but not that of any derivative. Constraint: IP(r) >= 0, for r=1,2,...,MF.

14: A(NROWS,KPLUS1) -- DOUBLE PRECISION array Output
On exit: A(i+1,j+1) contains the coefficient a in the
ij
approximating polynomial of degree i, for i=n,n+1,...,k; j=0,1,...,i.

15: S(KPLUS1) -- DOUBLE PRECISION array Output
On exit: S(i+1) contains s , for i=n,n+1,...,k, the root-
i
mean-square residual corresponding to the approximating polynomial of degree i. In the case where the number of data

points with non-zero weight is equal to k+1-n, s_i is indeterminate: the routine sets it to zero. For the interpretation of the values of s_i and their use in selecting an appropriate degree, see Section 3.1 of the Chapter Introduction.

16: NP1 -- INTEGER Output
On exit: n+1, where n is the total number of constraint conditions imposed: $n=MF+p_1+p_2+\ldots+p_{MF}$.

17: WRK(LWRK) -- DOUBLE PRECISION array Output
On exit: WRK contains weighted residuals of the highest degree of fit determined (k). The residual at x_i is in element 2(n+1)+3(m+k+1)+r, for r=1,2,...,m. The rest of the array is used as workspace.

18: LWRK -- INTEGER Input
On entry:
the dimension of the array WRK as declared in the (sub)program from which E02AGF is called.
Constraint: LWRK>=max(4*M+3*KPLUS1, 8*n+5*IPMAX+MF+10)+2*n+2, where IPMAX = max(IP(R)).

19: IWRK(LIWRK) -- INTEGER array Workspace

20: LIWRK -- INTEGER Input
On entry:
the dimension of the array IWRK as declared in the (sub)program from which E02AGF is called.
Constraint: LIWRK>=2*MF+2.

21: IFAIL -- INTEGER Input/Output
On entry: IFAIL must be set to 0, -1 or 1. For users not familiar with this parameter (described in the Essential Introduction) the recommended value is 0.

On exit: IFAIL = 0 unless the routine detects an error (see Section 6).

6. Error Indicators and Warnings

Errors detected by the routine:

```
IFAIL= 1
     On entry M < 1,

     or       KPLUS1 < n + 1,

     or       NROWS < KPLUS1,

     or       MF < 1,

     or       LYF < n,

     or       LWRK is too small (see Section 5),

     or       LIWRK<2*MF+2.
     (Here n is the total number of constraint conditions.)

IFAIL= 2
     IP(r) < 0 for some r = 1,2,...,MF.

IFAIL= 3
     XMIN >= XMAX, or XF(r) is not in the interval XMIN to XMAX
     for some r = 1,2,...,MF, or the XF(r) are not distinct.

IFAIL= 4
     X(r) is not in the interval XMIN to XMAX for some
     r=1,2,...,M.

IFAIL= 5
     X(r) < X(r-1) for some r=2,3,...,M.

IFAIL= 6
     KPLUS1>m''+n, where m'' is the number of data points with
     non-zero weight and distinct abscissae which do not coincide
     with any XF(r). Thus there is no unique solution.

IFAIL= 7
     The polynomials (mu)(x) and/or (nu)(x) cannot be determined.
     The problem supplied is too ill-conditioned. This may occur
     when the constraint points are very close together, or large
     in number, or when an attempt is made to constrain high-
     order derivatives.

7. Accuracy

No complete error analysis exists for either the interpolating
algorithm or the approximating algorithm. However, considerable
```

experience with the approximating algorithm shows that it is generally extremely satisfactory. Also the moderate number of constraints, of low order, which are typical of data fitting applications, are unlikely to cause difficulty with the interpolating routine.

8. Further Comments

The time taken by the routine to form the interpolating polynomial is approximately proportional to n^3, and that to form the approximating polynomials is very approximately proportional to m(k+1)(k+1-n).

To carry out a least-squares polynomial fit without constraints, use E02ADF. To carry out polynomial interpolation only, use E01AEF(*).

9. Example

The example program reads data in the following order, using the notation of the parameter list above:

MF

IP(i), XF(i), Y-value and derivative values (if any) at XF(i), for i= 1,2,...,MF

M

X(i), Y(i), W(i), for i=1,2,...,M

k, XMIN, XMAX

The output is:

the root-mean-square residual for each degree from n to k;

the Chebyshev coefficients for the fit of degree k;

the data points, and the fitted values and residuals for the fit of degree k.

The program is written in a generalized form which will read any number of data sets.

The data set supplied specifies 5 data points in the interval [0.
0,4.0] with unit weights, to which are to be fitted polynomials,
p, of degrees up to 4, subject to the 3 constraints:

p(0.0)=1.0, p'(0.0)=-2.0, p(4.0)=9.0.

The example program is not reproduced here. The source code for
all example programs is distributed with the NAG Foundation
Library software and should be available on-line.

%%%

E02 -- Curve and Surface Fitting E02AHF
E02AHF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for
your implementation to check implementation-dependent details.
The symbol (*) after a NAG routine name denotes a routine that is
not included in the Foundation Library.

1. Purpose

E02AHF determines the coefficients in the Chebyshev-series
representation of the derivative of a polynomial given in
Chebyshev-series form.

2. Specification

```
      SUBROUTINE E02AHF (NP1, XMIN, XMAX, A, IA1, LA, PATM1,
     1                   ADIF, IADIF1, LADIF, IFAIL)
      INTEGER          NP1, IA1, LA, IADIF1, LADIF, IFAIL
      DOUBLE PRECISION XMIN, XMAX, A(LA), PATM1, ADIF(LADIF)
```

3. Description

This routine forms the polynomial which is the derivative of a
given polynomial. Both the original polynomial and its derivative
are represented in Chebyshev-series form. Given the coefficients
a , for i=0,1,...,n, of a polynomial p(x) of degree n, where
 i

$$p(x) = \frac{1}{2}a_0 + a_1 T_1(x) + \ldots + a_n T_n(x)$$

the routine returns the coefficients a_i, for i=0,1,...,n-1, of the polynomial q(x) of degree n-1, where

$$q(x)=\frac{dp(x)}{dx}=-a_1+a_2T_1(x)+\ldots+a_{n-1}T_{n-1}(x).$$

Here $T_j(x)$ denotes the Chebyshev polynomial of the first kind of degree j with argument x. It is assumed that the normalised variable x in the interval [-1,+1] was obtained from the user's original variable x in the interval $[x_{min},x_{max}]$ by the linear transformation

$$x=\frac{2x-(x_{max}+x_{min})}{x_{max}-x_{min}}$$

and that the user requires the derivative to be with respect to the variable x. If the derivative with respect to x is required, set $x_{max}=1$ and $x_{min}=-1$.

Values of the derivative can subsequently be computed, from the coefficients obtained, by using E02AKF.

The method employed is that of [1] modified to obtain the derivative with respect to x. Initially setting $a_{n+1}=a_n=0$, the routine forms successively

$$a_{i-1} = a_{i+1} + \frac{2i}{x_{max} - x_{min}} a_i, \quad i = n, n-1, \ldots, 1.$$

4. References

[1] Unknown (1961) Chebyshev-series. Modern Computing Methods, Chapter 8. NPL Notes on Applied Science (2nd Edition). 16 HMSO.

5. Parameters

1: NP1 -- INTEGER Input
On entry: n+1, where n is the degree of the given polynomial p(x). Thus NP1 is the number of coefficients in this polynomial. Constraint: NP1 >= 1.

2: XMIN -- DOUBLE PRECISION Input

3: XMAX -- DOUBLE PRECISION Input
On entry: the lower and upper end-points respectively of the interval $[x_{min}, x_{max}]$. The Chebyshev-series

representation is in terms of the normalised variable x, where

$$x = \frac{2x - (x_{max} + x_{min})}{x_{max} - x_{min}}.$$

Constraint: XMAX > XMIN.

4: A(LA) -- DOUBLE PRECISION array Input
On entry: the Chebyshev coefficients of the polynomial p(x). Specifically, element 1 + i*IA1 of A must contain the coefficient a_i, for i=0,1,...,n. Only these n+1 elements will be accessed.

Unchanged on exit, but see ADIF, below.

5: IA1 -- INTEGER Input
On entry: the index increment of A. Most frequently the

Chebyshev coefficients are stored in adjacent elements of A, and IA1 must be set to 1. However, if, for example, they are stored in A(1),A(4),A(7),..., then the value of IA1 must be 3. See also Section 8. Constraint: IA1 >= 1.

6: LA -- INTEGER Input
On entry:
the dimension of the array A as declared in the (sub)program from which E02AHF is called.
Constraint: LA>=1+(NP1-1)*IA1.

7: PATM1 -- DOUBLE PRECISION Output
On exit: the value of p(x). If this value is passed to
min
the integration routine E02AJF with the coefficients of q(x) , then the original polynomial p(x) is recovered, including its constant coefficient.

8: ADIF(LADIF) -- DOUBLE PRECISION array Output
On exit: the Chebyshev coefficients of the derived polynomial q(x). (The differentiation is with respect to the variable x). Specifically, element 1+i*IADIF1 of ADIF

contains the coefficient a , i=0,1,...n-1. Additionally
i
element 1+n*IADIF1 is set to zero. A call of the routine may have the array name ADIF the same as A, provided that note is taken of the order in which elements are overwritten, when choosing the starting elements and increments IA1 and IADIF1: i.e., the coefficients a ,a ,...,a must be intact
0 1 i-1

after coefficient a is stored. In particular, it is
i
possible to overwrite the a completely by having IA1 =
i
IADIF1, and the actual arrays for A and ADIF identical.

9: IADIF1 -- INTEGER Input
On entry: the index increment of ADIF. Most frequently the Chebyshev coefficients are required in adjacent elements of ADIF, and IADIF1 must be set to 1. However, if, for example, they are to be stored in ADIF(1),ADIF(4),ADIF(7),..., then the value of IADIF1 must be 3. See Section 8. Constraint:

```
     IADIF1 >= 1.

10:  LADIF -- INTEGER                                          Input
     On entry:
     the dimension of the array ADIF as declared in the
     (sub)program from which E02AHF is called.
     Constraint: LADIF>=1+(NP1-1)*IADIF1.

11:  IFAIL -- INTEGER                                  Input/Output
     On entry: IFAIL must be set to 0, -1 or 1. For users not
     familiar with this parameter (described in the Essential
     Introduction) the recommended value is 0.

     On exit: IFAIL = 0 unless the routine detects an error (see
     Section 6).

6. Error Indicators and Warnings

Errors detected by the routine:

IFAIL= 1
     On entry NP1 < 1,

     or       XMAX <= XMIN,

     or       IA1 < 1,

     or       LA<=(NP1-1)*IA1,

     or       IADIF1 < 1,

     or       LADIF<=(NP1-1)*IADIF1.

7. Accuracy

There is always a loss of precision in numerical differentiation,
in this case associated with the multiplication by 2i in the
formula quoted in Section 3.

8. Further Comments

The time taken by the routine is approximately proportional to
n+1.

The increments IA1, IADIF1 are included as parameters to give a
degree of flexibility which, for example, allows a polynomial in
```

two variables to be differentiated with respect to either variable without rearranging the coefficients.

9. Example

Suppose a polynomial has been computed in Chebyshev-series form to fit data over the interval [-0.5,2.5]. The example program evaluates the 1st and 2nd derivatives of this polynomial at 4 equally spaced points over the interval. (For the purposes of this example, XMIN, XMAX and the Chebyshev coefficients are simply supplied in DATA statements. Normally a program would first read in or generate data and compute the fitted polynomial.)

The example program is not reproduced here. The source code for all example programs is distributed with the NAG Foundation Library software and should be available on-line.

%%%

E02 -- Curve and Surface Fitting E02AJF
E02AJF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for your implementation to check implementation-dependent details. The symbol (*) after a NAG routine name denotes a routine that is not included in the Foundation Library.

1. Purpose

E02AJF determines the coefficients in the Chebyshev-series representation of the indefinite integral of a polynomial given in Chebyshev-series form.

2. Specification

```
      SUBROUTINE E02AJF (NP1, XMIN, XMAX, A, IA1, LA, QATM1,
     1                   AINT, IAINT1, LAINT, IFAIL)
      INTEGER           NP1, IA1, LA, IAINT1, LAINT, IFAIL
      DOUBLE PRECISION XMIN, XMAX, A(LA), QATM1, AINT(LAINT)
```

3. Description

This routine forms the polynomial which is the indefinite integral of a given polynomial. Both the original polynomial and its integral are represented in Chebyshev-series form. If

supplied with the coefficients a_i, for i=0,1,...,n, of a polynomial p(x) of degree n, where

$$p(x) = \frac{1}{2}a_0 T_0(x) + a_1 T_1(x) + \ldots + a_n T_n(x),$$

the routine returns the coefficients a'_i, for i=0,1,...,n+1, of the polynomial q(x) of degree n+1, where

$$q(x) = \frac{1}{2}a'_0 T_0(x) + a'_1 T_1(x) + \ldots + a'_{n+1} T_{n+1}(x),$$

and

$$q(x) = \int p(x)\,dx.$$

Here $T_j(x)$ denotes the Chebyshev polynomial of the first kind of degree j with argument x. It is assumed that the normalised variable x in the interval [-1,+1] was obtained from the user's original variable x in the interval $[x_{min}, x_{max}]$ by the linear transformation

$$x = \frac{2x - (x_{max} + x_{min})}{x_{max} - x_{min}}$$

and that the user requires the integral to be with respect to the variable x. If the integral with respect to x is required, set $x_{max} = 1$ and $x_{min} = -1$.

Values of the integral can subsequently be computed, from the coefficients obtained, by using E02AKF.

The method employed is that of Chebyshev-series [1] modified for integrating with respect to x. Initially taking $a_{n+1} = a_{n+2} = 0$, the routine forms successively

$$a'_i = \frac{a_{i-1} - a_{i+1}}{2i} * \frac{x_{max} - x_{min}}{2}, \quad i = n+1, n, \ldots, 1.$$

The constant coefficient a'_0 is chosen so that q(x) is equal to a specified value, QATM1, at the lower end-point of the interval on

which it is defined, i.e., x=-1, which corresponds to $x = x_{min}$.

4. References

[1] Unknown (1961) Chebyshev-series. Modern Computing Methods, Chapter 8. NPL Notes on Applied Science (2nd Edition). 16 HMSO.

5. Parameters

1: NP1 -- INTEGER Input
On entry: n+1, where n is the degree of the given polynomial p(x). Thus NP1 is the number of coefficients in this polynomial. Constraint: NP1 >= 1.

2: XMIN -- DOUBLE PRECISION Input

3: XMAX -- DOUBLE PRECISION Input
On entry: the lower and upper end-points respectively of the interval $[x_{min}, x_{max}]$. The Chebyshev-series

representation is in terms of the normalised variable x, where

$$\frac{2x - (x_{max} + x_{min})}{}$$

$$x = \frac{\max + \min}{\max - \min}$$

Constraint: XMAX > XMIN.

4: A(LA) -- DOUBLE PRECISION array Input
On entry: the Chebyshev coefficients of the polynomial p(x). Specifically, element 1+i*IA1 of A must contain the coefficient a_i, for i=0,1,...,n. Only these n+1 elements will be accessed.

Unchanged on exit, but see AINT, below.

5: IA1 -- INTEGER Input
On entry: the index increment of A. Most frequently the Chebyshev coefficients are stored in adjacent elements of A, and IA1 must be set to 1. However, if for example, they are stored in A(1),A(4),A(7),..., then the value of IA1 must be 3. See also Section 8. Constraint: IA1 >= 1.

6: LA -- INTEGER Input
On entry:
the dimension of the array A as declared in the (sub)program from which E02AJF is called.
Constraint: LA>=1+(NP1-1)*IA1.

7: QATM1 -- DOUBLE PRECISION Input
On entry: the value that the integrated polynomial is required to have at the lower end-point of its interval of

definition, i.e., at x=-1 which corresponds to $x=x_{min}$. Thus, QATM1 is a constant of integration and will normally be set to zero by the user.

8: AINT(LAINT) -- DOUBLE PRECISION array Output
On exit: the Chebyshev coefficients of the integral q(x). (The integration is with respect to the variable x, and the constant coefficient is chosen so that $q(x_{min})$ equals QATM1)
Specifically, element 1+i*IAINT1 of AINT contains the coefficient a', for i=0,1,...,n+1. A call of the routine

may have the array name AINT the same as A, provided that note is taken of the order in which elements are overwritten when choosing starting elements and increments IA1 and IAINT1: i.e., the coefficients, $a_0, a_1, \ldots, a_{i-2}$ must be intact after coefficient a_i is stored. In particular it is possible to overwrite the a_i entirely by having IA1 = IAINT1, and the actual array for A and AINT identical.

9: IAINT1 -- INTEGER Input
On entry: the index increment of AINT. Most frequently the Chebyshev coefficients are required in adjacent elements of AINT, and IAINT1 must be set to 1. However, if, for example, they are to be stored in AINT(1),AINT(4),AINT(7),..., then the value of IAINT1 must be 3. See also Section 8.
Constraint: IAINT1 >= 1.

10: LAINT -- INTEGER Input
On entry:
the dimension of the array AINT as declared in the (sub)program from which E02AJF is called.
Constraint: LAINT>=1+NP1*IAINT1.

11: IFAIL -- INTEGER Input/Output
On entry: IFAIL must be set to 0, -1 or 1. For users not familiar with this parameter (described in the Essential Introduction) the recommended value is 0.

On exit: IFAIL = 0 unless the routine detects an error (see Section 6).

6. Error Indicators and Warnings

Errors detected by the routine:

IFAIL= 1
On entry NP1 < 1,

or XMAX <= XMIN,

or IA1 < 1,

or LA<=(NP1-1)*IA1,

```
     or       IAINT1 < 1,

     or       LAINT<=NP1*IAINT1.

7. Accuracy

In general there is a gain in precision in numerical integration,
in this case associated with the division by 2i in the formula
quoted in Section 3.

8. Further Comments

The time taken by the routine is approximately proportional to
n+1.

The increments IA1, IAINT1 are included as parameters to give a
degree of flexibility which, for example, allows a polynomial in
two variables to be integrated with respect to either variable
without rearranging the coefficients.

9. Example

Suppose a polynomial has been computed in Chebyshev-series form
to fit data over the interval [-0.5,2.5]. The example program
evaluates the integral of the polynomial from 0.0 to 2.0. (For
the purpose of this example, XMIN, XMAX and the Chebyshev
coefficients are simply supplied in DATA statements. Normally a
program would read in or generate data and compute the fitted
polynomial).

The example program is not reproduced here. The source code for
all example programs is distributed with the NAG Foundation
Library software and should be available on-line.
```

%%%

```
E02 -- Curve and Surface Fitting                              E02AKF
        E02AKF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for
your implementation to check implementation-dependent details.
The symbol (*) after a NAG routine name denotes a routine that is
not included in the Foundation Library.

1. Purpose
```

E02AKF evaluates a polynomial from its Chebyshev-series representation, allowing an arbitrary index increment for accessing the array of coefficients.

2. Specification

```
      SUBROUTINE E02AKF (NP1, XMIN, XMAX, A, IA1, LA, X, RESULT,
     1                   IFAIL)
      INTEGER           NP1, IA1, LA, IFAIL
      DOUBLE PRECISION XMIN, XMAX, A(LA), X, RESULT
```

3. Description

If supplied with the coefficients a_i, for $i=0,1,\ldots,n$, of a polynomial $p(x)$ of degree n, where

$$p(x) = \tfrac{1}{2}a_0 + a_1 T_1(x) + \ldots + a_n T_n(x),$$

this routine returns the value of $p(x)$ at a user-specified value of the variable x. Here $T_j(x)$ denotes the Chebyshev polynomial of the first kind of degree j with argument x. It is assumed that the independent variable x in the interval $[-1,+1]$ was obtained from the user's original variable x in the interval $[x_{min}, x_{max}]$ by the linear transformation

$$x = \frac{2x-(x_{max}+x_{min})}{x_{max}-x_{min}}.$$

The coefficients a may be supplied in the array A, with any
i
increment between the indices of array elements which contain successive coefficients. This enables the routine to be used in surface fitting and other applications, in which the array might have two or more dimensions.

The method employed is based upon the three-term recurrence relation due to Clenshaw [1], with modifications due to Reinsch and Gentleman (see [4]). For further details of the algorithm and its use see Cox [2] and Cox and Hayes [3].

4. References

[1] Clenshaw C W (1955) A Note on the Summation of Chebyshev-series. Math. Tables Aids Comput. 9 118--120.

[2] Cox M G (1973) A data-fitting package for the non-specialist user. Report NAC40. National Physical Laboratory.

[3] Cox M G and Hayes J G (1973) Curve fitting: a guide and suite of algorithms for the non-specialist user. Report NAC26. National Physical Laboratory.

[4] Gentlemen W M (1969) An Error Analysis of Goertzel's (Watt's) Method for Computing Fourier Coefficients. Comput. J. 12 160--165.

5. Parameters

1: NP1 -- INTEGER Input
On entry: n+1, where n is the degree of the given

polynomial p(x). Constraint: NP1 >= 1.

2: XMIN -- DOUBLE PRECISION Input

3: XMAX -- DOUBLE PRECISION Input
On entry: the lower and upper end-points respectively of the interval [x ,x]. The Chebyshev-series
min max

representation is in terms of the normalised variable x, where

$$x = \frac{2x - (x_{max} + x_{min})}{x_{max} - x_{min}}.$$

Constraint: XMIN < XMAX.

4: A(LA) -- DOUBLE PRECISION array Input
On entry: the Chebyshev coefficients of the polynomial p(x). Specifically, element 1+i*IA1 must contain the coefficient a_i, for i=0,1,...,n. Only these n+1 elements will be accessed.

5: IA1 -- INTEGER Input
On entry: the index increment of A. Most frequently, the Chebyshev coefficients are stored in adjacent elements of A, and IA1 must be set to 1. However, if, for example, they are stored in A(1),A(4),A(7),..., then the value of IA1 must be 3. Constraint: IA1 >= 1.

6: LA -- INTEGER Input
On entry:
the dimension of the array A as declared in the (sub)program from which E02AKF is called.
Constraint: LA>=(NP1-1)*IA1+1.

7: X -- DOUBLE PRECISION Input
On entry: the argument x at which the polynomial is to be evaluated. Constraint: XMIN <= X <= XMAX.

8: RESULT -- DOUBLE PRECISION Output

On exit: the value of the polynomial p(x).

9: IFAIL -- INTEGER Input/Output
On entry: IFAIL must be set to 0, -1 or 1. For users not familiar with this parameter (described in the Essential Introduction) the recommended value is 0.

On exit: IFAIL = 0 unless the routine detects an error (see Section 6).

6. Error Indicators and Warnings

```
Errors detected by the routine:

IFAIL= 1
     On entry NP1 < 1,

     or       IA1 < 1,

     or       LA<=(NP1-1)*IA1,

     or       XMIN >= XMAX.

IFAIL= 2
     X does not satisfy the restriction XMIN <= X <= XMAX.

7. Accuracy

The rounding errors are such that the computed value of the
polynomial is exact for a slightly perturbed set of coefficients
a +(delta)a . The ratio of the sum of the absolute values of the
 i         i
(delta)a  to the sum of the absolute values of the a  is less
       i                                            i
than a small multiple of (n+1)*machine precision.

8. Further Comments

The time taken by the routine is approximately proportional to
n+1.

9. Example

Suppose a polynomial has been computed in Chebyshev-series form
to fit data over the interval [-0.5,2.5]. The example program
evaluates the polynomial at 4 equally spaced points over the
interval. (For the purposes of this example, XMIN, XMAX and the
Chebyshev coefficients are supplied in DATA statements. Normally
a program would first read in or generate data and compute the
fitted polynomial.)

The example program is not reproduced here. The source code for
all example programs is distributed with the NAG Foundation
Library software and should be available on-line.
```

%%%

E02BAF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for your implementation to check implementation-dependent details. The symbol (*) after a NAG routine name denotes a routine that is not included in the Foundation Library.

1. Purpose

E02BAF computes a weighted least-squares approximation to an arbitrary set of data points by a cubic spline with knots prescribed by the user. Cubic spline interpolation can also be carried out.

2. Specification

```
      SUBROUTINE E02BAF (M, NCAP7, X, Y, W, LAMDA, WORK1, WORK2,
     1                   C, SS, IFAIL)
      INTEGER           M, NCAP7, IFAIL
      DOUBLE PRECISION X(M), Y(M), W(M), LAMDA(NCAP7), WORK1(M),
     1                 WORK2(4*NCAP7), C(NCAP7), SS
```

3. Description

This routine determines a least-squares cubic spline approximation s(x) to the set of data points (x_r, y_r) with weights w_r, for r=1,2,...,m. The value of NCAP7 = n+7, where n is the number of intervals of the spline (one greater than the number of interior knots), and the values of the knots $(\lambda)_5, (\lambda)_6, \ldots, (\lambda)_{n+3}$, interior to the data interval, are prescribed by the user.

s(x) has the property that it minimizes (theta), the sum of squares of the weighted residuals $(\epsilon)_r$, for r=1,2,...,m, where

$$(\epsilon)_r = w_r (y_r - s(x_r)).$$

The routine produces this minimizing value of (theta) and the

coefficients c ,c ,...,c , where q=n+3, in the B-spline
              1 2   q
representation

$$s(x) = \sum_{i=1}^{q} c_i N_i(x).$$

Here N (x) denotes the normalised B-spline of degree 3 defined
      i
upon the knots (lambda) ,(lambda)   ,...,(lambda)   .
                       i   i+1   i+4

In order to define the full set of B-splines required, eight
additional knots (lambda) ,(lambda) ,(lambda) ,(lambda) and
                         1   2   3   4
(lambda)   ,(lambda)- ,(lambda)   ,(lambda)   are inserted
        n+4   n+5   n+6   n+7
automatically by the routine. The first four of these are set
equal to the smallest x and the last four to the largest x .
                       r   r

The representation of s(x) in terms of B-splines is the most

compact form possible in that only n+3 coefficients, in addition

to the n+7 knots, fully define s(x).

The method employed involves forming and then computing the
least-squares solution of a set of m linear equations in the

coefficients c (i=1,2,...,n+3). The equations are formed using a
             i
recurrence relation for B-splines that is unconditionally stable
(Cox [1], de Boor [5]), even for multiple (coincident) knots. The
least-squares solution is also obtained in a stable manner by
using orthogonal transformations, viz. a variant of Givens
rotations (Gentleman [6] and [7]). This requires only one
equation to be stored at a time. Full advantage is taken of the
structure of the equations, there being at most four non-zero

values of N (x) for any value of x and hence at most four
          i
coefficients in each equation.

For further details of the algorithm and its use see Cox [2], [3] and [4].

Subsequent evaluation of s(x) from its B-spline representation may be carried out using E02BBF. If derivatives of s(x) are also required, E02BCF may be used. E02BDF can be used to compute the definite integral of s(x).

4. References

[1] Cox M G (1972) The Numerical Evaluation of B-splines. J. Inst. Math. Appl. 10 134--149.

[2] Cox M G (1974) A Data-fitting Package for the Non-specialist User. Software for Numerical Mathematics. (ed D J Evans) Academic Press.

[3] Cox M G (1975) Numerical methods for the interpolation and approximation of data by spline functions. PhD Thesis. City University, London.

[4] Cox M G and Hayes J G (1973) Curve fitting: a guide and suite of algorithms for the non-specialist user. Report NAC26. National Physical Laboratory.

[5] De Boor C (1972) On Calculating with B-splines. J. Approx. Theory. 6 50--62.

[6] Gentleman W M (1974) Algorithm AS 75. Basic Procedures for Large Sparse or Weighted Linear Least-squares Problems. Appl. Statist. 23 448--454.

[7] Gentleman W M (1973) Least-squares Computations by Givens Transformations without Square Roots. J. Inst. Math. Applic. 12 329--336.

[8] Schoenberg I J and Whitney A (1953) On Polya Frequency Functions III. Trans. Amer. Math. Soc. 74 246--259.

5. Parameters

1: M -- INTEGER Input

On entry: the number m of data points. Constraint: M >= MDIST >= 4, where MDIST is the number of distinct x values in the data.

2: NCAP7 -- INTEGER Input

On entry: n+7, where n is the number of intervals of the spline (which is one greater than the number of interior knots, i.e., the knots strictly within the range x_1 to x_m) over which the spline is defined. Constraint: 8 <= NCAP7 <= MDIST + 4, where MDIST is the number of distinct x values in the data.

3: X(M) -- DOUBLE PRECISION array Input
On entry: the values x_r of the independent variable (abscissa), for r=1,2,...,m. Constraint: $x_1 <= x_2 <= ... <= x_m$.

4: Y(M) -- DOUBLE PRECISION array Input
On entry: the values y_r of the dependent variable (ordinate), for r=1,2,...,m.

5: W(M) -- DOUBLE PRECISION array Input
On entry: the values w_r of the weights, for r=1,2,...,m. For advice on the choice of weights, see the Chapter Introduction. Constraint: W(r) > 0, for r=1,2,...,m.

6: LAMDA(NCAP7) -- DOUBLE PRECISION array Input/Output
On entry: LAMDA(i) must be set to the (i-4)th (interior) knot, $(\lambda)_i$, for i=5,6,...,n+3. Constraint: X(1) < LAMDA(5) <= LAMDA(6) <=... <= LAMDA(NCAP7-4) < X(M). On exit: the input values are unchanged, and LAMDA(i), for i = 1, 2, 3, 4, NCAP7-3, NCAP7-2, NCAP7-1, NCAP7 contains the additional (exterior) knots introduced by the routine. For advice on the choice of knots, see Section 3.3 of the Chapter Introduction.

7: WORK1(M) -- DOUBLE PRECISION array Workspace

8: WORK2(4*NCAP7) -- DOUBLE PRECISION array Workspace

9: C(NCAP7) -- DOUBLE PRECISION array Output
On exit: the coefficient c_i of the B-spline $N_i(x)$, for

$i=1,2,\ldots,n+3$. The remaining elements of the array are not used.

10: SS -- DOUBLE PRECISION Output
On exit: the residual sum of squares, (theta).

11: IFAIL -- INTEGER Input/Output
On entry: IFAIL must be set to 0, -1 or 1. For users not familiar with this parameter (described in the Essential Introduction) the recommended value is 0.

On exit: IFAIL = 0 unless the routine detects an error (see Section 6).

6. Error Indicators and Warnings

Errors detected by the routine:

IFAIL= 1
The knots fail to satisfy the condition

X(1) < LAMDA(5) <= LAMDA(6) <=... <= LAMDA(NCAP7-4) < X(M).
Thus the knots are not in correct order or are not interior to the data interval.

IFAIL= 2
The weights are not all strictly positive.

IFAIL= 3
The values of X(r), for r = 1,2,...,M are not in non-decreasing order.

IFAIL= 4
NCAP7 < 8 (so the number of interior knots is negative) or NCAP7 > MDIST + 4, where MDIST is the number of distinct x values in the data (so there cannot be a unique solution).

IFAIL= 5
The conditions specified by Schoenberg and Whitney [8] fail

```
     to hold for at least one subset of the distinct data
     abscissae. That is, there is no subset of NCAP7-4 strictly
     increasing values, X(R(1)),X(R(2)),...,X(R(NCAP7-4)), among
     the abscissae such that
          X(R(1)) < LAMDA(1) < X(R(5)),

          X(R(2)) < LAMDA(2) < X(R(6)),

          ...

          X(R(NCAP7-8)) < LAMDA(NCAP7-8) < X(R(NCAP7-4)).
     This means that there is no unique solution: there are
     regions containing too many knots compared with the number
     of data points.

7. Accuracy

The rounding errors committed are such that the computed
coefficients are exact for a slightly perturbed set of ordinates
y +(delta)y . The ratio of the root-mean-square value for the
 r         r
(delta)y  to the root-mean-square value of the y  can be expected
        r                                      r
to be less than a small multiple of (kappa)*m*machine precision,
where (kappa) is a condition number for the problem. Values of
(kappa) for 20-30 practical data sets all proved to lie between
4.5 and 7.8 (see Cox [3]). (Note that for these data sets,
replacing the coincident end knots at the end-points x  and x
                                                      1      m
used in the routine by various choices of non-coincident exterior
knots gave values of (kappa) between 16 and 180. Again see Cox
[3] for further details.) In general we would not expect (kappa)
to be large unless the choice of knots results in near-violation
of the Schoenberg-Whitney conditions.

A cubic spline which adequately fits the data and is free from
spurious oscillations is more likely to be obtained if the knots
are chosen to be grouped more closely in regions where the
function (underlying the data) or its derivatives change more
rapidly than elsewhere.

8. Further Comments



The time taken by the routine is approximately C*(2m+n+7)
```

seconds, where C is a machine-dependent constant.

Multiple knots are permitted as long as their multiplicity does not exceed 4, i.e., the complete set of knots must satisfy

(lambda) <(lambda) , for i=1,2,...,n+3, (cf. Section 6). At a
i i+4
knot of multiplicity one (the usual case), s(x) and its first two derivatives are continuous. At a knot of multiplicity two, s(x) and its first derivative are continuous. At a knot of multiplicity three, s(x) is continuous, and at a knot of multiplicity four, s(x) is generally discontinous.

The routine can be used efficiently for cubic spline

interpolation, i.e.,if m=n+3. The abscissae must then of course satisfy x <x <...<x . Recommended values for the knots in this
1 2 m

case are (lambda) =x , for i=5,6,...,n+3.
i i-2

9. Example

Determine a weighted least-squares cubic spline approximation with five intervals (four interior knots) to a set of 14 given data points. Tabulate the data and the corresponding values of the approximating spline, together with the residual errors, and also the values of the approximating spline at points half-way between each pair of adjacent data points.

The example program is written in a general form that will enable

a cubic spline approximation with n intervals (n-1 interior knots) to be obtained to m data points, with arbitrary positive weights, and the approximation to be tabulated. Note that E02BBF is used to evaluate the approximating spline. The program is self-starting in that any number of data sets can be supplied.

The example program is not reproduced here. The source code for all example programs is distributed with the NAG Foundation Library software and should be available on-line.

%%

E02 -- Curve and Surface Fitting E02BBF
E02BBF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for your implementation to check implementation-dependent details. The symbol (*) after a NAG routine name denotes a routine that is not included in the Foundation Library.

1. Purpose

E02BBF evaluates a cubic spline from its B-spline representation.

2. Specification

```
SUBROUTINE E02BBF (NCAP7, LAMDA, C, X, S, IFAIL)
INTEGER          NCAP7, IFAIL
DOUBLE PRECISION LAMDA(NCAP7), C(NCAP7), X, S
```

3. Description

This routine evaluates the cubic spline s(x) at a prescribed argument x from its augmented knot set $(\lambda)_i$, for $i=1,2,\ldots,n+7$, (see E02BAF) and from the coefficients c_i, for $i=1,2,\ldots,q$ in its B-spline representation

$$s(x)=\sum_{i=1}^{q} c_i N_i(x)$$

Here q=n+3, where n is the number of intervals of the spline, and $N_i(x)$ denotes the normalised B-spline of degree 3 defined upon the knots $(\lambda)_i,(\lambda)_{i+1},\ldots,(\lambda)_{i+4}$. The prescribed argument x must satisfy $(\lambda)_i \le x \le (\lambda)_{i+4}$.

4 n+4

It is assumed that (lambda) >=(lambda) , for j=2,3,...,n+7, and
j j-1
(lambda) >(lambda) .
4
n+4

The method employed is that of evaluation by taking convex combinations due to de Boor [4]. For further details of the algorithm and its use see Cox [1] and [3].

It is expected that a common use of E02BBF will be the evaluation of the cubic spline approximations produced by E02BAF. A generalization of E02BBF which also forms the derivative of s(x) is E02BCF. E02BCF takes about 50% longer than E02BBF.

4. References

[1] Cox M G (1972) The Numerical Evaluation of B-splines. J. Inst. Math. Appl. 10 134--149.

[2] Cox M G (1978) The Numerical Evaluation of a Spline from its B-spline Representation. J. Inst. Math. Appl. 21 135--143.

[3] Cox M G and Hayes J G (1973) Curve fitting: a guide and suite of algorithms for the non-specialist user. Report NAC26. National Physical Laboratory.

[4] De Boor C (1972) On Calculating with B-splines. J. Approx. Theory. 6 50--62.

5. Parameters

1: NCAP7 -- INTEGER Input

On entry: n+7, where n is the number of intervals (one greater than the number of interior knots, i.e., the knots strictly within the range (lambda) to (lambda)) over
4 n+4
which the spline is defined. Constraint: NCAP7 >= 8.

2: LAMDA(NCAP7) -- DOUBLE PRECISION array Input
On entry: LAMDA(j) must be set to the value of the jth member of the complete set of knots, (λ_j) for j=1,2,...,n+7. Constraint: the LAMDA(j) must be in non-decreasing order with LAMDA(NCAP7-3) > LAMDA(4).

3: C(NCAP7) -- DOUBLE PRECISION array Input
On entry: the coefficient c_i of the B-spline $N_i(x)$, for i=1,2,...,n+3. The remaining elements of the array are not used.

4: X -- DOUBLE PRECISION Input
On entry: the argument x at which the cubic spline is to be evaluated. Constraint: LAMDA(4) <= X <= LAMDA(NCAP7-3).

5: S -- DOUBLE PRECISION Output
On exit: the value of the spline, s(x).

6: IFAIL -- INTEGER Input/Output
On entry: IFAIL must be set to 0, -1 or 1. For users not familiar with this parameter (described in the Essential Introduction) the recommended value is 0.

On exit: IFAIL = 0 unless the routine detects an error (see Section 6).

6. Error Indicators and Warnings

Errors detected by the routine:

IFAIL= 1
The argument X does not satisfy LAMDA(4) <= X <= LAMDA(NCAP7-3).

In this case the value of S is set arbitrarily to zero.

IFAIL= 2
NCAP7 < 8, i.e., the number of interior knots is negative.

7. Accuracy

The computed value of s(x) has negligible error in most practical situations. Specifically, this value has an absolute error bounded in modulus by 18*c_max * machine precision, where c_max is the largest in modulus of c_j ,c_j+1 ,c_j+2 and c_j+3, and j is an integer such that (lambda)_j+3 <=x<=(lambda)_j+4. If c_j ,c_j+1 ,c_j+2 and c_j+3 are all of the same sign, then the computed value of s(x) has a relative error not exceeding 20*machine precision in modulus. For further details see Cox [2].

8. Further Comments

The time taken by the routine is approximately C*(1+0.1*log(n+7)) seconds, where C is a machine-dependent constant.

Note: the routine does not test all the conditions on the knots given in the description of LAMDA in Section 5, since to do this would result in a computation time approximately linear in n+7 instead of log(n+7). All the conditions are tested in E02BAF, however.

9. Example

Evaluate at 9 equally-spaced points in the interval 1.0<=x<=9.0 the cubic spline with (augmented) knots 1.0, 1.0, 1.0, 1.0, 3.0, 6.0, 8.0, 9.0, 9.0, 9.0, 9.0 and normalised cubic B-spline coefficients 1.0, 2.0, 4.0, 7.0, 6.0, 4.0, 3.0.

The example program is written in a general form that will enable

a cubic spline with n intervals, in its normalised cubic B-spline form, to be evaluated at m equally-spaced points in the interval

LAMDA(4) <= x <= LAMDA(n+4). The program is self-starting in that any number of data sets may be supplied.

The example program is not reproduced here. The source code for all example programs is distributed with the NAG Foundation

Library software and should be available on-line.

%%%

E02 -- Curve and Surface Fitting E02BCF
E02BCF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for your implementation to check implementation-dependent details. The symbol (*) after a NAG routine name denotes a routine that is not included in the Foundation Library.

1. Purpose

E02BCF evaluates a cubic spline and its first three derivatives from its B-spline representation.

2. Specification

```
SUBROUTINE E02BCF (NCAP7, LAMDA, C, X, LEFT, S, IFAIL)
INTEGER          NCAP7, LEFT, IFAIL
DOUBLE PRECISION LAMDA(NCAP7), C(NCAP7), X, S(4)
```

3. Description

This routine evaluates the cubic spline s(x) and its first three derivatives at a prescribed argument x. It is assumed that s(x) is represented in terms of its B-spline coefficients c_i, for $i=1,2,\ldots,n+3$ and (augmented) ordered knot set $(\lambda)_i$, for $i=1,2,\ldots,n+7$, (see E02BAF), i.e.,

$$s(x) = \sum_{i=1}^{q} c_i N_i(x)$$

Here q=n+3, n is the number of intervals of the spline and $N_i(x)$

                                                                    i
denotes the normalised B-spline of degree 3 (order 4) defined
upon the knots (lambda) ,(lambda)   ,...,(lambda)   . The
                       i   i+1   i+4
prescribed argument x must satisfy

                    (lambda) <=x<=(lambda)
                            4   n+4

At a simple knot (lambda) (i.e., one satisfying
                         i
(lambda)   <(lambda) <(lambda)   ), the third derivative of the
       i-1   i   i+1
spline is in general discontinuous. At a multiple knot (i.e., two or more knots with the same value), lower derivatives, and even the spline itself, may be discontinuous. Specifically, at a point x=u where (exactly) r knots coincide (such a point is termed a knot of multiplicity r), the values of the derivatives of order 4-j, for j=1,2,...,r, are in general discontinuous. (Here 1<=r<=4;r>4 is not meaningful.) The user must specify whether the value at such a point is required to be the left- or right-hand derivative.

The method employed is based upon:

(i) carrying out a binary search for the knot interval containing the argument x (see Cox [3]),

(ii) evaluating the non-zero B-splines of orders 1,2,3 and 4 by recurrence (see Cox [2] and [3]),

(iii) computing all derivatives of the B-splines of order 4 by applying a second recurrence to these computed B-spline values (see de Boor [1]),

(iv) multiplying the 4th-order B-spline values and their derivative by the appropriate B-spline coefficients, and summing, to yield the values of s(x) and its derivatives.

E02BCF can be used to compute the values and derivatives of cubic spline fits and interpolants produced by E02BAF.

If only values and not derivatives are required, E02BBF may be used instead of E02BCF, which takes about 50% longer than E02BBF.

4. References

[1] De Boor C (1972) On Calculating with B-splines. J. Approx. Theory. 6 50--62.

[2] Cox M G (1972) The Numerical Evaluation of B-splines. J. Inst. Math. Appl. 10 134--149.

[3] Cox M G (1978) The Numerical Evaluation of a Spline from its B-spline Representation. J. Inst. Math. Appl. 21 135--143.

5. Parameters

1: NCAP7 -- INTEGER Input

On entry: n+7, where n is the number of intervals of the spline (which is one greater than the number of interior knots, i.e., the knots strictly within the range (lambda) to (lambda) over which the spline is defined).

Constraint: NCAP7 >= 8.

2: LAMDA(NCAP7) -- DOUBLE PRECISION array Input
On entry: LAMDA(j) must be set to the value of the jth member of the complete set of knots, (lambda) , for j

j=1,2,...,n+7. Constraint: the LAMDA(j) must be in non-decreasing order with

LAMDA(NCAP7-3) > LAMDA(4).

3: C(NCAP7) -- DOUBLE PRECISION array Input
On entry: the coefficient c of the B-spline N (x), for i i

i=1,2,...,n+3. The remaining elements of the array are not used.

4: X -- DOUBLE PRECISION Input
On entry: the argument x at which the cubic spline and its derivatives are to be evaluated. Constraint: LAMDA(4) <= X <= LAMDA(NCAP7-3).

5: LEFT -- INTEGER Input
On entry: specifies whether left- or right-hand values of the spline and its derivatives are to be computed (see Section 3). Left- or right-hand values are formed according to whether LEFT is equal or not equal to 1. If x does not coincide with a knot, the value of LEFT is immaterial. If x = LAMDA(4), right-hand values are computed, and if x = LAMDA (NCAP7-3), left-hand values are formed, regardless of the value of LEFT.

6: S(4) -- DOUBLE PRECISION array Output
On exit: S(j) contains the value of the (j-1)th derivative of the spline at the argument x, for j = 1,2,3,4. Note that S(1) contains the value of the spline.

7: IFAIL -- INTEGER Input/Output
On entry: IFAIL must be set to 0, -1 or 1. For users not familiar with this parameter (described in the Essential Introduction) the recommended value is 0.

On exit: IFAIL = 0 unless the routine detects an error (see Section 6).

6. Error Indicators and Warnings

Errors detected by the routine:

IFAIL= 1
NCAP7 < 8, i.e., the number of intervals is not positive.

IFAIL= 2
Either LAMDA(4) >= LAMDA(NCAP7-3), i.e., the range over which s(x) is defined is null or negative in length, or X is an invalid argument, i.e., X < LAMDA(4) or X > LAMDA(NCAP7-3).

7. Accuracy

The computed value of s(x) has negligible error in most practical situations. Specifically, this value has an absolute error bounded in modulus by $18*c_{max}$ * machine precision, where c_{max} is the largest in modulus of c_j, c_{j+1}, c_{j+2} and c_{j+3}, and j is an integer such that $(\text{lambda})_{j+3} <= x <= (\text{lambda})_{j+4}$. If c_j, c_{j+1}, c_{j+2}

and c are all of the same sign, then the computed value of
j+3
s(x) has relative error bounded by 18*machine precision. For full details see Cox [3].

No complete error analysis is available for the computation of the derivatives of s(x). However, for most practical purposes the absolute errors in the computed derivatives should be small.

8. Further Comments

The time taken by this routine is approximately linear in

log(n+7).

Note: the routine does not test all the conditions on the knots given in the description of LAMDA in Section 5, since to do this

would result in a computation time approximately linear in n+7

instead of log(n+7). All the conditions are tested in E02BAF, however.

9. Example

Compute, at the 7 arguments x = 0, 1, 2, 3, 4, 5, 6, the left- and right-hand values and first 3 derivatives of the cubic spline defined over the interval 0<=x<=6 having the 6 interior knots x = 1, 3, 3, 3, 4, 4, the 8 additional knots 0, 0, 0, 0, 6, 6, 6, 6, and the 10 B-spline coefficients 10, 12, 13, 15, 22, 26, 24, 18, 14, 12.

The input data items (using the notation of Section 5) comprise the following values in the order indicated:

n m

LAMDA(j), for j= 1,2,...,NCAP7

C(j), for j= 1,2,...,NCAP7-4

```
    x(i),       for i=1,2,...,m
```

The example program is written in a general form that will enable the values and derivatives of a cubic spline having an arbitrary number of knots to be evaluated at a set of arbitrary points. Any number of data sets may be supplied. The only changes required to the program relate to the dimensions of the arrays LAMDA and C.

The example program is not reproduced here. The source code for all example programs is distributed with the NAG Foundation Library software and should be available on-line.

%%%

E02 -- Curve and Surface Fitting E02BDF
E02BDF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for your implementation to check implementation-dependent details. The symbol (*) after a NAG routine name denotes a routine that is not included in the Foundation Library.

1. Purpose

E02BDF computes the definite integral of a cubic spline from its B-spline representation.

2. Specification

```
SUBROUTINE E02BDF (NCAP7, LAMDA, C, DEFINT, IFAIL)
INTEGER          NCAP7, IFAIL
DOUBLE PRECISION LAMDA(NCAP7), C(NCAP7), DEFINT
```

3. Description

This routine computes the definite integral of the cubic spline s(x) between the limits x=a and x=b, where a and b are respectively the lower and upper limits of the range over which s(x) is defined. It is assumed that s(x) is represented in terms

of its B-spline coefficients c_i, for i=1,2,...,n+3 and

(augmented) ordered knot set $(\lambda)_i$, for i=1,2,...,n+7, with

(lambda) =a, for i = 1,2,3,4 and (lambda) =b, for
        i   i

i=n+4,n+5,n+6,n+7, (see E02BAF), i.e.,

$$s(x) = \sum_{i=1}^{q} c_i N_i(x).$$

Here q=n+3, n is the number of intervals of the spline and $N_i(x)$ denotes the normalised B-spline of degree 3 (order 4) defined upon the knots (lambda) ,(lambda) ,...,(lambda) .
                    i   i+1   i+4

The method employed uses the formula given in Section 3 of Cox [1].

E02BDF can be used to determine the definite integrals of cubic spline fits and interpolants produced by E02BAF.

4. References

[1] Cox M G (1975) An Algorithm for Spline Interpolation. J. Inst. Math. Appl. 15 95--108.

5. Parameters

1: NCAP7 -- INTEGER Input

On entry: n+7, where n is the number of intervals of the spline (which is one greater than the number of interior knots, i.e., the knots strictly within the range a to b) over which the spline is defined. Constraint: NCAP7 >= 8.

2: LAMDA(NCAP7) -- DOUBLE PRECISION array Input
On entry: LAMDA(j) must be set to the value of the jth member of the complete set of knots, (lambda) for
                                                    j

```
     j=1,2,...,n+7. Constraint: the LAMDA(j) must be in non-
     decreasing order with LAMDA(NCAP7-3) > LAMDA(4) and satisfy
                 LAMDA(1)=LAMDA(2)=LAMDA(3)=LAMDA(4)
     and

     LAMDA(NCAP7-3)=LAMDA(NCAP7-2)=LAMDA(NCAP7-1)=LAMDA(NCAP7).

 3:  C(NCAP7) -- DOUBLE PRECISION array                      Input
     On entry: the coefficient c  of the B-spline N (x), for
                                i                  i


     i=1,2,...,n+3. The remaining elements of the array are not
     used.

 4:  DEFINT -- DOUBLE PRECISION                              Output
     On exit: the value of the definite integral of s(x) between
     the limits x=a and x=b, where a=(lambda)  and b=(lambda)    .
                                           4               n+4

 5:  IFAIL -- INTEGER                                  Input/Output
     On entry: IFAIL must be set to 0, -1 or 1. For users not
     familiar with this parameter (described in the Essential
     Introduction) the recommended value is 0.

     On exit: IFAIL = 0 unless the routine detects an error (see
     Section 6).

6. Error Indicators and Warnings

Errors detected by the routine:

If on entry IFAIL = 0 or -1, explanatory error messages are
output on the current error message unit (as defined by X04AAF).

IFAIL= 1
     NCAP7 < 8, i.e., the number of intervals is not positive.

IFAIL= 2
     At least one of the following restrictions on the knots is
     violated:
          LAMDA(NCAP7-3) > LAMDA(4),

          LAMDA(j) >= LAMDA(j-1),
```

for j = 2,3,...,NCAP7, with equality in the cases
j=2,3,4,NCAP7-2,NCAP7-1, and NCAP7.

7. Accuracy

The rounding errors are such that the computed value of the integral is exact for a slightly perturbed set of B-spline coefficients c differing in a relative sense from those supplied
i
by no more than 2.2*(n+3)*machine precision.

8. Further Comments

The time taken by the routine is approximately proportional to

n+7.

9. Example

Determine the definite integral over the interval 0<=x<=6 of a cubic spline having 6 interior knots at the positions (lambda)=1, 3, 3, 3, 4, 4, the 8 additional knots 0, 0, 0, 0, 6, 6, 6, 6, and the 10 B-spline coefficients 10, 12, 13, 15, 22, 26, 24, 18, 14, 12.

The input data items (using the notation of Section 5) comprise the following values in the order indicated:

n

LAMDA(j) for j = 1,2,...,NCAP7
,

C(j), for j = 1,2,...,NCAP7-3

The example program is written in a general form that will enable the definite integral of a cubic spline having an arbitrary number of knots to be computed. Any number of data sets may be supplied. The only changes required to the program relate to the dimensions of the arrays LAMDA and C.

The example program is not reproduced here. The source code for all example programs is distributed with the NAG Foundation

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Library software and should be available on-line.

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

E02 -- Curve and Surface Fitting                                    E02BEF
        E02BEF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for
your implementation to check implementation-dependent details.
The symbol (*) after a NAG routine name denotes a routine that is
not included in the Foundation Library.

1. Purpose

E02BEF computes a cubic spline approximation to an arbitrary set
of data points. The knots of the spline are located
automatically, but a single parameter must be specified to
control the trade-off between closeness of fit and smoothness of
fit.

2. Specification

       SUBROUTINE E02BEF (START, M, X, Y, W, S, NEST, N, LAMDA,
      1                   C, FP, WRK, LWRK, IWRK, IFAIL)
       INTEGER          M, NEST, N, LWRK, IWRK(NEST), IFAIL
       DOUBLE PRECISION X(M), Y(M), W(M), S, LAMDA(NEST), C(NEST),
      1                 FP, WRK(LWRK)
       CHARACTER*1      START

3. Description

This routine determines a smooth cubic spline approximation s(x)
to the set of data points (x ,y ), with weights w , for
                            r  r                 r
r=1,2,...,m.

The spline is given in the B-spline representation

                            n-4
                            --
                     s(x)= >   c N (x)                          (1)
                            --  i i
                            i=1

where N (x) denotes the normalised cubic B-spline defined upon
       i
```

the knots $(\lambda)_i, (\lambda)_{i+1}, \ldots, (\lambda)_{i+4}$.

The total number n of these knots and their values
$(\lambda)_1, \ldots, (\lambda)_n$ are chosen automatically by the routine.
The knots $(\lambda)_5, \ldots, (\lambda)_{n-4}$ are the interior knots; they
divide the approximation interval $[x_1, x_m]$ into n-7 sub-intervals.
The coefficients $c_1, c_2, \ldots, c_{n-4}$ are then determined as the
solution of the following constrained minimization problem:

minimize

$$(\eta) = \sum_{i=5}^{n-4} (\delta)_i^2 \qquad (2)$$

subject to the constraint

$$(\theta) = \sum_{r=1}^{m} (\varepsilon)_r^2 \leq S \qquad (3)$$

where: $(\delta)_i$ stands for the discontinuity jump in the third
order derivative of s(x) at the interior knot
$(\lambda)_i$,

$(\varepsilon)_r$ denotes the weighted residual $w_r (y_r - s(x_r))$,

and S is a non-negative number to be specified by
the user.

The quantity (η) can be seen as a measure of the (lack of)
smoothness of s(x), while closeness of fit is measured through
(θ). By means of the parameter S, 'the smoothing factor', the
user will then control the balance between these two (usually
conflicting) properties. If S is too large, the spline will be

too smooth and signal will be lost (underfit); if S is too small, the spline will pick up too much noise (overfit). In the extreme cases the routine will return an interpolating spline ((theta)=0) if S is set to zero, and the weighted least-squares cubic polynomial ((eta)=0) if S is set very large. Experimenting with S values between these two extremes should result in a good compromise. (See Section 8.2 for advice on choice of S.)

The method employed is outlined in Section 8.3 and fully described in Dierckx [1], [2] and [3]. It involves an adaptive strategy for locating the knots of the cubic spline (depending on the function underlying the data and on the value of S), and an iterative method for solving the constrained minimization problem once the knots have been determined.

Values of the computed spline, or of its derivatives or definite integral, can subsequently be computed by calling E02BBF, E02BCF or E02BDF, as described in Section 8.4.

4. References

[1] Dierckx P (1975) An Algorithm for Smoothing, Differentiating and Integration of Experimental Data Using Spline Functions. J. Comput. Appl. Math. 1 165--184.

[2] Dierckx P (1982) A Fast Algorithm for Smoothing Data on a Rectangular Grid while using Spline Functions. SIAM J. Numer. Anal. 19 1286--1304.

[3] Dierckx P (1981) An Improved Algorithm for Curve Fitting with Spline Functions. Report TW54. Department of Computer Science, Katholieke Universiteit Leuven.

[4] Reinsch C H (1967) Smoothing by Spline Functions. Num. Math. 10 177--183.

5. Parameters

1: START -- CHARACTER*1 Input
On entry: START must be set to 'C' or 'W'.

If START = 'C' (Cold start), the routine will build up the knot set starting with no interior knots. No values need be assigned to the parameters N, LAMDA, WRK or IWRK.

If START = 'W' (Warm start), the routine will restart the

knot-placing strategy using the knots found in a previous call of the routine. In this case, the parameters N, LAMDA, WRK, and IWRK must be unchanged from that previous call. This warm start can save much time in searching for a satisfactory value of S. Constraint: START = 'C' or 'W'.

2: M -- INTEGER Input
On entry: m, the number of data points. Constraint: M >= 4.

3: X(M) -- DOUBLE PRECISION array Input
On entry: the values x_r of the independent variable (abscissa) x, for r=1,2,...,m. Constraint: $x_1 < x_2 < \ldots < x_m$.

4: Y(M) -- DOUBLE PRECISION array Input
On entry: the values y_r of the dependent variable (ordinate) y, for r=1,2,...,m.

5: W(M) -- DOUBLE PRECISION array Input
On entry: the values w_r of the weights, for r=1,2,...,m.
For advice on the choice of weights, see the Chapter Introduction, Section 2.1.2. Constraint: W(r) > 0, for r=1,2,...,m.

6: S -- DOUBLE PRECISION Input
On entry: the smoothing factor, S.

If S=0.0, the routine returns an interpolating spline.

If S is smaller than machine precision, it is assumed equal to zero.

For advice on the choice of S, see Section 3 and Section 8.2 Constraint: S >= 0.0.

7: NEST -- INTEGER Input
On entry: an over-estimate for the number, n, of knots required. Constraint: NEST >= 8. In most practical situations, NEST = M/2 is sufficient. NEST never needs to be larger than M + 4, the number of knots needed for interpolation (S = 0.0).

8: N -- INTEGER Input/Output

On entry: if the warm start option is used, the value of N must be left unchanged from the previous call. On exit: the total number, n, of knots of the computed spline.

9: LAMDA(NEST) -- DOUBLE PRECISION array Input/Output
On entry: if the warm start option is used, the values LAMDA(1), LAMDA(2),...,LAMDA(N) must be left unchanged from the previous call. On exit: the knots of the spline i.e., the positions of the interior knots LAMDA(5), LAMDA(6),... ,LAMDA(N-4) as well as the positions of the additional knots LAMDA(1) = LAMDA(2) = LAMDA(3) = LAMDA(4) = x_1 and

LAMDA(N-3) = LAMDA(N-2) = LAMDA(N-1) = LAMDA(N) = x_m needed for the B-spline representation.

10: C(NEST) -- DOUBLE PRECISION array Output
On exit: the coefficient c_i of the B-spline $N_i(x)$ in the spline approximation s(x), for i=1,2,...,n-4.

11: FP -- DOUBLE PRECISION Output
On exit: the sum of the squared weighted residuals, (theta), of the computed spline approximation. If FP = 0.0, this is an interpolating spline. FP should equal S within a relative tolerance of 0.001 unless n=8 when the spline has no interior knots and so is simply a cubic polynomial. For knots to be inserted, S must be set to a value below the value of FP produced in this case.

12: WRK(LWRK) -- DOUBLE PRECISION array Workspace
On entry: if the warm start option is used, the values WRK (1),...,WRK(n) must be left unchanged from the previous call.

13: LWRK -- INTEGER Input
On entry:
the dimension of the array WRK as declared in the (sub)program from which E02BEF is called.
Constraint: LWRK>=4*M+16*NEST+41.

14: IWRK(NEST) -- INTEGER array Workspace
On entry: if the warm start option is used, the values IWRK (1), ..., IWRK(n) must be left unchanged from the previous call.

```
     This array is used as workspace.

15:  IFAIL -- INTEGER                                    Input/Output
     On entry: IFAIL must be set to 0, -1 or 1. For users not
     familiar with this parameter (described in the Essential
     Introduction) the recommended value is 0.

     On exit: IFAIL = 0 unless the routine detects an error (see
     Section 6).

6. Error Indicators and Warnings

Errors detected by the routine:

If on entry IFAIL = 0 or -1, explanatory error messages are
output on the current error message unit (as defined by X04AAF).

IFAIL= 1
     On entry START /= 'C' or 'W',

     or       M < 4,

     or       S < 0.0,

     or       S = 0.0 and NEST < M + 4,

     or       NEST < 8,

     or       LWRK<4*M+16*NEST+41.

IFAIL= 2
     The weights are not all strictly positive.

IFAIL= 3
     The values of X(r), for r=1,2,...,M, are not in strictly
     increasing order.

IFAIL= 4
     The number of knots required is greater than NEST. Try
     increasing NEST and, if necessary, supplying larger arrays
     for the parameters LAMDA, C, WRK and IWRK. However, if NEST
     is already large, say NEST > M/2, then this error exit may
     indicate that S is too small.

IFAIL= 5
```

```
     The iterative process used to compute the coefficients of
     the approximating spline has failed to converge. This error
     exit may occur if S has been set very small. If the error
     persists with increased S, consult NAG.

If IFAIL = 4 or 5, a spline approximation is returned, but it
fails to satisfy the fitting criterion (see (2) and (3) in
Section 3) - perhaps by only a small amount, however.

7. Accuracy

On successful exit, the approximation returned is such that its
weighted sum of squared residuals FP is equal to the smoothing
factor S, up to a specified relative tolerance of 0.001 - except
that if n=8, FP may be significantly less than S: in this case
the computed spline is simply a weighted least-squares polynomial
approximation of degree 3, i.e., a spline with no interior knots.

8. Further Comments

8.1. Timing

The time taken for a call of E02BEF depends on the complexity of
the shape of the data, the value of the smoothing factor S, and
the number of data points. If E02BEF is to be called for
different values of S, much time can be saved by setting START =

8.2. Choice of S

If the weights have been correctly chosen (see Section 2.1.2 of
the Chapter Introduction), the standard deviation of w y  would
                                                      r r
be the same for all r, equal to (sigma), say. In this case,
                                                    2
choosing the smoothing factor S in the range (sigma) (m+-\/2m),
as suggested by Reinsch [4], is likely to give a good start in
the search for a satisfactory value. Otherwise, experimenting
with different values of S will be required from the start,
taking account of the remarks in Section 3.

In that case, in view of computation time and memory
requirements, it is recommended to start with a very large value
for S and so determine the least-squares cubic polynomial; the
value returned for FP, call it FP , gives an upper bound for S.
                                  0
Then progressively decrease the value of S to obtain closer fits
```

- say by a factor of 10 in the beginning, i.e., S=FP /10, S=FP
                                                  0   0
/100, and so on, and more carefully as the approximation shows
more details.

The number of knots of the spline returned, and their location, generally depend on the value of S and on the behaviour of the function underlying the data. However, if E02BEF is called with START = 'W', the knots returned may also depend on the smoothing factors of the previous calls. Therefore if, after a number of trials with different values of S and START = 'W', a fit can finally be accepted as satisfactory, it may be worthwhile to call E02BEF once more with the selected value for S but now using START = 'C'. Often, E02BEF then returns an approximation with the same quality of fit but with fewer knots, which is therefore better if data reduction is also important.

8.3. Outline of Method Used

If S=0, the requisite number of knots is known in advance, i.e., n=m+4; the interior knots are located immediately as (lambda) =
                                                                  i
x   , for i=5,6,...,n-4. The corresponding least-squares spline
 i-2
(see E02BAF) is then an interpolating spline and therefore a solution of the problem.

If S>0, a suitable knot set is built up in stages (starting with no interior knots in the case of a cold start but with the knot set found in a previous call if a warm start is chosen). At each stage, a spline is fitted to the data by least-squares (see E02BAF) and (theta), the weighted sum of squares of residuals, is computed. If (theta)>S, new knots are added to the knot set to reduce (theta) at the next stage. The new knots are located in intervals where the fit is particularly poor, their number depending on the value of S and on the progress made so far in reducing (theta). Sooner or later, we find that (theta)<=S and at that point the knot set is accepted. The routine then goes on to compute the (unique) spline which has this knot set and which satisfies the full fitting criterion specified by (2) and (3). The theoretical solution has (theta)=S. The routine computes the spline by an iterative scheme which is ended when (theta)=S within a relative tolerance of 0.001. The main part of each iteration consists of a linear least-squares computation of special form, done in a similarly stable and efficient manner as in E02BAF.

An exception occurs when the routine finds at the start that, even with no interior knots (n=8), the least-squares spline already has its weighted sum of squares of residuals <=S. In this case, since this spline (which is simply a cubic polynomial) also has an optimal value for the smoothness measure (eta), namely zero, it is returned at once as the (trivial) solution. It will usually mean that S has been chosen too large.

For further details of the algorithm and its use, see Dierckx [3]

8.4. Evaluation of Computed Spline

The value of the computed spline at a given value X may be obtained in the double precision variable S by the call:

```
CALL E02BBF(N,LAMDA,C,X,S,IFAIL)
```

where N, LAMDA and C are the output parameters of E02BEF.

The values of the spline and its first three derivatives at a given value X may be obtained in the double precision array SDIF of dimension at least 4 by the call:

```
CALL E02BCF(N,LAMDA,C,X,LEFT,SDIF,IFAIL)
```

where if LEFT = 1, left-hand derivatives are computed and if LEFT /= 1, right-hand derivatives are calculated. The value of LEFT is only relevant if X is an interior knot.

The value of the definite integral of the spline over the interval X(1) to X(M) can be obtained in the double precision variable SINT by the call:

```
CALL E02BDF(N,LAMDA,C,SINT,IFAIL)
```

9. Example

This example program reads in a set of data values, followed by a set of values of S. For each value of S it calls E02BEF to compute a spline approximation, and prints the values of the knots and the B-spline coefficients c_i.

The program includes code to evaluate the computed splines, by

calls to E02BBF, at the points x and at points mid-way between
                                r
them. These values are not printed out, however; instead the
results are illustrated by plots of the computed splines,
together with the data points (indicated by *) and the positions
of the knots (indicated by vertical lines): the effect of
decreasing S can be clearly seen. (The plots were obtained by
calling NAG Graphical Supplement routine J06FAF(*).)

Please see figures in printed Reference Manual

The example program is not reproduced here. The source code for
all example programs is distributed with the NAG Foundation
Library software and should be available on-line.

%%%

E02 -- Curve and Surface Fitting E02DAF
E02DAF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for
your implementation to check implementation-dependent details.
The symbol (*) after a NAG routine name denotes a routine that is
not included in the Foundation Library.

1. Purpose

E02DAF forms a minimal, weighted least-squares bicubic spline
surface fit with prescribed knots to a given set of data points.

2. Specification

```
 SUBROUTINE E02DAF (M, PX, PY, X, Y, F, W, LAMDA, MU,
1                   POINT, NPOINT, DL, C, NC, WS, NWS, EPS,
2                   SIGMA, RANK, IFAIL)
 INTEGER          M, PX, PY, POINT(NPOINT), NPOINT, NC, NWS,
1                 RANK, IFAIL
 DOUBLE PRECISION X(M), Y(M), F(M), W(M), LAMDA(PX), MU(PY),
1                 DL(NC), C(NC), WS(NWS), EPS, SIGMA
```

3. Description

This routine determines a bicubic spline fit s(x,y) to the set of
data points (x ,y ,f) with weights w , for r=1,2,...,m. The two
             r r r   r

sets of internal knots of the spline, {(lambda)} and {(mu)}, associated with the variables x and y respectively, are prescribed by the user. These knots can be thought of as dividing the data region of the (x,y) plane into panels (see diagram in Section 5). A bicubic spline consists of a separate bicubic polynomial in each panel, the polynomials joining together with continuity up to the second derivative across the panel boundaries.

s(x,y) has the property that (Sigma), the sum of squares of its weighted residuals $(\rho)_r$, for r=1,2,...,m, where

$$(\rho)_r = w_r (s(x_r, y_r) - f_r), \qquad (1)$$

is as small as possible for a bicubic spline with the given knot sets. The routine produces this minimized value of (Sigma) and the coefficients c_{ij} in the B-spline representation of s(x,y) - see Section 8. E02DEF and E02DFF are available to compute values of the fitted spline from the coefficients c_{ij}.

The least-squares criterion is not always sufficient to determine the bicubic spline uniquely: there may be a whole family of splines which have the same minimum sum of squares. In these cases, the routine selects from this family the spline for which the sum of squares of the coefficients c_{ij} is smallest: in other words, the minimal least-squares solution. This choice, although arbitrary, reduces the risk of unwanted fluctuations in the spline fit. The method employed involves forming a system of m linear equations in the coefficients c_{ij} and then computing its least-squares solution, which will be the minimal least-squares solution when appropriate. The basis of the method is described in Hayes and Halliday [4]. The matrix of the equation is formed using a recurrence relation for B-splines which is numerically stable (see Cox [1] and de Boor [2] - the former contains the more elementary derivation but, unlike [2], does not cover the case of coincident knots). The least-squares solution is also obtained in a stable manner by using orthogonal transformations, viz. a variant of Givens rotation (see Gentleman [3]). This requires only one row of the matrix to be stored at a time. Advantage is taken of the stepped-band structure which the matrix

possesses when the data points are suitably ordered, there being at most sixteen non-zero elements in any row because of the definition of B-splines. First the matrix is reduced to upper triangular form and then the diagonal elements of this triangle are examined in turn. When an element is encountered whose square, divided by the mean squared weight, is less than a threshold (epsilon), it is replaced by zero and the rest of the elements in its row are reduced to zero by rotations with the remaining rows. The rank of the system is taken to be the number of non-zero diagonal elements in the final triangle, and the non-zero rows of this triangle are used to compute the minimal least-squares solution. If all the diagonal elements are non-zero, the rank is equal to the number of coefficients c_{ij} and the solution obtained is the ordinary least-squares solution, which is unique in this case.

4. References

[1] Cox M G (1972) The Numerical Evaluation of B-splines. J. Inst. Math. Appl. 10 134--149.

[2] De Boor C (1972) On Calculating with B-splines. J. Approx. Theory. 6 50--62.

[3] Gentleman W M (1973) Least-squares Computations by Givens Transformations without Square Roots. J. Inst. Math. Applic. 12 329--336.

[4] Hayes J G and Halliday J (1974) The Least-squares Fitting of Cubic Spline Surfaces to General Data Sets. J. Inst. Math. Appl. 14 89--103.

5. Parameters

1: M -- INTEGER Input
On entry: the number of data points, m. Constraint: M > 1.

2: PX -- INTEGER Input

3: PY -- INTEGER Input
On entry: the total number of knots (lambda) and (mu) associated with the variables x and y, respectively.
Constraint: PX >= 8 and PY >= 8.

(They are such that PX-8 and PY-8 are the corresponding

numbers of interior knots.) The running time and storage required by the routine are both minimized if the axes are labelled so that PY is the smaller of PX and PY.

4: X(M) -- DOUBLE PRECISION array Input

5: Y(M) -- DOUBLE PRECISION array Input

6: F(M) -- DOUBLE PRECISION array Input
On entry: the co-ordinates of the data point (x ,y ,f), for
r r r
r=1,2,...,m. The order of the data points is immaterial, but see the array POINT, below.

7: W(M) -- DOUBLE PRECISION array Input
On entry: the weight w of the rth data point. It is
r
important to note the definition of weight implied by the equation (1) in Section 3, since it is also common usage to define weight as the square of this weight. In this routine, each w should be chosen inversely proportional to the
r
(absolute) accuracy of the corresponding f , as expressed,
r
for example, by the standard deviation or probable error of the f . When the f are all of the same accuracy, all the w
r r r
may be set equal to 1.0.

8: LAMDA(PX) -- DOUBLE PRECISION array Input/Output
On entry: LAMDA(i+4) must contain the ith interior knot (lambda) associated with the variable x, for
i+4
i=1,2,...,PX-8. The knots must be in non-decreasing order and lie strictly within the range covered by the data values of x. A knot is a value of x at which the spline is allowed to be discontinuous in the third derivative with respect to x, though continuous up to the second derivative. This degree of continuity can be reduced, if the user requires, by the use of coincident knots, provided that no more than four knots are chosen to coincide at any point. Two, or three, coincident knots allow loss of continuity in, respectively, the second and first derivative with respect to x at the value of x at which they coincide. Four coincident knots split the spline surface into two independent parts. For choice of knots see Section 8. On

exit: the interior knots LAMDA(5) to LAMDA(PX-4) are unchanged, and the segments LAMDA(1:4) and LAMDA(PX-3:PX) contain additional (exterior) knots introduced by the routine in order to define the full set of B-splines required. The four knots in the first segment are all set equal to the lowest data value of x and the other four additional knots are all set equal to the highest value: there is experimental evidence that coincident end-knots are best for numerical accuracy. The complete array must be left undisturbed if E02DEF or E02DFF is to be used subsequently.

9: MU(PY) -- DOUBLE PRECISION array Input
On entry: MU(i+4) must contain the ith interior knot (mu) $_{i+4}$ associated with the variable y, i=1,2,...,PY-8. The same remarks apply to MU as to LAMDA above, with Y replacing X, and y replacing x.

10: POINT(NPOINT) -- INTEGER array Input
On entry: indexing information usually provided by E02ZAF which enables the data points to be accessed in the order which produces the advantageous matrix structure mentioned in Section 3. This order is such that, if the (x,y) plane is thought of as being divided into rectangular panels by the two sets of knots, all data in a panel occur before data in succeeding panels, where the panels are numbered from bottom to top and then left to right with the usual arrangement of axes, as indicated in the diagram.

Please see figure in printed Reference Manual

A data point lying exactly on one or more panel sides is considered to be in the highest numbered panel adjacent to the point. E02ZAF should be called to obtain the array POINT, unless it is provided by other means.

11: NPOINT -- INTEGER Input
On entry:
the dimension of the array POINT as declared in the (sub)program from which E02DAF is called.
Constraint: NPOINT >= M + (PX-7)*(PY-7).

12: DL(NC) -- DOUBLE PRECISION array Output
On exit: DL gives the squares of the diagonal elements of the reduced triangular matrix, divided by the mean squared weight. It includes those elements, less than (epsilon),

which are treated as zero (see Section 3).

13: C(NC) -- DOUBLE PRECISION array Output
On exit: C gives the coefficients of the fit. C((PY-4)*(i-1)+j) is the coefficient c_{ij} of Section 3 and Section 8 for i=1,2,...,PX-4 and j=1,2,...,PY-4. These coefficients are used by E02DEF or E02DFF to calculate values of the fitted function.

14: NC -- INTEGER Input
On entry: the value (PX-4)*(PY-4).

15: WS(NWS) -- DOUBLE PRECISION array Workspace

16: NWS -- INTEGER Input
On entry:
the dimension of the array WS as declared in the (sub)program from which E02DAF is called.
Constraint: NWS>=(2*NC+1)*(3*PY-6)-2.

17: EPS -- DOUBLE PRECISION Input
On entry: a threshold (epsilon) for determining the effective rank of the system of linear equations. The rank is determined as the number of elements of the array DL (see below) which are non-zero. An element of DL is regarded as zero if it is less than (epsilon). Machine precision is a suitable value for (epsilon) in most practical applications which have only 2 or 3 decimals accurate in data. If some coefficients of the fit prove to be very large compared with the data ordinates, this suggests that (epsilon) should be increased so as to decrease the rank. The array DL will give a guide to appropriate values of (epsilon) to achieve this, as well as to the choice of (epsilon) in other cases where some experimentation may be needed to determine a value which leads to a satisfactory fit.

18: SIGMA -- DOUBLE PRECISION Output
On exit: (Sigma), the weighted sum of squares of residuals. This is not computed from the individual residuals but from the right-hand sides of the orthogonally-transformed linear equations. For further details see Hayes and Halliday [4] page 97. The two methods of computation are theoretically equivalent, but the results may differ because of rounding error.

```
19:  RANK -- INTEGER                                          Output
     On exit: the rank of the system as determined by the value
     of the threshold (epsilon). When RANK = NC, the least-
     squares solution is unique: in other cases the minimal
     least-squares solution is computed.

20:  IFAIL -- INTEGER                                  Input/Output
     On entry: IFAIL must be set to 0, -1 or 1. For users not
     familiar with this parameter (described in the Essential
     Introduction) the recommended value is 0.

     On exit: IFAIL = 0 unless the routine detects an error (see
     Section 6).

6. Error Indicators and Warnings

Errors detected by the routine:

IFAIL= 1
     At least one set of knots is not in non-decreasing order, or
     an interior knot is outside the range of the data values.

IFAIL= 2
     More than four knots coincide at a single point, possibly
     because all data points have the same value of x (or y) or
     because an interior knot coincides with an extreme data
     value.

IFAIL= 3
     Array POINT does not indicate the data points in panel
     order. Call E02ZAF to obtain a correct array.

IFAIL= 4
     On entry M <= 1,

     or       PX < 8,

     or       PY < 8,

     or       NC /= (PX-4)*(PY-4),

     or       NWS is too small,

     or       NPOINT is too small.

IFAIL= 5
```

All the weights w_r are zero or rank determined as zero.

7. Accuracy

The computation of the B-splines and reduction of the observation matrix to triangular form are both numerically stable.

8. Further Comments

The time taken by this routine is approximately proportional to the number of data points, m, and to $(3*(PY-4)+4)^2$.

The B-spline representation of the bicubic spline is

$$s(x,y) = \sum_{ij} c_{ij} M_i(x) N_j(y)$$

summed over i=1,2,...,PX-4 and over j=1,2,...,PY-4. Here $M_i(x)$ and $N_j(y)$ denote normalised cubic B-splines,the former defined on the knots $(\lambda)_i,(\lambda)_{i+1},\ldots,(\lambda)_{i+4}$ and the latter on the knots $(\mu)_j,(\mu)_{j+1},\ldots,(\mu)_{j+4}$. For further details, see Hayes and Halliday [4] for bicubic splines and de Boor [2] for normalised B-splines.

The choice of the interior knots, which help to determine the spline's shape, must largely be a matter of trial and error. It is usually best to start with a small number of knots and, examining the fit at each stage, add a few knots at a time at places where the fit is particularly poor. In intervals of x or y where the surface represented by the data changes rapidly, in function value or derivatives, more knots will be needed than elsewhere. In some cases guidance can be obtained by analogy with the case of coincident knots: for example, just as three coincident knots can produce a discontinuity in slope, three close knots can produce rapid change in slope. Of course, such rapid changes in behaviour must be adequately represented by the data points, as indeed must the behaviour of the surface generally, if a satisfactory fit is to be achieved. When there is

no rapid change in behaviour, equally-spaced knots will often suffice.

In all cases the fit should be examined graphically before it is accepted as satisfactory.

The fit obtained is not defined outside the rectangle

```
        (lambda) <=x<=(lambda)    ,   (mu) <=y<=(mu)
                4             PX-3        4         PY-3
```

The reason for taking the extreme data values of x and y for these four knots is that, as is usual in data fitting, the fit cannot be expected to give satisfactory values outside the data region. If, nevertheless, the user requires values over a larger rectangle, this can be achieved by augmenting the data with two artificial data points (a,c,0) and (b,d,0) with zero weight, where a<=x<=b, c<=y<=d defines the enlarged rectangle. In the case when the data are adequate to make the least-squares solution unique (RANK = NC), this enlargement will not affect the fit over the original rectangle, except for possibly enlarged rounding errors, and will simply continue the bicubic polynomials in the panels bordering the rectangle out to the new boundaries: in other cases the fit will be affected. Even using the original rectangle there may be regions within it, particularly at its corners, which lie outside the data region and where, therefore, the fit will be unreliable. For example, if there is no data point in panel 1 of the diagram in Section 5, the least-squares criterion leaves the spline indeterminate in this panel: the minimal spline determined by the subroutine in this case passes through the value zero at the point ((lambda) ,(mu)).

```
                                                4     4
```

9. Example

This example program reads a value for (epsilon), and a set of data points, weights and knot positions. If there are more y knots than x knots, it interchanges the x and y axes. It calls E02ZAF to sort the data points into panel order, E02DAF to fit a bicubic spline to them, and E02DEF to evaluate the spline at the data points.

Finally it prints:

- the weighted sum of squares of residuals computed from the linear equations;

```
     the rank determined by E02DAF;

     data points, fitted values and residuals in panel order;

     the weighted sum of squares of the residuals;

     the coefficients of the spline fit.

The program is written to handle any number of data sets.

Note: the data supplied in this example is not typical of a
realistic problem: the number of data points would normally be
much larger (in which case the array dimensions and the value of
NWS in the program would have to be increased); and the value of
(epsilon) would normally be much smaller on most machines (see
                                                    -6
Section 5; the relatively large value of 10   has been chosen in
order to illustrate a minimal least-squares solution when RANK <
NC; in this example NC = 24).

The example program is not reproduced here. The source code for
all example programs is distributed with the NAG Foundation
Library software and should be available on-line.
```

%%

```
E02 -- Curve and Surface Fitting                              E02DCF
        E02DCF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for
your implementation to check implementation-dependent details.
The symbol (*) after a NAG routine name denotes a routine that is
not included in the Foundation Library.

1. Purpose

E02DCF computes a bicubic spline approximation to a set of data
values, given on a rectangular grid in the x-y plane. The knots
of the spline are located automatically, but a single parameter
must be specified to control the trade-off between closeness of
fit and smoothness of fit.

2. Specification

       SUBROUTINE E02DCF (START, MX, X, MY, Y, F, S, NXEST,
```

```
      1                  NYEST, NX, LAMDA, NY, MU, C, FP, WRK,
      2                  LWRK, IWRK, LIWRK, IFAIL)
       INTEGER          MX, MY, NXEST, NYEST, NX, NY, LWRK, IWRK
      1                 (LIWRK), LIWRK, IFAIL
       DOUBLE PRECISION X(MX), Y(MY), F(MX*MY), S, LAMDA(NXEST),
      1                 MU(NYEST), C((NXEST-4)*(NYEST-4)), FP, WRK
      2                 (LWRK)
       CHARACTER*1      START
```

3. Description

This routine determines a smooth bicubic spline approximation $s(x,y)$ to the set of data points $(x_q, y_r, f_{q,r})$, for $q=1,2,\ldots,m_x$ and $r=1,2,\ldots,m_y$.

The spline is given in the B-spline representation

$$s(x,y)=\sum_{i=1}^{n_x-4}\sum_{j=1}^{n_y-4} c_{ij} M_i(x) N_j(y), \qquad (1)$$

where $M_i(x)$ and $N_j(y)$ denote normalised cubic B-splines, the former defined on the knots $(\lambda)_i$ to $(\lambda)_{i+4}$ and the latter on the knots $(\mu)_j$ to $(\mu)_{j+4}$. For further details, see Hayes and Halliday [4] for bicubic splines and de Boor [1] for normalised B-splines.

The total numbers n_x and n_y of these knots and their values $(\lambda)_1,\ldots,(\lambda)_{n_x}$ and $(\mu)_1,\ldots,(\mu)_{n_y}$ are chosen automatically by the routine. The knots $(\lambda)_5,\ldots,(\lambda)_{n_x-4}$ and $(\mu)_5,\ldots,(\mu)_{n_y-4}$ are the interior knots; they divide the approximation domain $[x_1,x_{n_x}]*[y_1,y_{n_y}]$ into (

$n_x-7)*(n_y-7)$ subpanels $[(\lambda)_i^{1\ m},(\lambda)_{i+1}^{m}]*[(\mu)_j^{1\ m},(\mu)_{j+1}^{m}]$, for $i=4,5,...,n_x-4$, $j=4,5,...,n_y-4$. Then, much as in the curve case (see E02BEF), the coefficients c_{ij} are determined as the solution of the following constrained minimization problem:

minimize

$$(\eta), \qquad (2)$$

subject to the constraint

$$(\theta)=\sum_{q=1}^{m_x}\sum_{r=1}^{m_y}(\epsilon)_{q,r}^2 \le S, \qquad (3)$$

where (η) is a measure of the (lack of) smoothness of $s(x,y)$. Its value depends on the discontinuity jumps in $s(x,y)$ across the boundaries of the subpanels. It is zero only when there are no discontinuities and is positive otherwise, increasing with the size of the jumps (see Dierckx [2] for details).

$(\epsilon)_{q,r}$ denotes the residual $f_{q,r}-s(x_q,y_r)$,

and S is a non-negative number to be specified by the user.

By means of the parameter S, 'the smoothing factor', the user will then control the balance between smoothness and closeness of fit, as measured by the sum of squares of residuals in (3). If S is too large, the spline will be too smooth and signal will be lost (underfit); if S is too small, the spline will pick up too much noise (overfit). In the extreme cases the routine will return an interpolating spline ($(\theta)=0$) if S is set to zero, and the least-squares bicubic polynomial ($(\eta)=0$) if S is set very large. Experimenting with S-values between these two extremes should result in a good compromise. (See Section 8.3 for advice on choice of S.)

The method employed is outlined in Section 8.5 and fully described in Dierckx [2] and [3]. It involves an adaptive strategy for locating the knots of the bicubic spline (depending on the function underlying the data and on the value of S), and an iterative method for solving the constrained minimization problem once the knots have been determined.

Values of the computed spline can subsequently be computed by calling E02DEF or E02DFF as described in Section 8.6.

4. References

[1] De Boor C (1972) On Calculating with B-splines. J. Approx. Theory. 6 50--62.

[2] Dierckx P (1982) A Fast Algorithm for Smoothing Data on a Rectangular Grid while using Spline Functions. SIAM J. Numer. Anal. 19 1286--1304.

[3] Dierckx P (1981) An Improved Algorithm for Curve Fitting with Spline Functions. Report TW54. Department of Computer Science, Katholieke Universiteit Leuven.

[4] Hayes J G and Halliday J (1974) The Least-squares Fitting of Cubic Spline Surfaces to General Data Sets. J. Inst. Math. Appl. 14 89--103.

[5] Reinsch C H (1967) Smoothing by Spline Functions. Num. Math. 10 177--183.

5. Parameters

1: START -- CHARACTER*1 Input
On entry: START must be set to 'C' or 'W'.

If START = 'C' (Cold start), the routine will build up the knot set starting with no interior knots. No values need be assigned to the parameters NX, NY, LAMDA, MU, WRK or IWRK.

If START = 'W' (Warm start), the routine will restart the knot-placing strategy using the knots found in a previous call of the routine. In this case, the parameters NX, NY, LAMDA, MU, WRK and IWRK must be unchanged from that previous call. This warm start can save much time in searching for a satisfactory value of S. Constraint: START = 'C' or 'W'.

2: MX -- INTEGER Input
On entry: m , the number of grid points along the x axis.
x
Constraint: MX >= 4.

3: X(MX) -- DOUBLE PRECISION array Input
On entry: X(q) must be set to x , the x co-ordinate of the
q
qth grid point along the x axis, for q=1,2,...,m .
x
Constraint: x <x <...<x .
1 2 m
x

4: MY -- INTEGER Input
On entry: m , the number of grid points along the y axis.
y
Constraint: MY >= 4.

5: Y(MY) -- DOUBLE PRECISION array Input
On entry: Y(r) must be set to y , the y co-ordinate of the
r
rth grid point along the y axis, for r=1,2,...,m .
y
Constraint: y <y <...<y .
1 2 m
y

6: F(MX*MY) -- DOUBLE PRECISION array Input
On entry: F(m *(q-1)+r) must contain the data value f ,
y q,r
for q=1,2,...,m and r=1,2,...,m .
x y

7: S -- DOUBLE PRECISION Input
On entry: the smoothing factor, S.

If S=0.0, the routine returns an interpolating spline.

If S is smaller than machine precision, it is assumed equal to zero.

For advice on the choice of S, see Section 3 and Section 8.3
Constraint: S >= 0.0.

8: NXEST -- INTEGER Input

9: NYEST -- INTEGER Input
On entry: an upper bound for the number of knots n_x and n_y required in the x- and y-directions respectively.

In most practical situations, NXEST =m_x /2 and NYEST m_y /2 is sufficient. NXEST and NYEST never need to be larger than m_x +4 and m_y +4 respectively, the numbers of knots needed for interpolation (S=0.0). See also Section 8.4. Constraint: NXEST >= 8 and NYEST >= 8.

10: NX -- INTEGER Input/Output
On entry: if the warm start option is used, the value of NX must be left unchanged from the previous call. On exit: the total number of knots, n_x, of the computed spline with respect to the x variable.

11: LAMDA(NXEST) -- DOUBLE PRECISION array Input/Output
On entry: if the warm start option is used, the values LAMDA(1), LAMDA(2),...,LAMDA(NX) must be left unchanged from the previous call. On exit: LAMDA contains the complete set of knots (lambda) associated with the x variable, i.e., the interior knots λ_5, λ_6, ..., λ_{NX-4} as well as the additional knots $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = X(1)$ and $\lambda_{NX-3} = \lambda_{NX-2} = \lambda_{NX-1} = \lambda_{NX} = X(MX)$ needed for the B-spline representation.

12: NY -- INTEGER Input/Output
On entry: if the warm start option is used, the value of NY must be left unchanged from the previous call. On exit: the total number of knots, n_y, of the computed spline with respect to the y variable.

13: MU(NYEST) -- DOUBLE PRECISION array Input/Output
On entry: if the warm start option is used, the values MU (1), MU(2),...,MU(NY) must be left unchanged from the previous call. On exit: MU contains the complete set of knots (mu) associated with the y variable, i.e., the

interior knots MU(5), MU(6),...,MU(NY-4) as well as the additional knots MU(1) = MU(2) = MU(3) = MU(4) = Y(1) and MU (NY-3) = MU(NY-2) = MU(NY-1) = MU(NY) = Y(MY) needed for the B-spline representation.

14: C((NXEST-4)*(NYEST-4)) -- DOUBLE PRECISION array Output
On exit: the coefficients of the spline approximation. C((n -4)*(i-1)+j) is the coefficient c_{ij} defined in Section 3.

15: FP -- DOUBLE PRECISION Output
On exit: the sum of squared residuals, (theta), of the computed spline approximation. If FP = 0.0, this is an interpolating spline. FP should equal S within a relative tolerance of 0.001 unless NX = NY = 8, when the spline has no interior knots and so is simply a bicubic polynomial. For knots to be inserted, S must be set to a value below the value of FP produced in this case.

16: WRK(LWRK) -- DOUBLE PRECISION array Workspace
On entry: if the warm start option is used, the values WRK (1),...,WRK(4) must be left unchanged from the previous call.

This array is used as workspace.

17: LWRK -- INTEGER Input
On entry:
the dimension of the array WRK as declared in the (sub)program from which E02DCF is called.
Constraint:

```
LWRK>=4*(MX+MY)+11*(NXEST+NYEST)+NXEST*MY

      +max(MY,NXEST)+54.
```

18: IWRK(LIWRK) -- INTEGER array Workspace
On entry: if the warm start option is used, the values IWRK (1), ..., IWRK(3) must be left unchanged from the previous call.

This array is used as workspace.

19: LIWRK -- INTEGER Input
On entry:
the dimension of the array IWRK as declared in the (sub)program from which E02DCF is called.

```
     Constraint: LIWRK >= 3 + MX + MY + NXEST + NYEST.

20:  IFAIL -- INTEGER                                    Input/Output
     On entry: IFAIL must be set to 0, -1 or 1. For users not
     familiar with this parameter (described in the Essential
     Introduction) the recommended value is 0.

     On exit: IFAIL = 0 unless the routine detects an error (see
     Section 6).

6. Error Indicators and Warnings

Errors detected by the routine:

If on entry IFAIL = 0 or -1, explanatory error messages are
output on the current error message unit (as defined by X04AAF).

IFAIL= 1
     On entry START /= 'C' or 'W',

     or       MX < 4,

     or       MY < 4,

     or       S < 0.0,

     or       S = 0.0 and NXEST < MX + 4,

     or       S = 0.0 and NYEST < MY + 4,

     or       NXEST < 8,

     or       NYEST < 8,

     or       LWRK < 4*(MX+MY)+11*(NXEST+NYEST)+NXEST*MY+
                     +max(MY,NXEST)+54

     or       LIWRK < 3 + MX + MY + NXEST + NYEST.

IFAIL= 2
     The values of X(q), for q = 1,2,...,MX, are not in strictly
     increasing order.

IFAIL= 3
     The values of Y(r), for r = 1,2,...,MY, are not in strictly
     increasing order.
```

IFAIL= 4
The number of knots required is greater than allowed by NXEST and NYEST. Try increasing NXEST and/or NYEST and, if necessary, supplying larger arrays for the parameters LAMDA, MU, C, WRK and IWRK. However, if NXEST and NYEST are already large, say NXEST > MX/2 and NYEST > MY/2, then this error exit may indicate that S is too small.

IFAIL= 5
The iterative process used to compute the coefficients of the approximating spline has failed to converge. This error exit may occur if S has been set very small. If the error persists with increased S, consult NAG.

If IFAIL = 4 or 5, a spline approximation is returned, but it fails to satisfy the fitting criterion (see (2) and (3) in Section 3) -- perhaps by only a small amount, however.

7. Accuracy

On successful exit, the approximation returned is such that its sum of squared residuals FP is equal to the smoothing factor S, up to a specified relative tolerance of 0.001 - except that if n =8 and n =8, FP may be significantly less than S: in this case
x y
the computed spline is simply the least-squares bicubic polynomial approximation of degree 3, i.e., a spline with no interior knots.

8. Further Comments

8.1. Timing

The time taken for a call of E02DCF depends on the complexity of the shape of the data, the value of the smoothing factor S, and the number of data points. If E02DCF is to be called for different values of S, much time can be saved by setting START =

8.2. Weighting of Data Points

E02DCF does not allow individual weighting of the data values. If these were determined to widely differing accuracies, it may be better to use E02DDF. The computation time would be very much longer, however.

8.3. Choice of S

If the standard deviation of f is the same for all q and r
q,r
(the case for which this routine is designed - see Section 8.2.)
and known to be equal, at least approximately, to (sigma), say,
then following Reinsch [5] and choosing the smoothing factor S in
2
the range (sigma) (m+-\/2m), where m=m m , is likely to give a
x y
good start in the search for a satisfactory value. If the
standard deviations vary, the sum of their squares over all the
data points could be used. Otherwise experimenting with different
values of S will be required from the start, taking account of
the remarks in Section 3.

In that case, in view of computation time and memory
requirements, it is recommended to start with a very large value
for S and so determine the least-squares bicubic polynomial; the
value returned for FP, call it FP , gives an upper bound for S.
0
Then progressively decrease the value of S to obtain closer fits
- say by a factor of 10 in the beginning, i.e., S=FP /10,
0
S=FP /100, and so on, and more carefully as the approximation
0
shows more details.

The number of knots of the spline returned, and their location,
generally depend on the value of S and on the behaviour of the
function underlying the data. However, if E02DCF is called with
START = 'W', the knots returned may also depend on the smoothing
factors of the previous calls. Therefore if, after a number of
trials with different values of S and START = 'W', a fit can
finally be accepted as satisfactory, it may be worthwhile to call
E02DCF once more with the selected value for S but now using
START = 'C'. Often, E02DCF then returns an approximation with the
same quality of fit but with fewer knots, which is therefore
better if data reduction is also important.

8.4. Choice of NXEST and NYEST

The number of knots may also depend on the upper bounds NXEST and
NYEST. Indeed, if at a certain stage in E02DCF the number of
knots in one direction (say n) has reached the value of its
x

upper bound (NXEST), then from that moment on all subsequent knots are added in the other (y) direction. Therefore the user has the option of limiting the number of knots the routine locates in any direction. For example, by setting NXEST = 8 (the lowest allowable value for NXEST), the user can indicate that he wants an approximation which is a simple cubic polynomial in the variable x.

8.5. Outline of Method Used

If S=0, the requisite number of knots is known in advance, i.e., n =m +4 and n =m +4; the interior knots are located immediately
 x x   y y
as (lambda) = x   and (mu) = y   , for i=5,6,...,n -4 and
           i   i-2   j   j-2   x
j=5,6,...,n -4. The corresponding least-squares spline is then an
           y
interpolating spline and therefore a solution of the problem.

If S>0, suitable knot sets are built up in stages (starting with no interior knots in the case of a cold start but with the knot set found in a previous call if a warm start is chosen). At each stage, a bicubic spline is fitted to the data by least-squares, and (theta), the sum of squares of residuals, is computed. If (theta)>S, new knots are added to one knot set or the other so as to reduce (theta) at the next stage. The new knots are located in intervals where the fit is particularly poor, their number depending on the value of S and on the progress made so far in reducing (theta). Sooner or later, we find that (theta)<=S and at that point the knot sets are accepted. The routine then goes on to compute the (unique) spline which has these knot sets and which satisfies the full fitting criterion specified by (2) and (3). The theoretical solution has (theta)=S. The routine computes the spline by an iterative scheme which is ended when (theta)=S within a relative tolerance of 0.001. The main part of each iteration consists of a linear least-squares computation of special form, done in a similarly stable and efficient manner as in E02BAF for least-squares curve fitting.

An exception occurs when the routine finds at the start that, even with no interior knots (n =n =8), the least-squares spline
                                x y
already has its sum of residuals <=S. In this case, since this spline (which is simply a bicubic polynomial) also has an optimal value for the smoothness measure (eta), namely zero, it is returned at once as the (trivial) solution. It will usually mean

that S has been chosen too large.

For further details of the algorithm and its use see Dierckx [2].

8.6. Evaluation of Computed Spline

The values of the computed spline at the points (TX(r),TY(r)), for r = 1,2,...,N, may be obtained in the double precision array FF, of length at least N, by the following code:

```
      IFAIL = 0
      CALL E02DEF(N,NX,NY,TX,TY,LAMDA,MU,C,FF,WRK,IWRK,IFAIL)
```

where NX, NY, LAMDA, MU and C are the output parameters of E02DCF , WRK is a double precision workspace array of length at least NY-4, and IWRK is an integer workspace array of length at least NY-4.

To evaluate the computed spline on a KX by KY rectangular grid of points in the x-y plane, which is defined by the x co-ordinates stored in TX(q), for q=1,2,...,KX, and the y co-ordinates stored in TY(r), for r=1,2,...,KY, returning the results in the double precision array FG which is of length at least KX*KY, the following call may be used:

```
      IFAIL = 0
      CALL E02DFF(KX,KY,NX,NY,TX,TY,LAMDA,MU,C,FG,WRK,LWRK,
     *            IWRK,LIWRK,IFAIL)
```

where NX, NY, LAMDA, MU and C are the output parameters of E02DCF , WRK is a double precision workspace array of length at least LWRK = min(NWRK1,NWRK2), NWRK1 = KX*4+NX, NWRK2 = KY*4+NY, and IWRK is an integer workspace array of length at least LIWRK = KY + NY - 4 if NWRK1 >= NWRK2, or KX + NX - 4 otherwise. The result of the spline evaluated at grid point (q,r) is returned in element (KY*(q-1)+r) of the array FG.

9. Example

This example program reads in values of MX, MY, x_q , for q = 1,2,. ordinates $f_{q,r}$ defined at the grid points (x_q, y_r). It then calls E02DCF to compute a bicubic spline approximation for one specified value of S, and prints the values of the computed knots

and B-spline coefficients. Finally it evaluates the spline at a small sample of points on a rectangular grid.

The example program is not reproduced here. The source code for all example programs is distributed with the NAG Foundation Library software and should be available on-line.

%%%

E02 -- Curve and Surface Fitting E02DDF
E02DDF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for your implementation to check implementation-dependent details. The symbol (*) after a NAG routine name denotes a routine that is not included in the Foundation Library.

1. Purpose

E02DDF computes a bicubic spline approximation to a set of scattered data. The knots of the spline are located automatically, but a single parameter must be specified to control the trade-off between closeness of fit and smoothness of fit.

2. Specification

```
      SUBROUTINE E02DDF (START, M, X, Y, F, W, S, NXEST, NYEST,
     1                   NX, LAMDA, NY, MU, C, FP, RANK, WRK,
     2                   LWRK, IWRK, LIWRK, IFAIL)
      INTEGER           M, NXEST, NYEST, NX, NY, RANK, LWRK, IWRK
     1                  (LIWRK), LIWRK, IFAIL
      DOUBLE PRECISION  X(M), Y(M), F(M), W(M), S, LAMDA(NXEST),
     1                  MU(NYEST), C((NXEST-4)*(NYEST-4)), FP, WRK
     2                  (LWRK)
      CHARACTER*1       START
```

3. Description

This routine determines a smooth bicubic spline approximation s(x,y) to the set of data points (x ,y ,f) with weights w , for
r r r r
r=1,2,...,m.

The approximation domain is considered to be the rectangle [x ,x]*[y ,y], where x (y) and x (y) denote

  min max   min max   min   min   max   max
the lowest and highest data values of x (y).

The spline is given in the B-spline representation

$$s(x,y)= \sum_{i=1}^{n_x-4}\sum_{j=1}^{n_y-4} c_{ij} M_i(x) N_j(y), \qquad (1)$$

where $M_i(x)$ and $N_j(y)$ denote normalised cubic B-splines, the former defined on the knots $(\lambda)_i$ to $(\lambda)_{i+4}$ and the latter on the knots $(\mu)_j$ to $(\mu)_{j+4}$. For further details, see Hayes and Halliday [4] for bicubic splines and de Boor [1] for normalised B-splines.

The total numbers n_x and n_y of these knots and their values $(\lambda)_1,\ldots,(\lambda)_{n_x}$ and $(\mu)_1,\ldots,(\mu)_{n_y}$ are chosen automatically by the routine. The knots $(\lambda)_5,\ldots,(\lambda)_{n_x-4}$ and $(\mu)_5,\ldots,(\mu)_{n_y-4}$ are the interior knots; they divide the approximation domain $[x_{min},x_{max}]*[y_{min},y_{max}]$ into $(n_x-7)*(n_y-7)$ subpanels $[(\lambda)_i,(\lambda)_{i+1}]*[(\mu)_j,(\mu)_{j+1}]$, for $i=4,5,\ldots,n_x-4$; $j=4,5,\ldots,n_y-4$. Then, much as in the curve case (see E02BEF), the coefficients c_{ij} are determined as the solution of the following constrained minimization problem:

minimize

$$(\eta), \qquad (2)$$

subject to the constraint

$$(\theta) = \sum_{r=1}^{m} (\epsilon)_r^2 \leq S \qquad (3)$$

where: (η) is a measure of the (lack of) smoothness of s(x,y). Its value depends on the discontinuity jumps in s(x,y) across the boundaries of the subpanels. It is zero only when there are no discontinuities and is positive otherwise, increasing with the size of the jumps (see Dierckx [2] for details).

$(\epsilon)_r$ denotes the weighted residual $w_r (f_r - s(x_r, y_r))$,

and S is a non-negative number to be specified by the user.

By means of the parameter S, 'the smoothing factor', the user will then control the balance between smoothness and closeness of fit, as measured by the sum of squares of residuals in (3). If S is too large, the spline will be too smooth and signal will be lost (underfit); if S is too small, the spline will pick up too much noise (overfit). In the extreme cases the method would return an interpolating spline ($(\theta)=0$) if S were set to zero, and returns the least-squares bicubic polynomial ($(\eta)=0$) if S is set very large. Experimenting with S-values between these two extremes should result in a good compromise. (See Section 8.2 for advice on choice of S.) Note however, that this routine, unlike E02BEF and E02DCF, does not allow S to be set exactly to zero: to compute an interpolant to scattered data, E01SAF or E01SEF should be used.

The method employed is outlined in Section 8.5 and fully described in Dierckx [2] and [3]. It involves an adaptive strategy for locating the knots of the bicubic spline (depending on the function underlying the data and on the value of S), and an iterative method for solving the constrained minimization problem once the knots have been determined.

Values of the computed spline can subsequently be computed by calling E02DEF or E02DFF as described in Section 8.6.

4. References

[1] De Boor C (1972) On Calculating with B-splines. J. Approx. Theory. 6 50--62.

[2] Dierckx P (1981) An Algorithm for Surface Fitting with Spline Functions. IMA J. Num. Anal. 1 267--283.

[3] Dierckx P (1981) An Improved Algorithm for Curve Fitting with Spline Functions. Report TW54. Department of Computer Science, Katholieke Universiteit Leuven.

[4] Hayes J G and Halliday J (1974) The Least-squares Fitting of Cubic Spline Surfaces to General Data Sets. J. Inst. Math. Appl. 14 89--103.

[5] Peters G and Wilkinson J H (1970) The Least-squares Problem and Pseudo-inverses. Comput. J. 13 309--316.

[6] Reinsch C H (1967) Smoothing by Spline Functions. Num. Math. 10 177--183.

5. Parameters

1: START -- CHARACTER*1 Input
On entry: START must be set to 'C' or 'W'.

If START = 'C' (Cold start), the routine will build up the knot set starting with no interior knots. No values need be assigned to the parameters NX, NY, LAMDA, MU or WRK.

If START = 'W' (Warm start), the routine will restart the knot-placing strategy using the knots found in a previous call of the routine. In this case, the parameters NX, NY, LAMDA, MU and WRK must be unchanged from that previous call. This warm start can save much time in searching for a satisfactory value of S. Constraint: START = 'C' or 'W'.

2: M -- INTEGER Input
On entry: m, the number of data points.

The number of data points with non-zero weight (see W below) must be at least 16.

3: X(M) -- DOUBLE PRECISION array Input

4: Y(M) -- DOUBLE PRECISION array Input

5: F(M) -- DOUBLE PRECISION array Input
On entry: X(r), Y(r), F(r) must be set to the co-ordinates of (x_r, y_r, f_r), the rth data point, for r=1,2,...,m. The order of the data points is immaterial.

6: W(M) -- DOUBLE PRECISION array Input
On entry: W(r) must be set to w_r, the rth value in the set of weights, for r=1,2,...,m. Zero weights are permitted and the corresponding points are ignored, except when determining x_{min}, x_{max}, y_{min} and y_{max} (see Section 8.4). For advice on the choice of weights, see Section 2.1.2 of the Chapter Introduction. Constraint: the number of data points with non-zero weight must be at least 16.

7: S -- DOUBLE PRECISION Input
On entry: the smoothing factor, S.

For advice on the choice of S, see Section 3 and Section 8.2 . Constraint: S > 0.0.

8: NXEST -- INTEGER Input

9: NYEST -- INTEGER Input
On entry: an upper bound for the number of knots n_x and n_y required in the x- and y-directions respectively.

In most practical situations, NXEST = NYEST = $4+\sqrt{m/2}$ is sufficient. See also Section 8.3. Constraint: NXEST >= 8 and NYEST >= 8.

10: NX -- INTEGER Input/Output
On entry: if the warm start option is used, the value of NX must be left unchanged from the previous call. On exit: the total number of knots, n_x, of the computed spline with respect to the x variable.

11: LAMDA(NXEST) -- DOUBLE PRECISION array Input/Output
On entry: if the warm start option is used, the values LAMDA (1), LAMDA(2),...,LAMDA(NX) must be left unchanged from the previous call. On exit: LAMDA contains the complete set of

knots (lambda associated with the x variable, i.e., the
                i
interior knots LAMDA(5), LAMDA(6),...,LAMDA(NX-4) as well as
the additional knots LAMDA(1) = LAMDA(2) = LAMDA(3) = LAMDA
(4) = x   and LAMDA(NX-3) = LAMDA(NX-2) = LAMDA(NX-1) =
       min
LAMDA(NX) = x   needed for the B-spline representation
             max
(where x   and x   are as described in Section 3).
        min   max

12: NY -- INTEGER Input/Output
On entry: if the warm start option is used, the value of NY must be left unchanged from the previous call. On exit: the total number of knots, n , of the computed spline with
                                 y
respect to the y variable.

13: MU(NYEST) -- DOUBLE PRECISION array Input/Output
On entry: if the warm start option is used, the values MU(1) MU(2),...,MU(NY) must be left unchanged from the previous call. On exit: MU contains the complete set of knots (mu)
                                                                  i
associated with the y variable, i.e., the interior knots MU (5), MU(6),...,MU(NY-4) as well as the additional knots MU (1) = MU(2) = MU(3) = MU(4) = y   and MU(NY-3) = MU(NY-2) =
                                    min
MU(NY-1) = MU(NY) = y   needed for the B-spline
                     max
representation (where y   and y   are as described in
                       min   max
Section 3).

14: C((NXEST-4)*(NYEST-4)) -- DOUBLE PRECISION array Output
On exit: the coefficients of the spline approximation. C((n -4)*(i-1)+j) is the coefficient c   defined in Section 3.
  y   ij

15: FP -- DOUBLE PRECISION Output
On exit: the weighted sum of squared residuals, (theta), of the computed spline approximation. FP should equal S within a relative tolerance of 0.001 unless NX = NY = 8, when the spline has no interior knots and so is simply a bicubic polynomial. For knots to be inserted, S must be set to a value below the value of FP produced in this case.

16: RANK -- INTEGER Output
On exit: RANK gives the rank of the system of equations used to compute the final spline (as determined by a suitable machine-dependent threshold). When RANK = (NX-4)*(NY-4), the solution is unique; otherwise the system is rank-deficient and the minimum-norm solution is computed. The latter case may be caused by too small a value of S.

17: WRK(LWRK) -- DOUBLE PRECISION array Workspace
On entry: if the warm start option is used, the value of WRK (1) must be left unchanged from the previous call.

This array is used as workspace.

18: LWRK -- INTEGER Input
On entry:
the dimension of the array WRK as declared in the (sub)program from which E02DDF is called.
Constraint: LWRK >= (7*u*v+25*w)*(w+1)+2*(u+v+4*M)+23*w+56,

where

u=NXEST-4, v=NYEST-4, and w=max(u,v).

For some problems, the routine may need to compute the minimal least-squares solution of a rank-deficient system of linear equations (see Section 3). The amount of workspace required to solve such problems will be larger than specified by the value given above, which must be increased by an amount, LWRK2 say. An upper bound for LWRK2 is given by 4*u*v*w+2*u*v+4*w, where u, v and w are as above. However, if there are enough data points, scattered uniformly over the approximation domain, and if the smoothing factor S is not too small, there is a good chance that this extra workspace is not needed. A lot of memory might therefore be saved by assuming LWRK2 = 0.

19: IWRK(LIWRK) -- INTEGER array Workspace

20: LIWRK -- INTEGER Input
On entry:
the dimension of the array IWRK as declared in the (sub)program from which E02DDF is called.
Constraint: LIWRK>=M+2*(NXEST-7)*(NYEST-7).

```
21:  IFAIL -- INTEGER                                    Input/Output
     On entry: IFAIL must be set to 0, -1 or 1. For users not
     familiar with this parameter (described in the Essential
     Introduction) the recommended value is 0.

     On exit: IFAIL = 0 unless the routine detects an error (see
     Section 6).

6. Error Indicators and Warnings

Errors detected by the routine:

If on entry IFAIL = 0 or -1, explanatory error messages are
output on the current error message unit (as defined by X04AAF).

IFAIL= 1
     On entry START /= 'C' or 'W',

     or       the number of data points with non-zero weight <
              16,

     or       S <= 0.0,

     or       NXEST < 8,

     or       NYEST < 8,

     or       LWRK < (7*u*v+25*w)*(w+1)+2*(u+v+4*M)+23*w+56,
              where u = NXEST - 4, v = NYEST - 4 and w=max(u,v),

     or       LIWRK <M+2*(NXEST-7)*(NYEST-7).

IFAIL= 2
     On entry either all the X(r), for r = 1,2,...,M, are equal,
     or all the Y(r), for r = 1,2,...,M, are equal.

IFAIL= 3
     The number of knots required is greater than allowed by
     NXEST and NYEST. Try increasing NXEST and/or NYEST and, if
     necessary, supplying larger arrays for the parameters LAMDA,
     MU, C, WRK and IWRK. However, if NXEST and NYEST are already


     large, say NXEST, NYEST > 4 + \/M/2, then this error exit
     may indicate that S is too small.
```

```
IFAIL= 4
     No more knots can be added because the number of B-spline
     coefficients (NX-4)*(NY-4) already exceeds the number of
     data points M. This error exit may occur if either of S or M
     is too small.

IFAIL= 5
     No more knots can be added because the additional knot would
     (quasi) coincide with an old one. This error exit may occur
     if too large a weight has been given to an inaccurate data
     point, or if S is too small.

IFAIL= 6
     The iterative process used to compute the coefficients of
     the approximating spline has failed to converge. This error
     exit may occur if S has been set very small. If the error
     persists with increased S, consult NAG.

IFAIL= 7
     LWRK is too small; the routine needs to compute the minimal
     least-squares solution of a rank-deficient system of linear
     equations, but there is not enough workspace. There is no
     approximation returned but, having saved the information
     contained in NX, LAMDA, NY, MU and WRK, and having adjusted
     the value of LWRK and the dimension of array WRK
     accordingly, the user can continue at the point the program
     was left by calling E02DDF with START = 'W'. Note that the
     requested value for LWRK is only large enough for the
     current phase of the algorithm. If the routine is restarted
     with LWRK set to the minimum value requested, a larger
     request may be made at a later stage of the computation. See
     Section 5 for the upper bound on LWRK. On soft failure, the
     minimum requested value for LWRK is returned in IWRK(1) and
     the safe value for LWRK is returned in IWRK(2).

If IFAIL = 3,4,5 or 6, a spline approximation is returned, but it
fails to satisfy the fitting criterion (see (2) and (3) in
Section 3 -- perhaps only by a small amount, however.

7. Accuracy

On successful exit, the approximation returned is such that its
weighted sum of squared residuals FP is equal to the smoothing
factor S, up to a specified relative tolerance of 0.001 - except
that if n =8 and n =8, FP may be significantly less than S: in
         x        y
```

this case the computed spline is simply the least-squares bicubic polynomial approximation of degree 3, i.e., a spline with no interior knots.

8. Further Comments

8.1. Timing

The time taken for a call of E02DDF depends on the complexity of the shape of the data, the value of the smoothing factor S, and the number of data points. If E02DDF is to be called for different values of S, much time can be saved by setting START = It should be noted that choosing S very small considerably increases computation time.

8.2. Choice of S

If the weights have been correctly chosen (see Section 2.1.2 of the Chapter Introduction), the standard deviation of w f would
r r
be the same for all r, equal to (sigma), say. In this case,
2
choosing the smoothing factor S in the range (sigma) (m+-\/2m), as suggested by Reinsch [6], is likely to give a good start in the search for a satisfactory value. Otherwise, experimenting with different values of S will be required from the start.

In that case, in view of computation time and memory requirements, it is recommended to start with a very large value for S and so determine the least-squares bicubic polynomial; the value returned for FP, call it FP , gives an upper bound for S.
0
Then progressively decrease the value of S to obtain closer fits - say by a factor of 10 in the beginning, i.e., S=FP /10,
0
S=FP /100, and so on, and more carefully as the approximation
0
shows more details.

To choose S very small is strongly discouraged. This considerably increases computation time and memory requirements. It may also cause rank-deficiency (as indicated by the parameter RANK) and endanger numerical stability.

The number of knots of the spline returned, and their location, generally depend on the value of S and on the behaviour of the

function underlying the data. However, if E02DDF is called with START = 'W', the knots returned may also depend on the smoothing factors of the previous calls. Therefore if, after a number of trials with different values of S and START = 'W', a fit can finally be accepted as satisfactory, it may be worthwhile to call E02DDF once more with the selected value for S but now using START = 'C'. Often, E02DDF then returns an approximation with the same quality of fit but with fewer knots, which is therefore better if data reduction is also important.

8.3. Choice of NXEST and NYEST

The number of knots may also depend on the upper bounds NXEST and NYEST. Indeed, if at a certain stage in E02DDF the number of knots in one direction (say n_x) has reached the value of its upper bound (NXEST), then from that moment on all subsequent knots are added in the other (y) direction. This may indicate that the value of NXEST is too small. On the other hand, it gives the user the option of limiting the number of knots the routine locates in any direction. For example, by setting NXEST = 8 (the lowest allowable value for NXEST), the user can indicate that he wants an approximation which is a simple cubic polynomial in the variable x.

8.4. Restriction of the approximation domain

The fit obtained is not defined outside the rectangle $[(\lambda)_4, (\lambda)_{n_x-3}] * [(\mu)_4, (\mu)_{n_y-3}]$. The reason for taking the extreme data values of x and y for these four knots is that, as is usual in data fitting, the fit cannot be expected to give satisfactory values outside the data region. If, nevertheless, the user requires values over a larger rectangle, this can be achieved by augmenting the data with two artificial data points (a,c,0) and (b,d,0) with zero weight, where [a,b]*[c,d] denotes the enlarged rectangle.

8.5. Outline of method used

First suitable knot sets are built up in stages (starting with no interior knots in the case of a cold start but with the knot set found in a previous call if a warm start is chosen). At each stage, a bicubic spline is fitted to the data by least-squares and (theta), the sum of squares of residuals, is computed. If

(theta)>S, a new knot is added to one knot set or the other so as to reduce (theta) at the next stage. The new knot is located in an interval where the fit is particularly poor. Sooner or later, we find that (theta)<=S and at that point the knot sets are accepted. The routine then goes on to compute a spline which has these knot sets and which satisfies the full fitting criterion specified by (2) and (3). The theoretical solution has (theta)=S. The routine computes the spline by an iterative scheme which is ended when (theta)=S within a relative tolerance of 0.001. The main part of each iteration consists of a linear least-squares computation of special form, done in a similarly stable and efficient manner as in E02DAF. As there also, the minimal least-squares solution is computed wherever the linear system is found to be rank-deficient.

An exception occurs when the routine finds at the start that, even with no interior knots (N = 8), the least-squares spline already has its sum of squares of residuals <=S. In this case, since this spline (which is simply a bicubic polynomial) also has an optimal value for the smoothness measure (eta), namely zero, it is returned at once as the (trivial) solution. It will usually mean that S has been chosen too large.

For further details of the algorithm and its use see Dierckx [2].

8.6. Evaluation of computed spline

The values of the computed spline at the points (TX(r),TY(r)), for r = 1,2,...,N, may be obtained in the double precision array FF, of length at least N, by the following code:

```
      IFAIL = 0
      CALL E02DEF(N,NX,NY,TX,TY,LAMDA,MU,C,FF,WRK,IWRK,IFAIL)
```

where NX, NY, LAMDA, MU and C are the output parameters of E02DDF , WRK is a double precision workspace array of length at least NY-4, and IWRK is an integer workspace array of length at least NY-4.

To evaluate the computed spline on a KX by KY rectangular grid of points in the x-y plane, which is defined by the x co-ordinates stored in TX(q), for q=1,2,...,KX, and the y co-ordinates stored in TY(r), for r=1,2,...,KY, returning the results in the double precision array FG which is of length at least KX*KY, the

following call may be used:

```
      IFAIL = 0
      CALL E02DFF(KX,KY,NX,NY,TX,TY,LAMDA,MU,C,FG,WRK,LWRK,
     *            IWRK,LIWRK,IFAIL)
```

where NX, NY, LAMDA, MU and C are the output parameters of E02DDF , WRK is a double precision workspace array of length at least LWRK = min(NWRK1,NWRK2), NWRK1 = KX*4+NX, NWRK2 = KY*4+NY, and IWRK is an integer workspace array of length at least LIWRK = KY + NY - 4 if NWRK1 >= NWRK2, or KX + NX - 4 otherwise. The result of the spline evaluated at grid point (q,r) is returned in element (KY*(q-1)+r) of the array FG.

9. Example

This example program reads in a value of M, followed by a set of M data points (x ,y ,f) and their weights w . It then calls
r r r r
E02DDF to compute a bicubic spline approximation for one specified value of S, and prints the values of the computed knots and B-spline coefficients. Finally it evaluates the spline at a small sample of points on a rectangular grid.

The example program is not reproduced here. The source code for all example programs is distributed with the NAG Foundation Library software and should be available on-line.

%%%

E02 -- Curve and Surface Fitting E02DEF
E02DEF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for your implementation to check implementation-dependent details. The symbol (*) after a NAG routine name denotes a routine that is not included in the Foundation Library.

1. Purpose

E02DEF calculates values of a bicubic spline from its B-spline representation.

2. Specification

```
      SUBROUTINE E02DEF (M, PX, PY, X, Y, LAMDA, MU, C, FF, WRK,
     1                   IWRK, IFAIL)
      INTEGER          M, PX, PY, IWRK(PY-4), IFAIL
      DOUBLE PRECISION X(M), Y(M), LAMDA(PX), MU(PY), C((PX-4)*
     1                 (PY-4)), FF(M), WRK(PY-4)
```

3. Description

This routine calculates values of the bicubic spline s(x,y) at prescribed points (x_r, y_r), for r=1,2,...,m, from its augmented knot sets $\{(\lambda)\}$ and $\{(\mu)\}$ and from the coefficients c_{ij}, for i=1,2,...,PX-4; j=1,2,...,PY-4, in its B-spline representation

$$s(x,y) = \sum_{ij} c_{ij} M_i(x) N_j(y).$$

Here $M_i(x)$ and $N_j(y)$ denote normalised cubic B-splines, the former defined on the knots $(\lambda)_i$ to $(\lambda)_{i+4}$ and the latter on the knots $(\mu)_j$ to $(\mu)_{j+4}$.

This routine may be used to calculate values of a bicubic spline given in the form produced by E01DAF, E02DAF, E02DCF and E02DDF. It is derived from the routine B2VRE in Anthony et al [1].

4. References

[1] Anthony G T, Cox M G and Hayes J G (1982) DASL - Data Approximation Subroutine Library. National Physical Laboratory.

[2] Cox M G (1978) The Numerical Evaluation of a Spline from its B-spline Representation. J. Inst. Math. Appl. 21 135--143.

5. Parameters

1: M -- INTEGER Input
On entry: m, the number of points at which values of the

spline are required. Constraint: M >= 1.

2: PX -- INTEGER Input

3: PY -- INTEGER Input
On entry: PX and PY must specify the total number of knots associated with the variables x and y respectively. They are such that PX-8 and PY-8 are the corresponding numbers of interior knots. Constraint: PX >= 8 and PY >= 8.

4: X(M) -- DOUBLE PRECISION array Input

5: Y(M) -- DOUBLE PRECISION array Input
On entry: X and Y must contain x_r and y_r, for r=1,2,...,m, respectively. These are the co-ordinates of the points at which values of the spline are required. The order of the points is immaterial. Constraint: X and Y must satisfy

LAMDA(4) <= X(r) <= LAMDA(PX-3)

and

MU(4) <= Y(r) <= MU(PY-3), for r=1,2,...,m.

The spline representation is not valid outside these intervals.

6: LAMDA(PX) -- DOUBLE PRECISION array Input

7: MU(PY) -- DOUBLE PRECISION array Input
On entry: LAMDA and MU must contain the complete sets of knots {(lambda)} and {(mu)} associated with the x and y variables respectively. Constraint: the knots in each set must be in non-decreasing order, with LAMDA(PX-3) > LAMDA(4) and MU(PY-3) > MU(4).

8: C((PX-4)*(PY-4)) -- DOUBLE PRECISION array Input
On entry: C((PY-4)*(i-1)+j) must contain the coefficient c_{ij} described in Section 3, for i=1,2,...,PX-4; j=1,2,...,PY-4.

9: FF(M) -- DOUBLE PRECISION array Output
On exit: FF(r) contains the value of the spline at the point (x_r, y_r), for r=1,2,...,m.

r r

10: WRK(PY-4) -- DOUBLE PRECISION array Workspace

11: IWRK(PY-4) -- INTEGER array Workspace

12: IFAIL -- INTEGER Input/Output
On entry: IFAIL must be set to 0, -1 or 1. For users not familiar with this parameter (described in the Essential Introduction) the recommended value is 0.

On exit: IFAIL = 0 unless the routine detects an error (see Section 6).

6. Error Indicators and Warnings

Errors detected by the routine:

If on entry IFAIL = 0 or -1, explanatory error messages are output on the current error message unit (as defined by X04AAF).

IFAIL= 1
On entry M < 1,

or PY < 8,

or PX < 8.

IFAIL= 2
On entry the knots in array LAMDA, or those in array MU, are not in non-decreasing order, or LAMDA(PX-3) <= LAMDA(4), or MU(PY-3) <= MU(4).

IFAIL= 3
On entry at least one of the prescribed points (x ,y) lies
r r
outside the rectangle defined by LAMDA(4), LAMDA(PX-3) and MU(4), MU(PY-3).

7. Accuracy

The method used to evaluate the B-splines is numerically stable, in the sense that each computed value of s(x ,y) can be regarded
r r
as the value that would have been obtained in exact arithmetic from slightly perturbed B-spline coefficients. See Cox [2] for

```
details.

8. Further Comments

Computation time is approximately proportional to the number of
points, m, at which the evaluation is required.

9. Example

This program reads in knot sets LAMDA(1),..., LAMDA(PX) and MU(1)
,..., MU(PY), and a set of bicubic spline coefficients c  .
                                                       ij
Following these are a value for m and the co-ordinates (x ,y ),
                                                         r  r
for r=1,2,...,m, at which the spline is to be evaluated.

The example program is not reproduced here. The source code for
all example programs is distributed with the NAG Foundation
Library software and should be available on-line.
```

%%%

```
E02 -- Curve and Surface Fitting                              E02DFF
        E02DFF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for
your implementation to check implementation-dependent details.
The symbol (*) after a NAG routine name denotes a routine that is
not included in the Foundation Library.

1. Purpose

E02DFF calculates values of a bicubic spline from its B-spline
representation. The spline is evaluated at all points on a
rectangular grid.

2. Specification

      SUBROUTINE E02DFF (MX, MY, PX, PY, X, Y, LAMDA, MU, C, FF,
     1                   WRK, LWRK, IWRK, LIWRK, IFAIL)
      INTEGER           MX, MY, PX, PY, LWRK, IWRK(LIWRK), LIWRK,
     1                  IFAIL
      DOUBLE PRECISION X(MX), Y(MY), LAMDA(PX), MU(PY), C((PX-4)*
     1                 (PY-4)), FF(MX*MY), WRK(LWRK)

3. Description
```

This routine calculates values of the bicubic spline s(x,y) on a rectangular grid of points in the x-y plane, from its augmented knot sets {(lambda)} and {(mu)} and from the coefficients c_{ij}, for i=1,2,...,PX-4; j=1,2,...,PY-4, in its B-spline representation

$$s(x,y) = \sum_{ij} c_{ij} M_i(x) N_j(y).$$

Here $M_i(x)$ and $N_j(y)$ denote normalised cubic B-splines, the former defined on the knots $(\lambda)_i$ to $(\lambda)_{i+4}$ and the latter on the knots $(\mu)_j$ to $(\mu)_{j+4}$.

The points in the grid are defined by co-ordinates x_q, for q=1,2,...,m$_x$, along the x axis, and co-ordinates y_r, for r=1,2,...,m$_y$ along the y axis.

This routine may be used to calculate values of a bicubic spline given in the form produced by E01DAF, E02DAF, E02DCF and E02DDF. It is derived from the routine B2VRE in Anthony et al [1].

4. References

[1] Anthony G T, Cox M G and Hayes J G (1982) DASL - Data Approximation Subroutine Library. National Physical Laboratory.

[2] Cox M G (1978) The Numerical Evaluation of a Spline from its B-spline Representation. J. Inst. Math. Appl. 21 135--143.

5. Parameters

1: MX -- INTEGER Input

2: MY -- INTEGER Input
On entry: MX and MY must specify m$_x$ and m$_y$ respectively,

x y
the number of points along the x and y axis that define the rectangular grid. Constraint: MX >= 1 and MY >= 1.

3: PX -- INTEGER Input

4: PY -- INTEGER Input
On entry: PX and PY must specify the total number of knots associated with the variables x and y respectively. They are such that PX-8 and PY-8 are the corresponding numbers of interior knots. Constraint: PX >= 8 and PY >= 8.

5: X(MX) -- DOUBLE PRECISION array Input

6: Y(MY) -- DOUBLE PRECISION array Input
On entry: X and Y must contain x , for q=1,2,...,m , and y ,
q x r
for r=1,2,...,m , respectively. These are the x and y co-
y
ordinates that define the rectangular grid of points at which values of the spline are required. Constraint: X and Y must satisfy

LAMDA(4) <= X(q) < X(q+1) <= LAMDA(PX-3), for q=1,2,...,m -1
x
and

MU(4) <= Y(r) < Y(r+1) <= MU(PY-3), for r=1,2,...,m -1.
y

The spline representation is not valid outside these intervals.

7: LAMDA(PX) -- DOUBLE PRECISION array Input

8: MU(PY) -- DOUBLE PRECISION array Input
On entry: LAMDA and MU must contain the complete sets of knots {(lambda)} and {(mu)} associated with the x and y variables respectively. Constraint: the knots in each set must be in non-decreasing order, with LAMDA(PX-3) > LAMDA(4) and MU(PY-3) > MU(4).

9: C((PX-4)*(PY-4)) -- DOUBLE PRECISION array Input
On entry: C((PY-4)*(i-1)+j) must contain the coefficient
c described in Section 3, for i=1,2,...,PX-4;
ij

```
     j=1,2,...,PY-4.

10:  FF(MX*MY) -- DOUBLE PRECISION array                    Output
     On exit: FF(MY*(q-1)+r) contains the value of the spline at
     the point (x ,y ), for q=1,2,...,m ; r=1,2,...,m .
                 q  r                  x             y

11:  WRK(LWRK) -- DOUBLE PRECISION array                 Workspace

12:  LWRK -- INTEGER                                        Input
     On entry:
     the dimension of the array WRK as declared in the
     (sub)program from which E02DFF is called.
     Constraint: LWRK >= min(NWRK1,NWRK2), where NWRK1=4*MX+PX,
     NWRK2=4*MY+PY.

13:  IWRK(LIWRK) -- INTEGER array                        Workspace

14:  LIWRK -- INTEGER                                       Input
     On entry:
     the dimension of the array IWRK as declared in the
     (sub)program from which E02DFF is called.
     Constraint: LIWRK >= MY + PY - 4 if NWRK1 > NWRK2, or MX +
     PX - 4 otherwise, where NWRK1 and NWRK2 are as defined in
     the description of argument LWRK.

15:  IFAIL -- INTEGER                                  Input/Output
     On entry: IFAIL must be set to 0, -1 or 1. For users not
     familiar with this parameter (described in the Essential
     Introduction) the recommended value is 0.

     On exit: IFAIL = 0 unless the routine detects an error (see
     Section 6).

6. Error Indicators and Warnings

Errors detected by the routine:

If on entry IFAIL = 0 or -1, explanatory error messages are
output on the current error message unit (as defined by X04AAF).

IFAIL= 1
     On entry MX < 1,

     or       MY < 1,
```

```
     or       PY < 8,

     or       PX < 8.

IFAIL= 2
     On entry LWRK is too small,

     or       LIWRK is too small.

IFAIL= 3
     On entry the knots in array LAMDA, or those in array MU, are
     not in non-decreasing order, or LAMDA(PX-3) <= LAMDA(4), or
     MU(PY-3) <= MU(4).

IFAIL= 4
     On entry the restriction LAMDA(4) <= X(1) <... < X(MX) <=
     LAMDA(PX-3), or the restriction MU(4) <= Y(1) <... < Y(MY)
     <= MU(PY-3), is violated.

7. Accuracy

The method used to evaluate the B-splines is numerically stable,
in the sense that each computed value of s(x ,y ) can be regarded
                                           r  r
as the value that would have been obtained in exact arithmetic
from slightly perturbed B-spline coefficients. See Cox [2] for
details.

8. Further Comments

Computation time is approximately proportional to m m +4(m +m ).
                                                   x y    x  y

9. Example

This program reads in knot sets LAMDA(1),..., LAMDA(PX) and MU(1)
,..., MU(PY), and a set of bicubic spline coefficients c  .
                                                        ij
Following these are values for m  and the x co-ordinates x , for
                               x                          q
q=1,2,...,m , and values for m  and the y co-ordinates y , for
           x                 y                          r
r=1,2,...,m , defining the grid of points on which the spline is
           y
to be evaluated.
```

The example program is not reproduced here. The source code for all example programs is distributed with the NAG Foundation Library software and should be available on-line.

%%%

E02 -- Curve and Surface Fitting E02GAF
E02GAF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for your implementation to check implementation-dependent details. The symbol (*) after a NAG routine name denotes a routine that is not included in the Foundation Library.

1. Purpose

E02GAF calculates an l_1 solution to an over-determined system of linear equations.

2. Specification

```
 SUBROUTINE E02GAF (M, A, LA, B, NPLUS2, TOLER, X, RESID,
1                   IRANK, ITER, IWORK, IFAIL)
 INTEGER          M, LA, NPLUS2, IRANK, ITER, IWORK(M),
1                 IFAIL
 DOUBLE PRECISION A(LA,NPLUS2), B(M), TOLER, X(NPLUS2),
1                 RESID
```

3. Description

Given a matrix A with m rows and n columns (m>=n) and a vector b with m elements, the routine calculates an l_1 solution to the over-determined system of equations

$$Ax=b.$$

That is to say, it calculates a vector x, with n elements, which minimizes the l_1 -norm (the sum of the absolute values) of the residuals

$$r(x)= \sum_{} |r_{}|,$$

```
                                  --   i
                                  i=1
```

where the residuals r are given by
                     i

```
                        n
                        --
                r =b - >  a  x ,   i=1,2,...,m.
                 i  i  --  ij j
                        j=1
```

Here a   is the element in row i and column j of A, b is the ith
      ij   i
element of b and x the jth element of x. The matrix A need not
              j
be of full rank.

Typically in applications to data fitting, data consisting of m
points with co-ordinates (t ,y) are to be approximated in the l
                          i i   1
-norm by a linear combination of known functions (phi) (t),
                                                       j

```
   (alpha) (phi) (t)+(alpha) (phi) (t)+...+(alpha) (phi) (t).
         1     1           2     2               n     n
```

This is equivalent to fitting an l solution to the over-
                                  1
determined system of equations

```
            n
            --
            >  (phi) (t )(alpha) =y ,   i=1,2,...,m.
            --      j  i        j  i
            j=1
```

Thus if, for each value of i and j, the element a   of the matrix
                                                 ij
A in the previous paragraph is set equal to the value of
(phi) (t) and b is set equal to y , the solution vector x will
     j i   i   i
contain the required values of the (alpha) . Note that the
                                          j
independent variable t above can, instead, be a vector of several
independent variables (this includes the case where each (phi)

i
is a function of a different variable, or set of variables).

The algorithm is a modification of the simplex method of linear programming applied to the primal formulation of the l problem
1
(see Barrodale and Roberts [1] and [2]). The modification allows several neighbouring simplex vertices to be passed through in a single iteration, providing a substantial improvement in efficiency.

4. References

[1] Barrodale I and Roberts F D K (1973) An Improved Algorithm for Discrete \\ll Linear Approximation. SIAM J. Numer.
1
Anal. 10 839--848.

[2] Barrodale I and Roberts F D K (1974) Solution of an Overdetermined System of Equations in the \\ll -norm. Comm.
1
ACM. 17, 6 319--320.

5. Parameters

1: M -- INTEGER Input
On entry: the number of equations, m (the number of rows of the matrix A). Constraint: M >= n >= 1.

2: A(LA,NPLUS2) -- DOUBLE PRECISION array Input/Output
On entry: A(i,j) must contain a , the element in the ith
ij
row and jth column of the matrix A, for i=1,2,...,m and j=1,2,...,n. The remaining elements need not be set. On exit: A contains the last simplex tableau generated by the simplex method.

3: LA -- INTEGER Input
On entry:
the first dimension of the array A as declared in the (sub)program from which E02GAF is called.
Constraint: LA >= M + 2.

4: B(M) -- DOUBLE PRECISION array Input/Output
On entry: b , the ith element of the vector b, for
i

i=1,2,...,m. On exit: the ith residual r corresponding to
i
the solution vector x, for i=1,2,...,m.

5: NPLUS2 -- INTEGER Input
On entry: n+2, where n is the number of unknowns (the number of columns of the matrix A). Constraint: 3 <= NPLUS2 <= M + 2.

6: TOLER -- DOUBLE PRECISION Input
On entry: a non-negative value. In general TOLER specifies a threshold below which numbers are regarded as zero. The
2/3
recommended threshold value is (epsilon) where (epsilon) is the machine precision. The recommended value can be computed within the routine by setting TOLER to zero. If premature termination occurs a larger value for TOLER may result in a valid solution. Suggested value: 0.0.

7: X(NPLUS2) -- DOUBLE PRECISION array Output
On exit: X(j) contains the jth element of the solution vector x, for j=1,2,...,n. The elements X(n+1) and X(n+2) are unused.

8: RESID -- DOUBLE PRECISION Output
On exit: the sum of the absolute values of the residuals for the solution vector x.

9: IRANK -- INTEGER Output
On exit: the computed rank of the matrix A.

10: ITER -- INTEGER Output
On exit: the number of iterations taken by the simplex method.

11: IWORK(M) -- INTEGER array Workspace

12: IFAIL -- INTEGER Input/Output
On entry: IFAIL must be set to 0, -1 or 1. For users not familiar with this parameter (described in the Essential Introduction) the recommended value is 0.

On exit: IFAIL = 0 unless the routine detects an error (see Section 6).

6. Error Indicators and Warnings

```
Errors detected by the routine:

IFAIL= 1
     An optimal solution has been obtained but this may not be
     unique.

IFAIL= 2
     The calculations have terminated prematurely due to rounding
     errors. Experiment with larger values of TOLER or try
     scaling the columns of the matrix (see Section 8).

IFAIL= 3
     On entry NPLUS2 < 3,

     or       NPLUS2 > M + 2,

     or       LA < M + 2.

7. Accuracy

Experience suggests that the computational accuracy of the
solution x is comparable with the accuracy that could be obtained
by applying Gaussian elimination with partial pivoting to the n
equations satisfied by this algorithm (i.e., those equations with
zero residuals). The accuracy therefore varies with the
conditioning of the problem, but has been found generally very
satisfactory in practice.

8. Further Comments

The effects of m and n on the time and on the number of
iterations in the Simplex Method vary from problem to problem,
but typically the number of iterations is a small multiple of n
and the total time taken by the routine is approximately
                  2
proportional to mn .

It is recommended that, before the routine is entered, the
columns of the matrix A are scaled so that the largest element in
each column is of the order of unity. This should improve the
conditioning of the matrix, and also enable the parameter TOLER
to perform its correct function. The solution x obtained will
then, of course, relate to the scaled form of the matrix. Thus if
the scaling is such that, for each j=1,2,...,n, the elements of
the jth column are multiplied by the constant k , the element x
```

of the solution vector x must be multiplied by k_j if it is desired to recover the solution corresponding to the original matrix A.

9. Example

Suppose we wish to approximate a set of data by a curve of the form

$$y = Ke^{t} + Le^{-t} + M$$

where K, L and M are unknown. Given values y_i at 5 points t_i we may form the over-determined set of equations for K, L and M

$$e^{x_i}K + e^{-x_i}L + M = y_i, \quad i=1,2,\ldots,5.$$

E02GAF is used to solve these in the l_1 sense.

The example program is not reproduced here. The source code for all example programs is distributed with the NAG Foundation Library software and should be available on-line.

%%%

E02 -- Curve and Surface Fitting E02ZAF
E02ZAF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for your implementation to check implementation-dependent details. The symbol (*) after a NAG routine name denotes a routine that is not included in the Foundation Library.

1. Purpose

E02ZAF sorts two-dimensional data into rectangular panels.

2. Specification

```
      SUBROUTINE E02ZAF (PX, PY, LAMDA, MU, M, X, Y, POINT,
     1                   NPOINT, ADRES, NADRES, IFAIL)
      INTEGER          PX, PY, M, POINT(NPOINT), NPOINT, ADRES
     1                 (NADRES), NADRES, IFAIL
      DOUBLE PRECISION LAMDA(PX), MU(PY), X(M), Y(M)
```

3. Description

A set of m data points with rectangular Cartesian co-ordinates x_r, y_r are sorted into panels defined by lines parallel to the y and x axes. The intercepts of these lines on the x and y axes are given in LAMDA(i), for i=5,6,...,PX-4 and MU(j), for j=5,6,...,PY-4, respectively. The subroutine orders the data so that all points in a panel occur before data in succeeding panels, where the panels are numbered from bottom to top and then left to right, with the usual arrangement of axes, as shown in the diagram. Within a panel the points maintain their original order.

Please see figure in printed Reference Manual

A data point lying exactly on one or more panel sides is taken to be in the highest-numbered panel adjacent to the point. The subroutine does not physically rearrange the data, but provides the array POINT which contains a linked list for each panel, pointing to the data in that panel. The total number of panels is (PX-7)*(PY-7).

4. References

None.

5. Parameters

1: PX -- INTEGER Input

2: PY -- INTEGER Input

On entry: PX and PY must specify eight more than the number of intercepts on the x axis and y axis, respectively.
Constraint: PX >= 8 and PY >= 8.

3: LAMDA(PX) -- DOUBLE PRECISION array Input
On entry: LAMDA(5) to LAMDA(PX-4) must contain, in non-decreasing order, the intercepts on the x axis of the sides

of the panels parallel to the y axis.

4: MU(PY) -- DOUBLE PRECISION array Input
On entry: MU(5) to MU(PY-4) must contain, in non-decreasing order, the intercepts on the y axis of the sides of the panels parallel to the x axis.

5: M -- INTEGER Input
On entry: the number m of data points.

6: X(M) -- DOUBLE PRECISION array Input

7: Y(M) -- DOUBLE PRECISION array Input
On entry: the co-ordinates of the rth data point (x_r, y_r), for r=1,2,...,m.

8: POINT(NPOINT) -- INTEGER array Output
On exit: for i = 1,2,...,NADRES, POINT(m+i) = I1 is the index of the first point in panel i, POINT(I1) = I2 is the index of the second point in panel i and so on.

POINT(IN) = 0 indicates that X(IN),Y(IN) was the last point in the panel.

The co-ordinates of points in panel i can be accessed in turn by means of the following instructions:

```
   IN = M + I
10 IN = POINT(IN)
   IF (IN.EQ. 0) GOTO 20
   XI = X(IN)
   YI = Y(IN)
   .
   .
   .
   GOTO 10
20...
```

9: NPOINT -- INTEGER Input
On entry:
the dimension of the array POINT as declared in the (sub)program from which E02ZAF is called.
Constraint: NPOINT >= M + (PX-7)*(PY-7).

10: ADRES(NADRES) -- INTEGER array Workspace

```
11:  NADRES -- INTEGER                                          Input
     On entry: the value (PX-7)*(PY-7), the number of panels
     into which the (x,y) plane is divided.

12:  IFAIL -- INTEGER                                     Input/Output
     On entry: IFAIL must be set to 0, -1 or 1. For users not
     familiar with this parameter (described in the Essential
     Introduction) the recommended value is 0.

     On exit: IFAIL = 0 unless the routine detects an error (see
     Section 6).

6. Error Indicators and Warnings

Errors detected by the routine:

If on entry IFAIL = 0 or -1, explanatory error messages are
output on the current error message unit (as defined by X04AAF).

IFAIL= 1
     The intercepts in the array LAMDA, or in the array MU, are
     not in non-decreasing order.

IFAIL= 2
     On entry PX < 8,

     or       PY < 8,

     or       M <= 0,

     or       NADRES /= (PX-7)*(PY-7),

     or       NPOINT < M + (PX-7)*(PY-7).

7. Accuracy

Not applicable.

8. Further Comments

The time taken by this routine is approximately proportional to
m*log(NADRES).

This subroutine was written to sort two dimensional data in the
manner required by routines E02DAF and E02DBF(*). The first 9
parameters of E02ZAF are the same as the parameters in E02DAF and
```

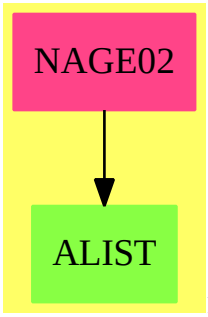
E02DBF(*) which have the same name.

9. Example

This example program reads in data points and the intercepts of the panel sides on the x and y axes; it calls E02ZAF to set up the index array POINT; and finally it prints the data points in panel order.

The example program is not reproduced here. The source code for all example programs is distributed with the NAG Foundation Library software and should be available on-line.

15.4 NagFittingPackage



1.00

Exports:

e02adf	e02aef	e02agf	e02ahf	e02ajf
e02akf	e02baf	e02bbf	e02bcf	e02bdf
e02bef	e02daf	e02dcf	e02ddf	e02def
e02dff	e02gaf	e02zaf		

⟨*package NAGE02 NagFittingPackage*⟩≡

```
)abbrev package NAGE02 NagFittingPackage
++ Author: Godfrey Nolan and Mike Dewar
++ Date Created: Jan 1994
++ Date Last Updated: Thu May 12 17:44:59 1994
++ Description:
++ This package uses the NAG Library to find a
++ function which approximates a set of data points. Typically the
++ data contain random errors, as of experimental measurement, which
++ need to be smoothed out. To seek an approximation to the data, it
++ is first necessary to specify for the approximating function a
++ mathematical form (a polynomial, for example) which contains a
++ number of unspecified coefficients: the appropriate fitting
++ routine then derives for the coefficients the values which
++ provide the best fit of that particular form. The package deals
++ mainly with curve and surface fitting (i.e., fitting with
++ functions of one and of two variables) when a polynomial or a
++ cubic spline is used as the fitting function, since these cover
++ the most common needs. However, fitting with other functions
++ and/or more variables can be undertaken by means of general
++ linear or nonlinear routines (some of which are contained in
++ other packages) depending on whether the coefficients in the
++ function occur linearly or nonlinearly. Cases where a graph
++ rather than a set of data points is given can be treated simply
++ by first reading a suitable set of points from the graph.
++ The package also contains routines for evaluating,
++ differentiating and integrating polynomial and spline curves and
++ surfaces, once the numerical values of their coefficients have
```

```
  ++ been determined.
  ++ See \downlink{Manual Page}{manpageXXe02}.


  NagFittingPackage(): Exports == Implementation where
    S ==> Symbol
    FOP ==> FortranOutputStackPackage

    Exports ==> with
      e02adf : (Integer,Integer,Integer,Matrix DoubleFloat,_
          Matrix DoubleFloat,Matrix DoubleFloat,Integer) -> Result
       ++ e02adf(m,kplus1,nrows,x,y,w,ifail)
       ++ computes weighted least-squares polynomial approximations
       ++ to an arbitrary set of data points.
       ++ See \downlink{Manual Page}{manpageXXe02adf}.
      e02aef : (Integer,Matrix DoubleFloat,DoubleFloat,Integer) -> Result
       ++ e02aef(nplus1,a,xcap,ifail)
       ++ evaluates a polynomial from its Chebyshev-series
       ++ representation.
       ++ See \downlink{Manual Page}{manpageXXe02aef}.
      e02agf : (Integer,Integer,Integer,DoubleFloat,_
          DoubleFloat,Matrix DoubleFloat,Matrix DoubleFloat,Matrix DoubleFloat,Inte
       ++ e02agf(m,kplus1,nrows,xmin,xmax,x,y,w,mf,xf,yf,lyf,ip,lwrk,liwrk,ifail)
       ++ computes constrained weighted least-squares polynomial
       ++ approximations in Chebyshev-series form to an arbitrary set of
       ++ data points. The values of the approximations and any number of
       ++ their derivatives can be specified at selected points.
       ++ See \downlink{Manual Page}{manpageXXe02agf}.
      e02ahf : (Integer,DoubleFloat,DoubleFloat,Matrix DoubleFloat,_
          Integer,Integer,Integer,Integer,Integer) -> Result
       ++ e02ahf(np1,xmin,xmax,a,ia1,la,iadif1,ladif,ifail)
       ++ determines the coefficients in the Chebyshev-series
       ++ representation of the derivative of a polynomial given in
       ++ Chebyshev-series form.
       ++ See \downlink{Manual Page}{manpageXXe02ahf}.
      e02ajf : (Integer,DoubleFloat,DoubleFloat,Matrix DoubleFloat,_
          Integer,Integer,DoubleFloat,Integer,Integer,Integer) -> Result
       ++ e02ajf(np1,xmin,xmax,a,ia1,la,qatm1,iaint1,laint,ifail)
       ++ determines the coefficients in the Chebyshev-series
       ++ representation of the indefinite integral of a polynomial given
       ++ in Chebyshev-series form.
       ++ See \downlink{Manual Page}{manpageXXe02ajf}.
      e02akf : (Integer,DoubleFloat,DoubleFloat,Matrix DoubleFloat,_
          Integer,Integer,DoubleFloat,Integer) -> Result
       ++ e02akf(np1,xmin,xmax,a,ia1,la,x,ifail)
       ++ evaluates a polynomial from its Chebyshev-series
```

```
 ++ representation, allowing an arbitrary index increment for
 ++ accessing the array of coefficients.
 ++ See \downlink{Manual Page}{manpageXXe02akf}.
e02baf : (Integer,Integer,Matrix DoubleFloat,Matrix DoubleFloat,_
    Matrix DoubleFloat,Matrix DoubleFloat,Integer) -> Result
 ++ e02baf(m,ncap7,x,y,w,lamda,ifail)
 ++ computes a weighted least-squares approximation to an
 ++ arbitrary set of data points by a cubic splines
 ++ prescribed by the user. Cubic spline can also be
 ++ carried out.
 ++ See \downlink{Manual Page}{manpageXXe02baf}.
e02bbf : (Integer,Matrix DoubleFloat,Matrix DoubleFloat,DoubleFloat,_
    Integer) -> Result
 ++ e02bbf(ncap7,lamda,c,x,ifail)
 ++ evaluates a cubic spline representation.
 ++ See \downlink{Manual Page}{manpageXXe02bbf}.
e02bcf : (Integer,Matrix DoubleFloat,Matrix DoubleFloat,DoubleFloat,_
    Integer,Integer) -> Result
 ++ e02bcf(ncap7,lamda,c,x,left,ifail)
 ++ evaluates a cubic spline and its first three derivatives
 ++ from its B-spline representation.
 ++ See \downlink{Manual Page}{manpageXXe02bcf}.
e02bdf : (Integer,Matrix DoubleFloat,Matrix DoubleFloat,Integer) -> Result
 ++ e02bdf(ncap7,lamda,c,ifail)
 ++ computes the definite integral from its
 ++ B-spline representation.
 ++ See \downlink{Manual Page}{manpageXXe02bdf}.
e02bef : (String,Integer,Matrix DoubleFloat,Matrix DoubleFloat,_
    Matrix DoubleFloat,DoubleFloat,Integer,Integer,Integer,Matrix DoubleFloat,Integer,Ma
 ++ e02bef(start,m,x,y,w,s,nest,lwrk,n,lamda,ifail,wrk,iwrk)
 ++ computes a cubic spline approximation to an arbitrary set
 ++ of data points. The knot are located
 ++ automatically, but a single parameter must be specified to
 ++ control the trade-off between closeness of fit and smoothness of
 ++ fit.
 ++ See \downlink{Manual Page}{manpageXXe02bef}.
e02daf : (Integer,Integer,Integer,Matrix DoubleFloat,_
    Matrix DoubleFloat,Matrix DoubleFloat,Matrix DoubleFloat,Matrix I
 ++ e02daf(m,px,py,x,y,f,w,mu,point,npoint,nc,nws,eps,lamda,ifail)
 ++ forms a minimal, weighted least-squares bicubic spline
 ++ surface fit with prescribed knots to a given set of data points.
 ++ See \downlink{Manual Page}{manpageXXe02daf}.
e02dcf : (String,Integer,Matrix DoubleFloat,Integer,_
    Matrix DoubleFloat,Matrix DoubleFloat,DoubleFloat,Integer,Integer,Integer,Integer,In
 ++ e02dcf(start,mx,x,my,y,f,s,nxest,nyest,lwrk,liwrk,nx,lamda,ny,mu,wrk,iwrk,ifail)
 ++ computes a bicubic spline approximation to a set of data
```

```
    ++ values, given on a rectangular grid in the x-y plane. The knots
    ++ of the spline are located automatically, but a single parameter
    ++ must be specified to control the trade-off between closeness of
    ++ fit and smoothness of fit.
    ++ See \downlink{Manual Page}{manpageXXe02dcf}.
   e02ddf : (String,Integer,Matrix DoubleFloat,Matrix DoubleFloat,_
       Matrix DoubleFloat,Matrix DoubleFloat,DoubleFloat,Integer,Integer,Integer
    ++ e02ddf(start,m,x,y,f,w,s,nxest,nyest,lwrk,liwrk,nx,lamda,ny,mu,wrk,ifail)
    ++ computes a bicubic spline approximation to a set of
    ++ scattered data are located
    ++ automatically, but a single parameter must be specified to
    ++ control the trade-off between closeness of fit and smoothness of
    ++ fit.
    ++ See \downlink{Manual Page}{manpageXXe02ddf}.
   e02def : (Integer,Integer,Integer,Matrix DoubleFloat,_
       Matrix DoubleFloat,Matrix DoubleFloat,Matrix DoubleFloat,Matrix DoubleFlo
    ++ e02def(m,px,py,x,y,lamda,mu,c,ifail)
    ++ calculates values of a bicubic spline
    ++ representation.
    ++ See \downlink{Manual Page}{manpageXXe02def}.
   e02dff : (Integer,Integer,Integer,Integer,_
       Matrix DoubleFloat,Matrix DoubleFloat,Matrix DoubleFloat,Matrix DoubleFlo
    ++ e02dff(mx,my,px,py,x,y,lamda,mu,c,lwrk,liwrk,ifail)
    ++ calculates values of a bicubic spline
    ++ representation. The spline is evaluated at all points on a
    ++ rectangular grid.
    ++ See \downlink{Manual Page}{manpageXXe02dff}.
   e02gaf : (Integer,Integer,Integer,DoubleFloat,_
       Matrix DoubleFloat,Matrix DoubleFloat,Integer) -> Result
    ++ e02gaf(m,la,nplus2,toler,a,b,ifail)
    ++ calculates an l  solution to an over-determined system of
    ++                       1
    ++ linear equations.
    ++ See \downlink{Manual Page}{manpageXXe02gaf}.
   e02zaf : (Integer,Integer,Matrix DoubleFloat,Matrix DoubleFloat,_
       Integer,Matrix DoubleFloat,Matrix DoubleFloat,Integer,Integer,Integer) ->
    ++ e02zaf(px,py,lamda,mu,m,x,y,npoint,nadres,ifail)
    ++ sorts two-dimensional data into rectangular panels.
    ++ See \downlink{Manual Page}{manpageXXe02zaf}.
Implementation ==> add

  import Lisp
  import DoubleFloat
  import Any
  import Record
  import Integer
```

```
import Matrix DoubleFloat
import Boolean
import NAGLinkSupportPackage
import AnyFunctions1(Integer)
import AnyFunctions1(Matrix DoubleFloat)
import AnyFunctions1(DoubleFloat)
import AnyFunctions1(Matrix Integer)
import AnyFunctions1(String)


e02adf(mArg:Integer,kplus1Arg:Integer,nrowsArg:Integer,_
    xArg:Matrix DoubleFloat,yArg:Matrix DoubleFloat,wArg:Matrix DoubleFloat,_
    ifailArg:Integer): Result ==
    [(invokeNagman(NIL$Lisp,_
    "e02adf",_
    ["m"::S,"kplus1"::S,"nrows"::S,"ifail"::S,"x"::S,"y"::S,"w"::S,"a"::S,"s"::S_
    ,"work1"::S,"work2"::S]$Lisp,_
    ["a"::S,"s"::S,"work1"::S,"work2"::S]$Lisp,_
    [["double"::S,["x"::S,"m"::S]$Lisp,["y"::S,"m"::S]$Lisp_
    ,["w"::S,"m"::S]$Lisp,["a"::S,"nrows"::S,"kplus1"::S]$Lisp,["s"::S,"kplus1"::S]$Lisp
    ,["work2"::S,["*"::S,2$Lisp,"kplus1"::S]$Lisp]$Lisp]$Lisp_
    ,["integer"::S,"m"::S,"kplus1"::S,"nrows"::S_
    ,"ifail"::S]$Lisp_
    ]$Lisp,_
    ["a"::S,"s"::S,"ifail"::S]$Lisp,_
    [([mArg::Any,kplus1Arg::Any,nrowsArg::Any,ifailArg::Any,xArg::Any,yArg::Any,wArg::An
    @List Any]$Lisp)$Lisp)_
    pretend List (Record(key:Symbol,entry:Any))]$Result

e02aef(nplus1Arg:Integer,aArg:Matrix DoubleFloat,xcapArg:DoubleFloat,_
    ifailArg:Integer): Result ==
    [(invokeNagman(NIL$Lisp,_
    "e02aef",_
    ["nplus1"::S,"xcap"::S,"p"::S,"ifail"::S,"a"::S]$Lisp,_
    ["p"::S]$Lisp,_
    [["double"::S,["a"::S,"nplus1"::S]$Lisp,"xcap"::S_
    ,"p"::S]$Lisp_
    ,["integer"::S,"nplus1"::S,"ifail"::S]$Lisp_
    ]$Lisp,_
    ["p"::S,"ifail"::S]$Lisp,_
    [([nplus1Arg::Any,xcapArg::Any,ifailArg::Any,aArg::Any ])_
    @List Any]$Lisp)$Lisp)_
    pretend List (Record(key:Symbol,entry:Any))]$Result

e02agf(mArg:Integer,kplus1Arg:Integer,nrowsArg:Integer,_
    xminArg:DoubleFloat,xmaxArg:DoubleFloat,xArg:Matrix DoubleFloat,_
```

```
    yArg:Matrix DoubleFloat,wArg:Matrix DoubleFloat,mfArg:Integer,_
    xfArg:Matrix DoubleFloat,yfArg:Matrix DoubleFloat,lyfArg:Integer,_
    ipArg:Matrix Integer,lwrkArg:Integer,liwrkArg:Integer,_
    ifailArg:Integer): Result ==
    [(invokeNagman(NIL$Lisp,_
    "e02agf",_
    ["m"::S,"kplus1"::S,"nrows"::S,"xmin"::S,"xmax"::S_
    ,"mf"::S,"lyf"::S,"lwrk"::S,"liwrk"::S,"np1"::S_
    ,"ifail"::S,"x"::S,"y"::S,"w"::S,"xf"::S,"yf"::S_
    ,"ip"::S,"a"::S,"s"::S,"wrk"::S,"iwrk"::S_
    ]$Lisp,_
    ["a"::S,"s"::S,"np1"::S,"wrk"::S,"iwrk"::S]$Lisp,_
    [["double"::S,"xmin"::S,"xmax"::S,["x"::S,"m"::S]$Lisp_
    ,["y"::S,"m"::S]$Lisp,["w"::S,"m"::S]$Lisp,["xf"::S,"mf"::S]$Lisp,["yf"::
    ,["s"::S,"kplus1"::S]$Lisp,["wrk"::S,"lwrk"::S]$Lisp]$Lisp_
    ,["integer"::S,"m"::S,"kplus1"::S,"nrows"::S_
    ,"mf"::S,"lyf"::S,["ip"::S,"mf"::S]$Lisp,"lwrk"::S,"liwrk"::S,"np1"::S,"i
    ]$Lisp,_
    ["a"::S,"s"::S,"np1"::S,"wrk"::S,"ifail"::S]$Lisp,_
    [([mArg::Any,kplus1Arg::Any,nrowsArg::Any,xminArg::Any,xmaxArg::Any,mfArg
    @List Any]$Lisp)$Lisp)_
    pretend List (Record(key:Symbol,entry:Any))]$Result

e02ahf(np1Arg:Integer,xminArg:DoubleFloat,xmaxArg:DoubleFloat,_
    aArg:Matrix DoubleFloat,ia1Arg:Integer,laArg:Integer,_
    iadif1Arg:Integer,ladifArg:Integer,ifailArg:Integer): Result ==
    [(invokeNagman(NIL$Lisp,_
    "e02ahf",_
    ["np1"::S,"xmin"::S,"xmax"::S,"ia1"::S,"la"::S_
    ,"iadif1"::S,"ladif"::S,"patm1"::S,"ifail"::S,"a"::S,"adif"::S]$Lisp,_
    ["patm1"::S,"adif"::S]$Lisp,_
    [["double"::S,"xmin"::S,"xmax"::S,["a"::S,"la"::S]$Lisp_
    ,"patm1"::S,["adif"::S,"ladif"::S]$Lisp]$Lisp_
    ,["integer"::S,"np1"::S,"ia1"::S,"la"::S,"iadif1"::S_
    ,"ladif"::S,"ifail"::S]$Lisp_
    ]$Lisp,_
    ["patm1"::S,"adif"::S,"ifail"::S]$Lisp,_
    [([np1Arg::Any,xminArg::Any,xmaxArg::Any,ia1Arg::Any,laArg::Any,iadif1Arg
    @List Any]$Lisp)$Lisp)_
    pretend List (Record(key:Symbol,entry:Any))]$Result

e02ajf(np1Arg:Integer,xminArg:DoubleFloat,xmaxArg:DoubleFloat,_
    aArg:Matrix DoubleFloat,ia1Arg:Integer,laArg:Integer,_
    qatm1Arg:DoubleFloat,iaint1Arg:Integer,laintArg:Integer,_
    ifailArg:Integer): Result ==
    [(invokeNagman(NIL$Lisp,_
```

```
        "e02ajf",_
        ["np1"::S,"xmin"::S,"xmax"::S,"ia1"::S,"la"::S_
        ,"qatm1"::S,"iaint1"::S,"laint"::S,"ifail"::S,"a"::S,"aint"::S]$Lisp,_
        ["aint"::S]$Lisp,_
        [["double"::S,"xmin"::S,"xmax"::S,["a"::S,"la"::S]$Lisp_
        ,"qatm1"::S,["aint"::S,"laint"::S]$Lisp]$Lisp_
        ,["integer"::S,"np1"::S,"ia1"::S,"la"::S,"iaint1"::S_
        ,"laint"::S,"ifail"::S]$Lisp_
        ]$Lisp,_
        ["aint"::S,"ifail"::S]$Lisp,_
        [([np1Arg::Any,xminArg::Any,xmaxArg::Any,ia1Arg::Any,laArg::Any,qatm1Arg::Any,iaint1
        @List Any]$Lisp)$Lisp)_
        pretend List (Record(key:Symbol,entry:Any))]$Result

    e02akf(np1Arg:Integer,xminArg:DoubleFloat,xmaxArg:DoubleFloat,_
        aArg:Matrix DoubleFloat,ia1Arg:Integer,laArg:Integer,_
        xArg:DoubleFloat,ifailArg:Integer): Result ==
        [(invokeNagman(NIL$Lisp,_
        "e02akf",_
        ["np1"::S,"xmin"::S,"xmax"::S,"ia1"::S,"la"::S_
        ,"x"::S,"result"::S,"ifail"::S,"a"::S]$Lisp,_
        ["result"::S]$Lisp,_
        [["double"::S,"xmin"::S,"xmax"::S,["a"::S,"la"::S]$Lisp_
        ,"x"::S,"result"::S]$Lisp_
        ,["integer"::S,"np1"::S,"ia1"::S,"la"::S,"ifail"::S_
        ]$Lisp_
        ]$Lisp,_
        ["result"::S,"ifail"::S]$Lisp,_
        [([np1Arg::Any,xminArg::Any,xmaxArg::Any,ia1Arg::Any,laArg::Any,xArg::Any,ifailArg:
        @List Any]$Lisp)$Lisp)_
        pretend List (Record(key:Symbol,entry:Any))]$Result

    e02baf(mArg:Integer,ncap7Arg:Integer,xArg:Matrix DoubleFloat,_
        yArg:Matrix DoubleFloat,wArg:Matrix DoubleFloat,lamdaArg:Matrix DoubleFloat,_
        ifailArg:Integer): Result ==
        [(invokeNagman(NIL$Lisp,_
        "e02baf",_
        ["m"::S,"ncap7"::S,"ss"::S,"ifail"::S,"x"::S,"y"::S,"w"::S,"c"::S,"lamda"::S_
        ,"work1"::S,"work2"::S]$Lisp,_
        ["c"::S,"ss"::S,"work1"::S,"work2"::S]$Lisp,_
        [["double"::S,["x"::S,"m"::S]$Lisp,["y"::S,"m"::S]$Lisp_
        ,["w"::S,"m"::S]$Lisp,["c"::S,"ncap7"::S]$Lisp,"ss"::S,["lamda"::S,"ncap7"::S]$Lisp
        ,["work2"::S,["*"::S,4$Lisp,"ncap7"::S]$Lisp]$Lisp]$Lisp_
        ,["integer"::S,"m"::S,"ncap7"::S,"ifail"::S_
        ]$Lisp_
        ]$Lisp,_
```

```
    ["c"::S,"ss"::S,"lamda"::S,"ifail"::S]$Lisp,_
    [([mArg::Any,ncap7Arg::Any,ifailArg::Any,xArg::Any,yArg::Any,wArg::Any,la
    @List Any]$Lisp)$Lisp)_
    pretend List (Record(key:Symbol,entry:Any))]$Result

e02bbf(ncap7Arg:Integer,lamdaArg:Matrix DoubleFloat,cArg:Matrix DoubleFloat,_
    xArg:DoubleFloat,ifailArg:Integer): Result ==
    [(invokeNagman(NIL$Lisp,_
    "e02bbf",_
    ["ncap7"::S,"x"::S,"s"::S,"ifail"::S,"lamda"::S,"c"::S]$Lisp,_
    ["s"::S]$Lisp,_
    [["double"::S,["lamda"::S,"ncap7"::S]$Lisp_
    ,["c"::S,"ncap7"::S]$Lisp,"x"::S,"s"::S]$Lisp_
    ,["integer"::S,"ncap7"::S,"ifail"::S]$Lisp_
    ]$Lisp,_
    ["s"::S,"ifail"::S]$Lisp,_
    [([ncap7Arg::Any,xArg::Any,ifailArg::Any,lamdaArg::Any,cArg::Any ])_
    @List Any]$Lisp)$Lisp)_
    pretend List (Record(key:Symbol,entry:Any))]$Result

e02bcf(ncap7Arg:Integer,lamdaArg:Matrix DoubleFloat,cArg:Matrix DoubleFloat,_
    xArg:DoubleFloat,leftArg:Integer,ifailArg:Integer): Result ==
    [(invokeNagman(NIL$Lisp,_
    "e02bcf",_
    ["ncap7"::S,"x"::S,"left"::S,"ifail"::S,"lamda"::S,"c"::S,"s"::S]$Lisp,_
    ["s"::S]$Lisp,_
    [["double"::S,["lamda"::S,"ncap7"::S]$Lisp_
    ,["c"::S,"ncap7"::S]$Lisp,"x"::S,["s"::S,4$Lisp]$Lisp]$Lisp_
    ,["integer"::S,"ncap7"::S,"left"::S,"ifail"::S_
    ]$Lisp_
    ]$Lisp,_
    ["s"::S,"ifail"::S]$Lisp,_
    [([ncap7Arg::Any,xArg::Any,leftArg::Any,ifailArg::Any,lamdaArg::Any,cArg:
    @List Any]$Lisp)$Lisp)_
    pretend List (Record(key:Symbol,entry:Any))]$Result

e02bdf(ncap7Arg:Integer,lamdaArg:Matrix DoubleFloat,cArg:Matrix DoubleFloat,_
    ifailArg:Integer): Result ==
    [(invokeNagman(NIL$Lisp,_
    "e02bdf",_
    ["ncap7"::S,"defint"::S,"ifail"::S,"lamda"::S,"c"::S]$Lisp,_
    ["defint"::S]$Lisp,_
    [["double"::S,["lamda"::S,"ncap7"::S]$Lisp_
    ,["c"::S,"ncap7"::S]$Lisp,"defint"::S]$Lisp_
    ,["integer"::S,"ncap7"::S,"ifail"::S]$Lisp_
    ]$Lisp,_
```

```
        ["defint"::S,"ifail"::S]$Lisp,_
        [([ncap7Arg::Any,ifailArg::Any,lamdaArg::Any,cArg::Any ])_
        @List Any]$Lisp)$Lisp)_
        pretend List (Record(key:Symbol,entry:Any))]$Result

    e02bef(startArg:String,mArg:Integer,xArg:Matrix DoubleFloat,_
        yArg:Matrix DoubleFloat,wArg:Matrix DoubleFloat,sArg:DoubleFloat,_
        nestArg:Integer,lwrkArg:Integer,nArg:Integer,_
        lamdaArg:Matrix DoubleFloat,ifailArg:Integer,wrkArg:Matrix DoubleFloat,_
        iwrkArg:Matrix Integer): Result ==
        [(invokeNagman(NIL$Lisp,_
        "e02bef",_
        ["start"::S,"m"::S,"s"::S,"nest"::S,"lwrk"::S_
        ,"fp"::S,"n"::S,"ifail"::S,"x"::S,"y"::S,"w"::S,"c"::S,"lamda"::S_
        ,"wrk"::S,"iwrk"::S]$Lisp,_
        ["c"::S,"fp"::S]$Lisp,_
        [["double"::S,["x"::S,"m"::S]$Lisp,["y"::S,"m"::S]$Lisp_
        ,["w"::S,"m"::S]$Lisp,"s"::S,["c"::S,"nest"::S]$Lisp,"fp"::S,["lamda"::S,"nest"::S]$
        ]$Lisp_
        ,["integer"::S,"m"::S,"nest"::S,"lwrk"::S,"n"::S_
        ,"ifail"::S,["iwrk"::S,"nest"::S]$Lisp]$Lisp_
        ,["character"::S,"start"::S]$Lisp_
        ]$Lisp,_
        ["c"::S,"fp"::S,"n"::S,"lamda"::S,"ifail"::S,"wrk"::S,"iwrk"::S]$Lisp,_
        [([startArg::Any,mArg::Any,sArg::Any,nestArg::Any,lwrkArg::Any,nArg::Any,ifailArg::A
        @List Any]$Lisp)$Lisp)_
        pretend List (Record(key:Symbol,entry:Any))]$Result

    e02daf(mArg:Integer,pxArg:Integer,pyArg:Integer,_
        xArg:Matrix DoubleFloat,yArg:Matrix DoubleFloat,fArg:Matrix DoubleFloat,_
        wArg:Matrix DoubleFloat,muArg:Matrix DoubleFloat,pointArg:Matrix Integer,_
        npointArg:Integer,ncArg:Integer,nwsArg:Integer,_
        epsArg:DoubleFloat,lamdaArg:Matrix DoubleFloat,ifailArg:Integer): Result ==
        [(invokeNagman(NIL$Lisp,_
        "e02daf",_
        ["m"::S,"px"::S,"py"::S,"npoint"::S,"nc"::S_
        ,"nws"::S,"eps"::S,"sigma"::S,"rank"::S,"ifail"::S_
        ,"x"::S,"y"::S,"f"::S,"w"::S,"mu"::S_
        ,"point"::S,"dl"::S,"c"::S,"lamda"::S,"ws"::S_
        ]$Lisp,_
        ["dl"::S,"c"::S,"sigma"::S,"rank"::S,"ws"::S]$Lisp,_
        [["double"::S,["x"::S,"m"::S]$Lisp,["y"::S,"m"::S]$Lisp_
        ,["f"::S,"m"::S]$Lisp,["w"::S,"m"::S]$Lisp,["mu"::S,"py"::S]$Lisp,"eps"::S,["dl"::S
        ,"sigma"::S,["lamda"::S,"px"::S]$Lisp,["ws"::S,"nws"::S]$Lisp]$Lisp_
        ,["integer"::S,"m"::S,"px"::S,"py"::S,["point"::S,"npoint"::S]$Lisp_
        ,"npoint"::S,"nc"::S,"nws"::S,"rank"::S,"ifail"::S]$Lisp_
```

```
    ]$Lisp,_
    ["dl"::S,"c"::S,"sigma"::S,"rank"::S,"lamda"::S,"ifail"::S]$Lisp,_
    [([mArg::Any,pxArg::Any,pyArg::Any,npointArg::Any,ncArg::Any,nwsArg::Any,
    @List Any]$Lisp)$Lisp)_
    pretend List (Record(key:Symbol,entry:Any))]$Result

e02dcf(startArg:String,mxArg:Integer,xArg:Matrix DoubleFloat,_
    myArg:Integer,yArg:Matrix DoubleFloat,fArg:Matrix DoubleFloat,_
    sArg:DoubleFloat,nxestArg:Integer,nyestArg:Integer,_
    lwrkArg:Integer,liwrkArg:Integer,nxArg:Integer,_
    lamdaArg:Matrix DoubleFloat,nyArg:Integer,muArg:Matrix DoubleFloat,_
    wrkArg:Matrix DoubleFloat,iwrkArg:Matrix Integer,ifailArg:Integer): Result
    [(invokeNagman(NIL$Lisp,_
    "e02dcf",_
    ["start"::S,"mx"::S,"my"::S,"s"::S,"nxest"::S_
    ,"nyest"::S,"lwrk"::S,"liwrk"::S,"fp"::S,"nx"::S_
    ,"ny"::S,"ifail"::S,"x"::S,"y"::S,"f"::S,"c"::S,"lamda"::S_
    ,"mu"::S,"wrk"::S,"iwrk"::S]$Lisp,_
    ["c"::S,"fp"::S]$Lisp,_
    [["double"::S,["x"::S,"mx"::S]$Lisp,["y"::S,"my"::S]$Lisp_
    ,["f"::S,["*"::S,"mx"::S,"my"::S]$Lisp]$Lisp,"s"::S,["c"::S,["*"::S,["-":
    ,"fp"::S,["lamda"::S,"nxest"::S]$Lisp,["mu"::S,"nyest"::S]$Lisp,["wrk"::S
    ]$Lisp_
    ,["integer"::S,"mx"::S,"my"::S,"nxest"::S,"nyest"::S_
    ,"lwrk"::S,"liwrk"::S,"nx"::S,"ny"::S,["iwrk"::S,"liwrk"::S]$Lisp,"ifail"
    ,["character"::S,"start"::S]$Lisp_
    ]$Lisp,_
    ["c"::S,"fp"::S,"nx"::S,"lamda"::S,"ny"::S,"mu"::S,"wrk"::S,"iwrk"::S,"if
    [([startArg::Any,mxArg::Any,myArg::Any,sArg::Any,nxestArg::Any,nyestArg::
    @List Any]$Lisp)$Lisp)_
    pretend List (Record(key:Symbol,entry:Any))]$Result

e02ddf(startArg:String,mArg:Integer,xArg:Matrix DoubleFloat,_
    yArg:Matrix DoubleFloat,fArg:Matrix DoubleFloat,wArg:Matrix DoubleFloat,_
    sArg:DoubleFloat,nxestArg:Integer,nyestArg:Integer,_
    lwrkArg:Integer,liwrkArg:Integer,nxArg:Integer,_
    lamdaArg:Matrix DoubleFloat,nyArg:Integer,muArg:Matrix DoubleFloat,_
    wrkArg:Matrix DoubleFloat,ifailArg:Integer): Result ==
    [(invokeNagman(NIL$Lisp,_
    "e02ddf",_
    ["start"::S,"m"::S,"s"::S,"nxest"::S,"nyest"::S_
    ,"lwrk"::S,"liwrk"::S,"fp"::S,"rank"::S,"nx"::S_
    ,"ny"::S,"ifail"::S,"x"::S,"y"::S,"f"::S,"w"::S,"c"::S_
    ,"iwrk"::S,"lamda"::S,"mu"::S,"wrk"::S]$Lisp,_
    ["c"::S,"fp"::S,"rank"::S,"iwrk"::S]$Lisp,_
    [["double"::S,["x"::S,"m"::S]$Lisp,["y"::S,"m"::S]$Lisp_
```

```
    ,["f"::S,"m"::S]$Lisp,["w"::S,"m"::S]$Lisp,"s"::S,["c"::S,["*"::S,["-"::S,"nxest"::S
    ,"fp"::S,["lamda"::S,"nxest"::S]$Lisp,["mu"::S,"nyest"::S]$Lisp,["wrk"::S,"lwrk"::S]
    ]$Lisp_
    ,["integer"::S,"m"::S,"nxest"::S,"nyest"::S_
    ,"lwrk"::S,"liwrk"::S,"rank"::S,["iwrk"::S,"liwrk"::S]$Lisp,"nx"::S,"ny"::S,"ifail"
    ,["character"::S,"start"::S]$Lisp_
    ]$Lisp,_
    ["c"::S,"fp"::S,"rank"::S,"iwrk"::S,"nx"::S,"lamda"::S,"ny"::S,"mu"::S,"wrk"::S,"ifa
    [([startArg::Any,mArg::Any,sArg::Any,nxestArg::Any,nyestArg::Any,lwrkArg::Any,liwrkA
    @List Any]$Lisp)$Lisp)_
    pretend List (Record(key:Symbol,entry:Any))]$Result

e02def(mArg:Integer,pxArg:Integer,pyArg:Integer,_
    xArg:Matrix DoubleFloat,yArg:Matrix DoubleFloat,lamdaArg:Matrix DoubleFloat,_
    muArg:Matrix DoubleFloat,cArg:Matrix DoubleFloat,ifailArg:Integer): Result ==
    [(invokeNagman(NIL$Lisp,_
    "e02def",_
    ["m"::S,"px"::S,"py"::S,"ifail"::S,"x"::S,"y"::S,"lamda"::S,"mu"::S,"c"::S_
    ,"ff"::S,"wrk"::S,"iwrk"::S]$Lisp,_
    ["ff"::S,"wrk"::S,"iwrk"::S]$Lisp,_
    [["double"::S,["x"::S,"m"::S]$Lisp,["y"::S,"m"::S]$Lisp_
    ,["lamda"::S,"px"::S]$Lisp,["mu"::S,"py"::S]$Lisp,["c"::S,["*"::S,["-"::S,"px"::S,4$
    ,["ff"::S,"m"::S]$Lisp,["wrk"::S,["-"::S,"py"::S,4$Lisp]$Lisp]$Lisp]$Lisp_
    ,["integer"::S,"m"::S,"px"::S,"py"::S,"ifail"::S_
    ,["iwrk"::S,["-"::S,"py"::S,4$Lisp]$Lisp]$Lisp]$Lisp_
    ]$Lisp,_
    ["ff"::S,"ifail"::S]$Lisp,_
    [([mArg::Any,pxArg::Any,pyArg::Any,ifailArg::Any,xArg::Any,yArg::Any,lamdaArg::Any,m
    @List Any]$Lisp)$Lisp)_
    pretend List (Record(key:Symbol,entry:Any))]$Result

e02dff(mxArg:Integer,myArg:Integer,pxArg:Integer,_
    pyArg:Integer,xArg:Matrix DoubleFloat,yArg:Matrix DoubleFloat,_
    lamdaArg:Matrix DoubleFloat,muArg:Matrix DoubleFloat,cArg:Matrix DoubleFloat,_
    lwrkArg:Integer,liwrkArg:Integer,ifailArg:Integer): Result ==
    [(invokeNagman(NIL$Lisp,_
    "e02dff",_
    ["mx"::S,"my"::S,"px"::S,"py"::S,"lwrk"::S_
    ,"liwrk"::S,"ifail"::S,"x"::S,"y"::S,"lamda"::S,"mu"::S,"c"::S_
    ,"ff"::S,"wrk"::S,"iwrk"::S]$Lisp,_
    ["ff"::S,"wrk"::S,"iwrk"::S]$Lisp,_
    [["double"::S,["x"::S,"mx"::S]$Lisp,["y"::S,"my"::S]$Lisp_
    ,["lamda"::S,"px"::S]$Lisp,["mu"::S,"py"::S]$Lisp,["c"::S,["*"::S,["-"::S,"px"::S,4$
    ,["ff"::S,["*"::S,"mx"::S,"my"::S]$Lisp]$Lisp,["wrk"::S,"lwrk"::S]$Lisp]$Lisp_
    ,["integer"::S,"mx"::S,"my"::S,"px"::S,"py"::S_
    ,"lwrk"::S,"liwrk"::S,"ifail"::S,["iwrk"::S,"liwrk"::S]$Lisp]$Lisp_
```

```
    ]$Lisp,_
    ["ff"::S,"ifail"::S]$Lisp,_
    [([mxArg::Any,myArg::Any,pxArg::Any,pyArg::Any,lwrkArg::Any,liwrkArg::Any
    @List Any]$Lisp)$Lisp)_
    pretend List (Record(key:Symbol,entry:Any))]$Result

e02gaf(mArg:Integer,laArg:Integer,nplus2Arg:Integer,_
    tolerArg:DoubleFloat,aArg:Matrix DoubleFloat,bArg:Matrix DoubleFloat,_
    ifailArg:Integer): Result ==
    [(invokeNagman(NIL$Lisp,_
    "e02gaf",_
    ["m"::S,"la"::S,"nplus2"::S,"toler"::S,"resid"::S_
    ,"irank"::S,"iter"::S,"ifail"::S,"x"::S,"a"::S,"b"::S,"iwork"::S]$Lisp,_
    ["x"::S,"resid"::S,"irank"::S,"iter"::S,"iwork"::S]$Lisp,_
    [["double"::S,"toler"::S,["x"::S,"nplus2"::S]$Lisp_
    ,"resid"::S,["a"::S,"la"::S,"nplus2"::S]$Lisp,["b"::S,"m"::S]$Lisp]$Lisp_
    ,["integer"::S,"m"::S,"la"::S,"nplus2"::S,"irank"::S_
    ,"iter"::S,"ifail"::S,["iwork"::S,"m"::S]$Lisp]$Lisp_
    ]$Lisp,_
    ["x"::S,"resid"::S,"irank"::S,"iter"::S,"a"::S,"b"::S,"ifail"::S]$Lisp,_
    [([mArg::Any,laArg::Any,nplus2Arg::Any,tolerArg::Any,ifailArg::Any,aArg:::
    @List Any]$Lisp)$Lisp)_
    pretend List (Record(key:Symbol,entry:Any))]$Result

e02zaf(pxArg:Integer,pyArg:Integer,lamdaArg:Matrix DoubleFloat,_
    muArg:Matrix DoubleFloat,mArg:Integer,xArg:Matrix DoubleFloat,_
    yArg:Matrix DoubleFloat,npointArg:Integer,nadresArg:Integer,_
    ifailArg:Integer): Result ==
    [(invokeNagman(NIL$Lisp,_
    "e02zaf",_
    ["px"::S,"py"::S,"m"::S,"npoint"::S,"nadres"::S_
    ,"ifail"::S,"lamda"::S,"mu"::S,"x"::S,"y"::S,"point"::S_
    ,"adres"::S]$Lisp,_
    ["point"::S,"adres"::S]$Lisp,_
    [["double"::S,["lamda"::S,"px"::S]$Lisp,["mu"::S,"py"::S]$Lisp_
    ,["x"::S,"m"::S]$Lisp,["y"::S,"m"::S]$Lisp]$Lisp_
    ,["integer"::S,"px"::S,"py"::S,"m"::S,"npoint"::S_
    ,"nadres"::S,["point"::S,"npoint"::S]$Lisp,"ifail"::S,["adres"::S,"nadres
    ]$Lisp,_
    ["point"::S,"ifail"::S]$Lisp,_
    [([pxArg::Any,pyArg::Any,mArg::Any,npointArg::Any,nadresArg::Any,ifailArg
    @List Any]$Lisp)$Lisp)_
    pretend List (Record(key:Symbol,entry:Any))]$Result
```

⟨*NAGE02.dotabb*⟩≡
```
"NAGE02" [color="#FF4488",href="bookvol10.4.pdf#nameddest=NAGE02"]
"ALIST" [color="#88FF44",href="bookvol10.3.pdf#nameddest=ALIST"]
"NAGE02" -> "ALIST"
```

15.5 package NAGF04 NagLinearEquationSolvingPackage

⟨*NagLinearEquationSolvingPackage.help*⟩≡

```
F04 -- Simultaneous Linear Equations          Introduction -- F04
                            Chapter F04
                  Simultaneous Linear Equations

1. Scope of the Chapter

This chapter, together with two routines in Chapter F07, is
concerned with the solution of the matrix equation AX=B, where B
may be a single vector or a matrix of multiple right-hand sides.
The matrix A may be real, complex, symmetric, Hermitian positive-
definite, or sparse. It may also be rectangular, in which case a
least-squares solution is obtained.

2. Background to the Problems

A set of linear equations may be written in the form

                                Ax=b

where the known matrix A, with real or complex coefficients, is
of size m by n, (m rows and n columns), the known right-hand
vector b has m components (m rows and one column), and the
required solution vector x has n components (n rows and one
column). There may sometimes be p vectors b , i=1,2,...,p on the
                                           i
right-hand side and the equations may then be written as

                                AX=B

the required matrix X having as its p columns the solutions of
Ax =b , i=1,2,...,p. Some routines deal with the latter case, but
  i  i
for clarity only the case p=1 is discussed here.

The most common problem, the determination of the unique solution
of Ax=b, occurs when m=n and A is non-singular, that is rank(A)=n
problem, discussed in Section 2.2 below, is the determination of
the least-squares solution of Ax~=b, i.e., the determination of a
vector x which minimizes the Euclidean length (two norm) of the
residual vector r=b-Ax. The usual case has m>n and rank(A)=n, in
```

which case x is unique.

2.1. Unique Solution of Ax=b

Most of the routines in this chapter, as well as two routines in Chapter F07, solve this particular problem. The solution is obtained by performing either an LU factorization, or a Cholesky factorization, as discussed in Section 2 of the F01 Chapter Introduction.

Two of the routines in this chapter use a process called iterative refinement to improve the initial solution in order to obtain a solution that is correct to working accuracy. It should be emphasised that if A and b are not known exactly then not all the figures in this solution may be meaningful. To be more precise, if x is the exact solution of the equations

$$Ax=b$$

and x is the solution of the perturbed equations

$$(A+E)x=b+e,$$

then, provided that $(\text{kappa})(A)\dfrac{||E||}{||A||}<=1$,

$$\frac{||x-x||}{||x||} <= \frac{(\text{kappa})(A)}{1-(\text{kappa})(A)\dfrac{||E||}{||A||}}\left(\frac{||E||}{||A||}+\frac{||e||}{||b||}\right),$$

where $(\text{kappa})(A)=||A||||A^{-1}||$ is the condition number of A with respect to inversion. Thus, if A is ill-conditioned (

(kappa)(A) is large), x may differ significantly from x. Often

```
           ||E||
(kappa)(A) -----<<1 in which case the above bound effectively
           ||A||
reduces to
```

$$\frac{\|x-x\|}{\|x\|} \leq (\text{kappa})(A)\left(\frac{\|E\|}{\|A\|} + \frac{\|e\|}{\|b\|}\right).$$

2.2. The Least-squares Solution of Ax~=b

The least-squares problem is to find a vector x to minimize

$$r^T r, \quad \text{where } r=b-Ax.$$

When m>=n and rank(A)=n then the solution vector x is unique. For the cases where x is not unique the routines in this chapter obtain the minimal length solution, that is the vector x for which $x^T x$ is a minimum.

2.3. Calculating the Inverse of a Matrix

The routines in this chapter can also be used to calculate the inverse of a square matrix A by solving the equation

$$AX=I,$$

where I is the identity matrix.

3. Recommendations on Choice and Use of Routines

3.1. General Purpose Routines

Many of the routines in this chapter perform the complete solution of the required equations, but some of the routines, as well as the routines in Chapter F07, assume that a prior factorization has been performed, using the appropriate factorization routine from Chapter F01 or Chapter F07. These, so-called, general purpose routines can be useful when explicit information on the factorization is required, as well as the

solution of the equations, or when the solution is required for multiple right-hand sides, or for a sequence of right-hand sides.

Note that some of the routines that perform a complete solution also allow multiple right-hand sides.

3.2. Iterative Refinement

The routines that perform iterative refinement are more costly than those that do not perform iterative refinement, both in terms of time and storage, and should only be used if the problem really warrants the additional accuracy provided by these routines. The storage requirements are approximately doubled, while the additional time is not usually prohibitive since the initial factorization is used at each iteration.

3.3. Sparse Matrix Routines

The routines for sparse matrices should usually be used only when the number of non-zero elements is very small, less than 10% of the total number of elements of A. Additionally, when the matrix is symmetric positive-definite the sparse routines should generally be used only when A does not have a (variable) band structure.

There are four routines for solving sparse linear equations, two for solving general real systems (F04AXF and F04QAF), one for solving symmetric positive-definite systems (F04MAF) and one for solving symmetric systems that may, or may not, be positive-definite (F04MBF). F04AXF and F04MAF utilise factorizations of the matrix A obtained by routines in Chapter F01, while the other two routines use iterative techniques and require a user-supplied function to compute matrix-vector products Ac and $A^T c$ for any given vector c. The routines requiring factorizations will usually be faster and the factorization can be utilised to solve for several right-hand sides, but the original matrix has to be explicitly supplied and is overwritten by the factorization, and the storage requirements will usually be substantially more than those of the iterative routines.

Routines F04MBF and F04QAF both allow the user to supply a pre-conditioner.

F04MBF can be used to solve systems of the form (A-(lambda)I)x=b, which can be useful in applications such as Rayleigh quotient

iteration.

F04QAF also solves sparse least-squares problems and allows the solution of damped (regularized) least-squares problems.

3.4. Decision Trees

If at any stage the answer to a question is 'Don't know' this should be read as 'No'.

For those routines that need to be preceded by a factorization routine, the appropriate routine name is given in brackets after the name of the routine for solving the equations. Note also that you may be directed to a routine in Chapter F07.

3.4.1. Routines for unique solution of Ax=b

Please see figure in printed Reference Manual

3.4.2. Routines for Least-squares problems

Please see figure in printed Reference Manual

F04 -- Simultaneous Linear Equations Contents -- F04
Chapter F04

Eigenvalues and Eigenvectors

F04ADF Approximate solution of complex simultaneous linear equations with multiple right-hand sides

F04ARF Approximate solution of real simultaneous linear equations, one right-hand side

F04ASF Accurate solution of real symmetric positive-definite simultaneous linear equations, one right-hand side

F04ATF Accurate solution of real simultaneous linear equations,

```
        one right-hand side

F04AXF  Approximate solution of real sparse simultaneous linear
        equations (coefficient matrix already factorized by
        F01BRF or F01BSF)

F04FAF  Approximate solution of real symmetric positive-definite
        tridiagonal simultaneous linear equations, one right-hand
        side

F04JGF  Least-squares (if rank = n) or minimal least-squares (if
        rank <n) solution of m real equations in n unknowns, rank
        <=n, m>=n

F04MAF  Real sparse symmetric positive-definite simultaneous
        linear equations (coefficient matrix already factorized)

F04MBF  Real sparse symmetric simultaneous linear equations

F04MCF  Approximate solution of real symmetric positive-definite
        variable-bandwidth simultaneous linear equations
        (coefficient matrix already factorized)

F04QAF  Sparse linear least-squares problem, m real equations in
        n unknowns
```

%%%

```
F04 -- Simultaneous Linear Equations                        F04ADF
        F04ADF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for
your implementation to check implementation-dependent details.
The symbol (*) after a NAG routine name denotes a routine that is
not included in the Foundation Library.

1. Purpose

F04ADF calculates the approximate solution of a set of complex
linear equations with multiple right-hand sides, using an LU
factorization with partial pivoting.

2. Specification

       SUBROUTINE F04ADF (A, IA, B, IB, N, M, C, IC, WKSPCE,
      1                   IFAIL)
```

```
      INTEGER              IA, IB, N, M, IC, IFAIL
      DOUBLE PRECISION     WKSPCE(*)
      COMPLEX(KIND(1.0D0)) A(IA,*), B(IB,*), C(IC,*)
```

3. Description

Given a set of complex linear equations AX=B, the routine first computes an LU factorization of A with partial pivoting, PA=LU, where P is a permutation matrix, L is lower triangular and U is unit upper triangular. The columns x of the solution X are found by forward and backward substitution in Ly=Pb and Ux=y, where b is a column of the right-hand side matrix B.

4. References

[1] Wilkinson J H and Reinsch C (1971) Handbook for Automatic Computation II, Linear Algebra. Springer-Verlag.

5. Parameters

1: A(IA,*) -- COMPLEX(KIND(1.0D)) array Input/Output
Note: the second dimension of the array A must be at least max(1,N).
On entry: the n by n matrix A. On exit: A is overwritten by the lower triangular matrix L and the off-diagonal elements of the upper triangular matrix U. The unit diagonal elements of U are not stored.

2: IA -- INTEGER Input
On entry:
the first dimension of the array A as declared in the (sub)program from which F04ADF is called.
Constraint: IA >= max(1,N).

3: B(IB,*) -- COMPLEX(KIND(1.0D)) array Input
Note: the second dimension of the array B must be at least max(1,M).
On entry: the n by m right-hand side matrix B. See also Section 8.

4: IB -- INTEGER Input
On entry:
the first dimension of the array B as declared in the (sub)program from which F04ADF is called.
Constraint: IB >= max(1,N).

```
 5:  N -- INTEGER                                                Input
     On entry: n, the order of the matrix A. Constraint: N >= 0.

 6:  M -- INTEGER                                                Input
     On entry: m, the number of right-hand sides. Constraint: M
     >= 0.

 7:  C(IC,*) -- COMPLEX(KIND(1.0D)) array                       Output
     Note: the second dimension of the array C must be at least
     max(1,M).
     On exit: the n by m solution matrix X. See also Section 8.

 8:  IC -- INTEGER                                               Input
     On entry:
     the first dimension of the array C as declared in the
     (sub)program from which F04ADF is called.
     Constraint: IC >= max(1,N).

 9:  WKSPCE(*) -- DOUBLE PRECISION array                      Workspace
     Note: the dimension of the array WKSPCE must be at least
     max(1,N).

10:  IFAIL -- INTEGER                                     Input/Output
     On entry: IFAIL must be set to 0, -1 or 1. For users not
     familiar with this parameter (described in the Essential
     Introduction) the recommended value is 0.

     On exit: IFAIL = 0 unless the routine detects an error (see
     Section 6).

6. Error Indicators and Warnings

Errors detected by the routine:

If on entry IFAIL = 0 or -1, explanatory error messages are
output on the current error message unit (as defined by X04AAF).

IFAIL= 1
     The matrix A is singular, possibly due to rounding errors.

IFAIL= 2
     On entry N < 0,

     or       M < 0,

     or       IA < max(1,N),
```

```
     or       IB < max(1,N),

     or       IC < max(1,N).
```

7. Accuracy

The accuracy of the computed solution depends on the conditioning of the original matrix. For a detailed error analysis see Wilkinson and Reinsch [1] page 106.

8. Further Comments

The time taken by the routine is approximately proportional to n^3

Unless otherwise stated in the Users' Note for your implementation, the routine may be called with the same actual array supplied for parameters B and C, in which case the solution vectors will overwrite the right-hand sides. However this is not standard Fortran 77, and may not work on all systems.

9. Example

To solve the set of linear equations AX=B where

$$A=\begin{pmatrix} 1 & 1+2i & 2+10i \\ 1+i & 3i & -5+14i \\ 1+i & 5i & -8+20i \end{pmatrix}$$

and

$$B=\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

The example program is not reproduced here. The source code for all example programs is distributed with the NAG Foundation Library software and should be available on-line.

%%%

F04 -- Simultaneous Linear Equations F04ARF

F04ARF -- NAG Foundation Library Routine Document

```
Note: Before using this routine, please read the Users' Note for
your implementation to check implementation-dependent details.
The symbol (*) after a NAG routine name denotes a routine that is
not included in the Foundation Library.

1. Purpose

F04ARF calculates the approximate solution of a set of real
linear equations with a single right-hand side, using an LU
factorization with partial pivoting.

2. Specification

       SUBROUTINE F04ARF (A, IA, B, N, C, WKSPCE, IFAIL)
       INTEGER          IA, N, IFAIL
       DOUBLE PRECISION A(IA,*), B(*), C(*), WKSPCE(*)

3. Description

Given a set of linear equations, Ax=b, the routine first computes
an LU factorization of A with partial pivoting, PA=LU, where P is
a permutation matrix, L is lower triangular and U is unit upper
triangular. The approximate solution x is found by forward and
backward substitution in Ly=Pb and Ux=y, where b is the right-
hand side.

4. References

[1]   Wilkinson J H and Reinsch C (1971) Handbook for Automatic
      Computation II, Linear Algebra. Springer-Verlag.

5. Parameters

 1:  A(IA,*) -- DOUBLE PRECISION array              Input/Output
     Note: the second dimension of the array A must be at least
     max(1,N).
     On entry: the n by n matrix A. On exit: A is overwritten by
     the lower triangular matrix L and the off-diagonal elements
     of the upper triangular matrix U. The unit diagonal elements
     of U are not stored.

 2:  IA -- INTEGER                                        Input
     On entry:
     the first dimension of the array A as declared in the
     (sub)program from which F04ARF is called.
     Constraint: IA >= max(1,N).
```

```
 3:  B(*) -- DOUBLE PRECISION array                          Input
     Note: the dimension of the array B must be at least
     max(1,N).
     On entry: the right-hand side vector b.

 4:  N -- INTEGER                                            Input
     On entry: n, the order of the matrix A. Constraint: N >= 0.

 5:  C(*) -- DOUBLE PRECISION array                         Output
     Note: the dimension of the array C must be at least
     max(1,N).
     On exit: the solution vector x.

 6:  WKSPCE(*) -- DOUBLE PRECISION array                 Workspace
     Note: the dimension of the array WKSPCE must be at least
     max(1,N).

 7:  IFAIL -- INTEGER                                  Input/Output
     On entry: IFAIL must be set to 0, -1 or 1. For users not
     familiar with this parameter (described in the Essential
     Introduction) the recommended value is 0.

     On exit: IFAIL = 0 unless the routine detects an error (see
     Section 6).

6. Error Indicators and Warnings

Errors detected by the routine:

If on entry IFAIL = 0 or -1, explanatory error messages are
output on the current error message unit (as defined by X04AAF).

IFAIL= 1
     The matrix A is singular, possibly due to rounding errors.

IFAIL= 2
     On entry N < 0,

     or       IA < max(1,N).

7. Accuracy

The accuracy of the computed solution depends on the conditioning
of the original matrix. For a detailed error analysis see
Wilkinson and Reinsch [1] page 107.
```

```
8. Further Comments

                                                                      3
The time taken by the routine is approximately proportional to n

Unless otherwise stated in the Users' Note for your
implementation, the routine may be called with the same actual
array supplied for parameters B and C, in which case the solution
vector will overwrite the right-hand side. However this is not
standard Fortran 77, and may not work on all systems.

9. Example

To solve the set of linear equations Ax=b where

                            ( 33  16  72)
                          A=(-24 -10 -57)
                            ( -8  -4 -17)

and

                              (-359)
                            b=( 281).
                              (  85)

The example program is not reproduced here. The source code for
all example programs is distributed with the NAG Foundation
Library software and should be available on-line.
```

%%%

```
F04 -- Simultaneous Linear Equations                          F04ASF
        F04ASF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for
your implementation to check implementation-dependent details.
The symbol (*) after a NAG routine name denotes a routine that is
not included in the Foundation Library.

1. Purpose

F04ASF calculates the accurate solution of a set of real
symmetric positive-definite linear equations with a single right-
hand side,  Ax=b, using a Cholesky factorization and iterative
```

refinement.

2. Specification

```
SUBROUTINE F04ASF (A, IA, B, N, C, WK1, WK2, IFAIL)
INTEGER          IA, N, IFAIL
DOUBLE PRECISION A(IA,*), B(*), C(*), WK1(*), WK2(*)
```

3. Description

Given a set of real linear equations Ax=b, where A is a symmetric positive-definite matrix, the routine first computes a Cholesky factorization of A as $A=LL^T$ where L is lower triangular. An approximation to x is found by forward and backward substitution. The residual vector r=b-Ax is then calculated using additional precision and a correction d to x is found by solving $LL^T d=r$. x is then replaced by x+d, and this iterative refinement of the solution is repeated until machine accuracy is obtained.

4. References

[1] Wilkinson J H and Reinsch C (1971) Handbook for Automatic Computation II, Linear Algebra. Springer-Verlag.

5. Parameters

1: A(IA,*) -- DOUBLE PRECISION array Input/Output
Note: the second dimension of the array A must be at least max(1,N).
On entry: the upper triangle of the n by n positive-definite symmetric matrix A. The elements of the array below the diagonal need not be set. On exit: the elements of the array below the diagonal are overwritten; the upper triangle of A is unchanged.

2: IA -- INTEGER Input
On entry:
the first dimension of the array A as declared in the (sub)program from which F04ASF is called.
Constraint: IA >= max(1,N).

3: B(*) -- DOUBLE PRECISION array Input
Note: the dimension of the array B must be at least max(1,N).

```
     On entry: the right-hand side vector b.

 4:  N -- INTEGER                                              Input
     On entry: n, the order of the matrix A. Constraint: N >= 0.

 5:  C(*) -- DOUBLE PRECISION array                            Output
     Note: the dimension of the array C must be at least
     max(1,N).
     On exit: the solution vector x.

 6:  WK1(*) -- DOUBLE PRECISION array                       Workspace
     Note: the dimension of the array WK1 must be at least
     max(1,N).

 7:  WK2(*) -- DOUBLE PRECISION array                       Workspace
     Note: the dimension of the array WK2 must be at least
     max(1,N).

 8:  IFAIL -- INTEGER                                    Input/Output
     On entry: IFAIL must be set to 0, -1 or 1. For users not
     familiar with this parameter (described in the Essential
     Introduction) the recommended value is 0.

     On exit: IFAIL = 0 unless the routine detects an error (see
     Section 6).

6. Error Indicators and Warnings

Errors detected by the routine:

If on entry IFAIL = 0 or -1, explanatory error messages are
output on the current error message unit (as defined by X04AAF).

IFAIL= 1
     The matrix A is not positive-definite, possibly due to
     rounding errors.

IFAIL= 2
     Iterative refinement fails to improve the solution, i.e.,
     the matrix A is too ill-conditioned.

IFAIL= 3
     On entry N < 0,

     or       IA < max(1,N).
```

```
7. Accuracy

The computed solutions should be correct to full machine
accuracy. For a detailed error analysis see Wilkinson and Reinsch
[1] page 39.

8. Further Comments

                                                                    3
The time taken by the routine is approximately proportional to n

The routine must not be called with the same name for parameters
B and C.

9. Example

To solve the set of linear equations Ax=b where

                          (5  7  6  5)
                          (7 10  8  7)
                        A=(6  8 10  9)
                          (5  7  9 10)

and


                             (23)
                             (32)
                           b=(33).
                             (31)

The example program is not reproduced here. The source code for
all example programs is distributed with the NAG Foundation
Library software and should be available on-line.
```

%%

```
F04 -- Simultaneous Linear Equations                        F04ATF
        F04ATF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for
your implementation to check implementation-dependent details.
The symbol (*) after a NAG routine name denotes a routine that is
not included in the Foundation Library.

1. Purpose
```

F04ATF calculates the accurate solution of a set of real linear equations with a single right-hand side, using an LU factorization with partial pivoting, and iterative refinement.

2. Specification

```
      SUBROUTINE F04ATF (A, IA, B, N, C, AA, IAA, WKS1, WKS2,
     1                   IFAIL)
      INTEGER           IA, N, IAA, IFAIL
      DOUBLE PRECISION A(IA,*), B(*), C(*), AA(IAA,*), WKS1(*),
     1                 WKS2(*)
```

3. Description

Given a set of real linear equations, Ax=b, the routine first computes an LU factorization of A with partial pivoting, PA=LU, where P is a permutation matrix, L is lower triangular and U is unit upper triangular. An approximation to x is found by forward and backward substitution in Ly=Pb and Ux=y. The residual vector r=b-Ax is then calculated using additional precision, and a correction d to x is found by solving LUd=r. x is replaced by x+d , and this iterative refinement of the solution is repeated until full machine accuracy is obtained.

4. References

[1] Wilkinson J H and Reinsch C (1971) Handbook for Automatic Computation II, Linear Algebra. Springer-Verlag.

5. Parameters

1: A(IA,*) -- DOUBLE PRECISION array Input
Note: the second dimension of the array A must be at least max(1,N).
On entry: the n by n matrix A.

2: IA -- INTEGER Input
On entry:
the first dimension of the array A as declared in the (sub)program from which F04ATF is called.
Constraint: IA >= max(1,N).

3: B(*) -- DOUBLE PRECISION array Input
Note: the dimension of the array B must be at least max(1,N).

```
    On entry: the right-hand side vector b.

 4: N -- INTEGER                                                  Input
    On entry: n, the order of the matrix A. Constraint: N >= 0.

 5: C(*) -- DOUBLE PRECISION array                                Output
    Note: the dimension of the array C must be at least
    max(1,N).
    On exit: the solution vector x.

 6: AA(IAA,*) -- DOUBLE PRECISION array                           Output
    Note: the second dimension of the array AA must be at least
    max(1,N).
    On exit: the triangular factors L and U, except that the
    unit diagonal elements of U are not stored.

 7: IAA -- INTEGER                                                Input
    On entry:
    the first dimension of the array AA as declared in the
    (sub)program from which F04ATF is called.
    Constraint: IAA >= max(1,N).

 8: WKS1(*) -- DOUBLE PRECISION array                         Workspace
    Note: the dimension of the array WKS1 must be at least
    max(1,N).

 9: WKS2(*) -- DOUBLE PRECISION array                         Workspace
    Note: the dimension of the array WKS2 must be at least
    max(1,N).

10: IFAIL -- INTEGER                                       Input/Output
    On entry: IFAIL must be set to 0, -1 or 1. For users not
    familiar with this parameter (described in the Essential
    Introduction) the recommended value is 0.

    On exit: IFAIL = 0 unless the routine detects an error (see
    Section 6).

6. Error Indicators and Warnings

Errors detected by the routine:

If on entry IFAIL = 0 or -1, explanatory error messages are
output on the current error message unit (as defined by X04AAF).

IFAIL= 1
```

```
     The matrix A is singular, possibly due to rounding errors.

IFAIL= 2

     Iterative refinement fails to improve the solution, i.e.,
     the matrix A is too ill-conditioned.

IFAIL= 3
     On entry N < 0,

     or       IA < max(1,N),

     or       IAA < max(1,N).

7. Accuracy

The computed solutions should be correct to full machine
accuracy. For a detailed error analysis see Wilkinson and Reinsch
[1] page 107.

8. Further Comments

                                                                    3
The time taken by the routine is approximately proportional to n

The routine must not be called with the same name for parameters
B and C.

9. Example

To solve the set of linear equations Ax=b where

                          ( 33  16  72)
                        A=(-24 -10 -57)
                          ( -8  -4 -17)

and


                            (-359)
                          b=( 281).
                            (  85)

The example program is not reproduced here. The source code for
all example programs is distributed with the NAG Foundation
Library software and should be available on-line.
```

%%

F04 -- Simultaneous Linear Equations F04AXF
F04AXF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for your implementation to check implementation-dependent details. The symbol (*) after a NAG routine name denotes a routine that is not included in the Foundation Library.

1. Purpose

F04AXF calculates the approximate solution of a set of real sparse linear equations with a single right-hand side, Ax=b or
T
A x=b, where A has been factorized by F01BRF or F01BSF.

2. Specification

```
 SUBROUTINE F04AXF (N, A, LICN, ICN, IKEEP, RHS, W, MTYPE,
1                   IDISP, RESID)
 INTEGER          N, LICN, ICN(LICN), IKEEP(5*N), MTYPE,
1                 IDISP(2)
 DOUBLE PRECISION A(LICN), RHS(N), W(N), RESID
```

3. Description

T
To solve a system of real linear equations Ax=b or A x=b, where A is a general sparse matrix, A must first be factorized by F01BRF or F01BSF. F04AXF then computes x by block forward or backward substitution using simple forward and backward substitution within each diagonal block.

The method is fully described in Duff [1].

4. References

[1] Duff I S (1977) MA28 -- a set of Fortran subroutines for sparse unsymmetric linear equations. A.E.R.E. Report R.8730. HMSO.

5. Parameters

1: N -- INTEGER Input

```
     On entry: n, the order of the matrix A.

 2:  A(LICN) -- DOUBLE PRECISION array                          Input
     On entry: the non-zero elements in the factorization of the
     matrix A, as returned by F01BRF or F01BSF.

 3:  LICN -- INTEGER                                            Input
     On entry:
     the dimension of the arrays A and ICN as declared in the
     (sub)program from which F04AXF is called.

 4:  ICN(LICN) -- INTEGER array                                 Input
     On entry: the column indices of the non-zero elements of
     the factorization, as returned by F01BRF or F01BSF.

 5:  IKEEP(5*N) -- INTEGER array                                Input
     On entry: the indexing information about the factorization,
     as returned by F01BRF or F01BSF.

 6:  RHS(N) -- DOUBLE PRECISION array                  Input/Output
     On entry: the right-hand side vector b. On exit: RHS is
     overwritten by the solution vector x.

 7:  W(N) -- DOUBLE PRECISION array                       Workspace

 8:  MTYPE -- INTEGER                                           Input
     On entry: MTYPE specifies the task to be performed:
          if MTYPE = 1, solve Ax=b,

                                T
          if MTYPE /= 1, solve A x=b.

 9:  IDISP(2) -- INTEGER array                                  Input
     On entry: the values returned in IDISP by F01BRF.

10:  RESID -- DOUBLE PRECISION                                 Output
     On exit: the value of the maximum residual,
              --
     max(|b - > a  x |), over all the unsatisfied equations, in
           i  -- ij j
              j
     case F01BRF or F01BSF has been used to factorize a singular
     or rectangular matrix.

6. Error Indicators and Warnings
```

None.

7. Accuracy

The accuracy of the computed solution depends on the conditioning of the original matrix. Since F04AXF is always used with either F01BRF or F01BSF, the user is recommended to set GROW = .TRUE. on entry to these routines and to examine the value of W(1) on exit (see the routine documents for F01BRF and F01BSF). For a detailed error analysis see Duff [1] page 17.

If storage for the original matrix is available then the error can be estimated by calculating the residual

$$r=b-Ax \quad (\text{or } b-A^T x)$$

and calling F04AXF again to find a correction (delta) for x by solving

$$A(\text{delta})=r \quad (\text{or } A^T(\text{delta})=r).$$

8. Further Comments

If the factorized form contains (tau) non-zeros (IDISP(2) = (tau)) then the time taken is very approximately 2(tau) times longer than the inner loop of full matrix code. Some advantage is taken of zeros in the right-hand side when solving $A^T x=b$ (MTYPE /= 1).

9. Example

To solve the set of linear equations Ax=b where

$$A=\begin{pmatrix} 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & -1 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ -2 & 0 & 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & -1 & 2 & -3 \\ -1 & -1 & 0 & 0 & 0 & 6 \end{pmatrix}$$

and

```
                                    (12)
                                    (18)
                                b=( 3).
                                    (-6)
                                    ( 0)

          The non-zero elements of A and indexing information are read in
          by the program, as described in the document for F01BRF.

          The example program is not reproduced here. The source code for
          all example programs is distributed with the NAG Foundation
          Library software and should be available on-line.

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

          F04 -- Simultaneous Linear Equations                      F04FAF
                  F04FAF -- NAG Foundation Library Routine Document

          Note: Before using this routine, please read the Users' Note for
          your implementation to check implementation-dependent details.
          The symbol (*) after a NAG routine name denotes a routine that is
          not included in the Foundation Library.

          1. Purpose

          F04FAF calculates the approximate solution of a set of real
          symmetric positive-definite tridiagonal linear equations.

          2. Specification

                 SUBROUTINE F04FAF (JOB, N, D, E, B, IFAIL)
                 INTEGER          JOB, N, IFAIL
                 DOUBLE PRECISION D(N), E(N), B(N)

          3. Description

          F04FAF is based upon the Linpack routine DPTSL (see Dongarra et
          al [1]) and solves the equations

                                       Tx=b,

          where T is a real n by n symmetric positive-definite tridiagonal
          matrix, using a modified symmetric Gaussian elimination algorithm
                                T
          to factorize T as T=MKM , where K is diagonal and M is a matrix
          of multipliers as described in Section 8.
```

When the input parameter JOB is supplied as 1, then the routine assumes that a previous call to F04FAF has already factorized T; otherwise JOB must be supplied as 0.

4. References

[1] Dongarra J J, Moler C B, Bunch J R and Stewart G W (1979) LINPACK Users' Guide. SIAM, Philadelphia.

5. Parameters

1: JOB -- INTEGER Input
On entry: specifies the job to be performed by F04FAF as follows:
JOB = 0
The matrix T is factorized and the equations Tx=b are solved for x.

JOB = 1
The matrix T is assumed to have already been factorized by a previous call to F04FAF with JOB = 0; the equations Tx=b are solved for x.

2: N -- INTEGER Input
On entry: n, the order of the matrix T. Constraint: N >= 1.

3: D(N) -- DOUBLE PRECISION array Input/Output
On entry: if JOB = 0, D must contain the diagonal elements of T. If JOB = 1, D must contain the diagonal matrix K, as returned by a previous call of F04FAF with JOB = 0. On exit: if JOB = 0, D is overwritten by the diagonal matrix K of the factorization. If JOB = 1, D is unchanged.

4: E(N) -- DOUBLE PRECISION array Input/Output
On entry: if JOB = 0, E must contain the super-diagonal elements of T, stored in E(2) to E(n). If JOB = 1, E must contain the off-diagonal elements of the matrix M, as returned by a previous call of F04FAF with JOB = 0. E(1) is not used. On exit: if JOB = 0, E(2) to E(n) are overwritten by the off-diagonal elements of the matrix M of the factorization. If JOB = 1, E is unchanged.

5: B(N) -- DOUBLE PRECISION array Input/Output
On entry: the right-hand side vector b. On exit: B is overwritten by the solution vector x.

```
 6:  IFAIL -- INTEGER                                    Input/Output
     On entry: IFAIL must be set to 0, -1 or 1. For users not
     familiar with this parameter (described in the Essential
     Introduction) the recommended value is 0.

     On exit: IFAIL = 0 unless the routine detects an error (see
     Section 6).

6. Error Indicators and Warnings

Errors detected by the routine:

IFAIL= 1
     On entry N < 1,

     or       JOB /= 0 or 1.

IFAIL= 2
     The matrix T is either not positive-definite or is nearly
     singular. This failure can only occur when JOB = 0 and
     inspection of the elements of D will give an indication of
     why failure has occurred. If an element of D is close to
     zero, then T is probably nearly singular; if an element of D
     is negative but not close to zero, then T is not positive-
     definite.

IFAILOverflow
     If overflow occurs during the execution of this routine,
     then either T is very nearly singular or an element of the
     right-hand side vector b is very large. In this latter case
     the equations should be scaled so that no element of b is
     very large. Note that to preserve symmetry it is necessary
                                                          T
     to scale by a transformation of the form (PTP )b=Px, where P
     is a diagonal matrix.

IFAILUnderflow
     Any underflows that occur during the execution of this
     routine are harmless.

7. Accuracy

The computed factorization (see Section 8) will satisfy the
equation
```

$$MKM^T = T+E$$

where $||E||_p <= 2(\text{epsilon})||T||_p$, p=1,F,infty,

(epsilon) being the machine precision. The computed solution of

the equations Tx=b, say x, will satisfy an equation of the form

$$(T+F)x=b,$$

where F can be expected to satisfy a bound of the form

$$||F|| <= (\text{alpha})(\text{epsilon})||T||,$$

(alpha) being a modest constant. This implies that the relative

error in x satisfies

$$\frac{||x-x||}{||x||} <= c(T)(\text{alpha})(\text{epsilon}),$$

where c(T) is the condition number of T with respect to

inversion. Thus if T is nearly singular, x can be expected to have a large relative error.

8. Further Comments

The time taken by the routine is approximately proportional to n.

The routine eliminates the off-diagonal elements of T by simultaneously performing symmetric Gaussian elimination from the top and the bottom of T. The result is that T is factorized as

$$T=MKM^T ,$$

where K is a diagonal matrix and M is a matrix of the form

```
                (1  0  0 ..  0    0 0    .. 0 0    0 )
                (m  1  0 ..  0    0 0    .. 0 0    0 )
                ( 2                                  )
                (0  m  1 ..  0    0 0    .. 0 0    0 )
                (    3                               )
                (.  .  . ..  .    . .    .. . .    . )
                (.  .  . ..  .    . .    .. . .    . )
                (0  0  0 ..  1    0 0    .. 0 0    0 )
              M=(0  0  0 ..  m    1 m    .. 0 0    0 )
                (             j+1    j+2             )
                (0  0  0 ..  0    0 ..   1 0  0    0 )
                (.  .  . ..  .    . .    .. . .    . )
                (.  .  . ..  .    . .    .. . .    . )
                (0  0  0 ..  0    0 0    .. 1 m    0 )
                (                               n-1  )
                (0  0  0 ..  0    0 0    .. 0 1    m )
                (                                   n)
                (0  0  0 ..  0    0 0 .    . 0 0    1 )
```

j being the integer part of n/2. (For example when n=5,j=2.) The diagonal elements of K are returned in D with k_i in the ith element of D and m_i is returned in the ith element of E.

The routine fails with IFAIL = 2 if any diagonal element of K is non-positive. It should be noted that T may be nearly singular even if all the diagonal elements of K are positive, but in this case at least one element of K is almost certain to be small relative to |||T|||. If there is any doubt as to whether or not T is nearly singular, then the user should consider examining the diagonal elements of K.

9. Example

To solve the symmetric positive-definite equations

$$Tx = b_1$$

and

$$Tx =$$

2 2

where

```
                ( 4 -2  0  0 0)       ( 6)      (10)
                (-2 10 -6  0 0)       ( 9)      ( 4)
              T=( 0 -6 29 15 0),   b =( 2),  b =( 9).
                ( 0  0 15 25 8)     1 (14)    2 (65)
                ( 0  0  0  8 5)       ( 7)      (23)
```

The equations are solved by two calls to F04FAF, the first with JOB = 0 and the second, using the factorization from the first call, with JOB = 1.

The example program is not reproduced here. The source code for all example programs is distributed with the NAG Foundation Library software and should be available on-line.

%%%

F04 -- Simultaneous Linear Equations F04JGF
F04JGF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for your implementation to check implementation-dependent details. The symbol (*) after a NAG routine name denotes a routine that is not included in the Foundation Library.

1. Purpose

F04JGF finds the solution of a linear least-squares problem, Ax=b , where A is a real m by n (m>=n) matrix and b is an m element vector. If the matrix of observations is not of full rank, then the minimal least-squares solution is returned.

2. Specification

```
      SUBROUTINE F04JGF (M, N, A, NRA, B, TOL, SVD, SIGMA,
     1                   IRANK, WORK, LWORK, IFAIL)
      INTEGER           M, N, NRA, IRANK, LWORK, IFAIL
      DOUBLE PRECISION A(NRA,N), B(M), TOL, SIGMA, WORK(LWORK)
      LOGICAL           SVD
```

3. Description

The minimal least-squares solution of the problem Ax=b is the

vector x of minimum (Euclidean) length which minimizes the length
of the residual vector r=b-Ax.

The real m by n (m>=n) matrix A is factorized as

$$A=Q\begin{pmatrix}U\\0\end{pmatrix}$$

where Q is an m by m orthogonal matrix and U is an n by n upper
triangular matrix. If U is of full rank, then the least-squares
solution is given by

$$x=(U^{-1}\ \ 0)Q^{T}b.$$

If U is not of full rank, then the singular value decomposition
of U is obtained so that U is factorized as

$$U=RDP^{T},$$

where R and P are n by n orthogonal matrices and D is the n by n
diagonal matrix

$$D=\mathrm{diag}(\sigma_1,\sigma_2,\ldots,\sigma_n),$$

with $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n \geq 0$, these being the singular
values of A. If the singular values $\sigma_{k+1},\ldots,\sigma_n$ are
negligible, but σ_k is not negligible, relative to the data
errors in A, then the rank of A is taken to be k and the minimal
least-squares solution is given by

$$x=P\begin{pmatrix}S^{-1} & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}R^{T}\\ 0\end{pmatrix}Q^{T}b,$$

where $S=\mathrm{diag}(\sigma_1,\sigma_2,\ldots,\sigma_k)$.

This routine obtains the factorizations by a call to F02WDF(*).

The routine also returns the value of the standard error

```
           /  T
          /  r r
(sigma)=  /   --- ,     if m>k,
        \/    m-k

                                T
       =  0,              if m=k, r r being the residual sum of
                          squares and k the rank of A.
```

4. References

[1] Lawson C L and Hanson R J (1974) Solving Least-squares Problems. Prentice-Hall.

5. Parameters

1: M -- INTEGER Input
On entry: m, the number of rows of A. Constraint: M >= N.

2: N -- INTEGER Input
On entry: n, the number of columns of A. Constraint: 1 <= N <= M.

3: A(NRA,N) -- DOUBLE PRECISION array Input/Output
On entry: the m by n matrix A. On exit: if SVD is returned as .FALSE., A} is overwritten by details of the QU factorization of A (see F02WDF(*) for further details). If SVD is returned as .TRUE., the first n rows of A are overwritten by the right-hand singular vectors, stored by rows; and the remaining rows of the array are used as workspace.

4: NRA -- INTEGER Input
On entry:
the first dimension of the array A as declared in the (sub)program from which F04JGF is called.
Constraint: NRA >= M.

5: B(M) -- DOUBLE PRECISION array Input/Output
On entry: the right-hand side vector b. On exit: the first n elements of B contain the minimal least-squares solution vector x. The remaining m-n elements are used for workspace.

6: TOL -- DOUBLE PRECISION Input

On entry: a relative tolerance to be used to determine the rank of A. TOL should be chosen as approximately the largest relative error in the elements of A. For example, if the elements of A are correct to about 4 significant figures
then TOL should be set to about 5*10^-4. See Section 8 for a description of how TOL is used to determine rank. If TOL is outside the range ((epsilon),1.0), where (epsilon) is the machine precision, then the value (epsilon) is used in place of TOL. For most problems this is unreasonably small.

7: SVD -- LOGICAL Output
On exit: SVD is returned as .FALSE. if the least-squares solution has been obtained from the QU factorization of A. In this case A is of full rank. SVD is returned as .TRUE. if the least-squares solution has been obtained from the singular value decomposition of A.

8: SIGMA -- DOUBLE PRECISION Output

On exit: the standard error, i.e., the value $\sqrt{r^T r/(m-k)}$ when m>k, and the value zero when m=k. Here r is the residual vector b-Ax and k is the rank of A.

9: IRANK -- INTEGER Output
On exit: k, the rank of the matrix A. It should be noted that it is possible for IRANK to be returned as n and SVD to be returned as .TRUE.. This means that the matrix U only just failed the test for non-singularity.

10: WORK(LWORK) -- DOUBLE PRECISION array Output
On exit: if SVD is returned as .FALSE., then the first n elements of WORK contain information on the QU factorization of A (see parameter A above and F02WDF(*)), and WORK(n+1) contains the condition number $||U||_E ||U^{-1}||_E$ of the upper triangular matrix U.

If SVD is returned as .TRUE., then the first n elements of WORK contain the singular values of A arranged in descending order and WORK(n+1) contains the total number of iterations taken by the QR algorithm. The rest of WORK is used as workspace.

```
11:  LWORK -- INTEGER                                          Input
     On entry:
     the dimension of the array WORK as declared in the
     (sub)program from which F04JGF is called.
     Constraint: LWORK >= 4*N.

12:  IFAIL -- INTEGER                                  Input/Output
     On entry: IFAIL must be set to 0, -1 or 1. For users not
     familiar with this parameter (described in the Essential
     Introduction) the recommended value is 0.

     On exit: IFAIL = 0 unless the routine detects an error (see
     Section 6).

6. Error Indicators and Warnings

Errors detected by the routine:

IFAIL= 1
     On entry N < 1,

     or       M < N,

     or       NRA < M,

     or       LWORK < 4*N.

IFAIL= 2
     The QR algorithm has failed to converge to the singular
     values in 50*N iterations. This failure can only happen when
     the singular value decomposition is employed, but even then
     it is not likely to occur.

7. Accuracy

                                      T
The computed factors Q, U, R, D and P  satisfy the relations

                   (U)        (R 0)(D) T
                 Q(0)=A+E , Q(0 I)(0)P =A+F,

where

                ||E|| <=c (epsilon)||A|| ,
                    2   1              2
```

```
                  ||F|| <=c (epsilon)||A|| ,
                       2   2              2

(epsilon) being the machine precision, and c  and c  being modest
                                            1      2
functions of m and n. Note that ||A|| =(sigma) .
                                     2        1

For a fuller discussion, covering the accuracy of the solution x
see Lawson and Hanson [1], especially pp 50 and 95.

8. Further Comments

If the least-squares solution is obtained from the QU
factorization then the time taken by the routine is approximately
                 2
proportional to n (3m-n). If the least-squares solution is
obtained from the singular value decomposition then the time
                                       2
taken is approximately proportional to n (3m+19n). The
approximate proportionality factor is the same in each case.

This routine is column biased and so is suitable for use in paged
environments.

Following the QU factorization of A the condition number

                                -1
                    c(U)=||U|| ||U  ||
                              E       E

is determined and if c(U) is such that

                          c(U)*TOL>1.0

then U is regarded as singular and the singular values of A are
computed. If this test is not satisfied, U is regarded as non-
singular and the rank of A is set to n. When the singular values
are computed the rank of A, say k, is returned as the largest
integer such that

                     (sigma) >TOL*(sigma) ,
                            k            1
```

```
unless (sigma) =0 in which case k is returned as zero. That is,
               1
singular values which satisfy (sigma) <=TOL*(sigma)  are regarded
                                     i             1
as negligible because relative perturbations of order TOL can
make such singular values zero.

9. Example

To obtain a least-squares solution for r=b-Ax, where

                (0.05  0.05 0.25 -0.25)      (1)
                (0.25  0.25 0.05 -0.05)      (2)
                (0.35  0.35 1.75 -1.75)      (3)
              A=(1.75  1.75 0.35 -0.35),   B=(4)
                (0.30 -0.30 0.30  0.30)      (5)
                (0.40 -0.40 0.40  0.40)      (6)

                                        -4
and the value TOL is to be taken as 5*10  .

The example program is not reproduced here. The source code for
all example programs is distributed with the NAG Foundation
Library software and should be available on-line.
```

%%

```
F04 -- Simultaneous Linear Equations                        F04MAF
        F04MAF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for
your implementation to check implementation-dependent details.
The symbol (*) after a NAG routine name denotes a routine that is
not included in the Foundation Library.

1. Purpose

To solve a sparse symmetric positive-definite system of linear
equations, Ax=b, using a pre-conditioned conjugate gradient
method, where A has been factorized by F01MAF.

2. Specification

       SUBROUTINE F04MAF (N, NZ, A, LICN, IRN, LIRN, ICN, B, ACC,
      1                   NOITS, WKEEP, WORK, IKEEP, INFORM,
      2                   IFAIL)
```

```
       INTEGER          N, NZ, LICN, IRN(LIRN), LIRN, ICN(LICN),
      1                 NOITS(2), IKEEP(2*N), INFORM(4), IFAIL
       DOUBLE PRECISION A(LICN), B(N), ACC(2), WKEEP(3*N), WORK
      1                 (3*N)
```

3. Description

F04MAF solves the n linear equations

$$Ax=b, \tag{1}$$

where A is a sparse symmetric positive-definite matrix, following the incomplete Cholesky factorization by F01MAF, given by

$$C=PLDL^{T}P^{T}, \quad WAW=C+E,$$

where P is a permutation matrix, L is a unit lower triangular matrix, D is a diagonal matrix with positive diagonal elements, E is an error matrix representing elements dropped during the factorization and diagonal elements that have been modified to ensure that C is positive-definite, and W is a diagonal matrix, chosen to make the diagonal elements of WAW unity.

Equation (1) is solved by applying a pre-conditioned conjugate gradient method to the equations

$$(WAW)(W^{-1}x)=Wb, \tag{2}$$

using C as the pre-conditioning matrix. Details of the conjugate gradient method are given in Munksgaard [1].

The iterative procedure is terminated if

$$||Wr||_2 <=(eta), \tag{3}$$

where r is the residual vector r=b-Ax, $||r||_2$ denotes the Euclidean length of r, (eta) is a user-supplied tolerance and x is the current approximation to the solution. Notice that

$$Wr=Wb-(WAW)(W^{-1}x)$$

so that Wr is the residual of the normalised equations (2).

F04MAF is based on the Harwell Library routine MA31B.

4. References

[1] Munksgaard N (1980) Solving Sparse Symmetric Sets of Linear Equations by Pre-conditioned Conjugate Gradients. ACM Trans. Math. Softw. 6 206--219.

5. Parameters

1: N -- INTEGER Input
On entry: n, the order of the matrix A. Constraint: N >= 1.

2: NZ -- INTEGER Input
On entry: the number of non-zero elements in the upper triangular part of the matrix A, including the number of elements on the leading diagonal. Constraint: NZ >= N.

3: A(LICN) -- DOUBLE PRECISION array Input
On entry: the first LROW elements, where LROW is the value supplied in INFORM(1), must contain details of the factorization, as returned by F01MAF.

4: LICN -- INTEGER Input
On entry: the length of the array A, as declared in the (sub)program from which F04MAF is called. It need never be larger than the value of LICN supplied to F01MAF. Constraint: LICN >= INFORM(1).

5: IRN(LIRN) -- INTEGER array Input
On entry: the first LCOL elements, where LCOL is the value supplied in INFORM(2), must contain details of the factorization, as returned by F01MAF.

6: LIRN -- INTEGER Input
On entry: the length of the array IRN, as declared in the (sub)program from which F04MAF is called. It need never be larger than the value of LIRN supplied to F01MAF. Constraint: LIRN >= INFORM(2).

7: ICN(LICN) -- INTEGER array Input
On entry: the first LROW elements, where LROW is the value supplied in INFORM(1), must contain details of the factorization, as returned by F01MAF.

8: B(N) -- DOUBLE PRECISION array Input/Output
On entry: the right-hand side vector b. On exit: B is overwritten by the solution vector x.

9: ACC(2) -- DOUBLE PRECISION array Input/Output
On entry: ACC(1) specifies the tolerance for convergence, (eta), in equation (3) of Section 3. If ACC(1) is outside the range [(epsilon),1], where (epsilon) is the machine precision, then the value (epsilon) is used in place of ACC (1). ACC(2) need not be set. On exit: ACC(2) contains the actual value of ||Wr|| at the final point. ACC(1) is
2
unchanged.

10: NOITS(2) -- INTEGER array Input/Output
On entry: NOITS(1) specifies the maximum permitted number of iterations. If NOITS(1) < 1, then the value 100 is used in its place. NOITS(2) need not be set. On exit: NOITS(2) contains the number of iterations taken to converge. NOITS (1) is unchanged.

11: WKEEP(3*N) -- DOUBLE PRECISION array Input
On entry: WKEEP must be unchanged from the previous call of F01MAF.

12: WORK(3*N) -- DOUBLE PRECISION array Output
On exit: WORK(1) contains a lower bound for the condition number of A. The rest of the array is used for workspace.

13: IKEEP(2*N) -- INTEGER array Input
On entry: IKEEP must be unchanged from the previous call of F01MAF.

14: INFORM(4) -- INTEGER array Input
On entry: INFORM must be unchanged from the previous call of F01MAF.

15: IFAIL -- INTEGER Input/Output
For this routine, the normal use of IFAIL is extended to control the printing of error and warning messages as well as specifying hard or soft failure (see the Essential Introduction).

Before entry, IFAIL must be set to a value with the decimal expansion cba, where each of the decimal digits c, b and a

```
     must have a value of 0 or 1.
     a=0   specifies hard failure, otherwise soft failure;

     b=0   suppresses error messages, otherwise error messages
           will be printed (see Section 6);

     c=0   suppresses warning messages, otherwise warning
           messages will be printed (see Section 6).
     The recommended value for inexperienced users is 110 (i.e.,
     hard failure with all messages printed).

     Unless the routine detects an error (see Section 6), IFAIL
     contains 0 on exit.

6. Error Indicators and Warnings

Errors detected by the routine:

For each error, an explanatory error message is output on the
current error message unit (as defined by X04AAF), unless
suppressed by the value of IFAIL on entry.

IFAIL= 1
     On entry N < 1,

     or       NZ < N,

     or       LICN < INFORM(1),

     or       LIRN < INFORM(2).

IFAIL= 2
     Convergence has not taken place within the requested NOITS
     (1) number of iterations. ACC(2) gives the value ||Wr|| ,
                                                        2
     for the final point. Either too few iterations have been
     allowed, or the requested convergence criterion cannot be
     met.

IFAIL= 3
     The matrix A is singular, or nearly singular. Singularity
     has been detected during the conjugate gradient iterations,
     so that the computations are not complete.

IFAIL= 4
     The matrix A is singular, or nearly singular. The message
```

output on the current error message channel will include an estimate of the condition number of A. In the case of soft failure an approximate solution is returned such that the value $||Wr||_2$ is given by ACC(2) and the estimate (a lower bound) of the condition number is returned in WORK(1).

7. Accuracy

On successful return, or on return with IFAIL = 2 or IFAIL = 4 the computed solution will satisfy equation (3) of Section 3, with (eta) = ACC(2).

8. Further Comments

The time taken by the routine will depend upon the sparsity of the factorization and the number of iterations required. The number of iterations will be affected by the nature of the factorization supplied by F01MAF. The more incomplete the factorization, the higher the number of iterations required by F04MAF.

When the solution of several systems of equations, all with the same matrix of coefficients, A, is required, then F01MAF need be called only once to factorize A. This is illustrated in the context of an eigenvalue problem in the example program for F02FJF.

9. Example

The example program illustrates the use of F01MAF in conjunction with F04MAF to solve the 16 linear equations Ax=b, where

$$
A=\begin{pmatrix}
1 & a & & a & & & & & & & & & & & & \\
a & 1 & a & & a & & & & & & & & & & & \\
 & a & 1 & a & & a & & & & & & & & & & \\
 & & a & 1 & 0 & & a & & & & & & & & & \\
a & & & 0 & 1 & a & & a & & & & & & & & \\
 & a & & & a & 1 & a & & a & & & & & & & \\
 & & a & & & a & 1 & a & & a & & & & & & \\
 & & & a & & & a & 1 & 0 & & a & & & & & \\
 & & & & a & & & 0 & 1 & a & & a & & & & \\
 & & & & & a & & & a & 1 & a & & a & & & \\
 & & & & & & a & & & a & 1 & a & & a & & \\
 & & & & & & & a & & & a & 1 & 0 & & a & \\
 & & & & & & & & a & & & 0 & 1 & a & &
\end{pmatrix}
$$

$$\begin{pmatrix} a & a & 1 & a & \\ a & a & 1 & a & \\ & a & a & 1 & a \\ & & a & a & 1 \end{pmatrix}$$

where $a = -\frac{1}{4}$.

$$b = \left(-\frac{1}{2}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{2}, -\frac{1}{4}, 0, 0, -\frac{1}{4}, -\frac{1}{4}, 0, 0, -\frac{1}{4}, -\frac{1}{2}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{2}\right)^T$$

The n by n matrix A arises in the solution of Laplace's equation in a unit-square, using a five-point formula with a 6 by 6 discretisation, with unity on the boundaries.

The drop tolerance, DROPTL, is taken as 0.1 and the density factor, DENSW, is taken as 0.8. The value IFAIL = 111 is used so that advisory and error messages will be printed, but soft failure would occur if IFAIL were returned as non-zero.

A relative accuracy of about 0.0001 is requested in the solution from F04MAF, with a maximum of 50 iterations.

The example program for F02FJF illustrates the use of routines F01MAF and F04MAF in solving an eigenvalue problem.

The example program is not reproduced here. The source code for all example programs is distributed with the NAG Foundation Library software and should be available on-line.

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F04 -- Simultaneous Linear Equations F04MBF

F04MBF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for your implementation to check implementation-dependent details. The symbol (*) after a NAG routine name denotes a routine that is not included in the Foundation Library.

1. Purpose

F04MBF solves a system of real sparse symmetric linear equations using a Lanczos algorithm.

2. Specification

```
 SUBROUTINE F04MBF (N, B, X, APROD, MSOLVE, PRECON, SHIFT,
1                   RTOL, ITNLIM, MSGLVL, ITN, ANORM,
2                   ACOND, RNORM, XNORM, WORK, RWORK,
3                   LRWORK, IWORK, LIWORK, INFORM, IFAIL)
 INTEGER           N, ITNLIM, MSGLVL, ITN, LRWORK, IWORK
1                  (LIWORK), LIWORK, INFORM, IFAIL
 DOUBLE PRECISION B(N), X(N), SHIFT, RTOL, ANORM, ACOND,
1                  RNORM, XNORM, WORK(N,5), RWORK(LRWORK)
 LOGICAL           PRECON
 EXTERNAL          APROD, MSOLVE
```

3. Description

F04MBF solves the system of linear equations

$$(A-(\lambda)I)x=b \qquad (3.1)$$

where A is an n by n sparse symmetric matrix and (lambda) is a scalar, which is of course zero if the solution of the equations

$$Ax=b$$

is required. It should be noted that neither A nor (A-(lambda)I) need be positive-definite.

(lambda) is supplied as the parameter SHIFT, and allows F04MBF to be used for finding eigenvectors of A in methods such as Rayleigh quotient iteration (see for example Lewis [1]), in which case (lambda) will be an approximation to an eigenvalue of A and b an approximation to an eigenvector of A.

The routine also provides an option to allow pre-conditioning and this will often reduce the number of iterations required by F04MBF.

F04MBF is based upon algorithm SYMMLQ (see Paige and Saunders [2]) and solves the equations by an algorithm based upon the

Lanczos process. Details of the method are given in Paige and Saunders [2]. The routine does not require A explicitly, but A is specified via a user-supplied routine APROD which, given an n element vector c, must return the vector z given by

$$z=Ac.$$

The pre-conditioning option is based on the following reasoning. If A can be expressed in the form

$$A=I+B$$

where B is of rank (rho), then the Lanczos process converges (in exact arithmetic) in at most (rho) iterations. If more generally A can be expressed in the form

$$A=M+C$$

where M is symmetric positive-definite and C has rank (rho), then

$$M^{-(1/2)}\,A\,M^{-(1/2)}=I+M^{-(1/2)}\,C\,M^{-(1/2)}$$

and $M^{-(1/2)}\,A\,M^{-(1/2)}$ also has rank (rho), and the Lanczos process applied to $M^{-(1/2)}\,A\,M^{-(1/2)}$ would again converge in at most (rho) iterations. On a computer, the number of iterations may be greater than (rho), but the Lanczos process may still be expected to converge rapidly. F04MBF does not require $M^{-(1/2)}\,A\,M^{-(1/2)}$ to be formed explicitly, but implicitly solves the equations

$$M^{-(1/2)}\,(A-(\mathrm{lambda})I)\,M^{-(1/2)}\,y=M^{-(1/2)}\,b\ ,\ y=M^{1/2}\,x \qquad (3.2)$$

with the user being required to supply a routine MSOLVE to solve the equations

$$Mz=c. \qquad (3.3)$$

For the pre-conditioning option to be effective, it is desirable that equations (3.3) can be solved efficiently. The example program in Section 9 illustrates the use of this option.

If we let r denote the residual vector

$$r=b-(A-(\mathrm{lambda})I)x$$

corresponding to an iterate x, then, when pre-conditioning has not been requested, the iterative procedure is terminated if it is estimated that

```
            ||r||<=tol.||A-(lambda)I||.||x||,                (3.4)
```

where tol is a user-supplied tolerance, ||r|| denotes the Euclidean length of the vector r and ||A|| denotes the Frobenius (Euclidean) norm of the matrix A. When pre-conditioning has been requested, the iterative procedure is terminated if it is estimated that

```
   -(1/2)            -(1/2)             -(1/2)      1/2
||M      r||<=tol.||M      (A-(lambda)I)M     ||.||M   x||. (3.5)
```

Note that

```
      -(1/2)    -(1/2)    -(1/2)              -(1/2)  1/2
     M      r=(M      b)-M      (A-(lambda)I)M      (M   x)
```

```
         -(1/2)
so that M      r is the residual vector corresponding to equation
(3.2). The routine will also terminate if it is estimated that

         ||A-(lambda)I||.||x||>=||b||/(epsilon),             (3.6)
```

where (epsilon) is the machine precision, when pre-conditioning has not been requested; or if it is estimated that

```
   -(1/2)              -(1/2)      1/2        -(1/2)
||M      (A-(lambda)I)M      ||.||M   x||>=||M      b||/(epsilon)

                                                            (3.7)
```

when pre-conditioning has been requested. If (3.6) is satisfied then x is almost certainly an eigenvector of A corresponding to the eigenvalue (lambda). If (lambda) was set to 0 (for the solution of Ax=b), then this condition simply means that A is effectively singular.

4. References

[1] Lewis J G (1977) Algorithms for sparse matrix eigenvalue problems. Technical Report STAN-CS-77-595. Computer Science Department, Stanford University.

[2] Paige C C and Saunders M A (1975) Solution of Sparse Indefinite Systems of Linear Equations. SIAM J. Numer. Anal.

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      12 617--629.

   5. Parameters

    1: N -- INTEGER                                              Input
       On entry: n, the order of the matrix A. Constraint: N >= 1.

    2: B(N) -- DOUBLE PRECISION array                            Input
       On entry: the right-hand side vector b.

    3: X(N) -- DOUBLE PRECISION array                           Output
       On exit: the solution vector x.

    4: APROD -- SUBROUTINE, supplied by the user.
                                                  External Procedure
       APROD must return the vector y=Ax for a given vector x.

       Its specification is:

             SUBROUTINE APROD (IFLAG, N, X, Y, RWORK, LRWORK,
            1                  IWORK,LIWORK)
             INTEGER          IFLAG, N, LRWORK, LIWORK, IWORK
            1                 (LIWORK)
             DOUBLE PRECISION X(N), Y(N), RWORK(LRWORK)

        1: IFLAG -- INTEGER                             Input/Output
           On entry: IFLAG is always non-negative. On exit: IFLAG
           may be used as a flag to indicate a failure in the
           computation of  Ax.  If IFLAG is negative on exit from
           APROD, F04MBF will exit immediately with IFAIL set to
           IFLAG.

        2: N -- INTEGER                                          Input
           On entry: n, the order of the matrix A.

        3: X(N) -- DOUBLE PRECISION array                        Input
           On entry: the vector x for which Ax is required.

        4: Y(N) -- DOUBLE PRECISION array                       Output
           On exit: the vector y=Ax.

        5: RWORK(LRWORK) -- DOUBLE PRECISION array  User Workspace

        6: LRWORK -- INTEGER                                     Input

        7: IWORK(LIWORK) -- INTEGER array           User Workspace
```

```
 8:  LIWORK -- INTEGER                                        Input
     APROD is called from F04MBF with the parameters RWORK,
     LRWORK, IWORK and LIWORK as supplied to F04MBF. The
     user is free to use the arrays RWORK and IWORK to
     supply information to APROD and MSOLVE as an
     alternative to using COMMON.
APROD must be declared as EXTERNAL in the (sub)program
from which F04MBF is called. Parameters denoted as
Input must not be changed by this procedure.

5:  MSOLVE -- SUBROUTINE, supplied by the user.
                                              External Procedure
MSOLVE is only referenced when PRECON is supplied as .TRUE..
When PRECON is supplied as .FALSE., then F04MBF may be
called with APROD as the actual argument for MSOLVE. When
PRECON is supplied as .TRUE., then MSOLVE must return the
solution y of the equations My=x for a given vector x, where
M must be symmetric positive-definite.

Its specification is:

       SUBROUTINE MSOLVE (IFLAG, N, X, Y, RWORK,
      1                   LRWORK, IWORK,LIWORK)
       INTEGER           IFLAG, N, LRWORK, LIWORK, IWORK
      1                  (LIWORK)
       DOUBLE PRECISION X(N), Y(N), RWORK(LRWORK)

 1:  IFLAG -- INTEGER                              Input/Output
     On entry: IFLAG is always non-negative. On exit: IFLAG
     may be used as a flag to indicate a failure in the
     solution of My=x.

     If IFLAG is negative on exit from MSOLVE, F04MBF will
     exit immediately with IFAIL set to IFLAG.

 2:  N -- INTEGER                                         Input
     On entry: n, the order of the matrix M.

 3:  X(N) -- DOUBLE PRECISION array                       Input
     On entry: the vector x for which the equations My=x are
     to be solved.

 4:  Y(N) -- DOUBLE PRECISION array                      Output
     On exit: the solution to the equations My=x.
```

```
      5:  RWORK(LRWORK) -- DOUBLE PRECISION array  User Workspace

      6:  LRWORK -- INTEGER                                Input

      7:  IWORK(LIWORK) -- INTEGER array           User Workspace

      8:  LIWORK -- INTEGER                                Input
          MSOLVE is called from F04MBF with the parameters RWORK,
          LRWORK, IWORK and LIWORK as supplied to F04MBF. The
          user is free to use the arrays RWORK and IWORK to
          supply information to APROD and MSOLVE as an
          alternative to using COMMON.
      MSOLVE must be declared as EXTERNAL in the (sub)program
      from which F04MBF is called. Parameters denoted as
      Input must not be changed by this procedure.

  6:  PRECON -- LOGICAL                                    Input
      On entry: PRECON specifies whether or not pre-conditioning
      is required. If PRECON = .TRUE., then pre-conditioning will
      be invoked and MSOLVE will be referenced by F04MBF; if
      PRECON = .FALSE., then MSOLVE is not referenced.

  7:  SHIFT -- DOUBLE PRECISION                            Input
      On entry: the value of (lambda). If the equations Ax=b are
      to be solved, then SHIFT must be supplied as zero.

  8:  RTOL -- DOUBLE PRECISION                             Input
      On entry: the tolerance for convergence, tol, of equation
      (3.4). RTOL should not normally be less than (epsilon),
      where (epsilon) is the machine precision.

  9:  ITNLIM -- INTEGER                                    Input
      On entry: an upper limit on the number of iterations. If
      ITNLIM <= 0, then the value N is used in place of ITNLIM.

 10:  MSGLVL -- INTEGER                                    Input
      On entry: the level of printing from F04MBF. If MSGLVL <= 0,
      then no printing occurs, but otherwise messages will be
      output on the advisory message channel (see X04ABF). A
      description of the printed output is given in Section 5.1
      below. The level of printing is determined as follows:
      MSGLVL <= 0
            No printing.

      MSGLVL = 1
            A brief summary is printed just prior to return from
```

F04MBF.

MSGLVL >= 2
A summary line is printed periodically to monitor the progress of F04MBF, together with a brief summary just prior to return from F04MBF.

11: ITN -- INTEGER Output
On exit: the number of iterations performed.

12: ANORM -- DOUBLE PRECISION Output
On exit: an estimate of ||A-(lambda)I|| when PRECON = .FALSE., and $||M^{-(1/2)}(A-(\lambda)I)M^{-(1/2)}||$ when PRECON = .TRUE..

13: ACOND -- DOUBLE PRECISION Output
On exit: an estimate of the condition number of (A-(lambda)I) when PRECON = .FALSE., and of $M^{-(1/2)}(A-(\lambda)I)M^{-(1/2)}$ when PRECON = .TRUE.. This will usually be a substantial under-estimate.

14: RNORM -- DOUBLE PRECISION Output
On exit: ||r||, where r=b-(A-(lambda)I)x and x is the solution returned in X.

15: XNORM -- DOUBLE PRECISION Output
On exit: ||x||, where x is the solution returned in X.

16: WORK(5*N) -- DOUBLE PRECISION array Workspace

17: RWORK(LRWORK) -- DOUBLE PRECISION array User Workspace
RWORK is not used by F04MBF, but is passed directly to routines APROD and MSOLVE and may be used to pass information to these routines.

18: LRWORK -- INTEGER Input
On entry: the length of the array RWORK as declared in the (sub)program from which F04MBF is called. Constraint: LRWORK >= 1.

19: IWORK(LIWORK) -- INTEGER array User Workspace
IWORK is not used by F04MBF, but is passed directly to routines APROD and MSOLVE and may be used to pass information to these routines.

20: LIWORK -- INTEGER Input
On entry: the length of the array IWORK as declared in the (sub)program from which F04MBF is called. Constraint: LIWORK >= 1.

21: INFORM -- INTEGER Output
On exit: the reason for termination of F04MBF as follows:
INFORM = 0
The right-hand side vector b=0 so that the exact solution is x=0. No iterations are performed in this case.

INFORM = 1
The termination criterion of equation (3.4) has been satisfied with tol as the value supplied in RTOL.

INFORM = 2
The termination criterion of equation (3.4) has been satisfied with tol equal to (epsilon), where (epsilon) is the machine precision. The value supplied in RTOL must have been less than (epsilon) and was too small for the machine.

INFORM = 3
The termination criterion of equation (3.5) has been satisfied so that X is almost certainly an eigenvector of A corresponding to the eigenvalue SHIFT.
The values INFORM = 4 and INFORM = 5 correspond to failure with IFAIL = 3 or IFAIL = 2 respectively (see Section 6) and when IFAIL is negative, INFORM will be set to the same negative value.

22: IFAIL -- INTEGER Input/Output
On entry: IFAIL must be set to 0, -1 or 1. For users not familiar with this parameter (described in the Essential Introduction) the recommended value is 0.

On exit: IFAIL = 0 unless the routine detects an error (see Section 6).

5.1. Description of the Printed Output

When MSGLVL > 0, then F04MBF will produce output (except in the case where the routine fails with IFAIL = 1) on the advisory message channel (see X04ABF).

The following notation is used in the output.

Output	Meaning
RBAR	$M^{-(1/2)}(b-(A-(\lambda)I)x)=r$
ABAR	$M^{-(1/2)}(A-(\lambda)I)M^{-(1/2)}=A$
Y	$M^{1/2}x$
R	$b-(A-(\lambda)I)x$
NORM(A)	$\|\|A\|\|$

Of course, when pre-conditioning has not been requested then the first three reduce to (b-(A-(lambda)I)x), (A-(lambda)I) and x respectively. When MSGLVL >= 2 then some initial information is printed and the following notation is used.

Output	Meaning
BETA1	$(b^T M^{-1} b)^{1/2} == (\beta)_1$
ALFA1	$(1/(\beta)_1)\,(M^{-1/2} b)\,(M^{-1/2} AM^{-1/2})(M^{1/2} b) == (\alpha)_1$

and a summary line is printed periodically giving the following information:

Output	Meaning
ITN	Iteration number, k.
X1(LQ)	The first element of the vector x_L, where x_k is the current iterate. See Paige and Saunders [2] for details.

X1(CG) The first element of the vector x_k^C, where x_k^C is the vector that would be obtained by conjugate gradients. See Paige and Saunders [2] for details.

NORM(RBAR) $||r||$, where r is as defined above and x is either x_k^L or x_k^C depending upon which is the best current approximation to the solution. (See LQ/CG below).

NORM(T) The value $||T_k||$, where T_k is the tridiagonal matrix of the Lanczos process. This increases monotonically and is a lower bound on $||A||$.

COND(L) A monotonically increasing lower bound on the condition number of A, $||A||||(A)^{-1}||$.

LQ/CG L is printed if x_k^L is the best current approximation to the solution and C is printed otherwise.

6. Error Indicators and Warnings

Errors detected by the routine:

IFAIL< 0
A negative value of IFAIL indicates an exit from F04MBF because the user has set IFLAG negative in APROD or MSOLVE. The value of IFAIL will be the same as the user's setting of IFLAG.

IFAIL= 1
On entry N < 1,

or LRWORK < 1,

```
     or       LIWORK < 1.

IFAIL= 2
     The pre-conditioning matrix M does not appear to be
     positive-definite. The user should check that MSOLVE is
     working correctly.

IFAIL= 3
     The limit on the number of iterations has been reached. If
     IFAIL = 1 on entry then the latest approximation to the
     solution is returned in X and the values ANORM, ACOND, RNORM
     and XNORM are also returned.

The value of INFORM contains additional information about the
termination of the routine and users must examine INFORM to judge
whether the routine has performed successfully for the problem in
hand. In particular INFORM = 3 denotes that the matrix A-
(lambda)I is effectively singular: if the purpose of calling
F04MBF is to solve a system of equations Ax=b, then this
condition must be regarded as a failure, but if the purpose is to
compute an eigenvector, this result would be very satisfactory.

7. Accuracy

The computed solution x will satisfy the equation

                        r=b-(A-(lambda)I)x

where the value ||r|| is as returned in the parameter RNORM.

8. Further Comments

The time taken by the routine is likely to be principally
determined by the time taken in APROD and, when pre-conditioning
has been requested, in MSOLVE. Each of these routines is called
once every iteration.

The time taken by the remaining operations in F04MBF is
approximately proportional to n.

9. Example

To solve the 10 equations Ax=b given by

                    (2 1 0 0 0 0 0 0 0 3)      (6)
```

```
                (1 2 1 0 0 0 0 0 0 0)      (4)
                (0 1 2 1 0 0 0 0 0 0)      (4)
                (0 0 1 2 1 0 0 0 0 0)      (4)
                (0 0 0 1 2 1 0 0 0 0)      (4)
           A=(0 0 0 0 1 2 1 0 0 0),   b=(4).
                (0 0 0 0 0 1 2 1 0 0)      (4)
                (0 0 0 0 0 0 1 2 1 0)      (4)
                (0 0 0 0 0 0 0 1 2 1)      (4)
                (3 0 0 0 0 0 0 0 1 2)      (6)
```

The tridiagonal part of A is positive-definite and such tridiagonal equations can be solved efficiently by F04FAF. The form of A suggests that this tridiagonal part is a good candidate for the pre-conditioning matrix M and so we illustrate the use of F04MBF by pre-conditioning with the 10 by 10 matrix

```
                  (2 1 0 ... 0)
                  (1 2 1 ... 0)
                  (0 1 2 ... 0)
              M=(. . .   .  ).
                  (. . .   .  )
                  (. . .   .  )
                  (0 0 0 ... 2)
```

Since A-M has only 2 non-zero elements and is obviously of rank 2, we can expect F04MBF to converge very quickly in this example. Of course, in practical problems we shall not usually be able to make such a good choice of M.

The example sets the tolerance RTOL = 10^{-5}.

The example program is not reproduced here. The source code for all example programs is distributed with the NAG Foundation Library software and should be available on-line.

%%%

F04 -- Simultaneous Linear Equations F04MCF

F04MCF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for your implementation to check implementation-dependent details. The symbol (*) after a NAG routine name denotes a routine that is not included in the Foundation Library.

1. Purpose

F04MCF computes the approximate solution of a system of real linear equations with multiple right-hand sides, AX=B, where A is a symmetric positive-definite variable-bandwidth matrix, which has previously been factorized by F01MCF. Related systems may also be solved.

2. Specification

```
      SUBROUTINE F04MCF (N, AL, LAL, D, NROW, IR, B, NRB,
     1                   ISELCT, X, NRX, IFAIL)
      INTEGER          N, LAL, NROW(N), IR, NRB, ISELCT, NRX,
     1                 IFAIL
      DOUBLE PRECISION AL(LAL), D(N), B(NRB,IR), X(NRX,IR)
```

3. Description

The normal use of this routine is the solution of the systems AX=B, following a call of F01MCF to determine the Cholesky factorization $A=LDL^T$ of the symmetric positive-definite variable-bandwidth matrix A.

However, the routine may be used to solve any one of the following systems of linear algebraic equations:

(1) $LDL^T X = B$ (usual system),

(2) $LDX = B$ (lower triangular system),

(3) $DL^T X = B$ (upper triangular system),

(4) $LL^T X = B$

(5) $LX = B$ (unit lower triangular system),

(6) $L^T X = B$ (unit upper triangular system).

L denotes a unit lower triangular variable-bandwidth matrix of order n, D a diagonal matrix of order n, and B a set of right-

hand sides.

The matrix L is represented by the elements lying within its envelope i.e., between the first non-zero of each row and the diagonal (see Section 9 for an example). The width NROW(i) of the ith row is the number of elements between the first non-zero element and the element on the diagonal inclusive.

4. References

[1] Wilkinson J H and Reinsch C (1971) Handbook for Automatic Computation II, Linear Algebra. Springer-Verlag.

5. Parameters

1: N -- INTEGER Input
On entry: n, the order of the matrix L. Constraint: N >= 1.

2: AL(LAL) -- DOUBLE PRECISION array Input
On entry: the elements within the envelope of the lower triangular matrix L, taken in row by row order, as returned by F01MCF. The unit diagonal elements of L must be stored explicitly.

3: LAL -- INTEGER Input
On entry:
the dimension of the array AL as declared in the (sub)program from which F04MCF is called.
Constraint: LAL >= NROW(1) + NROW(2) +... + NROW(n).

4: D(N) -- DOUBLE PRECISION array Input
On entry: the diagonal elements of the diagonal matrix D. D is not referenced if ISELCT >= 4.

5: NROW(N) -- INTEGER array Input
On entry: NROW(i) must contain the width of row i of L, i.e.,the number of elements between the first (leftmost) non-zero element and the element on the diagonal, inclusive.
Constraint: 1 <= NROW(i)<=i.

6: IR -- INTEGER Input
On entry: r, the number of right-hand sides. Constraint: IR >= 1.

7: B(NRB,IR) -- DOUBLE PRECISION array Input
On entry: the n by r right-hand side matrix B. See also

Section 8.

8: NRB -- INTEGER Input
On entry:
the first dimension of the array B as declared in the (sub)program from which F04MCF is called.
Constraint: NRB >= N.

9: ISELCT -- INTEGER Input
On entry: ISELCT must specify the type of system to be solved, as follows:

$$\text{ISELCT} = 1: \quad \text{solve } LDL^T X = B,$$

$$\text{ISELCT} = 2: \quad \text{solve } LDX = B,$$

$$\text{ISELCT} = 3: \quad \text{solve } DL^T X = B,$$

$$\text{ISELCT} = 4: \quad \text{solve } LL^T X = B,$$

$$\text{ISELCT} = 5: \quad \text{solve } LX = B,$$

$$\text{ISELCT} = 6: \quad \text{solve } L^T X = B.$$

10: X(NRX,IR) -- DOUBLE PRECISION array Output
On exit: the n by r solution matrix X. See also Section 8.

11: NRX -- INTEGER Input
On entry:
the first dimension of the array X as declared in the (sub)program from which F04MCF is called.
Constraint: NRX >= N.

12: IFAIL -- INTEGER Input/Output
On entry: IFAIL must be set to 0, -1 or 1. For users not familiar with this parameter (described in the Essential Introduction) the recommended value is 0.

On exit: IFAIL = 0 unless the routine detects an error (see Section 6).

6. Error Indicators and Warnings

```
Errors detected by the routine:

IFAIL= 1
     On entry N < 1,

     or       for some i, NROW(i)<1 or NROW(i) > i,

     or       LAL < NROW(1) + NROW(2) +... + NROW(N).

IFAIL= 2

     On entry IR < 1,

     or       NRB < N,

     or       NRX < N.

IFAIL= 3
     On entry ISELCT < 1,

     or       ISELCT > 6.

IFAIL= 4
     The diagonal matrix D is singular, i.e., at least one of the
     elements of D is zero. This can only occur if ISELCT <= 3.

IFAIL= 5
     At least one of the diagonal elements of L is not equal to
     unity.

7. Accuracy

The usual backward error analysis of the solution of triangular
system applies: each computed solution vector is exact for
slightly perturbed matrices L and D, as appropriate (cf.
Wilkinson and Reinsch [1] pp 25--27, 54--55).

8. Further Comments

The time taken by the routine is approximately proportional to
pr, where

                p=NROW(1)+NROW(2)+...+NROW(n).

Unless otherwise stated in the Users' Note for your
```

implementation, the routine may be called with the same actual array supplied for the parameters B and X, in which case the solution matrix will overwrite the right-hand side matrix. However this is not standard Fortran 77 and may not work in all implementations.

9. Example

To solve the system of equations AX=B, where

```
                        (1  2  0  0  5  0)
                        (2  5  3  0 14  0)
                        (0  3 13  0 18  0)
                    A=(0  0  0 16  8 24)
                        (5 14 18  8 55 17)
                        (0  0  0 24 17 77)
```

and

```
                             ( 6 -10)
                             (15 -21)
                             (11  -3)
                          B=( 0  24)
                             (51 -39)
                             (46  67)
```

Here A is symmetric and positive-definite and must first be factorized by F01MCF.

The example program is not reproduced here. The source code for all example programs is distributed with the NAG Foundation Library software and should be available on-line.

%%%

F04 -- Simultaneous Linear Equations F04QAF
F04QAF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for your implementation to check implementation-dependent details. The symbol (*) after a NAG routine name denotes a routine that is not included in the Foundation Library.

1. Purpose

F04QAF solves sparse unsymmetric equations, sparse linear least-

squares problems and sparse damped linear least-squares problems, using a Lanczos algorithm.

2. Specification

```
      SUBROUTINE F04QAF (M, N, B, X, SE, APROD, DAMP, ATOL,
     1                   BTOL, CONLIM, ITNLIM, MSGLVL, ITN,
     2                   ANORM, ACOND, RNORM, ARNORM, XNORM,
     3                   WORK, RWORK, LRWORK, IWORK, LIWORK,
     4                   INFORM, IFAIL)
      INTEGER           M, N, ITNLIM, MSGLVL, ITN, LRWORK, IWORK
     1                  (LIWORK), LIWORK, INFORM, IFAIL
      DOUBLE PRECISION B(M), X(N), SE(N), DAMP, ATOL, BTOL,
     1                  CONLIM, ANORM, ACOND, RNORM, ARNORM,
     2                  XNORM, WORK(N,2), RWORK(LRWORK)
      EXTERNAL          APROD
```

3. Description

F04QAF can be used to solve a system of linear equations

$$Ax=b \tag{3.1}$$

where A is an n by n sparse unsymmetric matrix, or can be used to solve linear least-squares problems, so that F04QAF minimizes the value (rho) given by

$$(rho)=||r||, \quad r=b-Ax \tag{3.2}$$

where A is an m by n sparse matrix and $||r||$ denotes the Euclidean length of r so that $||r||^2 = r^T r$. A damping parameter, (lambda), may be included in the least-squares problem in which case F04QAF minimizes the value (rho) given by

$$(rho)^2 = ||r||^2 + (lambda)^2 ||x||^2 \tag{3.3}$$

(lambda) is supplied as the parameter DAMP and should of course be zero if the solution to problems (3.1) or (3.2) is required. Minimizing (rho) in (3.3) is often called ridge regression.

F04QAF is based upon algorithm LSQR (see Paige and Saunders [1] and [2]) and solves the problems by an algorithm based upon the Lanczos process. Details of the method are given in [1]. The routine does not require A explicitly, but A is specified via a

user-supplied routine APROD which must perform the operations (T y+Ax) and (x+A y) for a given n element vector x and m element vector y. A parameter to APROD specifies which of the two operations is required on a given entry.

The routine also returns estimates of the standard errors of the sample regression coefficients (x_i, for i=1,2,...,n) given by the diagonal elements of the estimated variance-covariance matrix V. When problem (3.2) is being solved and A is of full rank, then V is given by

$$V=s^2 (A^T A)^{-1}, \quad s^2=(rho)^2/(m-n), \quad m>n$$

and when problem (3.3) is being solved then V is given by

$$V=s^2 (A^T A+(lambda)^2 I)^{-1}, \quad s^2=(rho)^2/m, \quad (lambda)/=0.$$

Let A denote the matrix

$$A=A, \quad (lambda)=0 \ ; \ A=\begin{pmatrix} A \\ (lambda)I \end{pmatrix}, \quad (lambda)/=0, \tag{3.4}$$

let r denote the residual vector

$$r=r^{(b)}, \quad (lambda)=0 \ ; \ r=(0)-Ax, \quad (lambda)/=0 \tag{3.5}$$

corresponding to an iterate x, so that (rho)=||r|| is the function being minimized, and let ||A|| denote the Frobenius (Euclidean) norm of A. Then the routine accepts x as a solution if it is estimated that one of the following two conditions is satisfied:

$$(rho)<=tol \ ||A||.||x||+tol \ ||b|| \tag{3.6}$$

```
                          1              2

                     T
                  ||A r||<=tol ||A||(rho)                    (3.7)
                             1
```

where tol and tol are user-supplied tolerances which estimate
         1   2
the relative errors in A and b respectively. Condition (3.6) is appropriate for compatible problems where, in theory, we expect the residual to be zero and will be satisfied by an acceptable solution x to a compatible problem. Condition (3.7) is appropriate for incompatible systems where we do not expect the residual to be zero and is based upon the observation that, in theory,

```
                              T
                             A r=0
```

when x is a solution to the least-squares problem, and so (3.7) will be satisfied by an acceptable solution x to a linear least-squares problem.

The routine also includes a test to prevent convergence to solutions, x, with unacceptably large elements. This can happen if A is nearly singular or is nearly rank deficient. If we let

the singular values of A be

```
              (sigma) >=(sigma) >=...>=(sigma) >=0
                     1         2              n
```

then the condition number of A is defined as

```
                    cond(A)=(sigma) /(sigma)
                                    1        k
```

where (sigma) is the smallest non-zero singular value of A and
            k

hence k is the rank of A. When k<n, then A is rank deficient, the least-squares solution is not unique and F04QAF will normally

converge to the minimal length solution. In practice A will not have exactly zero singular values, but may instead have small singular values that we wish to regard as zero.

The routine provides for this possibility by terminating if

$$\mathrm{cond}(A) \geq c_{\mathrm{lim}} \tag{3.8}$$

where c_{lim} is a user-supplied limit on the condition number of A. For problem (3.1) termination with this condition indicates that A is nearly singular and for problem (3.2) indicates that A is nearly rank deficient and so has near linear dependencies in its columns. In this case inspection of ||r||, $||A^T r||$ and ||x||, which are all returned by the routine, will indicate whether or not an acceptable solution has been found. Condition (3.8), perhaps in conjunction with (lambda)/=0, can be used to try and ' regularise' least-squares solutions. A full discussion of the stopping criteria is given in Section 6 of reference Paige and Saunders [1].

Introduction of a non-zero damping parameter (lambda) tends to reduce the size of the computed solution and to make its components less sensitive to changes in the data, and F04QAF is applicable when a value of (lambda) is known a priori. To have an

effect, (lambda) should normally be at least \/(epsilon)||A|| where (epsilon) is the machine precision. For further discussion see Paige and Saunders [2] and the references given there.

Whenever possible the matrix A should be scaled so that the relative errors in the elements of A are all of comparable size. Such a scaling helps to prevent the least-squares problem from

being unnecessarily sensitive to data errors and will normally reduce the number of iterations required. At the very least, in the absence of better information, the columns of A should be scaled to have roughly equal column length.

4. References

[1] Paige C C and Saunders M A (1982) LSQR: An Algorithm for Sparse Linear Equations and Sparse Least-squares. ACM Trans. Math. Softw. 8 43--71.

[2] Paige C C and Saunders M A (1982) ALGORITHM 583 LSQR: Sparse Linear Equations and Least-squares Problems. ACM Trans. Math. Softw. 8 195--209.

5. Parameters

1: M -- INTEGER Input
On entry: m, the number of rows of the matrix A.
Constraint: M >= 1.

2: N -- INTEGER Input
On entry: n, the number of columns of the matrix A.
Constraint: N >= 1.

3: B(M) -- DOUBLE PRECISION array Input/Output
On entry: the right-hand side vector b. On exit: the array is overwritten.

4: X(N) -- DOUBLE PRECISION array Output
On exit: the solution vector x.

5: SE(N) -- DOUBLE PRECISION array Output
On exit: the estimates of the standard errors of the components of x. Thus SE(i) contains an estimate of the element v_{ii} of the estimated variance-covariance matrix V.
The estimates returned in SE will be the lower bounds on the actual estimated standard errors, but will usually have at least one correct figure.

6: APROD -- SUBROUTINE, supplied by the user.
External Procedure
APROD must perform the operations $y:=y+Ax$ and $x:=x+A^T y$ for given vectors x and y.

Its specification is:

```
      SUBROUTINE APROD (MODE, M, N, X, Y, RWORK,
     1                  LRWORK, IWORK, LIWORK)
      INTEGER          MODE, M, N, LRWORK, LIWORK,
     1                 IWORK(LIWORK)
      DOUBLE PRECISION X(N), Y(M), RWORK(LRWORK)
```

1: MODE -- INTEGER Input/Output
On entry: MODE specifies which operation is to be performed:
If MODE = 1, then APROD must compute y+Ax.

If MODE = 2, then APROD must compute $x+A^T y$.
On exit: MODE may be used as a flag to indicate a failure in the computation of y+Ax or $x+A^T y$. If MODE is negative on exit from APROD, F04QAF will exit immediately with IFAIL set to MODE.

2: M -- INTEGER Input
On entry: m, the number of rows of A.

3: N -- INTEGER Input
On entry: n, the number of columns of A.

4: X(N) -- DOUBLE PRECISION array Input/Output
On entry: the vector x. On exit: if MODE = 1, X must be unchanged;

If MODE = 2, X must contain $x+A^T y$.

5: Y(M) -- DOUBLE PRECISION array Input/Output
On entry: the vector y. On exit: if MODE = 1, Y must contain y+Ax;

If MODE = 2, Y must be unchanged.

6: RWORK(LRWORK) -- DOUBLE PRECISION array User Workspace

7: LRWORK -- INTEGER Input

8: IWORK(LIWORK) -- INTEGER array User Workspace

9: LIWORK -- INTEGER Input
APROD is called from F04QAF with the parameters RWORK, LRWORK, IWORK and LIWORK as supplied to F04QAF. The user is free to use the arrays RWORK and IWORK to supply information to APROD as an alternative to using COMMON.
APROD must be declared as EXTERNAL in the (sub)program from which F04QAF is called. Parameters denoted as Input must not be changed by this procedure.

7: DAMP -- DOUBLE PRECISION Input
On entry: the value (lambda). If either problem (3.1) or problem (3.2) is to be solved, then DAMP must be supplied as zero.

8: ATOL -- DOUBLE PRECISION Input
On entry: the tolerance, tol_1, of the convergence criteria (3.6) and (3.7); it should be an estimate of the largest relative error in the elements of A. For example, if the elements of A are correct to about 4 significant figures, then ATOL should be set to about $5*10^{-4}$. If ATOL is supplied as less than (epsilon), where (epsilon) is the machine precision, then the value (epsilon) is used in place of ATOL.

9: BTOL -- DOUBLE PRECISION Input
On entry: the tolerance, tol_2, of the convergence criterion (3.6); it should be an estimate of the largest relative error in the elements of B. For example, if the elements of B are correct to about 4 significant figures, then BTOL should be set to about $5*10^{-4}$. If BTOL is supplied as less than (epsilon), where (epsilon) is the machine precision, then the value (epsilon) is used in place of BTOL.

10: CONLIM -- DOUBLE PRECISION Input
On entry: the value c_{lim} of equation (3.8); it should be an upper limit on the condition number of A. CONLIM should not normally be chosen much larger than 1.0/ATOL. If CONLIM is

```
     supplied as zero then the value 1.0/(epsilon), where
     (epsilon) is the machine precision, is used in place of
     CONLIM.

11:  ITNLIM -- INTEGER                                        Input
     On entry: an upper limit on the number of iterations. If
     ITNLIM <= 0, then the value N is used in place of ITNLIM,
     but for ill-conditioned problems a higher value of ITNLIM is
     likely to be necessary.

12:  MSGLVL -- INTEGER                                        Input
     On entry: the level of printing from F04QAF. If MSGLVL <= 0,
     then no printing occurs, but otherwise messages will be
     output on the advisory message channel (see X04ABF). A
     description of the printed output is given in Section 5.2
     below. The level of printing is determined as follows:
     MSGLVL <= 0
           No printing.

     MSGLVL = 1
           A brief summary is printed just prior to return from
           F04QAF.

     MSGLVL >= 2
           A summary line is printed periodically to monitor the
           progress of F04QAF, together with a brief summary just
           prior to return from F04QAF.

13:  ITN -- INTEGER                                           Output
     On exit: the number of iterations performed.

14:  ANORM -- DOUBLE PRECISION                                Output


     On exit: an estimate of ||A|| for the matrix A of equation
     (3.4).

15:  ACOND -- DOUBLE PRECISION                                Output


     On exit: an estimate of cond(A) which is a lower bound.

16:  RNORM -- DOUBLE PRECISION                                Output


     On exit: an estimate of ||r|| for the residual,  r, of
```

equation (3.5) corresponding to the solution x returned in

X. Note that ||r|| is the function being minimized.

17: ARNORM -- DOUBLE PRECISION Output
T
On exit: an estimate of the ||A r|| corresponding to the solution x returned in X.

18: XNORM -- DOUBLE PRECISION Output
On exit: an estimate of ||x|| for the solution x returned in X.

19: WORK(2*N) -- DOUBLE PRECISION array Workspace

20: RWORK(LRWORK) -- DOUBLE PRECISION array User Workspace
RWORK is not used by F04QAF, but is passed directly to routine APROD and may be used to pass information to that routine.

21: LRWORK -- INTEGER Input
On entry: the length of the array RWORK as declared in the (sub)program from which F04QAF is called. Constraint: LRWORK >= 1.

22: IWORK(LIWORK) -- INTEGER array User Workspace
IWORK is not used by F04QAF, but is passed directly to routine APROD and may be used to pass information to that routine.

23: LIWORK -- INTEGER Input
On entry: the length of the array IWORK as declared in the (sub)program from which F04QAF is called. Constraint: LIWORK >= 1.

24: INFORM -- INTEGER Output
On exit: the reason for termination of F04QAF as follows:
INFORM = 0
The exact solution is x=0. No iterations are performed in this case.

INFORM = 1
The termination criterion of equation (3.6) has been satisfied with tol and tol as the values supplied in
1 2

ATOL and BTOL respectively.

INFORM = 2
The termination criterion of equation (3.7) has been satisfied with tol $_1$ as the value supplied in ATOL.

INFORM = 3
The termination criterion of equation (3.6) has been satisfied with tol $_1$ and/or tol $_2$ as the value (epsilon), where (epsilon) is the machine precision. One or both of the values supplied in ATOL and BTOL must have been less than (epsilon) and was too small for this machine.

INFORM = 4
The termination criterion of equation (3.7) has been satisfied with tol $_1$ as the value (epsilon), where (epsilon) is the machine precision. The value supplied in ATOL must have been less than (epsilon) and was too small for this machine.

The values INFORM = 5, INFORM = 6 and INFORM = 7 correspond to failure with IFAIL = 2, IFAIL = 3 and IFAIL = 4 respectively (see Section 6) and when IFAIL is negative INFORM will be set to the same negative value.

25: IFAIL -- INTEGER Input/Output
On entry: IFAIL must be set to 0, -1 or 1. For users not familiar with this parameter (described in the Essential Introduction) the recommended value is 0.

On exit: IFAIL = 0 unless the routine detects an error (see Section 6).

5.1. Description of the printed output

When MSGLVL > 0, then F04QAF will produce output (except in the case where the routine fails with IFAIL = 1) on the advisory message channel (see X04ABF).

When MSGLVL >= 2 then a summary line is printed periodically giving the following information:

Output	Meaning

ITN	Iteration number, k.
X(1)	The first element of the current iterate x_k.
FUNCTION	The current value of the function, (rho), being minimized.
COMPAT	An estimate of $\|\|r_k\|\|/\|\|b\|\|$, where r_k is the residual corresponding to x_k. This value should converge to zero (in theory) if and only if the problem is compatible. COMPAT decreases monotonically.
INCOMPAT	An estimate of $\|\|A^T r_k\|\|/(\|\|A\|\|\|\|r_k\|\|)$ which should converge to zero if and only if at the solution (rho) is non-zero. INCOMPAT is not usually monotonic.
NRM(ABAR)	A monotonically increasing estimate of $\|\|A\|\|$.
COND(ABAR)	A monotonically increasing estimate of the condition number cond(A).

6. Error Indicators and Warnings

Errors detected by the routine:

IFAIL< 0
A negative value of IFAIL indicates an exit from F04QAF because the user has set MODE negative in APROD. The value of IFAIL will be the same as the user's setting of MODE.

IFAIL= 1
On entry M < 1,

```
     or       N < 1,

     or       LRWORK < 1,

     or       LIWORK < 1.

IFAIL= 2
     The condition of equation (3.8) has been satisfied for the
     value of c    supplied in CONLIM. If this failure is
              lim
     unexpected the user should check that APROD is working
     correctly. Although conditions (3.6) or (3.7) have not been
     satisfied, the values returned in RNORM, ARNORM and XNORM
     may nevertheless indicate that an acceptable solution has
     been reached.

IFAIL= 3
     The conditions of equation (3.8) has been satisified for the
     value c   =1.0/(epsilon), where (epsilon) is the machine
            lim


     precision. The matrix A is nearly singular or rank deficient
     and the problem is too ill-conditioned for this machine. If
     this failure is unexpected, the user should check that APROD
     is working correctly.

IFAIL= 4
     The limit on the number of iterations has been reached. The
     number of iterations required by F04QAF and the condition of


     the matrix A can depend strongly on the scaling of the
     problem. Poor scaling of the rows and columns of A should be
     avoided whenever possible.

7. Accuracy

When the problem is compatible, the computed solution x will
satisfy the equation

                                r=b-Ax,

where an estimate of ||r|| is returned in the parameter RNORM.
When the problem is incompatible, the computed solution x will
```

satisfy the equation

$$A^T r = e,$$

where an estimate of ||e|| is returned in the parameter ARNORM. See also Section 6.2 of Paige and Saunders [1].

8. Further Comments

The time taken by the routine is likely to be principally determined by the time taken in APROD, which is called twice on each iteration, once with MODE = 1 and once with MODE = 2. The time taken per iteration by the remaining operations in F04QAF is approximately proportional to max(m,n).

The Lanczos process will usually converge more quickly if A is pre-conditioned by a non-singular matrix M that approximates A in some sense and is also chosen so that equations of the form My=c can efficiently be solved for y. Some discussion of pre-conditioning in the context of symmetric matrices is given in Section 3 of the document for F04MBF. In the context of F04QAF, problem (3.1) is equivalent to

$$(AM^{-1})y=b, \quad Mx=y$$

and problem (3.2) is equivalent to minimizing

$$(rho)=||r||, \quad r=b-(AM^{-1})y, \quad Mx=y.$$

Note that the normal matrix $(AM^{-1})^T(AM^{-1})=M^{-T}(A^TA)M^{-1}$ so that the pre-conditioning AM^{-1} is equivalent to the pre-conditioning $M^{-T}(A^TA)M^{-1}$ of the normal matrix A^TA.

Pre-conditioning can be incorporated into F04QAF simply by coding the routine APROD to compute $y+AM^{-1}x$ and $x+M^{-T}A^Ty$ in place of $y+Ax$ and $x+A^Ty$ respectively, and then solving the equations $Mx=y$ for x on return from F04QAF. $y+AM^{-1}x$ should be computed by

solving Mz=x for z and then computing y+Az, and x+M^{-T} A^T y should be computed by solving M^T z=A y for z and then forming x+z.

9. Example

To solve the linear least-squares problem

$$\text{minimize } (\rho)=||r||, \quad r=b-Ax$$

where A is the 13 by 12 matrix and b is the 13 element vector given by

$$A=\begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 4 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 4 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & -1 & 4 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 4 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \end{pmatrix},$$

$$b=-h \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 2 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ -3 \\ -h \end{pmatrix}$$

with h=0.1.

Such a problem can arise by considering the Neumann problem on a

rectangle

$$\frac{\partial u}{\partial n}=0$$

$$\frac{\partial u}{\partial n}=0 \qquad \nabla^2 u=g(x,y) \qquad \frac{\partial u}{\partial n}=0 \qquad \Big/_{C} u=1$$

$$\frac{\partial u}{\partial n}=0$$

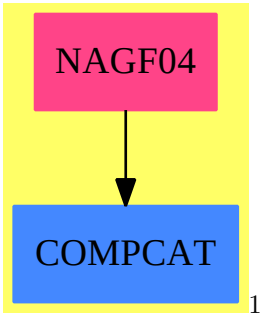
where C is the boundary of the rectangle, and discretising as illustrated below with the square mesh

Please see figure in printed Reference Manual

The 12 by 12 symmetric part of A represents the difference equations and the final row comes from the normalising condition. The example program has g(x,y)=1 at all the internal mesh points, but apart from this is written in a general manner so that the number of rows (NROWS) and columns (NCOLS) in the grid can readily be altered.

The example program is not reproduced here. The source code for all example programs is distributed with the NAG Foundation Library software and should be available on-line.

15.6 NagLinearEquationSolvingPackage



1.00

Exports:

f04adf	f04arf	f04asf	f04atf	f04axf
f04faf	f04jgf	f04maf	f04mbf	f04mcf
f04qaf				

⟨*package NAGF04 NagLinearEquationSolvingPackage*⟩≡

```
)abbrev package NAGF04 NagLinearEquationSolvingPackage
++ Author: Godfrey Nolan and Mike Dewar
++ Date Created: Jan 1994
++ Date Last Updated: Thu May 12 17:45:31 1994
++ Description:
++ This package uses the NAG Library to solve the matrix equation
++ \axiom{AX=B}, where \axiom{B}
++ may be a single vector or a matrix of multiple right-hand sides.
++ The matrix \axiom{A} may be real, complex, symmetric, Hermitian positive-
++ definite, or sparse. It may also be rectangular, in which case a
++ least-squares solution is obtained.
++ See \downlink{Manual Page}{manpageXXf04}.
NagLinearEquationSolvingPackage(): Exports == Implementation where
  S ==> Symbol
  FOP ==> FortranOutputStackPackage

  Exports ==> with
    f04adf : (Integer,Matrix Complex DoubleFloat,Integer,Integer,_
        Integer,Integer,Matrix Complex DoubleFloat,Integer) -> Result
     ++ f04adf(ia,b,ib,n,m,ic,a,ifail)
     ++ calculates the approximate solution of a set of complex
     ++ linear equations with multiple right-hand sides, using an LU
     ++ factorization with partial pivoting.
     ++ See \downlink{Manual Page}{manpageXXf04adf}.
    f04arf : (Integer,Matrix DoubleFloat,Integer,Matrix DoubleFloat,_
        Integer) -> Result
     ++ f04arf(ia,b,n,a,ifail)
     ++ calculates the approximate solution of a set of real
```

```
 ++ linear equations with a single right-hand side, using an LU
 ++ factorization with partial pivoting.
 ++ See \downlink{Manual Page}{manpageXXf04arf}.
f04asf : (Integer,Matrix DoubleFloat,Integer,Matrix DoubleFloat,_
    Integer) -> Result
 ++ f04asf(ia,b,n,a,ifail)
 ++ calculates the accurate solution of a set of real
 ++ symmetric positive-definite linear equations with a single right-
 ++ hand side,  Ax=b, using a Cholesky factorization and iterative
 ++ refinement.
 ++ See \downlink{Manual Page}{manpageXXf04asf}.
f04atf : (Matrix DoubleFloat,Integer,Matrix DoubleFloat,Integer,_
    Integer,Integer) -> Result
 ++ f04atf(a,ia,b,n,iaa,ifail)
 ++ calculates the accurate solution of a set of real linear
 ++ equations with a single right-hand side, using an LU
 ++ factorization with partial pivoting, and iterative refinement.
 ++ See \downlink{Manual Page}{manpageXXf04atf}.
f04axf : (Integer,Matrix DoubleFloat,Integer,Matrix Integer,_
    Matrix Integer,Integer,Matrix Integer,Matrix DoubleFloat) -> Result
 ++ f04axf(n,a,licn,icn,ikeep,mtype,idisp,rhs)
 ++ calculates the approximate solution of a set of real
 ++ sparse linear equations with a single right-hand side, Ax=b or
 ++  T
 ++ A x=b, where A has been factorized by F01BRF or F01BSF.
 ++ See \downlink{Manual Page}{manpageXXf04axf}.
f04faf : (Integer,Integer,Matrix DoubleFloat,Matrix DoubleFloat,_
    Matrix DoubleFloat,Integer) -> Result
 ++ f04faf(job,n,d,e,b,ifail)
 ++ calculates the approximate solution of a set of real
 ++ symmetric positive-definite tridiagonal linear equations.
 ++ See \downlink{Manual Page}{manpageXXf04faf}.
f04jgf : (Integer,Integer,Integer,DoubleFloat,_
    Integer,Matrix DoubleFloat,Matrix DoubleFloat,Integer) -> Result
 ++ f04jgf(m,n,nra,tol,lwork,a,b,ifail)
 ++ finds the solution of a linear least-squares problem, Ax=b
 ++ , where A is a real m by n (m>=n) matrix and b is an m element
 ++ vector. If the matrix of observations is not of full rank, then
 ++ the minimal least-squares solution is returned.
 ++ See \downlink{Manual Page}{manpageXXf04jgf}.
f04maf : (Integer,Integer,Matrix DoubleFloat,Integer,_
    Matrix Integer,Integer,Matrix Integer,Matrix DoubleFloat,Matrix Integer,M
 ++ f04maf(n,nz,avals,licn,irn,lirn,icn,wkeep,ikeep,inform,b,acc,noits,ifail)
 ++ e a sparse symmetric positive-definite system of linear
 ++ equations, Ax=b, using a pre-conditioned conjugate gradient
 ++ method, where A has been factorized by F01MAF.
```

```
    ++ See \downlink{Manual Page}{manpageXXf04maf}.
   f04mbf : (Integer,Matrix DoubleFloat,Boolean,DoubleFloat,_
       Integer,Integer,Integer,Integer,DoubleFloat,Integer,Union(fn:FileName,fp:Asp28(APROD
    ++ f04mbf(n,b,precon,shift,itnlim,msglvl,lrwork,liwork,rtol,ifail,aprod,msolve)
    ++ solves a system of real sparse symmetric linear equations
    ++ using a Lanczos algorithm.
    ++ See \downlink{Manual Page}{manpageXXf04mbf}.
   f04mcf : (Integer,Matrix DoubleFloat,Integer,Matrix DoubleFloat,_
       Matrix Integer,Integer,Matrix DoubleFloat,Integer,Integer,Integer,Integer) -> Result
    ++ f04mcf(n,al,lal,d,nrow,ir,b,nrb,iselct,nrx,ifail)
    ++ computes the approximate solution of a system of real
    ++ linear equations with multiple right-hand sides,  AX=B,  where A
    ++ is a symmetric positive-definite variable-bandwidth matrix, which
    ++ has previously been factorized by F01MCF. Related systems may
    ++ also be solved.
    ++ See \downlink{Manual Page}{manpageXXf04mcf}.
   f04qaf : (Integer,Integer,DoubleFloat,DoubleFloat,_
       DoubleFloat,DoubleFloat,Integer,Integer,Integer,Integer,Matrix DoubleFloat,Integer,U
    ++ f04qaf(m,n,damp,atol,btol,conlim,itnlim,msglvl,lrwork,liwork,b,ifail,aprod)
    ++ solves sparse unsymmetric equations, sparse linear least-
    ++ squares problems and sparse damped linear least-squares problems,
    ++ using a Lanczos algorithm.
    ++ See \downlink{Manual Page}{manpageXXf04qaf}.
  Implementation ==> add

   import Lisp
   import DoubleFloat
   import Any
   import Record
   import Integer
   import Matrix DoubleFloat
   import Boolean
   import NAGLinkSupportPackage
   import FortranPackage
   import AnyFunctions1(Integer)
   import AnyFunctions1(DoubleFloat)
   import AnyFunctions1(Boolean)
   import AnyFunctions1(Matrix Complex DoubleFloat)
   import AnyFunctions1(Matrix DoubleFloat)
   import AnyFunctions1(Matrix Integer)


   f04adf(iaArg:Integer,bArg:Matrix Complex DoubleFloat,ibArg:Integer,_
       nArg:Integer,mArg:Integer,icArg:Integer,_
       aArg:Matrix Complex DoubleFloat,ifailArg:Integer): Result ==
       [(invokeNagman(NIL$Lisp,_
```

```
        "f04adf",_
        ["ia"::S,"ib"::S,"n"::S,"m"::S,"ic"::S_
        ,"ifail"::S,"b"::S,"c"::S,"a"::S,"wkspce"::S]$Lisp,_
        ["c"::S,"wkspce"::S]$Lisp,_
        [["double"::S,["wkspce"::S,"n"::S]$Lisp]$Lisp_
        ,["integer"::S,"ia"::S,"ib"::S,"n"::S,"m"::S_
        ,"ic"::S,"ifail"::S]$Lisp_
        ,["double complex"::S,["b"::S,"ib"::S,"m"::S]$Lisp,["c"::S,"ic"::S,"m"::S
        ]$Lisp,_
        ["c"::S,"a"::S,"ifail"::S]$Lisp,_
        [([iaArg::Any,ibArg::Any,nArg::Any,mArg::Any,icArg::Any,ifailArg::Any,bAr
        @List Any]$Lisp)$Lisp)_
        pretend List (Record(key:Symbol,entry:Any))]$Result

    f04arf(iaArg:Integer,bArg:Matrix DoubleFloat,nArg:Integer,_
        aArg:Matrix DoubleFloat,ifailArg:Integer): Result ==
        [(invokeNagman(NIL$Lisp,_
        "f04arf",_
        ["ia"::S,"n"::S,"ifail"::S,"b"::S,"c"::S,"a"::S,"wkspce"::S]$Lisp,_
        ["c"::S,"wkspce"::S]$Lisp,_
        [["double"::S,["b"::S,"n"::S]$Lisp,["c"::S,"n"::S]$Lisp_
        ,["a"::S,"ia"::S,"n"::S]$Lisp,["wkspce"::S,"n"::S]$Lisp]$Lisp_
        ,["integer"::S,"ia"::S,"n"::S,"ifail"::S]$Lisp_
        ]$Lisp,_
        ["c"::S,"a"::S,"ifail"::S]$Lisp,_
        [([iaArg::Any,nArg::Any,ifailArg::Any,bArg::Any,aArg::Any ])_
        @List Any]$Lisp)$Lisp)_
        pretend List (Record(key:Symbol,entry:Any))]$Result

    f04asf(iaArg:Integer,bArg:Matrix DoubleFloat,nArg:Integer,_
        aArg:Matrix DoubleFloat,ifailArg:Integer): Result ==
        [(invokeNagman(NIL$Lisp,_
        "f04asf",_
        ["ia"::S,"n"::S,"ifail"::S,"b"::S,"c"::S,"a"::S,"wk1"::S,"wk2"::S_
        ]$Lisp,_
        ["c"::S,"wk1"::S,"wk2"::S]$Lisp,_
        [["double"::S,["b"::S,"n"::S]$Lisp,["c"::S,"n"::S]$Lisp_
        ,["a"::S,"ia"::S,"n"::S]$Lisp,["wk1"::S,"n"::S]$Lisp,["wk2"::S,"n"::S]$Li
        ,["integer"::S,"ia"::S,"n"::S,"ifail"::S]$Lisp_
        ]$Lisp,_
        ["c"::S,"a"::S,"ifail"::S]$Lisp,_
        [([iaArg::Any,nArg::Any,ifailArg::Any,bArg::Any,aArg::Any ])_
        @List Any]$Lisp)$Lisp)_
        pretend List (Record(key:Symbol,entry:Any))]$Result

    f04atf(aArg:Matrix DoubleFloat,iaArg:Integer,bArg:Matrix DoubleFloat,_
```

```
        nArg:Integer,iaaArg:Integer,ifailArg:Integer): Result ==
        [(invokeNagman(NIL$Lisp,_
        "f04atf",_
        ["ia"::S,"n"::S,"iaa"::S,"ifail"::S,"a"::S,"b"::S,"c"::S,"aa"::S,"wks1"::S_
        ,"wks2"::S]$Lisp,_
        ["c"::S,"aa"::S,"wks1"::S,"wks2"::S]$Lisp,_
        [["double"::S,["a"::S,"ia"::S,"n"::S]$Lisp_
        ,["b"::S,"n"::S]$Lisp,["c"::S,"n"::S]$Lisp,["aa"::S,"iaa"::S,"n"::S]$Lisp,["wks1"::S
        ]$Lisp_
        ,["integer"::S,"ia"::S,"n"::S,"iaa"::S,"ifail"::S_
        ]$Lisp_
        ]$Lisp,_
        ["c"::S,"aa"::S,"ifail"::S]$Lisp,_
        [([iaArg::Any,nArg::Any,iaaArg::Any,ifailArg::Any,aArg::Any,bArg::Any ])_
        @List Any]$Lisp)$Lisp)_
        pretend List (Record(key:Symbol,entry:Any))]$Result

    f04axf(nArg:Integer,aArg:Matrix DoubleFloat,licnArg:Integer,_
        icnArg:Matrix Integer,ikeepArg:Matrix Integer,mtypeArg:Integer,_
        idispArg:Matrix Integer,rhsArg:Matrix DoubleFloat): Result ==
        [(invokeNagman(NIL$Lisp,_
        "f04axf",_
        ["n"::S,"licn"::S,"mtype"::S,"resid"::S,"a"::S,"icn"::S,"ikeep"::S,"idisp"::S,"rhs"
        ,"w"::S]$Lisp,_
        ["resid"::S,"w"::S]$Lisp,_
        [["double"::S,["a"::S,"licn"::S]$Lisp,"resid"::S_
        ,["rhs"::S,"n"::S]$Lisp,["w"::S,"n"::S]$Lisp]$Lisp_
        ,["integer"::S,"n"::S,"licn"::S,["icn"::S,"licn"::S]$Lisp_
        ,["ikeep"::S,["*"::S,"n"::S,5$Lisp]$Lisp]$Lisp,"mtype"::S,["idisp"::S,2$Lisp]$Lisp]
        ]$Lisp,_
        ["resid"::S,"rhs"::S]$Lisp,_
        [([nArg::Any,licnArg::Any,mtypeArg::Any,aArg::Any,icnArg::Any,ikeepArg::Any,idispArg
        @List Any]$Lisp)$Lisp)_
        pretend List (Record(key:Symbol,entry:Any))]$Result

    f04faf(jobArg:Integer,nArg:Integer,dArg:Matrix DoubleFloat,_
        eArg:Matrix DoubleFloat,bArg:Matrix DoubleFloat,ifailArg:Integer): Result ==
        [(invokeNagman(NIL$Lisp,_
        "f04faf",_
        ["job"::S,"n"::S,"ifail"::S,"d"::S,"e"::S,"b"::S]$Lisp,_
        []$Lisp,_
        [["double"::S,["d"::S,"n"::S]$Lisp,["e"::S,"n"::S]$Lisp_
        ,["b"::S,"n"::S]$Lisp]$Lisp_
        ,["integer"::S,"job"::S,"n"::S,"ifail"::S]$Lisp_
        ]$Lisp,_
        ["d"::S,"e"::S,"b"::S,"ifail"::S]$Lisp,_
```

```
    [([jobArg::Any,nArg::Any,ifailArg::Any,dArg::Any,eArg::Any,bArg::Any ])_
    @List Any]$Lisp)$Lisp)_
    pretend List (Record(key:Symbol,entry:Any))]$Result

f04jgf(mArg:Integer,nArg:Integer,nraArg:Integer,_
    tolArg:DoubleFloat,lworkArg:Integer,aArg:Matrix DoubleFloat,_
    bArg:Matrix DoubleFloat,ifailArg:Integer): Result ==
    [(invokeNagman(NIL$Lisp,_
    "f04jgf",_
    ["m"::S,"n"::S,"nra"::S,"tol"::S,"lwork"::S_
    ,"svd"::S,"sigma"::S,"irank"::S,"ifail"::S,"work"::S,"a"::S,"b"::S]$Lisp,
    ["svd"::S,"sigma"::S,"irank"::S,"work"::S]$Lisp,_
    [["double"::S,"tol"::S,"sigma"::S,["work"::S,"lwork"::S]$Lisp_
    ,["a"::S,"nra"::S,"n"::S]$Lisp,["b"::S,"m"::S]$Lisp]$Lisp_
    ,["integer"::S,"m"::S,"n"::S,"nra"::S,"lwork"::S_
    ,"irank"::S,"ifail"::S]$Lisp_
    ,["logical"::S,"svd"::S]$Lisp_
    ]$Lisp,_
    ["svd"::S,"sigma"::S,"irank"::S,"work"::S,"a"::S,"b"::S,"ifail"::S]$Lisp,
    [([mArg::Any,nArg::Any,nraArg::Any,tolArg::Any,lworkArg::Any,ifailArg::An
    @List Any]$Lisp)$Lisp)_
    pretend List (Record(key:Symbol,entry:Any))]$Result

f04maf(nArg:Integer,nzArg:Integer,avalsArg:Matrix DoubleFloat,_
    licnArg:Integer,irnArg:Matrix Integer,lirnArg:Integer,_
    icnArg:Matrix Integer,wkeepArg:Matrix DoubleFloat,ikeepArg:Matrix Integer
    informArg:Matrix Integer,bArg:Matrix DoubleFloat,accArg:Matrix DoubleFloa
    noitsArg:Matrix Integer,ifailArg:Integer): Result ==
    [(invokeNagman(NIL$Lisp,_
    "f04maf",_
    ["n"::S,"nz"::S,"licn"::S,"lirn"::S,"ifail"::S_
    ,"avals"::S,"irn"::S,"icn"::S,"wkeep"::S,"ikeep"::S_
    ,"inform"::S,"work"::S,"b"::S,"acc"::S,"noits"::S_
    ]$Lisp,_
    ["work"::S]$Lisp,_
    [["double"::S,["avals"::S,"licn"::S]$Lisp,["wkeep"::S,["*"::S,3$Lisp,"n":
    ,["work"::S,["*"::S,3$Lisp,"n"::S]$Lisp]$Lisp,["b"::S,"n"::S]$Lisp,["acc"
    ]$Lisp_
    ,["integer"::S,"n"::S,"nz"::S,"licn"::S,["irn"::S,"lirn"::S]$Lisp_
    ,"lirn"::S,["icn"::S,"licn"::S]$Lisp,["ikeep"::S,["*"::S,2$Lisp,"n"::S]$L
    ,["noits"::S,2$Lisp]$Lisp,"ifail"::S]$Lisp_
    ]$Lisp,_
    ["work"::S,"b"::S,"acc"::S,"noits"::S,"ifail"::S]$Lisp,_
    [([nArg::Any,nzArg::Any,licnArg::Any,lirnArg::Any,ifailArg::Any,avalsArg:
    @List Any]$Lisp)$Lisp)_
    pretend List (Record(key:Symbol,entry:Any))]$Result
```

```
    f04mbf(nArg:Integer,bArg:Matrix DoubleFloat,preconArg:Boolean,_
        shiftArg:DoubleFloat,itnlimArg:Integer,msglvlArg:Integer,_
        lrworkArg:Integer,liworkArg:Integer,rtolArg:DoubleFloat,_
        ifailArg:Integer,aprodArg:Union(fn:FileName,fp:Asp28(APROD)),msolveArg:Union(fn:File
-- if both asps are AXIOM generated we do not need lrwork liwork
--   and will set to 1.
-- else believe the user but check that they are >0.
        if (aprodArg case fp) and (msolveArg case fp)
                then
                        lrworkArg:=1
                        liworkArg:=1
                else
                        lrworkArg:=max(1,lrworkArg)
                        liworkArg:=max(1,liworkArg)
        pushFortranOutputStack(aprodFilename := aspFilename "aprod")$FOP
        if aprodArg case fn
                  then outputAsFortran(aprodArg.fn)
                  else outputAsFortran(aprodArg.fp)
        popFortranOutputStack()$FOP
        pushFortranOutputStack(msolveFilename := aspFilename "msolve")$FOP
        if msolveArg case fn
                  then outputAsFortran(msolveArg.fn)
                  else outputAsFortran(msolveArg.fp)
        popFortranOutputStack()$FOP
        [(invokeNagman([aprodFilename,msolveFilename]$Lisp,_
        "f04mbf",_
        ["n"::S,"precon"::S,"shift"::S,"itnlim"::S,"msglvl"::S_
        ,"lrwork"::S,"liwork"::S,"itn"::S,"anorm"::S,"acond"::S_
        ,"rnorm"::S,"xnorm"::S,"inform"::S,"rtol"::S,"ifail"::S_
        ,"aprod"::S,"msolve"::S,"b"::S,"x"::S,"work"::S,"rwork"::S,"iwork"::S_
        ]$Lisp,_
        ["x"::S,"itn"::S,"anorm"::S,"acond"::S,"rnorm"::S,"xnorm"::S,"inform"::S,"work"::S,"
        [["double"::S,["b"::S,"n"::S]$Lisp,"shift"::S_
        ,["x"::S,"n"::S]$Lisp,"anorm"::S,"acond"::S,"rnorm"::S,"xnorm"::S,"rtol"::S,["work"
        ,"aprod"::S,"msolve"::S]$Lisp_
        ,["integer"::S,"n"::S,"itnlim"::S,"msglvl"::S_
        ,"lrwork"::S,"liwork"::S,"itn"::S,"inform"::S,"ifail"::S,["iwork"::S,"liwork"::S]$Li
        ,["logical"::S,"precon"::S]$Lisp_
        ]$Lisp,_
        ["x"::S,"itn"::S,"anorm"::S,"acond"::S,"rnorm"::S,"xnorm"::S,"inform"::S,"rtol"::S,"
        [([nArg::Any,preconArg::Any,shiftArg::Any,itnlimArg::Any,msglvlArg::Any,lrworkArg::A
        @List Any]$Lisp)$Lisp)_
        pretend List (Record(key:Symbol,entry:Any))]$Result

    f04mcf(nArg:Integer,alArg:Matrix DoubleFloat,lalArg:Integer,_
```

```
    dArg:Matrix DoubleFloat,nrowArg:Matrix Integer,irArg:Integer,_
    bArg:Matrix DoubleFloat,nrbArg:Integer,iselctArg:Integer,_
    nrxArg:Integer,ifailArg:Integer): Result ==
    [(invokeNagman(NIL$Lisp,_
    "f04mcf",_
    ["n"::S,"lal"::S,"ir"::S,"nrb"::S,"iselct"::S_
    ,"nrx"::S,"ifail"::S,"al"::S,"d"::S,"nrow"::S,"b"::S,"x"::S_
    ]$Lisp,_
    ["x"::S]$Lisp,_
    [["double"::S,["al"::S,"lal"::S]$Lisp,["d"::S,"n"::S]$Lisp_
    ,["b"::S,"nrb"::S,"ir"::S]$Lisp,["x"::S,"nrx"::S,"ir"::S]$Lisp]$Lisp_
    ,["integer"::S,"n"::S,"lal"::S,["nrow"::S,"n"::S]$Lisp_
    ,"ir"::S,"nrb"::S,"iselct"::S,"nrx"::S,"ifail"::S]$Lisp_
    ]$Lisp,_
    ["x"::S,"ifail"::S]$Lisp,_
    [([nArg::Any,lalArg::Any,irArg::Any,nrbArg::Any,iselctArg::Any,nrxArg::An
    @List Any]$Lisp)$Lisp)_
    pretend List (Record(key:Symbol,entry:Any))]$Result

f04qaf(mArg:Integer,nArg:Integer,dampArg:DoubleFloat,_
    atolArg:DoubleFloat,btolArg:DoubleFloat,conlimArg:DoubleFloat,_
    itnlimArg:Integer,msglvlArg:Integer,lrworkArg:Integer,_
    liworkArg:Integer,bArg:Matrix DoubleFloat,ifailArg:Integer,_
    aprodArg:Union(fn:FileName,fp:Asp30(APROD))): Result ==
    pushFortranOutputStack(aprodFilename := aspFilename "aprod")$FOP
    if aprodArg case fn
              then outputAsFortran(aprodArg.fn)
              else outputAsFortran(aprodArg.fp)
    popFortranOutputStack()$FOP
    [(invokeNagman([aprodFilename]$Lisp,_
    "f04qaf",_
    ["m"::S,"n"::S,"damp"::S,"atol"::S,"btol"::S_
    ,"conlim"::S,"itnlim"::S,"msglvl"::S,"lrwork"::S,"liwork"::S_
    ,"itn"::S,"anorm"::S,"acond"::S,"rnorm"::S,"arnorm"::S_
    ,"xnorm"::S,"inform"::S,"ifail"::S,"aprod"::S,"x"::S,"se"::S,"b"::S,"work
    ,"iwork"::S]$Lisp,_
    ["x"::S,"se"::S,"itn"::S,"anorm"::S,"acond"::S,"rnorm"::S,"arnorm"::S,"xn
    [["double"::S,"damp"::S,"atol"::S,"btol"::S_
    ,"conlim"::S,["x"::S,"n"::S]$Lisp,["se"::S,"n"::S]$Lisp,"anorm"::S,"acond
    ,["work"::S,"n"::S,2$Lisp]$Lisp,["rwork"::S,"lrwork"::S]$Lisp,"aprod"::S]
    ,["integer"::S,"m"::S,"n"::S,"itnlim"::S,"msglvl"::S_
    ,"lrwork"::S,"liwork"::S,"itn"::S,"inform"::S,"ifail"::S,["iwork"::S,"liw
    ]$Lisp,_
    ["x"::S,"se"::S,"itn"::S,"anorm"::S,"acond"::S,"rnorm"::S,"arnorm"::S,"xn
    [([mArg::Any,nArg::Any,dampArg::Any,atolArg::Any,btolArg::Any,conlimArg::
    @List Any]$Lisp)$Lisp)_
```

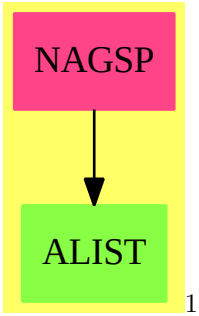
```
          pretend List (Record(key:Symbol,entry:Any))]$Result
```

⟨*NAGF04.dotabb*⟩≡

```
"NAGF04" [color="#FF4488",href="bookvol10.4.pdf#nameddest=NAGF04"]
"COMPCAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=COMPCAT"]
"NAGF04" -> "COMPCAT"
```

15.7 package NAGSP NAGLinkSupportPackage

15.8 NAGLinkSupportPackage



1.00

Exports:

aspFilename	checkPrecision	dimensionsOf	fortranCompilerName
fortranLinkerArgs	restorePrecision		

⟨*package NAGSP NAGLinkSupportPackage*⟩≡

```
)abbrev package NAGSP NAGLinkSupportPackage
++ Author: Mike Dewar and Godfrey Nolan
++ Date Created:  March 1993
++ Date Last Updated: March 4 1994
++                    October 6 1994
++ Basic Operations:
++ Related Domains:
++ Also See:
++ AMS Classifications:
++ Keywords:
++ Examples:
++ References:
++ Description: Support functions for the NAG Library Link functions
NAGLinkSupportPackage() : exports == implementation where

  exports ==> with
    fortranCompilerName : () -> String
      ++ fortranCompilerName() returns the name of the currently selected
      ++  Fortran compiler
    fortranLinkerArgs   : () -> String
      ++ fortranLinkerArgs() returns the current linker arguments
    aspFilename         : String -> String
      ++ aspFilename("f") returns a String consisting of "f" suffixed with
      ++  an extension identifying the current AXIOM session.
    dimensionsOf        : (Symbol, Matrix DoubleFloat) -> SExpression
```

```
        ++ dimensionsOf(s,m) \undocumented{}
    dimensionsOf        : (Symbol, Matrix Integer) -> SExpression
        ++ dimensionsOf(s,m) \undocumented{}
    checkPrecision      : () -> Boolean
        ++ checkPrecision() \undocumented{}
    restorePrecision    : () -> Void
        ++ restorePrecision() \undocumented{}

  implementation ==> add
    makeAs:                   (Symbol,Symbol) -> Symbol
    changeVariables:          (Expression Integer,Symbol) -> Expression Integer
    changeVariablesF:         (Expression Float,Symbol) -> Expression Float

    import String
    import Symbol

    checkPrecision():Boolean ==
      (_$fortranPrecision$Lisp = "single"::Symbol) and (_$nagEnforceDouble$Lisp)  =>
        systemCommand("set fortran precision double")$MoreSystemCommands
        if _$nagMessages$Lisp  then
          print("*** Warning: Resetting fortran precision to double")$PrintPackage
        true
      false

    restorePrecision():Void ==
      systemCommand("set fortran precision single")$MoreSystemCommands
      if _$nagMessages$Lisp  then
        print("** Warning: Restoring fortran precision to single")$PrintPackage
      void()$Void

    uniqueId : String := ""
    counter : Integer := 0
    getUniqueId():String ==
      if uniqueId = "" then
        uniqueId := concat(getEnv("HOST")$Lisp,getEnv("SPADNUM")$Lisp)
      concat(uniqueId,string (counter:=counter+1))

    fortranCompilerName() == string _$fortranCompilerName$Lisp
    fortranLinkerArgs() == string _$fortranLibraries$Lisp

    aspFilename(f:String):String == concat ["/tmp/",f,getUniqueId(),".f"]

    dimensionsOf(u:Symbol,m:Matrix DoubleFloat):SExpression ==
      [u,nrows m,ncols m]$Lisp
    dimensionsOf(u:Symbol,m:Matrix Integer):SExpression ==
      [u,nrows m,ncols m]$Lisp
```

⟨*NAGSP.dotabb*⟩≡

```
"NAGSP" [color="#FF4488",href="bookvol10.4.pdf#nameddest=NAGSP"]
"ALIST" [color="#88FF44",href="bookvol10.3.pdf#nameddest=ALIST"]
"NAGSP" -> "ALIST"
```

15.9 package NAGD01 NagIntegrationPackage

⟨*NagIntegrationPackage.help*⟩≡

```
D01(3NAG)        Foundation Library (12/10/92)        D01(3NAG)

     D01 -- Quadrature                              Introduction -- D01
                              Chapter D01
                              Quadrature

     1. Scope of the Chapter

     This chapter provides routines for the numerical evaluation of
     definite integrals in one or more dimensions and for evaluating
     weights and abscissae of integration rules.

     2. Background to the Problems

     The routines in this chapter are designed to estimate:

     (a)  the value of a one-dimensional definite integral of the
          form:
                                  b
                                  /
                                  |f(x)dx                          (1)
                                  /
                                  a
          where f(x) is defined by the user, either at a set of
          points (x ,f(x )), for i=1,2,...,n where a=x <x <...<x =b,
                   i    i                             1  2     n
          or in the form of a function; and the limits of integration
          a,b may be finite or infinite.

          Some methods are specially designed for integrands of the
          form
                             f(x)=w(x)g(x)                         (2)
          which contain a factor w(x), called the weight-function, of
          a specific form. These methods take full account of any
          peculiar behaviour attributable to the w(x) factor.

     (b)  the value of a multi-dimensional definite integral of the
          form:
                       /
                       | f(x ,x ,...,x )dx ...dx dx                (3)
                       /    1  2      n   n     2  1
                       R
```

$$n$$
where $f(x_1,x_2,\ldots,x_n)$ is a function defined by the user and R_n is some region of n-dimensional space.

The simplest form of R_n is the n-rectangle defined by:

$$a_i <= x_i <= b_i, \quad i=1,2,\ldots,n \tag{4}$$

where a_i and b_i are constants. When a_i and b_i are functions of x_j $(j<i)$, the region can easily be transformed to the rectangular form (see Davis and Rabinowitz [1] page 266). Some of the methods described incorporate the transformation procedure.

2.1. One-dimensional Integrals

To estimate the value of a one-dimensional integral, a quadrature rule uses an approximation in the form of a weighted sum of integrand values, i.e.,

$$\int_a^b f(x)dx \approx \sum_{i=1}^{N} w_i f(x_i). \tag{5}$$

The points x_i within the interval [a,b] are known as the abscissae, and the w_i are known as the weights.

More generally, if the integrand has the form (2), the corresponding formula is

$$\int_a^b w(x)g(x)dx \approx \sum_{i=1}^{N} w_i g(x_i). \tag{6}$$

If the integrand is known only at a fixed set of points, these points must be used as the abscissae, and the weighted sum is

```
calculated using finite-difference methods. However, if the
functional form of the integrand is known, so that its value at
any abscissa is easily obtained, then a wide variety of
quadrature rules are available, each characterised by its choice
of abscissae and the corresponding weights.

The appropriate rule to use will depend on the interval [a,b] -
whether finite or otherwise - and on the form of any w(x) factor
in the integrand. A suitable value of N depends on the general
behaviour of f(x); or of g(x), if there is a w(x) factor present.

Among possible rules, we mention particularly the Gaussian
formulae, which employ a distribution of abscissae which is
optimal for f(x) or g(x) of polynomial form.

The choice of basic rules constitutes one of the principles on
which methods for one-dimensional integrals may be classified.
The other major basis of classification is the implementation
strategy, of which some types are now presented.

(a)   Single rule evaluation procedures

      A fixed number of abscissae, N, is used. This number and
      the particular rule chosen uniquely determine the weights
      and abscissae. No estimate is made of the accuracy of the
      result.

(b)   Automatic procedures

      The number of abscissae, N, within [a,b] is gradually
      increased until consistency is achieved to within a level
      of accuracy (absolute or relative) requested by the user.
      There are essentially two ways of doing this; hybrid forms
      of these two methods are also possible:
      (i)   whole interval procedures (non-adaptive)

            A series of rules using increasing values of N are
            successively applied over the whole interval [a,b].
            It is clearly more economical if abscissae already
            used for a lower value of N can be used again as part
            of a higher-order formula. This principle is known as
             optimal extension. There is no overlap between the
            abscissae used in Gaussian formulae of different
            orders. However, the Kronrod formulae are designed to
            give an optimal (2N+1)-point formula by adding (N+1)
            points to an N-point Gauss formula. Further
```

extensions have been developed by Patterson.

(ii) adaptive procedures

The interval [a,b] is repeatedly divided into a number of sub-intervals, and integration rules are applied separately to each sub-interval. Typically, the subdivision process will be carried further in the neighbourhood of a sharp peak in the integrand, than where the curve is smooth. Thus, the distribution of abscissae is adapted to the shape of the integrand.

Subdivision raises the problem of what constitutes an acceptable accuracy in each sub-interval. The usual global acceptability criterion demands that the sum of the absolute values of the error estimates in the sub-intervals should meet the conditions required of the error over the whole interval. Automatic extrapolation over several levels of subdivision may eliminate the effects of some types of singularities.

An ideal general-purpose method would be an automatic method which could be used for a wide variety of integrands, was efficient (i.e., required the use of as few abscissae as possible), and was reliable (i.e., always gave results within the requested accuracy). Complete reliability is unobtainable, and generally higher reliability is obtained at the expense of efficiency, and vice versa. It must therefore be emphasised that the automatic routines in this chapter cannot be assumed to be 100% reliable. In general, however, the reliability is very high.

2.2. Multi-dimensional Integrals

A distinction must be made between cases of moderately low dimensionality (say, up to 4 or 5 dimensions), and those of higher dimensionality. Where the number of dimensions is limited, a one-dimensional method may be applied to each dimension, according to some suitable strategy, and high accuracy may be obtainable (using product rules). However, the number of integrand evaluations rises very rapidly with the number of dimensions, so that the accuracy obtainable with an acceptable amount of computational labour is limited; for example a product of 3-point rules in 20 dimensions would require more than 3^{20} integrand evaluations. Special techniques such as the Monte Carlo

methods can be used to deal with high dimensions.

(a) Products of one-dimensional rules

Using a two-dimensional integral as an example, we have

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2} f(x,y)\,dy\,dx \simeq \sum_{i=1}^{N} w_i \left[\int_{a_2}^{b_2} f(x_i,y)\,dy\right] \qquad (7)$$

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2} f(x,y)\,dy\,dx \simeq \sum_{i=1}^{N}\sum_{j=1}^{N} w_i v_j f(x_i,y_j) \qquad (8)$$

where (w_i, x_i) and (v_i, y_i) are the weights and abscissae of the rules used in the respective dimensions.

A different one-dimensional rule may be used for each dimension, as appropriate to the range and any weight function present, and a different strategy may be used, as appropriate to the integrand behaviour as a function of each independent variable.

For a rule-evaluation strategy in all dimensions, the formula (8) is applied in a straightforward manner. For automatic strategies (i.e., attempting to attain a requested accuracy), there is a problem in deciding what accuracy must be requested in the inner integral(s). Reference to formula (7) shows that the presence of a limited but random error in the y-integration for different values of x_i can produce a 'jagged' function of x, which may be difficult to integrate to the desired accuracy and for this reason products of automatic one-dimensional routines should be used with caution (see also Lyness [3]).

(b) Monte Carlo methods

These are based on estimating the mean value of the

integrand sampled at points chosen from an appropriate statistical distribution function. Usually a variance reducing procedure is incorporated to combat the fundamentally slow rate of convergence of the rudimentary form of the technique. These methods can be effective by comparison with alternative methods when the integrand contains singularities or is erratic in some way, but they are of quite limited accuracy.

(c) Number theoretic methods

These are based on the work of Korobov and Conroy and operate by exploiting implicitly the properties of the Fourier expansion of the integrand. Special rules, constructed from so-called optimal coefficients, give a particularly uniform distribution of the points throughout n-dimensional space and from their number theoretic properties minimize the error on a prescribed class of integrals. The method can be combined with the Monte Carlo procedure.

(d) Sag-Szekeres method

By transformation this method seeks to induce properties into the integrand which make it accurately integrable by the trapezoidal rule. The transformation also allows effective control over the number of integrand evaluations.

(e) Automatic adaptive procedures

An automatic adaptive strategy in several dimensions normally involves division of the region into subregions, concentrating the divisions in those parts of the region where the integrand is worst behaved. It is difficult to arrange with any generality for variable limits in the inner integral(s). For this reason, some methods use a region where all the limits are constants; this is called a hyper-rectangle. Integrals over regions defined by variable or infinite limits may be handled by transformation to a hyper-rectangle. Integrals over regions so irregular that such a transformation is not feasible may be handled by surrounding the region by an appropriate hyper-rectangle and defining the integrand to be zero outside the desired region. Such a technique should always be followed by a Monte Carlo method for integration.

```
      The method used locally in each subregion produced by the
      adaptive subdivision process is usually one of three types:
      Monte Carlo, number theoretic or deterministic.
      Deterministic methods are usually the most rapidly
      convergent but are often expensive to use for high
      dimensionality and not as robust as the other techniques.

2.3. References

Comprehensive reference:

[1]   Davis P J and Rabinowitz P (1975) Methods of Numerical
      Integration. Academic Press.

Special topics:

[2]   Gladwell I (1986) Vectorisation of one dimensional
      quadrature codes. Techincal Report. TR7/86  NAG.

[3]   Lyness J N (1983) When not to use an automatic quadrature
      routine. SIAM Review. 25 63--87.

[4]   Piessens R, De Doncker-Kapenga E, Uberhuber C and Kahaner D
      (1983) QUADPACK, A Subroutine Package for Automatic
      Integration. Springer-Verlag.

[5]   Sobol I M (1974) The Monte Carlo Method. The University of
      Chicago Press.

[6]   Stroud A H (1971) Approximate Calculation of Multiple
      Integrals. Prentice-Hall.

3. Recommendations on Choice and Use of Routines

The following three sub-sections consider in turn routines for:
one-dimensional integrals over a finite interval, and over a
semi-infinite or an infinite interval; and multi-dimensional
integrals. Within each sub-section, routines are classified by
the type of method, which ranges from simple rule evaluation to
automatic adaptive algorithms. The recommendations apply
particularly when the primary objective is simply to compute the
value of one or more integrals, and in these cases the automatic
adaptive routines are generally the most convenient and reliable,
although also the most expensive in computing time.

Note however that in some circumstances it may be counter-
```

productive to use an automatic routine. If the results of the quadrature are to be used in turn as input to a further computation (e.g. an 'outer' quadrature or an optimization problem), then this further computation may be adversely affected by the 'jagged performance profile' of an automatic routine; a simple rule-evaluation routine may provide much better overall performance. For further guidance, the article Lyness [3] is recommended.

3.1. One-dimensional Integrals over a Finite Interval

(a) Integrand defined as a set of points

If f(x) is defined numerically at four or more points, then the Gill-Miller finite difference method (D01GAF) should be used. The interval of integration is taken to coincide with the range of x-values of the points supplied. It is in the nature of this problem that any routine may be unreliable. In order to check results independently and so as to provide an alternative technique the user may fit the integrand by Chebyshev series using E02ADF and then use routines E02AJF and E02AKF to evaluate its integral (which need not be restricted to the range of the integration points, as is the case for D01GAF). A further alternative is to fit a cubic spline to the data using E02BAF and then to evaluate its integral using E02BDF.

(b) Integrand defined as a function

If the functional form of f(x) is known, then one of the following approaches should be taken. They are arranged in the order from most specific to most general, hence the first applicable procedure in the list will be the most efficient. However, if the user does not wish to make any assumptions about the integrand, the most reliable routine to use will be D01AJF, although this will in general be less efficient for simple integrals.

(i) Rule-evaluation routines

If f(x) is known to be sufficiently well-behaved (more precisely, can be closely approximated by a polynomial of moderate degree), a Gaussian routine with a suitable number of abscissae may be used.

D01BBF may be used if it is required to examine the weights and abscissae. In this case, the user should

write the code for the evaluation of quadrature summation (6).

(ii) Automatic adaptive routines

Firstly, several routines are available for integrands of the form w(x)g(x) where g(x) is a ' smooth' function (i.e., has no singularities, sharp peaks or violent oscillations in the interval of integration) and w(x) is a weight function of one of the following forms:

```
                (alpha)     (beta)          k          l
if w(x)=(b-x)       (x-a)      (log(b-x)) (log(x-a))
```

where k,l=0 or 1, (alpha),(beta)>-1: use D01APF;

if w(x)=1/(x-c): use D01AQF (this integral is called the Hilbert transform of g);

if w(x)=cos((omega)x) or sin((omega)x): use D01ANF (this routine can also handle certain types of singularities in g(x)).

Secondly, there are some routines for general f(x). If f(x) is known to be free of singularities, though it may be oscillatory, D01AKF may be used.

The most powerful of the finite interval integration routine is D01AJF (which can cope with singularities of several types). It may be used if none of the more specific situations described above applies. D01AJF is very reliable, particularly where the integrand has singularities other than at an end-point, or has discontinuities or cusps, and is therefore recommended where the integrand is known to be badly-behaved, or where its nature is completely unknown.

Most of the routines in this chapter require the user to supply a function or subroutine to evaluate the integrand at a single point.

If f(x) has singularities of certain types, discontinuities or sharp peaks occurring at known points, the integral should be evaluated separately over each of the subranges or D01ALF may be used.

3.2. One-dimensional Integrals over a Semi-infinite or Infinite Interval

(a) Integrand defined as a set of points

If f(x) is defined numerically at four or more points, and the portion of the integral lying outside the range of the points supplied may be neglected, then the Gill-Miller finite difference method, D01GAF, should be used.

(b) Integrand defined as a function

(i) Rule evaluation routines

If f(x) behaves approximately like a polynomial in x, apart from a weight function of the form

$e^{-(\beta)x}$ $(\beta)>0$ (semi-infinite interval, lower limit finite);

or $e^{-(\beta)x}$ $(\beta)<0$ (semi-infinite interval, upper limit finite);

or $e^{-(\beta)(x-(\alpha))^2}$ $(\beta)>0$ (infinite interval);

or if f(x) behaves approximately like a polynomial in $(x+B)^{-1}$ (semi-infinite range); then the Gaussian routines may be used.

D01BBF may be used if it is required to examine the weights and abscissae. In this case, the user should write the code for the evaluation of quadrature summation (6).

(ii) Automatic adaptive routines

D01AMF may be used, except for integrands which decay slowly towards an infinite end-point, and oscillate in sign over the entire range. For this class, it may be possible to calculate the integral by integrating between the zeros and invoking some extrapolation process.

D01ASF may be used for integrals involving weight functions of the form cos((omega)x) and sin((omega)x) over a semi-infinite interval (lower limit finite).

The following alternative procedures are mentioned for completeness, though their use will rarely be necessary:

(1) If the integrand decays rapidly towards an infinite end-point, a finite cut-off may be chosen, and the finite range methods applied.

(2) If the only irregularities occur in the finite part (apart from a singularity at the finite limit, with which D01AMF can cope), the range may be divided, with D01AMF used on the infinite part.

(3) A transformation to finite range may be employed, e.g.

$$x = \frac{1-t}{t} \quad \text{or} \quad x = -\log t$$

will transform (0,infty) to (1,0) while for infinite ranges we have

$$\int_{-\infty}^{+\infty} f(x)\,dx = \int_{0}^{\infty} [f(x)+f(-x)]\,dx.$$

If the integrand behaves badly on (-infty,0) and well on (0,infty) or vice versa it is better to compute it as

$$\int_{-\infty}^{+\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx.$$

This saves computing unnecessary function values in the semi-infinite range where the function is well behaved.

3.3. Multi-dimensional Integrals

A number of techniques are available in this area and the choice depends to a large extent on the dimension and the required

accuracy. It can be advantageous to use more than one technique as a confirmation of accuracy particularly for high dimensional integrations. Many of the routines incorporate the transformation procedure REGION which allows general product regions to be easily dealt with in terms of conversion to the standard n-cube region.

(a) Products of one-dimensional rules (suitable for up to about 5 dimensions)

If $f(x_1,x_2,\ldots,x_n)$ is known to be a sufficiently well-behaved function of each variable x_i, apart possibly from weight functions of the types provided, a product of Gaussian rules may be used. These are provided by D01BBF. In this case, the user should write the code for the evaluation of quadrature summation (6). Rules for finite, semi-infinite and infinite ranges are included.

The one-dimensional routines may also be used recursively. For example, the two-dimensional integral

$$I=\int_{a_1}^{b_1}\int_{a_2}^{b_2} f(x,y)\,dy\,dx$$

may be expressed as

$$I=\int_{a_1}^{b_1} F(x)\,dx,$$

where

$$F(x)=\int_{a_2}^{b_2} f(x,y)\,dy.$$

The user segment to evaluate F(x) will call the integration

routine for the y-integration, which will call another user segment for f(x,y) as a function of y (x being effectively a constant). Note that, as Fortran is not a recursive language, a different library integration routine must be used for each dimension. Apart from this restriction, the full range of one-dimensional routines are available, for finite/infinite intervals, constant/variable limits, rule evaluation/automatic strategies etc.

(b) Automatic routines (D01GBF and D01FCF)

Both routines are for integrals of the form

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\cdots\int_{a_n}^{b_n} f(x_1,x_2,\ldots,x_n)\,dx_n\,dx_{n-1}\ldots dx_1 .$$

D01GBF is an adaptive Monte Carlo routine. This routine is usually slow and not recommended for high accuracy work. It is a robust routine that can often be used for low accuracy results with highly irregular integrands or when n is large.

D01FCF is an adaptive deterministic routine. Convergence is fast for well-behaved integrands. Highly accurate results can often be obtained for n between 2 and 5, using significantly fewer integrand evaluations than would be required by D01GBF. The routine will usually work when the integrand is mildly singular and for n<=10 should be used before D01GBF. If it is known in advance that the integrand is highly irregular, it is best to compare results from at least two different routines.

There are many problems for which one or both of the routines will require large amounts of computing time to obtain even moderately accurate results. The amount of computing time is controlled by the number of integrand evaluations allowed by the user, and users should set this parameter carefully, with reference to the time available and the accuracy desired.

3.4. Decision Trees

(i) One-dimensional integrals over a finite interval. (If in doubt, follow the downward branch.)

Please see figure in printed Reference Manual

(ii) One-dimensional integrals over a semi-infinite or infinite interval. (If in doubt, follow the downward branch.)

Please see figure in printed Reference Manual

D01 -- Quadrature Contents -- D01
Chapter D01

Quadrature

D01AJF 1-D quadrature, adaptive, finite interval, strategy due to Piessens and de Doncker, allowing for badly-behaved integrands

D01AKF 1-D quadrature, adaptive, finite interval, method suitable for oscillating functions

D01ALF 1-D quadrature, adaptive, finite interval, allowing for singularities at user-specified break-points

D01AMF 1-D quadrature, adaptive, infinite or semi-infinite interval

D01ANF 1-D quadrature, adaptive, finite interval, weight function cos((omega)x) or sin((omega)x)

D01APF 1-D quadrature, adaptive, finite interval, weight function with end-point singularities of algebraico-logarithmic type

D01AQF 1-D quadrature, adaptive, finite interval, weight function 1/(x-c), Cauchy principal value (Hilbert transform)

D01ASF 1-D quadrature, adaptive, semi-infinite interval, weight

```
                function cos((omega)x) or sin((omega)x)

        D01BBF  Weights and abscissae for Gaussian quadrature rules

        D01FCF  Multi-dimensional adaptive quadrature over hyper-
                rectangle

        D01GAF  1-D quadrature, integration of function defined by data
                values, Gill-Miller method

        D01GBF  Multi-dimensional quadrature over hyper-rectangle, Monte
                Carlo method

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

  D01AJF(3NAG)      Foundation Library (12/10/92)      D01AJF(3NAG)

        D01 -- Quadrature                                          D01AJF
                D01AJF -- NAG Foundation Library Routine Document

        Note: Before using this routine, please read the Users' Note for
        your implementation to check implementation-dependent details.
        The symbol (*) after a NAG routine name denotes a routine that is
        not included in the Foundation Library.

        1. Purpose

        D01AJF is a general-purpose integrator which calculates an
        approximation to the integral of a function f(x) over a finite
        interval [a,b]:

                                      b
                                      /
                                   I= |f(x)dx.
                                      /
                                      a

        2. Specification

               SUBROUTINE D01AJF (F, A, B, EPSABS, EPSREL, RESULT,
              1                   ABSERR, W, LW, IW, LIW, IFAIL)
               INTEGER          LW, IW(LIW), LIW, IFAIL
               DOUBLE PRECISION F, A, B, EPSABS, EPSREL, RESULT, ABSERR, W
              1                 (LW)
               EXTERNAL         F
```

3. Description

D01AJF is based upon the QUADPACK routine QAGS (Piessens et al [3]). It is an adaptive routine, using the Gauss 10-point and Kronrod 21-point rules. The algorithm, described by de Doncker [1], incorporates a global acceptance criterion (as defined by Malcolm and Simpson [2]) together with the (epsilon)-algorithm (Wynn [4]) to perform extrapolation. The local error estimation is described by Piessens et al [3].

The routine is suitable as a general purpose integrator, and can be used when the integrand has singularities, especially when these are of algebraic or logarithmic type.

D01AJF requires the user to supply a function to evaluate the integrand at a single point.

The routine D01ATF(*) uses an identical algorithm but requires the user to supply a subroutine to evaluate the integrand at an array of points. Therefore D01ATF(*) will be more efficient if the evaluation can be performed in vector mode on a vector-processing machine.

4. References

[1] De Doncker E (1978) An Adaptive Extrapolation Algorithm for Automatic Integration. Signum Newsletter. 13 (2) 12--18.

[2] Malcolm M A and Simpson R B (1976) Local Versus Global Strategies for Adaptive Quadrature. ACM Trans. Math. Softw. 1 129--146.

[3] Piessens R, De Doncker-Kapenga E, Uberhuber C and Kahaner D (1983) QUADPACK, A Subroutine Package for Automatic Integration. Springer-Verlag.

[4] Wynn P (1956) On a Device for Computing the $e_m(S_n)$ Transformation. Math. Tables Aids Comput. 10 91--96.

5. Parameters

1: F -- DOUBLE PRECISION FUNCTION, supplied by the user.
External Procedure
F must return the value of the integrand f at a given point.

```
Its specification is:

       DOUBLE PRECISION FUNCTION F (X)
       DOUBLE PRECISION X

 1:  X -- DOUBLE PRECISION                                  Input
     On entry: the point at which the integrand f must be
     evaluated.
F must be declared as EXTERNAL in the (sub)program from
which D01AJF is called. Parameters denoted as Input
must not be changed by this procedure.

2:  A -- DOUBLE PRECISION                                  Input
    On entry: the lower limit of integration, a.

3:  B -- DOUBLE PRECISION                                  Input
    On entry: the upper limit of integration, b. It is not
    necessary that a<b.

4:  EPSABS -- DOUBLE PRECISION                             Input
    On entry: the absolute accuracy required. If EPSABS is
    negative, the absolute value is used. See Section 7.

5:  EPSREL -- DOUBLE PRECISION                             Input
    On entry: the relative accuracy required. If EPSREL is
    negative, the absolute value is used. See Section 7.

6:  RESULT -- DOUBLE PRECISION                            Output
    On exit: the approximation to the integral I.

7:  ABSERR -- DOUBLE PRECISION                            Output
    On exit: an estimate of the modulus of the absolute error,
    which should be an upper bound for |I-RESULT|.

8:  W(LW) -- DOUBLE PRECISION array                       Output
    On exit: details of the computation, as described in
    Section 8.

9:  LW -- INTEGER                                          Input
    On entry:
    the dimension of the array W as declared in the (sub)program
    from which D01AJF is called.
    The value of LW (together with that of LIW below) imposes a
    bound on the number of sub-intervals into which the interval
    of integration may be divided by the routine. The number of
    sub-intervals cannot exceed LW/4. The more difficult the
```

integrand, the larger LW should be. Suggested value: a value in the range 800 to 2000 is adequate for most problems. Constraint: LW >= 4.

10: IW(LIW) -- INTEGER array Output
On exit: IW(1) contains the actual number of sub-intervals used. The rest of the array is used as workspace.

11: LIW -- INTEGER Input
On entry:
the dimension of the array IW as declared in the (sub)program from which D01AJF is called.
The number of sub-intervals into which the interval of integration may be divided cannot exceed LIW. Suggested value: LIW = LW/4. Constraint: LIW >= 1.

12: IFAIL -- INTEGER Input/Output
On entry: IFAIL must be set to 0, -1 or 1. Users who are unfamiliar with this parameter should refer to the Essential Introduction for details.

On exit: IFAIL = 0 unless the routine detects an error or gives a warning (see Section 6).

For this routine, because the values of output parameters may be useful even if IFAIL /=0 on exit, users are recommended to set IFAIL to -1 before entry. It is then essential to test the value of IFAIL on exit.

6. Error Indicators and Warnings

Errors or warnings specified by the routine:

If on entry IFAIL = 0 or -1, explanatory error messages are output on the current error message unit (as defined by X04AAF).

IFAIL= 1
The maximum number of subdivisions allowed with the given workspace has been reached without the accuracy requirements being achieved. Look at the integrand in order to determine the integration difficulties. If the position of a local difficulty within the interval can be determined (e.g. a singularity of the integrand or its derivative, a peak, a discontinuity, etc) you will probably gain from splitting up the interval at this point and calling the integrator on the subranges. If necessary, another integrator, which is

```
     designed for handling the type of difficulty involved, must
     be used. Alternatively, consider relaxing the accuracy
     requirements specified by EPSABS and EPSREL, or increasing
     the amount of workspace.

IFAIL= 2
     Round-off error prevents the requested tolerance from being
     achieved. The error may be under-estimated. Consider
     requesting less accuracy.

IFAIL= 3
     Extremely bad local integrand behaviour causes a very strong
     subdivision around one (or more) points of the interval.

IFAIL= 4
     The requested tolerance cannot be achieved, because the
     extrapolation does not increase the accuracy satisfactorily;
     the returned result is the best which can be obtained. The
     same advice applies as in the case of IFAIL = 1.

IFAIL= 5
     The integral is probably divergent, or slowly convergent.
     Please note that divergence can occur with any non-zero
     value of IFAIL.

IFAIL= 6
     On entry LW < 4,

     or       LIW < 1.

7. Accuracy

The routine cannot guarantee, but in practice usually achieves,
the following accuracy:

                        |I-RESULT|<=tol,

where

                 tol=max{|EPSABS|,|EPSREL|*|I|},

and EPSABS and EPSREL are user-specified absolute and relative
error tolerance. Moreover it returns the quantity ABSERR which,
in normal circumstances, satisfies

                    |I-RESULT|<=ABSERR<=tol.
```

8. Further Comments

The time taken by the routine depends on the integrand and the accuracy required.

If IFAIL /= 0 on exit, then the user may wish to examine the contents of the array W, which contains the end-points of the sub-intervals used by D01AJF along with the integral contributions and error estimates over the sub-intervals.

Specifically, for i=1,2,...,n, let r_i denote the approximation to the value of the integral over the sub-interval $[a_i,b_i]$ in the partition of [a,b] and e_i be the corresponding absolute error estimate.

Then, $\int_{a_i}^{b_i} f(x)dx \tilde{=} r_i$ and RESULT= $\sum_{i=1}^{n} r_i$, unless D01AJF terminates while testing for divergence of the integral (see Piessens et al [3], Section 3.4.3). In this case, RESULT (and ABSERR) are taken to be the values returned from the extrapolation process. The value of n is returned in IW(1), and the values a_i, b_i, e_i and r_i are stored consecutively in the array W, that is:

a_i =W(i),

b_i =W(n+i),

e_i =W(2n+i) and

r_i =W(3n+i).

9. Example

```
To compute
                      2(pi)
                      /         x sin(30x)
                      |     ----------------dx.
                      /
                      0        /(  (  x  )2)
                              / (1-(-----) )
                             \/  (  (2(pi)) )

The example program is not reproduced here. The source code for
all example programs is distributed with the NAG Foundation
Library software and should be available on-line.

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     D01AKF(3NAG)      Foundation Library (12/10/92)      D01AKF(3NAG)

          D01 -- Quadrature                                          D01AKF
                  D01AKF -- NAG Foundation Library Routine Document

          Note: Before using this routine, please read the Users' Note for
          your implementation to check implementation-dependent details.
          The symbol (*) after a NAG routine name denotes a routine that is
          not included in the Foundation Library.

          1. Purpose

          D01AKF is an adaptive integrator, especially suited to
          oscillating, non-singular integrands, which calculates an
          approximation to the integral of a function f(x) over a finite
          interval [a,b]:

                                        b
                                        /
                                     I= |f(x)dx.
                                        /
                                        a

          2. Specification

                 SUBROUTINE D01AKF (F, A, B, EPSABS, EPSREL, RESULT,
                1                   ABSERR, W, LW, IW, LIW, IFAIL)
                 INTEGER          LW, IW(LIW), LIW, IFAIL
                 DOUBLE PRECISION F, A, B, EPSABS, EPSREL, RESULT, ABSERR, W
                1                 (LW)
```

```
      EXTERNAL         F
```

3. Description

D01AKF is based upon the QUADPACK routine QAG (Piessens et al [3]). It is an adaptive routine, using the Gauss 30-point and Kronrod 61-point rules. A 'global' acceptance criterion (as defined by Malcolm and Simpson [1]) is used. The local error estimation is described in Piessens et al [3].

Because this routine is based on integration rules of high order, it is especially suitable for non-singular oscillating integrands.

D01AKF requires the user to supply a function to evaluate the integrand at a single point.

The routine D01AUF(*) uses an identical algorithm but requires the user to supply a subroutine to evaluate the integrand at an array of points. Therefore D01AUF(*) will be more efficient if the evaluation can be performed in vector mode on a vector-processing machine.

D01AUF(*) also has an additional parameter KEY which allows the user to select from six different Gauss-Kronrod rules.

4. References

[1] Malcolm M A and Simpson R B (1976) Local Versus Global Strategies for Adaptive Quadrature. ACM Trans. Math. Softw. 1 129--146.

[2] Piessens R (1973) An Algorithm for Automatic Integration. Angewandte Informatik. 15 399--401.

[3] Piessens R, De Doncker-Kapenga E, Uberhuber C and Kahaner D (1983) QUADPACK, A Subroutine Package for Automatic Integration. Springer-Verlag.

5. Parameters

1: F -- DOUBLE PRECISION FUNCTION, supplied by the user.
External Procedure

F must return the value of the integrand f at a given point.

Its specification is:

```
      DOUBLE PRECISION FUNCTION F (X)
      DOUBLE PRECISION X
```

1: X -- DOUBLE PRECISION Input
On entry: the point at which the integrand f must be evaluated.

F must be declared as EXTERNAL in the (sub)program from which D01AKF is called. Parameters denoted as Input must not be changed by this procedure.

2: A -- DOUBLE PRECISION Input
On entry: the lower limit of integration, a.

3: B -- DOUBLE PRECISION Input
On entry: the upper limit of integration, b. It is not necessary that a<b.

4: EPSABS -- DOUBLE PRECISION Input
On entry: the absolute accuracy required. If EPSABS is negative, the absolute value is used. See Section 7.

5: EPSREL -- DOUBLE PRECISION Input
On entry: the relative accuracy required. If EPSREL is negative, the absolute value is used. See Section 7.

6: RESULT -- DOUBLE PRECISION Output
On exit: the approximation to the integral I.

7: ABSERR -- DOUBLE PRECISION Output
On exit: an estimate of the modulus of the absolute error, which should be an upper bound |I-RESULT|.

8: W(LW) -- DOUBLE PRECISION array Output
On exit: details of the computation, as described in Section 8.

9: LW -- INTEGER Input
On entry: the dimension of W, as declared in the (sub) program from which D01AKF is called. The value of LW (together with that of LIW below) imposes a bound on the number of sub-intervals into which the interval of integration may be divided by the routine. The number of sub-intervals cannot exceed LW/4. The more difficult the integrand, the larger LW should be. Suggested value: a value in the range 800 to 2000 is adequate for most problems.

```
     Constraint: LW >= 4. See IW below.

10:  IW(LIW) -- INTEGER array                                  Output
     On exit: IW(1) contains the actual number of sub-intervals
     used. The rest of the array is used as workspace.

11:  LIW -- INTEGER                                            Input
     On entry:
     the dimension of the array IW as declared in the
     (sub)program from which D01AKF is called.
     The number of sub-intervals into which the interval of
     integration may be divided cannot exceed LIW. Suggested
     value: LIW = LW/4. Constraint: LIW >= 1.

12:  IFAIL -- INTEGER                                  Input/Output
     On entry: IFAIL must be set to 0, -1 or 1. Users who are
     unfamiliar with this parameter should refer to the Essential
     Introduction for details.

     On exit: IFAIL = 0 unless the routine detects an error or
     gives a warning (see Section 6).

     For this routine, because the values of output parameters
     may be useful even if IFAIL /=0 on exit, users are
     recommended to set IFAIL to -1 before entry. It is then
     essential to test the value of IFAIL on exit.

6. Error Indicators and Warnings

Errors or warnings specified by the routine:

If on entry IFAIL = 0 or -1, explanatory error messages are
output on the current error message unit (as defined by X04AAF).

IFAIL= 1
     The maximum number of subdivisions allowed with the given
     workspace has been reached without the accuracy requirements
     being achieved. Look at the integrand in order to determine
     the integration difficulties. Probably another integrator
     which is designed for handling the type of difficulty
     involved must be used. Alternatively, consider relaxing the
     accuracy requirements specified by EPSABS and EPSREL, or
     increasing the amount of workspace.

IFAIL= 2
     Round-off error prevents the requested tolerance from being
```

```
     achieved. Consider  requesting less accuracy.

IFAIL= 3
     Extremely bad local integrand behaviour causes a very strong
     subdivision around one (or more) points of the interval.
     The same advice applies as in the case of IFAIL = 1.

IFAIL= 4
     On entry LW < 4,

     or       LIW < 1.

7. Accuracy

The routine cannot guarantee, but in practice usually achieves,
the following accuracy:

                        |I-RESULT|<=tol,

where

                 tol=max{|EPSABS|,|EPSREL|*|I|},

and EPSABS and EPSREL are user-specified absolute and relative
error tolerances. Moreover it returns the quantity ABSERR which,
in normal circumstances satisfies

                    |I-RESULT|<=ABSERR<=tol.

8. Further Comments

The time taken by the routine depends on the integrand and the
accuracy required.

If IFAIL /= 0 on exit, then the user may wish to examine the
contents of the array W, which contains the end-points of the
sub-intervals used by D01AKF along with the integral
contributions and error estimates over these sub-intervals.

Specifically, for i=1,2,...,n, let r  denote the approximation to
                                   i
the value of the integral over the sub-interval [a ,b ] in the
                                                 i  i
partition of [a,b] and e  be the corresponding absolute error
                       i
                 b
```

```
                    i                      n
                   /                       --
estimate. Then,    | f(x)dx~=r  and RESULT= >  r . The value of n
                   /          i            --  i
                   a                       i=1
                    i
is returned in IW(1), and the values a , b , e  and r  are stored
                                      i   i   i      i
consecutively in the array W, that is:

     a =W(i),
      i

     b =W(n+i),
      i

     e =W(2n+i) and
      i

     r =W(3n+i).
      i

9. Example

To compute

                        2(pi)
                        /
                        |    x sin(30x) cosx dx.
                        /
                        0

The example program is not reproduced here. The source code for
all example programs is distributed with the NAG Foundation
Library software and should be available on-line.
```

%%

```
     D01ALF(3NAG)      Foundation Library (12/10/92)      D01ALF(3NAG)

     D01 -- Quadrature                                          D01ALF
            D01ALF -- NAG Foundation Library Routine Document

     Note: Before using this routine, please read the Users' Note for
     your implementation to check implementation-dependent details.
     The symbol (*) after a NAG routine name denotes a routine that is
```

not included in the Foundation Library.

1. Purpose

D01ALF is a general purpose integrator which calculates an approximation to the integral of a function f(x) over a finite interval [a,b]:

$$I = \int_a^b f(x)\,dx$$

where the integrand may have local singular behaviour at a finite number of points within the integration interval.

2. Specification

```
 SUBROUTINE D01ALF (F, A, B, NPTS, POINTS, EPSABS, EPSREL,
1                   RESULT, ABSERR, W, LW, IW, LIW, IFAIL)
 INTEGER           NPTS, LW, IW(LIW), LIW, IFAIL
 DOUBLE PRECISION F, A, B, POINTS(*), EPSABS, EPSREL,
1                 RESULT, ABSERR, W(LW)
 EXTERNAL          F
```

3. Description

D01ALF is based upon the QUADPACK routine QAGP (Piessens et al [3]). It is very similar to D01AJF, but allows the user to supply difficult. It is an adaptive routine, using the Gauss 10-point and Kronrod 21-point rules. The algorithm described by de Doncker [1], incorporates a global acceptance criterion (as defined by Malcolm and Simpson [2]) together with the (epsilon)-algorithm (Wynn [4]) to perform extrapolation. The user-supplied 'break-points'always occur as the end-points of some sub-interval during the adaptive process. The local error estimation is described by Piessens et al [3].

4. References

[1] De Doncker E (1978) An Adaptive Extrapolation Algorithm for Automatic Integration. Signum Newsletter. 13 (2) 12--18.

[2] Malcolm M A and Simpson R B (1976) Local Versus Global Strategies for Adaptive Quadrature. ACM Trans. Math. Softw.

1 129--146.

[3] Piessens R, De Doncker-Kapenga E, Uberhuber C and Kahaner D (1983) QUADPACK, A Subroutine Package for Automatic Integration. Springer-Verlag.

[4] Wynn P (1956) On a Device for Computing the e (S) Transformation. Math. Tables Aids Comput. 10 91--96.

5. Parameters

1: F -- DOUBLE PRECISION FUNCTION, supplied by the user. External Procedure
F must return the value of the integrand f at a given point.

Its specification is:

```
DOUBLE PRECISION FUNCTION F (X)
DOUBLE PRECISION X
```

1: X -- DOUBLE PRECISION Input
On entry: the point at which the integrand f must be evaluated.

F must be declared as EXTERNAL in the (sub)program from which D01ALF is called. Parameters denoted as Input must not be changed by this procedure.

2: A -- DOUBLE PRECISION Input
On entry: the lower limit of integration, a.

3: B -- DOUBLE PRECISION Input
On entry: the upper limit of integration, b. It is not necessary that a<b.

4: NPTS -- INTEGER Input
On entry: the number of user-supplied break-points within the integration interval. Constraint: NPTS >= 0.

5: POINTS(NPTS) -- DOUBLE PRECISION array Input
On entry: the user-specified break-points. Constraint: the break-points must all lie within the interval of integration (but may be supplied in any order).

6: EPSABS -- DOUBLE PRECISION Input
On entry: the absolute accuracy required. If EPSABS is

```
     negative, the absolute value is used. See Section 7.

 7:  EPSREL -- DOUBLE PRECISION                                  Input
     On entry: the relative accuracy required. If EPSREL is
     negative, the absolute value is used. See Section 7.

 8:  RESULT -- DOUBLE PRECISION                                  Input
     On entry: the approximation to the integral I.

 9:  ABSERR -- DOUBLE PRECISION                                 Output
     On exit: an estimate of the modulus of the absolute error,
     which should be an upper bound for |I-RESULT|.

10:  W(LW) -- DOUBLE PRECISION array                            Output
     On exit: details of the computation, as described in
     Section 8.

11:  LW -- INTEGER                                              Input
     On entry:
     the dimension of the array W as declared in the (sub)program
     from which D01ALF is called.
     The value of LW (together with that of LIW below) imposes a
     bound on the number of sub-intervals into which the interval
     of integration may be divided by the routine. The number of
     sub-intervals cannot exceed (LW-2*NPTS-4)/4. The more
     difficult the integrand, the larger LW should be. Suggested
     value: a value in the range 800 to 2000 is adequate for most
     problems. Constraint: LW>=2*NPTS+8.

12:  IW(LIW) -- INTEGER array                                   Output
     On exit: IW(1) contains the actual number of sub-intervals
     used. The rest of the array is used as workspace.

13:  LIW -- INTEGER                                             Input
     On entry:
     the dimension of the array IW as declared in the
     (sub)program from which D01ALF is called.
     The number of sub-intervals into which the interval of
     integration may be divided cannot exceed (LIW-NPTS-2)/2.
     Suggested value: LIW = LW/2. Constraint: LIW >= NPTS + 4.

14:  IFAIL -- INTEGER                                   Input/Output
     On entry: IFAIL must be set to 0, -1 or 1. Users who are
     unfamiliar with this parameter should refer to the Essential
     Introduction for details.
```

```
     On exit: IFAIL = 0 unless the routine detects an error or
     gives a warning (see Section 6).

     For this routine, because the values of output parameters
     may be useful even if IFAIL /=0 on exit, users are
     recommended to set IFAIL to -1 before entry. It is then
     essential to test the value of IFAIL on exit.

6. Error Indicators and Warnings

Errors or warnings specified by the routine:

If on entry IFAIL = 0 or -1, explanatory error messages are
output on the current error message unit (as defined by X04AAF).

IFAIL= 1
     The maximum number of subdivisions allowed with the given
     workspace has been reached, without the accuracy
     requirements being achieved. Look at the integrand in order
     to determine the integration difficulties. If the position
     of a local difficulty within the interval can be determined
     (e.g. a singularity of the integrand or its derivative, a
     peak, a discontinuity, etc) it should be supplied to the
     routine as an element of the vector POINTS. If necessary,
     another integrator should be used, which is designed for
     handling the type of difficulty involved. Alternatively,
     consider relaxing the accuracy requirements specified by
     EPSABS and EPSREL, or increasing the amount of workspace.

IFAIL= 2
     Round-off error prevents the requested tolerance from being
     achieved. The error may be under-estimated. Consider
     requesting less accuracy.

IFAIL= 3
     Extremely bad local integrand behaviour causes a very strong
     subdivision around one (or more) points of the interval.
     The same advice applies as in the case of IFAIL = 1.

IFAIL= 4
     The requested tolerance cannot be achieved, because the
     extrapolation does not increase the accuracy satisfactorily;
     the result returned is the best which can be obtained.  The
     same advice applies as in the case IFAIL = 1.

IFAIL= 5
```

```
     The integral is probably divergent, or slowly convergent.
     Please note that divergence can also occur with any other
     non-zero value of IFAIL.

IFAIL= 6
     The input is invalid: break-points are specified outside the
     integration range, NPTS > LIMIT or NPTS < 0. RESULT and
     ABSERR are set to zero.

IFAIL= 7
     On entry LW<2*NPTS+8,

     or       LIW<NPTS+4.

7. Accuracy

The routine cannot guarantee, but in practice usually achieves,
the following accuracy:
                        |I-RESULT|<=tol,

where

                 tol=max{|EPSABS|,|EPSREL|*|I|},

and EPSABS and EPSREL are user-specified absolute and relative
error tolerances. Moreover it returns the quantity ABSERR which,
in normal circumstances, satisfies

                    |I-RESULT|<=ABSERR<=tol.

8. Further Comments

The time taken by the routine depends on the integrand and on the
accuracy required.

If IFAIL /= 0 on exit, then the user may wish to examine the
contents of the array W, which contains the end-points of the
sub-intervals used by D01ALF along with the integral
contributions and error estimates over these sub-intervals.

Specifically, for i=1,2,...,n, let r  denote the approximation to
                                   i
the value of the integral over the sub-interval [a ,b ] in the
                                                 i  i
partition of [a,b] and e  be the corresponding absolute error
                       i
```

```
                  b
                   i                      n
                  /                       --
estimate. Then,   | f(x)dx~=r  and RESULT= >  r  unless D01ALF
                  /          i            --  i
                  a                       i=1
                   i
```

terminates while testing for divergence of the integral (see Piessens et al [3] Section 3.4.3). In this case, RESULT (and ABSERR) are taken to be the values returned from the extrapolation process. The value of n is returned in IW(1), and the values a , b , e and r are stored consecutively in the
                  i   i   i   i
array W, that is:

a =W(i),
 i

b =W(n+i),
 i

e =W(2n+i) and
 i

r =W(3n+i).
 i

9. Example

To compute

```
                          1
                          /     1
                          | ---------dx.
                          /
                          0 \/|x-1/7|
```

A break-point is specified at x=1/7, at which point the integrand is infinite. (For definiteness the function FST returns the value 0.0 at this point.)

The example program is not reproduced here. The source code for all example programs is distributed with the NAG Foundation Library software and should be available on-line.

%%%

```
D01AMF(3NAG)      Foundation Library (12/10/92)      D01AMF(3NAG)

     D01 -- Quadrature                                          D01AMF
            D01AMF -- NAG Foundation Library Routine Document

     Note: Before using this routine, please read the Users' Note for
     your implementation to check implementation-dependent details.
     The symbol (*) after a NAG routine name denotes a routine that is
     not included in the Foundation Library.

     1. Purpose

     D01AMF calculates an approximation to the integral of a function
     f(x) over an infinite or semi-infinite interval [a,b]:

                                   b
                                   /
                                I= |f(x)dx
                                   /
                                   a

     2. Specification

           SUBROUTINE D01AMF (F, BOUND, INF, EPSABS, EPSREL, RESULT,
          1                   ABSERR, W, LW, IW, LIW, IFAIL)
           INTEGER          INF, LW, IW(LIW), LIW, IFAIL
           DOUBLE PRECISION F, BOUND, EPSABS, EPSREL, RESULT, ABSERR,
          1                 W(LW)
           EXTERNAL         F

     3. Description

     D01AMF is based on the QUADPACK routine QAGI (Piessens et al [3])
     [0,1] using one of the identities:

                     a             1
                     /             / (   1-t) 1
                     |     f(x)dx= |f(a- ---) --dt
                     /             / (    t )  2
                     -infty        0          t

                      infty        1
                      /            / (   1-t) 1
                      |    f(x)dx= |f(a+ ---) --dt
                      /            / (    t )  2
```

```
                  a           0          t

infty         infty                1
/             /                    /[ ( 1-t)  ( -1+t)] 1
|     f(x)dx= |    (f(x)+f(-x))dx= |[f( ---)+f( ----)] --dt
/             /                    /[ (  t )  (  t  )]  2
-infty        0                    0                   t
```

where a represents a finite integration limit. An adaptive procedure, based on the Gauss seven-point and Kronrod 15-point rules, is then employed on the transformed integral. The algorithm, described by de Doncker [1], incorporates a global acceptance criterion (as defined by Malcolm and Simpson [2]) together with the (epsilon)-algorithm (Wynn [4]) to perform extrapolation. The local error estimation is described by Piessens et al [3].

4. References

[1] De Doncker E (1978) An Adaptive Extrapolation Algorithm for Automatic Integration. Signum Newsletter. 13 (2) 12--18.

[2] Malcolm M A and Simpson R B (1976) Local Versus Global Strategies for Adaptive Quadrature. ACM Trans. Math. Softw. 1 129--146.

[3] Piessens R, De Doncker-Kapenga E, Uberhuber C and Kahaner D (1983) QUADPACK, A Subroutine Package for Automatic Integration. Springer-Verlag.

[4] Wynn P (1956) On a Device for Computing the e (S)
                                                    m n
Transformation. Math. Tables Aids Comput. 10 91--96.

5. Parameters

1: F -- DOUBLE PRECISION FUNCTION, supplied by the user.
External Procedure
On entry: the point at which the integrand f must be evaluated.

Its specification is:

```
DOUBLE PRECISION FUNCTION F (X)
DOUBLE PRECISION X
```

```
 1:  X -- DOUBLE PRECISION                                  Input
     On entry: the point at which the integrand f must be
     evaluated.
F must be declared as EXTERNAL in the (sub)program from
which D01AMF is called. Parameters denoted as Input
must not be changed by this procedure.

2:  BOUND -- DOUBLE PRECISION                              Input
    On entry: the finite limit of the integration range (if
    present). BOUND is not used if the interval is doubly
    infinite.

3:  INF -- INTEGER                                         Input
    On entry: indicates the kind of integration range:
         if INF =1, the range is [BOUND, +infty)

         if INF =-1, the range is (-infty, BOUND]

         if INF =+2, the range is (-infty, +infty).
     Constraint: INF =-1, 1 or 2.

4:  EPSABS -- DOUBLE PRECISION                             Input
    On entry: the absolute accuracy required. If EPSABS is
    negative, the absolute value is used. See Section 7.

5:  EPSREL -- DOUBLE PRECISION                             Input
    On entry: the relative accuracy required. If EPSREL is
    negative, the absolute value is used. See Section 7.

6:  RESULT -- DOUBLE PRECISION                            Output
    On exit: the approximation to the integral I.

7:  ABSERR -- DOUBLE PRECISION                            Output
    On exit: an estimate of the modulus of the absolute error,
    which should be an upper bound for |I-RESULT|.

8:  W(LW) -- DOUBLE PRECISION array                       Output
    On exit: details of the computation, as described in
    Section 8.

9:  LW -- INTEGER                                          Input
    On entry:
    the dimension of the array W as declared in the (sub)program
    from which D01AMF is called.
    The value of LW (together with that of LIW below) imposes a
    bound on the number of sub-intervals into which the interval
```

of integration may be divided by the routine. The number of sub-intervals cannot exceed LW/4. The more difficult the integrand, the larger LW should be. Suggested value: a value in the range 800 to 2000 is adequate for most problems. Constraint: LW >= 4.

10: IW(LIW) -- INTEGER array Output
On exit: IW(1) contains the actual number of sub-intervals used. The rest of the array is used as workspace.

11: LIW -- INTEGER Input
On entry:
the dimension of the array IW as declared in the (sub)program from which D01AMF is called.
The number of sub-intervals into which the interval of integration may be divided cannot exceed LIW. Suggested value: LIW = LW/4. Constraint: LIW >= 1.

12: IFAIL -- INTEGER Input/Output
On entry: IFAIL must be set to 0, -1 or 1. Users who are unfamiliar with this parameter should refer to the Essential Introduction for details.
On exit: IFAIL = 0 unless the routine detects an error or gives a warning (see Section 6).

For this routine, because the values of output parameters may be useful even if IFAIL /=0 on exit, users are recommended to set IFAIL to -1 before entry. It is then essential to test the value of IFAIL on exit.

6. Error Indicators and Warnings

Errors or warnings specified by the routine:

If on entry IFAIL = 0 or -1, explanatory error messages are output on the current error message unit (as defined by X04AAF).

IFAIL= 1
The maximum number of subdivisions allowed with the given workspace has been reached without the requested accuracy requirements being achieved. Look at the integrand in order to determine the integration difficulties. If the position of a local difficulty within the interval can be determined (e.g. a singularity of the integrand or its derivative, a peak, a discontinuity, etc) you will probably gain from splitting up the interval at this point and calling D01AMF

```
     on the infinite subrange and an appropriate integrator on
     the finite subrange. Alternatively, consider relaxing the
     accuracy requirements specified by EPSABS and EPSREL, or
     increasing the amount of workspace.

IFAIL= 2
     Round-off error prevents the requested tolerance from being
     achieved. The error may be underestimated. Consider
     requesting less accuracy.

IFAIL= 3
     Extremely bad local integrand behaviour causes a very strong
     subdivision around one (or more) points of the interval.
     The same advice applies as in the case of IFAIL = 1.

IFAIL= 4
     The requested tolerance cannot be achieved, because the
     extrapolation does not increase the accuracy satisfactorily;
     the returned result is the best which can be obtained.  The
     same advice applies as in the case of IFAIL = 1.

IFAIL= 5
     The integral is probably divergent, or slowly convergent. It
     must be noted that divergence can also occur with any other
     non-zero value of IFAIL.

IFAIL= 6
     On entry LW < 4,

     or       LIW < 1,

     or       INF /= -1, 1 or 2.
     Please note that divergence can occur with any non-zero
     value of IFAIL.

7. Accuracy

The routine cannot guarantee, but in practice usually achieves,
the following accuracy:

                         |I-RESULT|<=tol,

where

                  tol=max{|EPSABS|,|EPSREL|*|I|},
```

and EPSABS and EPSREL are user-specified absolute and relative error tolerances. Moreover it returns the quantity ABSERR, which, in normal circumstances, satisfies

|I-RESULT|<=ABSERR<=tol.

8. Further Comments

The time taken by the routine depends on the integrand and the accuracy required.

If IFAIL /= 0 on exit, then the user may wish to examine the contents of the array W, which contains the end-points of the sub-intervals used by D01AMF along with the integral contributions and error estimates over these sub-intervals.

Specifically, for i=1,2,...,n, let r_i denote the approximation to the value of the integral over the sub-interval $[a_i,b_i]$ in the partition of [a,b] and e_i be the corresponding absolute error estimate. Then, $\int_{a_i}^{b_i} f(x)dx \tilde{=} r_i$ and $RESULT = \sum_{i=1}^{n} r_i$ unless D01AMF terminates while testing for divergence of the integral (see Piessens et al [3] Section 3.4.3). In this case, RESULT (and ABSERR) are taken to be the values returned from the extrapolation process. The value of n is returned in IW(1), and the values a_i, b_i, e_i and r_i are stored consecutively in the array W, that is:

a_i =W(i), b_i =W(n+i), e_i =W(2n+i) and r_i =W(3n+i).

Note: that this information applies to the integral transformed to (0,1) as described in Section 3, not to the original integral.

9. Example

To compute

```
                                infty
                                /        1
                                |     --------dx.
                                /
                                0     (x+1)\/x
```

The exact answer is (pi).

The example program is not reproduced here. The source code for all example programs is distributed with the NAG Foundation Library software and should be available on-line.

%%%

D01ANF(3NAG) Foundation Library (12/10/92) D01ANF(3NAG)

D01 -- Quadrature D01ANF
D01ANF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for your implementation to check implementation-dependent details. The symbol (*) after a NAG routine name denotes a routine that is not included in the Foundation Library.

1. Purpose

D01ANF calculates an approximation to the sine or the cosine transform of a function g over [a,b]:

```
   b                                   b
   /                                   /
I= |g(x)||,sin((omega)x)||,dx  or  I= |g(x)||,cos((omega)x)||,dx
   /                                   /
   a                                   a
```

(for a user-specified value of (omega)).

2. Specification

```
      SUBROUTINE D01ANF (G, A, B, OMEGA, KEY, EPSABS, EPSREL,
     1                   RESULT, ABSERR, W, LW, IW, LIW, IFAIL)
      INTEGER           KEY, LW, IW(LIW), LIW, IFAIL
      DOUBLE PRECISION G, A, B, OMEGA, EPSABS, EPSREL, RESULT,
     1                 ABSERR, W(LW)
      EXTERNAL          G
```

3. Description

D01ANF is based upon the QUADPACK routine QFOUR (Piessens et al [3]). It is an adaptive routine, designed to integrate a function of the form g(x)w(x), where w(x) is either sin((omega)x) or cos((omega)x). If a sub-interval has length

$$L=|b-a|2^{-l}$$

then the integration over this sub-interval is performed by means of a modified Clenshaw-Curtis procedure (Piessens and Branders [2]) if L(omega)>4 and l<=20. In this case a Chebyshev-series approximation of degree 24 is used to approximate g(x), while an error estimate is computed from this approximation together with that obtained using Chebyshev-series of degree 12. If the above conditions do not hold then Gauss 7-point and Kronrod 15-point rules are used. The algorithm, described in [3], incorporates a global acceptance criterion (as defined in Malcolm and Simpson [1]) together with the (epsilon)-algorithm Wynn [4] to perform extrapolation. The local error estimation is described in [3].

4. References

[1] Malcolm M A and Simpson R B (1976) Local Versus Global Strategies for Adaptive Quadrature. ACM Trans. Math. Softw. 1 129--146.

[2] Piessens R and Branders M (1975) Algorithm 002. Computation of Oscillating Integrals. J. Comput. Appl. Math. 1 153--164.

[3] Piessens R, De Doncker-Kapenga E, Uberhuber C and Kahaner D (1983) QUADPACK, A Subroutine Package for Automatic Integration. Springer-Verlag.

[4] Wynn P (1956) On a Device for Computing the $e_m(S_n)$ Transformation. Math. Tables Aids Comput. 10 91--96.

5. Parameters

1: G -- DOUBLE PRECISION FUNCTION, supplied by the user. External Procedure

G must return the value of the function g at a given point.

```
Its specification is:

       DOUBLE PRECISION FUNCTION G (X)
       DOUBLE PRECISION X

 1:  X -- DOUBLE PRECISION                                  Input
     On entry: the point at which the function g must be
     evaluated.
G must be declared as EXTERNAL in the (sub)program from
which D01ANF is called. Parameters denoted as Input
must not be changed by this procedure.

 2:  A -- DOUBLE PRECISION                                  Input
     On entry: the lower limit of integration, a.

 3:  B -- DOUBLE PRECISION                                  Input
     On entry: the upper limit of integration, b. It is not
     necessary that a<b.

 4:  OMEGA -- DOUBLE PRECISION                              Input
     On entry: the parameter (omega) in the weight function of
     the transform.

 5:  KEY -- INTEGER                                         Input
     On entry: indicates which integral is to be computed:
          if KEY = 1, w(x)=cos((omega)x);

          if KEY = 2, w(x)=sin((omega)x).
      Constraint: KEY = 1 or 2.

 6:  EPSABS -- DOUBLE PRECISION                             Input
     On entry: the absolute accuracy required. If EPSABS is
     negative, the absolute value is used. See Section 7.

 7:  EPSREL -- DOUBLE PRECISION                             Input
     On entry: the relative accuracy required. If EPSREL is
     negative, the absolute value is used. See Section 7.

 8:  RESULT -- DOUBLE PRECISION                            Output
     On exit: the approximation to the integral I.

 9:  ABSERR -- DOUBLE PRECISION                            Output
     On exit: an estimate of the modulus of the absolute error,
     which should be an upper bound for |I-RESULT|.

10:  W(LW) -- DOUBLE PRECISION array                       Output
```

On exit: details of the computation, as described in Section 8.

11: LW -- INTEGER Input
On entry:
the dimension of the array W as declared in the (sub)program from which D01ANF is called.
The value of LW (together with that of LIW below) imposes a bound on the number of sub-intervals into which the interval of integration may be divided by the routine. The number of sub-intervals cannot exceed LW/4. The more difficult the integrand, the larger LW should be. Suggested value: a value in the range 800 to 2000 is adequate for most problems.
Constraint: LW >= 4.

12: IW(LIW) -- INTEGER array Output
On exit: IW(1) contains the actual number of sub-intervals used. The rest of the array is used as workspace.

13: LIW -- INTEGER Input
On entry:
the dimension of the array IW as declared in the (sub)program from which D01ANF is called.
The number of sub-intervals into which the interval of integration may be divided cannot exceed LIW/2. Suggested value: LIW = LW/2. Constraint: LIW >= 2.

14: IFAIL -- INTEGER Input/Output
On entry: IFAIL must be set to 0, -1 or 1. Users who are unfamiliar with this parameter should refer to the Essential Introduction for details.

On exit: IFAIL = 0 unless the routine detects an error or gives a warning (see Section 6).

For this routine, because the values of output parameters may be useful even if IFAIL /=0 on exit, users are recommended to set IFAIL to -1 before entry. It is then essential to test the value of IFAIL on exit.

6. Error Indicators and Warnings

Errors or warnings specified by the routine:

If on entry IFAIL = 0 or -1, explanatory error messages are output on the current error message unit (as defined by X04AAF).

```
IFAIL= 1
     The maximum number of subdivisions allowed with the given
     workspace has been reached without the accuracy requested
     being achieved. Look at the integrand in order to determine
     the integration difficulties. If the position of a local
     difficulty within the interval can be determined (e.g. a
     singularity of the integrand or its derivative, a peak, a
     discontinuity, etc) you will probably gain from splitting up
     the interval at this point and calling the integrator on the
     subranges. If necessary, another integrator, which is
     designed for handling the type of difficulty involved, must
     be used. Alternatively consider relaxing the accuracy
     requirements specified by EPSABS and EPSREL, or increasing
     amount of workspace.

IFAIL= 2
     Round-off error prevents the requested tolerance from being
     achieved. The error may be underestimated. Consider
     requesting less accuracy.

IFAIL= 3
     Extremely bad local behaviour of g(x) causes a very strong
     subdivision around one (or more) points of the interval.
     The same advice applies as in the case of IFAIL = 1.

IFAIL= 4
     The requested tolerance cannot be achieved because the
     extrapolation does not increase the accuracy satisfactorily;
     the returned result is the best which can be obtained.  The
     same advice applies as in the case of IFAIL = 1.

IFAIL= 5
     The integral is probably divergent, or slowly convergent. It
     must be noted that divergence can occur with any non-zero
     value of IFAIL.

IFAIL= 6
     On entry KEY < 1,

     or       KEY > 2.

IFAIL= 7
     On entry LW < 4,

     or       LIW < 2.
```

7. Accuracy

The routine cannot guarantee, but in practice usually achieves, the following accuracy:

$$|I-RESULT| \leq tol,$$

where

$$tol = \max\{|EPSABS|, |EPSREL| * |I|\},$$

and EPSABS and EPSREL are user-specified absolute and relative tolerances. Moreover it returns the quantity ABSERR, which, in normal circumstances, satisfies

$$|I-RESULT| \leq ABSERR \leq tol.$$

8. Further Comments

The time taken by the routine depends on the integrand and on the accuracy required.

If IFAIL /= 0 on exit, then the user may wish to examine the contents of the array W, which contains the end-points of the sub-intervals used by D01ANF along with the integral contributions and error estimates over these sub-intervals.

Specifically, for i=1,2,...,n, let r_i denote the approximation to the value of the integral over the sub-interval $[a_i, b_i]$ in the partition of [a,b] and e_i be the corresponding absolute error estimate. Then, $\int_{a_i}^{b_i} g(x)w(x)dx \tilde{=} r_i$ and $RESULT = \sum_{i=1}^{n} r_i$ unless D01ANF terminates while testing for divergence of the integral (see Piessens et al [3] Section 3.4.3). In this case, RESULT (and ABSERR) are taken to be the values returned from the extrapolation process. The value of n is returned in IW(1), and the values a_1, b_1, e_1 and r_1 are stored consecutively in the

```
                  i   i   i      i
array W, that is:

     a =W(i),
      i

     b =W(n+i),
      i

     e =W(2n+i) and
      i

     r =W(3n+i).
      i

9. Example

To compute

                          1
                          /
                          |ln(x)sin(10(pi)x)dx.
                          /
                          0

The example program is not reproduced here. The source code for
all example programs is distributed with the NAG Foundation
Library software and should be available on-line.
```

%%%

```
D01APF(3NAG)      Foundation Library (12/10/92)      D01APF(3NAG)

D01 -- Quadrature                                              D01APF
        D01APF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for
your implementation to check implementation-dependent details.
The symbol (*) after a NAG routine name denotes a routine that is
not included in the Foundation Library.

1. Purpose

D01APF is an adaptive integrator which calculates an
approximation to the integral of a function g(x)w(x) over a
finite interval [a,b]:
```

$$I = \int_a^b g(x)w(x)\,dx$$

where the weight function w has end-point singularities of algebraico-logarithmic type.

2. Specification

```
 SUBROUTINE D01APF (G, A, B, ALFA, BETA, KEY, EPSABS,
1                   EPSREL, RESULT, ABSERR, W, LW, IW, LIW,
2                   IFAIL)
 INTEGER           KEY, LW, IW(LIW), LIW, IFAIL
 DOUBLE PRECISION G, A, B, ALFA, BETA, EPSABS, EPSREL,
1                 RESULT, ABSERR, W(LW)
 EXTERNAL          G
```

3. Description

D01APF is based upon the QUADPACK routine QAWSE (Piessens et al [3]) and integrates a function of the form g(x)w(x), where the weight function w(x) may have algebraico-logarithmic singularities at the end-points a and/or b. The strategy is a modification of that in D01AKF. We start by bisecting the original interval and applying modified Clenshaw-Curtis integration of orders 12 and 24 to both halves. Clenshaw-Curtis integration is then used on all sub-intervals which have a or b as one of their end-points (Piessens et al [2]). On the other sub-intervals Gauss-Kronrod (7-15 point) integration is carried out.

A 'global' acceptance criterion (as defined by Malcolm and Simpson [1]) is used. The local error estimation control is described by Piessens et al [3].

4. References

[1] Malcolm M A and Simpson R B (1976) Local Versus Global Strategies for Adaptive Quadrature. ACM Trans. Math. Softw. 1 129--146.

[2] Piessens R, Mertens I and Branders M (1974) Integration of Functions having End-point Singularities. Angewandte

```
      Informatik. 16 65--68.

[3]   Piessens R, De Doncker-Kapenga E, Uberhuber C and Kahaner D
      (1983) QUADPACK, A Subroutine Package for Automatic
      Integration. Springer-Verlag.

5. Parameters

 1:  G -- DOUBLE PRECISION FUNCTION, supplied by the user.
                                                 External Procedure
     G must return the value of the function g at a given point
     X.

     Its specification is:

            DOUBLE PRECISION FUNCTION G (X)
            DOUBLE PRECISION X

      1:  X -- DOUBLE PRECISION                              Input
          On entry: the point at which the function g must be
          evaluated.
     G must be declared as EXTERNAL in the (sub)program from
     which D01APF is called. Parameters denoted as Input
     must not be changed by this procedure.

 2:  A -- DOUBLE PRECISION                                  Input
     On entry: the lower limit of integration, a.

 3:  B -- DOUBLE PRECISION                                  Input
     On entry: the upper limit of integration, b. Constraint: B
     > A.

 4:  ALFA -- DOUBLE PRECISION                               Input
     On entry: the parameter (alpha) in the weight function.
     Constraint: ALFA>-1.

 5:  BETA -- DOUBLE PRECISION                               Input
     On entry: the parameter (beta) in the weight function.
     Constraint: BETA>-1.

 6:  KEY -- INTEGER                                         Input
     On entry: indicates which weight function is to be used:
                                (alpha)     (beta)
          if KEY = 1, w(x)=(x-a)       (b-x)

                                (alpha)     (beta)
```

```
if KEY = 2, w(x)=(x-a)       (b-x)      ln(x-a)

                          (alpha)     (beta)
if KEY = 3, w(x)=(x-a)       (b-x)      ln(b-x)

                          (alpha)     (beta)
if KEY = 4, w(x)=(x-a)       (b-x)     ln(x-a)ln(b-x)
```

Constraint: KEY = 1, 2, 3 or 4

7: EPSABS -- DOUBLE PRECISION Input
On entry: the absolute accuracy required. If EPSABS is negative, the absolute value is used. See Section 7.

8: EPSREL -- DOUBLE PRECISION Input
On entry: the relative accuracy required. If EPSREL is negative, the absolute value is used. See Section 7.

9: RESULT -- DOUBLE PRECISION Output
On exit: the approximation to the integral I.

10: ABSERR -- DOUBLE PRECISION Output
On exit: an estimate of the modulus of the absolute error, which should be an upper bound for |I-RESULT|.

11: W(LW) -- DOUBLE PRECISION array Output
On exit: details of the computation, as described in Section 8.

12: LW -- INTEGER Input
On entry:
the dimension of the array W as declared in the (sub)program from which D01APF is called.
The value of LW (together with that of LIW below) imposes a bound on the number of sub-intervals into which the interval of integration may be divided by the routine. The number of sub-intervals cannot exceed LW/4. The more difficult the integrand, the larger LW should be. Suggested value: LW = 800 to 2000 is adequate for most problems. Constraint: LW >= 8.

13: IW(LIW) -- INTEGER array Output
On exit: IW(1) contains the actual number of sub-intervals used. The rest of the array is used as workspace.

14: LIW -- INTEGER Input

```
     On entry:
     the dimension of the array IW as declared in the
     (sub)program from which D01APF is called.
     The number of sub-intervals into which the interval of
     integration may be divided cannot exceed LIW. Suggested
     value: LIW = LW/4. Constraint: LIW >= 2.

15:  IFAIL -- INTEGER                                  Input/Output
     On entry: IFAIL must be set to 0, -1 or 1. Users who are
     unfamiliar with this parameter should refer to the Essential
     Introduction for details.

     On exit: IFAIL = 0 unless the routine detects an error or
     gives a warning (see Section 6).

     For this routine, because the values of output parameters
     may be useful even if IFAIL /=0 on exit, users are
     recommended to set IFAIL to -1 before entry. It is then
     essential to test the value of IFAIL on exit.

6. Error Indicators and Warnings

Errors or warnings specified by the routine:

If on entry IFAIL = 0 or -1, explanatory error messages are
output on the current error message unit (as defined by X04AAF).

IFAIL= 1
     The maximum number of subdivisions allowed with the given
     workspace has been reached without the accuracy requirements
     being achieved. Look at the integrand in order to determine
     the integration difficulties. If the position of a
     discontinuity or a singularity of algebraico-logarithmic
     type within the interval can be determined, the interval
     must be split up at this point and the integrator called on
     the subranges. If necessary, another integrator, which is
     designed for handling the difficulty involved, must be used.
     Alternatively consider relaxing the accuracy requirements
     specified by EPSABS and EPSREL, or increasing the amount of
     workspace.

IFAIL= 2
     Round-off error prevents the requested tolerance from being
     achieved. Consider  requesting less accuracy.

IFAIL= 3
```

```
     Extremely bad local integrand behaviour causes a very strong
     subdivision around one (or more) points of the interval.
     The same advice applies as in the case of IFAIL = 1.

IFAIL= 4
     On entry B <= A,

     or       ALFA <= -1,

     or       BETA <= -1,

     or       KEY < 1,

     or       KEY > 4.

IFAIL= 5
     On entry LW < 8,

     or       LIW < 2.

7. Accuracy

The routine cannot guarantee, but in practice usually achieves,
the following accuracy:

                        |I-RESULT|<=tol,

where

                 tol=max{|EPSABS|,|EPSREL|*|I|},

and EPSABS and EPSREL are user-specified absolute and relative
error tolerances.

Moreover it returns the quantity ABSERR which, in normal
circumstances, satisfies:

                    |I-RESULT|<=ABSERR<=tol.

8. Further Comments

The time taken by the routine depends on the integrand and on the
accuracy required.

If IFAIL /= 0 on exit, then the user may wish to examine the
contents of the array W, which contains the end-points of the
```

sub-intervals used by D01APF along with the integral
contributions and error estimates over these sub-intervals.

Specifically, for i=1,2,...,n, let r denote the approximation to
the value of the integral over the sub-interval [a ,b] in the
partition of [a,b] and e be the corresponding absolute error
estimate. Then,

$$\int_{a_i}^{b_i} f(x)w(x)\,dx \tilde{=} r_i \quad \text{and} \quad \text{RESULT} = \sum_{i=1}^{n} r_i .$$

The value of n is returned in IW(1), and the values a_i, b_i, e_i and r_i are
stored consecutively in the array W, that is:

a_i = W(i),

b_i = W(n+i),

e_i = W(2n+i),

r_i = W(3n+i).

9. Example

To compute:

$$\int_0^1 \ln(x)\cos(10(\pi)x)\,dx \quad \text{and}$$

$$\int_0^1 \frac{\sin(10x)}{}\,dx.$$

```
                                    0 \/x(1-x)
```

The example program is not reproduced here. The source code for all example programs is distributed with the NAG Foundation Library software and should be available on-line.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

```
     D01AQF(3NAG)      Foundation Library (12/10/92)      D01AQF(3NAG)

          D01 -- Quadrature                                          D01AQF
                  D01AQF -- NAG Foundation Library Routine Document
```

Note: Before using this routine, please read the Users' Note for your implementation to check implementation-dependent details. The symbol (*) after a NAG routine name denotes a routine that is not included in the Foundation Library.

1. Purpose

D01AQF calculates an approximation to the Hilbert transform of a function g(x) over [a,b]:

$$I = \int_a^b \frac{g(x)}{x-c}\,dx$$

for user-specified values of a, b and c.

2. Specification

```
       SUBROUTINE D01AQF (G, A, B, C, EPSABS, EPSREL, RESULT,
      1                   ABSERR, W, LW, IW, LIW, IFAIL)
       INTEGER          LW, IW(LIW), LIW, IFAIL
       DOUBLE PRECISION G, A, B, C, EPSABS, EPSREL, RESULT,
      1                 ABSERR, W(LW)
       EXTERNAL         G
```

3. Description

D01AQF is based upon the QUADPACK routine QAWC (Piessens et al [3]) and integrates a function of the form g(x)w(x), where the weight function

$$w(x) = \frac{1}{x-c}$$

is that of the Hilbert transform. (If a<c<b the integral has to be interpreted in the sense of a Cauchy principal value.) It is an adaptive routine which employs a 'global' acceptance criterion (as defined by Malcolm and Simpson [1]). Special care is taken to ensure that c is never the end-point of a sub-interval (Piessens et al[2]). On each sub-interval (c_1, c_2) modified Clenshaw-Curtis integration of orders 12 and 24 is performed if $c_1 - d \le c \le c_2 + d$ where $d = (c_2 - c_1)/20$. Otherwise the Gauss 7-point and Kronrod 15-point rules are used. The local error estimation is described by Piessens et al [3].

4. References

[1] Malcolm M A and Simpson R B (1976) Local Versus Global Strategies for Adaptive Quadrature. ACM Trans. Math. Softw. 1 129--146.

[2] Piessens R, Van Roy-Branders M and Mertens I (1976) The Automatic Evaluation of Cauchy Principal Value Integrals. Angewandte Informatik. 18 31--35.

[3] Piessens R, De Doncker-Kapenga E, Uberhuber C and Kahaner D (1983) QUADPACK, A Subroutine Package for Automatic Integration. Springer-Verlag.

5. Parameters

1: G -- DOUBLE PRECISION FUNCTION, supplied by the user. External Procedure

G must return the value of the function g at a given point.

Its specification is:

```
DOUBLE PRECISION FUNCTION G (X)
DOUBLE PRECISION X
```

1: X -- DOUBLE PRECISION Input

On entry: the point at which the function g must be evaluated.

G must be declared as EXTERNAL in the (sub)program from which D01AQF is called. Parameters denoted as Input must not be changed by this procedure.

2: A -- DOUBLE PRECISION Input
On entry: the lower limit of integration, a.

3: B -- DOUBLE PRECISION Input
On entry: the upper limit of integration, b. It is not necessary that a<b.

4: C -- DOUBLE PRECISION Input
On entry: the parameter c in the weight function.
Constraint: C must not equal A or B.

5: EPSABS -- DOUBLE PRECISION Input
On entry: the absolute accuracy required. If EPSABS is negative, the absolute value is used. See Section 7.

6: EPSREL -- DOUBLE PRECISION Input
On entry: the relative accuracy required. If EPSREL is negative, the absolute value is used. See Section 7.

7: RESULT -- DOUBLE PRECISION Output
On exit: the approximation to the integral I.

8: ABSERR -- DOUBLE PRECISION Output
On exit: an estimate of the modulus of the absolute error, which should be an upper bound for |I-RESULT|.

9: W(LW) -- DOUBLE PRECISION array Output
On exit: details of the computation, as described in Section 8.

10: LW -- INTEGER Input
On entry:
the dimension of the array W as declared in the (sub)program from which D01AQF is called.
The value of LW (together with that of LIW below) imposes a bound on the number of sub-intervals into which the interval of integration may be divided by the routine. The number of sub-intervals cannot exceed LW/4. The more difficult the integrand, the larger LW should be. Suggested value: LW = 800 to 2000 is adequate for most problems. Constraint: LW >= 4.

```
11:  IW(LIW) -- INTEGER array                                  Output
     On exit: IW(1) contains the actual number of sub-intervals
     used. The rest of the array is used as workspace.

12:  LIW -- INTEGER                                            Input
     On entry:
     the dimension of the array IW as declared in the
     (sub)program from which D01AQF is called.
     The number of sub-intervals into which the interval of
     integration may be divided cannot exceed LIW. Suggested
     value: LIW = LW/4. Constraint: LIW >= 1.

13:  IFAIL -- INTEGER                                  Input/Output
     On entry: IFAIL must be set to 0, -1 or 1. Users who are
     unfamiliar with this parameter should refer to the Essential
     Introduction for details.

     On exit: IFAIL = 0 unless the routine detects an error or
     gives a warning (see Section 6).

     For this routine, because the values of output parameters
     may be useful even if IFAIL /=0 on exit, users are
     recommended to set IFAIL to -1 before entry. It is then
     essential to test the value of IFAIL on exit.

6. Error Indicators and Warnings

Errors or warnings specified by the routine:

If on entry IFAIL = 0 or -1, explanatory error messages are
output on the current error message unit (as defined by X04AAF).

IFAIL= 1
     The maximum number of subdivisions allowed with the given
     workspace has been reached without the accuracy requirements
     being achieved. Look at the integrand in order to determine
     the integration difficulties. Another integrator which is
     designed for handling the type of difficulty involved, must
     be used. Alternatively consider relaxing the accuracy
     requirements specified by EPSABS and EPSREL, or increasing
     the workspace.

IFAIL= 2
     Round-off error prevents the requested tolerance from being
     achieved. Consider  requesting less accuracy.
```

IFAIL= 3
Extremely bad local behaviour of g(x) causes a very strong subdivision around one (or more) points of the interval. The same advice applies as in the case of IFAIL = 1.

IFAIL= 4
On entry C = A or C = B.

IFAIL= 5
On entry LW < 4,

or LIW < 1.

7. Accuracy

The routine cannot guarantee, but in practice usually achieves, the following accuracy:

|I-RESULT|<=tol,

where

tol=max{|EPSABS|,|EPSREL|*|I|},

and EPSABS and EPSREL are user-specified absolute and relative error tolerances. Moreover it returns the quantity ABSERR which, in normal circumstances satisfies:

|I-RESULT|<=ABSERR<=tol.

8. Further Comments

The time taken by the routine depends on the integrand and on the accuracy required.

If IFAIL /= 0 on exit, then the user may wish to examine the contents of the array W, which contains the end-points of the sub-intervals used by D01AQF along with the integral contributions and error estimates over these sub-intervals.

Specifically, for i=1,2,...,n, let r_i denote the approximation to the value of the integral over the sub-interval $[a_i,b_i]$ in the partition of [a,b] and e_i be the corresponding absolute error

```
                    b
                     i                          n
                    /                           --
estimate. Then,     | g(x)w(x)dx~=r  and RESULT= >  r . The value of
                    /              i            --  i
                    a                           i=1
                     i
```

n is returned in IW(1), and the values a_i, b_i, e_i and r_i are stored consecutively in the array W, that is:

a_i =W(i),

b_i =W(n+i),

e_i =W(2n+i) and

r_i =W(3n+i).

9. Example

To compute the Cauchy principal value of

$$\int_{-1}^{1} \frac{dx}{(x^2+0.01)(x-\frac{1}{2})}.$$

The example program is not reproduced here. The source code for all example programs is distributed with the NAG Foundation Library software and should be available on-line.

%%%

D01ASF(3NAG) Foundation Library (12/10/92) D01ASF(3NAG)

D01 -- Quadrature D01ASF
D01ASF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for

your implementation to check implementation-dependent details. The symbol (*) after a NAG routine name denotes a routine that is not included in the Foundation Library.

1. Purpose

D01ASF calculates an approximation to the sine or the cosine transform of a function g over [a,infty):

```
   infty
   /
I= |    g(x)||,sin((omega)x)dx  or  I=
   /
   a
 infty
 /
 |    g(x)||,cos((omega)x)dx
 /
 a
```

(for a user-specified value of (omega)).

2. Specification

```
       SUBROUTINE D01ASF (G, A, OMEGA, KEY, EPSABS, RESULT,
      1                   ABSERR, LIMLST, LST, ERLST, RSLST,
      2                   IERLST, W, LW, IW, LIW, IFAIL)
       INTEGER          KEY, LIMLST, LST, IERLST(LIMLST), LW, IW
      1                 (LIW), LIW, IFAIL
       DOUBLE PRECISION G, A, OMEGA, EPSABS, RESULT, ABSERR, ERLST
      1                 (LIMLST), RSLST(LIMLST), W(LW)
       EXTERNAL         G
```

3. Description

D01ASF is based upon the QUADPACK routine QAWFE (Piessens et al [2]). It is an adaptive routine, designed to integrate a function of the form g(x)w(x) over a semi-infinite interval, where w(x) is either sin((omega)x) or cos((omega)x). Over successive intervals

```
                C =[a+(k-1)c,a+kc],   k=1,2,...,LST
                 k
```

integration is performed by the same algorithm as is used by D01ANF. The intervals C are of constant length

k

c={2[|(omega)|]+1}(pi)/|(omega)|, (omega)/=0

where [|(omega)|] represents the largest integer less than or equal to |(omega)|. Since c equals an odd number of half periods, the integral contributions over succeeding intervals will alternate in sign when the function g is positive and monotonically decreasing over [a,infty). The algorithm, described by [2], incorporates a global acceptance criterion (as defined by Malcolm and Simpson [1]) together with the (epsilon)-algorithm (Wynn [3]) to perform extrapolation. The local error estimation is described by Piessens et al [2].

If (omega)=0 and KEY = 1, the routine uses the same algorithm as D01AMF (with EPSREL = 0.0).

In contrast to the other routines in Chapter D01, D01ASF works only with a user-specified absolute error tolerance (EPSABS). Over the interval C it attempts to satisfy the absolute accuracy
k
requirement

EPSA =U *EPSABS
k k

k-1
where U =(1-p)p , for k=1,2,... and p=0.9.
k

However, when difficulties occur during the integration over the kth sub-interval C such that the error flag IERLST(k) is non-
k
zero, the accuracy requirement over subsequent intervals is relaxed. See Piessens et al [2] for more details.

4. References

[1] Malcolm M A and Simpson R B (1976) Local Versus Global Strategies for Adaptive Quadrature. ACM Trans. Math. Softw. 1 129--146.

[2] Piessens R, De Doncker-Kapenga E, Uberhuber C and Kahaner D (1983) QUADPACK, A Subroutine Package for Automatic Integration. Springer-Verlag.

[3] Wynn P (1956) On a Device for Computing the e (S)
m n

Transformation. Math. Tables Aids Comput. 10 91--96.

5. Parameters

1: G -- DOUBLE PRECISION FUNCTION, supplied by the user. External Procedure
G must return the value of the function g at a given point.

Its specification is:

```
DOUBLE PRECISION FUNCTION G (X)
DOUBLE PRECISION X
```

1: X -- DOUBLE PRECISION Input
On entry: the point at which the function g must be evaluated.

G must be declared as EXTERNAL in the (sub)program from which D01ASF is called. Parameters denoted as Input must not be changed by this procedure.

2: A -- DOUBLE PRECISION Input
On entry: the lower limit of integration, a.

3: OMEGA -- DOUBLE PRECISION Input
On entry: the parameter (omega) in the weight function of the transform.

4: KEY -- INTEGER Input
On entry: indicates which integral is to be computed:
if KEY = 1, w(x)=cos((omega)x);

if KEY = 2, w(x)=sin((omega)x).
Constraint: KEY = 1 or 2.

5: EPSABS -- DOUBLE PRECISION Input
On entry: the absolute accuracy requirement. If EPSABS is negative, the absolute value is used. See Section 7.

6: RESULT -- DOUBLE PRECISION Output
On exit: the approximation to the integral I.

7: ABSERR -- DOUBLE PRECISION Output
On exit: an estimate of the modulus of the absolute error, which should be an upper bound for |I-RESULT|.

8: LIMLST -- INTEGER Input

On entry: an upper bound on the number of intervals C
k
needed for the integration. Suggested value: LIMLST = 50 is adequate for most problems. Constraint: LIMLST >= 3.

9: LST -- INTEGER Output
On exit: the number of intervals C actually used for the
k
integration.

10: ERLST(LIMLST) -- DOUBLE PRECISION array Output
On exit: ERLST(k) contains the error estimate corresponding to the integral contribution over the interval C , for
k
k=1,2,...,LST.

11: RSLST(LIMLST) -- DOUBLE PRECISION array Output
On exit: RSLST(k) contains the integral contribution over the interval C for k=1,2,...,LST.
k

12: IERLST(LIMLST) -- INTEGER array Output
On exit: IERLST(k) contains the error flag corresponding to RSLST(k), for k=1,2,...,LST. See Section 6.

13: W(LW) -- DOUBLE PRECISION array Workspace

14: LW -- INTEGER Input
On entry:
the dimension of the array W as declared in the (sub)program from which D01ASF is called.
The value of LW (together with that of LIW below) imposes a bound on the number of sub-intervals into which each interval C may be divided by the routine. The number of
k
sub-intervals cannot exceed LW/4. The more difficult the integrand, the larger LW should be. Suggested value: a value in the range 800 to 2000 is adequate for most problems.
Constraint: LW >= 4.

15: IW(LIW) -- INTEGER array Output
On exit: IW(1) contains the maximum number of sub-intervals actually used for integrating over any of the intervals C .
k
The rest of the array is used as workspace.

16: LIW -- INTEGER Input
On entry:
the dimension of the array IW as declared in the (sub)program from which D01ASF is called.
The number of sub-intervals into which each interval C_k may be divided cannot exceed LIW/2. Suggested value: LIW = LW/2.
Constraint: LIW >= 2.

17: IFAIL -- INTEGER Input/Output
On entry: IFAIL must be set to 0, -1 or 1. Users who are unfamiliar with this parameter should refer to the Essential Introduction for details.

On exit: IFAIL = 0 unless the routine detects an error or gives a warning (see Section 6).

For this routine, because the values of output parameters may be useful even if IFAIL /=0 on exit, users are recommended to set IFAIL to -1 before entry. It is then essential to test the value of IFAIL on exit.

6. Error Indicators and Warnings

Errors or warnings specified by the routine:

If on entry IFAIL = 0 or -1, explanatory error messages are output on the current error message unit (as defined by X04AAF).

IFAIL= 1
The maximum number of subdivisions allowed with the given workspace has been reached without the accuracy requirements being achieved. Look at the integrand in order to determine the integration difficulties. If the position of a local difficulty within the interval can be determined (e.g. a singularity of the integrand or its derivative, a peak, a discontinuity, etc) you will probably gain from splitting up the interval at this point and calling D01ASF on the infinite subrange and an appropriate integrator on the finite subrange. Alternatively, consider relaxing the accuracy requirements specified by EPSABS or increasing the amount of workspace.

IFAIL= 2
Round-off error prevents the requested tolerance from being achieved. The error may be underestimated. Consider

```
     requesting less accuracy.

IFAIL= 3
     Extremely bad local integrand behaviour causes a very strong
     subdivision around one (or more) points of the interval.
     The same advice applies as in the case of IFAIL = 1.

IFAIL= 4
     The requested tolerance cannot be achieved, because the
     extrapolation does not increase the accuracy satisfactorily;
     the returned result is the best which can be obtained.  The
     same advice applies as in the case of IFAIL = 1.

IFAIL= 5
     The integral is probably divergent, or slowly convergent.
     Please note that divergence can occur with any non-zero
     value of IFAIL.

IFAIL= 6
     On entry KEY < 1,

     or       KEY > 2,

     or       LIMLST < 3.

IFAIL= 7
     Bad integration behaviour occurs within one or more of the
     intervals C . Location and type of the difficulty involved
                k
     can be determined from the vector IERLST (see below).

IFAIL= 8
     Maximum number of intervals C  (= LIMLST) allowed has been
                                  k
     achieved. Increase the value of LIMLST to allow more cycles.

IFAIL= 9
     The extrapolation table constructed for convergence
     acceleration of the series formed by the integral
     contribution over the intervals C , does not converge to the
                                      k
     required accuracy.

IFAIL= 10
     On entry LW < 4,
```

```
     or       LIW < 2.

In the cases IFAIL = 7, 8 or 9, additional information about the
cause of the error can be obtained from the array IERLST, as
follows:

IERLST(k)=1
      The maximum number of subdivisions = min(LW/4,LIW/2) has
      been achieved on the kth interval.

IERLST(k)=2
      Occurrence of round-off error is detected and prevents the
      tolerance imposed on the kth interval from being achieved.

IERLST(k)=3
      Extremely bad integrand behaviour occurs at some points of
      the kth interval.

IERLST(k)=4
      The integration procedure over the kth interval does not
      converge (to within the required accuracy) due to round-off
      in the extrapolation procedure invoked on this interval. It
      is assumed that the result on this interval is the best
      which can be obtained.

IERLST(k)=5
      The integral over the kth interval is probably divergent or
      slowly convergent. It must be noted that divergence can
      occur with any other value of IERLST(k).

7. Accuracy

The routine cannot guarantee, but in practice usually achieves,
the following accuracy:

                        |I-RESULT|<=|EPSABS|,

where EPSABS is the user-specified absolute error tolerance.
Moreover, it returns the quantity ABSERR, which, in normal
circumstances, satisfies

                   |I-RESULT|<=ABSERR<=|EPSABS|.

8. Further Comments

None.
```

9. Example

To compute

```
                    infty
                    /      1
                    |     ---cos((pi)x/2)dx.
                    /
                    0     \/x
```

The example program is not reproduced here. The source code for all example programs is distributed with the NAG Foundation Library software and should be available on-line.

%%%

D01BBF(3NAG) D01BBF D01BBF(3NAG)

D01 -- Quadrature D01BBF
D01BBF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for your implementation to check implementation-dependent details. The symbol (*) after a NAG routine name denotes a routine that is not included in the Foundation Library.

Note for users via the AXIOM system: the interface to this routine has been enhanced for use with AXIOM and is slightly different to that offered in the standard version of the Foundation Library.

1. Purpose

D01BBF returns the weights and abscissae appropriate to a Gaussian quadrature formula with a specified number of abscissae. The formulae provided are Gauss-Legendre, Gauss-Rational, Gauss-Laguerre and Gauss-Hermite.

2. Specification

```
      SUBROUTINE D01BBF (A, B, ITYPE, N, WEIGHT, ABSCIS, GTYPE,
     1                   IFAIL)
      INTEGER          ITYPE, N, GTYPE, IFAIL
      DOUBLE PRECISION A, B, WEIGHT(N), ABSCIS(N)
```

3. Description

This routine returns the weights and abscissae for use in the Gaussian quadrature of a function f(x). The quadrature takes the form

$$S = \sum_{i=1}^{n} w_i f(x_i)$$

where w_i are the weights and x_i are the abscissae (see Davis and Rabinowitz [1], Froberg [2], Ralston [3] or Stroud and Secrest [4]).

Weights and abscissae are available for Gauss-Legendre, Gauss-Rational, Gauss-Laguerre and Gauss-Hermite quadrature, and for a selection of values of n (see Section 5).

(a) Gauss-Legendre Quadrature:

$$\tilde{S} = \int_a^b f(x)dx$$

where a and b are finite and it will be exact for any function of the form

$$f(x) = \sum_{i=0}^{2n-1} c_i x^i$$

(b) Gauss-Rational quadrature:

$$\tilde{S} = \int_a^{\infty} f(x)dx \quad (a+b>0) \quad \text{or} \quad \tilde{S} = \int_{-\infty}^{a} f(x)dx \quad (a+b<0)$$

and will be exact for any function of the form

$$\sum_{i=0}^{2n-1} \frac{c_i}{(x+b)^{2n+1-i}}$$

$$f(x) = \sum_{i=2}^{} \frac{i}{(x+b)} = \sum_{i=0}^{} \frac{i}{(x+b)^{2n+1}}$$

(c) Gauss-Laguerre quadrature, adjusted weights option:

$$\tilde{S} = \int_{a}^{\infty} f(x)dx \quad (b>0<) \quad \text{or} \quad \tilde{S} = \int_{-\infty}^{a} f(x)dx \quad (b<0)$$

and will be exact for any function of the form

$$f(x) = e^{-bx} \sum_{i=0}^{2n-1} c_i x^i$$

(d) Gauss-Hermite quadrature, adjusted weights option:

$$\tilde{S} = \int_{-\infty}^{+\infty} f(x)dx$$

and will be exact for any function of the form

$$f(x) = e^{-b(x-a)^2} \sum_{i=0}^{2n-1} c_i x^i \quad (b>0)$$

(e) Gauss-Laguerre quadrature, normal weights option:

$$\tilde{S} = \int_{a}^{\infty} e^{-bx} f(x)dx \quad (b>0) \quad \text{or} \quad \tilde{S} = \int_{-\infty}^{a} e^{-bx} f(x)dx$$

(b<0)

and will be exact for any function of the form

$$2n-1$$

$$f(x) = \sum_{i=0}^{\infty} c_i x^i$$

(f) Gauss-Hermite quadrature, normal weights option:

$$\tilde{S} = \int_{-\infty}^{+\infty} e^{-b(x-a)^2} f(x)dx$$

and will be exact for any function of the form:

$$f(x) = \sum_{i=0}^{2n-1} c_i x^i$$

Note: that the Gauss-Legendre abscissae, with a=-1, b=+1, are the zeros of the Legendre polynomials; the Gauss-Laguerre abscissae, with a=0, b=1, are the zeros of the Laguerre polynomials; and the Gauss-Hermite abscissae, with a=0, b=1, are the zeros of the Hermite polynomials.

4. References

[1] Davis P J and Rabinowitz P (1967) Numerical Integration. Blaisdell Publishing Company. 33--52.

[2] Froberg C E (1965) Introduction to Numerical Analysis. Addison-Wesley. 181--187.

[3] Ralston A (1965) A First Course in Numerical Analysis. McGraw-Hill. 87--90.

[4] Stroud A H and Secrest D (1966) Gaussian Quadrature Formulas. Prentice-Hall.

5. Parameters

1: A -- DOUBLE PRECISION Input

2: B -- DOUBLE PRECISION Input
On entry: the quantities a and b as described in the appropriate subsection of Section 3.

3: ITYPE -- INTEGER Input
On entry: indicates the type of weights for Gauss-Laguerre or Gauss-Hermite quadrature (see Section 3):

if ITYPE = 1, adjusted weights will be returned;

if ITYPE = 0, normal weights will be returned.

Constraint: ITYPE = 0 or 1.

For Gauss-Legendre or Gauss-Rational quadrature, this parameter is not used.

4: N -- INTEGER Input
On entry: the number of weights and abscissae to be returned, n. Constraint: N = 1,2,3,4,5,6,8,10,12,14,16,20,24,32,48 or 64.

5: WEIGHT(N) -- DOUBLE PRECISION array Output
On exit: the N weights. For Gauss-Laguerre and Gauss-Hermite quadrature, these will be the adjusted weights if ITYPE = 1, and the normal weights if ITYPE = 0.

6: ABSCIS(N) -- DOUBLE PRECISION array Output
On exit: the N abscissae.

7: GTYPE -- INTEGER Input
On entry: The value of GTYPE indicates which quadrature formula are to be used:
GTYPE = 0 for Gauss-Legendre weights and abscissae;

GTYPE = 1 for Gauss-Rational weights and abscissae;

GTYPE = 2 for Gauss-Laguerre weights and abscissae;

GTYPE = 3 for Gauss-Hermite weights and abscissae.

8: IFAIL -- INTEGER Input/Output
On entry: IFAIL must be set to 0, -1 or 1. For users not familiar with this parameter (described in the Essential Introduction) the recommended value is 0.

On exit: IFAIL = 0 unless the routine detects an error (see Section 6).

```
6. Error Indicators and Warnings

Errors detected by the routine:

IFAIL= 1
     The N-point rule is not among those stored. If the soft fail
     option is used, the weights and abscissae returned will be
     those for the largest valid value of N less than the
     requested value, and the excess elements of WEIGHT and
     ABSCIS (i.e., up to the requested N) will be filled with
     zeros.

IFAIL= 2
     The value of A and/or B is invalid.
          Gauss-Rational: A + B = 0

          Gauss-Laguerre: B = 0

          Gauss-Hermite: B <= 0
     If the soft fail option is used the weights and abscissae
     are returned as zero.

IFAIL= 3
     Laguerre and Hermite normal weights only: underflow is
     occurring in evaluating one or more of the normal weights.
     If the soft fail option is used, the underflowing weights
     are returned as zero. A smaller value of N must be used; or
     adjusted weights should be used (ITYPE = 1). In the latter
     case, take care that underflow does not occur when
     evaluating the integrand appropriate for adjusted weights.

IFAIL=4
    GTYPE < 0 or GTYPE > 3

7. Accuracy

The weights and abscissae are stored for standard values of A and
B to full machine accuracy.

8. Further Comments

Timing is negligible.

9. Example

This example program returns the abscissae and (adjusted) weights
```

```
      for the six-point Gauss-Laguerre formula.

      The example program is not reproduced here. The source code for
      all example programs is distributed with the NAG Foundation
      Library software and should be available on-line.
```

%%%

```
      D01FCF(3NAG)      Foundation Library (12/10/92)      D01FCF(3NAG)

           D01 -- Quadrature                                          D01FCF
                   D01FCF -- NAG Foundation Library Routine Document

           Note: Before using this routine, please read the Users' Note for
           your implementation to check implementation-dependent details.
           The symbol (*) after a NAG routine name denotes a routine that is
           not included in the Foundation Library.

           1. Purpose

           D01FCF attempts to evaluate a multi-dimensional integral (up to
           15 dimensions), with constant and finite limits, to a specified
           relative accuracy, using an adaptive subdivision strategy.

           2. Specification

                  SUBROUTINE D01FCF (NDIM, A, B, MINPTS, MAXPTS, FUNCTN,
                 1                   EPS, ACC, LENWRK, WRKSTR, FINVAL, IFAIL)
                  INTEGER          NDIM, MINPTS, MAXPTS, LENWRK, IFAIL
                  DOUBLE PRECISION A(NDIM), B(NDIM), FUNCTN, EPS, ACC, WRKSTR
                 1                 (LENWRK), FINVAL
                  EXTERNAL         FUNCTN

           3. Description

           The routine returns an estimate of a multi-dimensional integral
           over a hyper-rectangle (i.e., with constant limits), and also an
           estimate of the relative error. The user sets the relative
           accuracy required, supplies the integrand as a function
           subprogram (FUNCTN), and also sets the minimum and maximum
           acceptable number of calls to FUNCTN (in MINPTS and MAXPTS).

           The routine operates by repeated subdivision of the hyper-
           rectangular region into smaller hyper-rectangles. In each
           subregion, the integral is estimated using a seventh-degree rule,
           and an error estimate is obtained by comparison with a fifth-
```

degree rule which uses a subset of the same points. The fourth differences of the integrand along each co-ordinate axis are evaluated, and the subregion is marked for possible future subdivision in half along that co-ordinate axis which has the largest absolute fourth difference.

If the estimated errors, totalled over the subregions, exceed the requested relative error (or if fewer than MINPTS calls to FUNCTN have been made), further subdivision is necessary, and is performed on the subregion with the largest estimated error, that subregion being halved along the appropriate co-ordinate axis.

The routine will fail if the requested relative error level has not been attained by the time MAXPTS calls to FUNCTN have been made; or, if the amount LENWRK of working storage is insufficient. A formula for the recommended value of LENWRK is given in Section 5. If a smaller value is used, and is exhausted in the course of execution, the routine switches to a less efficient mode of operation; only if this mode also breaks down is insufficient storage reported.

D01FCF is based on the HALF subroutine developed by van Dooren and de Ridder [1]. It uses a different basic rule, described by Genz and Malik [2].

4. References

[1] Van Dooren P and De Ridder L (1976) An Adaptive Algorithm for Numerical Integration over an N-dimensional Cube. J. Comput. Appl. Math. 2 207--217.

[2] Genz A C and Malik A A (1980) An Adaptive Algorithm for Numerical Integration over an N-dimensional Rectangular Region. J. Comput. Appl. Math. 6 295--302.

5. Parameters

1: NDIM -- INTEGER Input
On entry: the number of dimensions of the integral, n.
Constraint: 2 <= NDIM <= 15.

2: A(NDIM) -- DOUBLE PRECISION array Input
On entry: the lower limits of integration, a_i, for i=1,2,...,n.

3: B(NDIM) -- DOUBLE PRECISION array Input
On entry: the upper limits of integration, b_i, for $i=1,2,\ldots,n$.

4: MINPTS -- INTEGER Input/Output
On entry: MINPTS must be set to the minimum number of integrand evaluations to be allowed. On exit: MINPTS contains the actual number of integrand evaluations used by D01FCF.

5: MAXPTS -- INTEGER Input
On entry: the maximum number of integrand evaluations to be allowed.
Constraints:

$$\text{MAXPTS} \geq \text{MINPTS}$$

$$\text{MAXPTS} \geq (\text{alpha}),$$

where $(\text{alpha}) = 2^{\text{NDIM}} + 2*\text{NDIM} + 2*\text{NDIM} + 1$.

6: FUNCTN -- DOUBLE PRECISION FUNCTION, supplied by the user.
External Procedure
FUNCTN must return the value of the integrand f at a given point.

Its specification is:

```
DOUBLE PRECISION FUNCTION FUNCTN (NDIM,Z)
INTEGER          NDIM
DOUBLE PRECISION Z(NDIM)
```

1: NDIM -- INTEGER Input
On entry: the number of dimensions of the integral, n.

2: Z(NDIM) -- DOUBLE PRECISION array Input
On entry: the co-ordinates of the point at which the integrand must be evaluated.

FUNCTN must be declared as EXTERNAL in the (sub)program from which D01FCF is called. Parameters denoted as Input must not be changed by this procedure.

7: EPS -- DOUBLE PRECISION Input
On entry: the relative error acceptable to the user. When

```
     the solution is zero or very small relative accuracy may not
     be achievable but the user may still set EPS to a reasonable
     value and check for the error exit IFAIL = 2. Constraint:
     EPS > 0.0.

 8:  ACC -- DOUBLE PRECISION                                  Output
     On exit: the estimated relative error in FINVAL.

 9:  LENWRK -- INTEGER                                         Input
     On entry:
     the dimension of the array WRKSTR as declared in the
     (sub)program from which D01FCF is called.
     Suggested value: for maximum efficiency, LENWRK >=
     (NDIM+2)*(1+MAXPTS/(alpha)) (see parameter MAXPTS for
     (alpha)).

     If LENWRK is less than this, the routine will usually run
     less efficiently and may fail. Constraint: LENWRK>=2*NDIM+4.

10:  WRKSTR(LENWRK) -- DOUBLE PRECISION array           Workspace

11:  FINVAL -- DOUBLE PRECISION                              Output
     On exit: the best estimate obtained for the integral.

12:  IFAIL -- INTEGER                                  Input/Output
     On entry: IFAIL must be set to 0, -1 or 1. Users who are
     unfamiliar with this parameter should refer to the Essential
     Introduction for details.
     On exit: IFAIL = 0 unless the routine detects an error or
     gives a warning (see Section 6).

     For this routine, because the values of output parameters
     may be useful even if IFAIL /=0 on exit, users are
     recommended to set IFAIL to -1 before entry. It is then
     essential to test the value of IFAIL on exit. To suppress
     the output of an error message when soft failure occurs, set
     IFAIL to 1.

6. Error Indicators and Warnings

Errors or warnings specified by the routine:

IFAIL= 1
     On entry NDIM < 2,

     or       NDIM > 15,
```

```
     or       MAXPTS is too small,

     or       LENWRK<2*NDIM+4,

     or       EPS <= 0.0.

IFAIL= 2
     MAXPTS was too small to obtain the required relative
     accuracy EPS. On soft failure, FINVAL and ACC contain
     estimates of the integral and the relative error, but ACC
     will be greater than EPS.

IFAIL= 3
     LENWRK was too small. On soft failure, FINVAL and ACC
     contain estimates of the integral and the relative error,
     but ACC will be greater than EPS.

7. Accuracy

A relative error estimate is output through the parameter ACC.

8. Further Comments

Execution time will usually be dominated by the time taken to
evaluate the integrand FUNCTN, and hence the maximum time that
could be taken will be proportional to MAXPTS.

9. Example

This example program estimates the integral

                    2
  1  1  1  1  4z z exp(2z z )
  /  /  /  /    1 3      1 3
  |  |  |  | ---------------dz dz dz dz =0.575364.
  /  /  /  /             2     4  3  2  1
  0  0  0  0    (1+z +z )
                   2  4

The accuracy requested is one part in 10,000.

The example program is not reproduced here. The source code for
all example programs is distributed with the NAG Foundation
Library software and should be available on-line.
```

%%

D01GAF(3NAG) Foundation Library (12/10/92) D01GAF(3NAG)

D01 -- Quadrature D01GAF
D01GAF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for your implementation to check implementation-dependent details. The symbol (*) after a NAG routine name denotes a routine that is not included in the Foundation Library.

1. Purpose

D01GAF integrates a function which is specified numerically at four or more points, over the whole of its specified range, using third-order finite-difference formulae with error estimates, according to a method due to Gill and Miller.

2. Specification

```
SUBROUTINE D01GAF (X, Y, N, ANS, ER, IFAIL)
INTEGER          N, IFAIL
DOUBLE PRECISION X(N), Y(N), ANS, ER
```

3. Description

This routine evaluates the definite integral

$$I = \int_{x_1}^{x_n} y(x)\,dx,$$

where the function y is specified at the n-points $x_1, x_2, \ldots, x_n$, which should be all distinct, and in either ascending or descending order. The integral between successive points is calculated by a four-point finite-difference formula centred on the interval concerned, except in the case of the first and last intervals, where four-point forward and backward difference formulae respectively are employed. If n is less than 4, the routine fails. An approximation to the truncation error is

```
integrated and added to the result. It is also returned
separately to give an estimate of the uncertainty in the result.
The method is due to Gill and Miller.

4. References

[1]   Gill P E and Miller G F (1972) An Algorithm for the
      Integration of Unequally Spaced Data. Comput. J. 15 80--83.

5. Parameters

 1:  X(N) -- DOUBLE PRECISION array                          Input
     On entry: the values of the independent variable, i.e., the
     x ,x ,...,x . Constraint: either X(1) < X(2) <... < X(N) or
      1  2      n
     X(1) > X(2) >... > X(N).

 2:  Y(N) -- DOUBLE PRECISION array                          Input
     On entry: the values of the dependent variable y  at the
                                                     i
     points x , for i=1,2,...,n.
             i

 3:  N -- INTEGER                                            Input
     On entry: the number of points, n. Constraint: N >= 4.

 4:  ANS -- DOUBLE PRECISION                                 Output
     On exit: the estimate of the integral.

 5:  ER -- DOUBLE PRECISION                                  Output
     On exit: an estimate of the uncertainty in ANS.

 6:  IFAIL -- INTEGER                                  Input/Output
     On entry: IFAIL must be set to 0, -1 or 1. For users not
     familiar with this parameter (described in the Essential
     Introduction) the recommended value is 0.

     On exit: IFAIL = 0 unless the routine detects an error (see
     Section 6).

6. Error Indicators and Warnings

Errors detected by the routine:

IFAIL= 1
     Indicates that fewer than four-points have been supplied to
```

```
     the routine.

IFAIL= 2
     Values of X are neither strictly increasing nor strictly
     decreasing.

IFAIL= 3
     Two points have the same X-value.

No error is reported arising from the relative magnitudes of ANS
and ER on return, due to the difficulty when the true answer is
zero.

7. Accuracy

No accuracy level is specified by the user before calling the
routine but on return ABS(ER) is an approximation to, but not
necessarily a bound for, |I-ANS|. If on exit IFAIL > 0, both ANS
and ER are returned as zero.

8. Further Comments

The time taken by the routine depends on the number of points
supplied, n.

In their paper, Gill and Miller [1] do not add the quantity ER to
ANS before return. However, extensive tests have shown that a
dramatic reduction in the error often results from such addition.
In other cases, it does not make an improvement, but these tend
to be cases of low accuracy in which the modified answer is not
significantly inferior to the unmodified one. The user has the
option of recovering the Gill-Miller answer by subtracting ER
from ANS on return from the routine.

9. Example

The example program evaluates the integral

                          1
                          /  4
                          | ----dx=(pi)
                          /    2
                          0 1+x

reading in the function values at 21 unequally-spaced points.
```

The example program is not reproduced here. The source code for all example programs is distributed with the NAG Foundation Library software and should be available on-line.

%%%

D01GBF(3NAG) Foundation Library (12/10/92) D01GBF(3NAG)

D01 -- Quadrature D01GBF
D01GBF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for your implementation to check implementation-dependent details. The symbol (*) after a NAG routine name denotes a routine that is not included in the Foundation Library.

1. Purpose

D01GBF returns an approximation to the integral of a function over a hyper-rectangular region, using a Monte Carlo method. An approximate relative error estimate is also returned. This routine is suitable for low accuracy work.

2. Specification

```
      SUBROUTINE D01GBF (NDIM, A, B, MINCLS, MAXCLS, FUNCTN,
     1                   EPS, ACC, LENWRK, WRKSTR, FINEST, IFAIL)
      INTEGER           NDIM, MINCLS, MAXCLS, LENWRK, IFAIL
      DOUBLE PRECISION A(NDIM), B(NDIM), FUNCTN, EPS, ACC, WRKSTR
     1                 (LENWRK), FINEST
      EXTERNAL          FUNCTN
```

3. Description

D01GBF uses an adaptive Monte Carlo method based on the algorithm described by Lautrup [1]. It is implemented for integrals of the form:

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\cdots\int_{a_n}^{b_n} f(x_1,x_2,\ldots,x_n)\,dx_n\ldots dx_2\,dx_1 .$$

Upon entry, unless LENWRK has been set to the minimum value 10*NDIM, the routine subdivides the integration region into a number of equal volume subregions. Inside each subregion the integral and the variance are estimated by means of pseudo-random sampling. All contributions are added together to produce an estimate for the whole integral and total variance. The variance along each co-ordinate axis is determined and the routine uses this information to increase the density and change the widths of the sub-intervals along each axis, so as to reduce the total variance. The total number of subregions is then increased by a factor of two and the program recycles for another iteration. The program stops when a desired accuracy has been reached or too many integral evaluations are needed for the next cycle.

4. References

[1] Lautrup B (1971) An Adaptive Multi-dimensional Integration Procedure. Proc. 2nd Coll. on Advanced Methods in Theoretical Physics, Marseille.

5. Parameters

1: NDIM -- INTEGER Input
On entry: the number of dimensions of the integral, n.
Constraint: NDIM >= 1.

2: A(NDIM) -- DOUBLE PRECISION array Input
On entry: the lower limits of integration, a_i, for $i=1,2,\ldots,n$.

3: B(NDIM) -- DOUBLE PRECISION array Input
On entry: the upper limits of integration, b_i, for $i=1,2,\ldots,n$.

4: MINCLS -- INTEGER Input/Output
On entry: MINCLS must be set:

either to the minimum number of integrand evaluations to be allowed, in which case MINCLS >= 0;

or to a negative value. In this case the routine assumes that a previous call had been made with the same parameters NDIM, A and B and with either the same integrand (in which case D01GBF continues calculation) or a similar integrand

```
    (in which case D01GBF begins the calculation with the
    subdivision used in the last iteration of the previous call)
    . See also WRKSTR. On exit: MINCLS contains the number of
    integrand evaluations actually used by D01GBF.

5:  MAXCLS -- INTEGER                                        Input
    On entry: the maximum number of integrand evaluations to be
    allowed. In the continuation case this is the number of new
    integrand evaluations to be allowed. These counts do not
    include zero integrand values.
    Constraints:
             MAXCLS > MINCLS,

             MAXCLS >= 4*(NDIM+1).

6:  FUNCTN -- DOUBLE PRECISION FUNCTION, supplied by the user.
                                                 External Procedure
    FUNCTN must return the value of the integrand f at a given
    point.

    Its specification is:

           DOUBLE PRECISION FUNCTION FUNCTN (NDIM, X)
           INTEGER          NDIM
           DOUBLE PRECISION X(NDIM)

     1:  NDIM -- INTEGER                                     Input
         On entry: the number of dimensions of the integral, n.

     2:  X(NDIM) -- DOUBLE PRECISION array                   Input
         On entry: the co-ordinates of the point at which the
         integrand must be evaluated.
    FUNCTN must be declared as EXTERNAL in the (sub)program
    from which D01GBF is called. Parameters denoted as
    Input must not be changed by this procedure.

7:  EPS -- DOUBLE PRECISION                                  Input
    On entry: the relative accuracy required. Constraint: EPS
    >= 0.0.

8:  ACC -- DOUBLE PRECISION                                 Output
    On exit: the estimated relative accuracy of FINEST.

9:  LENWRK -- INTEGER                                        Input
    On entry:
    the dimension of the array WRKSTR as declared in the
```

```
     (sub)program from which D01GBF is called.
     For maximum efficiency, LENWRK should be about
                                      1/NDIM
                  3*NDIM*(MAXCLS/4)      +7*NDIM.
     If LENWRK is given the value 10*NDIM then the subroutine
     uses only one iteration of a crude Monte Carlo method with
     MAXCLS sample points. Constraint: LENWRK >= 10*NDIM.

10:  WRKSTR(LENWRK) -- DOUBLE PRECISION array        Input/Output
     On entry: if MINCLS<0, WRKSTR must be unchanged from the
     previous call of D01GBF - except that for a new integrand
     WRKSTR(LENWRK) must be set to 0.0. See also MINCLS. On
     exit: WRKSTR contains information about the current sub-
     interval structure which could be used in later calls of
     D01GBF. In particular, WRKSTR(j) gives the number of sub-
     intervals used along the jth co-ordinate axis.

11:  FINEST -- DOUBLE PRECISION                          Output
     On exit: the best estimate obtained for the integral.

12:  IFAIL -- INTEGER                                Input/Output
     On entry: IFAIL must be set to 0, -1 or 1. Users who are
     unfamiliar with this parameter should refer to the Essential
     Introduction for details.

     On exit: IFAIL = 0 unless the routine detects an error or
     gives a warning (see Section 6).

     For this routine, because the values of output parameters
     may be useful even if IFAIL /=0 on exit, users are
     recommended to set IFAIL to -1 before entry. It is then
     essential to test the value of IFAIL on exit. To suppress
     the output of an error message when soft failure occurs, set
     IFAIL to 1.

6. Error Indicators and Warnings

Errors or warnings specified by the routine:

IFAIL= 1
     On entry NDIM < 1,

     or       MINCLS >= MAXCLS,

     or       LENWRK < 10*NDIM,
```

```
     or       MAXCLS < 4*(NDIM+1),

     or       EPS < 0.0.

IFAIL= 2
     MAXCLS was too small for D01GBF to obtain the required
     relative accuracy EPS. In this case D01GBF returns a value
     of FINEST with estimated relative error ACC, but ACC will be
     greater than EPS. This error exit may be taken before MAXCLS
     non-zero integrand evaluations have actually occurred, if
     the routine calculates that the current estimates could not
     be improved before MAXCLS was exceeded.

7. Accuracy

A relative error estimate is output through the parameter ACC.
The confidence factor is set so that the actual error should be
less than ACC 90% of the time. If a user desires a higher
confidence level then a smaller value of EPS should be used.

8. Further Comments

The running time for D01GBF will usually be dominated by the time
used to evaluate the integrand FUNCTN, so the maximum time that
could be used is approximately proportional to MAXCLS.

For some integrands, particularly those that are poorly behaved
in a small part of the integration region, D01GBF may terminate
with a value of ACC which is significantly smaller than the
actual relative error. This should be suspected if the returned
value of MINCLS is small relative to the expected difficulty of
the integral. Where this occurs, D01GBF should be called again,
but with a higher entry value of MINCLS (e.g. twice the returned
value) and the results compared with those from the previous
call.

The exact values of FINEST and ACC on return will depend (within
statistical limits) on the sequence of random numbers generated
within D01GBF by calls to G05CAF. Separate runs will produce
identical answers unless the part of the program executed prior
to calling D01GBF also calls (directly or indirectly) routines
from Chapter G05, and the series of such calls differs between
runs. If desired, the user may ensure the identity or difference
between runs of the results returned by D01GBF, by calling G05CBF
or G05CCF respectively, immediately before calling D01GBF.
```

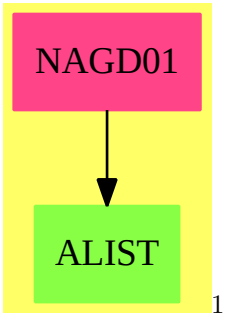
9. Example

This example program calculates the integral

$$\int_0^1\int_0^1\int_0^1\int_0^1 \frac{4x_1 x_3 \exp(2x_1 x_3)}{(1+x_2+x_4)^2}\,dx_1\,dx_2\,dx_3\,dx_4 = 0.575364.$$

The example program is not reproduced here. The source code for all example programs is distributed with the NAG Foundation Library software and should be available on-line.

15.10 NagIntegrationPackage



1.00

Exports:

d01ajf	d01akf	d01alf	d01amf	d01anf
d01apf	d01aqf	d01asf	d01bbf	d01fcf
d01gaf	d01gbf			

⟨*package NAGD01 NagIntegrationPackage*⟩≡

```
)abbrev package NAGD01 NagIntegrationPackage
++ Author: Godfrey Nolan and Mike Dewar
++ Date Created: Jan 1994
++ Date Last Updated: Thu May 12 17:44:37 1994
++ Description:
++ This package uses the NAG Library to calculate the numerical value of
++ definite integrals in one or more dimensions and to evaluate
++ weights and abscissae of integration rules.
++ See \downlink{Manual Page}{manpageXXd01}.

NagIntegrationPackage(): Exports == Implementation where
  S ==> Symbol
  FOP ==> FortranOutputStackPackage

  Exports ==> with
    d01ajf : (DoubleFloat,DoubleFloat,DoubleFloat,DoubleFloat,_
        Integer,Integer,Integer,Union(fn:FileName,fp:Asp1(F))) -> Result
     ++ d01ajf(a,b,epsabs,epsrel,lw,liw,ifail,f)
     ++ is a general-purpose integrator which calculates an
     ++ approximation to the integral of a function f(x) over a finite
     ++ interval [a,b]:
     ++ See \downlink{Manual Page}{manpageXXd01ajf}.
    d01akf : (DoubleFloat,DoubleFloat,DoubleFloat,DoubleFloat,_
        Integer,Integer,Integer,Union(fn:FileName,fp:Asp1(F))) -> Result
     ++ d01akf(a,b,epsabs,epsrel,lw,liw,ifail,f)
     ++ is an adaptive integrator, especially suited to
     ++ oscillating, non-singular integrands, which calculates an
     ++ approximation to the integral of a function f(x) over a finite
```

```
 ++ interval [a,b]:
 ++ See \downlink{Manual Page}{manpageXXd01akf}.
d01alf : (DoubleFloat,DoubleFloat,Integer,Matrix DoubleFloat,_
    DoubleFloat,DoubleFloat,Integer,Integer,Integer,Union(fn:FileName,fp:Asp1
 ++ d01alf(a,b,npts,points,epsabs,epsrel,lw,liw,ifail,f)
 ++ is a general purpose integrator which calculates an
 ++ approximation to the integral of a function f(x) over a finite
 ++ interval [a,b]:
 ++ See \downlink{Manual Page}{manpageXXd01alf}.
d01amf : (DoubleFloat,Integer,DoubleFloat,DoubleFloat,_
    Integer,Integer,Integer,Union(fn:FileName,fp:Asp1(F))) -> Result
 ++ d01amf(bound,inf,epsabs,epsrel,lw,liw,ifail,f)
 ++ calculates an approximation to the integral of a function
 ++ f(x) over an infinite or semi-infinite interval [a,b]:
 ++ See \downlink{Manual Page}{manpageXXd01amf}.
d01anf : (DoubleFloat,DoubleFloat,DoubleFloat,Integer,_
    DoubleFloat,DoubleFloat,Integer,Integer,Integer,Union(fn:FileName,fp:Asp1
 ++ d01anf(a,b,omega,key,epsabs,epsrel,lw,liw,ifail,g)
 ++ calculates an approximation to the sine or the cosine
 ++ transform of a function g over [a,b]:
 ++ See \downlink{Manual Page}{manpageXXd01anf}.
d01apf : (DoubleFloat,DoubleFloat,DoubleFloat,DoubleFloat,_
    Integer,DoubleFloat,DoubleFloat,Integer,Integer,Integer,Union(fn:FileName
 ++ d01apf(a,b,alfa,beta,key,epsabs,epsrel,lw,liw,ifail,g)
 ++ is an adaptive integrator which calculates an
 ++ approximation to the integral of a function g(x)w(x) over a
 ++ finite interval [a,b]:
 ++ See \downlink{Manual Page}{manpageXXd01apf}.
d01aqf : (DoubleFloat,DoubleFloat,DoubleFloat,DoubleFloat,_
    DoubleFloat,Integer,Integer,Integer,Union(fn:FileName,fp:Asp1(G))) -> Res
 ++ d01aqf(a,b,c,epsabs,epsrel,lw,liw,ifail,g)
 ++ calculates an approximation to the Hilbert transform of a
 ++ function g(x) over [a,b]:
 ++ See \downlink{Manual Page}{manpageXXd01aqf}.
d01asf : (DoubleFloat,DoubleFloat,Integer,DoubleFloat,_
    Integer,Integer,Integer,Integer,Union(fn:FileName,fp:Asp1(G))) -> Result
 ++ d01asf(a,omega,key,epsabs,limlst,lw,liw,ifail,g)
 ++ calculates an approximation to the sine or the cosine
 ++ transform of a function g over [a,infty):
 ++ See \downlink{Manual Page}{manpageXXd01asf}.
d01bbf : (DoubleFloat,DoubleFloat,Integer,Integer,_
    Integer,Integer) -> Result
 ++ d01bbf(a,b,itype,n,gtype,ifail)
 ++ returns the weight appropriate to a
 ++ Gaussian quadrature.
 ++ The formulae provided are Gauss-Legendre, Gauss-Rational, Gauss-
```

```
    ++ Laguerre and Gauss-Hermite.
    ++ See \downlink{Manual Page}{manpageXXd01bbf}.
   d01fcf : (Integer,Matrix DoubleFloat,Matrix DoubleFloat,Integer,_
      DoubleFloat,Integer,Integer,Integer,Union(fn:FileName,fp:Asp4(FUNCTN))) -> Result
    ++ d01fcf(ndim,a,b,maxpts,eps,lenwrk,minpts,ifail,functn)
    ++ attempts to evaluate a multi-dimensional integral (up to
    ++ 15 dimensions), with constant and finite limits, to a specified
    ++ relative accuracy, using an adaptive subdivision strategy.
    ++ See \downlink{Manual Page}{manpageXXd01fcf}.
   d01gaf : (Matrix DoubleFloat,Matrix DoubleFloat,Integer,Integer) -> Result
    ++ d01gaf(x,y,n,ifail)
    ++ integrates a function which is specified numerically at
    ++ four or more points, over the whole of its specified range, using
    ++ third-order finite-difference formulae with error estimates,
    ++ according to a method due to Gill and Miller.
    ++ See \downlink{Manual Page}{manpageXXd01gaf}.
   d01gbf : (Integer,Matrix DoubleFloat,Matrix DoubleFloat,Integer,_
      DoubleFloat,Integer,Integer,Matrix DoubleFloat,Integer,Union(fn:FileName,fp:Asp4(FUN
    ++ d01gbf(ndim,a,b,maxcls,eps,lenwrk,mincls,wrkstr,ifail,functn)
    ++ returns an approximation to the integral of a function
    ++ over a hyper-rectangular region, using a Monte Carlo method. An
    ++ approximate relative error estimate is also returned. This
    ++ routine is suitable for low accuracy work.
    ++ See \downlink{Manual Page}{manpageXXd01gbf}.
  Implementation ==> add

   import Lisp
   import DoubleFloat
   import Any
   import Record
   import Integer
   import Matrix DoubleFloat
   import Boolean
   import NAGLinkSupportPackage
   import FortranPackage
   import Union(fn:FileName,fp:Asp1(F))
   import AnyFunctions1(DoubleFloat)
   import AnyFunctions1(Integer)
   import AnyFunctions1(Matrix DoubleFloat)


   d01ajf(aArg:DoubleFloat,bArg:DoubleFloat,epsabsArg:DoubleFloat,_
       epsrelArg:DoubleFloat,lwArg:Integer,liwArg:Integer,_
       ifailArg:Integer,fArg:Union(fn:FileName,fp:Asp1(F))): Result ==
       pushFortranOutputStack(fFilename := aspFilename "f")$FOP
       if fArg case fn
```

```
                then outputAsFortran(fArg.fn)
                else outputAsFortran(fArg.fp)
        popFortranOutputStack()$FOP
        [(invokeNagman([fFilename]$Lisp,_
        "d01ajf",_
        ["a"::S,"b"::S,"epsabs"::S,"epsrel"::S,"lw"::S_
        ,"liw"::S,"result"::S,"abserr"::S,"ifail"::S,"f"::S_
        ,"w"::S,"iw"::S]$Lisp,_
        ["result"::S,"abserr"::S,"w"::S,"iw"::S,"f"::S]$Lisp,_
        [["double"::S,"a"::S,"b"::S,"epsabs"::S,"epsrel"::S_
        ,"result"::S,"abserr"::S,["w"::S,"lw"::S]$Lisp,"f"::S]$Lisp_
        ,["integer"::S,"lw"::S,"liw"::S,["iw"::S,"liw"::S]$Lisp_
        ,"ifail"::S]$Lisp_
        ]$Lisp,_
        ["result"::S,"abserr"::S,"w"::S,"iw"::S,"ifail"::S]$Lisp,_
        [([aArg::Any,bArg::Any,epsabsArg::Any,epsrelArg::Any,lwArg::Any,liwArg::A
        @List Any]$Lisp)$Lisp)_
        pretend List (Record(key:Symbol,entry:Any))]$Result

    d01akf(aArg:DoubleFloat,bArg:DoubleFloat,epsabsArg:DoubleFloat,_
        epsrelArg:DoubleFloat,lwArg:Integer,liwArg:Integer,_
        ifailArg:Integer,fArg:Union(fn:FileName,fp:Asp1(F))): Result ==
        pushFortranOutputStack(fFilename := aspFilename "f")$FOP
        if fArg case fn
                then outputAsFortran(fArg.fn)
                else outputAsFortran(fArg.fp)
        popFortranOutputStack()$FOP
        [(invokeNagman([fFilename]$Lisp,_
        "d01akf",_
        ["a"::S,"b"::S,"epsabs"::S,"epsrel"::S,"lw"::S_
        ,"liw"::S,"result"::S,"abserr"::S,"ifail"::S,"f"::S_
        ,"w"::S,"iw"::S]$Lisp,_
        ["result"::S,"abserr"::S,"w"::S,"iw"::S,"f"::S]$Lisp,_
        [["double"::S,"a"::S,"b"::S,"epsabs"::S,"epsrel"::S_
        ,"result"::S,"abserr"::S,["w"::S,"lw"::S]$Lisp,"f"::S]$Lisp_
        ,["integer"::S,"lw"::S,"liw"::S,["iw"::S,"liw"::S]$Lisp_
        ,"ifail"::S]$Lisp_
        ]$Lisp,_
        ["result"::S,"abserr"::S,"w"::S,"iw"::S,"ifail"::S]$Lisp,_
        [([aArg::Any,bArg::Any,epsabsArg::Any,epsrelArg::Any,lwArg::Any,liwArg::A
        @List Any]$Lisp)$Lisp)_
        pretend List (Record(key:Symbol,entry:Any))]$Result

    d01alf(aArg:DoubleFloat,bArg:DoubleFloat,nptsArg:Integer,_
        pointsArg:Matrix DoubleFloat,epsabsArg:DoubleFloat,epsrelArg:DoubleFloat,
        lwArg:Integer,liwArg:Integer,ifailArg:Integer,_
```

```
    fArg:Union(fn:FileName,fp:Asp1(F))): Result ==
    pushFortranOutputStack(fFilename := aspFilename "f")$FOP
    if fArg case fn
              then outputAsFortran(fArg.fn)
              else outputAsFortran(fArg.fp)
    popFortranOutputStack()$FOP
    [(invokeNagman([fFilename]$Lisp,_
    "d01alf",_
    ["a"::S,"b"::S,"npts"::S,"epsabs"::S,"epsrel"::S_
    ,"lw"::S,"liw"::S,"result"::S,"abserr"::S,"ifail"::S_
    ,"f"::S,"points"::S,"w"::S,"iw"::S]$Lisp,_
    ["result"::S,"abserr"::S,"w"::S,"iw"::S,"f"::S]$Lisp,_
    [["double"::S,"a"::S,"b"::S,["points"::S,"*"::S]$Lisp_
    ,"epsabs"::S,"epsrel"::S,"result"::S,"abserr"::S,["w"::S,"lw"::S]$Lisp,"f"::S]$Lisp_
    ,["integer"::S,"npts"::S,"lw"::S,"liw"::S,["iw"::S,"liw"::S]$Lisp_
    ,"ifail"::S]$Lisp_
    ]$Lisp,_
    ["result"::S,"abserr"::S,"w"::S,"iw"::S,"ifail"::S]$Lisp,_
    [([aArg::Any,bArg::Any,nptsArg::Any,epsabsArg::Any,epsrelArg::Any,lwArg::Any,liwArg
    @List Any]$Lisp)$Lisp)_
    pretend List (Record(key:Symbol,entry:Any))]$Result

d01amf(boundArg:DoubleFloat,infArg:Integer,epsabsArg:DoubleFloat,_
    epsrelArg:DoubleFloat,lwArg:Integer,liwArg:Integer,_
    ifailArg:Integer,fArg:Union(fn:FileName,fp:Asp1(F))): Result ==
    pushFortranOutputStack(fFilename := aspFilename "f")$FOP
    if fArg case fn
              then outputAsFortran(fArg.fn)
              else outputAsFortran(fArg.fp)
    popFortranOutputStack()$FOP
    [(invokeNagman([fFilename]$Lisp,_
    "d01amf",_
    ["bound"::S,"inf"::S,"epsabs"::S,"epsrel"::S,"lw"::S_
    ,"liw"::S,"result"::S,"abserr"::S,"ifail"::S,"f"::S_
    ,"w"::S,"iw"::S]$Lisp,_
    ["result"::S,"abserr"::S,"w"::S,"iw"::S,"f"::S]$Lisp,_
    [["double"::S,"bound"::S,"epsabs"::S,"epsrel"::S_
    ,"result"::S,"abserr"::S,["w"::S,"lw"::S]$Lisp,"f"::S]$Lisp_
    ,["integer"::S,"inf"::S,"lw"::S,"liw"::S,["iw"::S,"liw"::S]$Lisp_
    ,"ifail"::S]$Lisp_
    ]$Lisp,_
    ["result"::S,"abserr"::S,"w"::S,"iw"::S,"ifail"::S]$Lisp,_
    [([boundArg::Any,infArg::Any,epsabsArg::Any,epsrelArg::Any,lwArg::Any,liwArg::Any,i
    @List Any]$Lisp)$Lisp)_
    pretend List (Record(key:Symbol,entry:Any))]$Result
```

```
d01anf(aArg:DoubleFloat,bArg:DoubleFloat,omegaArg:DoubleFloat,_
    keyArg:Integer,epsabsArg:DoubleFloat,epsrelArg:DoubleFloat,_
    lwArg:Integer,liwArg:Integer,ifailArg:Integer,_
    gArg:Union(fn:FileName,fp:Asp1(G))): Result ==
    pushFortranOutputStack(gFilename := aspFilename "g")$FOP
    if gArg case fn
              then outputAsFortran(gArg.fn)
              else outputAsFortran(gArg.fp)
    popFortranOutputStack()$FOP
    [(invokeNagman([gFilename]$Lisp,_
    "d01anf",_
    ["a"::S,"b"::S,"omega"::S,"key"::S,"epsabs"::S_
    ,"epsrel"::S,"lw"::S,"liw"::S,"result"::S,"abserr"::S_
    ,"ifail"::S,"g"::S,"w"::S,"iw"::S]$Lisp,_
    ["result"::S,"abserr"::S,"w"::S,"iw"::S,"g"::S]$Lisp,_
    [["double"::S,"a"::S,"b"::S,"omega"::S,"epsabs"::S_
    ,"epsrel"::S,"result"::S,"abserr"::S,["w"::S,"lw"::S]$Lisp,"g"::S]$Lisp_
    ,["integer"::S,"key"::S,"lw"::S,"liw"::S,["iw"::S,"liw"::S]$Lisp_
    ,"ifail"::S]$Lisp_
    ]$Lisp,_
    ["result"::S,"abserr"::S,"w"::S,"iw"::S,"ifail"::S]$Lisp,_
    [([aArg::Any,bArg::Any,omegaArg::Any,keyArg::Any,epsabsArg::Any,epsrelArg
    @List Any]$Lisp)$Lisp)_
    pretend List (Record(key:Symbol,entry:Any))]$Result

d01apf(aArg:DoubleFloat,bArg:DoubleFloat,alfaArg:DoubleFloat,_
    betaArg:DoubleFloat,keyArg:Integer,epsabsArg:DoubleFloat,_
    epsrelArg:DoubleFloat,lwArg:Integer,liwArg:Integer,_
    ifailArg:Integer,gArg:Union(fn:FileName,fp:Asp1(G))): Result ==
    pushFortranOutputStack(gFilename := aspFilename "g")$FOP
    if gArg case fn
              then outputAsFortran(gArg.fn)
              else outputAsFortran(gArg.fp)
    popFortranOutputStack()$FOP
    [(invokeNagman([gFilename]$Lisp,_
    "d01apf",_
    ["a"::S,"b"::S,"alfa"::S,"beta"::S,"key"::S_
    ,"epsabs"::S,"epsrel"::S,"lw"::S,"liw"::S,"result"::S_
    ,"abserr"::S,"ifail"::S,"g"::S,"w"::S,"iw"::S]$Lisp,_
    ["result"::S,"abserr"::S,"w"::S,"iw"::S,"g"::S]$Lisp,_
    [["double"::S,"a"::S,"b"::S,"alfa"::S,"beta"::S_
    ,"epsabs"::S,"epsrel"::S,"result"::S,"abserr"::S,["w"::S,"lw"::S]$Lisp,"g
    ,["integer"::S,"key"::S,"lw"::S,"liw"::S,["iw"::S,"liw"::S]$Lisp_
    ,"ifail"::S]$Lisp_
    ]$Lisp,_
    ["result"::S,"abserr"::S,"w"::S,"iw"::S,"ifail"::S]$Lisp,_
```

```
    [([aArg::Any,bArg::Any,alfaArg::Any,betaArg::Any,keyArg::Any,epsabsArg::Any,epsrelAr
    @List Any]$Lisp)$Lisp)_
    pretend List (Record(key:Symbol,entry:Any))]$Result

d01aqf(aArg:DoubleFloat,bArg:DoubleFloat,cArg:DoubleFloat,_
    epsabsArg:DoubleFloat,epsrelArg:DoubleFloat,lwArg:Integer,_
    liwArg:Integer,ifailArg:Integer,gArg:Union(fn:FileName,fp:Asp1(G))): Result ==
    pushFortranOutputStack(gFilename := aspFilename "g")$FOP
    if gArg case fn
              then outputAsFortran(gArg.fn)
              else outputAsFortran(gArg.fp)
    popFortranOutputStack()$FOP
    [(invokeNagman([gFilename]$Lisp,_
    "d01aqf",_
    ["a"::S,"b"::S,"c"::S,"epsabs"::S,"epsrel"::S_
    ,"lw"::S,"liw"::S,"result"::S,"abserr"::S,"ifail"::S_
    ,"g"::S,"w"::S,"iw"::S]$Lisp,_
    ["result"::S,"abserr"::S,"w"::S,"iw"::S,"g"::S]$Lisp,_
    [["double"::S,"a"::S,"b"::S,"c"::S,"epsabs"::S_
    ,"epsrel"::S,"result"::S,"abserr"::S,["w"::S,"lw"::S]$Lisp,"g"::S]$Lisp_
    ,["integer"::S,"lw"::S,"liw"::S,["iw"::S,"liw"::S]$Lisp_
    ,"ifail"::S]$Lisp_
    ]$Lisp,_
    ["result"::S,"abserr"::S,"w"::S,"iw"::S,"ifail"::S]$Lisp,_
    [([aArg::Any,bArg::Any,cArg::Any,epsabsArg::Any,epsrelArg::Any,lwArg::Any,liwArg::An
    @List Any]$Lisp)$Lisp)_
    pretend List (Record(key:Symbol,entry:Any))]$Result

d01asf(aArg:DoubleFloat,omegaArg:DoubleFloat,keyArg:Integer,_
    epsabsArg:DoubleFloat,limlstArg:Integer,lwArg:Integer,_
    liwArg:Integer,ifailArg:Integer,gArg:Union(fn:FileName,fp:Asp1(G))): Result ==
    pushFortranOutputStack(gFilename := aspFilename "g")$FOP
    if gArg case fn
              then outputAsFortran(gArg.fn)
              else outputAsFortran(gArg.fp)
    popFortranOutputStack()$FOP
    [(invokeNagman([gFilename]$Lisp,_
    "d01asf",_
    ["a"::S,"omega"::S,"key"::S,"epsabs"::S,"limlst"::S_
    ,"lw"::S,"liw"::S,"result"::S,"abserr"::S,"lst"::S_
    ,"ifail"::S,"g"::S,"erlst"::S,"rslst"::S,"ierlst"::S,"iw"::S,"w"::S_
    ]$Lisp,_
    ["result"::S,"abserr"::S,"lst"::S,"erlst"::S,"rslst"::S,"ierlst"::S,"iw"::S,"w"::S,"
    [["double"::S,"a"::S,"omega"::S,"epsabs"::S_
    ,"result"::S,"abserr"::S,["erlst"::S,"limlst"::S]$Lisp,["rslst"::S,"limlst"::S]$Lisp
    ,["integer"::S,"key"::S,"limlst"::S,"lw"::S_
```

```
    ,"liw"::S,"lst"::S,["ierlst"::S,"limlst"::S]$Lisp,["iw"::S,"liw"::S]$Lisp
    ]$Lisp,_
    ["result"::S,"abserr"::S,"lst"::S,"erlst"::S,"rslst"::S,"ierlst"::S,"iw":
    [([aArg::Any,omegaArg::Any,keyArg::Any,epsabsArg::Any,limlstArg::Any,lwAr
    @List Any]$Lisp)$Lisp)_
    pretend List (Record(key:Symbol,entry:Any))]$Result

d01bbf(aArg:DoubleFloat,bArg:DoubleFloat,itypeArg:Integer,_
    nArg:Integer,gtypeArg:Integer,ifailArg:Integer): Result ==
    [(invokeNagman(NIL$Lisp,_
    "d01bbf",_
    ["a"::S,"b"::S,"itype"::S,"n"::S,"gtype"::S_
    ,"ifail"::S,"weight"::S,"abscis"::S]$Lisp,_
    ["weight"::S,"abscis"::S]$Lisp,_
    [["double"::S,"a"::S,"b"::S,["weight"::S,"n"::S]$Lisp_
    ,["abscis"::S,"n"::S]$Lisp]$Lisp_
    ,["integer"::S,"itype"::S,"n"::S,"gtype"::S_
    ,"ifail"::S]$Lisp_
    ]$Lisp,_
    ["weight"::S,"abscis"::S,"ifail"::S]$Lisp,_
    [([aArg::Any,bArg::Any,itypeArg::Any,nArg::Any,gtypeArg::Any,ifailArg::An
    @List Any]$Lisp)$Lisp)_
    pretend List (Record(key:Symbol,entry:Any))]$Result

d01fcf(ndimArg:Integer,aArg:Matrix DoubleFloat,bArg:Matrix DoubleFloat,_
    maxptsArg:Integer,epsArg:DoubleFloat,lenwrkArg:Integer,_
    minptsArg:Integer,ifailArg:Integer,functnArg:Union(fn:FileName,fp:Asp4(FU
    pushFortranOutputStack(functnFilename := aspFilename "functn")$FOP
    if functnArg case fn
              then outputAsFortran(functnArg.fn)
              else outputAsFortran(functnArg.fp)
    popFortranOutputStack()$FOP
    [(invokeNagman([functnFilename]$Lisp,_
    "d01fcf",_
    ["ndim"::S,"maxpts"::S,"eps"::S,"lenwrk"::S,"acc"::S_
    ,"finval"::S,"minpts"::S,"ifail"::S,"functn"::S,"a"::S,"b"::S,"wrkstr"::S
    ["acc"::S,"finval"::S,"wrkstr"::S,"functn"::S]$Lisp,_
    [["double"::S,["a"::S,"ndim"::S]$Lisp,["b"::S,"ndim"::S]$Lisp_
    ,"eps"::S,"acc"::S,"finval"::S,["wrkstr"::S,"lenwrk"::S]$Lisp,"functn"::S
    ,["integer"::S,"ndim"::S,"maxpts"::S,"lenwrk"::S_
    ,"minpts"::S,"ifail"::S]$Lisp_
    ]$Lisp,_
    ["acc"::S,"finval"::S,"minpts"::S,"ifail"::S]$Lisp,_
    [([ndimArg::Any,maxptsArg::Any,epsArg::Any,lenwrkArg::Any,minptsArg::Any,
    @List Any]$Lisp)$Lisp)_
    pretend List (Record(key:Symbol,entry:Any))]$Result
```

```
d01gaf(xArg:Matrix DoubleFloat,yArg:Matrix DoubleFloat,nArg:Integer,_
    ifailArg:Integer): Result ==
    [(invokeNagman(NIL$Lisp,_
    "d01gaf",_
    ["n"::S,"ans"::S,"er"::S,"ifail"::S,"x"::S,"y"::S]$Lisp,_
    ["ans"::S,"er"::S]$Lisp,_
    [["double"::S,["x"::S,"n"::S]$Lisp,["y"::S,"n"::S]$Lisp_
    ,"ans"::S,"er"::S]$Lisp_
    ,["integer"::S,"n"::S,"ifail"::S]$Lisp_
    ]$Lisp,_
    ["ans"::S,"er"::S,"ifail"::S]$Lisp,_
    [([nArg::Any,ifailArg::Any,xArg::Any,yArg::Any ])_
    @List Any]$Lisp)$Lisp)_
    pretend List (Record(key:Symbol,entry:Any))]$Result

d01gbf(ndimArg:Integer,aArg:Matrix DoubleFloat,bArg:Matrix DoubleFloat,_
    maxclsArg:Integer,epsArg:DoubleFloat,lenwrkArg:Integer,_
    minclsArg:Integer,wrkstrArg:Matrix DoubleFloat,ifailArg:Integer,_
    functnArg:Union(fn:FileName,fp:Asp4(FUNCTN))): Result ==
    pushFortranOutputStack(functnFilename := aspFilename "functn")$FOP
    if functnArg case fn
              then outputAsFortran(functnArg.fn)
              else outputAsFortran(functnArg.fp)
    popFortranOutputStack()$FOP
    [(invokeNagman([functnFilename]$Lisp,_
    "d01gbf",_
    ["ndim"::S,"maxcls"::S,"eps"::S,"lenwrk"::S,"acc"::S_
    ,"finest"::S,"mincls"::S,"ifail"::S,"functn"::S,"a"::S,"b"::S,"wrkstr"::S]$Lisp,_
    ["acc"::S,"finest"::S,"functn"::S]$Lisp,_
    [["double"::S,["a"::S,"ndim"::S]$Lisp,["b"::S,"ndim"::S]$Lisp_
    ,"eps"::S,"acc"::S,"finest"::S,["wrkstr"::S,"lenwrk"::S]$Lisp,"functn"::S]$Lisp_
    ,["integer"::S,"ndim"::S,"maxcls"::S,"lenwrk"::S_
    ,"mincls"::S,"ifail"::S]$Lisp_
    ]$Lisp,_
    ["acc"::S,"finest"::S,"mincls"::S,"wrkstr"::S,"ifail"::S]$Lisp,_
    [([ndimArg::Any,maxclsArg::Any,epsArg::Any,lenwrkArg::Any,minclsArg::Any,ifailArg::A
    @List Any]$Lisp)$Lisp)_
    pretend List (Record(key:Symbol,entry:Any))]$Result
```

⟨*NAGD01.dotabb*⟩≡

```
"NAGD01" [color="#FF4488",href="bookvol10.4.pdf#nameddest=NAGD01"]
"ALIST" [color="#88FF44",href="bookvol10.3.pdf#nameddest=ALIST"]
"NAGD01" -> "ALIST"
```

15.11 package NAGE01 NagInterpolationPackage

⟨*NagInterpolationPackage.help*⟩≡

```
E01 -- Interpolation                          Introduction -- E01
                          Chapter E01
                        Interpolation

1. Scope of the Chapter

This chapter is concerned with the interpolation of a function of
one or two variables. When provided with the value of the
function (and possibly one or more of its lowest-order
derivatives) at each of a number of values of the variable(s),
the routines provide either an interpolating function or an
interpolated value. For some of the interpolating functions,
there are supporting routines to evaluate, differentiate or
integrate them.

2. Background to the Problems

In motivation and in some of its numerical processes, this
chapter has much in common with Chapter E02 (Curve and Surface
Fitting). For this reason, we shall adopt the same terminology
and refer to dependent variable and independent variable(s)
instead of function and variable(s). Where there is only one
independent variable, we shall denote it by x and the dependent
variable by y. Thus, in the basic problem considered in this
chapter, we are given a set of distinct values x ,x ,...,x  of x
                                               1  2      m
and a corresponding set of values y ,y ,...,y  of y, and we shall
                                   1  2     m
describe the problem as being one of interpolating the data
points (x ,y ), rather than interpolating a function. In modern
         r  r
usage, however, interpolation can have either of two rather
different meanings, both relevant to routines in this chapter.
They are

(a) the determination of a function of x which takes the value y
                                                               r
    at x=x , for r=1,2,...,m (an interpolating function or
          r
    interpolant),
```

(b) the determination of the value (interpolated value or interpolate) of an interpolating function at any given value, $\hat{x}$, say x, of x within the range of the x_r (so as to estimate the value at $\hat{x}$ of the function underlying the data).

The latter is the older meaning, associated particularly with the use of mathematical tables. The term 'function underlying the data', like the other terminology described above, is used so as to cover situations additional to those in which the data points have been computed from a known function, as with a mathematical table. In some contexts, the function may be unknown, perhaps representing the dependency of one physical variable on another, say temperature upon time.

Whether the underlying function is known or unknown, the object of interpolation will usually be to approximate it to acceptable accuracy by a function which is easy to evaluate anywhere in some range of interest. Piecewise polynomials such as cubic splines (see Section 2.2 of the E02 Chapter Introduction for definitions of terms in this case), being easy to evaluate and also capable of approximating a wide variety of functions, are the types of function mostly used in this chapter as interpolating functions.

Piecewise polynomials also, to a large extent, avoid the well-known problem of large unwanted fluctuations which can arise when interpolating a data set with a simple polynomial. Fluctuations can still arise but much less frequently and much less severely than with simple polynomials. Unwanted fluctuations are avoided altogether by a routine using piecewise cubic polynomials having only first derivative continuity. It is designed especially for monotonic data, but for other data still provides an interpolant which increases, or decreases, over the same intervals as the data.

The concept of interpolation can be generalised in a number of ways. For example, we may be required to estimate the value of $\hat{x}$ the underlying function at a value x outside the range of the data. This is the process of extrapolation. In general, it is a good deal less accurate than interpolation and is to be avoided whenever possible.

Interpolation can also be extended to the case of two independent variables. We shall denote these by x and y, and the dependent variable by f. Methods used depend markedly on whether or not the data values of f are given at the intersections of a rectangular mesh in the (x,y)-plane. If they are, bicubic splines (see Section 2.3.2 of the E02 Chapter Introduction) are very suitable and usually very effective for the problem. For other cases, perhaps where the f values are quite arbitrarily scattered in the (x,y)-plane, polynomials and splines are not at all appropriate and special forms of interpolating function have to be employed. Many such forms have been devised and two of the most successful are in routines in this chapter. They both have continuity in first, but not higher, derivatives.

2.1. References

[1] Froberg C E (1970) Introduction to Numerical Analysis. Addison-Wesley (2nd Edition).

[2] Dahlquist G and Bjork A (1974) Numerical Methods. Prentice-Hall.

3. Recommendations on Choice and Use of Routines

3.1. General

Before undertaking interpolation, in other than the simplest cases, the user should seriously consider the alternative of using a routine from Chapter E02 to approximate the data by a polynomial or spline containing significantly fewer coefficients than the corresponding interpolating function. This approach is much less liable to produce unwanted fluctuations and so can often provide a better approximation to the function underlying the data.

When interpolation is employed to approximate either an underlying function or its values, the user will need to be satisfied that the accuracy of approximation achieved is adequate. There may be a means for doing this which is particular to the application, or the routine used may itself provide a means. In other cases, one possibility is to repeat the interpolation using one or more extra data points, if they are available, or otherwise one or more fewer, and to compare the results. Other possibilities, if it is an interpolating function which is determined, are to examine the function graphically, if that gives sufficient accuracy, or to observe the behaviour of

the differences in a finite-difference table, formed from evaluations of the interpolating function at equally-spaced values of x over the range of interest. The spacing should be small enough to cause the typical size of the differences to decrease as the order of difference increases.

3.2. One Independent Variable

E01BAF computes an interpolating cubic spline, using a particular choice for the set of knots which has proved generally satisfactory in practice. If the user wishes to choose a different set, a cubic spline routine from Chapter E02, namely E02BAF, may be used in its interpolating mode, setting NCAP7 = M+ 4 and all elements of the parameter W to unity. These routines provide the interpolating function in B-spline form (see Section 2.2.2 in the E02 Chapter Introduction). Routines for evaluating, differentiating and integrating this form are discussed in Section 3.7 of the E02 Chapter Introduction.

The cubic spline does not always avoid unwanted fluctuations, especially when the data show a steep slope close to a region of small slope, or when the data inadequately represent the underlying curve. In such cases, E01BEF can be very useful. It derives a piecewise cubic polynomial (with first derivative continuity) which, between any adjacent pair of data points, either increases all the way, or decreases all the way (or stays constant). It is especially suited to data which are monotonic over their whole range.

In this routine, the interpolating function is represented simply by its value and first derivative at the data points. Supporting routines compute its value and first derivative elsewhere, as well as its definite integral over an arbitary interval.

3.3. Two Independent Variables

3.3.1. Data on a rectangular mesh

Given the value f_{qr} of the dependent variable f at the point (x_q, y_r) in the plane of the independent variables x and y, for each q=1,2,...,m and r=1,2,...,n (so that the points (x_q, y_r) lie at the m*n intersections of a rectangular mesh), E01DAF computes an interpolating bicubic spline, using a particular choice for

each of the spline's knot-set. This choice, the same as in E01BAF , has proved generally satisfactory in practice. If, instead, the user wishes to specify his own knots, a routine from Chapter E02, namely E02DAF, may be adapted (it is more cumbersome for the purpose, however, and much slower for larger problems). Using m and n in the above sense, the parameter M must be set to m*n, PX and PY must be set to m+4 and n+4 respectively and all elements of W should be set to unity. The recommended value for EPS is zero.

3.3.2. Arbitrary data

As remarked at the end of Section 2, special types of interpolating are required for this problem, which can often be difficult to solve satisfactorily. Two of the most successful are employed in E01SAF and E01SEF, the two routines which (with their respective evaluation routines E01SBF and E01SFF) are provided for the problem. Definitions can be found in the routine documents. Both interpolants have first derivative continuity and are 'local', in that their value at any point depends only on data in the immediate neighbourhood of the point. This latter feature is necessary for large sets of data to avoid prohibitive computing time.

The relative merits of the two methods vary with the data and it is not possible to predict which will be the better in any particular case.

3.4. Index

```
E01 -- Interpolation                              Contents -- E01
Chapter E01

Interpolation

E01BAF  Interpolating functions, cubic spline interpolant, one
        variable

E01BEF  Interpolating functions, monotonicity-preserving,
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E01BFF  Interpolated values, interpolant computed by E01BEF,
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        definite integral, one variable

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E01SAF  Interpolating functions, method of Renka and Cline, two
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        E01SAF, two variables

E01SEF  Interpolating functions, modified Shepard's method, two
        variables

E01SFF  Interpolated values, evaluate interpolant computed by
        E01SEF, two variables
```

%%%

```
E01 -- Interpolation                                            E01BAF
        E01BAF -- NAG Foundation Library Routine Document
```

Note: Before using this routine, please read the Users' Note for your implementation to check implementation-dependent details. The symbol (*) after a NAG routine name denotes a routine that is not included in the Foundation Library.

1. Purpose

E01BAF determines a cubic spline interpolant to a given set of data.

2. Specification

```
      SUBROUTINE E01BAF (M, X, Y, LAMDA, C, LCK, WRK, LWRK,
     1                   IFAIL)
      INTEGER          M, LCK, LWRK, IFAIL
      DOUBLE PRECISION X(M), Y(M), LAMDA(LCK), C(LCK), WRK(LWRK)
```

3. Description

This routine determines a cubic spline s(x), defined in the range $x_1 \le x \le x_m$, which interpolates (passes exactly through) the set of data points (x_i, y_i), for $i=1,2,\ldots,m$, where $m \ge 4$ and $x_1 < x_2 < \ldots < x_m$ end conditions are not imposed. The spline interpolant chosen has $m-4$ interior knots $(\lambda)_5, (\lambda)_6, \ldots, (\lambda)_m$, which are set to the values of $x_3, x_4, \ldots, x_{m-2}$ respectively. This spline is represented in its B-spline form (see Cox [1]):

$$s(x) = \sum_{i=1}^{m} c_i N_i(x),$$

where $N_i(x)$ denotes the normalised B-Spline of degree 3, defined upon the knots $(\lambda)_i, (\lambda)_{i+1}, \ldots, (\lambda)_{i+4}$, and c_i denotes its coefficient, whose value is to be determined by the routine.

The use of B-splines requires eight additional knots $(\lambda)_1$,

(lambda) , (lambda) , (lambda) , (lambda) , (lambda) , (lambda) and (lambda) to be specified; the routine sets the first four of these to x and the last four to x .

The algorithm for determining the coefficients is as described in [1] except that QR factorization is used instead of LU decomposition. The implementation of the algorithm involves setting up appropriate information for the related routine E02BAF followed by a call of that routine. (For further details of E02BAF, see the routine document.)

Values of the spline interpolant, or of its derivatives or definite integral, can subsequently be computed as detailed in Section 8.

4. References

[1] Cox M G (1975) An Algorithm for Spline Interpolation. J. Inst. Math. Appl. 15 95--108.

[2] Cox M G (1977) A Survey of Numerical Methods for Data and Function Approximation. The State of the Art in Numerical Analysis. (ed D A H Jacobs) Academic Press. 627--668.

5. Parameters

1: M -- INTEGER Input
On entry: m, the number of data points. Constraint: M >= 4.

2: X(M) -- DOUBLE PRECISION array Input
On entry: X(i) must be set to x , the ith data value of the independent variable x, for i=1,2,...,m. Constraint: X(i) < X(i+1), for i=1,2,...,M-1.

3: Y(M) -- DOUBLE PRECISION array Input
On entry: Y(i) must be set to y , the ith data value of the dependent variable y, for i=1,2,...,m.

4: LAMDA(LCK) -- DOUBLE PRECISION array Output
On exit: the value of (lambda) , the ith knot, for

i
i=1,2,...,m+4.

5: C(LCK) -- DOUBLE PRECISION array Output
On exit: the coefficient c of the B-spline N (x), for
i i
i=1,2,...,m. The remaining elements of the array are not used.

6: LCK -- INTEGER Input
On entry:
the dimension of the arrays LAMDA and C as declared in the (sub)program from which E01BAF is called.
Constraint: LCK >= M + 4.

7: WRK(LWRK) -- DOUBLE PRECISION array Workspace

8: LWRK -- INTEGER Input
On entry:
the dimension of the array WRK as declared in the (sub)program from which E01BAF is called.
Constraint: LWRK>=6*M+16.

9: IFAIL -- INTEGER Input/Output
On entry: IFAIL must be set to 0, -1 or 1. For users not familiar with this parameter (described in the Essential Introduction) the recommended value is 0.

On exit: IFAIL = 0 unless the routine detects an error (see Section 6).

6. Error Indicators and Warnings

Errors detected by the routine:

IFAIL= 1
On entry M < 4,

or LCK<M+4,

or LWRK<6*M+16.

IFAIL= 2
The X-values fail to satisfy the condition

X(1) < X(2) < X(3) <... < X(M).

7. Accuracy

The rounding errors incurred are such that the computed spline is an exact interpolant for a slightly perturbed set of ordinates $y_i + (\delta)y_i$. The ratio of the root-mean-square value of the $(\delta)y_i$ to that of the y_i is no greater than a small multiple of the relative machine precision.

8. Further Comments

The time taken by the routine is approximately proportional to m.

All the x_i are used as knot positions except x_2 and x_{m-1}. This choice of knots (see Cox [2]) means that s(x) is composed of m-3 cubic arcs as follows. If m=4, there is just a single arc space spanning the whole interval x_1 to x_4. If m>=5, the first and last arcs span the intervals x_1 to x_3 and x_{m-2} to x_m respectively. Additionally if m>=6, the ith arc, for i=2,3,...,m-4 spans the interval x_{i+1} to x_{i+2}.

After the call

```
CALL E01BAF (M, X, Y, LAMDA, C, LCK, WRK, LWRK, IFAIL)
```

the following operations may be carried out on the interpolant s(x).

The value of s(x) at x = XVAL can be provided in the real variable SVAL by the call

```
CALL E02BBF (M+4, LAMDA, C, XVAL, SVAL, IFAIL)
```

The values of s(x) and its first three derivatives at x = XVAL can be provided in the real array SDIF of dimension 4, by the call

```
CALL E02BCF (M+4, LAMDA, C, XVAL, LEFT, SDIF, IFAIL)
```

Here LEFT must specify whether the left- or right-hand value of

the third derivative is required (see E02BCF for details).

The value of the integral of s(x) over the range x_1 to x_m can be provided in the real variable SINT by

```
CALL E02BDF (M+4, LAMDA, C, SINT, IFAIL)
```

9. Example

The example program sets up data from 7 values of the exponential function in the interval 0 to 1. E01BAF is then called to compute a spline interpolant to these data.

The spline is evaluated by E02BBF, at the data points and at points halfway between each adjacent pair of data points, and the spline values and the values of e^x are printed out.

The example program is not reproduced here. The source code for all example programs is distributed with the NAG Foundation Library software and should be available on-line.

%%%

E01 -- Interpolation E01BEF

E01BEF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for your implementation to check implementation-dependent details. The symbol (*) after a NAG routine name denotes a routine that is not included in the Foundation Library.

1. Purpose

E01BEF computes a monotonicity-preserving piecewise cubic Hermite interpolant to a set of data points.

2. Specification

```
SUBROUTINE E01BEF (N, X, F, D, IFAIL)
INTEGER          N, IFAIL
DOUBLE PRECISION X(N), F(N), D(N)
```

3. Description

This routine estimates first derivatives at the set of data points (x ,f), for r=1,2,...,n, which determine a piecewise
r r
cubic Hermite interpolant to the data, that preserves monotonicity over ranges where the data points are monotonic. If the data points are only piecewise monotonic, the interpolant will have an extremum at each point where monotonicity switches direction. The estimates of the derivatives are computed by a formula due to Brodlie, which is described in Fritsch and Butland [1], with suitable changes at the boundary points.

The routine is derived from routine PCHIM in Fritsch [2].

Values of the computed interpolant, and of its first derivative and definite integral, can subsequently be computed by calling E01BFF, E01BGF and E01BHF, as described in Section 8

4. References

[1] Fritsch F N and Butland J (1984) A Method for Constructing Local Monotone Piecewise Cubic Interpolants. SIAM J. Sci. Statist. Comput. 5 300--304.

[2] Fritsch F N (1982) PCHIP Final Specifications. Report UCID-30194. Lawrence Livermore National Laboratory.

5. Parameters

1: N -- INTEGER Input
On entry: n, the number of data points. Constraint: N >= 2.

2: X(N) -- DOUBLE PRECISION array Input
On entry: X(r) must be set to x , the rth value of the
r
independent variable (abscissa), for r=1,2,...,n.
Constraint: X(r) < X(r+1).

3: F(N) -- DOUBLE PRECISION array Input
On entry: F(r) must be set to f , the rth value of the
r
dependent variable (ordinate), for r=1,2,...,n.

4: D(N) -- DOUBLE PRECISION array Output
On exit: estimates of derivatives at the data points. D(r) contains the derivative at X(r).

```
 5:  IFAIL -- INTEGER                                    Input/Output
     On entry: IFAIL must be set to 0, -1 or 1. For users not
     familiar with this parameter (described in the Essential
     Introduction) the recommended value is 0.

     On exit: IFAIL = 0 unless the routine detects an error (see
     Section 6).

6. Error Indicators and Warnings

Errors detected by the routine:

If on entry IFAIL = 0 or -1, explanatory error messages are
output on the current error message unit (as defined by X04AAF).

IFAIL= 1
     On entry N < 2.

IFAIL= 2
     The values of X(r), for r=1,2,...,N, are not in strictly
     increasing order.

7. Accuracy

The computational errors in the array D should be negligible in
most practical situations.

8. Further Comments

The time taken by the routine is approximately proportional to n.

The values of the computed interpolant at the points PX(i), for
i=1,2,...,M, may be obtained in the real array PF, of length at
least M, by the call:


        CALL E01BFF(N,X,F,D,M,PX,PF,IFAIL)

where N, X and F are the input parameters to E01BEF and D is the
output parameter from E01BEF.

The values of the computed interpolant at the points PX(i), for i
= 1,2,...,M, together with its first derivatives, may be obtained
in the real arrays PF and PD, both of length at least M, by the
call:
```

```
        CALL E01BGF(N,X,F,D,M,PX,PF,PD,IFAIL)
```

where N, X, F and D are as described above.

The value of the definite integral of the interpolant over the interval A to B can be obtained in the real variable PINT by the call:

```
        CALL E01BHF(N,X,F,D,A,B,PINT,IFAIL)
```

where N, X, F and D are as described above.

9. Example

This example program reads in a set of data points, calls E01BEF to compute a piecewise monotonic interpolant, and then calls E01BFF to evaluate the interpolant at equally spaced points.

The example program is not reproduced here. The source code for all example programs is distributed with the NAG Foundation Library software and should be available on-line.

%%%

E01 -- Interpolation E01BFF

E01BFF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for your implementation to check implementation-dependent details. The symbol (*) after a NAG routine name denotes a routine that is not included in the Foundation Library.

1. Purpose

E01BFF evaluates a piecewise cubic Hermite interpolant at a set of points.

2. Specification

```
SUBROUTINE E01BFF (N, X, F, D, M, PX, PF, IFAIL)
INTEGER          N, M, IFAIL
DOUBLE PRECISION X(N), F(N), D(N), PX(M), PF(M)
```

3. Description

This routine evaluates a piecewise cubic Hermite interpolant, as

computed by E01BEF, at the points PX(i), for i=1,2,...,m. If any point lies outside the interval from X(1) to X(N), a value is extrapolated from the nearest extreme cubic, and a warning is returned.

The routine is derived from routine PCHFE in Fritsch [1].

4. References

[1] Fritsch F N (1982) PCHIP Final Specifications. Report UCID-30194. Lawrence Livermore National Laboratory.

5. Parameters

1: N -- INTEGER Input

2: X(N) -- DOUBLE PRECISION array Input

3: F(N) -- DOUBLE PRECISION array Input

4: D(N) -- DOUBLE PRECISION array Input
On entry: N, X, F and D must be unchanged from the previous call of E01BEF.

5: M -- INTEGER Input
On entry: m, the number of points at which the interpolant is to be evaluated. Constraint: M >= 1.

6: PX(M) -- DOUBLE PRECISION array Input
On entry: the m values of x at which the interpolant is to be evaluated.

7: PF(M) -- DOUBLE PRECISION array Output
On exit: PF(i) contains the value of the interpolant evaluated at the point PX(i), for i=1,2,...,m.

8: IFAIL -- INTEGER Input/Output
On entry: IFAIL must be set to 0, -1 or 1. For users not familiar with this parameter (described in the Essential Introduction) the recommended value is 0.

On exit: IFAIL = 0 unless the routine detects an error (see Section 6).

6. Error Indicators and Warnings

```
Errors detected by the routine:

If on entry IFAIL = 0 or -1, explanatory error messages are
output on the current error message unit (as defined by X04AAF).

IFAIL= 1
     On entry N < 2.

IFAIL= 2
     The values of X(r), for r = 1,2,...,N, are not in strictly
     increasing order.

IFAIL= 3
     On entry M < 1.

IFAIL= 4
     At least one of the points PX(i), for i = 1,2,...,M, lies
     outside the interval [X(1),X(N)], and extrapolation was
     performed at all such points. Values computed at such points
     may be very unreliable.

7. Accuracy

The computational errors in the array PF should be negligible in
most practical situations.

8. Further Comments

The time taken by the routine is approximately proportional to
the number of evaluation points, m. The evaluation will be most
efficient if the elements of PX are in non-decreasing order (or,
more generally, if they are grouped in increasing order of the
intervals [X(r-1),X(r)]). A single call of E01BFF with m>1 is
more efficient than several calls with m=1.

9. Example

This example program reads in values of N, X, F and D, and then
calls E01BFF to evaluate the interpolant at equally spaced
points.

The example program is not reproduced here. The source code for
all example programs is distributed with the NAG Foundation
Library software and should be available on-line.
```

%%%

E01 -- Interpolation E01BGF
E01BGF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for your implementation to check implementation-dependent details. The symbol (*) after a NAG routine name denotes a routine that is not included in the Foundation Library.

1. Purpose

E01BGF evaluates a piecewise cubic Hermite interpolant and its first derivative at a set of points.

2. Specification

```
SUBROUTINE E01BGF (N, X, F, D, M, PX, PF, PD, IFAIL)
INTEGER          N, M, IFAIL
DOUBLE PRECISION X(N), F(N), D(N), PX(M), PF(M), PD(M)
```

3. Description

This routine evaluates a piecewise cubic Hermite interpolant, as computed by E01BEF, at the points PX(i), for i=1,2,...,m. The first derivatives at the points are also computed. If any point lies outside the interval from X(1) to X(N), values of the interpolant and its derivative are extrapolated from the nearest extreme cubic, and a warning is returned.

If values of the interpolant only, and not of its derivative, are required, E01BFF should be used.

The routine is derived from routine PCHFD in Fritsch [1].

4. References

[1] Fritsch F N (1982) PCHIP Final Specifications. Report UCID-30194. Lawrence Livermore National Laboratory.

5. Parameters

1: N -- INTEGER Input

2: X(N) -- DOUBLE PRECISION array Input

3: F(N) -- DOUBLE PRECISION array Input

```
 4:  D(N) -- DOUBLE PRECISION array                          Input
     On entry: N, X, F and D must be unchanged from the previous
     call of E01BEF.

 5:  M -- INTEGER                                            Input
     On entry: m, the number of points at which the interpolant
     is to be evaluated. Constraint: M >= 1.

 6:  PX(M) -- DOUBLE PRECISION array                         Input
     On entry: the m values of x at which the interpolant is to
     be evaluated.

 7:  PF(M) -- DOUBLE PRECISION array                        Output
     On exit: PF(i) contains the value of the interpolant
     evaluated at the point PX(i), for i=1,2,...,m.

 8:  PD(M) -- DOUBLE PRECISION array                        Output
     On exit: PD(i) contains the first derivative of the
     interpolant evaluated at the point PX(i), for i=1,2,...,m.

 9:  IFAIL -- INTEGER                                  Input/Output
     On entry: IFAIL must be set to 0, -1 or 1. For users not
     familiar with this parameter (described in the Essential
     Introduction) the recommended value is 0.

     On exit: IFAIL = 0 unless the routine detects an error (see
     Section 6).

6. Error Indicators and Warnings

Errors detected by the routine:

If on entry IFAIL = 0 or -1, explanatory error messages are
output on the current error message unit (as defined by X04AAF).

IFAIL= 1
     On entry N < 2.

IFAIL= 2
     The values of X(r), for r = 1,2,...,N, are not in strictly
     increasing order.

IFAIL= 3
     On entry M < 1.
```

```
IFAIL= 4
     At least one of the points PX(i), for i = 1,2,...,M, lies
     outside the interval [X(1),X(N)], and extrapolation was
     performed at all such points. Values computed at these
     points may be very unreliable.

7. Accuracy

The computational errors in the arrays PF and PD should be
negligible in most practical situations.

8. Further Comments

The time taken by the routine is approximately proportional to
the number of evaluation points, m. The evaluation will be most
efficient if the elements of PX are in non-decreasing order (or,
more generally, if they are grouped in increasing order of the
intervals [X(r-1),X(r)]). A single call of E01BGF with m>1 is
more efficient than several calls with m=1.

9. Example

This example program reads in values of N, X, F and D, and calls
E01BGF to compute the values of the interpolant and its
derivative at equally spaced points.

The example program is not reproduced here. The source code for
all example programs is distributed with the NAG Foundation
Library software and should be available on-line.
```

%%%

```
E01 -- Interpolation                                          E01BHF
        E01BHF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for
your implementation to check implementation-dependent details.
The symbol (*) after a NAG routine name denotes a routine that is
not included in the Foundation Library.

1. Purpose

E01BHF evaluates the definite integral of a piecewise cubic
Hermite interpolant over the interval [a,b].

2. Specification
```

```
      SUBROUTINE E01BHF (N, X, F, D, A, B, PINT, IFAIL)
      INTEGER          N, IFAIL
      DOUBLE PRECISION X(N), F(N), D(N), A, B, PINT
```

3. Description

This routine evaluates the definite integral of a piecewise cubic Hermite interpolant, as computed by E01BEF, over the interval [a,b].

If either a or b lies outside the interval from X(1) to X(N) computation of the integral involves extrapolation and a warning is returned.

The routine is derived from routine PCHIA in Fritsch [1].

4. References

[1] Fritsch F N (1982) PCHIP Final Specifications. Report UCID-30194. Lawrence Livermore National Laboratory .

5. Parameters

1: N -- INTEGER Input

2: X(N) -- DOUBLE PRECISION array Input

3: F(N) -- DOUBLE PRECISION array Input

4: D(N) -- DOUBLE PRECISION array Input
On entry: N, X, F and D must be unchanged from the previous call of E01BEF.

5: A -- DOUBLE PRECISION Input

6: B -- DOUBLE PRECISION Input
On entry: the interval [a,b] over which integration is to be performed.

7: PINT -- DOUBLE PRECISION Output
On exit: the value of the definite integral of the interpolant over the interval [a,b].

8: IFAIL -- INTEGER Input/Output
On entry: IFAIL must be set to 0, -1 or 1. For users not

```
     familiar with this parameter (described in the Essential
     Introduction) the recommended value is 0.

     On exit: IFAIL = 0 unless the routine detects an error (see
     Section 6).

6. Error Indicators and Warnings

Errors detected by the routine:

If on entry IFAIL = 0 or -1, explanatory error messages are
output on the current error message unit (as defined by X04AAF).

IFAIL= 1
     On entry N < 2.

IFAIL= 2
     The values of X(r), for r = 1,2,...,N, are not in strictly
     increasing order.

IFAIL= 3
     On entry at least one of A or B lies outside the interval [X
     (1),X(N)], and extrapolation was performed to compute the
     integral. The value returned is therefore unreliable.

7. Accuracy

The computational error in the value returned for PINT should be
negligible in most practical situations.

8. Further Comments

The time taken by the routine is approximately proportional to
the number of data points included within the interval [a,b].

9. Example

This example program reads in values of N, X, F and D. It then
reads in pairs of values for A and B, and evaluates the definite
integral of the interpolant over the interval [A,B] until end-of-
file is reached.

The example program is not reproduced here. The source code for
all example programs is distributed with the NAG Foundation
Library software and should be available on-line.
```

%%

E01 -- Interpolation E01DAF
E01DAF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for your implementation to check implementation-dependent details. The symbol (*) after a NAG routine name denotes a routine that is not included in the Foundation Library.

1. Purpose

E01DAF computes a bicubic spline interpolating surface through a set of data values, given on a rectangular grid in the x-y plane.

2. Specification

```
      SUBROUTINE E01DAF (MX, MY, X, Y, F, PX, PY, LAMDA, MU, C,
     1                   WRK, IFAIL)
      INTEGER           MX, MY, PX, PY, IFAIL
      DOUBLE PRECISION X(MX), Y(MY), F(MX*MY), LAMDA(MX+4), MU(MX
     1                 +4), C(MX*MY), WRK((MX+6)*(MY+6))
```

3. Description

This routine determines a bicubic spline interpolant to the set of data points $(x_q, y_r, f_{q,r})$, for $q=1,2,\ldots,m$; $r=1,2,\ldots,m$. The spline is given in the B-spline representation

$$s(x,y)=\sum_{i=1}^{m_x}\sum_{j=1}^{m_y} c_{ij} M_i(x) N_j(y),$$

such that

$$s(x_q, y_r)=f_{q,r},$$

where $M_i(x)$ and $N_j(y)$ denote normalised cubic B-splines, the former defined on the knots $(\lambda)_i$ to $(\lambda)_{i+4}$ and the

latter on the knots (mu) to (mu) , and the c are the spline
j j+4 ij
coefficients. These knots, as well as the coefficients, are determined by the routine, which is derived from the routine B2IRE in Anthony et al[1]. The method used is described in Section 8.2.

For further information on splines, see Hayes and Halliday [4] for bicubic splines and de Boor [3] for normalised B-splines.

Values of the computed spline can subsequently be obtained by calling E02DEF or E02DFF as described in Section 8.3.

4. References

[1] Anthony G T, Cox M G and Hayes J G (1982) DASL - Data Approximation Subroutine Library. National Physical Laboratory.

[2] Cox M G (1975) An Algorithm for Spline Interpolation. J. Inst. Math. Appl. 15 95--108.

[3] De Boor C (1972) On Calculating with B-splines. J. Approx. Theory. 6 50--62.

[4] Hayes J G and Halliday J (1974) The Least-squares Fitting of Cubic Spline Surfaces to General Data Sets. J. Inst. Math. Appl. 14 89--103.

5. Parameters

1: MX -- INTEGER Input

2: MY -- INTEGER Input
On entry: MX and MY must specify m and m respectively,
x y
the number of points along the x and y axis that define the rectangular grid. Constraint: MX >= 4 and MY >= 4.

3: X(MX) -- DOUBLE PRECISION array Input

4: Y(MY) -- DOUBLE PRECISION array Input
On entry: X(q) and Y(r) must contain x , for q=1,2,...,m ,
q x
and y , for r=1,2,...,m , respectively. Constraints:
r y

X(q) < X(q+1), for q=1,2,...,m -1,
x

Y(r) < Y(r+1), for r=1,2,...,m -1.
y

5: F(MX*MY) -- DOUBLE PRECISION array Input
On entry: F(m *(q-1)+r) must contain f , for q=1,2,...,m ;
y q,r x
r=1,2,...,m .
y

6: PX -- INTEGER Output

7: PY -- INTEGER Output
On exit: PX and PY contain m +4 and m +4, the total number
x y
of knots of the computed spline with respect to the x and y variables, respectively.

8: LAMDA(MX+4) -- DOUBLE PRECISION array Output

9: MU(MY+4) -- DOUBLE PRECISION array Output
On exit: LAMDA contains the complete set of knots (lambda)
i
associated with the x variable, i.e., the interior knots LAMDA(5), LAMDA(6), ..., LAMDA(PX-4), as well as the additional knots LAMDA(1) = LAMDA(2) = LAMDA(3) = LAMDA(4) = X(1) and LAMDA(PX-3) = LAMDA(PX-2) = LAMDA(PX-1) = LAMDA(PX) = X(MX) needed for the B-spline representation. MU contains the corresponding complete set of knots (mu) associated
i
with the y variable.

10: C(MX*MY) -- DOUBLE PRECISION array Output
On exit: the coefficients of the spline interpolant. C(
m *(i-1)+j) contains the coefficient c described in
y ij
Section 3.

11: WRK((MX+6)*(MY+6)) -- DOUBLE PRECISION array Workspace

12: IFAIL -- INTEGER Input/Output
On entry: IFAIL must be set to 0, -1 or 1. For users not familiar with this parameter (described in the Essential Introduction) the recommended value is 0.

```
     On exit: IFAIL = 0 unless the routine detects an error (see
     Section 6).

6. Error Indicators and Warnings

Errors detected by the routine:

If on entry IFAIL = 0 or -1, explanatory error messages are
output on the current error message unit (as defined by X04AAF).

IFAIL= 1
     On entry MX < 4,

     or       MY < 4.

IFAIL= 2
     On entry either the values in the X array or the values in
     the Y array are not in increasing order.

IFAIL= 3
     A system of linear equations defining the B-spline
     coefficients was singular; the problem is too ill-
     conditioned to permit solution.

7. Accuracy

The main sources of rounding errors are in steps (2), (3), (6)
and (7) of the algorithm described in Section 8.2. It can be
shown (Cox [2]) that the matrix A  formed in step (2) has
                                 x
elements differing relatively from their true values by at most a
small multiple of 3(epsilon), where (epsilon) is the machine
precision. A  is 'totally positive', and a linear system with
           x
such a coefficient matrix can be solved quite safely by
elimination without pivoting. Similar comments apply to steps (6)
and (7). Thus the complete process is numerically stable.

8. Further Comments

8.1. Timing

The time taken by this routine is approximately proportional to
m m .
 x y
```

8.2. Outline of method used

The process of computing the spline consists of the following steps:

(1) choice of the interior x-knots (lambda) , (lambda) ,...,
(lambda) as (lambda) =x , for i=5,6,...,m ,
m i i-2 x
x

(2) formation of the system
A E=F,
x
where A is a band matrix of order m and bandwidth 4,
x x
containing in its qth row the values at x of the B-splines
q
in x, F is the m by m rectangular matrix of values f ,
x y q,r
and E denotes an m by m rectangular matrix of
x y
intermediate coefficients,

(3) use of Gaussian elimination to reduce this system to band triangular form,

(4) solution of this triangular system for E,

(5) choice of the interior y knots (mu) , (mu) ,...,(mu) as
5 6 m
y
(mu) =y , for i=5,6,...,m ,
i i-2 y

(6) formation of the system
T T
A C =E ,
y
where A is the counterpart of A for the y variable, and C
y x
denotes the m by m rectangular matrix of values of c ,
x y ij

(7) use of Gaussian elimination to reduce this system to band

triangular form,

(8) solution of this triangular system for C^T and hence C.

For computational convenience, steps (2) and (3), and likewise steps (6) and (7), are combined so that the formation of A_x and A_y and the reductions to triangular form are carried out one row at a time.

8.3. Evaluation of Computed Spline

The values of the computed spline at the points (TX(r),TY(r)), for r = 1,2,...,N, may be obtained in the double precision array FF, of length at least N, by the following call:

```
IFAIL = 0
CALL E02DEF(N,PX,PY,TX,TY,LAMDA,MU,C,FF,WRK,IWRK,IFAIL)
```

where PX, PY, LAMDA, MU and C are the output parameters of E01DAF , WRK is a double precision workspace array of length at least PY-4, and IWRK is an integer workspace array of length at least PY-4.

To evaluate the computed spline on an NX by NY rectangular grid of points in the x-y plane, which is defined by the x co-ordinates stored in TX(q), for q = 1,2,...,NX, and the y co-ordinates stored in TY(r), for r = 1,2,...,NY, returning the results in the double precision array FG which is of length at least NX*NY, the following call may be used:

```
 IFAIL = 0
 CALL E02DFF(NX,NY,PX,PY,TX,TY,LAMDA,MU,C,FG,WRK,LWRK,
*            IWRK,LIWRK,IFAIL)
```

where PX, PY, LAMDA, MU and C are the output parameters of E01DAF , WRK is a double precision workspace array of length at least LWRK = min(NWRK1,NWRK2), NWRK1 = NX*4+PX, NWRK2 = NY*4+PY, and IWRK is an integer workspace array of length at least LIWRK = NY + PY - 4 if NWRK1 > NWRK2, or NX + PX - 4 otherwise. The result of the spline evaluated at grid point (q,r) is returned in element (NY*(q-1)+r) of the array FG.

9. Example

This program reads in values of m , x for q=1,2,...,m , m and
x q x y
y for r=1,2,...,m , followed by values of the ordinates f
r y q,r
defined at the grid points (x ,y). It then calls E01DAF to
q r
compute a bicubic spline interpolant of the data values, and prints the values of the knots and B-spline coefficients. Finally it evaluates the spline at a small sample of points on a rectangular grid.

The example program is not reproduced here. The source code for all example programs is distributed with the NAG Foundation Library software and should be available on-line.

%%%

E01 -- Interpolation E01SAF
E01SAF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for your implementation to check implementation-dependent details. The symbol (*) after a NAG routine name denotes a routine that is not included in the Foundation Library.

1. Purpose

E01SAF generates a two-dimensional surface interpolating a set of scattered data points, using the method of Renka and Cline.

2. Specification

```
SUBROUTINE E01SAF (M, X, Y, F, TRIANG, GRADS, IFAIL)
INTEGER          M, TRIANG(7*M), IFAIL
DOUBLE PRECISION X(M), Y(M), F(M), GRADS(2,M)
```

3. Description

This routine constructs an interpolating surface F(x,y) through a set of m scattered data points (x ,y ,f), for r=1,2,...,m, using
r r r
a method due to Renka and Cline. In the (x,y) plane, the data points must be distinct. The constructed surface is continuous

and has continuous first derivatives.

The method involves firstly creating a triangulation with all the (x,y) data points as nodes, the triangulation being as nearly equiangular as possible (see Cline and Renka [1]). Then gradients in the x- and y-directions are estimated at node r, for r=1,2,...,m, as the partial derivatives of a quadratic function of x and y which interpolates the data value f_r, and which fits the data values at nearby nodes (those within a certain distance chosen by the algorithm) in a weighted least-squares sense. The weights are chosen such that closer nodes have more influence than more distant nodes on derivative estimates at node r. The computed partial derivatives, with the f_r values, at the three nodes of each triangle define a piecewise polynomial surface of a certain form which is the interpolant on that triangle. See Renka and Cline [4] for more detailed information on the algorithm, a development of that by Lawson [2]. The code is derived from Renka [3].

The interpolant F(x,y) can subsequently be evaluated at any point (x,y) inside or outside the domain of the data by a call to E01SBF. Points outside the domain are evaluated by extrapolation.

4. References

[1] Cline A K and Renka R L (1984) A Storage-efficient Method for Construction of a Thiessen Triangulation. Rocky Mountain J. Math. 14 119--139.

[2] Lawson C L (1977) Software for C^1 Surface Interpolation. Mathematical Software III. (ed J R Rice) Academic Press. 161--194.

[3] Renka R L (1984) Algorithm 624: Triangulation and Interpolation of Arbitrarily Distributed Points in the Plane. ACM Trans. Math. Softw. 10 440--442.

[4] Renka R L and Cline A K (1984) A Triangle-based C^1 Interpolation Method. Rocky Mountain J. Math. 14 223--237.

5. Parameters

```
 1:  M -- INTEGER                                                  Input
     On entry: m, the number of data points. Constraint: M >= 3.

 2:  X(M) -- DOUBLE PRECISION array                                Input

 3:  Y(M) -- DOUBLE PRECISION array                                Input

 4:  F(M) -- DOUBLE PRECISION array                                Input
     On entry: the co-ordinates of the rth data point, for
     r=1,2,...,m. The data points are accepted in any order, but
     see Section 8. Constraint: The (x,y) nodes must not all be
     collinear, and each node must be unique.

 5:  TRIANG(7*M) -- INTEGER array                                  Output
     On exit: a data structure defining the computed
     triangulation, in a form suitable for passing to E01SBF.

 6:  GRADS(2,M) -- DOUBLE PRECISION array                          Output
     On exit: the estimated partial derivatives at the nodes, in
     a form suitable for passing to E01SBF. The derivatives at
     node r with respect to x and y are contained in GRADS(1,r)
     and GRADS(2,r) respectively, for r=1,2,...,m.

 7:  IFAIL -- INTEGER                                        Input/Output
     On entry: IFAIL must be set to 0, -1 or 1. For users not
     familiar with this parameter (described in the Essential
     Introduction) the recommended value is 0.

     On exit: IFAIL = 0 unless the routine detects an error (see
     Section 6).

6. Error Indicators and Warnings

Errors detected by the routine:

If on entry IFAIL = 0 or -1, explanatory error messages are
output on the current error message unit (as defined by X04AAF).

IFAIL= 1
     On entry M < 3.

IFAIL= 2
     On entry all the (X,Y) pairs are collinear.

IFAIL= 3
     On entry (X(i),Y(i)) = (X(j),Y(j)) for some i/=j.
```

```
7. Accuracy

On successful exit, the computational errors should be negligible
in most situations but the user should always check the computed
surface for acceptability, by drawing contours for instance. The
surface always interpolates the input data exactly.

8. Further Comments

The time taken for a call of E01SAF is approximately proportional
to the number of data points, m. The routine is more efficient
if, before entry, the values in X, Y, F are arranged so that the
X array is in ascending order.

9. Example

This program reads in a set of 30 data points and calls E01SAF to
construct an interpolating surface. It then calls E01SBF to
evaluate the interpolant at a sample of points on a rectangular
grid.

Note that this example is not typical of a realistic problem: the
number of data points would normally be larger, and the
interpolant would need to be evaluated on a finer grid to obtain
an accurate plot, say.

The example program is not reproduced here. The source code for
all example programs is distributed with the NAG Foundation
Library software and should be available on-line.
```

%%%

```
E01 -- Interpolation                                          E01SBF
        E01SBF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for
your implementation to check implementation-dependent details.
The symbol (*) after a NAG routine name denotes a routine that is
not included in the Foundation Library.

1. Purpose

E01SBF evaluates at a given point the two-dimensional interpolant
function computed by E01SAF.
```

2. Specification

```
      SUBROUTINE E01SBF (M, X, Y, F, TRIANG, GRADS, PX, PY, PF,
     1                   IFAIL)
      INTEGER          M, TRIANG(7*M), IFAIL
      DOUBLE PRECISION X(M), Y(M), F(M), GRADS(2,M), PX, PY, PF
```

3. Description

This routine takes as input the parameters defining the interpolant F(x,y) of a set of scattered data points (x_r, y_r, f_r), for r=1,2,...,m, as computed by E01SAF, and evaluates the interpolant at the point (px,py).

If (px,py) is equal to (x_r, y_r) for some value of r, the returned value will be equal to f_r.

If (px,py) is not equal to (x_r, y_r) for any r, the derivatives in GRADS will be used to compute the interpolant. A triangle is sought which contains the point (px,py), and the vertices of the triangle along with the partial derivatives and f_r values at the vertices are used to compute the value F(px,py). If the point (px,py) lies outside the triangulation defined by the input parameters, the returned value is obtained by extrapolation. In this case, the interpolating function F is extended linearly beyond the triangulation boundary. The method is described in more detail in Renka and Cline [2] and the code is derived from Renka [1].

E01SBF must only be called after a call to E01SAF.

4. References

[1] Renka R L (1984) Algorithm 624: Triangulation and Interpolation of Arbitrarily Distributed Points in the Plane. ACM Trans. Math. Softw. 10 440--442.

[2] Renka R L and Cline A K (1984) A Triangle-based C^1 Interpolation Method. Rocky Mountain J. Math. 14 223--237.

```
5. Parameters

 1:  M -- INTEGER                                                  Input

 2:  X(M) -- DOUBLE PRECISION array                                Input

 3:  Y(M) -- DOUBLE PRECISION array                                Input

 4:  F(M) -- DOUBLE PRECISION array                                Input

 5:  TRIANG(7*M) -- INTEGER array                                  Input

 6:  GRADS(2,M) -- DOUBLE PRECISION array                          Input
     On entry: M, X, Y, F, TRIANG and GRADS must be unchanged
     from the previous call of E01SAF.

 7:  PX -- DOUBLE PRECISION                                        Input

 8:  PY -- DOUBLE PRECISION                                        Input
     On entry: the point (px,py) at which the interpolant is to
     be evaluated.

 9:  PF -- DOUBLE PRECISION                                       Output
     On exit: the value of the interpolant evaluated at the
     point (px,py).

10:  IFAIL -- INTEGER                                       Input/Output
     On entry: IFAIL must be set to 0, -1 or 1. For users not
     familiar with this parameter (described in the Essential
     Introduction) the recommended value is 0.

     On exit: IFAIL = 0 unless the routine detects an error (see
     Section 6).

6. Error Indicators and Warnings

Errors detected by the routine:

If on entry IFAIL = 0 or -1, explanatory error messages are
output on the current error message unit (as defined by X04AAF).

IFAIL= 1
     On entry M < 3.

IFAIL= 2
     On entry the triangulation information held in the array
```

```
     TRIANG does not specify a valid triangulation of the data
     points. TRIANG may have been corrupted since the call to
     E01SAF.

IFAIL= 3
     The evaluation point (PX,PY) lies outside the nodal
     triangulation, and the value returned in PF is computed by
     extrapolation.

7. Accuracy

Computational errors should be negligible in most practical
situations.

8. Further Comments

The time taken for a call of E01SBF is approximately proportional
to the number of data points, m.

The results returned by this routine are particularly suitable
for applications such as graph plotting, producing a smooth
surface from a number of scattered points.

9. Example

See the example for E01SAF.

The example program is not reproduced here. The source code for
all example programs is distributed with the NAG Foundation
Library software and should be available on-line.
```

%%%

```
E01 -- Interpolation                                          E01SEF
        E01SEF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for
your implementation to check implementation-dependent details.
The symbol (*) after a NAG routine name denotes a routine that is
not included in the Foundation Library.

1. Purpose

E01SEF generates a two-dimensional surface interpolating a set of
scattered data points, using a modified Shepard method.
```

2. Specification

```
 SUBROUTINE E01SEF (M, X, Y, F, RNW, RNQ, NW, NQ, FNODES,
1                   MINNQ, WRK, IFAIL)
 INTEGER          M, NW, NQ, MINNQ, IFAIL
 DOUBLE PRECISION X(M), Y(M), F(M), RNW, RNQ, FNODES(5*M),
1                 WRK(6*M)
```

3. Description

This routine constructs an interpolating surface F(x,y) through a set of m scattered data points (x_r, y_r, f_r), for r=1,2,...,m, using a modification of Shepard's method. The surface is continuous and has continuous first derivatives.

The basic Shepard method, described in [2], interpolates the input data with the weighted mean

$$F(x,y)= \frac{\sum_{r=1}^{m} w_r(x,y) f_r}{\sum_{r=1}^{m} w_r(x,y)},$$

$$\text{where } w_r(x,y)= \frac{1}{d_r^2} \text{ and } d_r^2=(x-x_r)^2+(y-y_r)^2 .$$

The basic method is global in that the interpolated value at any point depends on all the data, but this routine uses a modification due to Franke and Nielson described in [1], whereby the method becomes local by adjusting each $w_r(x,y)$ to be zero outside a circle with centre (x_r, y_r) and some radius R_w. Also, to improve the performance of the basic method, each f_r above is

replaced by a function f (x,y), which is a quadratic fitted by
r
weighted least-squares to data local to (x ,y) and forced to
r r
interpolate (x ,y ,f). In this context, a point (x,y) is defined
r r r
to be local to another point if it lies within some distance R
q
of it. Computation of these quadratics constitutes the main work done by this routine. If there are less than 5 other points within distance R from (x ,y), the quadratic is replaced by a
q r r
linear function. In cases of rank-deficiency, the minimum norm solution is computed.

The user may specify values for R and R , but it is usually
w q
easier to choose instead two integers N and N , from which the
w q
routine will compute R and R . These integers can be thought of
w q
as the average numbers of data points lying within distances R
w
and R respectively from each node. Default values are provided,
q
and advice on alternatives is given in Section 8.2.

The interpolant F(x,y) generated by this routine can subsequently be evaluated for any point (x,y) in the domain of the data by a call to E01SFF.

4. References

[1] Franke R and Nielson G (1980) Smooth Interpolation of Large Sets of Scattered Data. Internat. J. Num. Methods Engrg. 15 1691--1704.

[2] Shepard D (1968) A Two-dimensional Interpolation Function for Irregularly Spaced Data. Proc. 23rd Nat. Conf. ACM. Brandon/Systems Press Inc., Princeton. 517--523.

5. Parameters

1: M -- INTEGER Input
On entry: m, the number of data points. Constraint: M >= 3.

2: X(M) -- DOUBLE PRECISION array Input

3: Y(M) -- DOUBLE PRECISION array Input

4: F(M) -- DOUBLE PRECISION array Input
On entry: the co-ordinates of the rth data point, for r=1,2,...,m. The order of the data points is immaterial. Constraint: each of the (X(r),Y(r)) pairs must be unique.

5: RNW -- DOUBLE PRECISION Input/Output

6: RNQ -- DOUBLE PRECISION Input/Output
On entry: suitable values for the radii R_w and R_q, described in Section 3. Constraint: RNQ <= 0 or 0 < RNW <= RNQ. On exit: if RNQ is set less than or equal to zero on entry, then default values for both of them will be computed from the parameters NW and NQ, and RNW and RNQ will contain these values on exit.

7: NW -- INTEGER Input

8: NQ -- INTEGER Input
On entry: if RNQ > 0.0 and RNW > 0.0 then NW and NQ are not referenced by the routine. Otherwise, NW and NQ must specify suitable values for the integers N_w and N_q described in Section 3.

If NQ is less than or equal to zero on entry, then default values for both of them, namely NW = 9 and NQ = 18, will be used. Constraint: NQ <= 0 or 0 < NW <= NQ.

9: FNODES(5*M) -- DOUBLE PRECISION array Output
On exit: the coefficients of the constructed quadratic nodal functions. These are in a form suitable for passing to E01SFF.

10: MINNQ -- INTEGER Output
On exit: the minimum number of data points that lie within radius RNQ of any node, and thus define a nodal function. If MINNQ is very small (say, less than 5), then the interpolant may be unsatisfactory in regions where the data points are sparse.

11: WRK(6*M) -- DOUBLE PRECISION array Workspace

12: IFAIL -- INTEGER Input/Output
On entry: IFAIL must be set to 0, -1 or 1. For users not familiar with this parameter (described in the Essential Introduction) the recommended value is 0.

On exit: IFAIL = 0 unless the routine detects an error (see Section 6).

6. Error Indicators and Warnings

Errors detected by the routine:

If on entry IFAIL = 0 or -1, explanatory error messages are output on the current error message unit (as defined by X04AAF).

IFAIL= 1
On entry M < 3.

IFAIL= 2
On entry RNQ > 0 and either RNW > RNQ or RNW <= 0.

IFAIL= 3
On entry NQ > 0 and either NW > NQ or NW <= 0.

IFAIL= 4
On entry (X(i),Y(i)) is equal to (X(j),Y(j)) for some i/=j.

7. Accuracy

On successful exit, the computational errors should be negligible in most situations but the user should always check the computed surface for acceptability, by drawing contours for instance. The surface always interpolates the input data exactly.

8. Further Comments

8.1. Timing

The time taken for a call of E01SEF is approximately proportional to the number of data points, m, provided that N_q is of the same order as its default value (18). However if N_q is increased so that the method becomes more global, the time taken becomes

approximately proportional to m .
m N} {{\rm w}}$ and ${\rm N} {{\rm q}}$
8.2. Choice of ${\
Note first that the radii R and R , described in Section 3, are
w q

/ N / N
D / w D / q
computed as - / -- and - / -- respectively, where D is
2\/ m 2/ m
the maximum distance between any pair of data points.
Default values N =9 and N =18 work quite well when the data
w q
points are fairly uniformly distributed. However, for data having
some regions with relatively few points or for small data sets
(m<25), a larger value of N may be needed. This is to ensure a
w
reasonable number of data points within a distance R of each
w
node, and to avoid some regions in the data area being left
outside all the discs of radius R on which the weights w (x,y)
w r
are non-zero. Maintaining N approximately equal to 2N is
q w
usually an advantage.

Note however that increasing N and N does not improve the
w q
quality of the interpolant in all cases. It does increase the
computational cost and makes the method less local.

9. Example

This program reads in a set of 30 data points and calls E01SEF to
construct an interpolating surface. It then calls E01SFF to
evaluate the interpolant at a sample of points on a rectangular
grid.

Note that this example is not typical of a realistic problem: the
number of data points would normally be larger, and the
interpolant would need to be evaluated on a finer grid to obtain
an accurate plot, say.

The example program is not reproduced here. The source code for
all example programs is distributed with the NAG Foundation

```
Library software and should be available on-line.

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

E01 -- Interpolation                                              E01SFF
        E01SFF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for
your implementation to check implementation-dependent details.
The symbol (*) after a NAG routine name denotes a routine that is
not included in the Foundation Library.

1. Purpose

E01SFF evaluates at a given point the two-dimensional
interpolating function computed by E01SEF.

2. Specification

       SUBROUTINE E01SFF (M, X, Y, F, RNW, FNODES, PX, PY, PF,
      1                   IFAIL)
       INTEGER          M, IFAIL
       DOUBLE PRECISION X(M), Y(M), F(M), RNW, FNODES(5*M), PX,
      1                 PY, PF

3. Description

This routine takes as input the interpolant F(x,y) of a set of
scattered data points (x ,y ,f ), for r=1,2,...,m, as computed by
                       r  r  r
E01SEF, and evaluates the interpolant at the point (px,py).

If (px,py) is equal to (x ,y ) for some value of r, the returned
                        r  r
value will be equal to f .
                       r

If (px,py) is not equal to (x ,y ) for any r, all points that are
                             r  r
within distance RNW of (px,py), along with the corresponding
nodal functions given by FNODES, will be used to compute a value
of the interpolant.

E01SFF must only be called after a call to E01SEF.

4. References
```

[1] Franke R and Nielson G (1980) Smooth Interpolation of Large Sets of Scattered Data. Internat. J. Num. Methods Engrg. 15 1691--1704.

[2] Shepard D (1968) A Two-dimensional Interpolation Function for Irregularly Spaced Data. Proc. 23rd Nat. Conf. ACM. Brandon/Systems Press Inc., Princeton. 517--523.

5. Parameters

1: M -- INTEGER Input

2: X(M) -- DOUBLE PRECISION array Input

3: Y(M) -- DOUBLE PRECISION array Input

4: F(M) -- DOUBLE PRECISION array Input

5: RNW -- DOUBLE PRECISION Input

6: FNODES(5*M) -- DOUBLE PRECISION array Input
On entry: M, X, Y, F, RNW and FNODES must be unchanged from the previous call of E01SEF.

7: PX -- DOUBLE PRECISION Input

8: PY -- DOUBLE PRECISION Input
On entry: the point (px,py) at which the interpolant is to be evaluated.

9: PF -- DOUBLE PRECISION Output
On exit: the value of the interpolant evaluated at the point (px,py).

10: IFAIL -- INTEGER Input/Output
On entry: IFAIL must be set to 0, -1 or 1. For users not familiar with this parameter (described in the Essential Introduction) the recommended value is 0.

On exit: IFAIL = 0 unless the routine detects an error (see Section 6).

6. Error Indicators and Warnings

Errors detected by the routine:

```
If on entry IFAIL = 0 or -1, explanatory error messages are
output on the current error message unit (as defined by X04AAF).

IFAIL= 1
     On entry M < 3.

IFAIL= 2
     The interpolant cannot be evaluated because the evaluation
     point (PX,PY) lies outside the support region of the data
     supplied in X, Y and F. This error exit will occur if
     (PX,PY) lies at a distance greater than or equal to RNW from
     every point given by arrays X and Y.

     The value 0.0 is returned in PF. This value will not provide
     continuity with values obtained at other points (PX,PY),
     i.e., values obtained when IFAIL = 0 on exit.

7. Accuracy

Computational errors should be negligible in most practical
situations.

8. Further Comments

The time taken for a call of E01SFF is approximately proportional
to the number of data points, m.

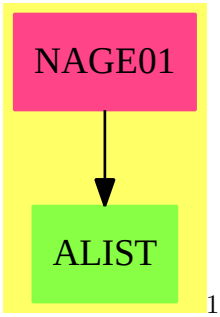
The results returned by this routine are particularly suitable
for applications such as graph plotting, producing a smooth
surface from a number of scattered points.

9. Example

See the example for E01SEF.

The example program is not reproduced here. The source code for
all example programs is distributed with the NAG Foundation
Library software and should be available on-line.
```

15.12 NagInterpolationPackage



1.00

Exports:

e01baf	e01bef	e01bff	e01bgf	e01bhf
e01daf	e01saf	e01sbf	e01sef	e01sff

⟨*package NAGE01 NagInterpolationPackage*⟩≡

```
)abbrev package NAGE01 NagInterpolationPackage
++ Author: Godfrey Nolan and Mike Dewar
++ Date Created: Jan 1994
++ Date Last Updated: Thu May 12 17:44:53 1994
++ Description:
++ This package uses the NAG Library to calculate the interpolation of a
++ function of one or two variables. When provided with the value of the
++ function (and possibly one or more of its lowest-order
++ derivatives) at each of a number of values of the variable(s),
++ the routines provide either an interpolating function or an
++ interpolated value. For some of the interpolating functions,
++ there are supporting routines to evaluate, differentiate or
++ integrate them.
++ See \downlink{Manual Page}{manpageXXe01}.


NagInterpolationPackage(): Exports == Implementation where
  S ==> Symbol
  FOP ==> FortranOutputStackPackage

  Exports ==> with
    e01baf : (Integer,Matrix DoubleFloat,Matrix DoubleFloat,Integer,_
        Integer,Integer) -> Result
     ++ e01baf(m,x,y,lck,lwrk,ifail)
     ++ determines a cubic spline to a given set of
     ++ data.
     ++ See \downlink{Manual Page}{manpageXXe01baf}.
    e01bef : (Integer,Matrix DoubleFloat,Matrix DoubleFloat,Integer) -> Result
     ++ e01bef(n,x,f,ifail)
```

```
 ++ computes a monotonicity-preserving piecewise cubic Hermite
 ++ interpolant to a set of data points.
 ++ See \downlink{Manual Page}{manpageXXe01bef}.
e01bff : (Integer,Matrix DoubleFloat,Matrix DoubleFloat,Matrix DoubleFloat,_
    Integer,Matrix DoubleFloat,Integer) -> Result
 ++ e01bff(n,x,f,d,m,px,ifail)
 ++ evaluates a piecewise cubic Hermite interpolant at a set
 ++ of points.
 ++ See \downlink{Manual Page}{manpageXXe01bff}.
e01bgf : (Integer,Matrix DoubleFloat,Matrix DoubleFloat,Matrix DoubleFloat,_
    Integer,Matrix DoubleFloat,Integer) -> Result
 ++ e01bgf(n,x,f,d,m,px,ifail)
 ++ evaluates a piecewise cubic Hermite interpolant and its
 ++ first derivative at a set of points.
 ++ See \downlink{Manual Page}{manpageXXe01bgf}.
e01bhf : (Integer,Matrix DoubleFloat,Matrix DoubleFloat,Matrix DoubleFloat,_
    DoubleFloat,DoubleFloat,Integer) -> Result
 ++ e01bhf(n,x,f,d,a,b,ifail)
 ++ evaluates the definite integral of a piecewise cubic
 ++ Hermite interpolant over the interval [a,b].
 ++ See \downlink{Manual Page}{manpageXXe01bhf}.
e01daf : (Integer,Integer,Matrix DoubleFloat,Matrix DoubleFloat,_
    Matrix DoubleFloat,Integer) -> Result
 ++ e01daf(mx,my,x,y,f,ifail)
 ++ computes a bicubic spline interpolating surface through a
 ++ set of data values, given on a rectangular grid in the x-y plane.
 ++ See \downlink{Manual Page}{manpageXXe01daf}.
e01saf : (Integer,Matrix DoubleFloat,Matrix DoubleFloat,Matrix DoubleFloat,_
    Integer) -> Result
 ++ e01saf(m,x,y,f,ifail)
 ++ generates a two-dimensional surface interpolating a set of
 ++ scattered data points, using the method of Renka and Cline.
 ++ See \downlink{Manual Page}{manpageXXe01saf}.
e01sbf : (Integer,Matrix DoubleFloat,Matrix DoubleFloat,Matrix DoubleFloat,_
    Matrix Integer,Matrix DoubleFloat,DoubleFloat,DoubleFloat,Integer) -> Result
 ++ e01sbf(m,x,y,f,triang,grads,px,py,ifail)
 ++ evaluates at a given point the two-dimensional interpolant
 ++ function computed by E01SAF.
 ++ See \downlink{Manual Page}{manpageXXe01sbf}.
e01sef : (Integer,Matrix DoubleFloat,Matrix DoubleFloat,Matrix DoubleFloat,_
    Integer,Integer,DoubleFloat,DoubleFloat,Integer) -> Result
 ++ e01sef(m,x,y,f,nw,nq,rnw,rnq,ifail)
 ++ generates a two-dimensional surface interpolating a set of
 ++ scattered data points, using a modified Shepard method.
 ++ See \downlink{Manual Page}{manpageXXe01sef}.
e01sff : (Integer,Matrix DoubleFloat,Matrix DoubleFloat,Matrix DoubleFloat,_
```

```
        DoubleFloat,Matrix DoubleFloat,DoubleFloat,DoubleFloat,Integer) -> Result
     ++ e01sff(m,x,y,f,rnw,fnodes,px,py,ifail)
     ++ evaluates at a given point the two-dimensional
     ++ interpolating function computed by E01SEF.
     ++ See \downlink{Manual Page}{manpageXXe01sff}.
  Implementation ==> add

    import Lisp
    import DoubleFloat
    import Any
    import Record
    import Integer
    import Matrix DoubleFloat
    import Boolean
    import NAGLinkSupportPackage
    import AnyFunctions1(Integer)
    import AnyFunctions1(Matrix DoubleFloat)
    import AnyFunctions1(Matrix Integer)
    import AnyFunctions1(DoubleFloat)


    e01baf(mArg:Integer,xArg:Matrix DoubleFloat,yArg:Matrix DoubleFloat,_
        lckArg:Integer,lwrkArg:Integer,ifailArg:Integer): Result ==
        [(invokeNagman(NIL$Lisp,_
        "e01baf",_
        ["m"::S,"lck"::S,"lwrk"::S,"ifail"::S,"x"::S,"y"::S,"lamda"::S,"c"::S,"wr
        ]$Lisp,_
        ["lamda"::S,"c"::S,"wrk"::S]$Lisp,_
        [["double"::S,["x"::S,"m"::S]$Lisp,["y"::S,"m"::S]$Lisp_
        ,["lamda"::S,"lck"::S]$Lisp,["c"::S,"lck"::S]$Lisp,["wrk"::S,"lwrk"::S]$L
        ,["integer"::S,"m"::S,"lck"::S,"lwrk"::S,"ifail"::S_
        ]$Lisp_
        ]$Lisp,_
        ["lamda"::S,"c"::S,"ifail"::S]$Lisp,_
        [([mArg::Any,lckArg::Any,lwrkArg::Any,ifailArg::Any,xArg::Any,yArg::Any ]
        @List Any]$Lisp)$Lisp)_
        pretend List (Record(key:Symbol,entry:Any))]$Result

    e01bef(nArg:Integer,xArg:Matrix DoubleFloat,fArg:Matrix DoubleFloat,_
        ifailArg:Integer): Result ==
        [(invokeNagman(NIL$Lisp,_
        "e01bef",_
        ["n"::S,"ifail"::S,"x"::S,"f"::S,"d"::S]$Lisp,_
        ["d"::S]$Lisp,_
        [["double"::S,["x"::S,"n"::S]$Lisp,["f"::S,"n"::S]$Lisp_
        ,["d"::S,"n"::S]$Lisp]$Lisp_
```

```
    ,["integer"::S,"n"::S,"ifail"::S]$Lisp_
    ]$Lisp,_
    ["d"::S,"ifail"::S]$Lisp,_
    [([nArg::Any,ifailArg::Any,xArg::Any,fArg::Any ])_
    @List Any]$Lisp)$Lisp)_
    pretend List (Record(key:Symbol,entry:Any))]$Result

e01bff(nArg:Integer,xArg:Matrix DoubleFloat,fArg:Matrix DoubleFloat,_
    dArg:Matrix DoubleFloat,mArg:Integer,pxArg:Matrix DoubleFloat,_
    ifailArg:Integer): Result ==
    [(invokeNagman(NIL$Lisp,_
    "e01bff",_
    ["n"::S,"m"::S,"ifail"::S,"x"::S,"f"::S,"d"::S,"px"::S,"pf"::S_
    ]$Lisp,_
    ["pf"::S]$Lisp,_
    [["double"::S,["x"::S,"n"::S]$Lisp,["f"::S,"n"::S]$Lisp_
    ,["d"::S,"n"::S]$Lisp,["px"::S,"m"::S]$Lisp,["pf"::S,"m"::S]$Lisp]$Lisp_
    ,["integer"::S,"n"::S,"m"::S,"ifail"::S]$Lisp_
    ]$Lisp,_
    ["pf"::S,"ifail"::S]$Lisp,_
    [([nArg::Any,mArg::Any,ifailArg::Any,xArg::Any,fArg::Any,dArg::Any,pxArg::Any ])_
    @List Any]$Lisp)$Lisp)_
    pretend List (Record(key:Symbol,entry:Any))]$Result

e01bgf(nArg:Integer,xArg:Matrix DoubleFloat,fArg:Matrix DoubleFloat,_
    dArg:Matrix DoubleFloat,mArg:Integer,pxArg:Matrix DoubleFloat,_
    ifailArg:Integer): Result ==
    [(invokeNagman(NIL$Lisp,_
    "e01bgf",_
    ["n"::S,"m"::S,"ifail"::S,"x"::S,"f"::S,"d"::S,"px"::S,"pf"::S_
    ,"pd"::S]$Lisp,_
    ["pf"::S,"pd"::S]$Lisp,_
    [["double"::S,["x"::S,"n"::S]$Lisp,["f"::S,"n"::S]$Lisp_
    ,["d"::S,"n"::S]$Lisp,["px"::S,"m"::S]$Lisp,["pf"::S,"m"::S]$Lisp,["pd"::S,"m"::S]$L
    ,["integer"::S,"n"::S,"m"::S,"ifail"::S]$Lisp_
    ]$Lisp,_
    ["pf"::S,"pd"::S,"ifail"::S]$Lisp,_
    [([nArg::Any,mArg::Any,ifailArg::Any,xArg::Any,fArg::Any,dArg::Any,pxArg::Any ])_
    @List Any]$Lisp)$Lisp)_
    pretend List (Record(key:Symbol,entry:Any))]$Result

e01bhf(nArg:Integer,xArg:Matrix DoubleFloat,fArg:Matrix DoubleFloat,_
    dArg:Matrix DoubleFloat,aArg:DoubleFloat,bArg:DoubleFloat,_
    ifailArg:Integer): Result ==
    [(invokeNagman(NIL$Lisp,_
    "e01bhf",_
```

```
    ["n"::S,"a"::S,"b"::S,"pint"::S,"ifail"::S_
    ,"x"::S,"f"::S,"d"::S]$Lisp,_
    ["pint"::S]$Lisp,_
    [["double"::S,["x"::S,"n"::S]$Lisp,["f"::S,"n"::S]$Lisp_
    ,["d"::S,"n"::S]$Lisp,"a"::S,"b"::S,"pint"::S]$Lisp_
    ,["integer"::S,"n"::S,"ifail"::S]$Lisp_
    ]$Lisp,_
    ["pint"::S,"ifail"::S]$Lisp,_
    [([nArg::Any,aArg::Any,bArg::Any,ifailArg::Any,xArg::Any,fArg::Any,dArg::
    @List Any]$Lisp)$Lisp)_
    pretend List (Record(key:Symbol,entry:Any))]$Result

e01daf(mxArg:Integer,myArg:Integer,xArg:Matrix DoubleFloat,_
    yArg:Matrix DoubleFloat,fArg:Matrix DoubleFloat,ifailArg:Integer): Result
    [(invokeNagman(NIL$Lisp,_
    "e01daf",_
    ["mx"::S,"my"::S,"px"::S,"py"::S,"ifail"::S_
    ,"x"::S,"y"::S,"f"::S,"lamda"::S,"mu"::S_
    ,"c"::S,"wrk"::S]$Lisp,_
    ["px"::S,"py"::S,"lamda"::S,"mu"::S,"c"::S,"wrk"::S]$Lisp,_
    [["double"::S,["x"::S,"mx"::S]$Lisp,["y"::S,"my"::S]$Lisp_
    ,["f"::S,["*"::S,"mx"::S,"my"::S]$Lisp]$Lisp,["lamda"::S,["+"::S,"mx"::S,
    ,["c"::S,["*"::S,"mx"::S,"my"::S]$Lisp]$Lisp,["wrk"::S,["*"::S,["+"::S,"m
    ]$Lisp_
    ,["integer"::S,"mx"::S,"my"::S,"px"::S,"py"::S_
    ,"ifail"::S]$Lisp_
    ]$Lisp,_
    ["px"::S,"py"::S,"lamda"::S,"mu"::S,"c"::S,"ifail"::S]$Lisp,_
    [([mxArg::Any,myArg::Any,ifailArg::Any,xArg::Any,yArg::Any,fArg::Any ])_
    @List Any]$Lisp)$Lisp)_
    pretend List (Record(key:Symbol,entry:Any))]$Result

e01saf(mArg:Integer,xArg:Matrix DoubleFloat,yArg:Matrix DoubleFloat,_
    fArg:Matrix DoubleFloat,ifailArg:Integer): Result ==
    [(invokeNagman(NIL$Lisp,_
    "e01saf",_
    ["m"::S,"ifail"::S,"x"::S,"y"::S,"f"::S,"triang"::S,"grads"::S_
    ]$Lisp,_
    ["triang"::S,"grads"::S]$Lisp,_
    [["double"::S,["x"::S,"m"::S]$Lisp,["y"::S,"m"::S]$Lisp_
    ,["f"::S,"m"::S]$Lisp,["grads"::S,2$Lisp,"m"::S]$Lisp]$Lisp_
    ,["integer"::S,"m"::S,["triang"::S,["*"::S,7$Lisp,"m"::S]$Lisp]$Lisp_
    ,"ifail"::S]$Lisp_
    ]$Lisp,_
    ["triang"::S,"grads"::S,"ifail"::S]$Lisp,_
    [([mArg::Any,ifailArg::Any,xArg::Any,yArg::Any,fArg::Any ])_
```

```
        @List Any]$Lisp)$Lisp)_
        pretend List (Record(key:Symbol,entry:Any))]$Result

    e01sbf(mArg:Integer,xArg:Matrix DoubleFloat,yArg:Matrix DoubleFloat,_
        fArg:Matrix DoubleFloat,triangArg:Matrix Integer,gradsArg:Matrix DoubleFloat,_
        pxArg:DoubleFloat,pyArg:DoubleFloat,ifailArg:Integer): Result ==
        [(invokeNagman(NIL$Lisp,_
        "e01sbf",_
        ["m"::S,"px"::S,"py"::S,"pf"::S,"ifail"::S_
        ,"x"::S,"y"::S,"f"::S,"triang"::S,"grads"::S_
        ]$Lisp,_
        ["pf"::S]$Lisp,_
        [["double"::S,["x"::S,"m"::S]$Lisp,["y"::S,"m"::S]$Lisp_
        ,["f"::S,"m"::S]$Lisp,["grads"::S,2$Lisp,"m"::S]$Lisp,"px"::S,"py"::S,"pf"::S]$Lisp_
        ,["integer"::S,"m"::S,["triang"::S,["*"::S,7$Lisp,"m"::S]$Lisp]$Lisp_
        ,"ifail"::S]$Lisp_
        ]$Lisp,_
        ["pf"::S,"ifail"::S]$Lisp,_
        [([mArg::Any,pxArg::Any,pyArg::Any,ifailArg::Any,xArg::Any,yArg::Any,fArg::Any,trian
        @List Any]$Lisp)$Lisp)_
        pretend List (Record(key:Symbol,entry:Any))]$Result

    e01sef(mArg:Integer,xArg:Matrix DoubleFloat,yArg:Matrix DoubleFloat,_
        fArg:Matrix DoubleFloat,nwArg:Integer,nqArg:Integer,_
        rnwArg:DoubleFloat,rnqArg:DoubleFloat,ifailArg:Integer): Result ==
        [(invokeNagman(NIL$Lisp,_
        "e01sef",_
        ["m"::S,"nw"::S,"nq"::S,"minnq"::S,"rnw"::S_
        ,"rnq"::S,"ifail"::S,"x"::S,"y"::S,"f"::S,"fnodes"::S,"wrk"::S_
        ]$Lisp,_
        ["fnodes"::S,"minnq"::S,"wrk"::S]$Lisp,_
        [["double"::S,["x"::S,"m"::S]$Lisp,["y"::S,"m"::S]$Lisp_
        ,["f"::S,"m"::S]$Lisp,["fnodes"::S,["*"::S,5$Lisp,"m"::S]$Lisp]$Lisp,"rnw"::S,"rnq"
        ]$Lisp_
        ,["integer"::S,"m"::S,"nw"::S,"nq"::S,"minnq"::S_
        ,"ifail"::S]$Lisp_
        ]$Lisp,_
        ["fnodes"::S,"minnq"::S,"rnw"::S,"rnq"::S,"ifail"::S]$Lisp,_
        [([mArg::Any,nwArg::Any,nqArg::Any,rnwArg::Any,rnqArg::Any,ifailArg::Any,xArg::Any,y
        @List Any]$Lisp)$Lisp)_
        pretend List (Record(key:Symbol,entry:Any))]$Result

    e01sff(mArg:Integer,xArg:Matrix DoubleFloat,yArg:Matrix DoubleFloat,_
        fArg:Matrix DoubleFloat,rnwArg:DoubleFloat,fnodesArg:Matrix DoubleFloat,_
        pxArg:DoubleFloat,pyArg:DoubleFloat,ifailArg:Integer): Result ==
        [(invokeNagman(NIL$Lisp,_
```

```
"e01sff",_
["m"::S,"rnw"::S,"px"::S,"py"::S,"pf"::S_
,"ifail"::S,"x"::S,"y"::S,"f"::S,"fnodes"::S]$Lisp,_
["pf"::S]$Lisp,_
[["double"::S,["x"::S,"m"::S]$Lisp,["y"::S,"m"::S]$Lisp_
,["f"::S,"m"::S]$Lisp,"rnw"::S,["fnodes"::S,["*"::S,5$Lisp,"m"::S]$Lisp]$
,["integer"::S,"m"::S,"ifail"::S]$Lisp_
]$Lisp,_
["pf"::S,"ifail"::S]$Lisp,_
[([mArg::Any,rnwArg::Any,pxArg::Any,pyArg::Any,ifailArg::Any,xArg::Any,yA
@List Any]$Lisp)$Lisp)_
pretend List (Record(key:Symbol,entry:Any))]$Result
```

⟨*NAGE01.dotabb*⟩≡

```
"NAGE01" [color="#FF4488",href="bookvol10.4.pdf#nameddest=NAGE01"]
"ALIST" [color="#88FF44",href="bookvol10.3.pdf#nameddest=ALIST"]
"NAGE01" -> "ALIST"
```

15.13 package NAGF07 NagLapack

⟨*NagLapack.help*⟩≡

```
F07 -- Linear Equations (LAPACK)              Introduction -- F07
                            Chapter F07
                   Linear Equations (LAPACK)

1. Scope of the Chapter

This chapter provides four routines concerned with matrix
factorization, and the solution of systems of linear equations
following the matrix factorizations.

2. Background to the Problems

Background material, together with pointers to the routines in
this chapter, are to be found in the F01 and F04 Chapter
Introductions.

3. Recommendations on Choice and Use of Routines

The routines in this chapter are derived from the LAPACK project
and may also be called using the LAPACK name, which is given in
brackets following the F07 name in the following descriptions.

Routine F07ADF (DGETRF) performs an LU factorization of a real m
by n matrix A. Following the use of this routine, F07AEF (DGETRS)
may be used to solve a system of n non-singular linear equations,
with one or more right-hand sides.

Routine F07FDF (DPOTRF) performs the Cholesky factorization of a
real symmetric positive-definite matrix A. Following the use of
this routine, F07FEF (DPOTRS) may be used to solve a system of
symmetric positive-definite linear equations, with one or more
right-hand sides.


F07 -- Linear Equations (LAPACK)                  Contents -- F07
Chapter F07

Linear Equations (LAPACK)

F07ADF  (DGETRF) LU factorization of real m by n matrix

F07AEF  (DGETRS) Solution of real system of linear equations,
```

```
        multiple right-hand sides, matrix already factorized by
        F07ADF

F07FDF  (DPOTRF) Cholesky factorization of real symmetric
        positive-definite matrix

F07FEF  (DPOTRS) Solution of real symmetric positive-definite
        system of linear equations, multiple right-hand sides,
        matrix already factorized by F07FDF
```

%%%

```
F07 -- Linear Equations (LAPACK)                          F07ADF
        F07ADF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for
your implementation to check implementation-dependent details.
The symbol (*) after a NAG routine name denotes a routine that is
not included in the Foundation Library.

1. Purpose

F07ADF (DGETRF) computes the LU factorization of a real m by n
matrix.

2. Specification

       SUBROUTINE F07ADF (M, N, A, LDA, IPIV, INFO)
       ENTRY            M, N, A, LDA, IPIV, INFO
       INTEGER          M, N, LDA, IPIV(*), INFO
       DOUBLE PRECISION A(LDA,*)

The ENTRY statement enables the routine to be called by its
LAPACK name.

3. Description

This routine forms the LU factorization of a real m by n matrix A
as A=PLU, where P is a permutation matrix, L is lower triangular
with unit diagonal elements (lower trapezoidal if m>n) and U is
upper triangular (upper trapezoidal if m<n). Usually A is square
(m=n), and both L and U are triangular. The routine uses partial
pivoting, with row interchanges.

4. References
```

[1] Golub G H and Van Loan C F (1989) Matrix Computations (2nd Edition). Johns Hopkins University Press, Baltimore, Maryland.

5. Parameters

1: M -- INTEGER Input
On entry: m, the number of rows of the matrix A.
Constraint: M >= 0.

2: N -- INTEGER Input
On entry: n, the number of columns of the matrix A.
Constraint: N >= 0.

3: A(LDA,*) -- DOUBLE PRECISION array Input/Output
Note: the second dimension of the array A must be at least max(1,N).
On entry: the m by n matrix A. On exit: A is overwritten by the factors L and U; the unit diagonal elements of L are not stored.

4: LDA -- INTEGER Input
On entry:
the first dimension of the array A as declared in the (sub)program from which F07ADF is called.
Constraint: LDA >= max(1,M).

5: IPIV(*) -- INTEGER array Output
Note: the dimension of the array IPIV must be at least max(1,min(M,N)).
On exit: the pivot indices. Row i of the matrix A was interchanged with row IPIV(i) for i=1,2,...,min(m,n).

6: INFO -- INTEGER Output
On exit: INFO = 0 unless the routine detects an error (see Section 6).

6. Error Indicators and Warnings

INFO < 0
If INFO = -i, the ith parameter has an illegal value. An explanatory message is output, and execution of the program is terminated.

INFO > 0
If INFO = i, u is exactly zero. The factorization has been

```
                ii
     completed but the factor U is exactly singular, and division
     by zero will occur if it is subsequently used to solve a
                                                  -1
     system of linear equations or to compute A   .

7. Accuracy

The computed factors L and U are the exact factors of a perturbed
matrix A+E, where

                |E|<=c(min(m,n))(epsilon)P|L||U|,

c(n) is a modest linear function of n, and (epsilon) is the
machine precision.

8. Further Comments

The total number of floating-point operations is approximately
 2 3                          1 2                  1 2
 -n  if m=n (the usual case),  -n (3m-n) if m>n and  -m (3n-m) if
 3                            3                    3
m<n.

A call to this routine with m=n may be followed by calls to the
routines:

                                        T
     F07AEF (DGETRS) to solve AX=B or A X=B;

     F07AGF (DGECON)(*) to estimate the condition number of A;

     F07AJF (DGETRI)(*) to compute the inverse of A.

The complex analogue of this routine is F07ARF (ZGETRF)(*).

9. Example

To compute the LU factorization of the matrix A, where

                  ( 1.80  2.88  2.05 -0.89)
                  ( 5.25 -2.95 -0.95 -3.80)
                A=( 1.58 -2.69 -2.90 -1.04).
                  (-1.11 -0.66 -0.59  0.80)

The example program is not reproduced here. The source code for
```

```
all example programs is distributed with the NAG Foundation
Library software and should be available on-line.

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

F07 -- Linear Equations (LAPACK)                              F07AEF
        F07AEF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for
your implementation to check implementation-dependent details.
The symbol (*) after a NAG routine name denotes a routine that is
not included in the Foundation Library.

1. Purpose

F07AEF (DGETRS) solves a real system of linear equations with
                                        T
multiple right-hand sides, AX=B or A X=B, where A has been
factorized by F07ADF (DGETRF).

2. Specification

       SUBROUTINE F07AEF (TRANS, N, NRHS, A, LDA, IPIV, B, LDB,
      1                   INFO)
       ENTRY             TRANS, N, NRHS, A, LDA, IPIV, B, LDB, INFO
       INTEGER           N, NRHS, LDA, IPIV(*), LDB, INFO
       DOUBLE PRECISION A(LDA,*), B(LDB,*)
       CHARACTER*1       TRANS

The ENTRY statement enables the routine to be called by its
LAPACK name.

3. Description

                                                        T
To solve a real system of linear equations AX=B or A X=B, this
routine must be preceded by a call to F07ADF (DGETRF)which
computes the LU factorization of A as A=PLU. The solution is
computed by forward and backward substitution.

If TRANS = 'N', the solution is computed by solving PLY=B and
then UX=Y.

                                                                T
If TRANS = 'T' or 'C', the solution is computed by solving U Y=B
          T T
```

```
and then L P X=Y.

4. References

[1]   Golub G H and Van Loan C F (1989) Matrix Computations (2nd
      Edition). Johns Hopkins University Press, Baltimore,
      Maryland.

5. Parameters

 1:  TRANS -- CHARACTER*1                                        Input
     On entry: indicates the form of the equations as follows:
          if TRANS = 'N', then AX=B is solved for X;

                                          T
          if TRANS = 'T' or 'C', then A X=B is solved for X.
      Constraint: TRANS = 'N', 'T' or 'C'.

 2:  N -- INTEGER                                                Input
     On entry: n, the order of the matrix A. Constraint: N >= 0.

 3:  NRHS -- INTEGER                                             Input
     On entry: r, the number of right-hand sides. Constraint:
     NRHS >= 0.

 4:  A(LDA,*) -- DOUBLE PRECISION array                          Input
     Note: the second dimension of the array A must be at least
     max(1,N).
     On entry: the LU factorization of A, as returned by F07ADF
     (DGETRF).

 5:  LDA -- INTEGER                                              Input
     On entry:
     the first dimension of the array A as declared in the
     (sub)program from which F07AEF is called.
     Constraint: LDA >= max(1,N).

 6:  IPIV(*) -- INTEGER array                                    Input
     Note: the dimension of the array IPIV must be at least
     max(1,N).
     On entry: the pivot indices, as returned by F07ADF (DGETRF).

 7:  B(LDB,*) -- DOUBLE PRECISION array                    Input/Output
     Note: the second dimension of the array B must be at least
     max(1,NRHS).
     On entry: the n by r right-hand side matrix B. On exit: the
```

n by r solution matrix X.

8: LDB -- INTEGER Input
On entry:
the first dimension of the array B as declared in the (sub)program from which F07AEF is called.
Constraint: LDB >= max(1,N).

9: INFO -- INTEGER Output
On exit: INFO = 0 unless the routine detects an error (see Section 6).

6. Error Indicators and Warnings

INFO < 0
If INFO = -i, the ith parameter has an illegal value. An explanatory message is output, and execution of the program is terminated.

7. Accuracy

For each right-hand side vector b, the computed solution x is the exact solution of a perturbed system of equations (A+E)x=b, where

|E|<=c(n)(epsilon)P|L||U|,

c(n) is a modest linear function of n, and (epsilon) is the machine precision.

If $\hat{x}$ is the true solution, then the computed solution x satisfies a forward error bound of the form

$$\frac{||\hat{x}-x||_{\infty}}{||x||_{\infty}} \leq c(n)\,\mathrm{cond}(A,x)(\mathrm{epsilon})$$

where $\mathrm{cond}(A,x)=|||A^{-1}||A||x|||_{\infty} / ||x||_{\infty} \leq \mathrm{cond}(A)=|||A^{-1}||A|||_{\infty} \leq (\mathrm{kappa})_{\infty}(A)$. Note that cond(A,x)

```
                                                    T
can be much smaller than cond(A), and cond(A ) can be much larger
(or smaller) than cond(A).

Forward and backward error bounds can be computed by calling
F07AHF (DGERFS)(*), and an estimate for (kappa)     (A) can be
                                                      infty
obtained by calling F07AGF (DGECON)(*) with NORM ='I'.

8. Further Comments

The total number of floating-point operations is approximately
  2
2n r.

This routine may be followed by a call to F07AHF (DGERFS)(*) to
refine the solution and return an error estimate.

The complex analogue of this routine is F07ASF (ZGETRS)(*).

9. Example

To solve the system of equations AX=B, where

                      ( 1.80  2.88  2.05 -0.89)
                      ( 5.25 -2.95 -0.95 -3.80)
                  A=( 1.58 -2.69 -2.90 -1.04)
                      (-1.11 -0.66 -0.59  0.80)

and


                           ( 9.52  18.47)
                           (24.35   2.25)
                       B=( 0.77 -13.28).
                           (-6.22  -6.21)

Here A is unsymmetric and must first be factorized by F07ADF
(DGETRF)).

The example program is not reproduced here. The source code for
all example programs is distributed with the NAG Foundation
Library software and should be available on-line.
```

%%%

F07 -- Linear Equations (LAPACK) F07FDF
F07FDF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for your implementation to check implementation-dependent details. The symbol (*) after a NAG routine name denotes a routine that is not included in the Foundation Library.

1. Purpose

F07FDF (DPOTRF) computes the Cholesky factorization of a real symmetric positive-definite matrix.

2. Specification

```
SUBROUTINE F07FDF (UPLO, N, A, LDA, INFO)
ENTRY            UPLO, N, A, LDA, INFO
INTEGER          N, LDA, INFO
DOUBLE PRECISION A(LDA,*)
CHARACTER*1      UPLO
```

The ENTRY statement enables the routine to be called by its LAPACK name.

3. Description

This routine forms the Cholesky factorization of a real symmetric positive-definite matrix A either as $A=U^{T}U$ if UPLO = 'U' or $A=LL^{T}$ if UPLO = 'L', where U is an upper triangular matrix and L is lower triangular.

4. References

[1] Demmel J W (1989) On Floating-point Errors in Cholesky. LAPACK Working Note No. 14. University of Tennessee, Knoxville.

[2] Golub G H and Van Loan C F (1989) Matrix Computations (2nd Edition). Johns Hopkins University Press, Baltimore, Maryland.

5. Parameters

1: UPLO -- CHARACTER*1 Input
On entry: indicates whether the upper or lower triangular

```
     part of A is stored and how A is factorized, as follows:
          if UPLO = 'U', then the upper triangular part of A is
                                                T
          stored and A is factorized as U U, where U is upper
          triangular;

          if UPLO = 'L', then the lower triangular part of A is
                                                 T
          stored and A is factorized as LL , where L is lower
          triangular.
      Constraint: UPLO = 'U' or 'L'.

 2:  N -- INTEGER                                              Input
     On entry: n, the order of the matrix A. Constraint: N >= 0.

 3:  A(LDA,*) -- DOUBLE PRECISION array                  Input/Output
     Note: the second dimension of the array A must be at least
     max(1,N).
     On entry: the n by n symmetric positive-definite matrix A.
     If UPLO = 'U', the upper triangle of A must be stored and
     the elements of the array below the diagonal are not
     referenced; if UPLO = 'L', the lower triangle of A must be
     stored and the elements of the array above the diagonal are
     not referenced. On exit: the upper or lower triangle of A is
     overwritten by the Cholesky factor U or L as specified by
     UPLO.

 4:  LDA -- INTEGER                                            Input
     On entry:
     the first dimension of the array A as declared in the
     (sub)program from which F07FDF is called.
     Constraint: LDA >= max(1,N).

 5:  INFO -- INTEGER                                          Output
     On exit: INFO = 0 unless the routine detects an error (see
     Section 6).

6. Error Indicators and Warnings

INFO < 0
     If INFO = -i, the ith parameter has an illegal value. An
     explanatory message is output, and execution of the program
     is terminated.

INFO > 0
     If INFO = i, the leading minor of order i is not positive-
```

```
     definite and the factorization could not be completed. Hence
     A itself is not positive-definite. This may indicate an
     error in forming the matrix A. To factorize a symmetric
     matrix which is not positive-definite, call F07MDF
     (DSYTRF)(*) instead.

7. Accuracy

If UPLO = 'U', the computed factor U is the exact factor of a
perturbed matrix A+E, where

                                        T
                   |E|<=c(n)(epsilon)|U ||U|,

c(n) is a modest linear function of n, and (epsilon) is the
machine precision. If UPLO = 'L', a similar statement holds for
the computed factor L. It follows that


|e  |<=c(n)(epsilon)  /a  a  .
  ij                \/  ii jj

8. Further Comments

The total number of floating-point operations is approximately
 1 3
 -n .
 3

A call to this routine may be followed by calls to the routines:

     F07FEF (DPOTRS) to solve AX=B;

     F07FGF (DPOCON)(*) to estimate the condition number of A;

     F07FJF (DPOTRI)(*) to compute the inverse of A.

The complex analogue of this routine is F07FRF (ZPOTRF)(*).

9. Example

To compute the Cholesky factorization of the matrix A, where

                    ( 4.16 -3.12  0.56 -0.10)
                    (-3.12  5.03 -0.83  1.18)
                  A=( 0.56 -0.83  0.76  0.34).
```

```
                                (-0.10  1.18  0.34  1.18)

          The example program is not reproduced here. The source code for
          all example programs is distributed with the NAG Foundation
          Library software and should be available on-line.

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

          F07 -- Linear Equations (LAPACK)                          F07FEF
                  F07FEF -- NAG Foundation Library Routine Document

          Note: Before using this routine, please read the Users' Note for
          your implementation to check implementation-dependent details.
          The symbol (*) after a NAG routine name denotes a routine that is
          not included in the Foundation Library.

          1. Purpose

          F07FEF (DPOTRS) solves a real symmetric positive-definite system
          of linear equations with multiple right-hand sides, AX=B, where A
          has been factorized by F07FDF (DPOTRF).

          2. Specification

                 SUBROUTINE F07FEF (UPLO, N, NRHS, A, LDA, B, LDB, INFO)
                 ENTRY            UPLO, N, NRHS, A, LDA, B, LDB, INFO
                 INTEGER          N, NRHS, LDA, LDB, INFO
                 DOUBLE PRECISION A(LDA,*), B(LDB,*)
                 CHARACTER*1      UPLO

          The ENTRY statement enables the routine to be called by its
          LAPACK name.

          3. Description

          To solve a real symmetric positive-definite system of linear
          equations AX=B, this routine must be preceded by a call to
          F07FDF (DPOTRF) which computes the Cholesky factorization of A.
          The solution X is computed by forward and backward substitution.

                                                  T
          If UPLO = 'U', A=U U, where U is upper triangular; the solution X
                                                T
          is computed by solving U Y=B and then UX=Y.

                                                T
```

If UPLO = 'L', A=LL , where L is lower triangular; the solution X
is computed by solving LY=B and then $L^T X=Y$.

4. References

[1] Golub G H and Van Loan C F (1989) Matrix Computations (2nd Edition). Johns Hopkins University Press, Baltimore, Maryland.

5. Parameters

1: UPLO -- CHARACTER*1 Input
On entry: indicates whether the upper or lower triangular part of A is stored and how A is factorized, as follows:
if UPLO = 'U', then $A=U^T U$ where U is upper triangular;
if UPLO = 'L', then $A=LL^T$ where L is lower triangular.
Constraint: UPLO = 'U' or 'L'.

2: N -- INTEGER Input
On entry: n, the order of the matrix A. Constraint: N >= 0.

3: NRHS -- INTEGER Input
On entry: r, the number of right-hand sides. Constraint: NRHS >= 0.

4: A(LDA,*) -- DOUBLE PRECISION array Input
Note: the second dimension of the array A must be at least max(1,N).
On entry: the Cholesky factor of A, as returned by F07FDF (DPOTRF).

5: LDA -- INTEGER Input
On entry:
the first dimension of the array A as declared in the (sub)program from which F07FEF is called.
Constraint: LDA >=max(1,N).

6: B(LDB,*) -- DOUBLE PRECISION array Input/Output
Note: the second dimension of the array B must be at least max(1,NRHS).
On entry: the n by r right-hand side matrix B.

```
 7:  LDB -- INTEGER                                              Input
     On entry:
     the first dimension of the array B as declared in the
     (sub)program from which F07FEF is called.
     Constraint: LDB >=max(1,N).

 8:  INFO -- INTEGER                                             Output
     On exit: INFO = 0 unless the routine detects an error (see
     Section 6).

6. Error Indicators and Warnings

INFO < 0
     If INFO = -i, the ith parameter has an illegal value. An
     explanatory message is output, and execution of the program
     is terminated.

7. Accuracy

For each right-hand side vector b, the computed solution x is the
exact solution of a perturbed system of equations (A+E)x=b, where

                          T
     |E|<=c(n)(epsilon)|U ||U| if UPLO = 'U',

                             T
     |E|<=c(n)(epsilon)|L||L | if UPLO = 'L',

c(n) is a modest linear function of n, and (epsilon) is the
machine precision.

   ^
If x is the true solution, then the computed solution x satisfies
a forward bound of the form

                    ^
              ||x-x||
                     infty
              ------------<=c(n)cond(A,x)(epsilon)
               ||x||
                    infty

                    -1
where cond(A,x)=|||A  ||A||x|||     /||x||      <=
                                infty      infty
            -1
```

cond(A)=|||A ||A||| <=(kappa) (A). Note that cond(A,x)
infty infty
can be much smaller than cond(A).

Forward and backward error bounds can be computed by calling
F07FHF (DPORFS)(*), and an estimate for (kappa) (A) (
infty
=(kappa) (A)) can be obtained by calling F07FGF (DPOCON)(*).
1

8. Further Comments

The total number of floating-point operations is approximately
2
2n r.

This routine may be followed by a call to F07FHF (DPORFS)(*) to refine the solution and return an error estimate.

The complex analogue of this routine is F07FSF (ZPOTRS)(*).

9. Example

To compute the Cholesky factorization of the matrix A, where

$$A = \begin{pmatrix} 4.16 & -3.12 & 0.56 & -0.10 \\ -3.12 & 5.03 & -0.83 & 1.18 \\ 0.56 & -0.83 & 0.76 & 0.34 \\ -0.10 & 1.18 & 0.34 & 1.18 \end{pmatrix}.$$

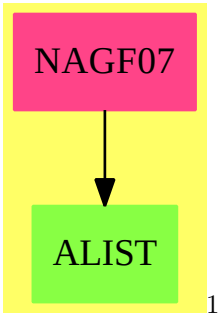
and

$$B = \begin{pmatrix} 8.70 & 8.30 \\ -13.35 & 2.13 \\ 1.89 & 1.61 \\ -4.14 & 5.00 \end{pmatrix}.$$

Here A is symmetric positive-definite and must first be factorized by F07FDF (DPOTRF).

The example program is not reproduced here. The source code for all example programs is distributed with the NAG Foundation Library software and should be available on-line.

15.14 NagLapack



1.00

Exports:
f07adf f07aef f07fdf f07fef

⟨*package NAGF07 NagLapack*⟩≡
```
)abbrev package NAGF07 NagLapack
++ Author: Godfrey Nolan and Mike Dewar
++ Date Created: Jan 1994
++ Date Last Updated: Thu May 12 17:45:42 1994
++ Description:
++ This package uses the NAG Library to compute matrix
++ factorizations, and to solve systems of linear equations
++ following the matrix factorizations.
++ See \downlink{Manual Page}{manpageXXf07}.
NagLapack(): Exports == Implementation where
  S ==> Symbol
  FOP ==> FortranOutputStackPackage

  Exports ==> with
    f07adf : (Integer,Integer,Integer,Matrix DoubleFloat) -> Result
     ++ f07adf(m,n,lda,a)
     ++ (DGETRF) computes the LU factorization of a real m by n
     ++ matrix.
     ++ See \downlink{Manual Page}{manpageXXf07adf}.
    f07aef : (String,Integer,Integer,Matrix DoubleFloat,_
        Integer,Matrix Integer,Integer,Matrix DoubleFloat) -> Result
     ++ f07aef(trans,n,nrhs,a,lda,ipiv,ldb,b)
     ++ (DGETRS) solves a real system of linear equations with
     ++                                      T
     ++ multiple right-hand sides, AX=B or A X=B, where A has been
     ++ factorized by F07ADF (DGETRF).
     ++ See \downlink{Manual Page}{manpageXXf07aef}.
    f07fdf : (String,Integer,Integer,Matrix DoubleFloat) -> Result
     ++ f07fdf(uplo,n,lda,a)
     ++ (DPOTRF) computes the Cholesky factorization of a real
```

```
     ++ symmetric positive-definite matrix.
     ++ See \downlink{Manual Page}{manpageXXf07fdf}.
    f07fef : (String,Integer,Integer,Matrix DoubleFloat,_
        Integer,Integer,Matrix DoubleFloat) -> Result
     ++ f07fef(uplo,n,nrhs,a,lda,ldb,b)
     ++ (DPOTRS) solves a real symmetric positive-definite system
     ++ of linear equations with multiple right-hand sides, AX=B, where A
     ++ has been factorized by F07FDF (DPOTRF).
     ++ See \downlink{Manual Page}{manpageXXf07fef}.
  Implementation ==> add

    import Lisp
    import DoubleFloat
    import Any
    import Record
    import Integer
    import Matrix DoubleFloat
    import Boolean
    import NAGLinkSupportPackage
    import AnyFunctions1(Integer)
    import AnyFunctions1(Matrix DoubleFloat)
    import AnyFunctions1(String)
    import AnyFunctions1(Matrix Integer)


    f07adf(mArg:Integer,nArg:Integer,ldaArg:Integer,_
        aArg:Matrix DoubleFloat): Result ==
        [(invokeNagman(NIL$Lisp,_
        "f07adf",_
        ["m"::S,"n"::S,"lda"::S,"info"::S,"ipiv"::S,"a"::S]$Lisp,_
        ["ipiv"::S,"info"::S]$Lisp,_
        [["double"::S,["a"::S,"lda"::S,"n"::S]$Lisp_
        ]$Lisp_
        ,["integer"::S,"m"::S,"n"::S,"lda"::S,["ipiv"::S,"m"::S]$Lisp_
        ,"info"::S]$Lisp_
        ]$Lisp,_
        ["ipiv"::S,"info"::S,"a"::S]$Lisp,_
        [([mArg::Any,nArg::Any,ldaArg::Any,aArg::Any ])_
        @List Any]$Lisp)$Lisp)_
        pretend List (Record(key:Symbol,entry:Any))]$Result

    f07aef(transArg:String,nArg:Integer,nrhsArg:Integer,_
        aArg:Matrix DoubleFloat,ldaArg:Integer,ipivArg:Matrix Integer,_
        ldbArg:Integer,bArg:Matrix DoubleFloat): Result ==
        [(invokeNagman(NIL$Lisp,_
        "f07aef",_
```

```
        ["trans"::S,"n"::S,"nrhs"::S,"lda"::S,"ldb"::S_
        ,"info"::S,"a"::S,"ipiv"::S,"b"::S]$Lisp,_
        ["info"::S]$Lisp,_
        [["double"::S,["a"::S,"lda"::S,"n"::S]$Lisp_
        ,["b"::S,"ldb"::S,"nrhs"::S]$Lisp]$Lisp_
        ,["integer"::S,"n"::S,"nrhs"::S,"lda"::S,["ipiv"::S,"n"::S]$Lisp_
        ,"ldb"::S,"info"::S]$Lisp_
        ,["character"::S,"trans"::S]$Lisp_
        ]$Lisp,_
        ["info"::S,"b"::S]$Lisp,_
        [([transArg::Any,nArg::Any,nrhsArg::Any,ldaArg::Any,ldbArg::Any,aArg::Any
        @List Any]$Lisp)$Lisp)_
        pretend List (Record(key:Symbol,entry:Any))]$Result

    f07fdf(uploArg:String,nArg:Integer,ldaArg:Integer,_
        aArg:Matrix DoubleFloat): Result ==
        [(invokeNagman(NIL$Lisp,_
        "f07fdf",_
        ["uplo"::S,"n"::S,"lda"::S,"info"::S,"a"::S]$Lisp,_
        ["info"::S]$Lisp,_
        [["double"::S,["a"::S,"lda"::S,"n"::S]$Lisp_
        ]$Lisp_
        ,["integer"::S,"n"::S,"lda"::S,"info"::S]$Lisp_
        ,["character"::S,"uplo"::S]$Lisp_
        ]$Lisp,_
        ["info"::S,"a"::S]$Lisp,_
        [([uploArg::Any,nArg::Any,ldaArg::Any,aArg::Any ])_
        @List Any]$Lisp)$Lisp)_
        pretend List (Record(key:Symbol,entry:Any))]$Result

    f07fef(uploArg:String,nArg:Integer,nrhsArg:Integer,_
        aArg:Matrix DoubleFloat,ldaArg:Integer,ldbArg:Integer,_
        bArg:Matrix DoubleFloat): Result ==
        [(invokeNagman(NIL$Lisp,_
        "f07fef",_
        ["uplo"::S,"n"::S,"nrhs"::S,"lda"::S,"ldb"::S_
        ,"info"::S,"a"::S,"b"::S]$Lisp,_
        ["info"::S]$Lisp,_
        [["double"::S,["a"::S,"lda"::S,"n"::S]$Lisp_
        ,["b"::S,"ldb"::S,"nrhs"::S]$Lisp]$Lisp_
        ,["integer"::S,"n"::S,"nrhs"::S,"lda"::S,"ldb"::S_
        ,"info"::S]$Lisp_
        ,["character"::S,"uplo"::S]$Lisp_
        ]$Lisp,_
        ["info"::S,"b"::S]$Lisp,_
        [([uploArg::Any,nArg::Any,nrhsArg::Any,ldaArg::Any,ldbArg::Any,aArg::Any,
```

```
          @List Any]$Lisp)$Lisp)_
          pretend List (Record(key:Symbol,entry:Any))]$Result
```

⟨*NAGF07.dotabb*⟩≡

```
"NAGF07" [color="#FF4488",href="bookvol10.4.pdf#nameddest=NAGF07"]
"ALIST" [color="#88FF44",href="bookvol10.3.pdf#nameddest=ALIST"]
"NAGF07" -> "ALIST"
```

15.15 package NAGF01 NagMatrixOperationsPackage

⟨*NagMatrixOperationsPackage.help*⟩≡

```
F01 -- Matrix Factorization                    Introduction -- F01
                          Chapter F01
                     Matrix Factorization

1. Scope of the Chapter

This chapter provides facilities for matrix factorizations and
associated transformations.

2. Background to the Problems

An n by n matrix may be factorized as

                                  T
                              A=PLUQ ,

where L and U are respectively lower and uper triangular
matrices, and P and Q are permutation matrices. This is called an
LU factorization. For general dense matrices it is usual to
choose Q=I and to then choose P to ensure that the factorization
is numerically stable. For sparse matrices, judicious choice of P
and Q ensures numerical stability as well as maintaining as much
sparsity as possible in the factors L and U. The LU factorization
is normally used in connection with the solution of the linear
equations


                                 Ax=b,

whose solution, x, may then be obtained by solving in succession
the simpler equations

                              T
                     Ly=P b,  Uz=y,  x=Qz

the first by forward substitution and the second by backward
substitution. Routines to perform this solution are to be found
in Chapter F04.

                                                                T
```

When A is symmetric positive-definite then we can choose U=L and Q=P, to give the Cholesky factorization. This factorization is numerically stable without permutations, but in the sparse case the permutations can again be used to try to maintain sparsity. The Cholesky factorization is sometimes expressed as

$$A=PLDL^{T}P^{T},$$

where D is a diagonal matrix with positive diagonal elements and L is unit lower triangular.

The LU factorization can also be performed on rectangular matrices, but in this case it is more usual to perform a QR factorization. When A is an m by n (m>=n) matrix this is given by

$$A=Q\begin{pmatrix}R\\(0)\end{pmatrix},$$

where R is an n by n upper triangular matrix and Q is an orthogonal (unitary in the complex case) matrix.

3. Recommendations on Choice and Use of Routines

Routine F07ADF performs the LU factorization of a real m by n dense matrix.

The LU factorization of a sparse matrix is performed by routine F01BRF. Following the use of F01BRF, matrices with the same sparsity pattern may be factorized by routine F01BSF.

The Cholesky factorization of a real symmetric positive-definite dense matrix is performed by routine F07FDF.

Routine F01MCF performs the Cholesky factorization of a real symmetric positive-definite variable band (skyline) matrix, and a general sparse symmetric positive-definite matrix may be factorized using routine F01MAF.

The QR factorization of an m by n (m>=n) matrix is performed by routine F01QCF in the real case, and F01RCF in the complex case. Following the use of F01QCF, operations with Q may be performed using routine F01QDF and some, or all, of the columns of Q may be formed using routine F01QEF. Routines F01RDF and F01REF perform the same tasks following the use of F01RCF.

```
F01 -- Matrix Factorizations                          Contents -- F01
Chapter F01

Matrix Factorizations

F01BRF  LU factorization of real sparse matrix

F01BSF  LU factorization of real sparse matrix with known
        sparsity pattern

          T
F01MAF  LL  factorization of real sparse symmetric positive-
        definite matrix

           T
F01MCF  LDL  factorization of real symmetric positive-definite
        variable-bandwidth matrix

F01QCF  QR factorization of real m by n matrix (m>=n)

                                                              T
F01QDF  Operations with orthogonal matrices, compute QB or Q B
        after factorization by F01QCF

F01QEF  Operations with orthogonal matrices, form columns of Q
        after factorization by F01QCF

F01RCF  QR factorization of complex m by n matrix (m>=n)

                                                            H
F01RDF  Operations with unitary matrices, compute QB or Q B after
        factorization by F01RCF

F01REF  Operations with unitary matrices, form columns of Q after
        factorization by F01RCF
```

%%%

```
F01 -- Matrix Factorizations                                F01BRF
        F01BRF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for
your implementation to check implementation-dependent details.
The symbol (*) after a NAG routine name denotes a routine that is
```

not included in the Foundation Library.

1. Purpose

F01BRF factorizes a real sparse matrix. The routine either forms the LU factorization of a permutation of the entire matrix, or, optionally, first permutes the matrix to block lower triangular form and then only factorizes the diagonal blocks.

2. Specification

```
      SUBROUTINE F01BRF (N, NZ, A, LICN, IRN, LIRN, ICN, PIVOT,
     1                   IKEEP, IW, W, LBLOCK, GROW, ABORT,
     2                   IDISP, IFAIL)
      INTEGER           N, NZ, LICN, IRN(LIRN), LIRN, ICN(LICN),
     1                  IKEEP(5*N), IW(8*N), IDISP(10), IFAIL
      DOUBLE PRECISION  A(LICN), PIVOT, W(N)
      LOGICAL           LBLOCK, GROW, ABORT(4)
```

3. Description

Given a real sparse matrix A, this routine may be used to obtain the LU factorization of a permutation of A,

$$PAQ=LU$$

where P and Q are permutation matrices, L is unit lower triangular and U is upper triangular. The routine uses a sparse variant of Gaussian elimination, and the pivotal strategy is designed to compromise between maintaining sparsity and controlling loss of accuracy through round-off.

Optionally the routine first permutes the matrix into block lower triangular form and then only factorizes the diagonal blocks. For some matrices this gives a considerable saving in storage and execution time.

Extensive data checks are made; duplicated non-zeros can be accumulated.

The factorization is intended to be used by F04AXF to solve sparse systems of linear equations $Ax=b$ or $A^{T}x=b$. If several matrices of the same sparsity pattern are to be factorized, F01BSF should be used for the second and subsequent matrices.

The method is fully described by Duff [1].

4. References

[1] Duff I S (1977) MA28 -- a set of Fortran subroutines for sparse unsymmetric linear equations. A.E.R.E. Report R.8730. HMSO.

5. Parameters

1: N -- INTEGER Input
On entry: n, the order of the matrix A. Constraint: N > 0.

2: NZ -- INTEGER Input
On entry: the number of non-zero elements in the matrix A. Constraint: NZ > 0.

3: A(LICN) -- DOUBLE PRECISION array Input/Output
On entry: A(i), for i = 1,2,...,NZ must contain the non-zero elements of the sparse matrix A. They can be in any order since the routine will reorder them. On exit: the non-zero elements in the LU factorization. The array must not be changed by the user between a call of this routine and a call of F04AXF.

4: LICN -- INTEGER Input
On entry:
the dimension of the arrays A and ICN as declared in the (sub)program from which F01BRF is called.
Since the factorization is returned in A and ICN, LICN should be large enough to accommodate this and should ordinarily be 2 to 4 times as large as NZ. Constraint: LICN >= NZ.

5: IRN(LIRN) -- INTEGER array Input/Output
On entry: IRN(i), for i = 1,2,...,NZ must contain the row index of the non-zero element stored in A(i). On exit: the array is overwritten and is not needed for subsequent calls of F01BSF or F04AXF.

6: LIRN -- INTEGER Input
On entry:
the dimension of the array IRN as declared in the (sub)program from which F01BRF is called.
It need not be as large as LICN; normally it will not need to be very much greater than NZ. Constraint: LIRN >= NZ.

7: ICN(LICN) -- INTEGER array Input/Output
On entry: ICN(i), for i = 1,2,...,NZ must contain the column index of the non-zero element stored in A(i). On exit: the column indices of the non-zero elements in the factorization. The array must not be changed by the user between a call of this routine and subsequent calls of F01BSF or F04AXF.

8: PIVOT -- DOUBLE PRECISION Input
On entry: PIVOT should have a value in the range 0.0 <= PIVOT <= 0.9999 and is used to control the choice of pivots. If PIVOT < 0.0, the value 0.0 is assumed, and if PIVOT > 0.9999, the value 0.9999 is assumed. When searching a row for a pivot, any element is excluded which is less than PIVOT times the largest of those elements in the row available as pivots. Thus decreasing PIVOT biases the algorithm to maintaining sparsity at the expense of stability. Suggested value: PIVOT = 0.1 has been found to work well on test examples.

9: IKEEP(5*N) -- INTEGER array Output
On exit: indexing information about the factorization. The array must not be changed by the user between a call of this routine and calls of F01BSF or F04AXF.

10: IW(8*N) -- INTEGER array Workspace

11: W(N) -- DOUBLE PRECISION array Output
On exit: if GROW = .TRUE., W(1) contains an estimate (an upper bound) of the increase in size of elements encountered during the factorization (see GROW); the rest of the array is used as workspace.

If GROW = .FALSE., the array is not used.

12: LBLOCK -- LOGICAL Input
On entry: if LBLOCK = .TRUE., the matrix is pre-ordered into block lower triangular form before the LU factorization is performed; otherwise the entire matrix is factorized. Suggested value: LBLOCK = .TRUE. unless the matrix is known to be irreducible.

13: GROW -- LOGICAL Input
On entry: if GROW = .TRUE., then on exit W(1) contains an estimate (an upper bound) of the increase in size of

elements encountered during the factorization. If the matrix is well-scaled (see Section 8.2), then a high value for W(1) indicates that the LU factorization may be inaccurate and the user should be wary of the results and perhaps increase the parameter PIVOT for subsequent runs (see Section 7). Suggested value: GROW = .TRUE..

14: ABORT(4) -- LOGICAL array Input
On entry:
if ABORT(1) = .TRUE., the routine will exit immediately on detecting a structural singularity (one that depends on the pattern of non-zeros) and return

IFAIL = 1; otherwise it will complete the factorization (see Section 8.3).

If ABORT(2) = .TRUE., the routine will exit immediately on detecting a numerical singularity (one that depends on the numerical values) and return IFAIL = 2; otherwise it will complete the factorization (see Section 8.3).

If ABORT(3) = .TRUE., the routine will exit immediately (with IFAIL = 5) when the arrays A and ICN are filled up by the previously factorized, active and unfactorized parts of the matrix; otherwise it continues so that better guidance on necessary array sizes can be given in IDISP(6) and IDISP(7), and will exit with IFAIL in the range 4 to 6. Note that there is always an immediate error exit if the array IRN is too small.

If ABORT(4) = .TRUE., the routine exits immediately (with IFAIL = 13) if it finds duplicate elements in the input matrix. If ABORT(4) = .FALSE., the routine proceeds using a value equal to the sum of the duplicate elements. In either case details of each duplicate element are output on the current advisory message unit (see X04ABF), unless suppressed by the value of IFAIL on entry.
Suggested values:
ABORT(1) = .TRUE.

ABORT(2) = .TRUE.

ABORT(3) = .FALSE.

ABORT(4) = .TRUE..

15: IDISP(10) -- INTEGER array Output
On exit: IDISP is used to communicate information about the factorization to the user and also between a call of F01BRF and subsequent calls to F01BSF or F04AXF.

IDISP(1) and IDISP(2), indicate the position in arrays A and ICN of the first and last elements in the LU factorization of the diagonal blocks. (IDISP(2) gives the number of non-zeros in the factorization.)

IDISP(3) and IDISP(4), monitor the adequacy of 'elbow room' in the arrays IRN and A/ICN respectively, by giving the number of times that the data in these arrays has been compressed during the factorization to release more storage. If either IDISP(3) or IDISP(4) is quite large (say greater than 10), it will probably pay the user to increase the size of the corresponding array(s) for subsequent runs. If either is very low or zero, then the user can perhaps save storage by reducing the size of the corresponding array(s).

IDISP(5), gives an upper bound on the rank of the matrix.

IDISP(6) and IDISP(7), give the minimum size of arrays IRN and A/ICN respectively which would enable a successful run on an identical matrix (but some ' elbow-room' should be allowed - see Section 8).

IDISP(8) to (10), are only used if LBLOCK = .TRUE..

IDISP(8), gives the structural rank of the matrix.

IDISP(9), gives the number of diagonal blocks.

IDISP(10), gives the size of the largest diagonal block.

IDISP(1) and IDISP(2), must not be changed by the user between a call of F01BRF and subsequent calls to F01BSF or F04AXF.

16: IFAIL -- INTEGER Input/Output
For this routine, the normal use of IFAIL is extended to

control the printing of error and warning messages as well as specifying hard or soft failure (see the Essential Introduction).

Before entry, IFAIL must be set to a value with the decimal expansion cba, where each of the decimal digits c, b and a must have a value of 0 or 1.
a=0 specifies hard failure, otherwise soft failure;

b=0 suppresses error messages, otherwise error messages will be printed (see Section 6);

c=0 suppresses warning messages, otherwise warning messages will be printed (see Section 6).
The recommended value for inexperienced users is 110 (i.e., hard failure with all messages printed).

Unless the routine detects an error (see Section 6), IFAIL contains 0 on exit.

6. Error Indicators and Warnings

Errors detected by the routine:

For each error, an explanatory error message is output on the current error message unit (as defined by X04AAF), unless suppressed by the value of IFAIL on entry.

IFAIL=-2
Successful factorization of a numerically singular matrix (which may also be structurally singular) (see Section 8.3).

IFAIL=-1
Successful factorization of a structurally singular matrix (see Section 8.3).

IFAIL= 1
The matrix is structurally singular and the factorization has been abandoned (ABORT(1) was .TRUE. on entry).

IFAIL= 2
The matrix is numerically singular and the factorization has been abandoned (ABORT(2) was .TRUE. on entry).

IFAIL= 3
LIRN is too small: there is not enough space in the array

```
          IRN to continue the factorization. The user is recommended
          to try again with LIRN (and the length of IRN) equal to at
          least IDISP(6) + N/2.

     IFAIL= 4
          LICN is much too small: there is much too little space in
          the arrays A and ICN to continue the factorization.

     IFAIL= 5
          LICN is too small: there is not enough space in the arrays A
          and ICN to store the factorization. If ABORT(3) was .FALSE.
          on entry, the factorization has been completed but some of
          the LU factors have been discarded to create space, IDISP(7)
          then gives the minimum value of LICN (i.e., the minimum
          length of A and ICN) required for a successful factorization
          of the same matrix.

     IFAIL= 6
          LICN and LIRN are both too small: effectively this is a
          combination of IFAIL = 3 and IFAIL = 5 (with ABORT(3) = .
          FALSE.).

     IFAIL= 7
          LICN is too small: there is not enough space in the arrays A
          and ICN for the permutation to block triangular form.

     IFAIL= 8
          On entry N <= 0.

     IFAIL= 9
          On entry NZ <= 0.

     IFAIL= 10
          On entry LICN < NZ.

     IFAIL= 11
          On entry LIRN < NZ.

     IFAIL= 12
          On entry an element of the input matrix has a row or column
          index (i.e., an element of IRN or ICN) outside the range 1
          to N.

     IFAIL= 13
          Duplicate elements have been found in the input matrix and
          the factorization has been abandoned (ABORT(4) = .TRUE. on
```

entry).

7. Accuracy

The factorization obtained is exact for a perturbed matrix whose (i,j)th element differs from a_{ij} by less than $3(\text{epsilon})(\text{rho})m_{ij}$ where (epsilon) is the machine precision, (rho) is the growth value returned in W(1) if GROW = .TRUE., and m_{ij} the number of Gaussian elimination operations applied to element (i,j). The value of m_{ij} is not greater than n and is usually much less. Small (rho) values therefore guarantee accurate results, but unfortunately large (rho) values may give a very pessimistic indication of accuracy.

8. Further Comments

8.1. Timing

The time required may be estimated very roughly from the number (tau) of non-zeros in the factorized form (output as IDISP(2)) and for this routine and its associates is

F01BRF: $5(\text{tau})^2/n$ units

F01BSF: $(\text{tau})^2/n$ units

F04AXF: $2(\text{tau})$ units

where our unit is the time for the inner loop of a full matrix code (e.g. solving a full set of equations takes about $\frac{1}{3}n^3$ units). Note that the faster F01BSF time makes it well worthwhile to use this for a sequence of problems with the same pattern.

It should be appreciated that (tau) varies widely from problem to problem. For network problems it may be little greater than NZ, the number of non-zeros in A; for discretisation of 2-dimensional and 3-dimensional partial differential equations it may be about $3n\log n$ and $n^{5/3}$, respectively.

2 2

The time taken to find the block lower triangular form (LBLOCK = it is not found (LBLOCK = .FALSE.). If the matrix is irreducible (IDISP(9) = 1 after a call with LBLOCK = .TRUE.) then this time is wasted. Otherwise, particularly if the largest block is small (IDISP(10)<<n), the consequent savings are likely to be greater.

The time taken to estimate growth (GROW = .TRUE.) is typically under 2% of the overall time.

The overall time may be substantially increased if there is inadequate 'elbow-room' in the arrays A, IRN and ICN. When the sizes of the arrays are minimal (IDISP(6) and IDISP(7)) it can execute as much as three times slower. Values of IDISP(3) and IDISP(4) greater than about 10 indicate that it may be worthwhile to increase array sizes.

8.2. Scaling

The use of a relative pivot tolerance PIVOT essentially presupposes that the matrix is well-scaled, i.e., that the matrix elements are broadly comparable in size. Practical problems are often naturally well-scaled but particular care is needed for problems containing mixed types of variables (for example millimetres and neutron fluxes).

8.3. Singular and Rectangular Systems

It is envisaged that this routine will almost always be called for square non-singular matrices and that singularity indicates an error condition. However, even if the matrix is singular it is possible to complete the factorization. It is even possible for F04AXF to solve a set of equations whose matrix is singular provided the set is consistent.

Two forms of singularity are possible. If the matrix would be singular for any values of the non-zeros (e.g. if it has a whole row of zeros), then we say it is structurally singular, and continue only if ABORT(1) = .FALSE.. If the matrix is non-singular by virtue of the particular values of the non-zeros, then we say that it is numerically singular and continue only if ABORT(2) = .FALSE..

Rectangular matrices may be treated by setting N to the larger of the number of rows and numbers of columns and setting ABORT(1) =

Note: the soft failure option should be used (last digit of IFAIL = 1) if the user wishes to factorize singular matrices with ABORT (1) or ABORT(2) set to .FALSE..

8.4. Duplicated Non-zeros

The matrix A may consist of a sum of contributions from different sub-systems (for example finite elements). In such cases the user may rely on this routine to perform assembly, since duplicated elements are summed.

9. Example

To factorize the real sparse matrix:

```
( 5  0  0  0 0  0)
( 0  2 -1  2 0  0)
( 0  0  3  0 0  0)
(-2  0  0  1 1  0).
(-1  0  0 -1 2 -3)
(-1 -1  0  0 0  6)
```

This example program simply prints out some information about the factorization as returned by F01BRF in W(1) and IDISP. Normally the call of F01BRF would be followed by a call of F04AXF (see Example for F04AXF).

The example program is not reproduced here. The source code for all example programs is distributed with the NAG Foundation Library software and should be available on-line.

%%%

F01 -- Matrix Factorizations F01BSF
F01BSF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for your implementation to check implementation-dependent details. The symbol (*) after a NAG routine name denotes a routine that is not included in the Foundation Library.

1. Purpose

F01BSF factorizes a real sparse matrix using the pivotal sequence previously obtained by F01BRF when a matrix of the same sparsity

pattern was factorized.

2. Specification

```
      SUBROUTINE F01BSF (N, NZ, A, LICN, IVECT, JVECT, ICN,
     1                   IKEEP, IW, W, GROW, ETA, RPMIN, ABORT,
     2                   IDISP, IFAIL)
      INTEGER           N, NZ, LICN, IVECT(NZ), JVECT(NZ), ICN
     1                  (LICN), IKEEP(5*N), IW(8*N), IDISP(2),
     2                  IFAIL
      DOUBLE PRECISION A(LICN), W(N), ETA, RPMIN
      LOGICAL           GROW, ABORT
```

3. Description

This routine accepts as input a real sparse matrix of the same sparsity pattern as a matrix previously factorized by a call of F01BRF. It first applies to the matrix the same permutations as were used by F01BRF, both for permutation to block triangular form and for pivoting, and then performs Gaussian elimination to obtain the LU factorization of the diagonal blocks.

Extensive data checks are made; duplicated non-zeros can be accumulated.

The factorization is intended to be used by F04AXF to solve
T
sparse systems of linear equations $Ax=b$ or $A\ x=b$.

F01BSF is much faster than F01BRF and in some applications it is expected that there will be many calls of F01BSF for each call of F01BRF.

The method is fully described in Duff [1].

4. References

[1] Duff I S (1977) MA28 -- a set of Fortran subroutines for sparse unsymmetric linear equations. A.E.R.E. Report R.8730. HMSO.

5. Parameters

1: N -- INTEGER Input
On entry: n, the order of the matrix A. Constraint: N > 0.

2: NZ -- INTEGER Input
On entry: the number of non-zeros in the matrix A.
Constraint: NZ > 0.

3: A(LICN) -- DOUBLE PRECISION array Input/Output
On entry: A(i), for i = 1,2,...,NZ must contain the non-zero elements of the sparse matrix A. They can be in any order since the routine will reorder them. On exit: the non-zero elements in the factorization. The array must not be changed by the user between a call of this routine and a call of F04AXF.

4: LICN -- INTEGER Input
On entry:
the dimension of the arrays A and ICN as declared in the (sub)program from which F01BSF is called.
It should have the same value as it had for F01BRF.
Constraint: LICN >= NZ.

5: IVECT(NZ) -- INTEGER array Input

6: JVECT(NZ) -- INTEGER array Input
On entry: IVECT(i) and JVECT(i), for i = 1,2,...,NZ must contain the row index and the column index respectively of the non-zero element stored in A(i).

7: ICN(LICN) -- INTEGER array Input
On entry: the same information as output by F01BRF. It must not be changed by the user between a call of this routine and a call of F04AXF.

8: IKEEP(5*N) -- INTEGER array Input
On entry: the same indexing information about the factorization as output from F01BRF. It must not be changed between a call of this routine and a call of F04AXF.

9: IW(8*N) -- INTEGER array Workspace

10: W(N) -- DOUBLE PRECISION array Output
On exit: if GROW = .TRUE., W(1) contains an estimate (an upper bound) of the increase in size of elements encountered during the factorization (see GROW); the rest of the array is used as workspace.

If GROW = .FALSE., the array is not used.

11: GROW -- LOGICAL Input
On entry: if GROW = .TRUE., then on exit W(1) contains an estimate (an upper bound) of the increase in size of elements encountered during the factorization. If the matrix is well-scaled (see Section 8.2), then a high value for W(1) indicates that the LU factorization may be inaccurate and the user should be wary of the results and perhaps increase the parameter PIVOT for subsequent runs (see Section 7).

12: ETA -- DOUBLE PRECISION Input
On entry: the relative pivot threshold below which an error diagnostic is provoked and IFAIL is set to 7. If ETA is greater than 1.0, then no check on pivot size is made.
Suggested value: ETA = 10^{-4}.

13: RPMIN -- DOUBLE PRECISION Output
On exit: if ETA is less than 1.0, then RPMIN gives the smallest ratio of the pivot to the largest element in the row of the corresponding upper triangular factor thus monitoring the stability of the factorization. If RPMIN is very small it may be advisable to perform a new factorization using F01BRF.

14: ABORT -- LOGICAL Input
On entry: if ABORT = .TRUE., the routine exits immediately (with IFAIL = 8) if it finds duplicate elements in the input matrix. If ABORT = .FALSE., the routine proceeds using a value equal to the sum of the duplicate elements. In either case details of each duplicate element are output on the current advisory message unit (see X04ABF), unless suppressed by the value of IFAIL on entry. Suggested value: ABORT = .TRUE..

15: IDISP(2) -- INTEGER array Input
On entry: IDISP(1) and IDISP(2) must be unchanged since the previous call of F01BRF.

16: IFAIL -- INTEGER Input/Output
For this routine, the normal use of IFAIL is extended to control the printing of error and warning messages as well as specifying hard or soft failure (see the Essential Introduction).

Before entry, IFAIL must be set to a value with the decimal expansion cba, where each of the decimal digits c, b and a

```
     must have a value of 0 or 1.
     a=0   specifies hard failure, otherwise soft failure;

     b=0   suppresses error messages, otherwise error messages
           will be printed (see Section 6);

     c=0   suppresses warning messages, otherwise warning
           messages will be printed (see Section 6).
     The recommended value for inexperienced users is 110 (i.e.,
     hard failure with all messages printed).

     Unless the routine detects an error (see Section 6), IFAIL
     contains 0 on exit.

6. Error Indicators and Warnings

Errors detected by the routine:

For each error, an explanatory error message is output on the
current error message unit (as defined by X04AAF), unless
suppressed by the value of IFAIL on entry.

IFAIL= 1
     On entry N <= 0.

IFAIL= 2
     On entry NZ <= 0.

IFAIL= 3
     On entry LICN < NZ.

IFAIL= 4
     On entry an element of the input matrix has a row or column
     index (i.e., an element of IVECT or JVECT) outside the range
     1 to N.

IFAIL= 5
     The input matrix is incompatible with the matrix factorized
     by the previous call of F01BRF (see Section 8).

IFAIL= 6
     The input matrix is numerically singular.

IFAIL= 7
     A very small pivot has been detected (see Section 5, ETA).
     The factorization has been completed but is potentially
```

unstable.

IFAIL= 8
Duplicate elements have been found in the input matrix and the factorization has been abandoned (ABORT = .TRUE. on entry).

7. Accuracy

The factorization obtained is exact for a perturbed matrix whose (i,j)th element differs from a_{ij} by less than $3(\text{epsilon})(\text{rho})m_{ij}$ where (epsilon) is the machine precision, (rho) is the growth value returned in W(1) if GROW = .TRUE., and m_{ij} the number of Gaussian elimination operations applied to element (i,j).

If (rho) = W(1) is very large or RPMIN is very small, then a fresh call of F01BRF is recommended.

8. Further Comments

If the user has a sequence of problems with the same sparsity pattern then this routine is recommended after F01BRF has been called for one such problem. It is typically 4 to 7 times faster but is potentially unstable since the previous pivotal sequence is used. Further details on timing are given in document F01BRF.

If growth estimation is performed (GROW = .TRUE.), then the time increases by between 5% and 10%. Pivot size monitoring (ETA <= 1. 0) involves a similar overhead.

We normally expect this routine to be entered with a matrix having the same pattern of non-zeros as was earlier presented to F01BRF. However there is no record of this pattern, but rather a record of the pattern including all fill-ins. Therefore we permit additional non-zeros in positions corresponding to fill-ins.

If singular matrices are being treated then it is also required that the present matrix be sufficiently like the previous one for the same permutations to be suitable for factorization with the same set of zero pivots.

9. Example

To factorize the real sparse matrices

```
( 5  0  0  0 0  0)
( 0  2 -1  2 0  0)
( 0  0  3  0 0  0)
(-2  0  0  1 1  0)
(-1  0  0 -1 2 -3)
(-1 -1  0  0 0  6)
```

and

```
(10  0  0  0  0  0)
( 0 12 -3 -1  0  0)
( 0  0 15  0  0  0)
(-2  0  0 10 -1  0).
(-1  0  0 -5  1 -1)
(-1 -2  0  0  0  6)
```

This example program simply prints the values of W(1) and RPMIN returned by F01BSF. Normally the calls of F01BRF and F01BSF would be followed by calls of F04AXF.

The example program is not reproduced here. The source code for all example programs is distributed with the NAG Foundation Library software and should be available on-line.

%%%

F01 -- Matrix Factorizations F01MAF
F01MAF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for your implementation to check implementation-dependent details. The symbol (*) after a NAG routine name denotes a routine that is not included in the Foundation Library.

1. Purpose

F01MAF computes an incomplete Cholesky factorization of a real sparse symmetric positive-definite matrix A.

2. Specification

```
 SUBROUTINE F01MAF (N, NZ, A, LICN, IRN, LIRN, ICN, DROPTL,
1                   DENSW, WKEEP, IKEEP, IWORK, ABORT,
2                   INFORM, IFAIL)
 INTEGER          N, NZ, LICN, IRN(LIRN), LIRN, ICN(LICN),
```

```
      1                 IKEEP(2*N), IWORK(6*N), INFORM(4), IFAIL
       DOUBLE PRECISION A(LICN), DROPTL, DENSW, WKEEP(3*N)
       LOGICAL          ABORT(3)
```

3. Description

F01MAF computes an incomplete Cholesky factorization

$$C=PLDL^{T}P^{T}, \quad WAW=C+E$$

for the sparse symmetric positive-definite matrix A, where P is a permutation matrix, L is a unit lower triangular matrix, D is a diagonal matrix with positive diagonal elements, E is an error matrix representing elements dropped during the factorization and diagonal elements that have been modified to ensure that C is positive-definite, and W is a diagonal matrix, chosen to make the diagonal elements of WAW unity.

$W^{-1}CW^{-1}$ is a pre-conditioning matrix for A, and the factorization of C is intended to be used by F04MAF to solve systems of linear equations Ax=b.

The permutation matrix P is chosen to reduce the amount of fill-in that occurs in L and the user-supplied parameter DROPTL can also be used to control the amount of fill-in that occurs.

Full details on the factorization can be found in Munksgaard [1].

F01MAF is based on the Harwell Library routine MA31A.

4. References

[1] Munksgaard N (1980) Solving Sparse Symmetric Sets of Linear Equations by Pre-conditioned Conjugate Gradients. ACM Trans. Math. Softw. 6 206--219.

5. Parameters

1: N -- INTEGER Input
On entry: n, the order of the matrix A. Constraint: N >= 1.

2: NZ -- INTEGER Input
On entry: the number of non-zero elements in the upper triangular part of the matrix A, including the number of

elements on the leading diagonal. Constraint: NZ >= N.

3: A(LICN) -- DOUBLE PRECISION array Input/Output
On entry: the first NZ elements of the array A must contain the non-zero elements of the upper triangular part of the sparse positive-definite symmetric matrix A, including the elements on the leading diagonal. On exit: the first (NZ-N) elements of A contain the elements above the diagonal of the matrix WAW, where W is a diagonal matrix whose ith diagonal element is $w_i = a_{ii}^{-1/2}$. These elements are returned in order by rows and the value returned in ICN(k) gives the column index of the element returned in A(k). The value w_i is returned in the ith element of the array WKEEP. The remaining LROW-NZ+N elements of A, where LROW is the value returned in INFORM(1), return details of the factorization for use by F04MAF.

4: LICN -- INTEGER Input
On entry:
the dimension of the array A as declared in the (sub)program from which F01MAF is called.
If fill-in is expected during the factorization, then a larger value of LICN will allow fewer elements to be dropped during the factorization, thus giving a more accurate factorization, which in turn will almost certainly mean that fewer iterations will be required by F04MAF. Constraint: LICN>=2*NZ.

5: IRN(LIRN) -- INTEGER array Input/Output
On entry: IRN(k), for k = 1,2,...,NZ must contain the row index of the non-zero element of the matrix A supplied in A (k). On exit: the first LCOL elements of IRN, where LCOL is the value returned in INFORM(2), return details of the factorization for use by F04MAF.

6: LIRN -- INTEGER Input
On entry:
the dimension of the array IRN as declared in the (sub)program from which F01MAF is called.
LIRN must be at least NZ, but, as with LICN, if fill-in is expected then a larger value of LIRN will allow a more accurate factorization. For this purpose LIRN should exceed NZ by the same amount that LICN exceeds 2*NZ. Constraint: LIRN >= NZ.

7: ICN(LICN) -- INTEGER array Input/Output
On entry: ICN(k), for k = 1,2,...,NZ must contain the column index of the non-zero element of the matrix A supplied in A (k). Thus a_{ij} =A(k), where i = IRN(k) and j = ICN(k). On exit: the first (NZ-N) elements of ICN give the column indices of the first (NZ-N) elements returned in A. The remaining LROW - NZ + N elements of ICN return details of the factorization for use by F04MAF.

8: DROPTL -- DOUBLE PRECISION Input/Output
On entry: a value in the range [-1.0,1.0] to be used as a tolerance in deciding whether or not to drop elements during the factorization. At the kth pivot step the element $a_{ij}^{(k+1)}$ is dropped if it would cause fill-in and if

$$|a_{ij}^{(k+1)}| < |\mathrm{DROPTL}| * \sqrt{|a_{ii}^{(k)} a_{jj}^{(k)}|}.$$

If DROPTL is supplied as negative, then it is not altered during the factorization and so is unchanged on exit, but if DROPTL is supplied as positive then it may be altered by the routine with the aim of obtaining an accurate factorization in the space available. If DROPTL is supplied as -1.0, then no fill-in will occur during the factorization; and if DROPTL is supplied as 0.0 then a complete factorization is performed. On exit: may be overwritten with the value used by the routine in order to obtain an accurate factorization in the space available, if DROPTL > 0.0 on entry.

9: DENSW -- DOUBLE PRECISION Input/Output
On entry: a value in the range [0.0,1.0] to be used in deciding whether or not to regard the active part of the matrix at the kth pivot step as being full. If the ratio of non-zero elements to the total number of elements is greater than or equal to DENSW, then the active part is regarded as full. If DENSW < 1.0, then the storage used is likely to increase compared to the case where DENSW = 0, but the execution time is likely to decrease. Suggested value: DENSW = 0.8. On exit: if on entry DENSW is not in the range [0.0,1.0], then it is set to 0.8. Otherwise it is unchanged.

10: WKEEP(3*N) -- DOUBLE PRECISION array Output
On exit: information which must be passed unchanged to F04MAF. The first N elements contain the values w_i, for i=1,2,...,n, and the next N elements contain the diagonal elements of D.

11: IKEEP(2*N) -- INTEGER array Output
On exit: information which must be passed unchanged to F04MAF.

12: IWORK(6*N) -- INTEGER array Workspace

13: ABORT(3) -- LOGICAL array Input
On entry:
if ABORT(1) = .TRUE., the routine will exit immediately on detecting duplicate elements and return IFAIL = 5. Otherwise when ABORT(1) = .FALSE., the calculations will continue using the sum of the duplicate entries. In either case details of the duplicate elements are output on the current advisory message unit (see X04ABF), unless suppressed by the value of IFAIL on entry.

If ABORT(2) = .TRUE., the routine will exit immediately on detecting a zero or negative pivot element and return IFAIL = 6. Otherwise when ABORT(2) = .FALSE., the zero or negative pivot element will be modified to ensure positive-definiteness and a message will be printed on the current advisory message unit, unless suppressed by the value of IFAIL on entry.

If ABORT(3) = .TRUE., the routine will exit immediately if the arrays A and ICN have been filled up and return IFAIL = 7. Otherwise when ABORT(3) = .FALSE., the data in the arrays is compressed to release more storage and a message will be printed on the current advisory message unit, unless suppressed by the value of IFAIL on entry. If DROPTL is positive on entry, it may be modified in order to allow a factorization to be completed in the available space.
Suggested values:
ABORT(1) = .TRUE.,

ABORT(2) = .TRUE.,

```
          ABORT(3) = .TRUE..

14:  INFORM(4) -- INTEGER array                                  Output
     On exit:
          INFORM(1) returns the number of elements of A and ICN
          that have been used by the routine. Thus at least the
          first INFORM(1) elements of A and of ICN must be
          supplied to F04MAF.

          Similarly, INFORM(2) returns the number of elements of
          IRN that have been used by the routine and so at least
          the first INFORM(2) elements must be supplied to
          F04MAF.

          INFORM(3) returns the number of entries supplied in A
          that corresponded to diagonal and duplicate elements.
          If no duplicate entries were found, then INFORM(3)
          will return the value of N.

          INFORM(4) returns the value k of the pivot step from
          which the active matrix was regarded as full.
     INFORM must be passed unchanged to F04MAF.

15:  IFAIL -- INTEGER                                      Input/Output
     For this routine, the normal use of IFAIL is extended to
     control the printing of error and warning messages as well
     as specifying hard or soft failure (see the Essential
     Introduction).

     Before entry, IFAIL must be set to a value with the decimal
     expansion cba, where each of the decimal digits c, b and a
     must have a value of 0 or 1.
     a=0   specifies hard failure, otherwise soft failure;

     b=0   suppresses error messages, otherwise error messages
           will be printed (see Section 6);

     c=0   suppresses warning messages, otherwise warning
           messages will be printed (see Section 6).
     The recommended value for inexperienced users is 110 (i.e.,
     hard failure with all messages printed).

     Unless the routine detects an error (see Section 6), IFAIL
     contains 0 on exit.

6. Error Indicators and Warnings
```

```
Errors detected by the routine:

For each error, an explanatory error message is output on the
current error message unit (as defined by X04AAF), unless
suppressed by the value of IFAIL on entry.

IFAIL= 1
     On entry N < 1,

     or       NZ < N,

     or       LIRN < NZ,

     or       LICN<2*NZ.

IFAIL= 2
     One of the conditions 0 < IRN(k) <= ICN(k) <= N is not
     satisfied so that A(k) is not in the upper triangle of the
     matrix. No further computation is attempted.

IFAIL= 3
     One of the diagonal elements of the matrix A is zero or
     negative so that A is not positive-definite. No further
     computation is attempted.

IFAIL= 4
     The available space has been used and no further
     compressions are possible. The user should either increase
     DROPTL, or allocate more space to A, IRN and ICN.

     For all the remaining values of IFAIL the computations will
     continue in the case of soft failure, so that more than one
     advisory message may be printed.

IFAIL= 5
     Duplicate elements have been detected and ABORT(1) = .TRUE..

IFAIL= 6
     A zero or negative pivot element has been detected during
     the factorization and ABORT(2) = .TRUE..

     This should not happen if A is an M-matrix (see Munksgaard
     [1]), but may occur for other types of positive-definite
     matrix.
```

IFAIL= 7
The available space has been used and ABORT(3) = .TRUE..

7. Accuracy

The accuracy of the factorization will be determined by the size of the elements that are dropped and the size of the modifications made to the diagonal elements. If these sizes are small then the computed factors will correspond to a matrix close to A and the number of iterations required by F04MAF will be small. The more incomplete the factorization, the higher the number of iterations required by F04MAF.

8. Further Comments

The time taken by the routine will depend upon the sparsity pattern of the matrix and the number of fill-ins that occur during the factorization. At the very least the time taken can be expected to be roughly proportional to n(tau), where (tau) is the number of non-zeros.
The routine is intended for use with positive-definite matrices, but the user is warned that it will not necessarily detect non-positive-definiteness. Indeed the routine may return a factorization that can satisfactorily be used by F04MAF even when A is not positive-definite, but this should not be relied upon as F04MAF may not converge.

9. Example

The example program illustrates the use of F01MAF in conjunction with F04MAF to solve the 16 linear equations Ax=b, where

$$
A=\begin{pmatrix}
1 & z & & z & & & & & & & & & & & & \\
z & 1 & z & & z & & & & & & & & & & & \\
 & z & 1 & z & & z & & & & & & & & & & \\
 & & z & 1 & 0 & & z & & & & & & & & & \\
z & & & 0 & 1 & z & & z & & & & & & & & \\
 & z & & & z & 1 & z & & z & & & & & & & \\
 & & z & & & z & 1 & z & & z & & & & & & \\
 & & & z & & & z & 1 & 0 & & z & & & & & \\
 & & & & z & & & 0 & 1 & z & & z & & & & \\
 & & & & & z & & & z & 1 & z & & z & & & \\
 & & & & & & z & & & z & 1 & z & & z & & \\
 & & & & & & & z & & & z & 1 & 0 & & z & \\
 & & & & & & & & z & & & 0 & 1 & z & & \\
 & & & & & & & & & z & & & z & 1 & z & \\
 & & & & & & & & & & z & & & z & 1 & z \\
 & & & & & & & & & & & z & & & z & 1
\end{pmatrix}.
$$

```
                    (                    z     z 1 z)
                    (                      z     z 1)

 T ( 1  1  1  1  1      1  1      1  1  1  1  1)
b =( -  -  -  -  - 0 0  -  - 0 0  -  -  -  -  -),
   ( 2  4  4  2  4      4  4      4  2  4  4  2)

          1
where z=- -.
          4
```

The n by n matrix A arises in the solution of Laplace's equation in a unit square, using a 5-point formula with a 6 by 6 discretisation, with unity on the boundaries.

The drop tolerance, DROPTL, is taken as 0.1, and the density factor, DENSW, is taken as 0.8. The value IFAIL = 111 is used so that advisory and error messages will be printed, but soft failure would occur if IFAIL were returned as non-zero.

A relative accuracy of about 0.0001 is requested in the solution from F04MAF, with a maximum of 50 iterations.

The example program for F02FJF illustrates the use of F01MAF and F04MAF in solving an eigenvalue problem.

The example program is not reproduced here. The source code for all example programs is distributed with the NAG Foundation Library software and should be available on-line.

%%%

F01 -- Matrix Factorizations F01MCF

F01MCF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for your implementation to check implementation-dependent details. The symbol (*) after a NAG routine name denotes a routine that is not included in the Foundation Library.

1. Purpose

F01MCF computes the Cholesky factorization of a real symmetric positive-definite variable-bandwidth matrix.

2. Specification

```
      SUBROUTINE F01MCF (N, A, LAL, NROW, AL, D, IFAIL)
      INTEGER          N, LAL, NROW(N), IFAIL
      DOUBLE PRECISION A(LAL), AL(LAL), D(N)
```

3. Description

This routine determines the unit lower triangular matrix L and
                                                          T
the diagonal matrix D in the Cholesky factorization A=LDL of a symmetric positive-definite variable-bandwidth matrix A of order n. (Such a matrix is sometimes called a 'sky-line' matrix.)

The matrix A is represented by the elements lying within the envelope of its lower triangular part, that is, between the first non-zero of each row and the diagonal (see Section 9 for an example). The width NROW(i) of the ith row is the number of elements between the first non-zero element and the element on the diagonal, inclusive. Although, of course, any matrix possesses an envelope as defined, this routine is primarily intended for the factorization of symmetric positive-definite matrices with an average bandwidth which is small compared with n (also see Section 8).

The method is based on the property that during Cholesky factorization there is no fill-in outside the envelope.

The determination of L and D is normally the first of two steps in the solution of the system of equations Ax=b. The remaining
                                T
step, viz. the solution of LDL x=b may be carried out using F04MCF.

4. References

[1] Jennings A (1966) A Compact Storage Scheme for the Solution of Symmetric Linear Simultaneous Equations. Comput. J. 9 281--285.

[2] Wilkinson J H and Reinsch C (1971) Handbook for Automatic Computation II, Linear Algebra. Springer-Verlag.

5. Parameters

1: N -- INTEGER Input
On entry: n, the order of the matrix A. Constraint: N >= 1.

2: A(LAL) -- DOUBLE PRECISION array Input
On entry: the elements within the envelope of the lower triangle of the positive-definite symmetric matrix A, taken in row by row order. The following code assigns the matrix elements within the envelope to the correct elements of the array:

```
K = 0
DO 20 I = 1, N
DO 10 J = I-NROW(I)+1, I
K = K + 1
A(K) = matrix (I,J)
10    CONTINUE
20 CONTINUE
```

See also Section 8.

3: LAL -- INTEGER Input
On entry: the smaller of the dimensions of the arrays A and AL as declared in the calling (sub)program from which F01MCF is called. Constraint: LAL >= NROW(1) + NROW(2) +... + NROW(n).

4: NROW(N) -- INTEGER array Input
On entry: NROW(i) must contain the width of row i of the matrix A, i.e., the number of elements between the first (leftmost) non-zero element and the element on the diagonal, inclusive. Constraint: 1 <= NROW(i) <= i, for i=1,2,...,n.

5: AL(LAL) -- DOUBLE PRECISION array Output
On exit: the elements within the envelope of the lower triangular matrix L, taken in row by row order. The envelope of L is identical to that of the lower triangle of A. The unit diagonal elements of L are stored explicitly. See also Section 8.

6: D(N) -- DOUBLE PRECISION array Output
On exit: the diagonal elements of the the diagonal matrix D . Note that the determinant of A is equal to the product of these diagonal elements. If the value of the determinant is required it should not be determined by forming the product explicitly, because of the possibility of overflow or underflow. The logarithm of the determinant may safely be formed from the sum of the logarithms of the diagonal elements.

```
 7:  IFAIL -- INTEGER                                    Input/Output
     On entry: IFAIL must be set to 0, -1 or 1. For users not
     familiar with this parameter (described in the Essential
     Introduction) the recommended value is 0.

     On exit: IFAIL = 0 unless the routine detects an error (see
     Section 6).

6. Error Indicators and Warnings

Errors detected by the routine:

IFAIL= 1
     On entry N < 1,

     or       for some i, NROW(i) < 1 or NROW(i) > i,

     or       LAL < NROW(1) + NROW(2) +... + NROW(N).

IFAIL= 2
     A is not positive-definite, or this property has been
     destroyed by rounding errors. The factorization has not been
     completed.

IFAIL= 3
     A is not positive-definite, or this property has been
     destroyed by rounding errors. The factorization has been
     completed but may be very inaccurate (see Section 7).

7. Accuracy

If IFAIL = 0 on exit, then the computed L and D satisfy the
            T
relation LDL =A+F, where

                                 2
                    ||F|| <=km (epsilon)max a
                         2                 i ii

and

                               2
                   ||F|| <=km (epsilon)||A|| ,
                        2                   2
```

where k is a constant of order unity, m is the largest value of NROW(i), and (epsilon) is the machine precision. See Wilkinson and Reinsch [2], pp 25--27, 54--55. If IFAIL = 3 on exit, then the factorization has been completed although the matrix was not positive-definite. However the factorization may be very inaccurate and should be used only with great caution. For instance, if it is used to solve a set of equations Ax=b using F04MCF, the residual vector b-Ax should be checked.

8. Further Comments

The time taken by the routine is approximately proportional to the sum of squares of the values of NROW(i).

The distribution of row widths may be very non-uniform without undue loss of efficiency. Moreover, the routine has been designed to be as competitive as possible in speed with routines designed for full or uniformly banded matrices, when applied to such matrices.

Unless otherwise stated in the Users' Note for your implementation, the routine may be called with the same actual array supplied for parameters A and AL, in which case L overwrites the lower triangle of A. However this is not standard Fortran 77 and may not work in all implementations.

9. Example

To obtain the Cholesky factorization of the symmetric matrix, whose lower triangle is:

```
(1                )
(2  5             )
(0  3 13          )
(0  0  0 16       ).
(5 14 18  8 55    )
(0  0  0 24 17 77)
```

For this matrix, the elements of NROW must be set to 1, 2, 2, 1, 5, 3, and the elements within the envelope must be supplied in row order as:

1, 2, 5, 3, 13, 16, 5, 14, 18, 8, 55, 24, 17, 77.

The example program is not reproduced here. The source code for

all example programs is distributed with the NAG Foundation Library software and should be available on-line.

%%%

F01 -- Matrix Factorizations F01QCF
F01QCF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for your implementation to check implementation-dependent details. The symbol (*) after a NAG routine name denotes a routine that is not included in the Foundation Library.

1. Purpose

F01QCF finds the QR factorization of the real m by n matrix A, where m>=n.

2. Specification

```
SUBROUTINE F01QCF (M, N, A, LDA, ZETA, IFAIL)
INTEGER          M, N, LDA, IFAIL
DOUBLE PRECISION A(LDA,*), ZETA(*)
```

3. Description

The m by n matrix A is factorized as

$$A = Q \begin{pmatrix} R \\ 0 \end{pmatrix} \quad \text{when } m>n,$$

$$A = QR \quad \text{when } m=n,$$

where Q is an m by m orthogonal matrix and R is an n by n upper triangular matrix. The factorization is obtained by Householder's method. The kth transformation matrix, Q_k, which is used to introduce zeros into the kth column of A is given in the form

$$Q_k = \begin{pmatrix} I & 0 \\ 0 & T \end{pmatrix}_k$$

where

$$T$$

$$T_k = I - u_k u_k^T,$$

$$u_k = \begin{pmatrix} \zeta_k \\ z_k \end{pmatrix},$$

ζ_k is a scalar and z_k is an (m-k) element vector. ζ_k and z_k are chosen to annihilate the elements below the triangular part of A.

The vector u_k is returned in the kth element of the array ZETA and in the kth column of A, such that ζ_k is in ZETA(k) and the elements of z_k are in A(k+1,k),...,A(m,k). The elements of R are returned in the upper triangular part of A.

Q is given by

$$Q = (Q_n Q_{n-1} \ldots Q_1)^T.$$

Good background descriptions to the QR factorization are given in Dongarra et al [1] and Golub and Van Loan [2], but note that this routine is not based upon LINPACK routine DQRDC.

4. References

[1] Dongarra J J, Moler C B, Bunch J R and Stewart G W (1979) LINPACK Users' Guide. SIAM, Philadelphia.

[2] Golub G H and Van Loan C F (1989) Matrix Computations (2nd Edition). Johns Hopkins University Press, Baltimore, Maryland.

[3] Wilkinson J H (1965) The Algebraic Eigenvalue Problem. Oxford University Press.

5. Parameters

```
 1:  M -- INTEGER                                                  Input
     On entry: m, the number of rows of A. Constraint: M >= N.

 2:  N -- INTEGER                                                  Input
     On entry: n, the number of columns of A.

     When N = 0 then an immediate return is effected.
     Constraint: N >= 0.

 3:  A(LDA,*) -- DOUBLE PRECISION array              Input/Output
     Note: the second dimension of the array A must be at least
     max(1,n).
     On entry: the leading m by n part of the array A must
     contain the matrix to be factorized. On exit: the n by n
     upper triangular part of A will contain the upper triangular
     matrix R and the m by n strictly lower triangular part of A
     will contain details of the factorization as described in
     Section 3.

 4:  LDA -- INTEGER                                                Input
     On entry:
     the first dimension of the array A as declared in the
     (sub)program from which F01QCF is called.
     Constraint: LDA >= max(1,M).

 5:  ZETA(*) -- DOUBLE PRECISION array                        Output
     Note: the dimension of the array ZETA must be at least max
     (1,n) On exit: ZETA(k) contains the scalar (zeta)  for the k
                                                     k
     th transformation. If T =I then ZETA(k)=0.0, otherwise ZETA(
                            k
     k) contains (zeta)  as described in Section 3 and (zeta)  is
                     k                                      k

     always in the range (1.0, \/2.0).

 6:  IFAIL -- INTEGER                                  Input/Output
     On entry: IFAIL must be set to 0, -1 or 1. For users not
     familiar with this parameter (described in the Essential
     Introduction) the recommended value is 0.

     On exit: IFAIL = 0 unless the routine detects an error (see
     Section 6).

6. Error Indicators and Warnings
```

Errors detected by the routine:

If on entry IFAIL = 0 or -1, explanatory error messages are
output on the current error message unit (as defined by X04AAF).

IFAIL=-1
     On entry M < N,

     or   N < 0,

     or   LDA < M.

7. Accuracy

The computed factors Q and R satisfy the relation

$$Q(0)=A+E,$$

where

$$|||E|||<=c(epsilon)|||A|||,$$

and (epsilon) is the machine precision (see X02AJF(*)), c is a
modest function of m and n and |||.||| denotes the spectral (two)
norm.

8. Further Comments

The approximate number of floating-point operations is given by
$2n^2(3m-n)/3$.

Following the use of this routine the operations

$$B:=QB \quad \text{and} \quad B:=Q^T B,$$

where B is an m by k matrix, can be performed by calls to F01QDF.
The operation B:=QB can be obtained by the call:

```
      IFAIL = 0
      CALL F01QDF('No transpose', 'Separate', M, N, A, LDA, ZETA,
     *            K, B, LDB, WORK, IFAIL)
```

T
and B:=Q B can be obtained by the call:

```
      IFAIL = 0
      CALL F01QDF('Transpose', 'Separate', M, N, A, LDA, ZETA,
     *            K, B, LDB, WORK, IFAIL)
```

In both cases WORK must be a k element array that is used as workspace. If B is a one-dimensional array (single column) then the parameter LDB can be replaced by M. See F01QDF for further details.

The first k columns of the orthogonal matrix Q can either be obtained by setting B to the first k columns of the unit matrix and using the first of the above two calls, or by calling F01QEF, which overwrites the k columns of Q on the first k columns of the array A. Q is obtained by the call:

```
     CALL F01QEF('Separate', M, N, K, A, LDA, ZETA, WORK, IFAIL)
```

As above WORK must be a k element array. If k is larger than N, then A must have been declared to have at least k columns.

Operations involving the matrix R can readily be performed by the Level 2 BLAS routines DTRSV and DTRMV (see Chapter F06), but note that no test for near singularity of R is incorporated in DTRSV. If R is singular, or nearly singular then F02WUF(*) can be used to determine the singular value decomposition of R.

9. Example

To obtain the QR factorization of the 5 by 3 matrix

$$A = \begin{pmatrix} 2.0 & 2.5 & 2.5 \\ 2.0 & 2.5 & 2.5 \\ 1.6 & -0.4 & 2.8 \\ 2.0 & -0.5 & 0.5 \\ 1.2 & -0.3 & -2.9 \end{pmatrix}.$$

The example program is not reproduced here. The source code for all example programs is distributed with the NAG Foundation Library software and should be available on-line.

%%%

F01 -- Matrix Factorizations F01QDF
F01QDF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for your implementation to check implementation-dependent details. The symbol (*) after a NAG routine name denotes a routine that is not included in the Foundation Library.

1. Purpose

F01QDF performs one of the transformations

$$B:=QB \text{ or } B:=Q^T B,$$

where B is an m by ncolb real matrix and Q is an m by m orthogonal matrix, given as the product of Householder transformation matrices.

This routine is intended for use following F01QCF or F01QFF(*).

2. Specification

```
      SUBROUTINE F01QDF (TRANS, WHERET, M, N, A, LDA, ZETA,
     1                   NCOLB, B, LDB, WORK, IFAIL)
      INTEGER           M, N, LDA, NCOLB, LDB, IFAIL
      DOUBLE PRECISION A(LDA,*), ZETA(*), B(LDB,*), WORK(*)
      CHARACTER*1       TRANS, WHERET
```

3. Description

Q is assumed to be given by

$$Q=(Q_n Q_{n-1} \ldots Q_1)^T,$$

Q_k being given in the form

$$Q_k = \begin{pmatrix} I & 0 \\ 0 & T \end{pmatrix}$$

where

$$T_k = I - u_k u_k^T,$$

$$u_k = \begin{pmatrix} (\zeta)_k \\ z_k \end{pmatrix},$$

$(\zeta)_k$ is a scalar and z_k is an $(m-k)$ element vector. z_k must be supplied in the kth column of A in elements A(k+1,k),...,A(m,k) and $(\zeta)_k$ must be supplied either in A(k,k) or in ZETA(k), depending upon the parameter WHERET.

To obtain Q explicitly B may be set to I and pre-multiplied by Q. This is more efficient than obtaining Q^T.

4. References

[1] Golub G H and Van Loan C F (1989) Matrix Computations (2nd Edition). Johns Hopkins University Press, Baltimore, Maryland.

[2] Wilkinson J H (1965) The Algebraic Eigenvalue Problem. Oxford University Press.

5. Parameters

1: TRANS -- CHARACTER*1 Input
On entry: the operation to be performed as follows:
TRANS = 'N' (No transpose)
Perform the operation B:=QB.

TRANS = 'T' or 'C' (Transpose)
Perform the operation $B:=Q^T B$.
Constraint: TRANS must be one of 'N', 'T' or 'C'.

2: WHERET -- CHARACTER*1 Input
On entry: indicates where the elements of (zeta) are to be found as follows:
WHERET = 'I' (In A)

```
            The elements of (zeta) are in A.

      WHERET = 'S' (Separate)
            The elements of (zeta) are separate from A, in ZETA.
      Constraint: WHERET must be one of 'I' or 'S'.

3:    M -- INTEGER                                                Input
      On entry: m, the number of rows of A. Constraint: M >= N.

4:    N -- INTEGER                                                Input
      On entry: n, the number of columns of A.

      When N = 0 then an immediate return is effected.
      Constraint: N >= 0.

5:    A(LDA,*) -- DOUBLE PRECISION array                          Input
      Note: the second dimension of the array A must be at least
      max(1,N).
      On entry: the leading m by n strictly lower triangular part
      of the array A must contain details of the matrix Q. In
      addition, when WHERET = 'I', then the diagonal elements of A
      must contain the elements of (zeta) as described under the
      argument ZETA below.

      When WHERET = 'S', the diagonal elements of the array A are
      referenced, since they are used temporarily to store the
      (zeta) , but they contain their original values on return.
            k

6:    LDA -- INTEGER                                              Input
      On entry:
      the first dimension of the array A as declared in the
      (sub)program from which F01QDF is called.
      Constraint: LDA >= max(1,M).

7:    ZETA(*) -- DOUBLE PRECISION array                           Input
      Note: when WHERET = 'S', the dimension of the array ZETA
      must be greater than or equal to max(1,N). On entry: if
      WHERET = 'S', the array ZETA must contain the elements of
      (zeta). If ZETA(k) = 0.0 then T  is assumed to be I
                                     k
      otherwise ZETA(k) is assumed to contain (zeta) .
                                                    k

      When WHERET = 'I', ZETA is not referenced.
```

```
 8:  NCOLB -- INTEGER                                          Input
     On entry: ncolb, number of columns of B.

     When NCOLB = 0 then an immediate return is effected.
     Constraint: NCOLB >= 0.

 9:  B(LDB,*) -- DOUBLE PRECISION array              Input/Output
     Note: the second dimension of the array B must be at least
     max(1,NCOLB).
     On entry: the leading m by ncolb part of the array B must
     contain the matrix to be transformed. On exit: B is
     overwritten by the transformed matrix.

10:  LDB -- INTEGER                                           Input
     On entry:
     the first dimension of the array B as declared in the
     (sub)program from which F01QDF is called.
     Constraint: LDB >= max(1,M).

11:  WORK(*) -- DOUBLE PRECISION array                    Workspace
     Note: the dimension of the array WORK must be at least
     max(1,NCOLB).

12:  IFAIL -- INTEGER                                 Input/Output
     On entry: IFAIL must be set to 0, -1 or 1. For users not
     familiar with this parameter (described in the Essential
     Introduction) the recommended value is 0.

     On exit: IFAIL = 0 unless the routine detects an error (see
     Section 6).

6. Error Indicators and Warnings

Errors detected by the routine:

If on entry IFAIL = 0 or -1, explanatory error messages are
output on the current error message unit (as defined by X04AAF).

IFAIL=-1
     On entry TRANS /= 'N', 'T' or 'C',

     or       WHERET /= 'I' or 'S',

     or       M < N,

     or       N < 0,
```

or LDA < M,

or NCOLB < 0,

or LDB < M.

7. Accuracy

Letting C denote the computed matrix $Q^T B$, C satisfies the relation

$$QC=B+E,$$

where

$$||E||<=c(epsilon)||B||,$$

and (epsilon) the machine precision (see X02AJF(*)), c is a modest function of m and |||.||| denotes the spectral (two) norm. An equivalent result holds for the computed matrix QB. See also Section 7 of F01QCF.

8. Further Comments

The approximate number of floating-point operations is given by 2n(2m-n)ncolb.

9. Example

To obtain the matrix $Q^T B$ for the matrix B given by

$$B=\begin{pmatrix} 1.1 & 0.00 \\ 0.9 & 0.00 \\ 0.6 & 1.32 \\ 0.0 & 1.10 \\ -0.8 & -0.26 \end{pmatrix}$$

following the QR factorization of the 5 by 3 matrix A given by

$$A=\begin{pmatrix} 2.0 & 2.5 & 2.5 \\ 2.0 & 2.5 & 2.5 \\ 1.6 & -0.4 & 2.8 \\ 2.0 & -0.5 & 0.5 \end{pmatrix}.$$

```
                                (1.2 -0.3 -2.9)

          The example program is not reproduced here. The source code for
          all example programs is distributed with the NAG Foundation
          Library software and should be available on-line.

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

          F01 -- Matrix Factorizations                                  F01QEF
                  F01QEF -- NAG Foundation Library Routine Document

          Note: Before using this routine, please read the Users' Note for
          your implementation to check implementation-dependent details.
          The symbol (*) after a NAG routine name denotes a routine that is
          not included in the Foundation Library.

          1. Purpose

          F01QEF returns the first ncolq columns of the real m by m
          orthogonal matrix Q, where Q is given as the product of
          Householder transformation matrices.

          This routine is intended for use following F01QCF or F01QFF(*).

          2. Specification

                 SUBROUTINE F01QEF (WHERET, M, N, NCOLQ, A, LDA, ZETA,
                1                   WORK, IFAIL)
                 INTEGER          M, N, NCOLQ, LDA, IFAIL
                 DOUBLE PRECISION A(LDA,*), ZETA(*), WORK(*)
                 CHARACTER*1      WHERET

          3. Description

          Q is assumed to be given by

                                                 T
                                 Q=(Q Q   ...Q ) ,
                                     n n-1    1

          Q  being given in the form
           k

                                        (I 0 )
                                       Q =(0 T )
                                        k (   k)
```

where

$$T_k = I - u_k u_k^T,$$

$$u_k = \begin{pmatrix} (\zeta)_k \\ z_k \end{pmatrix},$$

$(\zeta)_k$ is a scalar and z_k is an $(m-k)$ element vector. z_k must be supplied in the kth column of A in elements A(k+1,k),...,A(m,k) and $(\zeta)_k$ must be supplied either in A(k,k) or in ZETA(k), depending upon the parameter WHERET.

4. References

[1] Golub G H and Van Loan C F (1989) Matrix Computations (2nd Edition). Johns Hopkins University Press, Baltimore, Maryland.

[2] Wilkinson J H (1965) The Algebraic Eigenvalue Problem. Oxford University Press.

5. Parameters

1: WHERET -- CHARACTER*1 Input
On entry: indicates where the elements of (zeta) are to be found as follows:
WHERET = 'I' (In A)
The elements of (zeta) are in A.

WHERET = 'S' (Separate)
The elements of (zeta) are separate from A, in ZETA.
Constraint: WHERET must be one of 'I' or 'S'.

2: M -- INTEGER Input
On entry: m, the number of rows of A. Constraint: M >= N.

3: N -- INTEGER Input
On entry: n, the number of columns of A. Constraint: N >= 0.

4: NCOLQ -- INTEGER Input
On entry: ncolq, the required number of columns of Q.
Constraint: 0 <= NCOLQ <= M.

When NCOLQ = 0 then an immediate return is effected.

5: A(LDA,*) -- DOUBLE PRECISION array Input/Output
Note: the second dimension of the array A must be at least max(1,N,NCOLQ).
On entry: the leading m by n strictly lower triangular part of the array A must contain details of the matrix Q. In addition, when WHERET = 'I', then the diagonal elements of A must contain the elements of (zeta) as described under the argument ZETA below. On exit: the first NCOLQ columns of the array A are overwritten by the first NCOLQ columns of the m by m orthogonal matrix Q. When N = 0 then the first NCOLQ columns of A are overwritten by the first NCOLQ columns of the identity matrix.

6: LDA -- INTEGER Input
On entry:
the first dimension of the array A as declared in the (sub)program from which F01QEF is called.
Constraint: LDA >= max(1,M).

7: ZETA(*) -- DOUBLE PRECISION array Input
Note: the dimension of the array ZETA must be at least max(1,N).
On entry: with WHERET = 'S', the array ZETA must contain the elements of (zeta). If ZETA(k) = 0.0 then T_k is assumed to be I, otherwise ZETA(k) is assumed to contain $(\text{zeta})_k$.

When WHERET = 'I', the array ZETA is not referenced.

8: WORK(*) -- DOUBLE PRECISION array Workspace
Note: the dimension of the array WORK must be at least max(1,NCOLQ).

9: IFAIL -- INTEGER Input/Output
On entry: IFAIL must be set to 0, -1 or 1. For users not familiar with this parameter (described in the Essential Introduction) the recommended value is 0.

```
     On exit: IFAIL = 0 unless the routine detects an error (see
     Section 6).

6. Error Indicators and Warnings

Errors detected by the routine:

If on entry IFAIL = 0 or -1, explanatory error messages are
output on the current error message unit (as defined by X04AAF).

IFAIL=-1
     On entry WHERET /= 'I' or 'S',

     or       M < N,

     or       N < 0,

     or       NCOLQ < 0 or NCOLQ > M,

     or       LDA < M.

7. Accuracy

The computed matrix Q satisfies the relation

                              Q=P+E,

where P is an exactly orthogonal matrix and

                         ||E||<=c(epsilon)

(epsilon) is the machine precision (see X02AJF(*)), c is a modest
function of m and |||.||| denotes the spectral (two) norm. See
also Section 7 of F01QCF.

8. Further Comments

The approximate number of floating-point operations required is
given by

     2
     -n{(3m-n)(2ncolq-n)-n(ncolq-n)},    ncolq>n,
     3

     2     2
     -ncolq (3m-ncolq),                 ncolq<=n.
```

```
     3

9. Example

To obtain the 5 by 5 orthogonal matrix Q following the QR
factorization of the 5 by 3 matrix A given by

                          (2.0  2.5  2.5)
                          (2.0  2.5  2.5)
                        A=(1.6 -0.4  2.8).
                          (2.0 -0.5  0.5)
                          (1.2 -0.3 -2.9)

The example program is not reproduced here. The source code for
all example programs is distributed with the NAG Foundation
Library software and should be available on-line.
```

%%%

```
F01 -- Matrix Factorizations                                    F01RCF
        F01RCF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for
your implementation to check implementation-dependent details.
The symbol (*) after a NAG routine name denotes a routine that is
not included in the Foundation Library.

1. Purpose

F01RCF finds the QR factorization of the complex m by n matrix A,
where m>=n.

2. Specification

       SUBROUTINE   F01RCF (M, N, A, LDA, THETA, IFAIL)
       INTEGER              M, N, LDA, IFAIL
       COMPLEX(KIND(1.0D0)) A(LDA,*), THETA(*)

3. Description

The m by n matrix A is factorized as

          (R)
       A=Q(0)       when m>n,

       A=QR         when m=n,
```

where Q is an m by m unitary matrix and R is an n by n upper triangular matrix with real diagonal elements.

The factorization is obtained by Householder's method. The kth transformation matrix, Q , which is used to introduce zeros into
k
the kth column of A is given in the form

$$Q_k = \begin{pmatrix} I & 0 \\ 0 & T_k \end{pmatrix},$$

where

$$T_k = I - (\text{gamma})_k \, u_k \, u_k^H,$$

$$u_k = \begin{pmatrix} (\text{zeta})_k \\ z_k \end{pmatrix},$$

(gamma) is a scalar for which Re (gamma) =1.0, (zeta) is a real
k k k
scalar and z is an (m-k) element vector. (gamma) , (zeta) and
k k k
z are chosen to annihilate the elements below the triangular
k
part of A and to make the diagonal elements real.

The scalar (gamma) and the vector u are returned in the kth
k k
element of the array THETA and in the kth column of A, such that
(theta) , given by
k

$$(\text{theta})_k = ((\text{zeta})_k, \text{Im}(\text{gamma})_k),$$

is in THETA(k) and the elements of z are in a ,...,a . The
k k+1,k m,k
elements of R are returned in the upper triangular part of A.

Q is given by

$$Q=(Q_n Q_{n-1} \ldots Q_1)^H .$$

A good background description to the QR factorization is given in Dongarra et al [1], but note that this routine is not based upon LINPACK routine ZQRDC.

4. References

[1] Dongarra J J, Moler C B, Bunch J R and Stewart G W (1979) LINPACK Users' Guide. SIAM, Philadelphia.

[2] Wilkinson J H (1965) The Algebraic Eigenvalue Problem. Oxford University Press.

5. Parameters

1: M -- INTEGER Input
On entry: m, the number of rows of A. Constraint: M >= N.

2: N -- INTEGER Input
On entry: n, the number of columns of A. Constraint: N >= 0.

When N = 0 then an immediate return is effected.

3: A(LDA,*) -- COMPLEX(KIND(1.0D0)) array Input/Output
Note: the second dimension of the array A must be at least max(1,N).
On entry: the leading m by n part of the array A must contain the matrix to be factorized. On exit: the n by n upper triangular part of A will contain the upper triangular matrix R, with the imaginary parts of the diagonal elements set to zero, and the m by n strictly lower triangular part of A will contain details of the factorization as described above.

4: LDA -- INTEGER Input
On entry:
the first dimension of the array A as declared in the (sub)program from which F01RCF is called.
Constraint: LDA >= max(1,M).

5: THETA(*) -- COMPLEX(KIND(1.0D)) array Output

Note: the dimension of the array THETA must be at least max(1,N).
On exit: the scalar (theta) for the kth transformation. If
k
T =I then THETA(k) = 0.0; if
k

((alpha) 0)
T =(0 I) Re(alpha)<0.0,
k

then THETA(k)=(alpha); otherwise THETA(k) contains THETA(k) as described in Section 3 and Re(THETA(k)) is always in the

range (1.0, \/2.0).

6: IFAIL -- INTEGER Input/Output
On entry: IFAIL must be set to 0, -1 or 1. For users not familiar with this parameter (described in the Essential Introduction) the recommended value is 0.

On exit: IFAIL = 0 unless the routine detects an error (see Section 6).

6. Error Indicators and Warnings

Errors detected by the routine:

If on entry IFAIL = 0 or -1, explanatory error messages are output on the current error message unit (as defined by X04AAF).

IFAIL=-1
On entry M < N,

or N < 0,

or LDA < M.

7. Accuracy
The computed factors Q and R satisfy the relation

(R)
Q(0)=A+E,

where

||E||<=c(epsilon)||A||,

```
(epsilon) being the machine precision, c is a modest function of
m and n and ||.|| denotes the spectral (two) norm.

8. Further Comments

The approximate number of real floating-point operations is given
     2
by 8n (3m-n)/3.

Following the use of this routine the operations

                                        H
                      B:=QB  and  B:=Q B,

where B is an m by k matrix, can be performed by calls to F01RDF.
The operation B:=QB can be obtained by the call:


      IFAIL = 0
      CALL F01RDF('No conjugate', 'Separate', M, N, A, LDA, THETA,
     *            K, B, LDB, WORK, IFAIL)

        H
and B:=Q B can be obtained by the call:


      IFAIL = 0
      CALL F01RDF('Conjugate', 'Separate', M, N, A, LDA, THETA,
     *            K, B, LDB, WORK, IFAIL)

In both cases WORK must be a k element array that is used as
workspace. If B is a one-dimensional array (single column) then
the parameter LDB can be replaced by M. See F01RDF for further
details.

The first k columns of the unitary matrix Q can either be
obtained by setting B to the first k columns of the unit matrix
and using the first of the above two calls, or by calling F01REF,
which overwrites the k columns of Q on the first k columns of the
array A. Q is obtained by the call:


      CALL F01REF('Separate', M, N, K, A, LDA, THETA, WORK, IFAIL)

As above, WORK must be a k element array. If k is larger than n,
```

then A must have been declared to have at least k columns.

Operations involving the matrix R can readily be performed by the Level 2 BLAS routines ZTRSV and ZTRMV (see Chapter F06), but note that no test for near singularity of R is incorporated in ZTRSV. If R is singular, or nearly singular, then F02XUF(*) can be used to determine the singular value decomposition of R.

9. Example

To obtain the QR factorization of the 5 by 3 matrix

$$A=\begin{pmatrix} 0.5i & -0.5+1.5i & -1.0+1.0i \\ 0.4+0.3i & 0.9+1.3i & 0.2+1.4i \\ 0.4 & -0.4+0.4i & 1.8 \\ 0.3-0.4i & 0.1+0.7i & 0.0 \\ -0.3i & 0.3+0.3i & 2.4i \end{pmatrix}.$$

The example program is not reproduced here. The source code for all example programs is distributed with the NAG Foundation Library software and should be available on-line.

%%%

F01 -- Matrix Factorizations F01RDF

F01RDF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for your implementation to check implementation-dependent details. The symbol (*) after a NAG routine name denotes a routine that is not included in the Foundation Library.

1. Purpose

F01RDF performs one of the transformations

$$B:=QB \text{ or } B:=Q^H B,$$

where B is an m by ncolb complex matrix and Q is an m by m unitary matrix, given as the product of Householder transformation matrices.

This routine is intended for use following F01RCF or F01RFF(*).

2. Specification

```
      SUBROUTINE F01RDF (TRANS, WHERET, M, N, A, LDA, THETA,
     1                   NCOLB, B, LDB, WORK, IFAIL)
      INTEGER              M, N, LDA, NCOLB, LDB, IFAIL
      COMPLEX(KIND(1.0D0)) A(LDA,*), THETA(*), B(LDB,*), WORK(*)
      CHARACTER*1          TRANS, WHERET
```

3. Description

The unitary matrix Q is assumed to be given by

$$Q=(Q_n Q_{n-1} \ldots Q_1)^H,$$

Q_k being given in the form

$$Q_k = \begin{pmatrix} I & 0 \\ 0 & T_k \end{pmatrix},$$

where

$$T_k = I-(\gamma)_k u_k u_k^H,$$

$$u_k = \begin{pmatrix} (\zeta)_k \\ z_k \end{pmatrix},$$

$(\gamma)_k$ is a scalar for which Re $(\gamma)_k$ =1.0, $(\zeta)_k$ is a real scalar and z_k is an (m-k) element vector.

z_k must be supplied in the kth column of A in elements $a_{k+1,k}, \ldots, a_{m,k}$ and $(\theta)_k$, given by

$$(\theta)_k = (\zeta)_k, \mathrm{Im}\,(\gamma)_k),$$

must be supplied either in a or in THETA(k), depending upon
k,k
the parameter WHERET.

To obtain Q explicitly B may be set to I and pre-multiplied by Q.
H
This is more efficient than obtaining Q . Alternatively, F01REF may be used to obtain Q overwritten on A.

4. References

[1] Wilkinson J H (1965) The Algebraic Eigenvalue Problem. Oxford University Press.

5. Parameters

1: TRANS -- CHARACTER*1 Input
On entry: the operation to be performed as follows:
TRANS = 'N' (No transpose)
Perform the operation B:=QB.

TRANS = 'C' (Conjugate transpose)
H
Perform the operation B:=Q B.
Constraint: TRANS must be one of 'N' or 'C'.

2: WHERET -- CHARACTER*1 Input
On entry: the elements of (theta) are to be found as follows:
WHERET = 'I' (In A)
The elements of (theta) are in A.

WHERET = 'S' (Separate)
The elements of (theta) are separate from A, in THETA.
Constraint: WHERET must be one of 'I' or 'S'.

3: M -- INTEGER Input
On entry: m, the number of rows of A. Constraint: M >= N.

4: N -- INTEGER Input
On entry: n, the number of columns of A. Constraint: N >= 0.

When N = 0 then an immediate return is effected.

5: A(LDA,*) -- COMPLEX(KIND(1.0D)) array Input

```
    Note: the second dimension of the array A must be at least
    max(1,N).
    On entry: the leading m by n strictly lower triangular part
    of the array A must contain details of the matrix Q. In
    addition, when WHERET = 'I', then the diagonal elements of A
    must contain the elements of (theta) as described under the
    argument THETA below.

    When WHERET = 'S', then the diagonal elements of the array A
    are referenced, since they are used temporarily to store the
    (zeta) , but they contain their original values on return.
          k

6:  LDA -- INTEGER                                          Input
    On entry:
    the first dimension of the array A as declared in the
    (sub)program from which F01RDF is called.
    Constraint: LDA >= max(1,M).

7:  THETA(*) -- COMPLEX(KIND(1.0D)) array                   Input
    Note: the dimension of the array THETA must be at least
    max(1,N).
    On entry: with WHERET = 'S', the array THETA must contain
    the elements of (theta). If THETA(k)=0.0 then T  is assumed
                                                   k
    to be I; if THETA(k)=(alpha), with Re(alpha)<0.0, then T  is
                                                            k
    assumed to be of the form
                              ((alpha) 0)
                         T =(0       I);
                           k
    otherwise THETA(k) is assumed to contain (theta)  given by
                                                  k
    (theta) =((zeta) ,Im(gamma) ).
           k        k          k

    When WHERET = 'I', the array THETA is not referenced, and
    may be dimensioned of length 1.

8:  NCOLB -- INTEGER                                        Input
    On entry: ncolb, the number of columns of B. Constraint:
    NCOLB >= 0.
    When NCOLB = 0 then an immediate return is effected.

9:  B(LDB,*) -- COMPLEX(KIND(1.0D)) array           Input/Output
    Note: the second dimension of the array B must be at least
```

```
     max(1,NCOLB).
     On entry: the leading m by ncolb part of the array B must
     contain the matrix to be transformed. On exit: B is
     overwritten by the transformed matrix.

10:  LDB -- INTEGER                                          Input
     On entry:
     the first dimension of the array B as declared in the
     (sub)program from which F01RDF is called.
     Constraint: LDB >= max(1,M).

11:  WORK(*) -- COMPLEX(KIND(1.0D)) array                 Workspace
     Note: the dimension of the array WORK must be at least
     max(1,NCOLB).

12:  IFAIL -- INTEGER                                    Input/Output
     On entry: IFAIL must be set to 0, -1 or 1. For users not
     familiar with this parameter (described in the Essential
     Introduction) the recommended value is 0.

     On exit: IFAIL = 0 unless the routine detects an error (see
     Section 6).

6. Error Indicators and Warnings

Errors detected by the routine:

If on entry IFAIL = 0 or -1, explanatory error messages are
output on the current error message unit (as defined by X04AAF).

IFAIL=-1
     On entry TRANS /= 'N' or 'C',

     or       WHERET /= 'I' or 'S',

     or       M < N,

     or       N < 0,

     or       LDA < M,

     or       NCOLB < 0,

     or       LDB < M.

7. Accuracy
```

Letting C denote the computed matrix Q B, C satisfies the relation

QC=B+E,

where

||E||<=c(epsilon)||B||,

(epsilon) being the machine precision, c is a modest function of m and |||.||| denotes the spectral (two) norm. An equivalent result holds for the computed matrix QB. See also Section 7 of F01RCF.

8. Further Comments

The approximate number of real floating-point operations is given by 8n(2m-n)ncolb.

9. Example

To obtain the matrix $Q^H B$ for the matrix B given by

$$B = \begin{pmatrix} (-0.55+1.05i & 0.45+1.05i) \\ (0.49+0.93i & 1.09+0.13i) \\ (0.56-0.16i & 0.64+0.16i) \\ (0.39+0.23i & -0.39-0.23i) \\ (1.13+0.83i & -1.13+0.77i) \end{pmatrix}$$

following the QR factorization of the 5 by 3 matrix A given by

$$A = \begin{pmatrix} (0.5i & -0.5+1.5i & -1.0+1.0i) \\ (0.4+0.3i & 0.9+1.3i & 0.2+1.4i) \\ (0.4 & -0.4+0.4i & 1.8) \\ (0.3-0.4i & 0.1+0.7i & 0.0) \\ (-0.3i & 0.3+0.3i & 2.4i) \end{pmatrix}.$$

The example program is not reproduced here. The source code for all example programs is distributed with the NAG Foundation Library software and should be available on-line.

%%%

F01REF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for your implementation to check implementation-dependent details. The symbol (*) after a NAG routine name denotes a routine that is not included in the Foundation Library.

1. Purpose

F01REF returns the first ncolq columns of the complex m by m unitary matrix Q, where Q is given as the product of Householder transformation matrices.

This routine is intended for use following F01RCF.

2. Specification

```
      SUBROUTINE F01REF (WHERET, M, N, NCOLQ, A, LDA, THETA,
     1                   WORK, IFAIL)
      INTEGER              M, N, NCOLQ, LDA, IFAIL
      COMPLEX(KIND(1.0D0)) A(LDA,*), THETA(*), WORK(*)
      CHARACTER*1          WHERET
```

3. Description

The unitary matrix Q is assumed to be given by

$$Q = (Q_n Q_{n-1} \ldots Q_1)^H,$$

Q_k being given in the form

$$Q_k = \begin{pmatrix} I & 0 \\ 0 & T_k \end{pmatrix},$$

where

$$T_k = I - (\text{gamma})_k u_k u_k^H,$$

$$\begin{pmatrix} (\text{zeta})_k \\ \end{pmatrix}$$

$$u_k = (z_k \quad),$$
$$(\quad)$$

(gamma) is a scalar for which Re (gamma) =1.0, (zeta) is a real
γ_k ζ_k k

scalar and z_k is an (m-k) element vector.

z_k must be supplied in the kth column of A in elements
$a_{k+1,k}, \ldots, a_{m,k}$ and θ_k, given by

$$\theta_k = (\zeta_k, \mathrm{Im}(\gamma_k)),$$

must be supplied either in $a_{k,k}$ or in THETA(k) depending upon the parameter WHERET.

4. References

[1] Wilkinson J H (1965) The Algebraic Eigenvalue Problem. Oxford University Press.

5. Parameters

1: WHERET -- CHARACTER*1 Input
On entry: the elements of (theta) are to be found as follows:
WHERET = 'I' (In A)
The elements of (theta) are in A.

WHERET = 'S' (Separate)
The elements of (theta) are separate from A, in THETA.
Constraint: WHERET must be one of 'I' or 'S'.

2: M -- INTEGER Input
On entry: m, the number of rows of A. Constraint: M >= N.

3: N -- INTEGER Input
On entry: n, the number of columns of A. Constraint: N >= 0.

4: NCOLQ -- INTEGER Input

On entry: ncolq, the required number of columns of Q.
Constraint: 0 <= NCOLQ <= M.

When NCOLQ = 0 then an immediate return is effected.

5: A(LDA,*) -- COMPLEX(KIND(1.0D)) array Input/Output
Note: the second dimension of the array A must be at least max(1,N,NCOLQ).
On entry: the leading m by n strictly lower triangular part of the array A must contain details of the matrix Q. In addition, when WHERET = 'I', then the diagonal elements of A must contain the elements of (theta) as described under the argument THETA below. On exit: the first NCOLQ columns of the array A are overwritten by the first NCOLQ columns of the m by m unitary matrix Q. When N = 0 then the first NCOLQ columns of A are overwritten by the first NCOLQ columns of the unit matrix.

6: LDA -- INTEGER Input
On entry:
the first dimension of the array A as declared in the (sub)program from which F01REF is called.
Constraint: LDA >= max(1,M).

7: THETA(*) -- COMPLEX(KIND(1.0D)) array Input
Note: the dimension of the array THETA must be at least max(1,N).
On entry: if WHERET = 'S', the array THETA must contain the elements of (theta). If THETA(k)=0.0 then T_k is assumed to be I; if THETA(k)=(alpha), with Re(alpha)<0.0, then T_k is assumed to be of the form

$$T_k = \begin{pmatrix} (\alpha) & 0 \\ 0 & I \end{pmatrix};$$

otherwise THETA(k) is assumed to contain $(\theta)_k$ given by $(\theta)_k = (\zeta_k, \mathrm{Im}(\gamma_k))$.

When WHERET = 'I', the array THETA is not referenced.

8: WORK(*) -- COMPLEX(KIND(1.0D)) array Workspace
Note: the dimension of the array WORK must be at least max(1,NCOLQ).

```
 9:  IFAIL -- INTEGER                                    Input/Output
     On entry: IFAIL must be set to 0, -1 or 1. For users not
     familiar with this parameter (described in the Essential
     Introduction) the recommended value is 0.

     On exit: IFAIL = 0 unless the routine detects an error (see
     Section 6).

6. Error Indicators and Warnings

Errors detected by the routine:

If on entry IFAIL = 0 or -1, explanatory error messages are
output on the current error message unit (as defined by X04AAF).

IFAIL=-1
     On entry WHERET /= 'I' or 'S',

     or       M < N,

     or       N < 0,

     or       NCOLQ < 0 or NCOLQ > M,

     or       LDA < M.

7. Accuracy

The computed matrix Q satisfies the relation

                              Q=P+E,

where P is an exactly unitary matrix and

                        ||E||<=c(epsilon),

(epsilon) being the machine precision, c is a modest function of
m and |||.||| denotes the spectral (two) norm. See also Section 7
of F01RCF.

8. Further Comments

The approximate number of real floating-point operations required
is given by
```

$$
\begin{cases}
\frac{8}{3}n\{(3m-n)(2ncolq-n)-n(ncolq-n)\}, & ncolq>n \\
\frac{8}{3}ncolq\,(3m-ncolq), & ncolq\leq n
\end{cases}
$$

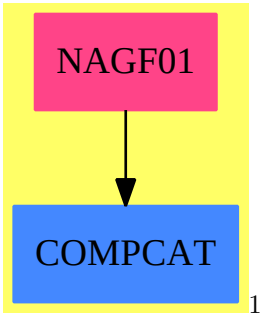
9. Example

To obtain the 5 by 5 unitary matrix Q following the QR factorization of the 5 by 3 matrix A given by

$$
A=\begin{pmatrix}
0.5i & -0.5+1.5i & -1.0+1.4i \\
0.4+0.3i & 0.9+1.3i & 0.2+1.4i \\
0.4 & -0.4+0.4i & 1.8 \\
0.3-0.4i & 0.1+0.7i & 0.0 \\
-0.3i & 0.3+0.3i & 2.4i
\end{pmatrix}.
$$

The example program is not reproduced here. The source code for all example programs is distributed with the NAG Foundation Library software and should be available on-line.

15.16 NagMatrixOperationsPackage



1.00

Exports:

f01brf	f01bsf	f01maf	f01mcf	f01qcf
f01qdf	f01qef	f01rcf	f01rdf	f01ref

⟨*package NAGF01 NagMatrixOperationsPackage*⟩≡

```
)abbrev package NAGF01 NagMatrixOperationsPackage
++ Author: Godfrey Nolan and Mike Dewar
++ Date Created: Jan 1994
++ Date Last Updated: Thu May 12 17:45:15 1994
++ Description:
++ This package uses the NAG Library to provide facilities for matrix
++ factorizations and associated transformations.
++ See \downlink{Manual Page}{manpageXXf01}.
NagMatrixOperationsPackage(): Exports == Implementation where
  S ==> Symbol
  FOP ==> FortranOutputStackPackage

  Exports ==> with
    f01brf : (Integer,Integer,Integer,Integer,_
        DoubleFloat,Boolean,Boolean,List Boolean,Matrix DoubleFloat,Matrix Integer,Matrix In
     ++ f01brf(n,nz,licn,lirn,pivot,lblock,grow,abort,a,irn,icn,ifail)
     ++ factorizes a real sparse matrix. The routine either forms
     ++ the LU factorization of a permutation of the entire matrix, or,
     ++ optionally, first permutes the matrix to block lower triangular
     ++ form and then only factorizes the diagonal blocks.
     ++ See \downlink{Manual Page}{manpageXXf01brf}.
    f01bsf : (Integer,Integer,Integer,Matrix Integer,_
        Matrix Integer,Matrix Integer,Matrix Integer,Boolean,DoubleFloat,Boolean,Matrix Inte
     ++ f01bsf(n,nz,licn,ivect,jvect,icn,ikeep,grow,eta,abort,idisp,avals,ifail)
     ++ factorizes a real sparse matrix using the pivotal sequence
     ++ previously obtained by F01BRF when a matrix of the same sparsity
     ++ pattern was factorized.
     ++ See \downlink{Manual Page}{manpageXXf01bsf}.
    f01maf : (Integer,Integer,Integer,Integer,_
```

```
    List Boolean,Matrix DoubleFloat,Matrix Integer,Matrix Integer,DoubleFloat
 ++ f01maf(n,nz,licn,lirn,abort,avals,irn,icn,droptl,densw,ifail)
 ++ computes an incomplete Cholesky factorization of a real
 ++ sparse symmetric positive-definite matrix A.
 ++ See \downlink{Manual Page}{manpageXXf01maf}.
f01mcf : (Integer,Matrix DoubleFloat,Integer,Matrix Integer,_
    Integer) -> Result
 ++ f01mcf(n,avals,lal,nrow,ifail)
 ++ computes the Cholesky factorization of a real symmetric
 ++ positive-definite variable-bandwidth matrix.
 ++ See \downlink{Manual Page}{manpageXXf01mcf}.
f01qcf : (Integer,Integer,Integer,Matrix DoubleFloat,_
    Integer) -> Result
 ++ f01qcf(m,n,lda,a,ifail)
 ++ finds the QR factorization of the real m by n matrix A,
 ++ where m>=n.
 ++ See \downlink{Manual Page}{manpageXXf01qcf}.
f01qdf : (String,String,Integer,Integer,_
    Matrix DoubleFloat,Integer,Matrix DoubleFloat,Integer,Integer,Matrix Doub
 ++ f01qdf(trans,wheret,m,n,a,lda,zeta,ncolb,ldb,b,ifail)
 ++ performs one of the transformations
 ++ See \downlink{Manual Page}{manpageXXf01qdf}.
f01qef : (String,Integer,Integer,Integer,_
    Integer,Matrix DoubleFloat,Matrix DoubleFloat,Integer) -> Result
 ++ f01qef(wheret,m,n,ncolq,lda,zeta,a,ifail)
 ++ returns the first ncolq columns of the real m by m
 ++ orthogonal matrix Q, where Q is given as the product of
 ++ Householder transformation matrices.
 ++ See \downlink{Manual Page}{manpageXXf01qef}.
f01rcf : (Integer,Integer,Integer,Matrix Complex DoubleFloat,_
    Integer) -> Result
 ++ f01rcf(m,n,lda,a,ifail)
 ++ finds the QR factorization of the complex m by n matrix A,
 ++ where m>=n.
 ++ See \downlink{Manual Page}{manpageXXf01rcf}.
f01rdf : (String,String,Integer,Integer,_
    Matrix Complex DoubleFloat,Integer,Matrix Complex DoubleFloat,Integer,Int
 ++ f01rdf(trans,wheret,m,n,a,lda,theta,ncolb,ldb,b,ifail)
 ++ performs one of the transformations
 ++ See \downlink{Manual Page}{manpageXXf01rdf}.
f01ref : (String,Integer,Integer,Integer,_
    Integer,Matrix Complex DoubleFloat,Matrix Complex DoubleFloat,Integer) ->
 ++ f01ref(wheret,m,n,ncolq,lda,theta,a,ifail)
 ++ returns the first ncolq columns of the complex m by m
 ++ unitary matrix Q, where Q is given as the product of Householder
 ++ transformation matrices.
```

```
    ++ See \downlink{Manual Page}{manpageXXf01ref}.
  Implementation ==> add

    import Lisp
    import DoubleFloat
    import Any
    import Record
    import Integer
    import Matrix DoubleFloat
    import Boolean
    import NAGLinkSupportPackage
    import AnyFunctions1(Integer)
    import AnyFunctions1(DoubleFloat)
    import AnyFunctions1(Boolean)
    import AnyFunctions1(String)
    import AnyFunctions1(List Boolean)
    import AnyFunctions1(Matrix DoubleFloat)
    import AnyFunctions1(Matrix Complex DoubleFloat)
    import AnyFunctions1(Matrix Integer)


    f01brf(nArg:Integer,nzArg:Integer,licnArg:Integer,_
        lirnArg:Integer,pivotArg:DoubleFloat,lblockArg:Boolean,_
        growArg:Boolean,abortArg:List Boolean,aArg:Matrix DoubleFloat,_
        irnArg:Matrix Integer,icnArg:Matrix Integer,ifailArg:Integer): Result ==
        [(invokeNagman(NIL$Lisp,_
        "f01brf",_
        ["n"::S,"nz"::S,"licn"::S,"lirn"::S,"pivot"::S_
        ,"lblock"::S,"grow"::S,"ifail"::S,"abort"::S,"ikeep"::S,"w"::S,"idisp"::S,"a"::S_
        ,"irn"::S,"icn"::S,"iw"::S]$Lisp,_
        ["ikeep"::S,"w"::S,"idisp"::S,"iw"::S]$Lisp,_
        [["double"::S,"pivot"::S,["w"::S,"n"::S]$Lisp_
        ,["a"::S,"licn"::S]$Lisp]$Lisp_
        ,["integer"::S,"n"::S,"nz"::S,"licn"::S,"lirn"::S_
        ,["ikeep"::S,["*"::S,5$Lisp,"n"::S]$Lisp]$Lisp,["idisp"::S,10$Lisp]$Lisp,["irn"::S,"
        ,"ifail"::S,["iw"::S,["*"::S,8$Lisp,"n"::S]$Lisp]$Lisp]$Lisp_
        ,["logical"::S,"lblock"::S,"grow"::S,["abort"::S,4$Lisp]$Lisp]$Lisp_
        ]$Lisp,_
        ["ikeep"::S,"w"::S,"idisp"::S,"a"::S,"irn"::S,"icn"::S,"ifail"::S]$Lisp,_
        [([nArg::Any,nzArg::Any,licnArg::Any,lirnArg::Any,pivotArg::Any,lblockArg::Any,growA
        @List Any]$Lisp)$Lisp)_
        pretend List (Record(key:Symbol,entry:Any))]$Result

    f01bsf(nArg:Integer,nzArg:Integer,licnArg:Integer,_
        ivectArg:Matrix Integer,jvectArg:Matrix Integer,icnArg:Matrix Integer,_
        ikeepArg:Matrix Integer,growArg:Boolean,etaArg:DoubleFloat,_
```

```
        abortArg:Boolean,idispArg:Matrix Integer,avalsArg:Matrix DoubleFloat,_
        ifailArg:Integer): Result ==
        [(invokeNagman(NIL$Lisp,_
        "f01bsf",_
        ["n"::S,"nz"::S,"licn"::S,"grow"::S,"eta"::S_
        ,"abort"::S,"rpmin"::S,"ifail"::S,"ivect"::S,"jvect"::S,"icn"::S,"ikeep":
        ,"w"::S,"avals"::S,"iw"::S]$Lisp,_
        ["w"::S,"rpmin"::S,"iw"::S]$Lisp,_
        [["double"::S,"eta"::S,["w"::S,"n"::S]$Lisp_
        ,"rpmin"::S,["avals"::S,"licn"::S]$Lisp]$Lisp_
        ,["integer"::S,"n"::S,"nz"::S,"licn"::S,["ivect"::S,"nz"::S]$Lisp_
        ,["jvect"::S,"nz"::S]$Lisp,["icn"::S,"licn"::S]$Lisp,["ikeep"::S,["*"::S,
        ,["idisp"::S,2$Lisp]$Lisp,"ifail"::S,["iw"::S,["*"::S,8$Lisp,"n"::S]$Lisp
        ,["logical"::S,"grow"::S,"abort"::S]$Lisp_
        ]$Lisp,_
        ["w"::S,"rpmin"::S,"avals"::S,"ifail"::S]$Lisp,_
        [([nArg::Any,nzArg::Any,licnArg::Any,growArg::Any,etaArg::Any,abortArg::A
        @List Any]$Lisp)$Lisp)_
        pretend List (Record(key:Symbol,entry:Any))]$Result

    f01maf(nArg:Integer,nzArg:Integer,licnArg:Integer,_
        lirnArg:Integer,abortArg:List Boolean,avalsArg:Matrix DoubleFloat,_
        irnArg:Matrix Integer,icnArg:Matrix Integer,droptlArg:DoubleFloat,_
        denswArg:DoubleFloat,ifailArg:Integer): Result ==
        [(invokeNagman(NIL$Lisp,_
        "f01maf",_
        ["n"::S,"nz"::S,"licn"::S,"lirn"::S,"droptl"::S_
        ,"densw"::S,"ifail"::S,"abort"::S,"wkeep"::S,"ikeep"::S,"inform"::S,"aval
        ,"irn"::S,"icn"::S,"iwork"::S]$Lisp,_
        ["wkeep"::S,"ikeep"::S,"inform"::S,"iwork"::S]$Lisp,_
        [["double"::S,["wkeep"::S,["*"::S,3$Lisp,"n"::S]$Lisp]$Lisp_
        ,["avals"::S,"licn"::S]$Lisp,"droptl"::S,"densw"::S]$Lisp_
        ,["integer"::S,"n"::S,"nz"::S,"licn"::S,"lirn"::S_
        ,["ikeep"::S,["*"::S,2$Lisp,"n"::S]$Lisp]$Lisp,["inform"::S,4$Lisp]$Lisp,
        ,"ifail"::S,["iwork"::S,["*"::S,6$Lisp,"n"::S]$Lisp]$Lisp]$Lisp_
        ,["logical"::S,["abort"::S,3$Lisp]$Lisp]$Lisp_
        ]$Lisp,_
        ["wkeep"::S,"ikeep"::S,"inform"::S,"avals"::S,"irn"::S,"icn"::S,"droptl":
        [([nArg::Any,nzArg::Any,licnArg::Any,lirnArg::Any,droptlArg::Any,denswArg
        @List Any]$Lisp)$Lisp)_
        pretend List (Record(key:Symbol,entry:Any))]$Result

    f01mcf(nArg:Integer,avalsArg:Matrix DoubleFloat,lalArg:Integer,_
        nrowArg:Matrix Integer,ifailArg:Integer): Result ==
        [(invokeNagman(NIL$Lisp,_
        "f01mcf",_
```

```
        ["n"::S,"lal"::S,"ifail"::S,"avals"::S,"nrow"::S,"al"::S,"d"::S]$Lisp,_
        ["al"::S,"d"::S]$Lisp,_
        [["double"::S,["avals"::S,"lal"::S]$Lisp,["al"::S,"lal"::S]$Lisp_
        ,["d"::S,"n"::S]$Lisp]$Lisp_
        ,["integer"::S,"n"::S,"lal"::S,["nrow"::S,"n"::S]$Lisp_
        ,"ifail"::S]$Lisp_
        ]$Lisp,_
        ["al"::S,"d"::S,"ifail"::S]$Lisp,_
        [([nArg::Any,lalArg::Any,ifailArg::Any,avalsArg::Any,nrowArg::Any ])_
        @List Any]$Lisp)$Lisp)_
        pretend List (Record(key:Symbol,entry:Any))]$Result

    f01qcf(mArg:Integer,nArg:Integer,ldaArg:Integer,_
        aArg:Matrix DoubleFloat,ifailArg:Integer): Result ==
        [(invokeNagman(NIL$Lisp,_
        "f01qcf",_
        ["m"::S,"n"::S,"lda"::S,"ifail"::S,"zeta"::S,"a"::S]$Lisp,_
        ["zeta"::S]$Lisp,_
        [["double"::S,["zeta"::S,"n"::S]$Lisp,["a"::S,"lda"::S,"n"::S]$Lisp_
        ]$Lisp_
        ,["integer"::S,"m"::S,"n"::S,"lda"::S,"ifail"::S_
        ]$Lisp_
        ]$Lisp,_
        ["zeta"::S,"a"::S,"ifail"::S]$Lisp,_
        [([mArg::Any,nArg::Any,ldaArg::Any,ifailArg::Any,aArg::Any ])_
        @List Any]$Lisp)$Lisp)_
        pretend List (Record(key:Symbol,entry:Any))]$Result

    f01qdf(transArg:String,wheretArg:String,mArg:Integer,_
        nArg:Integer,aArg:Matrix DoubleFloat,ldaArg:Integer,_
        zetaArg:Matrix DoubleFloat,ncolbArg:Integer,ldbArg:Integer,_
        bArg:Matrix DoubleFloat,ifailArg:Integer): Result ==
        [(invokeNagman(NIL$Lisp,_
        "f01qdf",_
        ["trans"::S,"wheret"::S,"m"::S,"n"::S,"lda"::S_
        ,"ncolb"::S,"ldb"::S,"ifail"::S,"a"::S,"zeta"::S,"b"::S,"work"::S]$Lisp,_
        ["work"::S]$Lisp,_
        [["double"::S,["a"::S,"lda"::S,"n"::S]$Lisp_
        ,["zeta"::S,"n"::S]$Lisp,["b"::S,"ldb"::S,"ncolb"::S]$Lisp,["work"::S,"ncolb"::S]$Li
        ,["integer"::S,"m"::S,"n"::S,"lda"::S,"ncolb"::S_
        ,"ldb"::S,"ifail"::S]$Lisp_
        ,["character"::S,"trans"::S,"wheret"::S]$Lisp_
        ]$Lisp,_
        ["b"::S,"ifail"::S]$Lisp,_
        [([transArg::Any,wheretArg::Any,mArg::Any,nArg::Any,ldaArg::Any,ncolbArg::Any,ldbArg
        @List Any]$Lisp)$Lisp)_
```

```
    pretend List (Record(key:Symbol,entry:Any))]$Result

f01qef(wheretArg:String,mArg:Integer,nArg:Integer,_
    ncolqArg:Integer,ldaArg:Integer,zetaArg:Matrix DoubleFloat,_
    aArg:Matrix DoubleFloat,ifailArg:Integer): Result ==
    [(invokeNagman(NIL$Lisp,_
    "f01qef",_
    ["wheret"::S,"m"::S,"n"::S,"ncolq"::S,"lda"::S_
    ,"ifail"::S,"zeta"::S,"a"::S,"work"::S]$Lisp,_
    ["work"::S]$Lisp,_
    [["double"::S,["zeta"::S,"n"::S]$Lisp,["a"::S,"lda"::S,"ncolq"::S]$Lisp_
    ,["work"::S,"ncolq"::S]$Lisp]$Lisp_
    ,["integer"::S,"m"::S,"n"::S,"ncolq"::S,"lda"::S_
    ,"ifail"::S]$Lisp_
    ,["character"::S,"wheret"::S]$Lisp_
    ]$Lisp,_
    ["a"::S,"ifail"::S]$Lisp,_
    [([wheretArg::Any,mArg::Any,nArg::Any,ncolqArg::Any,ldaArg::Any,ifailArg:
    @List Any]$Lisp)$Lisp)_
    pretend List (Record(key:Symbol,entry:Any))]$Result

f01rcf(mArg:Integer,nArg:Integer,ldaArg:Integer,_
    aArg:Matrix Complex DoubleFloat,ifailArg:Integer): Result ==
    [(invokeNagman(NIL$Lisp,_
    "f01rcf",_
    ["m"::S,"n"::S,"lda"::S,"ifail"::S,"theta"::S,"a"::S]$Lisp,_
    ["theta"::S]$Lisp,_
    [["integer"::S,"m"::S,"n"::S,"lda"::S,"ifail"::S_
    ]$Lisp_
    ,["double complex"::S,["theta"::S,"n"::S]$Lisp,["a"::S,"lda"::S,"n"::S]$L
    ]$Lisp,_
    ["theta"::S,"a"::S,"ifail"::S]$Lisp,_
    [([mArg::Any,nArg::Any,ldaArg::Any,ifailArg::Any,aArg::Any ])_
    @List Any]$Lisp)$Lisp)_
    pretend List (Record(key:Symbol,entry:Any))]$Result

f01rdf(transArg:String,wheretArg:String,mArg:Integer,_
    nArg:Integer,aArg:Matrix Complex DoubleFloat,ldaArg:Integer,_
    thetaArg:Matrix Complex DoubleFloat,ncolbArg:Integer,ldbArg:Integer,_
    bArg:Matrix Complex DoubleFloat,ifailArg:Integer): Result ==
    [(invokeNagman(NIL$Lisp,_
    "f01rdf",_
    ["trans"::S,"wheret"::S,"m"::S,"n"::S,"lda"::S_
    ,"ncolb"::S,"ldb"::S,"ifail"::S,"a"::S,"theta"::S,"b"::S,"work"::S]$Lisp,
    ["work"::S]$Lisp,_
    [["integer"::S,"m"::S,"n"::S,"lda"::S,"ncolb"::S_
```

```
        ,"ldb"::S,"ifail"::S]$Lisp_
        ,["character"::S,"trans"::S,"wheret"::S]$Lisp_
        ,["double complex"::S,["a"::S,"lda"::S,"n"::S]$Lisp,["theta"::S,"n"::S]$Lisp,["b"::S
        ]$Lisp,_
        ["b"::S,"ifail"::S]$Lisp,_
        [([transArg::Any,wheretArg::Any,mArg::Any,nArg::Any,ldaArg::Any,ncolbArg::Any,ldbArg
        @List Any]$Lisp)$Lisp)_
        pretend List (Record(key:Symbol,entry:Any))]$Result

    f01ref(wheretArg:String,mArg:Integer,nArg:Integer,_
        ncolqArg:Integer,ldaArg:Integer,thetaArg:Matrix Complex DoubleFloat,_
        aArg:Matrix Complex DoubleFloat,ifailArg:Integer): Result ==
        [(invokeNagman(NIL$Lisp,_
        "f01ref",_
        ["wheret"::S,"m"::S,"n"::S,"ncolq"::S,"lda"::S_
        ,"ifail"::S,"theta"::S,"a"::S,"work"::S]$Lisp,_
        ["work"::S]$Lisp,_
        [["integer"::S,"m"::S,"n"::S,"ncolq"::S,"lda"::S_
        ,"ifail"::S]$Lisp_
        ,["character"::S,"wheret"::S]$Lisp_
        ,["double complex"::S,["theta"::S,"n"::S]$Lisp,["a"::S,"lda"::S,"n"::S]$Lisp,["work"
        ]$Lisp,_
        ["a"::S,"ifail"::S]$Lisp,_
        [([wheretArg::Any,mArg::Any,nArg::Any,ncolqArg::Any,ldaArg::Any,ifailArg::Any,thetaA
        @List Any]$Lisp)$Lisp)_
        pretend List (Record(key:Symbol,entry:Any))]$Result
```

⟨*NAGF01.dotabb*⟩≡

```
"NAGF01" [color="#FF4488",href="bookvol10.4.pdf#nameddest=NAGF01"]
"COMPCAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=COMPCAT"]
"NAGF01" -> "COMPCAT"
```

15.17 package NAGE04 NagOptimisationPackage

⟨*NagOptimisationPackage.help*⟩≡

```
E04 -- Minimizing or Maximizing a Function    Introduction -- E04
                          Chapter E04
              Minimizing or Maximizing a Function

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2.3.4. Nonlinearly-constrained minimization
```

1. Scope of the Chapter

An optimization problem involves minimizing a function (called the objective function) of several variables, possibly subject to restrictions on the values of the variables defined by a set of constraint functions. The routines in the NAG Foundation Library

are concerned with function minimization only, since the problem of maximizing a given function can be transformed into a minimization problem simply by multiplying the function by -1.

This introduction is only a brief guide to the subject of optimization designed for the casual user. Anyone with a difficult or protracted problem to solve will find it beneficial to consult a more detailed text, such as Gill et al [5] or Fletcher [3].

Readers who are unfamiliar with the mathematics of the subject may find some sections difficult at first reading; if so, they should concentrate on Sections 2.1, 2.2, 2.5, 2.6 and 3.

2. Background to the Problems

2.1. Types of Optimization Problems

Solution of optimization problems by a single, all-purpose, method is cumbersome and inefficient. Optimization problems are therefore classified into particular categories, where each category is defined by the properties of the objective and constraint functions, as illustrated by some examples below.

Properties of Objective Function	Properties of Constraints
Nonlinear	Nonlinear
Sums of squares of nonlinear functions	Sparse linear
Quadratic	Linear
Sums of squares of linear functions	Bounds
Linear	None

For instance, a specific problem category involves the minimization of a nonlinear objective function subject to bounds on the variables. In the following sections we define the particular categories of problems that can be solved by routines contained in this Chapter.

2.1.1. Unconstrained minimization

In unconstrained minimization problems there are no constraints on the variables. The problem can be stated mathematically as follows:

$$\underset{x}{\text{minimize}}\ F(x)$$

where x is in R^n, that is, $x=(x_1,x_2,\ldots,x_n)^T$.

2.1.2. Nonlinear least-squares problems

Special consideration is given to the problem for which the function to be minimized can be expressed as a sum of squared functions. The least-squares problem can be stated mathematically as follows:

$$\underset{x}{\text{minimize}}\ \left\{f : f=\frac{1}{2}\sum_{i=1}^{m} f_i(x)^2\right\},\quad x \text{ is in } R^n$$

where the ith element of the m-vector f is the function $f_i(x)$.

2.1.3. Minimization subject to bounds on the variables

These problems differ from the unconstrained problem in that at least one of the variables is subject to a simple restriction on its value, e.g.$x_5 \le 10$, but no constraints of a more general form are present.

The problem can be stated mathematically as follows:

$$\underset{x}{\text{minimize}}\ F(x),\quad x \text{ is in } R^n$$

subject to $l_i \le x_i \le u_i$, $i=1,2,\ldots,n$.

This format assumes that upper and lower bounds exist on all the

variables. By conceptually allowing u =infty and l =-infty all
                                     i   i
the variables need not be restricted.

2.1.4. Minimization subject to linear constraints

A general linear constraint is defined as a constraint function
that is linear in more than one of the variables, e.g. 3x +2x >=4
                                                       1   2
The various types of linear constraint are reflected in the
following mathematical statement of the problem:

$$\underset{x}{\text{minimize}}\ F(x),\quad x \text{ is in } R^n$$

subject to the

equality constraints:	$a_i^T x = b_i$	$i=1,2,\ldots,m_1$;
inequality constraints:	$a_i^T x \geq b_i$	$i=m_1+1,m_1+2,\ldots,m_2$;
	$a_i^T x \leq b_i$	$i=m_2+1,m_2+2,\ldots,m_3$;
range constraints:	$s_j \leq a_i^T x \leq t_j$	$i=m_3+1,m_3+2,\ldots,m_4$; $j=1,2,\ldots,m_4-m_3$;
bounds constraints:	$l_i \leq x_i \leq u_i$	$i=1,2,\ldots,n$

where each a_i is a vector of length n; b_i, s_j and t_j are constant
scalars; and any of the categories may be empty.

Although the bounds on x_i could be included in the definition of

general linear constraints, we prefer to distinguish between them for reasons of computational efficiency.

If F(x) is a linear function, the linearly-constrained problem is termed a linear programming problem (LP problem); if F(x) is a quadratic function, the problem is termed a quadratic programming problem (QP problem). For further discussion of LP and QP problems, including the dual formulation of such problems, see Dantzig [2].

2.1.5. Minimization subject to nonlinear constraints

A problem is included in this category if at least one constraint
                                2
function is nonlinear, e.g. x +x +x -2>=0. The mathematical
                               1 3 4
statement of the problem is identical to that for the linearly-constrained case, except for the addition of the following constraints:

equality constraints:	$c_i(x)=0$	$i=1,2,\ldots,m_5$;
inequality constraints:	$c_i(x)\geq 0$	$i=m_5+1,m_5+2,\ldots,m_6$;
range constraints:	$v_j \leq c_i(x) \leq w_j$	$i=m_6+1,m_6+2,\ldots,m_7$, $j=1,2,\ldots,m_7-m_6$

where each c_i is a nonlinear function; v_j and w_j are constant scalars; and any category may be empty. Note that we do not include a separate category for constraints of the form $c_i(x)\leq 0$, since this is equivalent to $-c_i(x)\geq 0$.

2.2. Geometric Representation and Terminology

To illustrate the nature of optimization problems it is useful to consider the following example in two dimensions

                        x
                         1   2   2

$$F(x)=e^{(4x_1^2+2x_2^2+4x_1x_2+2x_2+1)}.$$

(This function is used as the example function in the documentation for the unconstrained routines.)

Figure 1
Please see figure in printed Reference Manual

Figure 1 is a contour diagram of F(x). The contours labelled $F_0, F_1, \ldots, F_4$ are isovalue contours, or lines along which the function F(x) takes specific constant values. The point x^* is a local unconstrained minimum, that is, the value of $F(x^*)$ is less than at all the neighbouring points. A function may have several such minima. The lowest of the local minima is termed a global minimum. In the problem illustrated in Figure 1, x^* is the only local minimum. The point x is said to be a saddle point because it is a minimum along the line AB, but a maximum along CD.

If we add the constraint $x_1 \geq 0$ to the problem of minimizing F(x), the solution remains unaltered. In Figure 1 this constraint is represented by the straight line passing through $x_1 = 0$, and the shading on the line indicates the unacceptable region. The region in R^n satisfying the constraints of an optimization problem is termed the feasible region. A point satisfying the constraints is defined as a feasible point.

If we add the nonlinear constraint $x_1^2 + x_2^2 - x_1 x_2 - 1.5 \geq 0$, represented by the curved shaded line in Figure 1, then x^* is not a feasible point. The solution of the new constrained problem is $\hat{x}$, the feasible point with the smallest function value.

2.2.1. Gradient vector

The vector of first partial derivatives of F(x) is called the gradient vector, and is denoted by g(x), i.e.,

$$g(x)=\left[\frac{dF(x)}{dx_1}, \frac{dF(x)}{dx_2}, \ldots, \frac{dF(x)}{dx_n}\right]^T .$$

For the function illustrated in Figure 1,

$$g(x)=\begin{bmatrix} x_1 \\ F(x)+e^{x_1}(8x_1+4x_2) \\ x_1 \\ e^{x_1}(4x_1^2+4x_2+2) \end{bmatrix}.$$

The gradient vector is of importance in optimization because it must be zero at an unconstrained minimum of any function with continuous first derivatives.

2.2.2. Hessian matrix

The matrix of second partial derivatives of a function is termed its Hessian matrix. The Hessian matrix of F(x) is denoted by G(x) and its (i,j)th element is given by $\frac{d^2F(x)}{dx_i dx_j}$. If F(x) has continuous second derivatives, then G(x) must be positive semi-definite at any unconstrained minimum of F.

2.2.3. Jacobian matrix; matrix of constraint normals

In nonlinear least-squares problems, the matrix of first partial derivatives of the vector-valued function f(x) is termed the Jacobian matrix of f(x) and its (i,j)th component is $\frac{df_i}{dx_j}$.

The vector of first partial derivatives of the constraint $c_i(x)$ is denoted by

$$\left[\frac{dc_i(x)}{dx_1} \quad \frac{dc_i(x)}{dx_2}\right]^T$$

$$a_i(x) = \left[\frac{\partial^i}{\partial x_1^i}, \ldots, \frac{\partial^i}{\partial x_n^i} \right] .$$

At a point, x, the vector $\hat{a}_i(x)$ is orthogonal (normal) to the isovalue contour of $\hat{c}_i(x)$ passing through x; this relationship is illustrated for a two-dimensional function in Figure 2.

Figure 2
Please see figure in printed Reference Manual

The matrix whose columns are the vectors $\{a_i\}$ is termed the matrix of constraint normals. Note that if $c_i(x)$ is a linear constraint involving $a^T x$, then its vector of first partial derivatives is simply the vector a_i.

2.3. Sufficient Conditions for a Solution

All nonlinear functions will be assumed to have continuous second derivatives in the neighbourhood of the solution.

2.3.1. Unconstrained minimization

The following conditions are sufficient for the point x^* to be an unconstrained local minimum of $F(x)$:

(i) $|||g(x^*)|||=0$; and

(ii) $G(x^*)$ is positive-definite,

where $|||g|||$ denotes the Euclidean length of g.

2.3.2. Minimization subject to bounds on the variables

At the solution of a bounds-constrained problem, variables which are not on their bounds are termed free variables. If it is known in advance which variables are on their bounds at the solution, the problem can be solved as an unconstrained problem in just the free variables; thus, the sufficient conditions for a solution are similar to those for the unconstrained case, applied only to the free variables.

Sufficient conditions for a feasible point x^* to be the solution of a bound-constrained problem are as follows:

(i) $|||g(x^*)|||=0$; and

(ii) $G(x^*)$ is positive-definite; and

(iii) $g_j(x^*)<0, x_j=u_j$; $g_j(x^*)>0, x_j=l_j$,

where g(x) is the gradient of F(x) with respect to the free

variables, and G(x) is the Hessian matrix of F(x) with respect to the free variables. The extra condition (iii) ensures that F(x) cannot be reduced by moving off one or more of the bounds.

2.3.3. Linearly-constrained minimization

For the sake of simplicity, the following description does not include a specific treatment of bounds or range constraints, since the results for general linear inequality constraints can be applied directly to these cases.

At a solution x^*, of a linearly-constrained problem, the constraints which hold as equalities are called the active or binding constraints. Assume that there are t active constraints at the solution x^*, and let $\hat{A}$ denote the matrix whose columns are

the columns of A corresponding to the active constraints, with b
the vector similarly obtained from b; then

$$\hat{A}^T * \hat{x} = \hat{b}.$$

The matrix Z is defined as an n by (n-t) matrix satisfying:

$$\hat{A}^T Z = 0; \quad Z^T Z = I.$$

The columns of Z form an orthogonal basis for the set of vectors
orthogonal to the columns of A.

Define

$g_z(x) = Z^T g(x)$, the projected gradient vector of F(x);

$G_z(x) = Z^T G(x) Z$, the projected Hessian matrix of F(x).

At the solution of a linearly-constrained problem, the projected
gradient vector must be zero, which implies that the gradient
vector $g(x^*)$ can be written as a linear combination of the
columns of $\hat{A}$, i.e., $g(x^*) = \sum_{i=1}^{t} \lambda_i \hat{a}_i = \hat{A}\lambda$. The scalar
λ_i is defined as the Lagrange multiplier corresponding to
the ith active constraint. A simple interpretation of the ith
Lagrange multiplier is that it gives the gradient of F(x) along
the ith active constraint normal; a convenient definition of the
Lagrange multiplier vector (although not a recommended method for
computation) is:

$$\lambda = (\hat{A}^T \hat{A})^{-1} \hat{A}^T g(x^*).$$

```
                          *
Sufficient conditions for x  to be the solution of a linearly-
constrained problem are:

       *                 ^T * ^
(i)   x  is feasible, and A x =b; and

             *                              *  ^
(ii)  |||g (x )|||=0, or equivalently, g(x )=A(lambda); and
          z

          *
(iii) G (x ) is positive-definite; and
       z

(iv)  (lambda) >0 if (lambda)  corresponds to a constraint
               i              i
      ^T *  ^
      a x >=b ;
       i     i

     (lambda) <0 if (lambda)  corresponds to a constraint
              i              i
     ^T *  ^
     a x <=b .
      i     i

     The sign of (lambda)  is immaterial for equality
                         i
     constraints, which by definition are always active.
```

2.3.4. Nonlinearly-constrained minimization

For nonlinearly-constrained problems, much of the terminology is defined exactly as in the linearly-constrained case. The set of active constraints at x again means the set of constraints that hold as equalities at x, with corresponding definitions of $\hat{c}$ and $\hat{A}$: the vector $\hat{c}(x)$ contains the active constraint functions, and the columns of $\hat{A}(x)$ are the gradient vectors of the active constraints. As before, Z is defined in terms of $\hat{A}(x)$ as a matrix such that:

$$\hat{A}^T Z = 0; \quad Z^T Z = I$$

where the dependence on x has been suppressed for compactness.

The projected gradient vector $g_z(x)$ is the vector $Z^T g(x)$. At the solution x^* of a nonlinearly-constrained problem, the projected gradient must be zero, which implies the existence of Lagrange multipliers corresponding to the active constraints, i.e., $g(x^*) = \hat{A}(x^*)(\lambda)$.

The Lagrangian function is given by:

$$L(x,(\lambda)) = F(x) - (\lambda)^{\hat{T}} c(x).$$

We define $g_L(x)$ as the gradient of the Lagrangian function; $G_L(x)$ as its Hessian matrix, and $\hat{G}_L(x)$ as its projected Hessian matrix, i.e., $\hat{G}_L = Z^T G_L Z$.

Sufficient conditions for x^* to be a solution of nonlinearly-constrained problem are:

(i) x^* is feasible, and $c(x^*) = 0$; and

(ii) $|||g_z(x^*)||| = 0$, or, equivalently, $g(x^*) = \hat{A}(x^*)(\lambda)$; and

(iii) $\hat{G}_L(x^*)$ is positive-definite; and

(iv) (lambda) >0 if (lambda) corresponds to a constraint of the
          i   i
       ^
     form c >=0; the sign of (lambda) is immaterial for an
           i   i
     equality constraint.

Note that condition (ii) implies that the projected gradient of
                                                *
the Lagrangian function must also be zero at x , since the
               T   ^ *
application of Z annihilates the matrix A(x).

2.4. Background to Optimization Methods

All the algorithms contained in this Chapter generate an
                     (k)   *
iterative sequence {x } that converges to the solution x in
the limit, except for some special problem categories (i.e.,
linear and quadratic programming). To terminate computation of
the sequence, a convergence test is performed to determine
whether the current estimate of the solution is an adequate
approximation. The convergence tests are discussed in Section 2.6

                                          (k)
Most of the methods construct a sequence {x } satisfying:

                   (k+1) (k)   (k) (k)
                  x   =x   +(alpha)   p   ,

                 (k)
where the vector p   is termed the direction of search, and
       (k)   (k)
(alpha)   is the steplength. The steplength (alpha)   is chosen
          (k+1)   (k)
so that F(x   )<F(x   ).

2.4.1. Methods for unconstrained optimization

The distinctions among methods arise primarily from the need to
use varying levels of information about derivatives of F(x) in
defining the search direction. We describe three basic approaches
to unconstrained problems, which may be extended to other problem
categories. Since a full description of the methods would fill
several volumes, the discussion here can do little more than

allude to the processes involved, and direct the reader to other sources for a full explanation.

(a) Newton-type Methods (Modified Newton Methods)

Newton-type methods use the Hessian matrix $G(x^{(k)})$, or a finite difference approximation to $G(x^{(k)})$, to define the search direction. The routines in the Library either require a subroutine that computes the elements of $G(x^{(k)})$, or they approximate $G(x^{(k)})$ by finite differences.

Newton-type methods are the most powerful methods available for general problems and will find the minimum of a quadratic function in one iteration. See Sections 4.4. and 4.5.1. of Gill et al [5].

(b) Quasi-Newton Methods

Quasi-Newton methods approximate the Hessian $G(x^{(k)})$ by a matrix $B^{(k)}$ which is modified at each iteration to include information obtained about the curvature of F along the latest search direction. Although not as robust as Newton-type methods, quasi-Newton methods can be more efficient because $G(x^{(k)})$ is not computed, or approximated by finite-differences. Quasi-Newton methods minimize a quadratic function in n iterations. See Section 4.5.2 of Gill et al [5].

(c) Conjugate-Gradient Methods

Unlike Newton-type and quasi-Newton methods, conjugate gradient methods do not require the storage of an n by n matrix and so are ideally suited to solve large problems. Conjugate-gradient type methods are not usually as reliable or efficient as Newton-type, or quasi-Newton methods. See Section 4.8.3 of Gill et al [5].

2.4.2. Methods for nonlinear least-squares problems

These methods are similar to those for unconstrained optimization, but exploit the special structure of the Hessian matrix to give improved computational efficiency.

Since

$$F(x)= \sum_{i=1}^{m} f_i^2(x)$$

the Hessian matrix G(x) is of the form

$$G(x)=2[J(x)^T J(x)+ \sum_{i=1}^{m} f_i(x)G_i(x)],$$

where J(x) is the Jacobian matrix of f(x), and $G_i(x)$ is the Hessian matrix of $f_i(x)$.

In the neighbourhood of the solution, $|||f(x)|||$ is often small compared to $|||J(x)^T J(x)|||$ (for example, when f(x) represents the goodness of fit of a nonlinear model to observed data). In such cases, $2J(x)^T J(x)$ may be an adequate approximation to G(x), thereby avoiding the need to compute or approximate second derivatives of $\{f_i(x)\}$. See Section 4.7 of Gill et al [5].

2.4.3. Methods for handling constraints

Bounds on the variables are dealt with by fixing some of the variables on their bounds and adjusting the remaining free variables to minimize the function. By examining estimates of the Lagrange multipliers it is possible to adjust the set of variables fixed on their bounds so that eventually the bounds active at the solution should be correctly identified. This type of method is called an active set method. One feature of such methods is that, given an initial feasible point, all

approximations $x^{(k)}$ are feasible. This approach can be extended to general linear constraints. At a point, x, the set of constraints which hold as equalities being used to predict, or approximate, the set of active constraints is called the working set.

Nonlinear constraints are more difficult to handle. If at all possible, it is usually beneficial to avoid including nonlinear constraints during the formulation of the problem. The methods currently implemented in the Library handle nonlinearly constrained problems either by transforming them into a sequence of bound constraint problems, or by transforming them into a sequence of quadratic programming problems. A feature of almost all methods for nonlinear constraints is that $x^{(k)}$ is not guaranteed to be feasible except in the limit, and this is certainly true of the routines currently in the Library. See Chapter 6, particularly Section 6.4 and Section 6.5 of Gill et al [5].

Anyone interested in a detailed description of methods for optimization should consult the references.

2.5. Scaling

Scaling (in a broadly defined sense) often has a significant influence on the performance of optimization methods. Since convergence tolerances and other criteria are necessarily based on an implicit definition of 'small' and 'large', problems with unusual or unbalanced scaling may cause difficulties for some algorithms. Nonetheless, there are currently no scaling routines in the Library, although the position is under constant review. In light of the present state of the art, it is considered that sensible scaling by the user is likely to be more effective than any automatic routine. The following sections present some general comments on problem scaling.

2.5.1. Transformation of variables

One method of scaling is to transform the variables from their original representation, which may reflect the physical nature of the problem, to variables that have certain desirable properties in terms of optimization. It is generally helpful for the following conditions to be satisfied:

(i) the variables are all of similar magnitude in the region of interest;

(ii) a fixed change in any of the variables results in similar changes in F(x). Ideally, a unit change in any variable produces a unit change in F(x);

(iii) the variables are transformed so as to avoid cancellation error in the evaluation of F(x).

Normally, users should restrict themselves to linear transformations of variables, although occasionally nonlinear transformations are possible. The most common such transformation (and often the most appropriate) is of the form

$$x_{new} = Dx_{old},$$

where D is a diagonal matrix with constant coefficients. Our experience suggests that more use should be made of the transformation

$$x_{new} = Dx_{old} + v,$$

where v is a constant vector.

Consider, for example, a problem in which the variable x_3 represents the position of the peak of a Gaussian curve to be fitted to data for which the extreme values are 150 and 170; therefore x_3 is known to lie in the range 150--170. One possible scaling would be to define a new variable x_3, given by

$$x_3 = \frac{x_3}{170}.$$

A better transformation, however, is given by defining x_3 as

$$x_3 = \frac{x_3 - 160}{10}.$$

Frequently, an improvement in the accuracy of evaluation of F(x) can result if the variables are scaled before the routines to evaluate F(x) are coded. For instance, in the above problem just mentioned of Gaussian curve fitting, x_3 may always occur in terms of the form $(x_3 - x_{3m})$, where x_{3m} is a constant representing the mean peak position.

2.5.2. Scaling the objective function

The objective function has already been mentioned in the discussion of scaling the variables. The solution of a given problem is unaltered if F(x) is multiplied by a positive constant, or if a constant value is added to F(x). It is generally preferable for the objective function to be of the order of unity in the region of interest; thus, if in the original formulation F(x) is always of the order of 10^{+5} (say), then the value of F(x) should be multiplied by 10^{-5} when evaluating the function within the optimization routines. If a constant is added or subtracted in the computation of F(x), usually it should be omitted - i.e., it is better to formulate $F(x)$ as $x_2^2 + x_1^2$ rather than as $x_2^2 + x_1^2 + 1000$ or even $x_2^2 + x_1^2 + 1$. The inclusion of such a constant in the calculation of F(x) can result in a loss of significant figures.

2.5.3. Scaling the constraints

The solution of a nonlinearly-constrained problem is unaltered if the ith constraint is multiplied by a positive weight w_i. At the approximation of the solution determined by a Library routine, the active constraints will not be satisfied exactly, but will 10^{-8} 10^{-6}

have 'small' values (for example, c =10 , c =10 , etc.). In
1 2
general, this discrepancy will be minimized if the constraints are weighted so that a unit change in x produces a similar change in each constraint.

A second reason for introducing weights is related to the effect of the size of the constraints on the Lagrange multiplier estimates and, consequently, on the active set strategy. Additional discussion is given in Gill et al [5].

2.6. Analysis of Computed Results

2.6.1. Convergence criteria

The convergence criteria inevitably vary from routine to routine, since in some cases more information is available to be checked (for example, is the Hessian matrix positive-definite?), and different checks need to be made for different problem categories (for example, in constrained minimization it is necessary to verify whether a trial solution is feasible). Nonetheless, the underlying principles of the various criteria are the same; in non-mathematical terms, they are:

(i) is the sequence $\{x^{(k)}\}$ converging?

(ii) is the sequence $\{F^{(k)}\}$ converging?

(iii) are the necessary and sufficient conditions for the solution satisfied?

The decision as to whether a sequence is converging is necessarily speculative. The criterion used in the present routines is to assume convergence if the relative change occurring between two successive iterations is less than some prescribed quantity. Criterion (iii) is the most reliable but often the conditions cannot be checked fully because not all the required information may be available.

2.6.2. Checking results

Little a priori guidance can be given as to the quality of the solution found by a nonlinear optimization algorithm, since no guarantees can be given that the methods will always work.

Therefore, it is necessary for the user to check the computed solution even if the routine reports success. Frequently a ' solution' may have been found even when the routine does not report a success. The reason for this apparent contradiction is that the routine needs to assess the accuracy of the solution. This assessment is not an exact process and consequently may be unduly pessimistic. Any 'solution' is in general only an approximation to the exact solution, and it is possible that the accuracy specified by the user is too stringent.

Further confirmation can be sought by trying to check whether or not convergence tests are almost satisfied, or whether or not some of the sufficient conditions are nearly satisfied. When it is thought that a routine has returned a non-zero value of IFAIL only because the requirements for 'success' were too stringent it may be worth restarting with increased convergence tolerances.

For nonlinearly-constrained problems, check whether the solution returned is feasible, or nearly feasible; if not, the solution returned is not an adequate solution.

Confidence in a solution may be increased by resolving the problem with a different initial approximation to the solution. See Section 8.3 of Gill et al [5] for further information.

2.6.3. Monitoring progress

Many of the routines in the Chapter have facilities to allow the user to monitor the progress of the minimization process, and users are encouraged to make use of these facilities. Monitoring information can be a great aid in assessing whether or not a satisfactory solution has been obtained, and in indicating difficulties in the minimization problem or in the routine's ability to cope with the problem.

The behaviour of the function, the estimated solution and first derivatives can help in deciding whether a solution is acceptable and what to do in the event of a return with a non-zero value of IFAIL.

2.6.4. Confidence intervals for least-squares solutions

When estimates of the parameters in a nonlinear least-squares problem have been found, it may be necessary to estimate the variances of the parameters and the fitted function. These can be calculated from the Hessian of F(x) at the solution.

In many least-squares problems, the Hessian is adequately
T
approximated at the solution by G=2J J (see Section 2.4.3). The Jacobian, J, or a factorization of J is returned by all the comprehensive least-squares routines and, in addition, a routine is supplied in the Library to estimate variances of the parameters following the use of most of the nonlinear least-
T
squares routines, in the case that G=2J J is an adequate approximation.

Let H be the inverse of G, and S be the sum of squares, both

calculated at the solution x; an unbiased estimate of the variance of the ith parameter x is
i

$$\text{var } x_i = \frac{2S}{m-n} H_{ii}$$

and an unbiased estimate of the covariance of x and x is
i j

$$\text{covar}(x_i, x_j) = \frac{2S}{m-n} H_{ij}.$$

*
If x is the true solution, then the 100(1-(beta)) confidence

interval on x is

$$x_i - \sqrt{\text{var } x_i}\, t_{(1-(\beta)/2,m-n)} < x_i^* < x_i$$

```
    /
+  / var x .t                ,   i=1,2,...,n
 \/       i  (1-(beta)/2,m-n)
```

where t is the 100(1-(beta))/2 percentage point
(1-(beta)/2,m-n)
of the t-distribution with m-n degrees of freedom.

In the majority of problems, the residuals f , for i=1,2,...,m,
i
contain the difference between the values of a model function (phi)(z,x) calculated for m different values of the independent variable z, and the corresponding observed values at these points. The minimization process determines the parameters, or

constants x, of the fitted function (phi)(z,x). For any value, z, of the independent variable z, an unbiased estimate of the variance of (phi) is

$$\text{var}(\text{phi}) = \frac{2S}{m-n} \sum_{i=1}^{n} \sum_{j=1}^{n} \left[\frac{d(\text{phi})}{dx_i}\right]_z \left[\frac{d(\text{phi})}{dx_j}\right]_z H_{ij} .$$

The 100(1-(beta)) confidence interval on F at the point z is

$$(\text{phi})(z,x) - \sqrt{\text{var}(\text{phi})}\, t_{((\text{beta})/2,m-n)} < (\text{phi})(z,x)^* < (\text{phi})(z,x) + \sqrt{\text{var}(\text{phi})}\, t_{((\text{beta})/2,m-n)} .$$

For further details on the analysis of least-squares solutions see Bard [1] and Wolberg [7].

2.7. References

[1] Bard Y (1974) Nonlinear Parameter Estimation. Academic

Press.

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[7] Wolberg J R (1967) Prediction Analysis. Van Nostrand.

3. Recommendations on Choice and Use of Routines

The choice of routine depends on several factors: the type of problem (unconstrained, etc.); the level of derivative information available (function values only, etc.); the experience of the user (there are easy-to-use versions of some routines); whether or not storage is a problem; and whether computational time has a high priority.

3.1. Choice of Routine

Routines are provided to solve the following types of problem:

Nonlinear Programming	E04UCF
Quadratic Programming	E04NAF
Linear Programming	E04MBF
Nonlinear Function (using 1st derivatives)	E04DGF
Nonlinear Function, unconstrained or simple bounds (using function values only)	E04JAF
Nonlinear least-squares (using function values only)	E04FDF
Nonlinear least-squares (using function values and 1st derivatives)	E04GCF

E04UCF can be used to solve unconstrained, bound-constrained and

linearly-constrained problems.

E04NAF can be used as a comprehensive linear programming solver; however, in most cases the easy-to-use routine E04MBFwill be adequate.

E04MBF can be used to obtain a feasible point for a set of linear constraints.

E04DGF can be used to solve large scale unconstrained problems.

The routines can be used to solve problems in a single variable.

3.2. Service Routines

One of the most common errors in use of optimization routines is that the user's subroutines incorrectly evaluate the relevant partial derivatives. Because exact gradient information normally enhances efficiency in all areas of optimization, the user should be encouraged to provide analytical derivatives whenever possible. However, mistakes in the computation of derivatives can result in serious and obscure run-time errors, as well as complaints that the Library routines are incorrect.

E04UCF incorporates a check on the gradients being supplied and users are encouraged to utilize this option; E04GCF also incorporates a call to a derivative checker.

E04YCF estimates selected elements of the variance-covariance matrix for the computed regression parameters following the use of a nonlinear least-squares routine.

3.3. Function Evaluations at Infeasible Points

Users must not assume that the routines for constrained problems will require the objective function to be evaluated only at points which satisfy the constraints, i.e., feasible points. In the first place some of the easy-to-use routines call a service routine which will evaluate the objective function at the user-supplied initial point, and at neighbouring points (to check user-supplied derivatives or to estimate intervals for finite differencing). Apart from this, all routines will ensure that any evaluations of the objective function occur at points which approximately satisfy any simple bounds or linear constraints. Satisfaction of such constraints is only approximate because:

(a) routines which have a parameter FEATOL may allow such constraints to be violated by a margin specified by FEATOL;

(b) routines which estimate derivatives by finite differences may require function evaluations at points which just violate such constraints even though the current iteration just satisfies them.

There is no attempt to ensure that the current iteration satisfies any nonlinear constraints. Users who wish to prevent their objective function being evaluated outside some known region (where it may be undefined or not practically computable), may try to confine the iteration within this region by imposing suitable simple bounds or linear constraints (but beware as this may create new local minima where these constraints are active).

Note also that some routines allow the user-supplied routine to return a parameter (MODE) with a negative value to force an immediate clean exit from the minimization when the objective function cannot be evaluated.

3.4. Related Problems

Apart from the standard types of optimization problem, there are other related problems which can be solved by routines in this or other chapters of the Library.

E04MBF can be used to find a feasible point for a set of linear constraints and simple bounds.

Two routines in Chapter F04 solve linear least-squares problems,

$$\text{i.e., minimize } \sum_{i=1}^{m} r_i(x)^2 \quad \text{where } r_i(x) = b_i - \sum_{j=1}^{n} a_{ij} x_j .$$

E02GAF solves an overdetermined system of linear equations in the

$$l_1 \text{ norm, i.e., minimizes } \sum_{i=1}^{m} |r_i(x)|, \text{ with } r_i \text{ as above.}$$

```
Chapter E04

Minimizing or Maximizing a Function

E04DGF  Unconstrained minimum, pre-conditioned conjugate gradient
        algorithm, function of several variables using 1st
        derivatives

E04DJF  Read optional parameter values for E04DGF from external
        file

E04DKF  Supply optional parameter values to E04DGF

E04FDF  Unconstrained minimum of a sum of squares, combined
        Gauss-Newton and modified Newton algorithm using function
        values only

E04GCF  Unconstrained minimum of a sum of squares, combined
        Gauss-Newton and quasi-Newton algorithm, using 1st
        derivatives

E04JAF  Minimum, function of several variables, quasi-Newton
        algorithm, simple bounds, using function values only

E04MBF  Linear programming problem

E04NAF  Quadratic programming problem

E04UCF  Minimum, function of several variables, sequential QP
        method, nonlinear constraints, using function values and
        optionally 1st derivatives

E04UDF  Read optional parameter values for E04UCF from external
        file

E04UEF  Supply optional parameter values to E04UCF

E04YCF  Covariance matrix for nonlinear least-squares problem
```

%%%

```
E04 -- Minimizing or Maximizing a Function                E04DGF
       E04DGF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for
your implementation to check implementation-dependent details.
```

```
The symbol (*) after a NAG routine name denotes a routine that is
not included in the Foundation Library.

Note for users via the AXIOM system: the interface to this routine
has been enhanced for use with AXIOM and is slightly different to
that offered in the standard version of the Foundation Library.  In
particular, the optional parameters of the NAG routine are now
included in the parameter list.  These are described in section
5.1.2, below.

1. Purpose

E04DGF minimizes an unconstrained nonlinear function of several
variables using a pre-conditioned, limited memory quasi-Newton
conjugate gradient method. First derivatives are required. The
routine is intended for use on large scale problems.

2. Specification

       SUBROUTINE E04DGF(N,OBJFUN,ITER,OBJF,OBJGRD,X,IWORK,WORK,IUSER,
      1                  USER,ES,FU,IT,LIN,LIST,MA,OP,PR,STA,STO,
      2                  VE,IFAIL)
       INTEGER           N, ITER, IWORK(N+1), IUSER(*),
      1                  IT, PR, STA, STO, VE, IFAIL
       DOUBLE PRECISION OBJF, OBJGRD(N), X(N), WORK(13*N), USER(*)
      1                  ES, FU, LIN, OP, MA
       LOGICAL           LIST
       EXTERNAL          OBJFUN

3. Description

E04DGF uses a pre-conditioned conjugate gradient method and is
based upon algorithm PLMA as described in Gill and Murray [1] and
Gill et al [2] Section 4.8.3.

The algorithm proceeds as follows:

Let x  be a given starting point and let k denote the current
     0
iteration, starting with k=0. The iteration requires g , the
                                                      k
gradient vector evaluated at x , the kth estimate of the minimum.
                              k
At each iteration a vector p  (known as the direction of search)
                            k
is computed and the new estimate x    is given by x +(alpha) p
```

k+1 k k k

where α_k (the step length) minimizes the function $F(x_k+\alpha p_k)$ with respect to the scalar α. A choice of initial step α_0 is taken as

$$\alpha_0=\min\{1,2|F_k-F_{est}|/g_k^T g_k\}$$

where F_{est} is a user-supplied estimate of the function value at the solution. If F_{est} is not specified, the software always chooses the unit step length for α_0. Subsequent step length estimates are computed using cubic interpolation with safeguards.

A quasi-Newton method can be used to compute the search direction p_k by updating the inverse of the approximate Hessian (H_k) and computing

$$p_{k+1}=-H_{k+1}g_{k+1} \qquad (1)$$

The updating formula for the approximate inverse is given by

$$H_{k+1}=H_k-\frac{1}{y_k^Ts_k}\left(H_ky_ks_k^T+s_ky_k^TH_k\right)+\frac{1}{y_k^Ts_k}\left(1+\frac{y_k^TH_ky_k}{y_k^Ts_k}\right)s_ks_k^T \qquad (2)$$

where $y_k=g_{k-1}-g_k$ and $s_k=x_{k+1}-x_k=\alpha_kp_k$.

The method used by E04DGF to obtain the search direction is based upon computing p_{k+1} as $-H_{k+1}g_{k+1}$ where H_{k+1} is a matrix obtained by updating the identity matrix with a limited number of quasi-Newton corrections. The storage of an n by n matrix is avoided by

storing only the vectors that define the rank two corrections - hence the term limited-memory quasi-Newton method. The precise method depends upon the number of updating vectors stored. For example, the direction obtained with the 'one-step' limited memory update is given by (1) using (2) with H_k equal to the identity matrix, viz.

$$p_{k+1} = -g_{k+1} + \frac{1}{y_k^T s_k}(s_k^T g_{k+1} + y_k^T g_{k+1})s_k - \frac{1}{y_k^T s_k}(1 + \frac{y_k^T g_{k+1}}{y_k^T s_k})s_k$$

E04DGF uses a two-step method described in detail in Gill and Murray [1] in which restarts and pre-conditioning are incorporated. Using a limited-memory quasi-Newton formula, such as the one above, guarantees p_{k+1} to be a descent direction if all the inner products y_k^T are positive for all vectors y_k and s_k used in the updating formula.

The termination criterion of E04DGF is as follows:

Let τ_F specify a parameter that indicates the number of correct figures desired in F_k (τ_F is equivalent to Optimality Tolerance in the optional parameter list, see Section 5.1). If the following three conditions are satisfied

(i) $F_{k-1} - F_k < \tau_F (1+|F_k|)$

(ii) $\|x_{k-1} - x_k\| < \sqrt{\tau_F}\,(1+\|x_k\|)$

(iii) $\|g_k\| \leq 3\sqrt{\tau_F}\,(1+|F_k|)$ or $\|g_k\| < \epsilon_A$,

where ϵ_A is the absolute error associated with

```
     computing the objective function
```

then the algorithm is considered to have converged. For a full discussion on termination criteria see Gill et al [2] Chapter 8.

4. References

[1] Gill P E and Murray W (1979) Conjugate-gradient Methods for Large-scale Nonlinear Optimization. Technical Report SOL 79-15. Department of Operations Research, Stanford University.

[2] Gill P E, Murray W and Wright M H (1981) Practical Optimization. Academic Press.

5. Parameters

1: N -- INTEGER Input
On entry: the number n of variables. Constraint: N >= 1.

2: OBJFUN -- SUBROUTINE, supplied by the user. External Procedure
OBJFUN must calculate the objective function F(x) and its gradient for a specified n element vector x.

Its specification is:

```
      SUBROUTINE OBJFUN (MODE, N, X, OBJF, OBJGRD,
     1                   NSTATE, IUSER, USER)
      INTEGER          MODE, N, NSTATE, IUSER(*)
      DOUBLE PRECISION X(N), OBJF, OBJGRD(N), USER(*)
```

1: MODE -- INTEGER Input/Output
MODE is a flag that the user may set within OBJFUN to indicate a failure in the evaluation of the objective function. On entry: MODE is always non-negative. On exit: if MODE is negative the execution of E04DGF is terminated with IFAIL set to MODE.

2: N -- INTEGER Input
On entry: the number n of variables.

3: X(N) -- DOUBLE PRECISION array Input
On entry: the point x at which the objective function is required.

4: OBJF -- DOUBLE PRECISION Output

```
        On exit: the value of the objective function F at the
        current point x.

    5:  OBJGRD(N) -- DOUBLE PRECISION array                Output
                                                          ddF
        On exit: OBJGRD(i) must contain the value of      ---- at
                                                          ddx
                                                             i
        the point x, for i=1,2,...,n.

    6:  NSTATE -- INTEGER                                  Input
        On entry: NSTATE will be 1 on the first call of OBJFUN
        by E04DGF, and is 0 for all subsequent calls. Thus, if
        the user wishes, NSTATE may be tested within OBJFUN in
        order to perform certain calculations once only. For
        example the user may read data or initialise COMMON
        blocks when NSTATE = 1.

    7:  IUSER(*) -- INTEGER array                 User Workspace

    8:  USER(*) -- DOUBLE PRECISION array         User Workspace
        OBJFUN is called from E04DGF with the parameters IUSER
        and USER as supplied to E04DGF. The user is free to use
        arrays IUSER and USER to supply information to OBJFUN
        as an alternative to using COMMON.
    OBJFUN must be declared as EXTERNAL in the (sub)program
    from which E04DGF is called. Parameters denoted as
    Input must not be changed by this procedure.

3:  ITER -- INTEGER                                        Output
    On exit: the number of iterations performed.

4:  OBJF -- DOUBLE PRECISION                               Output
    On exit: the value of the objective function F(x) at the
    final iterate.

5:  OBJGRD(N) -- DOUBLE PRECISION array                    Output
    On exit: the objective gradient at the final iterate.

6:  X(N) -- DOUBLE PRECISION array                   Input/Output
    On entry: an initial estimate of the solution. On exit: the
    final estimate of the solution.

7:  IWORK(N+1) -- INTEGER array                          Workspace

8:  WORK(13*N) -- DOUBLE PRECISION array                 Workspace
```

9: IUSER(*) -- INTEGER array User Workspace
Note: the dimension of the array IUSER must be at least 1. This array is not used by E04DGF, but is passed directly to routine OBJFUN and may be used to supply information to OBJFUN.

10: USER(*) -- DOUBLE PRECISION array User Workspace
Note: the dimension of the array USER must be at least 1. This array is not used by E04DGF, but is passed directly to routine OBJFUN and may be used to supply information to OBJFUN.

11: IFAIL -- INTEGER Input/Output
On entry: IFAIL must be set to 0, -1 or 1. Users who are unfamiliar with this parameter should refer to the Essential Introduction for details.

On exit: IFAIL = 0 unless the routine detects an error or gives a warning (see Section 6).

For this routine, because the values of output parameters may be useful even if IFAIL /=0 on exit, users are recommended to set IFAIL to -1 before entry. It is then essential to test the value of IFAIL on exit.

5.1. Optional Input Parameters

Several optional parameters in E04DGF define choices in the behaviour of the routine. In order to reduce the number of formal parameters of E04DGF these optional parameters have associated default values (see Section 5.1.3) that are appropriate for most problems. Therefore the user need only specify those optional parameters whose values are to be different from their default values.

The remainder of this section can be skipped by users who wish to use the default values for all optional parameters. A complete list of optional parameters and their default values is given in Section 5.1.3.

5.1.1. Specification of the Optional Parameters

Optional parameters may be specified by calling one, or both, of E04DJF and E04DKF prior to a call to E04DGF.

E04DJF reads options from an external options file, with Begin and End as the first and last lines respectively and each intermediate line defining a single optional parameter. For example,

```
Begin
  Print Level = 1
End
```

The call

```
CALL E04DJF(IOPTNS, INFORM)
```

can then be used to read the file on unit IOPTNS. INFORM will be zero on successful exit. E04DJF should be consulted for a full description of this method of supplying optional parameters.

E04DKF can be called to supply options directly, one call being necessary for each optional parameter.

For example,

```
CALL E04DKF('Print level = 1')
```

E04DKF should be consulted for a full description of this method of supplying optional parameters.

All optional parameters not specified by the user are set to their default values. Optional parameters specified by the user are unaltered by E04DGF (unless they define invalid values) and so remain in effect for subsequent calls to E04DGF, unless altered by the user.

5.1.2. Description of the Optional Parameters

The following list (in alphabetical order) gives the valid options. For each option, we give the keyword, any essential optional qualifiers, the default value, and the definition. The minimum valid abbreviation of each keyword is underlined. If no characters of an optional qualifier are underlined, the qualifier may be omitted. The letter a denotes a phrase (character string) that qualifies an option. The letters i and r denote INTEGER and real values required with certain options. The number (epsilon)

is a generic notation for machine precision, and (epsilon)
R
denotes the relative precision of the objective function (the optional parameter Function Precision; see below).

Defaults

This special keyword may be used to reset the default values following a call to E04DGF.

Estimated Optimal Function Value r

(Axiom parameter ES)

This value of r specifies the user-supplied guess of the optimum objective function value. This value is used by E04DGF to calculate an initial step length (see Section 3). If the value of r is not specified by the user (the default), then this has the effect of setting the initial step length to unity. It should be noted that for badly scaled functions a unit step along the steepest descent direction will often compute the function at very large values of x.

0.9
Function Precision r Default = (epsilon)

(Axiom parameter FU)

The parameter defines (epsilon) , which is intended to be a
R
measure of the accuracy with which the problem function F can be computed. The value of (epsilon) should reflect the relative
R
precision of 1+|F(x)|; i.e. (epsilon) acts as a relative
R
precision when |F| is large, and as an absolute precision when |F| is small. For example, if F(x) is typically of order 1000 and the first six significant digits are known to be correct, an appropriate value for (epsilon) would be 1.0E-6. In contrast, if
R
-4
F(x) is typically of order 10 and the first six significant digits are known to be correct, an appropriate value for (epsilon) would be 1.0E-10. The choice of (epsilon) can be
R R
quite complicated for badly scaled problems; see Chapter 8 of

```
Gill and Murray [2], for a discussion of scaling techniques. The
default value is appropriate for most simple functions that are
computed with full accuracy. However when the accuracy of the
computed function values is known to be significantly worse than
full precision, the value of (epsilon)  should be large enough so
                                       R
that E04DGF will not attempt to distinguish between function
values that differ by less than the error inherent in the
calculation. If 0<=r<(epsilon), where (epsilon) is the machine
precision then the default value is used.

Iteration Limit  i Default = max(50,5n)

Iters

Itns

   (Axiom parameter IT)

The value i (i>=0) specifies the maximum number of iterations
allowed before termination. If i<0 the default value is used. See
Section 8 for further information.

Linesearch Tolerance  r Default = 0.9

   (Axiom parameter LIN)

The value r (0<=r<1) controls the accuracy with which the step
(alpha) taken during each iteration approximates a minimum of the
function along the search direction (the smaller the value of r,
the more accurate the linesearch). The default value r=0.9
requests an inaccurate search, and is appropriate for most
problems. A more accurate search may be appropriate when it is
desirable to reduce the number of iterations - for example, if
the objective function is cheap to evaluate.

List     Default =  List
Nolist

   (Axiom parameter LIST)

Normally each optional parameter specification is printed as it
is supplied. Nolist may be used to suppress the printing and List
may be used to restore printing.
                                      10
Maximum Step Length  r Default = 10
```

(Axiom parameter MA)

The value r (r>0) defines the maximum allowable step length for the line search. If r<=0 the default value is used.

Optimality Tolerance r Default = (epsilon) 0.8

(Axiom parameter OP)

The parameter r ((epsilon) <=r<1) specifies the accuracy to which R the user wishes the final iterate to approximate a solution of the problem. Broadly speaking, r indicates the number of correct figures desired in the objective function at the solution. For example, if r is 10 - 6 and E04DGF terminates successfully, the final value of F should have approximately six correct figures. E04DGF will terminate successfully if the iterative sequence of x -values is judged to have converged and the final point satisfies the termination criteria (see Section 3, where (tau) F represents Optimality Tolerance).

Print Level i Default = 10

(Axiom parameter PR)

The value i controls the amount of printout produced by E04DGF. The following levels of printing are available.

i Output.

0 No output.

1 The final solution.

5 One line of output for each iteration.

10 The final solution and one line of output for each iteration.

Start Objective Check at Variable i Default = 1

(Axiom parameter STA)

Stop Objective Check at Variable i Default = n

(Axiom parameter STO)

These keywords take effect only if Verify Level > 0 (see below). They may be used to control the verification of gradient elements computed by subroutine OBJFUN. For example if the first 30 variables appear linearly in the objective, so that the corresponding gradient elements are constant, then it is reasonable to specify Start Objective Check at Variable 31.

Verify Level i Default = 0

Verify No

Verify Level -1

Verify Level 0

Verify

Verify Yes

Verify Objective Gradients

Verify Gradients

Verify Level 1

(Axiom parameter VE)

These keywords refer to finite-difference checks on the gradient elements computed by the user-provided subroutine OBJFUN. It is possible to set Verify Level in several ways, as indicated above. For example, the gradients will be verified if Verify, Verify Yes, Verify Gradients, Verify Objective Gradients or Verify Level = 1 is specified.

If i<0 then no checking will be performed. If i>0 then the gradients will be verified at the user-supplied point. If i=0 only a 'cheap' test will be performed, requiring one call to OBJFUN. If i=1, a more reliable (but more expensive) check will be made on individual gradient components, within the ranges specified by the Start and Stop keywords as described above. A

result of the form OK or BAD? is printed by E04DGF to indicate whether or not each component appears to be correct.

5.1.3. Optional parameter checklist and default values

For easy reference, the following sample list shows all valid keywords and their default values. The default options Function Precision and Optimality Tolerance depend upon (epsilon), the machine precision.

Optional Parameters	Default Values
Estimated Optimal Function Value	
Function precision	0.9 (epsilon)
Iterations	max(50,5n)
Linesearch Tolerance	0.9
Maximum Step Length	10 10
List/Nolist	List
Optimality Tolerance	0.8 (epsilon)
Print Level	10
Start Objective Check at Variable	1
Stop Objective Check at Variable	n
Verify Level	0

5.2. Description of Printed Output

The level of printed output from E04DGF is controlled by the user (see the description of Print Level in Section 5.1).

When Print Level >= 5, the following line of output is produced

at each iteration.

Itn is the iteration count.

Step is the step (alpha) taken along the computed search direction. On reasonably well-behaved problems, the unit step will be taken as the solution is approached.

Nfun is the cumulated number of evaluations of the objective function needed for the linesearch. Evaluations needed for the estimation of the gradients by finite differences are not included. Nfun is printed as a guide to the amount of work required for the linesearch. E04DGF will perform at most 16 function evaluations per iteration.

Objective is the value of the objective function.

Norm G is the Euclidean norm of the gradient of the objective function.

Norm X is the Euclidean norm of x.

Norm (X(k-1)-X(k)) is the Euclidean norm of $x_{k-1} - x_k$.

When Print Level = 1 or Print Level >= 10 then the solution at the end of execution of E04DGF is printed out.

The following describes the printout for each variable:

Variable gives the name (VARBL) and index j (j = 1 to n) of the variable

Value is the value of the variable at the final iterate

Gradient Value is the value of the gradient of the objective function with respect to the jth variable at the final iterate

6. Error Indicators and Warnings

Errors or warnings specified by the routine:

If on entry IFAIL = 0 or -1, explanatory error messages are

output on the current error message unit (as defined by X04AAF).

On exit from E04DGF, IFAIL should be tested. If Print Level > 0 then a short description of IFAIL is printed.

Errors and diagnostics indicated by IFAIL from E04DGF are as follows:

IFAIL< 0
A negative value of IFAIL indicates an exit from E04DGF because the user set MODE negative in routine OBJFUN. The value of IFAIL will be the same as the user's setting of MODE.

IFAIL= 1
Not used by this routine.

IFAIL= 2
Not used by this routine.

IFAIL= 3
The maximum number of iterations has been performed. If the algorithm appears to be making progress the iterations value may be too small (see Section 5.1.2) so the user should increase iterations and rerun E04DGF. If the algorithm seems to be 'bogged down',the user should check for incorrect gradients or ill-conditioning as described below under IFAIL = 6.

IFAIL= 4
The computed upper bound on the step length taken during the linesearch was too small. A rerun with an increased value of the Maximum Step Length ((rho) say) may be successful unless
10
(rho)>=10 (the default value), in which case the current point cannot be improved upon.

IFAIL= 5
Not used by this routine.

IFAIL= 6
A sufficient decrease in the function value could not be attained during the final linesearch. If the subroutine OBJFUN computes the function and gradients correctly, then this may occur because an overly stringent accuracy has been requested, i.e., Optimality Tolerance is too small or if the

minimum lies close to a step length of zero. In this case the user should apply the four tests described in Section 3 to determine whether or not the final solution is acceptable (the user will need to set Print Level >= 5). For a discussion of attainable accuracy see Gill and Murray [2].

If many iterations have occurred in which essentially no progress has been made or E04DGF has failed to move from the initial point, subroutine OBJFUN may be incorrect. The user should refer to the comments below under IFAIL = 7 and check the gradients using the Verify parameter. Unfortunately, there may be small errors in the objective gradients that cannot be detected by the verification process. Finite-difference approximations to first derivatives are catastrophically affected by even small inaccuracies.

IFAIL= 7

Large errors were found in the derivatives of the objective function. This value of IFAIL will occur if the verification process indicated that at least one gradient component had no correct figures. The user should refer to the printed output to determine which elements are suspected to be in error.

As a first step, the user should check that the code for the objective values is correct - for example, by computing the function at a point where the correct value is known. However, care should be taken that the chosen point fully tests the evaluation of the function. It is remarkable how often the values x=0 or x=1 are used to test function evaluation procedures, and how often the special properties of these numbers make the test meaningless.

Special care should be used in this test if computation of the objective function involves subsidiary data communicated in COMMON storage. Although the first evaluation of the function may be correct, subsequent calculations may be in error because some of the subsidiary data has accidentally been overwritten.

Errors in programming the function may be quite subtle in that the function value is 'almost' correct. For example, the function may not be accurate to full precision because of the inaccurate calculation of a subsidiary quantity, or the limited accuracy of data upon which the function depends. A common error on machines where numerical

calculations are usually performed in double precision is to include even one single-precision constant in the calculation of the function; since some compilers do not convert such constants to double precision, half the correct figures may be lost by such a seemingly trivial error.

IFAIL= 8

The gradient (g) at the starting point is too small. The value $g^T g$ is less than (epsilon) $|F(x_m)|$, where (epsilon) is the machine precision.

The problem should be rerun at a different starting point.

IFAIL= 9

On entry N < 1.

7. Accuracy

On successful exit the accuracy of the solution will be as defined by the optional parameter Optimality Tolerance.

8. Further Comments

Problems whose Hessian matrices at the solution contain sets of clustered eigenvalues are likely to be minimized in significantly fewer than n iterations. Problems without this property may require anything between n and 5n iterations, with approximately 2n iterations being a common figure for moderately difficult problems.

9. Example

To find a minimum of the function

$$F = e^{x_1}\,(4x_1^2 + 2x_2^2 + 4x_1x_2 + 2x_2 + 1).$$

The example program is not reproduced here. The source code for all example programs is distributed with the NAG Foundation Library software and should be available on-line.

%%%

```
E04 -- Minimizing or Maximizing a Function                E04DJF
        E04DJF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for
your implementation to check implementation-dependent details.
The symbol (*) after a NAG routine name denotes a routine that is
not included in the Foundation Library.

1. Purpose

To supply optional parameters to E04DGF from an external file.

2. Specification

       SUBROUTINE E04DJF (IOPTNS, INFORM)
       INTEGER          IOPTNS, INFORM

3. Description

E04DJF may be used to supply values for optional parameters to
E04DGF. E04DJF reads an external file and each line of the file
defines a single optional parameter. It is only necessary to
supply values for those parameters whose values are to be
different from their default values.

Each optional parameter is defined by a single character string
of up to 72 characters, consisting of one or more items. The
items associated with a given option must be separated by spaces,
or equal signs (=). Alphabetic characters may be upper or lower
case. The string

      Print level = 1

is an example of a string used to set an optional parameter. For
each option the string contains one or more of the following
items:

(a)   A mandatory keyword.

(b)   A phrase that qualifies the keyword.

(c)   A number that specifies an INTEGER or real value. Such
      numbers may be up to 16 contiguous characters in Fortran
      77's I, F, E or D formats, terminated by a space if this is
      not the last item on the line.
```

Blank strings and comments are ignored. A comment begins with an asterisk (*) and all subsequent characters in the string are regarded as part of the comment.

The file containing the options must start with begin and must finish with end An example of a valid options file is:

```
Begin * Example options file
  Print level = 10
End
```

Normally each line of the file is printed as it is read, on the current advisory message unit (see X04ABF), but printing may be suppressed using the keyword nolist. To suppress printing of begin, nolist must be the first option supplied as in the file:

```
Begin
  Nolist
  Print level = 10
End
```

Printing will automatically be turned on again after a call to E04DGF and may be turned on again at any time by the user by using the keyword list.

Optional parameter settings are preserved following a call to E04DGF, and so the keyword defaults is provided to allow the user to reset all the optional parameters to their default values prior to a subsequent call to E04DGF.

A complete list of optional parameters, their abbreviations, synonyms and default values is given in Section 5.1 of the routine document for E04DGF.

4. References

None.

5. Parameters

1: IOPTNS -- INTEGER Input
On entry: IOPTNS must be the unit number of the options file. Constraint: 0 <= IOPTNS <= 99.

```
 2:  INFORM -- INTEGER                                          Output
     On exit: INFORM will be zero if an options file with the
     correct structure has been read. Otherwise INFORM will be
     positive. Positive values of INFORM indicate that an options
     file may not have been successfully read as follows:
     INFORM = 1
           IOPTNS is not in the range [0,99].

     INFORM = 2
           begin was found, but end-of-file was found before end
           was found.

     INFORM = 3
           end-of-file was found before begin was found.

6. Error Indicators and Warnings

If a line is not recognised as a valid option, then a warning
message is output on the current advisory message unit (see
X04ABF).

7. Accuracy

Not applicable.

8. Further Comments

E04DKF may also be used to supply optional parameters to E04DGF.

9. Example

See the example for E04DGF.
```

%%%

```
E04 -- Minimizing or Maximizing a Function                      E04DKF
        E04DKF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for
your implementation to check implementation-dependent details.
The symbol (*) after a NAG routine name denotes a routine that is
not included in the Foundation Library.

1. Purpose

To supply individual optional parameters to E04DGF.
```

2. Specification

```
SUBROUTINE E04DKF (STRING)
CHARACTER*(*)    STRING
```

3. Description

E04DKF may be used to supply values for optional parameters to E04DGF. It is only necessary to call E04DKF for those parameters whose values are to be different from their default values. One call to E04DKF sets one parameter value.

Each optional parameter is defined by a single character string of up to 72 characters, consisting of one or more items. The items associated with a given option must be separated by spaces, or equal signs (=). Alphabetic characters may be upper or lower case. The string

```
Print Level = 1
```

is an example of a string used to set an optional parameter. For each option the string contains one or more of the following items:

(a) A mandatory keyword.

(b) A phrase that qualifies the keyword.

(c) A number that specifies an INTEGER or real value. Such numbers may be up to 16 contiguous characters in Fortran 77's I, F, E or D formats, terminated by a space if this is not the last item on the line.

Blank strings and comments are ignored. A comment begins with an asterisk (*) and all subsequent characters in the string are regarded as part of the comment.

Normally, each user-specified option is printed as it is defined, on the current advisory message unit (see X04ABF), but this printing may be suppressed using the keyword nolist Thus the statement

```
CALL E04DKF ('Nolist')
```

suppresses printing of this and subsequent options. Printing will

automatically be turned on again after a call to E04DGF, and may be turned on again at any time by the user, by using the keyword list.

Optional parameter settings are preserved following a call to E04DGF, and so the keyword defaults is provided to allow the user to reset all the optional parameters to their default values by the statement,

```
CALL E04DKF ('Defaults')
```

prior to a subsequent call to E04DGF.

A complete list of optional parameters, their abbreviations, synonyms and default values is given in Section 5.1 of the routine document for E04DGF.

4. References

None.

5. Parameters

1: STRING -- CHARACTER*(*) Input
On entry: STRING must be a single valid option string. See Section 3 above, and Section 5.1 of the routine document for E04DGF.

6. Error Indicators and Warnings

If the parameter STRING is not recognised as a valid option string, then a warning message is output on the current advisory message unit (see X04ABF).

7. Accuracy

Not applicable.

8. Further Comments

E04DJF may also be used to supply optional parameters to E04DGF.

9. Example

See the example for E04DGF.

%%

E04 -- Minimizing or Maximizing a Function E04FDF
E04FDF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for your implementation to check implementation-dependent details. The symbol (*) after a NAG routine name denotes a routine that is not included in the Foundation Library.

1. Purpose

E04FDF is an easy-to-use algorithm for finding an unconstrained minimum of a sum of squares of m nonlinear functions in n variables (m>=n). No derivatives are required.

It is intended for functions which are continuous and which have continuous first and second derivatives (although it will usually work even if the derivatives have occasional discontinuities).

2. Specification

```
SUBROUTINE E04FDF (M, N, X, FSUMSQ, IW, LIW, W, LW, IFAIL)
INTEGER          M, N, IW(LIW), LIW, LW, IFAIL
DOUBLE PRECISION X(N), FSUMSQ, W(LW)
```

3. Description

This routine is essentially identical to the subroutine LSNDN1 in the National Physical Laboratory Algorithms Library. It is applicable to problems of the form

$$\text{Minimize } F(x) = \sum_{i=1}^{m} [f_i(x)]^2$$

where $x=(x_1, x_2, \ldots, x_n)^T$ and m>=n. (The functions $f_i(x)$ are often referred to as 'residuals'.) The user must supply a subroutine LSFUN1 to evaluate functions $f_i(x)$ at any point x.

From a starting point supplied by the user, a sequence of points

is generated which is intended to converge to a local minimum of the sum of squares. These points are generated using estimates of the curvature of F(x).

4. References

[1] Gill P E and Murray W (1978) Algorithms for the Solution of the Nonlinear Least-squares Problem. SIAM J. Numer. Anal. 15 977--992.

5. Parameters

1: M -- INTEGER Input

2: N -- INTEGER Input
On entry: the number m of residuals $f_i(x)$, and the number n of variables, x_j. Constraint: 1 <= N <= M.

3: X(N) -- DOUBLE PRECISION array Input/Output
On entry: X(j) must be set to a guess at the jth component of the position of the minimum, for j=1,2,...,n. On exit: the lowest point found during the calculations. Thus, if IFAIL = 0 on exit, X(j) is the jth component of the position of the minimum.

4: FSUMSQ -- DOUBLE PRECISION Output
On exit: the value of the sum of squares, F(x), corresponding to the final point stored in X.

5: IW(LIW) -- INTEGER array Workspace

6: LIW -- INTEGER Input
On entry: the length of IW as declared in the (sub)program from which E04FDF has been called. Constraint: LIW >= 1.

7: W(LW) -- DOUBLE PRECISION array Workspace

8: LW -- INTEGER Input
On entry: the length of W as declared in the (sub)program from which E04FDF is called. Constraints:
LW >= N*(7 + N + 2*M + (N-1)/2) + 3*M, if N > 1,

LW >= 9 + 5*>M, if N = 1.

9: IFAIL -- INTEGER Input/Output
On entry: IFAIL must be set to 0, -1 or 1. Users who are unfamiliar with this parameter should refer to the Essential Introduction for details.

On exit: IFAIL = 0 unless the routine detects an error or gives a warning (see Section 6).

For this routine, because the values of output parameters may be useful even if IFAIL /=0 on exit, users are recommended to set IFAIL to -1 before entry. It is then essential to test the value of IFAIL on exit.

5.1. Optional Parameters

LSFUN1 -- SUBROUTINE, supplied by the user.
External Procedure
This routine must be supplied by the user to calculate the vector of values $f_i(x)$ at any point x. Since the routine is not a parameter to E04FDF, it must be called LSFUN1. It should be tested separately before being used in conjunction with E04FDF (see the Chapter Introduction).

Its specification is:

```
SUBROUTINE LSFUN1 (M, N, XC, FVECC)
INTEGER          M, N
DOUBLE PRECISION XC(N), FVECC(M)
```

1: M -- INTEGER Input

2: N -- INTEGER Input
On entry: the numbers m and n of residuals and variables, respectively.

3: XC(N) -- DOUBLE PRECISION array Input
On entry: the point x at which the values of the f_i are required.

4: FVECC(M) -- DOUBLE PRECISION array Output
On exit: FVECC(i) must contain the value of f_i at the point x, for i=1,2,...,m.

LSFUN1 must be declared as EXTERNAL in the (sub)program

```
     from which E04FDF is called. Parameters denoted as
     Input must not be changed by this procedure.

6. Error Indicators and Warnings

Errors or warnings specified by the routine:

If on entry IFAIL = 0 or -1, explanatory error messages are
output on the current error message unit (as defined by X04AAF).

IFAIL= 1
     On entry N < 1,

     or       M < N,

     or       LIW < 1,

     or       LW < N*(7 + N + 2*M + (N-1)/2) + 3*M, when N > 1,

     or       LW < 9 + 5*>M, when N = 1.

IFAIL= 2
     There have been 400*n calls of LSFUN1, yet the algorithm
     does not seem to have converged. This may be due to an
     awkward function or to a poor starting point, so it is worth
     restarting E04FDF from the final point held in X.

IFAIL= 3
     The final point does not satisfy the conditions for
     acceptance as a minimum, but no lower point could be found.

IFAIL= 4
     An auxiliary routine has been unable to complete a singular
     value decomposition in a reasonable number of sub-
     iterations.

IFAIL= 5

IFAIL= 6

IFAIL= 7

IFAIL= 8
     There is some doubt about whether the point x found by
     E04FDF is a minimum of F(x). The degree of confidence in the
     result decreases as IFAIL increases. Thus when IFAIL = 5, it
```

is probable that the final x gives a good estimate of the position of a minimum, but when IFAIL = 8 it is very unlikely that the routine has found a minimum.

If the user is not satisfied with the result (e.g. because IFAIL lies between 3 and 8), it is worth restarting the calculations from a different starting point (not the point at which the failure occurred) in order to avoid the region which caused the failure. Repeated failure may indicate some defect in the formulation of the problem.

7. Accuracy

If the problem is reasonably well scaled and a successful exit is made, then, for a computer with a mantissa of t decimals, one would expect to get about t/2-1 decimals accuracy in the components of x and between t-1 (if F(x) is of order 1 at the minimum) and 2t-2 (if F(x) is close to zero at the minimum) decimals accuracy in F(x).

8. Further Comments

The number of iterations required depends on the number of variables, the number of residuals and their behaviour, and the distance of the starting point from the solution. The number of multiplications performed per iteration of E04FDF varies, but for
$$2 \quad 3$$
m>>n is approximately n*m +O(n). In addition, each iteration makes at least n+1 calls of LSFUN1. So, unless the residuals can be evaluated very quickly, the run time will be dominated by the time spent in LSFUN1.

Ideally, the problem should be scaled so that the minimum value of the sum of squares is in the range (0,1), and so that at points a unit distance away from the solution the sum of squares is approximately a unit value greater than at the minimum. It is unlikely that the user will be able to follow these recommendations very closely, but it is worth trying (by guesswork), as sensible scaling will reduce the difficulty of the minimization problem, so that E04FDF will take less computer time.

When the sum of squares represents the goodness of fit of a nonlinear model to observed data, elements of the variance-covariance matrix of the estimated regression coefficients can be computed by a subsequent call to E04YCF, using information

returned in segments of the workspace array W. See E04YCF for further details.

9. Example

To find least-squares estimates of x_1, x_2 and x_3 in the model

$$y = x_1 + \frac{t_1}{x_2 t_1 + x_3 t_2}$$

using the 15 sets of data given in the following table.

y	t_1	t_2	t_3
0.14	1.0	15.0	1.0
0.18	2.0	14.0	2.0
0.22	3.0	13.0	3.0
0.25	4.0	12.0	4.0
0.29	5.0	11.0	5.0
0.32	6.0	10.0	6.0
0.35	7.0	9.0	7.0
0.39	8.0	8.0	8.0
0.37	9.0	7.0	7.0
0.58	10.0	6.0	6.0
0.73	11.0	5.0	5.0
0.96	12.0	4.0	4.0
1.34	13.0	3.0	3.0
2.10	14.0	2.0	2.0
4.39	15.0	1.0	1.0

The program uses (0.5, 1.0, 1.5) as the initial guess at the position of the minimum.

The example program is not reproduced here. The source code for all example programs is distributed with the NAG Foundation Library software and should be available on-line.

%%%

E04 -- Minimizing or Maximizing a Function E04GCF

E04GCF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for your implementation to check implementation-dependent details. The symbol (*) after a NAG routine name denotes a routine that is not included in the Foundation Library.

1. Purpose

E04GCF is an easy-to-use quasi-Newton algorithm for finding an unconstrained minimum of a sum of squares of m nonlinear functions in n variables (m>=n). First derivatives are required.

It is intended for functions which are continuous and which have continuous first and second derivatives (although it will usually work even if the derivatives have occasional discontinuities).

2. Specification

```
SUBROUTINE E04GCF (M, N, X, FSUMSQ, IW, LIW, W, LW, IFAIL)
INTEGER          M, N, IW(LIW), LIW, LW, IFAIL
DOUBLE PRECISION X(N), FSUMSQ, W(LW)
```

3. Description

This routine is essentially identical to the subroutine LSFDQ2 in the National Physical Laboratory Algorithms Library. It is applicable to problems of the form

$$\text{Minimize} \quad F(x) = \sum_{i=1}^{m} [f_i(x)]^2$$

where $x=(x_1, x_2, \ldots, x_n)^T$ and m>=n. (The functions $f_i(x)$ are often referred to as 'residuals'.) The user must supply a subroutine LSFUN2 to evaluate the residuals and their first derivatives at any point x.

Before attempting to minimize the sum of squares, the algorithm checks LSFUN2 for consistency. Then, from a starting point supplied by the user, a sequence of points is generated which is intended to converge to a local minimum of the sum of squares. These points are generated using estimates of the curvature of F(x).

4. References

[1] Gill P E and Murray W (1978) Algorithms for the Solution of the Nonlinear Least-squares Problem. SIAM J. Numer. Anal. 15 977--992.

5. Parameters

1: M -- INTEGER Input

2: N -- INTEGER Input
On entry: the number m of residuals $f_i(x)$, and the number n of variables, x_j. Constraint: 1 <= N <= M.

3: X(N) -- DOUBLE PRECISION array Input/Output
On entry: X(j) must be set to a guess at the jth component of the position of the minimum, for j=1,2,...,n. The routine checks the first derivatives calculated by LSFUN2 at the starting point, and so is more likely to detect an error in the user's routine if the initial X(j) are non-zero and mutually distinct. On exit: the lowest point found during the calculations. Thus, if IFAIL = 0 on exit, X(j) is the j th component of the position of the minimum.

4: FSUMSQ -- DOUBLE PRECISION Output
On exit: the value of the sum of squares, F(x), corresponding to the final point stored in X.

5: IW(LIW) -- INTEGER array Workspace

6: LIW -- INTEGER Input
On entry: the length of IW as declared in the (sub)program from which E04GCF is called. Constraint: LIW >= 1.

7: W(LW) -- DOUBLE PRECISION array Workspace

8: LW -- INTEGER Input
On entry: the length of W as declared in the (sub)program from which E04GCF is called. Constraints:
LW >= 2*N*(4 + N + M) + 3*M, if N > 1,

LW >= 11 + 5*M, if N = 1.

9: IFAIL -- INTEGER Input/Output
On entry: IFAIL must be set to 0, -1 or 1. Users who are unfamiliar with this parameter should refer to the Essential Introduction for details.

On exit: IFAIL = 0 unless the routine detects an error or gives a warning (see Section 6).

For this routine, because the values of output parameters may be useful even if IFAIL /=0 on exit, users are recommended to set IFAIL to -1 before entry. It is then essential to test the value of IFAIL on exit.

5.1. Optional Parameters

LSFUN2 -- SUBROUTINE, supplied by the user.
External Procedure
This routine must be supplied by the user to calculate the vector of values $f_i(x)$ and the Jacobian matrix of first derivatives $\frac{\partial^2 f_i}{\partial x_j^2}$ at any point x. Since the routine is not a parameter to E04GCF, it must be called LSFUN2. It should be tested separately before being used in conjunction with E04GCF (see the Chapter Introduction).

Its specification is:

```
SUBROUTINE LSFUN2 (M, N, XC, FVECC, FJACC, LJC)
INTEGER          M, N, LJC
DOUBLE PRECISION XC(N), FVECC(M), FJACC(LJC,N)
```

Important: The dimension declaration for FJACC must contain the variable LJC, not an integer constant.

1: M -- INTEGER Input

2: N -- INTEGER Input
On entry: the numbers m and n of residuals and variables, respectively.

3: XC(N) -- DOUBLE PRECISION array Input
On entry: the point x at which the values of the f_i

```
                ddf
                   i
    and the     ---- are required.
                ddx
                   j

     4: FVECC(M) -- DOUBLE PRECISION array                Output
        On exit: FVECC(i) must contain the value of f  at the
                                                      i
        point x, for i=1,2,...,m.

     5: FJACC(LJC,N) -- DOUBLE PRECISION array             Output
                                                          ddf
                                                             i
        On exit: FJACC(i,j) must contain the value of    ---- at
                                                          ddx
                                                             j
        the point x, for i=1,2,...,m; j=1,2,...,n.

     6: LJC -- INTEGER                                      Input
        On entry: the first dimension of the array FJACC.
    LSFUN2 must be declared as EXTERNAL in the (sub)program
    from which E04GCF is called. Parameters denoted as
    Input must not be changed by this procedure.

6. Error Indicators and Warnings

Errors or warnings specified by the routine:

If on entry IFAIL = 0 or -1, explanatory error messages are
output on the current error message unit (as defined by X04AAF).

IFAIL= 1
    On entry N < 1,

    or       M < N,

    or       LIW < 1,

    or       LW < 2*N*(4 + N + M) + 3*M, when N > 1,

    or       LW < 9 + 5*>M, when N = 1.

IFAIL= 2
    There have been 50*n calls of LSFUN2, yet the algorithm does
    not seem to have converged. This may be due to an awkward
```

function or to a poor starting point, so it is worth restarting E04GCF from the final point held in X.

IFAIL= 3
The final point does not satisfy the conditions for acceptance as a minimum, but no lower point could be found.

IFAIL= 4
An auxiliary routine has been unable to complete a singular value decomposition in a reasonable number of sub-iterations.

IFAIL= 5

IFAIL= 6

IFAIL= 7

IFAIL= 8
There is some doubt about whether the point X found by E04GCF is a minimum of F(x). The degree of confidence in the result decreases as IFAIL increases. Thus, when IFAIL = 5, it is probable that the final x gives a good estimate of the position of a minimum, but when IFAIL = 8 it is very unlikely that the routine has found a minimum.

IFAIL= 9
It is very likely that the user has made an error in forming the derivatives $\frac{\partial^2 f}{\partial x_i \partial x_j}$ in LSFUN2.

If the user is not satisfied with the result (e.g. because IFAIL lies between 3 and 8), it is worth restarting the calculations from a different starting point (not the point at which the failure occurred) in order to avoid the region which caused the failure. Repeated failure may indicate some defect in the formulation of the problem.

7. Accuracy

If the problem is reasonably well scaled and a successful exit is made then, for a computer with a mantissa of t decimals, one would expect to get t/2-1 decimals accuracy in the components of

x and between t-1 (if F(x) is of order 1 at the minimum) and 2t-2 (if F(x) is close to zero at the minimum) decimals accuracy in F(x).

8. Further Comments

The number of iterations required depends on the number of variables, the number of residuals and their behaviour, and the distance of the starting point from the solution. The number of multiplications performed per iteration of E04GCF varies, but for $m \gg n$ is approximately $n*m + O(n^2)$. In addition, each iteration makes at least one call of LSFUN2. So, unless the residuals and their derivatives can be evaluated very quickly, the run time will be dominated by the time spent in LSFUN2.

Ideally the problem should be scaled so that the minimum value of the sum of squares is in the range (0,1) and so that at points a unit distance away from the solution the sum of squares is approximately a unit value greater than at the minimum. It is unlikely that the user will be able to follow these recommendations very closely, but it is worth trying (by guesswork), as sensible scaling will reduce the difficulty of the minimization problem, so that E04GCF will take less computer time.

When the sum of squares represents the goodness of fit of a nonlinear model to observed data, elements of the variance-covariance matrix of the estimated regression coefficients can be computed by a subsequent call to E04YCF, using information returned in segments of the workspace array W. See E04YCF for further details.

9. Example

To find the least-squares estimates of x_1, x_2 and x_3 in the model

$$y = x_1 + \frac{t_1}{x_2 t_1 + x_3 t_2}$$

using the 15 sets of data given in the following table.

```
 y    t   t    t
       1   2    3
0.14  1.0 15.0 1.0
0.18  2.0 14.0 2.0
0.22  3.0 13.0 3.0
0.25  4.0 12.0 4.0
0.29  5.0 11.0 5.0
0.32  6.0 10.0 6.0
0.35  7.0  9.0 7.0
0.39  8.0  8.0 8.0
0.37  9.0  7.0 7.0
0.58 10.0  6.0 6.0
0.73 11.0  5.0 5.0
0.96 12.0  4.0 4.0
1.34 13.0  3.0 3.0
2.10 14.0  2.0 2.0
4.39 15.0  1.0 1.0
```

The program uses (0.5, 1.0, 1.5) as the initial guess at the position of the minimum.

The example program is not reproduced here. The source code for all example programs is distributed with the NAG Foundation Library software and should be available on-line.

%%%

E04 -- Minimizing or Maximizing a Function E04JAF
E04JAF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for your implementation to check implementation-dependent details. The symbol (*) after a NAG routine name denotes a routine that is not included in the Foundation Library.

1. Purpose

E04JAF is an easy-to-use quasi-Newton algorithm for finding a minimum of a function $F(x_1, x_2, \ldots, x_n)$, subject to fixed upper and lower bounds of the independent variables $x_1, x_2, \ldots, x_n$, using function values only.

It is intended for functions which are continuous and which have continuous first and second derivatives (although it will usually

work even if the derivatives have occasional discontinuities).

2. Specification

```
      SUBROUTINE E04JAF (N, IBOUND, BL, BU, X, F, IW, LIW, W,
     1                   LW, IFAIL)
      INTEGER          N, IBOUND, IW(LIW), LIW, LW, IFAIL
      DOUBLE PRECISION BL(N), BU(N), X(N), F, W(LW)
```

3. Description

This routine is applicable to problems of the form:

$$\text{Minimize } F(x_1, x_2, \ldots, x_n) \text{ subject to } l_j \leq x_j \leq u_j, \quad j = 1, 2, \ldots, n$$

when derivatives of F(x) are unavailable.

Special provision is made for problems which actually have no bounds on the x_j, problems which have only non-negativity bounds and problems in which $l_1 = l_2 = \ldots = l_n$ and $u_1 = u_2 = \ldots = u_n$. The user must supply a subroutine FUNCT1 to calculate the value of F(x) at any point x.

From a starting point supplied by the user there is generated, on the basis of estimates of the gradient and the curvature of F(x), a sequence of feasible points which is intended to converge to a local minimum of the constrained function. An attempt is made to verify that the final point is a minimum.

4. References

[1] Gill P E and Murray W (1976) Minimization subject to bounds on the variables. Report NAC 72. National Physical Laboratory.

5. Parameters

1: N -- INTEGER Input
On entry: the number n of independent variables.
Constraint: N >= 1.

2: IBOUND -- INTEGER Input
On entry: indicates whether the facility for dealing with

bounds of special forms is to be used.

It must be set to one of the following values:
IBOUND = 0
if the user will be supplying all the l_j and u_j individually.

IBOUND = 1
if there are no bounds on any x_j.

IBOUND = 2
if all the bounds are of the form $0 \leq x_j$.

IBOUND = 3
if $l_1 = l_2 = \ldots = l_n$ and $u_1 = u_2 = \ldots = u_n$.

3: BL(N) -- DOUBLE PRECISION array Input/Output
On entry: the lower bounds l_j.

If IBOUND is set to 0, the user must set BL(j) to l_j, for j=1,2,...,n. (If a lower bound is not specified for a particular x_j, the corresponding BL(j) should be set to -10.)

If IBOUND is set to 3, the user must set BL(1) to l_1; E04JAF will then set the remaining elements of BL equal to BL(1).
On exit: the lower bounds actually used by E04JAF.

4: BU(N) -- DOUBLE PRECISION array Input/Output
On entry: the upper bounds u_j.

If IBOUND is set to 0, the user must set BU(j) to u_j, for j=1,2,...,n. (If an upper bound is not specified for a particular x_j, the corresponding BU(j) should be set to 10.)

If IBOUND is set to 3, the user must set BU(1) to u_1; E04JAF will then set the remaining elements of BU equal to BU(1).
On exit: the upper bounds actually used by E04JAF.

5: X(N) -- DOUBLE PRECISION array Input/Output
On entry: X(j) must be set to an estimate of the jth component of the position of the minimum, for j=1,2,...,n.
On exit: the lowest point found during the calculations. Thus, if IFAIL = 0 on exit, X(j) is the jth component of the position of the minimum.

6: F -- DOUBLE PRECISION Output
On exit: the value of F(x) corresponding to the final point stored in X.

7: IW(LIW) -- INTEGER array Workspace

8: LIW -- INTEGER Input
On entry: the length of IW as declared in the (sub)program from which E04JAF is called. Constraint: LIW >= N + 2.

9: W(LW) -- DOUBLE PRECISION array Workspace

10: LW -- INTEGER Input
On entry: the length of W as declared in the (sub)program from which E04JAF is called. Constraint: LW>=max(N*(N-1)/2+12*N,13).

11: IFAIL -- INTEGER Input/Output
On entry: IFAIL must be set to 0, -1 or 1. Users who are unfamiliar with this parameter should refer to the Essential Introduction for details.

On exit: IFAIL = 0 unless the routine detects an error or gives a warning (see Section 6).

For this routine, because the values of output parameters may be useful even if IFAIL /=0 on exit, users are recommended to set IFAIL to -1 before entry. It is then essential to test the value of IFAIL on exit. To suppress the output of an error message when soft failure occurs, set IFAIL to 1.

5.1. Optional Parameters

```
     FUNCT1 -- SUBROUTINE, supplied by the user.
                                          External Procedure
     This routine must be supplied by the user to calculate the
     value of the function F(x) at any point x. Since this
     routine is not a parameter to E04JAF, it must be called
     FUNCT1. It should be tested separately before being used in
     conjunction with E04JAF (see the Chapter Introduction).

     Its specification is:

           SUBROUTINE FUNCT1 (N, XC, FC)
           INTEGER          N
           DOUBLE PRECISION XC(N), FC

      1:  N -- INTEGER                                        Input
          On entry: the number n of variables.

      2:  XC(N) -- DOUBLE PRECISION array                     Input
          On entry: the point x at which the function value is
          required.

      3:  FC -- DOUBLE PRECISION                             Output
          On exit: the value of the function F at the current
          point x.
     FUNCT1 must be declared as EXTERNAL in the (sub)program
     from which E04JAF is called. Parameters denoted as
     Input must not be changed by this procedure.

6. Error Indicators and Warnings

Errors or warnings specified by the routine:

IFAIL= 1
     On entry N < 1,

     or       IBOUND < 0,

     or       IBOUND > 3,

     or       IBOUND = 0 and BL(j) > BU(j) for some j,

     or       IBOUND = 3 and BL(1) > BU(1),

     or       LIW < N + 2,
```

```
     or       LW<max(13,12*N+N*(N-1)/2).

IFAIL= 2
     There have been 400*n function evaluations, yet the
     algorithm does not seem to be converging. The calculations
     can be restarted from the final point held in X. The error
     may also indicate that F(x) has no minimum.

IFAIL= 3
     The conditions for a minimum have not all been met but a
     lower point could not be found and the algorithm has failed.

IFAIL= 4
     An overflow has occurred during the computation. This is an
     unlikely failure, but if it occurs the user should restart
     at the latest point given in X.

IFAIL= 5

IFAIL= 6

IFAIL= 7

IFAIL= 8
     There is some doubt about whether the point x found by
     E04JAF is a minimum. The degree of confidence in the result
     decreases as IFAIL increases. Thus, when IFAIL = 5 it is
     probable that the final x gives a good estimate of the
     position of a minimum, but when IFAIL = 8 it is very
     unlikely that the routine has found a minimum.

IFAIL= 9
     In the search for a minimum, the modulus of one of the
                                              6
     variables has become very large  (~10 ).  This indicates
     that there is a mistake in FUNCT1, that the user's problem
     has no finite solution, or that the problem needs rescaling
     (see Section 8).

If the user is dissatisfied with the result (e.g. because IFAIL =
5, 6, 7 or 8), it is worth restarting the calculations from a
different starting point (not the point at which the failure
occurred) in order to avoid the region which caused the failure.
If persistent trouble occurs and the gradient can be calculated,
it may be advisable to change to a routine which uses gradients
(see the Chapter Introduction).
```

7. Accuracy

When a successful exit is made then, for a computer with a mantissa of t decimals, one would expect to get about t/2-1 decimals accuracy in x and about t-1 decimals accuracy in F, provided the problem is reasonably well scaled.

8. Further Comments

The number of iterations required depends on the number of variables, the behaviour of F(x) and the distance of the starting point from the solution. The number of operations performed in an iteration of E04JAF is roughly proportional to n^2. In addition, each iteration makes at least m+1 calls of FUNCT1, where m is the number of variables not fixed on bounds. So, unless F(x) can be evaluated very quickly, the run time will be dominated by the time spent in FUNCT1.

Ideally the problem should be scaled so that at the solution the value of F(x) and the corresponding values of $x_1, x_2, \ldots, x_n$ are each in the range (-1,+1), and so that at points a unit distance away from the solution, F is approximately a unit value greater than at the minimum. It is unlikely that the user will be able to follow these recommendations very closely, but it is worth trying (by guesswork), as sensible scaling will reduce the difficulty of the minimization problem, so that E04JAF will take less computer time.

9. Example

To minimize

$$F=(x_1+10x_2)^2+5(x_3-x_4)^2+(x_2-2x_3)^4+10(x_1-x_4)^4$$

subject to

$$1 \leq x_1 \leq 3$$

$$-2 \leq x_2 \leq 0$$

```
                                  1<=x <=3,
                                      4
```

starting from the initial guess (3, - 1, 0, 1).

The example program is not reproduced here. The source code for all example programs is distributed with the NAG Foundation Library software and should be available on-line.

%%%

E04 -- Minimizing or Maximizing a Function E04MBF
E04MBF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for your implementation to check implementation-dependent details. The symbol (*) after a NAG routine name denotes a routine that is not included in the Foundation Library.

1. Purpose

E04MBF is an easy-to-use routine for solving linear programming problems, or for finding a feasible point for such problems. It is not intended for large sparse problems.

2. Specification

```
 SUBROUTINE E04MBF (ITMAX, MSGLVL, N, NCLIN, NCTOTL, NROWA,
1                   A, BL, BU, CVEC, LINOBJ, X, ISTATE,
2                   OBJLP, CLAMDA, IWORK, LIWORK, WORK,
3                   LWORK, IFAIL)
 INTEGER           ITMAX, MSGLVL, N, NCLIN, NCTOTL, NROWA,
1                  ISTATE(NCTOTL), IWORK(LIWORK), LIWORK,
2                  LWORK, IFAIL
 DOUBLE PRECISION A(NROWA,N), BL(NCTOTL), BU(NCTOTL), CVEC
1                  (N), X(N), OBJLP, CLAMDA(NCTOTL), WORK
2                  (LWORK)
 LOGICAL           LINOBJ
```

3. Description

E04MBF solves linear programming (LP) problems of the form

$$\text{Minimize} \quad c^T x \quad \text{subject to} \quad l \leq \begin{pmatrix} x \\ Ax \end{pmatrix} \leq u \qquad \text{(LP)}$$

$$x \text{ is in } R^n$$

where c is an n element vector and A is an m by n matrix i.e., there are n variables and m general linear constraints. m may be zero in which case the LP problem is subject only to bounds on the variables. Notice that upper and lower bounds are specified for all the variables and constraints. This form allows full generality in specifying other types of constraints. For example the ith constraint may be specified as equality by setting $l_i = u_i$. If certain bounds are not present the associated elements of l or u can be set to special values that will be treated as -infty or +infty.

The routine allows the linear objective function to be omitted in which case a feasible point for the set of constraints is sought.

The user must supply an initial estimate of the solution.

Users who wish to exercise additional control and users with problems whose solution would benefit from additional flexibility should consider using the comprehensive routine E04NAF.

4. References

[1] Gill P E, Murray W and Wright M H (1981) Practical Optimization. Academic Press.

[2] Gill P E, Murray W, Saunders M A and Wright M H (1983) User's Guide for SOL/QPSOL. Report SOL 83-7. Department of Operations Research, Stanford University.

5. Parameters

1: ITMAX -- INTEGER Input
On entry: an upper bound on the number of iterations to be taken. If ITMAX is not positive, then the value 50 is used in place of ITMAX.

2: MSGLVL -- INTEGER Input
On entry: indicates whether or not printout is required at the final solution. When printing occurs the output is on the advisory message channel (see X04ABF). A description of the printed output is given in Section 5.1. The level of printing is determined as follows:

```
MSGLVL < 0
      No printing.

MSGLVL = 0
      Printing only if an input parameter is incorrect, or
      if the problem is so ill-conditioned that subsequent
      overflow is likely. This setting is strongly
      recommended in preference to MSGLVL < 0.

MSGLVL = 1
      Printing at the solution.

MSGLVL > 1
      Values greater than 1 should normally be used only at
      the direction of NAG; such values may generate large
      amounts of printed output.
```

3: N -- INTEGER Input
On entry: the number n of variables. Constraint: N >= 1.

4: NCLIN -- INTEGER Input
On entry: the number of general linear constraints in the problem. Constraint: NCLIN >= 0.

5: NCTOTL -- INTEGER Input
On entry: the value (N+NCLIN).

6: NROWA -- INTEGER Input
On entry:
the first dimension of the array A as declared in the (sub)program from which E04MBF is called.
Constraint: NROWA >= max(1,NCLIN).

7: A(NROWA,N) -- DOUBLE PRECISION array Input
On entry: the leading NCLIN by n part of A must contain the NCLIN general constraints, with the coefficients of the ith constraint in the ith row of A. If NCLIN = 0, then A is not referenced.

8: BL(NCTOTL) -- DOUBLE PRECISION array Input
On entry: the first n elements of BL must contain the lower bounds on the n variables, and when NCLIN > 0, the next NCLIN elements of BL must contain the lower bounds on the NCLIN general linear constraints. To specify a non-existent lower bound (l =-infty), set BL(j)<=-1.0E+20.
j

9: BU(NCTOTL) -- DOUBLE PRECISION array Input
On entry: the first n elements of BU must contain the upper bounds on the n variables, and when NCLIN > 0, the next NCLIN elements of BU must contain the upper bounds on the NCLIN general linear constraints. To specify a non-existent upper bound (u =+infty), set BU(j)>=1.0E+20. Constraint:
BL(j)<=BU(j), for j=1,2,...,NCTOTL.

10: CVEC(N) -- DOUBLE PRECISION array Input
On entry: with LINOBJ = .TRUE., CVEC must contain the coefficients of the objective function. If LINOBJ = .FALSE., then CVEC is not referenced.

11: LINOBJ -- LOGICAL Input
On entry: indicates whether or not a linear objective function is present. If LINOBJ = .TRUE., then the full LP problem is solved, but if LINOBJ = .FALSE., only a feasible point is found and the array CVEC is not referenced.

12: X(N) -- DOUBLE PRECISION array Input/Output
On entry: an estimate of the solution, or of a feasible point. Even when LINOBJ = .TRUE. it is not necessary for the point supplied in X to be feasible. In the absence of better information all elements of X may be set to zero. On exit: the solution to the LP problem when LINOBJ = .TRUE., or a feasible point when LINOBJ = .FALSE..

When no feasible point exists (see IFAIL = 1 in Section 6) then X contains the point for which the sum of the infeasibilities is a minimum. On return with IFAIL = 2, 3 or 4, X contains the point at which E04MBF terminated.

13: ISTATE(NCTOTL) -- INTEGER array Output
On exit: with IFAIL < 5, ISTATE indicates the status of every constraint at the final point. The first n elements of ISTATE refer to the upper and lower bounds on the variables and when NCLIN > 0 the next NCLIN elements refer to the general constraints.

Their meaning is:
ISTATE(j) Meaning

-2 The constraint violates its lower bound. This value cannot occur for any element of ISTATE when

a feasible point has been found.

-1 The constraint violates its upper bound. This value cannot occur for any element of ISTATE when a feasible point has been found.

0 The constraint is not in the working set (is not active) at the final point. Usually this means that the constraint lies strictly between its bounds.

1 This inequality constraint is in the working set (is active) at its lower bound.

2 This inequality constraint is in the working set (is active) at its upper bound.

3 This constraint is included in the working set (is active) as an equality. This value can only occur when BL(j) = BU(j).

14: OBJLP -- DOUBLE PRECISION Output
On exit: when LINOBJ = .TRUE., then on successful exit, OBJLP contains the value of the objective function at the solution, and on exit with IFAIL = 2, 3 or 4, OBJLP contains the value of the objective function at the point returned in X.

When LINOBJ = .FALSE., then on successful exit OBJLP will be zero and on return with IFAIL = 1, OBJLP contains the minimum sum of the infeasibilities corresponding to the point returned in X.

15: CLAMDA(NCTOTL) -- DOUBLE PRECISION array Output
On exit: when LINOBJ = .TRUE., then on successful exit, or on exit with IFAIL = 2, 3, or 4, CLAMDA contains the Lagrange multipliers (reduced costs) for each constraint with respect to the working set. The first n components of CLAMDA contain the multipliers for the bound constraints on the variables and the remaining NCLIN components contain the multipliers for the general linear constraints.

If ISTATE(j) = 0 so that the jth constraint is not in the working set then CLAMDA(j) is zero. If X is optimal and ISTATE(j) = 1, then CLAMDA(j) should be non-negative, and if ISTATE(j) = 2, then CLAMDA(j) should be non-positive.

When LINOBJ = .FALSE., all NCTOTL elements of CLAMDA are returned as zero.

16: IWORK(LIWORK) -- INTEGER array Workspace

17: LIWORK -- INTEGER Input
On entry: the length of the array IWORK as declared in the (sub)program from which E04MBF is called. Constraint: LIWORK>=2*N.

18: WORK(LWORK) -- DOUBLE PRECISION array Workspace

19: LWORK -- INTEGER Input
On entry: the length of the array WORK as declared in the (sub)program from which E04MBF is called. Constraints:
when N <= NCLIN then

$$\text{LWORK} \geq 2*N^2 + 6*N + 4*\text{NCLIN} + \text{NROWA};$$

when 0 <= NCLIN < N then

$$\text{LWORK} \geq 2*(\text{NCLIN}+1)^2 + 4*\text{NCLIN} + 6*N + \text{NROWA}.$$

20: IFAIL -- INTEGER Input/Output
On entry: IFAIL must be set to 0, -1 or 1. Users who are unfamiliar with this parameter should refer to the Essential Introduction for details.

On exit: IFAIL = 0 unless the routine detects an error or gives a warning (see Section 6).

For this routine, because the values of output parameters may be useful even if IFAIL /=0 on exit, users are recommended to set IFAIL to -1 before entry. It is then essential to test the value of IFAIL on exit. To suppress the output of an error message when soft failure occurs, set IFAIL to 1.

5.1. Description of the Printed Output

When MSGLVL = 1, then E04MBF will produce output on the advisory message channel (see X04ABF), giving information on the final point. The following describes the printout associated with each variable.

Output	Meaning
VARBL	The name (V) and index j, for j=1,2,...,n, of the variable.
STATE	The state of the variable. (FR if neither bound is in the working set, EQ for a fixed variable, LL if on its lower bound, UL if on its upper bound and TB if held on a temporary bound.) If the value of the variable lies outside the upper or lower bound then STATE will be ++ or -- respectively.
VALUE	The value of the variable at the final iteration.
LOWER BOUND	The lower bound specified for the variable.
UPPER BOUND	The upper bound specified for the variable.
LAGR MULT	The value of the Lagrange multiplier for the associated bound.
RESIDUAL	The difference between the value of the variable and the nearer of its bounds.

For each of the general constraints the printout is as above with refers to the jth element of Ax, except that VARBL is replaced by:

LNCON	The name (L) and index j, for j = 1,2,...,NCLIN of the constraint.

6. Error Indicators and Warnings

Errors or warnings specified by the routine:

Note: when MSGLVL=1 a short description of the error is printed.

IFAIL= 1
No feasible point could be found. Moving violated constraints so that they are satisfied at the point returned in X gives the minimum moves necessary to make the LP problem feasible.

IFAIL= 2
The solution to the LP problem is unbounded.

IFAIL= 3
A total of 50 changes were made to the working set without altering x. Cycling is probably occurring. The user should consider using E04NAF with MSGLVL >= 5 to monitor constraint additions and deletions in order to determine whether or not cycling is taking place.

IFAIL= 4
The limit on the number of iterations has been reached. Increase ITMAX or consider using E04NAF to monitor progress.

IFAIL= 5
An input parameter is invalid. Unless MSGLVL < 0 a message will be printed.

IFAILOverflow
If the printed output before the overflow occurred contains a warning about serious ill-conditioning in the working set when adding the jth constraint, then either the user should try using E04NAF and experiment with the magnitude of FEATOL (j) in that routine, or the offending linearly dependent constraint (with index j) should be removed from the problem.

7. Accuracy

The routine implements a numerically stable active set strategy and returns solutions that are as accurate as the condition of the LP problem warrants on the machine.

8. Further Comments

The time taken by each iteration is approximately proportional to $\min(n^2, NCLIN^2)$.

Sensible scaling of the problem is likely to reduce the number of iterations required and make the problem less sensitive to perturbations in the data, thus improving the condition of the LP problem. In the absence of better information it is usually sensible to make the Euclidean lengths of each constraint of comparable magnitude. See Gill et al [1] for further information and advice.

Note that the routine allows constraints to be violated by an absolute tolerance equal to the machine precision (see X02AJF(*))

9. Example

To minimize the function

$$-0.02x_1 - 0.2x_2 - 0.2x_3 - 0.2x_4 - 0.2x_5 + 0.04x_6 + 0.04x_7$$

subject to the bounds

$$-0.01 \leq x_1 \leq 0.01$$
$$-0.1 \leq x_2 \leq 0.15,$$
$$-0.01 \leq x_3 \leq 0.03,$$
$$-0.04 \leq x_4 \leq 0.02,$$
$$-0.1 \leq x_5 \leq 0.05,$$
$$-0.01 \leq x_6$$
$$-0.01 \leq x_7$$

and the general constraints

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 = -0.13$$

$$0.15x_1 + 0.04x_2 + 0.02x_3 + 0.04x_4 + 0.02x_5 + 0.01x_6 + 0.03x_7 \leq -0.0049$$

$$0.03x_1 + 0.05x_2 + 0.08x_3 + 0.02x_4 + 0.06x_5 + 0.01x_6 \leq -0.0064$$

$$0.02x_1 + 0.04x_2 + 0.01x_3 + 0.02x_4 + 0.02x_5 \leq -0.0037$$

$$0.02x_1 + 0.03x_2 + 0.01x_3 \leq -0.0012$$

$$-0.0992 \leq 0.70x_1 + 0.75x_2 + 0.80x_3 + 0.75x_4 + 0.80x_5 + 0.97x_6$$

$$-0.003 \leq 0.02x_1 + 0.06x_2 + 0.08x_3 + 0.12x_4 + 0.02x_5 + 0.01x_6 + 0.97x_7 \leq 0.002$$

```
                1      2      3      4      5      6      7
```

The initial point, which is infeasible, is

$$x_0 = (-0.01, -0.03, 0.0, -0.01, -0.1, 0.02, 0.01)^T.$$

The example program is not reproduced here. The source code for all example programs is distributed with the NAG Foundation Library software and should be available on-line.

%%%

E04 -- Minimizing or Maximizing a Function E04NAF

E04NAF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for your implementation to check implementation-dependent details. The symbol (*) after a NAG routine name denotes a routine that is not included in the Foundation Library.

1. Purpose

E04NAF is a comprehensive routine for solving quadratic programming (QP) or linear programming (LP) problems. It is not intended for large sparse problems.

2. Specification

```
 SUBROUTINE E04NAF (ITMAX, MSGLVL, N, NCLIN, NCTOTL, NROWA,
1                   NROWH, NCOLH, BIGBND, A, BL, BU, CVEC,
2                   FEATOL, HESS, QPHESS, COLD, LP, ORTHOG,
3                   X, ISTATE, ITER, OBJ, CLAMDA, IWORK,
4                   LIWORK, WORK, LWORK, IFAIL)
 INTEGER           ITMAX, MSGLVL, N, NCLIN, NCTOTL, NROWA,
1                  NROWH, NCOLH, ISTATE(NCTOTL), ITER, IWORK
2                  (LIWORK), LIWORK, LWORK, IFAIL
 DOUBLE PRECISION BIGBND, A(NROWA,N), BL(NCTOTL),
1                  BU(NCTOTL), CVEC(N), FEATOL(NCTOTL), HESS
2                  (NROWH,NCOLH), X(N), OBJ, CLAMDA(NCTOTL),
3                  WORK(LWORK)
 LOGICAL           COLD, LP, ORTHOG
 EXTERNAL          QPHESS
```

3. Description

E04NAF is essentially identical to the subroutine SOL/QPSOL described in Gill et al [1].

E04NAF is designed to solve the quadratic programming (QP) problem - the minimization of a quadratic function subject to a set of linear constraints on the variables. The problem is assumed to be stated in the following form:

$$\text{Minimize} \quad c^T x + \frac{1}{2} x^T H x \quad \text{subject to} \quad l \leq \begin{pmatrix} x \\ Ax \end{pmatrix} \leq u , \qquad (1)$$

where c is a constant n-vector and H is a constant n by n symmetric matrix; note that H is the Hessian matrix (matrix of second partial derivatives) of the quadratic objective function. The matrix A is m by n, where m may be zero; A is treated as a dense matrix.

The constraints involving A will be called the general constraints. Note that upper and lower bounds are specified for all the variables and for all the general constraints. The form of (1) allows full generality in specifying other types of constraints. In particular, an equality constraint is specified by setting $l_i = u_i$. If certain bounds are not present, the associated elements of l or u can be set to special values that will be treated as -infty or +infty.

The user must supply an initial estimate of the solution to (1), and a subroutine that computes the product Hx for any given vector x. If H is positive-definite or positive semi-definite, E04NAF will obtain a global minimum; otherwise, the solution obtained will be a local minimum (which may or may not be a global minimum). If H is defined as the zero matrix, E04NAF will solve the resulting linear programming (LP) problem; however, this can be accomplished more efficiently by setting a logical variable in the call of the routine (see the parameter LP in Section 5).

E04NAF allows the user to provide the indices of the constraints that are believed to be exactly satisfied at the solution. This facility, known as a warm start, can lead to significant savings in computational effort when solving a sequence of related problems.

The method has two distinct phases. In the first (the LP phase), an iterative procedure is carried out to determine a feasible point. In this context, feasibility is defined by a user-provided array FEATOL; the jth constraint is considered satisfied if its violation does not exceed FEATOL(j). The second phase (the QP phase) generates a sequence of feasible iterates in order to minimize the quadratic objective function. In both phases, a subset of the constraints - called the working set - is used to define the search direction at each iteration; typically, the working set includes constraints that are satisfied to within the corresponding tolerances in the FEATOL array.

We now briefly describe a typical iteration in the QP phase. Let x_k denote the estimate of the solution at the kth iteration; the next iterate is defined by

$$x_{k+1} = x_k + (\alpha)_k p_k$$

where p_k is an n-dimensional search direction and $(\alpha)_k$ is a scalar step length. Assume that the working (active) set contains t_k linearly independent constraints, and let C_k denote the matrix of coefficients of the bounds and general constraints in the current working set.

Let Z_k denote a matrix whose columns form a basis for the null space of C_k, so that $C_k Z_k = 0$. (Note that Z_k has n_z columns, where $n_z = n - t_k$.) The vector $Z_k^T (c + Hx_k)$ is called the projected gradient at x_k. If the projected gradient is zero at x_k (i.e., x_k is a constrained stationary point in the subspace defined by Z_k), Lagrange multipliers $(\lambda)_k$ are defined as the solution of the compatible overdetermined system

$$C_k^T (\lambda)_k = c + Hx_k \tag{2}$$

The Lagrange multiplier (lambda) corresponding to an inequality constraint in the working set is said to be optimal if (lambda)<=0 when the associated constraint is at its upper bound, or if (lambda)>=0 when the associated constraint is at its lower bound. If a multiplier is non-optimal, the objective function can be reduced by deleting the corresponding constraint (with index JDEL, see Section 5.1) from the working set.

If the projected gradient at x_k is non-zero, the search direction p_k is defined as

$$p_k = Z_k p_z \tag{3}$$

where p_z is an n_z-vector. In effect, the constraints in the working set are treated as equalities, by constraining p_k to lie within the subspace of vectors orthogonal to the rows of C_k. This definition ensures that $C_k p_k = 0$, and hence the values of the constraints in the working set are not altered by any move along p_k.

The vector p_z is obtained by solving the equations

$$Z_k^T H Z_k p_z = -Z_k^T (c + H x_k) \tag{4}$$

(The matrix $Z_k^T H Z_k$ is called the projected Hessian matrix.) If the projected Hessian is positive-definite, the vector defined by (3) and (4) is the step to the minimum of the quadratic function in the subspace defined by Z_k.

If the projected Hessian is positive-definite and $x_k + p_k$ is

feasible, (alpha) will be taken as unity. In this case, the
k
projected gradient at x will be zero (see NORM ZTG in Section
k+1
5.1), and Lagrange multipliers can be computed (see Gill et al
[2]). Otherwise, (alpha) is set to the step to the 'nearest'
k
constraint (with index JADD, see Section 5.1), which is added to
the working set at the next iteration.

The matrix Z is obtained from the TQ factorization of C , in
k k
which C is represented as
k

$$C_k Q = (0 \; T_k) \tag{5}$$

where T is reverse-triangular. It follows from (5) that Z may
k k
be taken as the first n columns of Q. If the projected Hessian
z
is positive-definite, (3) is solved using the Cholesky
factorization

$$Z_k^T H Z_k = R_k^T R_k$$

where R is upper triangular. These factorizations are updated as
k
constraints enter or leave the working set (see Gill et al [2]
for further details).

An important feature of E04NAF is the treatment of indefiniteness
in the projected Hessian. If the projected Hessian is positive-
definite, it may become indefinite only when a constraint is
deleted from the working set. In this case, a temporary
modification (of magnitude HESS MOD, see Section 5.1) is added to
the last diagonal element of the Cholesky factor. Once a
modification has occurred, no further constraints are deleted
from the working set until enough constraints have been added so
that the projected Hessian is again positive-definite. If
equation (1) has a finite solution, a move along the direction
obtained by solving (4) with the modified Cholesky factor must
encounter a constraint that is not already in the working set.

In order to resolve indefiniteness in this way, we must ensure that the projected Hessian is positive-definite at the first iterate in the QP phase. Given the n by n projected Hessian, a
z z
step-wise Cholesky factorization is performed with symmetric interchanges (and corresponding rearrangement of the columns of Z), terminating if the next step would cause the matrix to become indefinite. This determines the largest possible positive-definite principal sub-matrix of the (permuted) projected Hessian. If n steps of the Cholesky factorization have been
R
successfully completed, the relevant projected Hessian is an n
R
T
by n positive-definite matrix Z HZ , where Z comprises the
R R R R
first n columns of Z. The quadratic function will subsequently
R
be minimized within subspaces of reduced dimension until the full projected Hessian is positive-definite.

If a linear program is being solved and there are fewer general constraints than variables, the method moves from one vertex to another while minimizing the objective function. When necessary, an initial vertex is defined by temporarily fixing some of the variables at their initial values.

Several strategies are used to control ill-conditioning in the working set. One such strategy is associated with the FEATOL array. Allowing the jth constraint to be violated by as much as FEATOL(j) often provides a choice of constraints that could be added to the working set. When a choice exists, the decision is based on the conditioning of the working set. Negative steps are occasionally permitted, since x may violate the constraint to be
k
added.

4. References

[1] Gill P E, Murray W, Saunders M A and Wright M H (1983) User's Guide for SOL/QPSOL. Report SOL 83-7. Department of Operations Research, Stanford University.

[2] Gill P E, Murray W, Saunders M A and Wright M H (1982) The design and implementation of a quadratic programming

algorithm. Report SOL 82-7. Department of Operations Research, Stanford University.

[3] Gill P E, Murray W and Wright M H (1981) Practical Optimization. Academic Press.

5. Parameters

1: ITMAX -- INTEGER Input
On entry: an upper bound on the number of iterations to be taken during the LP phase or the QP phase. If ITMAX is not positive, then the value 50 is used in place of ITMAX.

2: MSGLVL -- INTEGER Input
On entry: MSGLVL must indicate the amount of intermediate output desired (see Section 5.1 for a description of the printed output). All output is written to the current advisory message unit (see X04ABF). For MSGLVL >= 10, each level includes the printout for all lower levels.

Value	Definition
<0	No printing.
0	Printing only if an input parameter is incorrect, or if the working set is so ill-conditioned that subsequent overflow is likely. This setting is strongly recommended in preference to MSGLVL < 0.
1	The final solution only.
5	One brief line of output for each constraint addition or deletion (no printout of the final solution).
>=10	The final solution and one brief line of output for each constraint addition or deletion.
>=15	At each iteration, X, ISTATE, and the indices of the free variables (i.e.,the variables not currently held on a bound).
>=20	At each iteration, the Lagrange multiplier estimates and the general constraint values.
>=30	At each iteration, the diagonal elements of the matrix T associated with the TQ factorization of the

```
                working set, and the diagonal elements of the
                Cholesky factor R of the projected Hessian.

        >=80    Debug printout.

        99      The arrays CVEC and HESS.

    3:  N -- INTEGER                                              Input
        On entry: the number, n, of variables. Constraint: N >= 1.

    4:  NCLIN -- INTEGER                                          Input
        On entry: the number of general linear constraints in the
        problem. Constraint: NCLIN >= 0.

    5:  NCTOTL -- INTEGER                                         Input
        On entry: the value (N+NCLIN).

    6:  NROWA -- INTEGER                                          Input
        On entry:
        the first dimension of the array A as declared in the
        (sub)program from which E04NAF is called.
        Constraint: NROWA >= max(1,NCLIN).

    7:  NROWH -- INTEGER                                          Input
        On entry: the first dimension of the array HESS as declared
        in the (sub)program from which E04NAF is called.
        Constraint: NROWH >= 1.

    8:  NCOLH -- INTEGER                                          Input
        On entry: the column dimension of the array HESS as declared
        in the (sub)program from which E04NAF is called.
        Constraint: NCOLH >= 1.

    9:  BIGBND -- DOUBLE PRECISION                                Input
        On entry: BIGBND must denote an 'infinite' component of l
        and u. Any upper bound greater than or equal to BIGBND will
        be regarded as plus infinity, and a lower bound less than or
        equal to -BIGBND will be regarded as minus infinity.
        Constraint: BIGBND > 0.0.

   10:  A(NROWA,N) -- DOUBLE PRECISION array                      Input
        On entry: the leading NCLIN by n part of A must contain the
        NCLIN general constraints, with the ith constraint in the i
        th row of A. If NCLIN = 0, then A is not referenced.

   11:  BL(NCTOTL) -- DOUBLE PRECISION array                      Input
```

On entry: the lower bounds for all the constraints, in the following order. The first n elements of BL must contain the lower bounds on the variables. If NCLIN > 0, the next NCLIN elements of BL must contain the lower bounds for the general linear constraints. To specify a non-existent lower bound (i.e., l =-infty), the value used must satisfy BL(j)<=-
j
BIGBND To specify the jth constraint as an equality, the user must set BL(j) = BU(j). Constraint: BL(j) <= BU(j), j=1,2,...,NCTOTL.

12: BU(NCTOTL) -- DOUBLE PRECISION array Input
On entry: the upper bounds for all the constraints, in the following order. The first n elements of BU must contain the upper bounds on the variables. If NCLIN > 0, the next NCLIN elements of BU must contain the upper bounds for the general linear constraints. To specify a non-existent upper bound (i.e., u =+infty), the value used must satisfy BU(j) >=
j
BIGBND. To specify the jth constraint as an equality, the user must set BU(j) = BL(j). Constraint: BU(j) >= BL(j), j=1,2,...,NCTOTL.

13: CVEC(N) -- DOUBLE PRECISION array Input
On entry: the coefficients of the linear term of the objective function (the vector c in equation (1)).

14: FEATOL(NCTOTL) -- DOUBLE PRECISION array Input
On entry: a set of positive tolerances that define the maximum permissible absolute violation in each constraint in order for a point to be considered feasible, i.e., if the violation in constraint j is less than FEATOL(j), the point is considered to be feasible with respect to the jth constraint. The ordering of the elements of FEATOL is the same as that described above for BL.

The elements of FEATOL should not be too small and a warning message will be printed on the current advisory message channel if any element of FEATOL is less than the machine precision (see X02AJF(*)). As the elements of FEATOL increase, the algorithm is less likely to encounter difficulties with ill-conditioning and degeneracy. However, larger values of FEATOL(j) mean that constraint j could be violated by a significant amount. It is recommended that FEATOL(j) be set to a value equal to the largest acceptable violation for constraint j. For example, if the data

defining the constraints are of order unity and are correct to about 6 decimal digits, it would be appropriate to choose
$$-6$$
FEATOL(j) as 10 for all relevant j. Often the square root of the machine precision is a reasonable choice if the constraint is well scaled.

15: HESS(NROWH,NCOLH) -- DOUBLE PRECISION array Input
On entry: HESS may be used to store the Hessian matrix H of equation (1) if desired. HESS is accessed only by the subroutine QPHESS and is not accessed if LP = .TRUE.. Refer to the specification of QPHESS (below) for further details of how HESS may be used to pass data to QPHESS.

16: QPHESS -- SUBROUTINE, supplied by the user.
External Procedure
QPHESS must define the product of the Hessian matrix H and a vector x. The elements of H need not be defined explicitly. QPHESS is not accessed if LP is set to .TRUE. and in this case QPHESS may be the dummy routine E04NAN. (E04NAN is included in the NAG Foundation Library and so need not be supplied by the user. Its name may be implementation-dependent: see the Users' Note for your implementation for details.)

Its specification is:

```
SUBROUTINE QPHESS (N, NROWH, NCOLH, JTHCOL,
1                  HESS, X, HX)
INTEGER          N, NROWH, NCOLH, JTHCOL
DOUBLE PRECISION HESS(NROWH,NCOLH), X(N), HX(N)
```

1: N -- INTEGER Input
On entry: the number n of variables.

2: NROWH -- INTEGER Input
On entry: the row dimension of the array HESS.

3: NCOLH -- INTEGER Input
On entry: the column dimension of the array HESS.

4: JTHCOL -- INTEGER Input
The input parameter JTHCOL is included to allow flexibility for the user in the special situation when x is the jth co-ordinate vector (i.e.,the jth column of the identity matrix). This may be of interest because

the product Hx is then the jth column of H, which can sometimes be computed very efficiently. The user may code QPHESS to take advantage of this case. On entry: if JTHCOL = j, where j>0, HX must contain column JTHCOL of H, and hence special code may be included in QPHESS to test JTHCOL if desired. However, special code is not necessary, since the vector x always contains column JTHCOL of the identity matrix whenever QPHESS is called with JTHCOL > 0.

5: HESS(NROWH,NCOLH) -- DOUBLE PRECISION array Input
On entry: the Hessian matrix H.

In some cases, it may be desirable to use a one-dimensional array to transmit data or workspace to QPHESS; HESS should then be declared with dimension (NROWH) in the (sub)program from which E04NAF is called and the parameter NCOLH must be 1.

In other situations, it may be desirable to compute Hx without accessing HESS - for example, if H is sparse or has special structure. (This is illustrated in the subroutine QPHES1 in the example program in Section 9.) The parameters HESS, NROWH and NCOLH may then refer to any convenient array.

When MSGLVL = 99, the (possibly undefined) contents of HESS will be printed, except if NROWH and NCOLH are both 1. Also printed are the results of calling QPHESS with JTHCOL = 1,2,...,n.

6: X(N) -- DOUBLE PRECISION array Input
On entry: the vector x.

7: HX(N) -- DOUBLE PRECISION array Output
On exit: HX must contain the product Hx.

QPHESS must be declared as EXTERNAL in the (sub)program from which E04NAF is called. Parameters denoted as Input must not be changed by this procedure.

17: COLD -- LOGICAL Input
On entry: COLD must indicate whether the user has specified an initial estimate of the active set of constraints. If COLD is set to .TRUE., the initial working set is determined by E04NAF. If COLD is set to .FALSE. (a 'warm start'), the user must define the ISTATE array which gives the status of

each constraint with respect to the working set. E04NAF will override the user's specification of ISTATE if necessary, so that a poor choice of working set will not cause a fatal error.

The warm start option is particularly useful when E04NAF is called repeatedly to solve related problems.

18: LP -- LOGICAL Input
On entry: if LP = .FALSE., E04NAF will solve the specified quadratic programming problem. If LP = .TRUE., E04NAF will treat H as zero and solve the resulting linear programming problem; in this case, the parameters HESS and QPHESS will not be referenced.

19: ORTHOG -- LOGICAL Input
On entry: ORTHOG must indicate whether orthogonal transformations are to be used in computing and updating the TQ factorization of the working set

$$A_s Q=(0\ T),$$

where A_s is a sub-matrix of A and T is reverse-triangular.
If ORTHOG = .TRUE., the TQ factorization is computed using Householder reflections and plane rotations, and the matrix Q is orthogonal. If ORTHOG = .FALSE., stabilized elementary transformations are used to maintain the factorization, and Q is not orthogonal. A rule of thumb in making the choice is that orthogonal transformations require more work, but provide greater numerical stability. Thus, we recommend setting ORTHOG to .TRUE. if the problem is reasonably small or the active set is ill-conditioned. Otherwise, setting ORTHOG to .FALSE. will often lead to a reduction in solution time with negligible loss of reliability.

20: X(N) -- DOUBLE PRECISION array Input/Output
On entry: an estimate of the solution. In the absence of better information all elements of X may be set to zero. On exit: from E04NAF, X contains the best estimate of the solution.

21: ISTATE(NCTOTL) -- INTEGER array Input/Output
On entry: with COLD as .FALSE., ISTATE must indicate the status of every constraint with respect to the working set. The ordering of ISTATE is as follows; the first n elements of ISTATE refer to the upper and lower bounds on the

```
variables and elements n+1 through n + NCLIN refer to the
upper and lower bounds on Ax. The significance of each
possible value of ISTATE(j) is as follows:
ISTATE(j) Meaning

-2        The constraint violates its lower bound by more
          than FEATOL(j). This value of ISTATE cannot occur
          after a feasible point has been found.

-1        The constraint violates its upper bound by more
          than FEATOL(j). This value of ISTATE cannot occur
          after a feasible point has been found.

0         The constraint is not in the working set. Usually,
          this means that the constraint lies strictly
          between its bounds.

1         This inequality constraint is included in the
          working set at its lower bound. The value of the
          constraint is within FEATOL(j) of its lower bound.

2         This inequality constraint is included in the
          working set at its upper bound. The value of the
          constraint is within FEATOL(j) of its upper bound.

3         The constraint is included in the working set as
          an equality. This value of ISTATE can occur only
          when BL(j) = BU(j). The corresponding constraint
          is within FEATOL(j) of its required value.
If COLD = .TRUE., ISTATE need not be set by the user.
However, when COLD = .FALSE., every element of ISTATE must
be set to one of the values given above to define a
suggested initial working set (which will be changed by
E04NAF if necessary). The most likely values are:
ISTATE(j) Meaning

0         The corresponding constraint should not be in the
          initial working set.

1         The constraint should be in the initial working
          set at its lower bound.

2         The constraint should be in the initial working
          set at its upper bound.

3         The constraint should be in the initial working
```

```
                set as an equality. This value must not be
                specified unless BL(j) = BU(j). The values 1, 2 or
                3 all have the same effect when BL(j) = BU(j).
        Note that if E04NAF has been called previously with the same
        values of N and NCLIN, ISTATE already contains satisfactory
        values. On exit: when E04NAF exits with IFAIL set to 0, 1 or
        3, the values in the array ISTATE indicate the status of the
        constraints in the active set at the solution. Otherwise,
        ISTATE indicates the composition of the working set at the
        final iterate.

22:  ITER -- INTEGER                                          Output
        On exit: the number of iterations performed in either the LP
        phase or the QP phase, whichever was last entered.

        Note that ITER is reset to zero after the LP phase.

23:  OBJ -- DOUBLE PRECISION                                  Output
        On exit: the value of the quadratic objective function at x
        if x is feasible (IFAIL <= 5), or the sum of infeasibilities
        at x otherwise (6 <= IFAIL <= 8).

24:  CLAMDA(NCTOTL) -- DOUBLE PRECISION array                 Output
        On exit: the values of the Lagrange multiplier for each
        constraint with respect to the current working set. The
        ordering of CLAMDA is as follows; the first n components
        contain the multipliers for the bound constraints on the
        variables, and the remaining components contain the
        multipliers for the general linear constraints. If ISTATE(j)
        = 0 (i.e.,constraint j is not in the working set), CLAMDA(j)
        is zero. If x is optimal and ISTATE(j) = 1, CLAMDA(j) should
        be non-negative; if ISTATE(j) = 2, CLAMDA(j) should be non-
        positive.

25:  IWORK(LIWORK) -- INTEGER array                        Workspace

26:  LIWORK -- INTEGER                                        Input
        On entry:
        the dimension of the array IWORK as declared in the
        (sub)program from which E04NAF is called.
        Constraint: LIWORK>=2*N.

27:  WORK(LWORK) -- DOUBLE PRECISION array                 Workspace

28:  LWORK -- INTEGER                                         Input
        On entry:
```

the dimension of the array WORK as declared in the (sub)program from which E04NAF is called.
Constrai if LP = .FALSE. or NCLIN >= N then
nts: 2
LWORK>=2*N +4*N*NCLIN+NROWA.

if LP = .TRUE. and NCLIN < N then
2
LWORK>=2*(NCLIN+1) +4*N+2*NCLIN+NROWA.
If MSGLVL > 0, the amount of workspace provided and the amount of workspace required are output on the current advisory message unit (as defined by X04ABF). As an alternative to computing LWORK from the formula given above, the user may prefer to obtain an appropriate value from the output of a preliminary run with a positive value of MSGLVL and LWORK set to 1 (E04NAF will then terminate with IFAIL = 9).

29: IFAIL -- INTEGER Input/Output
On entry: IFAIL must be set to 0, -1 or 1. Users who are unfamiliar with this parameter should refer to the Essential Introduction for details.

On exit: IFAIL = 0 unless the routine detects an error or gives a warning (see Section 6).

For this routine, because the values of output parameters may be useful even if IFAIL /=0 on exit, users are recommended to set IFAIL to -1 before entry. It is then essential to test the value of IFAIL on exit. To suppress the output of an error message when soft failure occurs, set IFAIL to 1.

IFAIL contains zero on exit if x is a strong local minimum. i.e., the projected gradient is neglible, the Lagrange multipliers are optimal, and the projected Hessian is positive-definite. In some cases, a zero value of IFAIL means that x is a global minimum (e.g. when the Hessian matrix is positive-definite).

5.1. Description of the Printed Output

When MSGLVL >= 5, a line of output is produced for every change in the working set (thus, several lines may be printed during a single iteration).

To aid interpretation of the printed results, we mention the convention for numbering the constraints: indices 1 through to n refer to the bounds on the variables, and when NCLIN > 0 indices n+1 through to n + NCLIN refer to the general constraints. When the status of a constraint changes, the index of the constraint is printed, along with the designation L (lower bound), U (upper bound) or E (equality).

In the LP phase, the printout includes the following:

ITN is the iteration count.

JDEL is the index of the constraint deleted from the working set. If JDEL is zero, no constraint was deleted.

JADD is the index of the constraint added to the working set. If JADD is zero, no constraint was added.

STEP is the step taken along the computed search direction.

COND T is a lower bound on the condition number of the matrix of predicted active constraints.

NUMINF is the number of violated constraints (infeasibilities).

SUMINF is a weighted sum of the magnitudes of the constraint violations.

LPOBJ is the value of the linear objective function $c^T x$. It is printed only if LP = .TRUE..

During the QP phase, the printout includes the following:

ITN is the iteration count (reset to zero after the LP phase).

JDEL is the index of the constraint deleted from the working set. If JDEL is zero, no constraint was deleted.

JADD is the index of the constraint added to the

working set. If JADD is zero, no constraint was added.

STEP — is the step (alpha $_k$) taken along the direction of search (if STEP is 1.0, the current point is a minimum in the subspace defined by the current working set).

NHESS — is the number of calls to subroutine QPHESS.

OBJECTIVE — is the value of the quadratic objective function.

NCOLZ — is the number of columns of Z (see Section 3). In general, it is the dimension of the subspace in which the quadratic is currently being minimized.

NORM GFREE — is the Euclidean norm of the gradient of the objective function with respect to the free variables, i.e. variables not currently held at a bound (NORM GFREE is not printed if ORTHOG = .FALSE.). In some cases, the objective function and gradient are updated rather than recomputed. If so, this entry will be -- to indicate that the gradient with respect to the free variables has not been computed.

NORM QTG — is a weighted norm of the gradient of the objective function with respect to the free variables (NORM QTG is not printed if ORTHOG = .TRUE.). In some cases, the objective function and gradient are updated rather than recomputed. If so, this entry will be -- to indicate that the gradient with respect to the free variables has not been computed.

NORM ZTG — is the Euclidean norm of the projected gradient (see Section 3).

COND T — is a lower bound on the condition number of the matrix of constraints in the working set.

COND ZHZ — is a lower bound on the condition number of the projected Hessian matrix.

HESS MOD — is the correction added to the diagonal of the

projected Hessian to ensure that a satisfactory Cholesky factorization exists (see Section 3). When the projected Hessian is sufficiently positive-definite, HESS MOD will be zero.

When MSGLVL = 1 or MSGLVL >= 10, the summary printout at the end of execution of E04NAF includes a listing of the status of every constraint. Note that default names are assigned to all variables and constraints.

The following describes the printout for each variable.

VARBL — is the name (V) and index j, j=1,2,...,n, of the variable.

STATE — gives the state of the variable (FR if neither bound is in the working set, EQ if a fixed variable, LL if on its lower bound, UL if on its upper bound, TB if held on a temporary bound). If VALUE lies outside the upper or lower bounds by more than FEATOL(j), STATE will be ++ or -- respectively.

VALUE — is the value of the variable at the final iteration.

LOWER BOUND — is the lower bound specified for the variable.

UPPER BOUND — is the upper bound specified for the variable.

LAGR MULT — is the value of the Lagrange multiplier for the associated bound constraint. This will be zero if STATE is FR. If x is optimal and STATE is LL, the multiplier should be non-negative; if STATE is UL, the multiplier should be non-positive.

RESIDUAL — is the difference between the variable and the nearer of its bounds BL(j) and BU(j).

For each of the general constraints the printout is as above with refers to the jth element of Ax, except that VARBL is replaced by

LNCON — The name (L) and index j, j=1,2,...,NCLIN, of the constraint.

6. Error Indicators and Warnings

Errors or warnings specified by the routine:

IFAIL= 1
x is a weak local minimum (the projected gradient is negligible, the Lagrange multipliers are optimal, but the projected Hessian is only semi-definite). This means that the solution is not unique.

IFAIL= 2
The solution appears to be unbounded, i.e., the quadratic function is unbounded below in the feasible region. This value of IFAIL occurs when a step of infinity would have to be taken in order to continue the algorithm.

IFAIL= 3
x appears to be a local minimum, but optimality cannot be verified because some of the Lagrange multipliers are very small in magnitude.

E04NAF has probably found a solution. However, the presence of very small Lagrange multipliers means that the predicted active set may be incorrect, or that x may be only a constrained stationary point rather than a local minimum. The method in E04NAF is not guaranteed to find the correct active set when there are very small multipliers. E04NAF attempts to delete constraints with zero multipliers, but this does not necessarily resolve the issue. The determination of the correct active set is a combinatorial problem that may require an extremely large amount of time. The occurrence of small multipliers often (but not always) indicates that there are redundant constraints.

IFAIL= 4
The iterates of the QP phase could be cycling, since a total of 50 changes were made to the working set without altering x.

This value will occur if 50 iterations are performed in the QP phase without changing x. The user should check the printed output for a repeated pattern of constraint deletions and additions. If a sequence of constraint changes is being repeated, the iterates are probably cycling. (E04NAF does not contain a method that is guaranteed to avoid cycling, which would be combinatorial in nature.) Cycling may occur in two circumstances: at a constrained

stationary point where there are some small or zero Lagrange multipliers (see the discussion of IFAIL = 3); or at a point (usually a vertex) where the constraints that are satisfied exactly are nearly linearly dependent. In the latter case, the user has the option of identifying the offending dependent constraints and removing them from the problem, or restarting the run with larger values of FEATOL for nearly dependent constraints. If E04NAF terminates with IFAIL = 4, but no suspicious pattern of constraint changes can be observed, it may be worthwhile to restart with the final x (with or without the warm start option).

IFAIL= 5

The limit of ITMAX iterations was reached in the QP phase before normal termination occurred.

The value of ITMAX may be too small. If the method appears to be making progress (e.g. the objective function is being satisfactorily reduced), increase ITMAX and rerun E04NAF (possibly using the warm start facility to specify the initial working set). If ITMAX is already large, but some of the constraints could be nearly linearly dependent, check the output for a repeated pattern of constraints entering and leaving the working set. (Near-dependencies are often indicated by wide variations in size in the diagonal elements of the T matrix, which will be printed if MSGLVL >= 30.) In this case, the algorithm could be cycling (see the comments for IFAIL = 4).

IFAIL= 6

The LP phase terminated without finding a feasible point, and hence it is not possible to satisfy all the constraints to within the tolerances specified by the FEATOL array. In this case, the final iterate will reveal values for which there will be a feasible point (e.g. a feasible point will exist if the feasibility tolerance for each violated constraint exceeds its RESIDUAL at the final point). The modified problem (with altered values in FEATOL) may then be solved using a warm start.

The user should check that there are no constraint redundancies. If the data for the jth constraint are accurate only to the absolute precision (delta), the user should ensure that the value of FEATOL(j) is greater than (delta). For example, if all elements of A are of order unity and are accurate only to three decimal places, every

component of FEATOL should be at least 10^{-3}.

IFAIL= 7
The iterates may be cycling during the LP phase; see the comments above under IFAIL = 4.

IFAIL= 8
The limit of ITMAX iterations was reached during the LP phase. See comments above under IFAIL = 5.

IFAIL= 9
An input parameter is invalid.

Overflow
If the printed output before the overflow error contains a warning about serious ill-conditioning in the working set when adding the jth constraint, it may be possible to avoid the difficulty by increasing the magnitude of FEATOL(j) and rerunning the program. If the message recurs even after this change, the offending linearly dependent constraint (with index j) must be removed from the problem. If a warning message did not precede the fatal overflow, the user should contact NAG.

7. Accuracy

The routine implements a numerically stable active set strategy and returns solutions that are as accurate as the condition of the QP problem warrants on the machine.

8. Further Comments

The number of iterations depends upon factors such as the number of variables and the distances of the starting point from the solution. The number of operations performed per iteration is roughly proportional to $(\text{NFREE})^2$, where NFREE (NFREE<=n) is the number of variables fixed on their upper or lower bounds.

Sensible scaling of the problem is likely to reduce the number of iterations required and make the problem less sensitive to perturbations in the data, thus improving the condition of the QP problem. See the Chapter Introduction and Gill et al [1] for further information and advice.

9. Example

$$\text{To minimize the function } c\,x + \frac{1}{2}\,x^T H x, \text{ where}$$

$$c=[-0.02,-0.2,-0.2,-0.2,-0.2,0.04,0.04]^T$$

$$H=\begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & -2 \\ 0 & 0 & 0 & 0 & 0 & -2 & -2 \end{bmatrix}$$

subject to the bounds

%%%

E04 -- Minimizing or Maximizing a Function E04UCF
E04UCF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for your implementation to check implementation-dependent details. The symbol (*) after a NAG routine name denotes a routine that is not included in the Foundation Library.

Note for users via the AXIOM system: the interface to this routine has been enhanced for use with AXIOM and is slightly different to that offered in the standard version of the Foundation Library. In particular, the optional parameters of the NAG routine are now included in the parameter list. These are described in section 5.1.2, below.

1. Purpose

E04UCF is designed to minimize an arbitrary smooth function subject to constraints, which may include simple bounds on the variables, linear constraints and smooth nonlinear constraints. (E04UCF may be used for unconstrained, bound-constrained and linearly constrained optimization.) The user must provide subroutines that define the objective and constraint functions and as many of their first partial derivatives as possible. Unspecified derivatives are approximated by finite differences.

All matrices are treated as dense, and hence E04UCF is not intended for large sparse problems.

E04UCF uses a sequential quadratic programming (SQP) algorithm in which the search direction is the solution of a quadratic programming (QP) problem. The algorithm treats bounds, linear constraints and nonlinear constraints separately.

2. Specification

```
      SUBROUTINE E04UCF (N, NCLIN, NCNLN, NROWA, NROWJ, NROWR,
     1                   A, BL, BU, CONFUN, OBJFUN, ITER,
     2                   ISTATE, C, CJAC, CLAMDA, OBJF, OBJGRD,
     3                   R, X, IWORK, LIWORK, WORK, LWORK,
     4                   IUSER, USER, STA, CRA, DER, FEA, FUN,
     5                   HES, INFB, INFS, LINF, LINT, LIST,
     6                   MAJI, MAJP, MINI, MINP, MON, NONF,
     7                   OPT, STE, STAO, STAC, STOO, STOC, VE,
     8                   IFAIL)
      INTEGER           N, NCLIN, NCNLN, NROWA, NROWJ, NROWR,
     1                  ITER, ISTATE(N+NCLIN+NCNLN), IWORK(LIWORK)
     2                  , LIWORK, LWORK, IUSER(*), DER, MAJI,
     3                  MAJP, MINI, MINP, MON, STAO, STAC, STOO,
     4                  STOC, VE, IFAIL
      DOUBLE PRECISION  A(NROWA,*), BL(N+NCLIN+NCNLN), BU
     1                  (N+NCLIN+NCNLN), C(*), CJAC(NROWJ,*),
     2                  CLAMDA(N+NCLIN+NCNLN), OBJF, OBJGRD(N), R
     3                  (NROWR,N), X(N), WORK(LWORK), USER(*),
     4                  CRA, FEA, FUN,  INFB, INFS, LINF, LINT,
     5                  NONF, OPT, STE
      LOGICAL           LIST, STA, HES
      EXTERNAL          CONFUN, OBJFUN
```

3. Description

E04UCF is designed to solve the nonlinear programming problem -- the minimization of a smooth nonlinear function subject to a set of constraints on the variables. The problem is assumed to be stated in the following form:

$$\underset{x \text{ is in } R^n}{\text{Minimize}} \quad F(x) \quad \text{subject to} \quad l \leq \begin{Bmatrix} x \\ Ax \\ L \\ c(x) \end{Bmatrix} \leq u, \tag{1}$$

where F(x), the objective function, is a nonlinear function, A

L
is an n by n constant matrix, and c(x) is an n element vector
L N
of nonlinear constraint functions. (The matrix A and the vector
L
c(x) may be empty.) The objective function and the constraint functions are assumed to be smooth, i.e., at least twice-continuously differentiable. (The method of E04UCF will usually solve (1) if there are only isolated discontinuities away from the solution.)

This routine is essentially identical to the subroutine SOL/NPSOL described in Gill et al [8].

Note that upper and lower bounds are specified for all the variables and for all the constraints.

An equality constraint can be specified by setting l =u . If
i i
certain bounds are not present, the associated elements of l or u can be set to special values that will be treated as -infty or +infty.

If there are no nonlinear constraints in (1) and F is linear or quadratic then one of E04MBF, E04NAF or E04NCF(*) will generally be more efficient. If the problem is large and sparse the MINOS package (see Murtagh and Saunders [13]) should be used, since E04UCF treats all matrices as dense.

The user must supply an initial estimate of the solution to (1), together with subroutines that define F(x), c(x) and as many first partial derivatives as possible; unspecified derivatives are approximated by finite differences.

The objective function is defined by subroutine OBJFUN, and the nonlinear constraints are defined by subroutine CONFUN. On every call, these subroutines must return appropriate values of the objective and nonlinear constraints. The user should also provide the available partial derivatives. Any unspecified derivatives are approximated by finite differences; see Section 5.1 for a discussion of the optional parameter Derivative Level. Just before either OBJFUN or CONFUN is called, each element of the current gradient array OBJGRD or CJAC is initialised to a special value. On exit, any element that retains the value is estimated by finite differences. Note that if there are nonlinear costraints, then the first call to CONFUN will precede the first

call to OBJFUN.

For maximum reliability, it is preferable for the user to provide all partial derivatives (see Chapter 8 of Gill et al [10], for a detailed discussion). If all gradients cannot be provided, it is similarly advisable to provide as many as possible. While developing the subroutines OBJFUN and CONFUN, the optional parameter Verify (see Section 5.1) should be used to check the calculation of any known gradients.

E04UCF implements a sequential quadratic programming (SQP) method. The document for E04NCF(*) should be consulted in conjunction with this document.

In the rest of this section we briefly summarize the main features of the method of E04UCF. Where possible, explicit reference is made to the names of variables that are parameters of subroutines E04UCF or appear in the printed output (see Section 5.2).

At a solution of (1), some of the constraints will be active, i.e., satisfied exactly. An active simple bound constraint implies that the corresponding variable is fixed at its bound, and hence the variables are partitioned into fixed and free variables. Let C denote the m by n matrix of gradients of the active general linear and nonlinear constraints. The number of fixed variables will be denoted by n_{FX}, with n_{FR} ($n_{FR}=n-n_{FX}$) the number of free variables. The subscripts 'FX' and 'FR' on a vector or matrix will denote the vector or matrix composed of the components corresponding to fixed or free variables.

A point x is a first-order Kuhn-Tucker point for (1) (see, e.g., Powell [14]) if the following conditions hold:

(i) x is feasible;

(ii) there exist vectors (xi) and (lambda) (the Lagrange multiplier vectors for the bound and general constraints) such that

$$g = C^T(\lambda) + (\xi), \qquad (2)$$

where g is the gradient of F evaluated at x, and $(\xi)_j = 0$ if the jth variable is free.

(iii) The Lagrange multiplier corresponding to an inequality constraint active at its lower bound must be non-negative, and non-positive for an inequality constraint active at its upper bound.

Let Z denote a matrix whose columns form a basis for the set of vectors orthogonal to the rows of C_{FR} ; i.e., $C_{FR} Z = 0$. An equivalent statement of the condition (2) in terms of Z is

$$Z^T g_{FR} = 0.$$

The vector $Z^T g_{FR}$ is termed the projected gradient of F at x.

Certain additional conditions must be satisfied in order for a first-order Kuhn-Tucker point to be a solution of (1) (see, e.g., Powell [14]).

The method of E04UCF is a sequential quadratic programming (SQP) method. For an overview of SQP methods, see, for example, Fletcher [5], Gill et al [10] and Powell [15].

The basic structure of E04UCF involves major and minor iterations. The major iterations generate a sequence of iterates $\{x_k\}$ that converge to x^*, a first-order Kuhn-Tucker point of (1).

At a typical major iteration, the new iterate $\bar{x}$ is defined by

$$\bar{x} = x + (\alpha) p \qquad (3)$$

where x is the current iterate, the non-negative scalar (alpha) is the step length, and p is the search direction. (For simplicity, we shall always consider a typical iteration and avoid reference to the index of the iteration.) Also associated with each major iteration are estimates of the Lagrange multipliers and a prediction of the active set.

The search direction p in (3) is the solution of a quadratic programming subproblem of the form

$$\underset{p}{\text{Minimize}} \quad g^T p + \frac{1}{2} p^T H p, \qquad \text{subject to} \quad \bar{l} \leq \begin{Bmatrix} p \\ A_L p \\ A_N p \end{Bmatrix} \leq u, \tag{4}$$

where g is the gradient of F at x, the matrix H is a positive-definite quasi-Newton approximation to the Hessian of the Lagrangian function (see Section 8.3), and A_N is the Jacobian matrix of c evaluated at x. (Finite-difference estimates may be used for g and A_N; see the optional parameter Derivative Level in Section 5.1.) Let l in (1) be partitioned into three sections: l_B, l_L and l_N, corresponding to the bound, linear and nonlinear constraints. The vector $\bar{l}$ in (4) is similarly partitioned, and is defined as

$$\bar{l}_B = l_B - x, \quad \bar{l}_L = l_L - A_L x, \quad \text{and} \quad \bar{l}_N = l_N - c,$$

where c is the vector of nonlinear constraints evaluated at x. The vector $\bar{u}$ is defined in an analogous fashion.

The estimated Lagrange multipliers at each major iteration are the Lagrange multipliers from the subproblem (4) (and similarly for the predicted active set). (The numbers of bounds, general linear and nonlinear constraints in the QP active set are the quantities Bnd, Lin and Nln in the printed output of E04UCF.) In E04UCF, (4) is solved using E04NCF(*). Since solving a quadratic program as an iterative procedure, the minor iterations of E04UCF are the iterations of E04NCF(*). (More details about solving the subproblem are given in Section 8.1.)

Certain matrices associated with the QP subproblem are relevant in the major iterations. Let the subscripts 'FX' and 'FR' refer to the predicted fixed and free variables, and let C denote the m by n matrix of gradients of the general linear and nonlinear constraints in the predicted active set. First, we have available the TQ factorization of C_{FR}:

$$C_{FR} Q_{FR} = (0\ T), \tag{5}$$

where T is a nonsingular m by m reverse-triangular matrix (i.e., $t_{ij} = 0$ if $i+j<m$), and the non-singular n by n matrix Q_{FR} is the product of orthogonal transformations (see Gill et al [6]). Second, we have the upper-triangular Cholesky factor R of the transformed and re-ordered Hessian matrix

$$R_{FR}^{T} R_{FR} = H_{Q}^{\tilde{}} = Q^{T} H Q, \tag{6}$$

where $\tilde{H}$ is the Hessian H with rows and columns permuted so that the free variables are first, and Q is the n by n matrix

$$Q = \begin{pmatrix} Q_{FR} \\ I_{FX} \end{pmatrix}, \tag{7}$$

with I_{FX} the identity matrix of order n_{FX}. If the columns of Q are partitioned so that

$$Q_{FR} = (Z\ Y),$$

the n_z ($n_z = n - m$) columns of Z form a basis for the null space of C_{FR}. The matrix Z is used to compute the projected gradient $Z^{T} g_{FR}$ at the current iterate. (The values Nz, Norm Gf and Norm Gz printed by E04UCF give n_z and the norms of g_{FR} and $Z^{T} g_{FR}$.)

A theoretical characteristic of SQP methods is that the predicted active set from the QP subproblem (4) is identical to the correct active set in a neighbourhood of x^{*}. In E04UCF, this feature is exploited by using the QP active set from the previous iteration as a prediction of the active set for the next QP subproblem,

which leads in practice to optimality of the subproblems in only one iteration as the solution is approached. Separate treatment of bound and linear constraints in E04UCF also saves computation in factorizing C_{FR} and H_Q.

Once p has been computed, the major iteration proceeds by determining a step length (alpha) that produces a 'sufficient decrease' in an augmented Lagrangian merit function (see Section 8.2). Finally, the approximation to the transformed Hessian matrix H_Q is updated using a modified BFGS quasi-Newton update (see Section 8.3) to incorporate new curvature information obtained in the move from x to $\bar{x}$.

On entry to E04UCF, an iterative procedure from E04NCF(*) is executed, starting with the user-provided initial point, to find a point that is feasible with respect to the bounds and linear constraints (using the tolerance specified by Linear Feasibility Tolerance see Section 5.1). If no feasible point exists for the bound and linear constraints, (1) has no solution and E04UCF terminates. Otherwise, the problem functions will thereafter be evaluated only at points that are feasible with respect to the bounds and linear constraints. The only exception involves variables whose bounds differ by an amount comparable to the finite-difference interval (see the discussion of Difference Interval in Section 5.1). In contrast to the bounds and linear constraints, it must be emphasised that the nonlinear constraints will not generally be satisfied until an optimal point is reached.

Facilities are provided to check whether the user-provided gradients appear to be correct (see the optional parameter Verify in Section 5.1). In general, the check is provided at the first point that is feasible with respect to the linear constraints and bounds. However, the user may request that the check be performed at the initial point.

In summary, the method of E04UCF first determines a point that satisfies the bound and linear constraints. Thereafter, each iteration includes:

(a) the solution of a quadratic programming subproblem;

(b) a linesearch with an augmented Lagrangian merit function;

and

(c) a quasi-Newton update of the approximate Hessian of the Lagrangian function.

These three procedures are described in more detail in Section 8.

4. References

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5. Parameters

1: N -- INTEGER Input
On entry: the number, n, of variables in the problem.
Constraint: N > 0.

2: NCLIN -- INTEGER Input
On entry: the number, n_L, of general linear constraints in the problem. Constraint: NCLIN >= 0.

3: NCNLN -- INTEGER Input
On entry: the number, n_N, of nonlinear constraints in the problem. Constraint: NCNLN >= 0.

4: NROWA -- INTEGER Input
On entry:
the first dimension of the array A as declared in the (sub)program from which E04UCF is called.

Constraint: NROWA >= max(1,NCLIN).

5: NROWJ -- INTEGER Input
On entry:
the first dimension of the array CJAC as declared in the (sub)program from which E04UCF is called.
Constraint: NROWJ >= max(1,NCNLN).

6: NROWR -- INTEGER Input
On entry:
the first dimension of the array R as declared in the (sub)program from which E04UCF is called.
Constraint: NROWR >= N.

7: A(NROWA,*) -- DOUBLE PRECISION array Input
The second dimension of the array A must be >= N for NCLIN > 0. On entry: the ith row of the array A must contain the ith row of the matrix A_L of general linear constraints in (1).
That is, the ith row contains the coefficients of the ith general linear constraint, for i = 1,2,...,NCLIN.

If NCLIN = 0 then the array A is not referenced.

8: BL(N+NCLIN+NCNLN) -- DOUBLE PRECISION array Input
On entry: the lower bounds for all the constraints, in the following order. The first n elements of BL must contain the lower bounds on the variables. If NCLIN > 0, the next n_L elements of BL must contain the lower bounds on the general linear constraints. If NCNLN > 0, the next n_N elements of BL must contain the lower bounds for the nonlinear constraints. To specify a non-existent lower bound (i.e., l_j =-infty), the value used must satisfy BL(j)<=-BIGBND, where BIGBND is the value of the optional parameter Infinite Bound Size whose default value is 10^{10} (see Section 5.1). To specify the jth constraint as an equality, the user must set BL(j) = BU(j) = (beta), say, where |(beta)|<BIGBND. Constraint: BL(j) <= BU(j), for j=1,2,...,N+NCLIN+NCNLN.

9: BU(N+NCLIN+NCNLN) -- DOUBLE PRECISION array Input
On entry: the upper bounds for all the constraints in the following order. The first n elements of BU must contain the

upper bounds on the variables. If NCLIN > 0, the next n
L
elements of BU must contain the upper bounds on the general linear constraints. If NCNLN > 0, the next n elements of BU
N
must contain the upper bounds for the nonlinear constraints. To specify a non-existent upper bound (i.e., u =+infty), the
j
value used must satisfy BU(j) >= BIGBND, where BIGBND is the value of the optional parameter Infinite Bound Size, whose
10
default value is 10 (see Section 5.1). To specify the jth constraint as an equality, the user must set BU(j) = BL(j) = (beta), say, where |(beta)| < BIGBND. Constraint: BU(j) >= BL(j), for j=1,2,...,N+NCLIN+NCNLN.

10: CONFUN -- SUBROUTINE, supplied by the user.
External Procedure
CONFUN must calculate the vector c(x) of nonlinear constraint functions and (optionally) its Jacobian for a specified n element vector x. If there are no nonlinear constraints (NCNLN=0), CONFUN will never be called by E04UCF and CONFUN may be the dummy routine E04UDM. (E04UDM is included in the NAG Foundation Library and so need not be supplied by the user. Its name may be implementation-dependent: see the Users' Note for your implementation for details.) If there are nonlinear constraints, the first call to CONFUN will occur before the first call to OBJFUN.

Its specification is:

```
      SUBROUTINE CONFUN (MODE, NCNLN, N, NROWJ, NEEDC,
     1                   X, C, CJAC, NSTATE, IUSER,
     2                   USER)
      INTEGER           MODE, NCNLN, N, NROWJ, NEEDC
     1                  (NCNLN), NSTATE, IUSER(*)
      DOUBLE PRECISION X(N), C(NCNLN), CJAC(NROWJ,N),
     1                  USER(*)
```

1: MODE -- INTEGER Input/Output
On entry: MODE indicates the values that must be assigned during each call of CONFUN. MODE will always have the value 2 if all elements of the Jacobian are available, i.e., if Derivative Level is either 2 or 3 (see Section 5.1). If some elements of CJAC are unspecified, E04UCF will call CONFUN with MODE = 0, 1,

```
     or 2:

     If MODE = 2, only the elements of C corresponding to
     positive values of NEEDC must be set (and similarly for
     the available components of the rows of CJAC).

     If MODE = 1, the available components of the rows of
     CJAC corresponding to positive values in NEEDC must be
     set. Other rows of CJAC and the array C will be
     ignored.

     If MODE = 0, the components of C corresponding to
     positive values in NEEDC must be set. Other components
     and the array CJAC are ignored. On exit: MODE may be
     set to a negative value if the user wishes to terminate
     the solution to the current problem. If MODE is
     negative on exit from CONFUN then E04UCF will terminate
     with IFAIL set to MODE.

2:   NCNLN -- INTEGER                                    Input
     On entry: the number, n , of nonlinear constraints.
                            N

3:   N -- INTEGER                                       Input
     On entry: the number, n, of variables.

4:   NROWJ -- INTEGER                                   Input
     On entry: the first dimension of the array CJAC.

5:   NEEDC(NCNLN) -- INTEGER array                      Input
     On entry: the indices of the elements of C or CJAC that
     must be evaluated by CONFUN. If NEEDC(i)>0 then the ith
     element of C and/or the ith row of CJAC (see parameter
     MODE above) must be evaluated at x.

6:   X(N) -- DOUBLE PRECISION array                     Input
     On entry: the vector x of variables at which the
     constraint functions are to be evaluated.

7:   C(NCNLN) -- DOUBLE PRECISION array                 Output
     On exit: if NEEDC(i)>0 and MODE = 0 or 2, C(i) must
     contain the value of the ith constraint at x. The
     remaining components of C, corresponding to the non-
     positive elements of NEEDC, are ignored.

8:   CJAC(NROWJ,N) -- DOUBLE PRECISION array            Output
```

On exit: if NEEDC(i)>0 and MODE = 1 or 2, the ith row of CJAC must contain the available components of the vector (nabla)c given by

$$(\nabla)c_i = \left(\frac{\partial c_i}{\partial x_1}, \frac{\partial c_i}{\partial x_2}, \ldots, \frac{\partial c_i}{\partial x_n} \right)^T ,$$

where $\frac{\partial c_i}{\partial x_j}$ is the partial derivative of the ith constraint with respect to the jth variable, evaluated at the point x. See also the parameter NSTATE below. The remaining rows of CJAC, corresponding to non-positive elements of NEEDC, are ignored.

If all constraint gradients (Jacobian elements) are known (i.e., Derivative Level = 2 or 3; see Section 5.1) any constant elements may be assigned to CJAC one time only at the start of the optimization. An element of CJAC that is not subsequently assigned in CONFUN will retain its initial value throughout. Constant elements may be loaded into CJAC either before the call to E04UCF or during the first call to CONFUN (signalled by the value NSTATE = 1). The ability to preload constants is useful when many Jacobian elements are identically zero, in which case CJAC may be initialised to zero and non-zero elements may be reset by CONFUN.

Note that constant non-zero elements do affect the values of the constraints. Thus, if CJAC(i,j) is set to a constant value, it need not be reset in subsequent calls to CONFUN, but the value CJAC(i,j)*X(j) must nonetheless be added to C(i).

It must be emphasized that, if Derivative Level < 2, unassigned elements of CJAC are not treated as constant; they are estimated by finite differences, at non-trivial expense. If the user does not supply a value for Difference Interval (see Section 5.1), an interval for each component of x is computed automatically at the start of the optimization. The automatic procedure can usually identify constant

```
     elements of CJAC, which are then computed once only by
     finite differences.

 9:  NSTATE -- INTEGER                                    Input
     On entry: if NSTATE = 1 then E04UCF is calling CONFUN
     for the first time. This parameter setting allows the
     user to save computation time if certain data must be
     read or calculated only once.

10:  IUSER(*) -- INTEGER array                User Workspace

11:  USER(*) -- DOUBLE PRECISION array        User Workspace
     CONFUN is called from E04UCF with the parameters IUSER
     and USER as supplied to E04UCF. The user is free to use
     the arrays IUSER and USER to supply information to
     CONFUN as an alternative to using COMMON.
CONFUN must be declared as EXTERNAL in the (sub)program
from which E04UCF is called. Parameters denoted as
Input must not be changed by this procedure.

11:  OBJFUN -- SUBROUTINE, supplied by the user.
                                              External Procedure
OBJFUN must calculate the objective function F(x) and
(optionally) the gradient g(x) for a specified n element
vector x.

Its specification is:

       SUBROUTINE OBJFUN (MODE, N, X, OBJF, OBJGRD,
      1                   NSTATE, IUSER, USER)
       INTEGER          MODE, N, NSTATE, IUSER(*)
       DOUBLE PRECISION X(N), OBJF, OBJGRD(N), USER(*)

 1:  MODE -- INTEGER                              Input/Output
     On entry: MODE indicates the values that must be
     assigned during each call of OBJFUN.

     MODE will always have the value 2 if all components of
     the objective gradient are specified by the user, i.e.,
     if Derivative Level is either 1 or 3. If some gradient
     elements are unspecified, E04UCF will call OBJFUN with
     MODE = 0, 1 or 2.
          If MODE = 2, compute OBJF and the available
          components of OBJGRD.

          If MODE = 1, compute all available components of
```

```
                    OBJGRD; OBJF is not required.

                    If MODE = 0, only OBJF needs to be computed;
                    OBJGRD is ignored.
                On exit: MODE may be set to a negative value if the
                user wishes to terminate the solution to the current
                problem. If MODE is negative on exit from OBJFUN, then
                E04UCF will terminate with IFAIL set to MODE.

            2:  N -- INTEGER                                        Input
                On entry: the number, n, of variables.

            3:  X(N) -- DOUBLE PRECISION array                      Input
                On entry: the vector x of variables at which the
                objective function is to be evaluated.

            4:  OBJF -- DOUBLE PRECISION                           Output
                On exit: if MODE = 0 or 2, OBJF must be set to the
                value of the objective function at x.

            5:  OBJGRD(N) -- DOUBLE PRECISION array                Output
                On exit: if MODE = 1 or 2, OBJGRD must return the
                available components of the gradient evaluated at x.

            6:  NSTATE -- INTEGER                                   Input
                On entry: if NSTATE = 1 then E04UCF is calling OBJFUN
                for the first time. This parameter setting allows the
                user to save computation time if certain data must be
                read or calculated only once.

            7:  IUSER(*) -- INTEGER array                  User Workspace

            8:  USER(*) -- DOUBLE PRECISION array          User Workspace
                OBJFUN is called from E04UCF with the parameters IUSER
                and USER as supplied to E04UCF. The user is free to use
                the arrays IUSER and USER to supply information to
                OBJFUN as an alternative to using COMMON.
            OBJFUN must be declared as EXTERNAL in the (sub)program
            from which E04UCF is called. Parameters denoted as
            Input must not be changed by this procedure.

           12:  ITER -- INTEGER                                     Output
                On exit: the number of iterations performed.

           13:  ISTATE(N+NCLIN+NCNLN) -- INTEGER array          Input/Output
                On entry: ISTATE need not be initialised if E04UCF is called
```

with (the default) Cold Start option. The ordering of ISTATE is as follows. The first n elements of ISTATE refer to the upper and lower bounds on the variables, elements n+1 through n+n refer to the upper and lower bounds on A x, and elements n+n +1 through n+n +n refer to the upper and lower bounds on c(x). When a Warm Start option is chosen, the elements of ISTATE corresponding to the bounds and linear constraints define the initial working set for the procedure that finds a feasible point for the linear constraints and bounds. The active set at the conclusion of this procedure and the elements of ISTATE corresponding to nonlinear constraints then define the initial working set for the first QP subproblem. Possible values for ISTATE(j) are:

ISTATE(j)	Meaning
0	The corresponding constraint is not in the initial QP working set.
1	This inequality constraint should be in the working set at its lower bound.
2	This inequality constraint should be in the working set at its upper bound.
3	This equality constraint should be in the initial working set. This value must not be specified unless BL(j) = BU(j). The values 1,2 or 3 all have the same effect when BL(j) = BU(j).

Note that if E04UCF has been called previously with the same values of N, NCLIN and NCNLN, ISTATE already contains satisfactory values. If necessary, E04UCF will override the user's specification of ISTATE so that a poor choice will not cause the algorithm to fail. On exit: with IFAIL = 0 or 1, the values in the array ISTATE correspond to the active set of the final QP subproblem, and are a prediction of the status of the constraints at the solution of the problem. Otherwise, ISTATE indicates the composition of the QP working set at the final iterate. The significance of each possible value of ISTATE(j) is as follows:

-2	This constraint violates its lower bound by more than the appropriate feasibility tolerance (see the optional parameters LinearFeasibility Tolerance and Nonlinear Feasibility Tolerance in

Section 5.1). This value can occur only when no feasible point can be found for a QP subproblem.

-1 This constraint violates its upper bound by more than the appropriate feasibility tolerance (see the optional parameters Linearear Feasibility Tolerance and Nonlinear Feasibility Tolerance in Section 5.1). This value can occur only when no feasible point can be found for a QP subproblem.

0 The constraint is satisfied to within the feasibility tolerance, but is not in the working set.

1 This inequality constraint is included in the QP working set at its upper bound.

2 This inequality constraint is included in the QP working set at its upper bound.

3 This constraint is included in the QP working set as an equality. This value of ISTATE can occur only when BL(j) = BU(j).

14: C(*) -- DOUBLE PRECISION array Output
Note: the dimension of the array C must be at least max(1,NCNLN).
On exit: if NCNLN > 0, C(i) contains the value of the ith nonlinear constraint function c_i at the final iterate, for i=1,2,...,NCNLN. If NCNLN = 0, then the array C is not referenced.

15: CJAC(NROWJ,*) -- DOUBLE PRECISION array Input/Output
Note: the second dimension of the array CJAC must be at least N for NCNLN >0 and 1 otherwise On entry: in general, CJAC need not be initialised before the call to E04UCF. However, if Derivative Level = 3, the user may optionally set the constant elements of CJAC (see parameter NSTATE in the description of CONFUN). Such constant elements need not be re-assigned on subsequent calls to CONFUN. If NCNLN = 0, then the array CJAC is not referenced. On exit: if NCNLN > 0, CJAC contains the Jacobian matrix of the nonlinear constraint functions at the final iterate, i.e., CJAC(i,j) contains the partial derivative of the ith constraint function with respect to the jth variable, for i=1,2,...,

NCNLN; j = 1,2,...,N. (See the discussion of parameter CJAC under CONFUN.)

16: CLAMDA(N+NCLIN+NCNLN) -- DOUBLE PRECISION array Input/Output
On entry: CLAMDA need not be initialised if E04UCF is called with the (default) Cold Start option. With the Warm Start option, CLAMDA must contain a multiplier estimate for each nonlinear constraint with a sign that matches the status of the constraint specified by the ISTATE array (as above). The ordering of CLAMDA is as follows; the first n elements contain the multipliers for the bound constraints on the variables, elements n+1 through n+n contain the multipliers
L
for the general linear constraints, and elements n+n +1
L
through n+n +n contain the multipliers for the nonlinear
L N
constraints. If the jth constraint is defined as 'inactive' by the initial value of the ISTATE array, CLAMDA(j) should be zero; if the jth constraint is an inequality active at its lower bound, CLAMDA(j) should be non-negative; if the j th constraint is an inequality active at its upper bound, CLAMDA(j) should be non-positive. On exit: the values of the QP multipliers from the last QP subproblem. CLAMDA(j) should be non-negative if ISTATE(j) = 1 and non-positive if ISTATE(j) = 2.

17: OBJF -- DOUBLE PRECISION Output
On exit: the value of the objective function, F(x), at the final iterate.

18: OBJGRD(N) -- DOUBLE PRECISION array Output
On exit: the gradient (or its finite-difference approximation) of the objective function at the final iterate.

19: R(NROWR,N) -- DOUBLE PRECISION array Input/Output
On entry: R need not be initialised if E04UCF is called with a Cold Start option (the default), and will be taken as the identity. With a Warm Start R must contain the upper-triangular Cholesky factor R of the initial approximation of the Hessian of the Lagrangian function, with the variables in the natural order. Elements not in the upper-triangular part of R are assumed to be zero and need not be assigned. On exit: if Hessian = No, (the default; see Section 5.1), R
T~

```
     contains the upper-triangular Cholesky factor R of Q HQ, an
     estimate of the transformed and re-ordered Hessian of the
     Lagrangian at x (see (6) in Section 3). If Hessian = Yes, R
     contains the upper-triangular Cholesky factor R of H, the
     approximate (untransformed) Hessian of the Lagrangian, with
     the variables in the natural order.

20:  X(N) -- DOUBLE PRECISION array                    Input/Output
     On entry: an initial estimate of the solution. On exit: the
     final estimate of the solution.

21:  IWORK(LIWORK) -- INTEGER array                      Workspace

22:  LIWORK -- INTEGER                                       Input
     On entry:
     the dimension of the array IWORK as declared in the
     (sub)program from which E04UCF is called.
     Constraint: LIWORK>=3*N+NCLIN+2*NCNLN.

23:  WORK(LWORK) -- DOUBLE PRECISION array               Workspace

24:  LWORK -- INTEGER                                        Input
     On entry:
     the dimension of the array WORK as declared in the
     (sub)program from which E04UCF is called.
     Constraints:
     if NCLIN = NCNLN = 0 then
           LWORK >=20*N

     if NCNLN = 0 and NCLIN > 0 then
                      2
           LWORK >=2*N +20*N+11*NCLIN

     if NCNLN > 0 and NCLIN >= 0 then
                    2
           LWORK>=2*N +N*NCLIN+20*N*NCNLN+20*N+ 11*NCLIN+21*NCNLN

     If Major Print Level > 0, the required amounts of workspace
     are output on the current advisory message channel (see
     X04ABF). As an alternative to computing LIWORK and LWORK
     from the formulas given above, the user may prefer to obtain
     appropriate values from the output of a preliminary run with
     a positive value of Major Print Level and LIWORK and LWORK
     set to 1. (E04UCF will then terminate with IFAIL = 9.)

25:  IUSER(*) -- INTEGER array                       User Workspace
```

Note: the dimension of the array IUSER must be at least 1. IUSER is not used by E04UCF, but is passed directly to routines CONFUN and OBJFUN and may be used to pass information to those routines.

26: USER(*) -- DOUBLE PRECISION array User Workspace
Note: the dimension of the array USER must be at least 1. USER is not used by E04UCF, but is passed directly to routines CONFUN and OBJFUN and may be used to pass information to those routines.

27: IFAIL -- INTEGER Input/Output
On entry: IFAIL must be set to 0, -1 or 1. Users who are unfamiliar with this parameter should refer to the Essential Introduction for details.

On exit: IFAIL = 0 unless the routine detects an error or gives a warning (see Section 6).

For this routine, because the values of output parameters may be useful even if IFAIL /=0 on exit, users are recommended to set IFAIL to -1 before entry. It is then essential to test the value of IFAIL on exit.

E04UCF returns with IFAIL = 0 if the iterates have converged to a point x that satisfies the first-order Kuhn-Tucker conditions to the accuracy requested by the optional parameter Optimality Tolerance (see Section 5.1), i.e., the projected gradient and active constraint residuals are negligible at x.

The user should check whether the following four conditions are satisfied:
(i) the final value of Norm Gz is significantly less than that at the starting point;

(ii) during the final major iterations, the values of Step and ItQP are both one;

(iii) the last few values of both Norm Gz and Norm C become small at a fast linear rate;

(iv) Cond Hz is small.
If all these conditions hold, x is almost certainly a local minimum of (1). (See Section 9 for a specific example.)

5.1. Optional Input Parameters

Several optional parameters in E04UCF define choices in the behaviour of the routine. In order to reduce the number of formal parameters of E04UCF these optional parameters have associated default values (see Section 5.1.3) that are appropriate for most problems. Therefore the user need only specify those optional parameters whose values are to be different from their default values.

The remainder of this section can be skipped by users who wish to use the default values for all optional prameters. A complete list of optional parameters and their default values is given in Section 5.1.3

5.1.1. Specification of the optional parameters

Optional parameters may be specified by calling one, or both, of E04UDF and E04UEF prior to a call to E04UCF.

E04UDF reads options from an external options file, with Begin and End as the first and last lines respectively and each intermediate line defining a single optional parameter. For example,

```
Begin
  Print Level = 1
End
```

The call

```
CALL E04UDF (IOPTNS, INFORM)
```

can then be used to read the file on unit IOPTNS. INFORM will be zero on successful exit. E04UDF should be consulted for a full description of this method of supplying optional parameters.

E04UEF can be called directly to supply options, one call being necessary for each optional parameter. For example,

```
CALL E04UEF ('Print level = 1')
```

E04UEF should be consulted for a full description of this method of supplying optional parameters.

All optional parameters not specified by the user are set to

their default values. Optional parameters specified by the user are unaltered by E04UCF (unless they define invalid values) and so remain in effect for subsequent calls to E04UCF, unless altered by the user.

5.1.2. Description of the optional parameters

The following list (in alphabetical order) gives the valid options. For each option, we give the keyword, any essential optional qualifiers, the default value, and the definition. The minimum valid abbreviation of each keyword is underlined. If no characters of an optional qualifier are underlined, the qualifier may be omitted. The letter a denotes a phrase (character string) that qualifies an option. The letters i and r denote INTEGER and DOUBLE PRECISION values required with certain options. The number (epsilon) is a generic notation for machine precision (see X02AJF(*)), and (epsilon) denotes the relative precision of the
R
objective function (the optional parameter Function Precision see below).

Central Difference Interval r Default values are computed

If the algorithm switches to central differences because the forward-difference approximation is not sufficiently accurate, the value of r is used as the difference interval for every component of x. The use of finite-differences is discussed further below under the optional parameter Difference Interval.

Cold Start Default = Cold Start

Warm Start

(AXIOM parameter STA, warm start when .TRUE.)

This option controls the specification of the initial working set in both the procedure for finding a feasible point for the linear constraints and bounds, and in the first QP subproblem thereafter. With a Cold Start, the first working set is chosen by E04UCF based on the values of the variables and constraints at the initial point. Broadly speaking, the initial working set will include equality constraints and bounds or inequality constraints that violate or 'nearly' satisfy their bounds (within Crash Tolerance; see below). With a Warm Start, the user must set the ISTATE array and define CLAMDA and R as discussed in Section 5. ISTATE values associated with bounds and linear constraints

determine the initial working set of the procedure to find a feasible point with respect to the bounds and linear constraints. ISTATE values associated with nonlinear constraints determine the initial working set of the first QP subproblem after such a feasible point has been found. E04UCF will override the user's specification of ISTATE if necessary, so that a poor choice of the working set will not cause a fatal error. A warm start will be advantageous if a good estimate of the initial working set is available - for example, when E04UCF is called repeatedly to solve related problems.

Crash Tolerance r Default = 0.01

(AXIOM parameter CRA)

This value is used in conjunction with the optional parameter Cold Start (the default value). When making a cold start, the QP algorithm in E04UCF must select an initial working set. When r>=0 , the initial working set will include (if possible) bounds or general inequality constraints that lie within r of their bounds.
                                                T
In particular, a constraint of the form a x>=l will be included
                                                j
                                     T
in the initial working set if |a x-l|<=r(1+|l|). If r<0 or r>1,
                                     j
the default value is used.

Defaults

This special keyword may be used to reset the default values following a call to E04UCF.

Derivative Level i Default = 3

(AXIOM parameter DER)

This parameter indicates which derivatives are provided by the user in subroutines OBJFUN and CONFUN. The possible choices for i are the following.

i	Meaning
3	All objective and constraint gradients are provided by the user.

2 All of the Jacobian is provided, but some components of the objective gradient are not specified by the user.

1 All elements of the objective gradient are known, but some elements of the Jacobian matrix are not specified by the user.

0 Some elements of both the objective gradient and the Jacobian matrix are not specified by the user.

The value i=3 should be used whenever possible, since E04UCF is more reliable and will usually be more efficient when all derivatives are exact.

If i=0 or 2, E04UCF will estimate the unspecified components of the objective gradient, using finite differences. The computation of finite-difference approximations usually increases the total run-time, since a call to OBJFUN is required for each unspecified element. Furthermore, less accuracy can be attained in the solution (see Chapter 8 of Gill et al [10], for a discussion of limiting accuracy).

If i=0 or 1, E04UCF will approximate unspecified elements of the Jacobian. One call to CONFUN is needed for each variable for which partial derivatives are not available. For example, if the Jacobian has the form

```
(* * * *)
(* ? ? *)
(* * ? *)
(* * * *)
```

where '*' indicates an element provided by the user and '?' indicates an unspecified element, E04UCF will call CONFUN twice: once to estimate the missing element in column 2, and again to estimate the two missing elements in column 3. (Since columns 1 and 4 are known, they require no calls to CONFUN.)

At times, central differences are used rather than forward differences, in which case twice as many calls to OBJFUN and CONFUN are needed. (The switch to central differences is not under the user's control.)

Difference Interval r Default values are computed

(AXIOM parameter DIF)

This option defines an interval used to estimate gradients by finite differences in the following circumstances:

(a) For verifying the objective and/or constraint gradients (see the description of Verify, below).

(b) For estimating unspecified elements of the objective gradient of the Jacobian matrix.

In general, a derivative with respect to the jth variable is approximated using the interval $\hat{}$(delta) , where $\hat{}$(delta) =r(1+|$\hat{}$x |) with $\hat{}$x the first point feasible with respect to the bounds and linear constraints. If the functions are well scaled, the resulting derivative approximation should be accurate to O(r).

See Gill et al [10] for a discussion of the accuracy in finite-difference approximations.

If a difference interval is not specified by the user, a finite-difference interval will be computed automatically for each variable by a procedure that requires up to six calls of CONFUN and OBJFUN for each component. This option is recommended if the function is badly scaled or the user wishes to have E04UCF determine constant elements in the objective and constraint gradients (see the descriptions of CONFUN and OBJFUN in Section 5).

Feasibility Tolerance r Default = $\sqrt{\epsilon}$

(AXIOM parameter FEA)

The scalar r defines the maximum acceptable absolute violations in linear and nonlinear constraints at a 'feasible' point; i.e., a constraint is considered satisfied if its violation does not exceed r. If r<(epsilon) or r>=1, the default value is used. Using this keyword sets both optional parameters Linear Feasibility Tolerance and Nonlinear Feasibility Tolerance to r, if (epsilon)<=r<1. (Additional details are given below under the descriptions of these parameters.)

0.9

```
Function Precision r Default = (epsilon)

   (AXIOM parameter FUN)

This parameter defines (epsilon) , which is intended to be a
                                R
measure of the accuracy with which the problem functions f and c
can be computed. If r<(epsilon) or r>=1, the default value is
used. The value of (epsilon)  should reflect the relative
                           R
precision of 1+|F(x)|; i.e., (epsilon)  acts as a relative
                                      R
precision when |F| is large, and as an absolute precision when
|F| is small. For example, if F(x) is typically of order 1000 and
the first six significant digits are known to be correct, an
appropriate value for (epsilon)  would be 1.0E-6. In contrast, if
                             R
                            -4
F(x) is typically of order 10   and the first six significant
digits are known to be correct, an appropriate value for
(epsilon)  would be 1.0E-10. The choice of (epsilon)  can be
         R                                          R
quite complicated for badly scaled problems; see Chapter 8 of
Gill et al [10] for a discussion of scaling techniques. The
default value is appropriate for most simple functions that are
computed with full accuracy. However, when the accuracy of the
computed function values is known to be significantly worse than
full precision, the value of (epsilon)  should be large enough so
                                      R
that E04UCF will not attempt to distinguish between function
values that differ by less than the error inherent in the
calculation.

Hessian No Default = No

Hessian Yes

   (No AXIOM parameter - fixed as Yes)

This option controls the contents of the upper-triangular matrix
R (see Section 5). E04UCF works exclusively with the transformed
and re-ordered Hessian H  (6), and hence extra computation is
                        Q
required to form the Hessian itself. If Hessian = No, R contains
the Cholesky factor of the transformed and re-ordered Hessian. If
Hessian = Yes the Cholesky factor of the approximate Hessian
```

itself is formed and stored in R. The user should select Hessian = Yes if a warm start will be used for the next call to E04UCF.

Infinite Bound Size r Default = 10^{10}

(AXIOM parameter INFB)

If r>0, r defines the 'infinite' bound BIGBND in the definition of the problem constraints. Any upper bound greater than or equal to BIGBND will be regarded as plus infinity (and similarly for a lower bound less than or equal to -BIGBND). If r<=0, the default value is used.

Infinite Step Size r Default = max(BIGBND,10^{10})

(AXIOM parameter INFS)

If r>0, r specifies the magnitude of the change in variables that is treated as a step to an unbounded solution. If the change in x during an iteration would exceed the value of Infinite Step Size, the objective function is considered to be unbounded below in the feasible region. If r<=0, the default value is used.

Iteration limit i Default = max(50,$3(n+n_L)+10n_N$)

See Major Iteration Limit below.

Linear Feasibility Tolerance r_1 Default = $\sqrt{(epsilon)}$

(AXIOM parameter LINF)

Nonlinear Feasibility Tolerance r_2 Default = $\sqrt{(epsilon)}$ if

(AXIOM parameter NONF)

Derivative Level >= 2 and (epsilon)$^{0.33}$ otherwise

The scalars r_1 and r_2 define the maximum acceptable absolute

violations in linear and nonlinear constraints at a 'feasible' point; i.e., a linear constraint is considered satisfied if its violation does not exceed r_1, and similarly for a nonlinear constraint and r_2. If $r_i <$ (epsilon) or $r_i >= 1$, the default value is used, for i=1,2.

On entry to E04UCF, an iterative procedure is executed in order to find a point that satisfies the linear constraint and bounds on the variables to within the tolerance r_1. All subsequent iterates will satisfy the linear constraints to within the same tolerance (unless r_1 is comparable to the finite-difference interval).

For nonlinear constraints, the feasibility tolerance r_2 defines the largest constraint violation that is acceptable at an optimal point. Since nonlinear constraints are generally not satisfied until the final iterate, the value of Nonlinear Feasibility Tolerance acts as a partial termination criterion for the iterative sequence generated by E04UCF (see the discussion of Optimality Tolerance).

These tolerances should reflect the precision of the corresponding constraints. For example, if the variables and the coefficients in the linear constraints are of order unity, and the latter are correct to about 6 decimal digits, it would be appropriate to specify r_1 as 10^{-6}.

Linesearch Tolerance r Default = 0.9

(AXIOM parameter LINT)

The value r ($0 <= r < 1$) controls the accuracy with which the step (alpha) taken during each iteration approximates a minimum of the merit function along the search direction (the smaller the value of r, the more accurate the linesearch). The default value r=0.9 requests an inaccurate search, and is appropriate for most problems, particularly those with any nonlinear constraints.

If there are no nonlinear constraints, a more accurate search may be appropriate when it is desirable to reduce the number of major iterations - for example, if the objective function is cheap to evaluate, or if a substantial number of gradients are unspecified.

List Default = List

Nolist

(AXIOM parameter LIST)

Normally each optional parameter specification is printed as it is supplied. Nolist may be used to suppress the printing and List may be used to restore printing.

Major Iteration Limit i Default = max(50,3(n+n)+10n)
L N

Iteration Limit

Iters

Itns

(AXIOM parameter MAJI)

The value of i specifies the maximum number of major iterations allowed before termination. Setting i=0 and Major Print Level> 0 means that the workspace needed will be computed and printed, but no iterations will be performed.

Major Print level i Default = 10

Print Level

(AXIOM parameter MAJP)

The value of i controls the amount of printout produced by the major iterations of E04UCF. (See also Minor Print level below.) The levels of printing are indicated below.

i Output

0 No output.

1 The final solution only.

5 One line for each major iteration (no printout of the final solution).

>=10 The final solution and one line of output for each iteration.

>=20 At each major iteration, the objective function, the Euclidean norm of the nonlinear constraint violations, the values of the nonlinear constraints (the vector c), the values of the linear constraints (the vector A x), and the current values of the variables (the vector x).
L

>=30 At each major iteration, the diagonal elements of the matrix T associated with the TQ factorization (5) of the QP working set, and the diagonal elements of R, the triangular factor of the transformed and re-ordered Hessian (6).

Minor Iteration Limit i Default = max(50,3(n+n +n))
L N

(AXIOM parameter MINI)

The value of i specifies the maximum number of iterations for the optimality phase of each QP subproblem.

Minor Print Level i Default = 0

(AXIOM parameter MINP)

The value of i controls the amount of printout produced by the minor iterations of E04UCF, i.e., the iterations of the quadratic programming algorithm. (See also Major Print Level, above.) The following levels of printing are available.

i Output

0 No output.

1 The final QP solution.

5 One line of output for each minor iteration (no printout of the final QP solution).

>=10 The final QP solution and one brief line of output for each minor iteration.

>=20 At each minor iteration, the current estimates of the QP multipliers, the current estimate of the QP search direction, the QP constraint values, and the status of each QP constraint.

>=30 At each minor iteration, the diagonal elements of the matrix T associated with the TQ factorization (5) of the QP working set, and the diagonal elements of the Cholesky factor R of the transformed Hessian (6).

Nonlinear Feasibility Tolerance r Default = $\sqrt{\varepsilon}$

See Linear Feasibility Tolerance, above.

Optimality Tolerance r Default = $\varepsilon^{0.8}$

(AXIOM parameter OPT)

The parameter r ($\varepsilon \le r < 1$) specifies the accuracy to which the user wishes the final iterate to approximate a solution of the problem. Broadly speaking, r indicates the number of correct figures desired in the objective function at the solution. For example, if r is 10^{-6} and E04UCF terminates successfully, the final value of F should have approximately six correct figures. If $r < \varepsilon_R$ or $r \ge 1$ the default value is used.

E04UCF will terminate successfully if the iterative sequence of x-values is judged to have converged and the final point satisfies the first-order Kuhn-Tucker conditions (see Section 3). The sequence of iterates is considered to have converged at x if

$$\alpha\,||p|| \le \sqrt{r}(1+||x||), \tag{8a}$$

where p is the search direction and α the step length from (3). An iterate is considered to satisfy the first-order conditions for a minimum if

$$||Z_{FR}^T g_{FR}|| <= \sqrt{r}(1+\max(1+|F(x)|, ||\bar{g}_{FR}||)) \quad (8b)$$

and

$$|res_j| <= ftol \text{ for all } j, \quad (8c)$$

where $Z_{FR}^T g_{FR}$ is the projected gradient (see Section 3), g_{FR} is the gradient of $F(x)$ with respect to the free variables, res_j is the violation of the jth active nonlinear constraint, and ftol is the Nonlinear Feasibility Tolerance.

Step Limit r Default = 2.0

(AXIOM parameter STE)

If r>0, r specifies the maximum change in variables at the first step of the linesearch. In some cases, such as $F(x)=ae^{bx}$ or $F(x)=ax^b$, even a moderate change in the components of x can lead to floating-point overflow. The parameter r is therefore used to encourage evaluation of the problem functions at meaningful points. Given any major iterate x, the first point $\tilde{x}$ at which F and c are evaluated during the linesearch is restricted so that

$$||\tilde{x}-x||_2 <= r(1+||x||_2).$$

The linesearch may go on and evaluate F and c at points further from x if this will result in a lower value of the merit function. In this case, the character L is printed at the end of the optional line of printed output, (see Section 5.2). If L is printed for most of the iterations, r should be set to a larger value.

Wherever possible, upper and lower bounds on x should be used to prevent evaluation of nonlinear functions at wild values. The

```
default value Step Limit = 2.0 should not affect progress on
well-behaved functions, but values 0.1 or 0.01 may be helpful
when rapidly varying functions are present. If a small value of
Step Limit is selected, a good starting point may be required. An
important application is to the class of nonlinear least-squares
problems. If r<=0, the default value is used.

Start Objective Check At Variable k Default = 1

   (AXIOM parameter STAO)

Start Constraint Check At Variable k Default = 1

   (AXIOM parameter STAC)

Stop Objective Check At Variable l Default = n

   (AXIOM parameter STOO)

Stop Constraint Check At Variable l Default = n

   (AXIOM parameter STOC)

These keywords take effect only if Verify Level > 0 (see below).
They may be used to control the verification of gradient elements
computed by subroutines OBJFUN and CONFUN. For example, if the
first 30 components of the objective gradient appeared to be
correct in an earlier run, so that only component 31 remains
questionable, it is reasonable to specify Start Objective Check
At Variable 31. If the first 30 variables appear linearly in the
objective, so that the corresponding gradient elements are
constant, the above choice would also be appropriate.

Verify Level i Default = 0

Verify No

Verify Level - 1

Verify Level 0

Verify Objective Gradients

Verify Level 1

Verify Constraint Gradients
```

Verify Level 2

Verify

Verify Yes

Verify Gradients

Verify Level 3

(AXIOM parameter VE)

These keywords refer to finite-difference checks on the gradient elements computed by the user-provided subroutines OBJFUN and CONFUN. (Unspecified gradient components are not checked.) It is possible to specify Verify Levels 0-3 in several ways, as indicated above. For example, the nonlinear objective gradient (if any) will be verified if either Verify Objective Gradients or Verify Level 1 is specified. Similarly, the objective and the constraint gradients will be verified if Verify Yes or Verify Level 3 or Verify is specified.

If 0<=i<=3, gradients will be verified at the first point that satisfies the linear constraints and bounds. If i=0, only a ' cheap' test will be performed, requiring one call to OBJFUN and one call to CONFUN. If 1<=i<=3, a more reliable (but more expensive) check will be made on individual gradient components, within the ranges specified by the Start and Stop keywords described above. A result of the form OK or BAD? is printed by E04UCF to indicate whether or not each component appears to be correct.

If 10<=i<=13, the action is the same as for i - 10, except that it will take place at the user-specified initial value of x.

We suggest that Verify Level 3 be specified whenever a new function routine is being developed.

5.1.3. Optional parameter checklist and default values

For easy reference, the following list shows all the valid keywords and their default values. The symbol (epsilon) represents the machine precision (see X02AJF(*)).

Optional Parameters Default Values

Central difference interval	Computed automatically
Cold/Warm start	Cold start
Crash tolerance	0.01
Defaults	
Derivative level	3
Difference interval	Computed automatically
Feasibility tolerance	$\sqrt{\epsilon}$
Function precision	$\epsilon^{0.9}$
Hessian	No
Infinite bound size	10^{10}
Infinite step size	10^{10}
Linear feasibility tolerance	$\sqrt{\epsilon}$
Linesearch tolerance	0.9
List/Nolist	List
Major iteration limit	$\max(50,3(n_L+n_N)+10n_N)$
Major print level	10
Minor iteration limit	$\max(50,3(n_L+n_N))$

Minor print level	0
Nonlinear feasibility tolerance	$\sqrt{(\text{epsilon})}$ if Derivative Level >= 2 $(\text{epsilon})^{0.33}$ otherwise (epsilon)
Optimality tolerance	$(\text{epsilon})^{0.8}$ R
Step limit	2.0
Start objective check	1
Start constraint check	1
Stop objective check	n
Stop constraint check	n
Verify level	0

5.2. Description of Printed Output

The level of printed output from E04UCF is controlled by the user (see the description of Major Print Level and Minor Print Level in Section 5.1). If Minor Print Level > 0, output is obtained from the subroutines that solve the QP subproblem. For a detailed description of this information the reader should refer to E04NCF(*).

When Major Print Level >= 5, the following line of output is produced at every major iteration of E04UCF. In all cases, the values of the quantities printed are those in effect on completion of the given iteration.

Itn	is the iteration count.
ItQP	is the sum of the iterations required by the feasibility and optimality phases of the QP subproblem. Generally, ItQP will be 1 in the later iterations, since theoretical analysis predicts that the correct active set will be identified near the solution (see Section 3).

Note that ItQP may be greater than the Minor Iteration Limit if some iterations are required for the feasibility phase.

Step — is the step (alpha) taken along the computed search direction. On reasonably well-behaved problems, the unit step will be taken as the solution is approached.

Nfun — is the cumulative number of evaluations of the objective function needed for the linesearch. Evaluations needed for the estimation of the gradients by finite differences are not included. Nfun is printed as a guide to the amount of work required for the linesearch.

Merit — is the value of the augmented Lagrangian merit function (12) at the current iterate. This function will decrease at each iteration unless it was necessary to increase the penalty parameters (see Section 8.2). As the solution is approached, Merit will converge to the value of the objective function at the solution.

If the QP subproblem does not have a feasible point (signified by I at the end of the current output line), the merit function is a large multiple of the constraint violations, weighted by the penalty parameters. During a sequence of major iterations with infeasible subproblems, the sequence of Merit values will decrease monotonically until either a feasible subproblem is obtained or E04UCF terminates with IFAIL = 3 (no feasible point could be found for the nonlinear constraints).

If no nonlinear constraints are present (i.e., NCNLN = 0), this entry contains Objective, the value of the objective function F(x). The objective function will decrease monotonically to its optimal value when there are no nonlinear constraints.

Bnd — is the number of simple bound constraints in the predicted active set.

Lin — is the number of general linear constraints in the predicted active set.

Nln — is the number of nonlinear constraints in the predicted active set (not printed if NCNLN is zero).

Nz — is the number of columns of Z (see Section 8.1). The value of Nz is the number of variables minus the number of constraints in the predicted active set; i.e., Nz = n-(Bnd + Lin + Nln).

Norm Gf — is the Euclidean norm of g_{FR}, the gradient of the objective function with respect to the free variables, i.e.,variables not currently held at a bound.

Norm Gz — is $||Z^T g_{FR}||$, the Euclidean norm of the projected gradient (see Section 8.1). Norm Gz will be approximately zero in the neighbourhood of a solution.

Cond H — is a lower bound on the condition number of the Hessian approximation H.

Cond Hz — is a lower bound on the condition number of the projected Hessian approximation H_z ($H_z = Z^T H Z = R_z^T R_z$; see (6) and (12) in Sections 3 and 8.1). The larger this number, the more difficult the problem.

Cond T — is a lower bound on the condition number of the matrix of predicted active constraints.

Norm C — is the Euclidean norm of the residuals of constraints that are violated or in the predicted active set (not printed if NCNLN is zero). Norm C will be approximately zero in the neighbourhood of a solution.

Penalty — is the Euclidean norm of the vector of penalty parameters used in the augumented Lagrangian merit function (not printed if NCNLN is zero).

Conv — is a three-letter indication of the status of the three convergence tests (8a)-(8c) defined in the description of the optional parameter Optimality Tolerance in Section 5.1 Each letter is T if the test is satisfied, and F otherwise. The three tests indicate whether:
(a) the sequence of iterates has converged;

(b) the projected gradient (Norm Gz) is sufficiently small; and

(c) the norm of the residuals of constraints in the predicted active set (Norm C) is small enough.
If any of these indicators is F when E04UCF terminates with IFAIL = 0, the user should check the solution carefully.

M — is printed if the Quasi-Newton update was modified to ensure that the Hessian approximation is positive-definite (see Section 8.3).

I — is printed if the QP subproblem has no feasible point.

C — is printed if central differences were used to compute the unspecified objective and constraint gradients. If the value of Step is zero, the switch to central differences was made because no lower point could be found in the linesearch. (In this case, the QP subproblem is resolved with the central-difference gradient and Jacobian.) If the value of Step is non-zero, central differences were computed because Norm Gz and Norm C imply that x is close to a Kuhn-Tucker point.

L — is printed if the linesearch has produced a relative change in x greater than the value defined by the optional parameter Step Limit. If this output occurs frequently during later iterations of the run, Step Limit should be set to a larger value.

R is printed if the approximate Hessian has been refactorized. If the diagonal condition estimator of R indicates that the approximate Hessian is badly conditioned, the approximate Hessian is refactorized using column interchanges. If necessary, R is modified so that its diagonal condition estimator is bounded.

When Major Print Level = 1 or Major Print Level >= 10, the summary printout at the end of execution of E04UCF includes a listing of the status of every variable and constraint. Note that default names are assigned to all variables and constraints.

The following describes the printout for each variable.

Varbl gives the name (V) and index j=1,2,...,n of the variable.

State gives the state of the variable in the predicted active set (FR if neither bound is in the active set, EQ if a fixed variable, LL if on its lower bound, UL if on its upper bound). If the variable is predicted to lie outside its upper or lower bound by more than the feasibility tolerance, State will be ++ or -- respectively. (The latter situation can occur only when there is no feasible point for the bounds and linear constraints.)

Value is the value of the variable at the final iteration.

Lower bound is the lower bound specified for the variable. (None indicates that BL(j)<=- BIGBND.)

Upper bound is the upper bound specified for the variable. (None indicates that BL(j)>=BIGBND.)

Lagr Mult is the value of the Lagrange-multiplier for the associated bound constraint. This will be zero if State is FR. If x is optimal, the multiplier should be non-negative if State is LL, and non-positive if State is UL.

Residual is the difference between the variable Value and the nearer of its bounds BL(j) and BU(j).

```
The printout for general constraints is the same as for
variables, except for the following:

L Con      is the name (L) and index i, for i = 1,2,...,NCLIN of
           a linear constraint.

N Con      is the name (N) and index i, for i = 1,2,...,NCNLN of
           a nonlinear constraint.

6. Error Indicators and Warnings

Errors or warnings specified by the routine:

If on entry IFAIL = 0 or -1, explanatory error messages are
output on the current error message unit (as defined by X04AAF).

The input data for E04UCF should always be checked (even if
E04UCF terminates with IFAIL=0).

Note that when Print Level>0, a short description of IFAIL is
printed.

Errors and diagnostics indicated by IFAIL, together with some
recommendations for recovery are indicated below.

IFAIL= 1
     The final iterate x satisfies the first-order Kuhn-Tucker
     conditions to the accuracy requested, but the sequence of
     iterates has not yet converged. E04UCF was terminated
     because no further improvement could be made in the merit
     function.

     This value of IFAIL may occur in several circumstances. The
     most common situation is that the user asks for a solution
     with accuracy that is not attainable with the given
     precision of the problem (as specified by Function Precision
     see Section 5). This condition will also occur if, by
     chance, an iterate is an 'exact' Kuhn-Tucker point, but the
     change in the variables was significant at the previous
     iteration. (This situation often happens when minimizing
     very simple functions, such as quadratics.)

     If the four conditions listed in Section 5 for IFAIL = 0 are
     satisfied, x is likely to be a solution of (1) even if IFAIL
     = 1.
```

IFAIL= 2
E04UCF has terminated without finding a feasible point for the linear constraints and bounds, which means that no feasible point exists for the given value of Linear Feasibility Tolerance (see Section 5.1). The user should check that there are no constraint redundancies. If the data for the constraints are accurate only to an absolute precision (sigma), the user should ensure that the value of the optional parameter Linear Feasibility Tolerance is greater than (sigma). For example, if all elements of A are of order unity and are accurate to only three decimal places, Linear Feasibility Tolerance should be at least 10^{-3}.

IFAIL= 3
No feasible point could be found for the nonlinear constraints. The problem may have no feasible solution. This means that there has been a sequence of QP subproblems for which no feasible point could be found (indicated by I at the end of each terse line of output). This behaviour will occur if there is no feasible point for the nonlinear constraints. (However, there is no general test that can determine whether a feasible point exists for a set of nonlinear constraints.) If the infeasible subproblems occur from the very first major iteration, it is highly likely that no feasible point exists. If infeasibilities occur when earlier subproblems have been feasible, small constraint inconsistencies may be present. The user should check the validity of constraints with negative values of ISTATE. If the user is convinced that a feasible point does exist, E04UCF should be restarted at a different starting point.

IFAIL= 4
The limiting number of iterations (determined by the optional parameter Major Iteration Limit see Section 5.1) has been reached.

If the algorithm appears to be making progress, Major Iteration Limit may be too small. If so, increase its value and rerun E04UCF (possibly using the Warm Start option). If the algorithm seems to be 'bogged down', the user should check for incorrect gradients or ill-conditioning as described below under IFAIL = 6.

Note that ill-conditioning in the working set is sometimes

resolved automatically by the algorithm, in which case performing additional iterations may be helpful. However, ill-conditioning in the Hessian approximation tends to persist once it has begun, so that allowing additional iterations without altering R is usually inadvisable. If the quasi-Newton update of the Hessian approximation was modified during the latter iterations (i.e., an M occurs at the end of each terse line), it may be worthwhile to try a warm start at the final point as suggested above.

IFAIL= 6

x does not satisfy the first-order Kuhn-Tucker conditions, and no improved point for the merit function could be found during the final line search.

A sufficient decrease in the merit function could not be attained during the final line search. This sometimes occurs because an overly stringent accuracy has been requested, i.e., Optimality Tolerance is too small. In this case the user should apply the four tests described under IFAIL = 0 above to determine whether or not the final solution is acceptable (see Gill et al [10], for a discussion of the attainable accuracy).

If many iterations have occurred in which essentially no progress has been made and E04UCF has failed completely to move from the initial point then subroutines OBJFUN or CONFUN may be incorrect. The user should refer to comments below under IFAIL = 7 and check the gradients using the Verify parameter. Unfortunately, there may be small errors in the objective and constraint gradients that cannot be detected by the verification process. Finite-difference approximations to first derivatives are catastrophically affected by even small inaccuracies. An indication of this situation is a dramatic alteration in the iterates if the finite-difference interval is altered. One might also suspect this type of error if a switch is made to central differences even when Norm Gz and Norm C are large.

Another possibility is that the search direction has become inaccurate because of ill-conditioning in the Hessian approximation or the matrix of constraints in the working set; either form of ill-conditioning tends to be reflected in large values of ItQP (the number of iterations required to solve each QP subproblem).

If the condition estimate of the projected Hessian (Cond Hz) is extremely large, it may be worthwhile to rerun E04UCF from the final point with the Warm Start option. In this situation, ISTATE should be left unaltered and R should be reset to the identity matrix.

If the matrix of constraints in the working set is ill-conditioned (i.e., Cond T is extremely large), it may be helpful to run E04UCF with a relaxed value of the Feasibility Tolerance (Constraint dependencies are often indicated by wide variations in size in the diagonal elements of the matrix T, whose diagonals will be printed for Major Print Level >= 30).

IFAIL= 7

The user-provided derivatives of the objective function and/or nonlinear constraints appear to be incorrect.

Large errors were found in the derivatives of the objective function and/or nonlinear constraints. This value of IFAIL will occur if the verification process indicated that at least one gradient or Jacobian component had no correct figures. The user should refer to the printed output to determine which elements are suspected to be in error.

As a first-step, the user should check that the code for the objective and constraint values is correct - for example, by computing the function at a point where the correct value is known. However, care should be taken that the chosen point fully tests the evaluation of the function. It is remarkable how often the values x=0 or x=1 are used to test function evaluation procedures, and how often the special properties of these numbers make the test meaningless.

Special care should be used in this test if computation of the objective function involves subsidiary data communicated in COMMON storage. Although the first evaluation of the function may be correct, subsequent calculations may be in error because some of the subsidiary data has accidently been overwritten.

Errors in programming the function may be quite subtle in that the function value is 'almost' correct. For example, the function may not be accurate to full precision because of the inaccurate calculation of a subsidiary quantity, or the limited accuracy of data upon which the function

depends. A common error on machines where numerical calculations are usually performed in double precision is to include even one single-precision constant in the calculation of the function; since some compilers do not convert such constants to double precision, half the correct figures may be lost by such a seemingly trivial error.

IFAIL= 9
An input parameter is invalid. The user should refer to the printed output to determine which parameter must be redefined.

IFAILOverflow
If the printed output before the overflow error contains a warning about serious ill-conditioning in the working set when adding the jth constraint, it may be possible to avoid the difficulty by increasing the magnitude of the optional parameter Linear Feasiblity Tolerance or Nonlinear Feasiblity Tolerance, and rerunning the program. If the message recurs even after this change, the offending linearly dependent constraint (with index 'j') must be removed from the problem. If overflow occurs in one of the user-supplied routines (e.g. if the nonlinear functions involve exponentials or singularities), it may help to specify tighter bounds for some of the variables (i.e., reduce the gap between appropriate l_j and u_j).

7. Accuracy

If IFAIL = 0 on exit then the vector returned in the array X is an estimate of the solution to an accuracy of approximately Feasiblity Tolerance (see Section 5.1), whose default value is $(\text{epsilon})^{0.8}$, where (epsilon) is the machine precision (see X02AJF(*)).

8. Further Comments

In this section we give some further details of the method used by E04UCF.

8.1. Solution of the Quadratic Programming Subproblem

The search direction p is obtained by solving (4) using the method of E04NCF(*) (Gill et al [8]), which was specifically

designed to be used within an SQP algorithm for nonlinear programming.

The method of E04UCF is a two-phase (primal) quadratic programming method. The two phases of the method are: finding an initial feasible point by minimizing the sum of infeasibilities (the feasibility phase), and minimizing the quadratic objective function within the feasible region (the optimality phase). The computations in both phases are perfomed by the same subroutines. The two-phase nature of the algorithm is reflected by changing the function being minimized from the sum of infeasibilities to the quadratic objective function.

In general, a quadratic program must be solved by iteration. Let p denote the current estimate of the solution of (4); the new

iterate $\bar{p}$ is defined by

$$\bar{p}=p+(sigma)d, \tag{9}$$

where, as in (3), (sigma) is a non-negative step length and d is a search direction.

At the beginning of each iteration of E04UCF, a working set is defined of constraints (general and bound) that are satisfied exactly. The vector d is then constructed so that the values of constraints in the working set remain unaltered for any move along d. For a bound constraint in the working set, this property is achieved by setting the corresponding component of d to zero, i.e., by fixing the variable at its bound. As before, the subscripts 'FX' and 'FR' denote selection of the components associated with the fixed and free variables.

Let C denote the sub-matrix of rows of

$$\begin{pmatrix} A \\ L \\ A \\ N \end{pmatrix}$$

corresponding to general constraints in the working set. The general constraints in the working set will remain unaltered if

$$C_{FR}\, d_{FR} = 0, \tag{10}$$

which is equivalent to defining d_{FR} as

$$d_{FR} = Z d_z \tag{11}$$

for some vector d_z, where Z is the matrix associated with the TQ factorization (5) of C_{FR}.

The definition of d_z in (11) depends on whether the current p is feasible. If not, d_z is zero except for a component (gamma) in the jth position, where j and (gamma) are chosen so that the sum of infeasibilities is decreasing along d. (For further details, see Gill et al [8].) In the feasible case, d_z satisfies the equations

$$R_z^T R_z d_z = -Z^T q_{FR}, \tag{12}$$

where R_z is the Cholesky factor of $Z^T H_{FR} Z$ and q is the gradient of the quadratic objective function (q=g+Hp). (The vector $Z^T q_{FR}$ is the projected gradient of the QP.) With (12), P+d is the minimizer of the quadratic objective function subject to treating the constraints in the working set as equalities.

If the QP projected gradient is zero, the current point is a constrained stationary point in the subspace defined by the working set. During the feasiblity phase, the projected gradient will usually be zero only at a vertex (although it may vanish at non-vertices in the presence of constraint dependencies). During the optimality phase, a zero projected gradient implies that p minimizes the quadratic objective function when the constraints in the working set are treated as equalities. In either case, Lagrange multipliers are computed. Given a positive constant (delta) of the order of the machine precision, the Lagrange

multiplier (mu) corresponding to an inequality constraint in the
j
working set at its upper bound is said to be optimal if
(mu) <=(delta) when the jth constraint is at its upper bound, or
j
if (mu) >=-(delta) when the associated constraint is at its lower
j
bound. If any multiplier is non-optimal, the current objective
function (either the true objective or the sum of
infeasibilities) can be reduced by deleting the corresponding
constraint from the working set.

If optimal multipliers occur during the feasibility phase and the
sum of infeasibilities is non-zero, no feasible point exists. The
QP algorithm will then continue iterating to determine the
minimum sum of infeasibilities. At this point, the Lagrange
multiplier (mu) will satisfy -(1+(delta))<=(mu) <=(delta) for an
j j
inequality constraint at its upper bound, and
-(delta)<=(mu) <=1+(delta) for an inequality at its lower bound.
j
The Lagrange multiplier for an equality constraint will satisfy
|(mu) |<=1+(delta).
j

The choice of step length (sigma) in the QP iteration (9) is
based on remaining feasible with respect to the satisfied
constraints. During the optimality phase, if p+d is feasible,
(sigma) will be taken as unity. (In this case, the projected
-
gradient at p will be zero.) Otherwise, (sigma) is set to
(sigma) , the step to the 'nearest'constraint, which is added to
M
the working set at the next iteration.

Each change in the working set leads to a simple change to C :
FR
if the status of a general constraint changes, a row of C is
FR
altered; if a bound constraint enters or leaves the working set,
a column of C changes. Explicit representations are recurred of
FR
T T
the matrices T, Q and R, and of the vectors Q q and Q g.
FR

8.2. The Merit Function

After computing the search direction as described in Section 3, each major iteration proceeds by determining a step length (alpha) in (3) that produces a 'sufficient decrease' in the augmented Lagrangian merit function

$$L(x,(\lambda),s) = F(x) - \sum_i (\lambda)_i (c_i(x) - s_i) + \frac{1}{2}\sum_i (\rho)_i (c_i(x) - s_i)^2, \qquad (13)$$

where x, (lambda) and s vary during the linsearch. The summation terms in (13) involve only the nonlinear constraints. The vector (lambda) is an estimate of the Lagrange multipliers for the nonlinear constraints of (1). The non-negative slack variables $\{s_i\}$ allow nonlinear inequality constraints to be treated without introducing discontinuities. The solution of the QP subproblem (4) provides a vector triple that serves as a direction of search for the three sets of variables. The non-negative vector (rho) of penalty parameters is initialised to zero at the beginning of the first major iteration. Thereafter, selected components are increased whenever necessary to ensure descent for the merit function. Thus, the sequence of norms of (rho) (the printed quantity Penalty, see Section 5.2) is generally non-decreasing, although each $(\rho)_i$ may be reduced a limited number of times.

The merit function (13) and its global convergence properties are described in Gill et al [9].

8.3. The Quasi-Newton Update

The matrix H in (4) is a positive-definite quasi-Newton approximation to the Hessian of the Lagrangian function. (For a review of quasi-Newton methods, see Dennis and Schnabel [3].) At the end of each major iteration, a new Hessian approximation $\bar{H}$ is defined as a rank-two modification of H. In E04UCF, the BFGS

quasi-Newton update is used:

$$\bar{H}=H-\frac{1}{T}Hss^{T}H+\frac{1}{T}yy^{T}, \qquad (14)$$
$$s^{T}Hs \qquad y^{T}s$$

where $s=\bar{x}-x$ (the change in x).

In E04UCF, H is required to be positive-definite. If H is positive-definite, $\bar{H}$ defined by (14) will be positive-definite if and only if $y^{T}s$ is positive (see, e.g. Dennis and More [1]). Ideally, y in (14) would be taken as y_{L}, the change in gradient of the Lagrangian function

$$y_{L}=g-A_{N}^{T}(\bar{\mu})-g+A_{N}^{T}(\mu), \qquad (15)$$

where $(\mu)_{N}$ denotes the QP multipliers associated with the nonlinear constraints of the original problem. If $y^{T}s_{L}$ is not sufficiently positive, an attempt is made to perform the update with a vector y of the form

$$y=y_{L}+\sum_{i=1}^{m_{N}}(\omega)_{i}(a_{i}(\bar{x})c_{i}(\bar{x})-a_{i}(x)c_{i}(x)),$$

where $(\omega)_{i}\geq 0$. If no such vector can be found, the update is perfomed with a scaled y_{L}; in this case, M is printed to indicate that the update is modified.

Rather than modifying H itself, the Cholesky factor of the transformed Hessian H (6) is updated, where Q is the matrix from

(5) associated with the active set of the QP subproblem. The update (13) is equivalent to the following update to H :

$$\bar{H} = H - \frac{1}{s^T H s} H s s^T H + \frac{1}{y^T s} y y^T, \qquad (16)$$

where $y = Q^T y$, and $s = Q^T s$. This update may be expressed as a rank-one update to R (see Dennis and Schnabel [2]).

9. Example

This section describes one version of the so-called 'hexagon' problem (a different formulation is given as Problem 108 in Hock and Schittkowski [11]). The problem is to determine the hexagon of maximum area such that no two of its vertices are more than one unit apart (the solution is not a regular hexagon).

All constraint types are included (bounds, linear, nonlinear), and the Hessian of the Lagrangian function is not positive-definite at the solution. The problem has nine variables, non-infinite bounds on seven of the variables, four general linear constraints, and fourteen nonlinear constraints.

The objective function is

$$F(x) = -x_2 x_6 + x_1 x_7 - x_3 x_7 + x_5 x_8 - x_4 x_9 + x_3 x_8 .$$

The bounds on the variables are

$$x_1 \geq 0,\ -1 \leq x_3 \leq 1,\ x_5 \geq 0,\ x_6 \geq 0,\ x_7 \geq 0,\ x_8 \leq 0,\ \text{and}\ x_9 \leq 0.$$

Thus,

$$l_B = (0, -\infty, -1, -\infty, 0, 0, 0, -\infty, -\infty)^T$$

```
                                                                  T
          u =(infty,infty,1,infty,infty,infty,infty,0,0)
           B
```

The general linear constraints are

```
     x -x >=0,x -x >=0, x -x >=0,and x -x >=0.
      2  1     3  2      3  4         4  5
```

Hence,

```
          (0)     (-1  1 0  0  0 0 0 0 0)        (infty)
          (0)     ( 0 -1 1  0  0 0 0 0 0)        (infty)
       l =(0), A =( 0  0 1 -1  0 0 0 0 0) and u =(infty).
        L (0)   L ( 0  0 0  1 -1 0 0 0 0)      L (infty)
```

The nonlinear constraints are all of the form c (x)<=1, for
                                                i
i=1,2,...,14; hence, all components of l are -infty, and all
                                       N
components of u are 1. The fourteen functions {c (x)} are
              N   i

$$c_1(x) = x_1^2 + x_6^2,$$

$$c_2(x) = (x_2 - x_1)^2 + (x_7 - x_6)^2,$$

$$c_3(x) = (x_3 - x_1)^2 + x_6^2,$$

$$c_4(x) = (x_4 - x_1)^2 + (x_8 - x_6)^2,$$

$$c_5(x) = (x_5 - x_1)^2 + (x_9 - x_6)^2,$$

$$c_6(x) = x_2^2 + x_7^2,$$

```
                2  2
c (x)=(x -x ) +x ,
 7      3  2    7

                2       2
c (x)=(x -x ) +(x -x ) ,
 8      4  2     8  7

                2       2
c (x)=(x -x ) +(x -x ) ,
 9      2  5     7  9

                2  2
c  (x)=(x -x ) +x ,
 10      4  3    8

                2  2
c  (x)=(x -x ) +x ,
 11      5  3    9

          2  2
c  (x)=x +x ,
 12      4  8

                2       2
c  (x)=(x -x ) +(x -x ) ,
 13      4  5     9  8

          2  2
c  (x)=x +x .
 14      5  9
```

An optimal solution (to five figures) is

$$x^* = (0.060947, 0.59765, 1.0, 0.59765, 0.060947, 0.34377, 0.5, -0.5, 0.34377)^T,$$

and $F(x^*) = -1.34996$. (The optimal objective function is unique, but is achieved for other values of x.) Five nonlinear constraints and one simple bound are active at x^*. The sample

solution output is given later in this section, following the sample main program and problem definition.

Two calls are made to E04UCF in order to demonstrate some of its features. For the first call, the starting point is:

$$x_0 = (0.1, 0.125, 0.666666, 0.142857, 0.111111, 0.2, 0.25, -0.2, -0.25)^T.$$

All objective and constraint derivatives are specified in the user-provided subroutines OBJFN1 and CONFN1, i.e., the default option Derivative Level =3 is used.

On completion of the first call to E04UCF, the optimal variables are perturbed to produce the initial point for a second run in which the problem functions are defined by the subroutines OBJFN2 and CONFN2. To illustrate one of the finite-difference options in E04UCF, these routines are programmed so that the first six components of the objective gradient and the constant elements of the Jacobian matrix are not specified; hence, the option Derivative Level =0 is chosen. During computation of the finite-difference intervals, the constant Jacobian elements are identified and set, and E04UCF automatically increases the derivative level to 2.

The second call to E04UCF illustrates the use of the Warm Start Level option to utilize the final active set, nonlinear multipliers and approximate Hessian from the first run. Note that Hessian = Yes was specified for the first run so that the array R would contain the Cholesky factor of the approximate Hessian of the Lagrangian.

The two calls to E04UCF illustrate the alternative methods of assigning default parameters. (There is no special significance in the order of these assignments; an options file may just as easily be used to modify parameters set by E04UEF.)

The results are typical of those obtained from E04UCF when solving well behaved (non-trivial) nonlinear problems. The approximate Hessian and working set remain relatively well-conditioned. Similarly the penalty parameters remain small and approximately constant. The numerical results illustrate much of the theoretically predicted behaviour of a quasi-Newton SQP method. As x approaches the solution, only one minor iteration is perfomed per major iteration, and the Norm Gz and Norm C columns

exhibit the fast linear convergence rate mentioned in Sections 5 and 6. Note that the constraint violations converge earlier than the projected gradient. The final values of the project gradient norm and constraint norm reflect the limiting accuracy of the two quantities. It is possible to achieve almost full precision in the constraint norm but only half precision in the projected gradient norm. Note that the final accuracy in the nonlinear constraints is considerably better than the feasibility tolerance, because the constraint violations are being refined during the last few iterations while the algorithm is working to reduce the projected gradient norm. In this problem, the constraint values and Lagrange multipliers at the solution are ' well balanced', i.e., all the multipliers are approximately the same order of magnitude. The behaviour is typical of a well-scaled problem.

The example program is not reproduced here. The source code for all example programs is distributed with the NAG Foundation Library software and should be available on-line.

%%%

E04 -- Minimizing or Maximizing a Function E04UDF
E04UDF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for your implementation to check implementation-dependent details. The symbol (*) after a NAG routine name denotes a routine that is not included in the Foundation Library.

1. Purpose

To supply optional parameters to E04UCF from an external file.

2. Specification

```
SUBROUTINE E04UDF (IOPTNS, INFORM)
INTEGER          IOPTNS, INFORM
```

3. Description

E04UDF may be used to supply values for optional parameters to E04UCF. E04UDF reads an external file and each line of the file defines a single optional parameter. It is only necessary to supply values for those parameters whose values are to be different from their default values.

Each optional parameter is defined by a single character string of up to 72 characters, consisting of one or more items. The items associated with a given option must be separated by spaces, or equal signs (=). Alphabetic characters may be upper or lower case. The string

```
 Print level = 1
```

is an example of a string used to set an optional parameter. For each option the string contains one or more of the following items:

(a) A mandatory keyword.

(b) A phrase that qualifies the keyword.

(c) A number that specifies an INTEGER or real value. Such numbers may be up to 16 contiguous characters in Fortran 77's I, F, E or D formats, terminated by a space if this is not the last item on the line.

Blank strings and comments are ignored. A comment begins with an asterisk (*) and all subsequent characters in the string are regarded as part of the comment.

The file containing the options must start with begin and must finish with end An example of a valid options file is:

```
Begin * Example options file
 Print level =10
End
```

Normally each line of the file is printed as it is read, on the current advisory message unit (see X04ABF), but printing may be suppressed using the keyword nolist To suppress printing of begin, nolist must be the first option supplied as in the file:

```
Begin
  Nolist
  Print level = 10
End
```

Printing will automatically be turned on again after a call to E04UCF and may be turned on again at any time by the user by using the keyword list.

Optional parameter settings are preserved following a call to E04UCF, and so the keyword defaults is provided to allow the user to reset all the optional parameters to their default values prior to a subsequent call to E04UCF.

A complete list of optional parameters, their abbreviations, synonyms and default values is given in Section 5.1 of the document for E04UCF.

4. References

None.

5. Parameters

1: IOPTNS -- INTEGER Input
On entry: IOPTNS must be the unit number of the options file. Constraint: 0 <= IOPTNS <= 99.

2: INFORM -- INTEGER Output
On exit: INFORM will be zero, if an options file with the current structure has been read. Otherwise INFORM will be positive. Positive values of INFORM indicate that an options file may not have been successfully read as follows:
INFORM = 1
IOPTNS is not in the range [0,99].

INFORM = 2
begin was found, but end-of-file was found before end was found.

INFORM = 3
end-of-file was found before begin was found.

6. Error Indicators and Warnings

If a line is not recognised as a valid option, then a warning message is output on the current advisory message unit (X04ABF).

7. Accuracy

Not applicable.

8. Further Comments

E04UEF may also be used to supply optional parameters to E04UCF.

9. Example

See the example for E04UCF.

%%%

E04 -- Minimizing or Maximizing a Function E04UEF
E04UEF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for your implementation to check implementation-dependent details. The symbol (*) after a NAG routine name denotes a routine that is not included in the Foundation Library.

1. Purpose

To supply individual optional parameters to E04UCF.

2. Specification

```
SUBROUTINE E04UEF (STRING)
CHARACTER*(*)    STRING
```

3. Description

E04UEF may be used to supply values for optional parameters to E04UCF. It is only necessary to call E04UEF for those parameters whose values are to be different from their default values. One call to E04UEF sets one parameter value.

Each optional parameter is defined by a single character string of up to 72 characters, consisting of one or more items. The items associated with a given option must be separated by spaces, or equal signs (=). Alphabetic characters may be upper or lower case. The string

```
Print level = 1
```

is an example of a string used to set an optional parameter. For each option the string contains one or more of the following items:

(a) A mandatory keyword.

(b) A phrase that qualifies the keyword.

(c) A number that specifies an INTEGER or real value. Such numbers may be up to 16 contiguous characters in Fortran 77's I, F, E or D formats, terminated by a space if this is not the last item on the line.

Blank strings and comments are ignored. A comment begins with an asterisk (*) and all subsequent characters in the string are regarded as part of the comment.

Normally, each user-specified option is printed as it is defined, on the current advisory message unit (see X04ABF), but this printing may be suppressed using the keyword nolist Thus the statement

```
CALL E04UEF ('Nolist')
```

suppresses printing of this and subsequent options. Printing will automatically be turned on again after a call to E04UCF, and may be turned on again at any time by the user, by using the keyword list.

Optional parameter settings are preserved following a call to E04UCF, and so the keyword defaults is provided to allow the user to reset all the optional parameters to their default values by the statement,

```
CALL E04UEF ('Defaults')
```

prior to a subsequent call to E04UCF.

A complete list of optional parameters, their abbreviations, synonyms and default values is given in Section 5.1 of the document for E04UCF.

4. References

None.

5. Parameters

1: STRING -- CHARACTER*(*) Input
On entry: STRING must be a single valid option string. See Section 3 above and Section 5.1 of the routine document for E04UCF. On entry: STRING must be a single valid option

```
     string. See Section 3 above and Section 5.1 of the routine
     document for E04UCF.

6. Error Indicators and Warnings

If the parameter STRING is not recognised as a valid option
string, then a warning message is output on the current advisory
message unit (X04ABF).

7. Accuracy

Not applicable.

8. Further Comments

E04UDF may also be used to supply optional parameters to E04UCF.

9. Example

See the example for E04UCF.
```

%%

```
E04 -- Minimizing or Maximizing a Function                E04YCF
        E04YCF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for
your implementation to check implementation-dependent details.
The symbol (*) after a NAG routine name denotes a routine that is
not included in the Foundation Library.

1. Purpose

E04YCF returns estimates of elements of the variance-covariance
matrix of the estimated regression coefficients for a nonlinear
least squares problem. The estimates are derived from the
Jacobian of the function f(x) at the solution.

This routine may be used following any one of the nonlinear
least-squares routines E04FCF(*), E04FDF, E04GBF(*), E04GCF,
E04GDF(*), E04GEF(*), E04HEF(*), E04HFF(*).

2. Specification

       SUBROUTINE E04YCF (JOB, M, N, FSUMSQ, S, V, LV, CJ, WORK,
      1                   IFAIL)
```

```
      INTEGER          JOB, M, N, LV, IFAIL
      DOUBLE PRECISION FSUMSQ, S(N), V(LV,N), CJ(N), WORK(N)
```

3. Description

E04YCF is intended for use when the nonlinear least-squares
                T
function, F(x)=f (x)f(x), represents the goodness of fit of a nonlinear model to observed data. The routine assumes that the Hessian of F(x), at the solution, can be adequately approximated
     T
by 2J J, where J is the Jacobian of f(x) at the solution. The estimated variance-covariance matrix C is then given by

$$C=(\sigma)^2 (J^T J)^{-1} \quad J^T J \text{ non-singular},$$

where $(\sigma)^2$ is the estimated variance of the residual at the

solution, x, given by

$$(\sigma)^2 = \frac{F(x)}{m-n},$$

m being the number of observations and n the number of variables.

The diagonal elements of C are estimates of the variances of the estimated regression coefficients. See the Chapter Introduction E04 and Bard [1] and Wolberg [2] for further information on the use of C.

      T
When J J is singular then C is taken to be

$$C=(\sigma)^2 (J^T J)^{*},$$

        T *   T
where (J J) is the pseudo-inverse of J J, but in this case the parameter IFAIL is returned as non-zero as a warning to the user that J has linear dependencies in its columns. The assumed rank

of J can be obtained from IFAIL.

The routine can be used to find either the diagonal elements of C, or the elements of the jth column of C, or the whole of C.

E04YCF must be preceded by one of the nonlinear least-squares routines mentioned in Section 1, and requires the parameters FSUMSQ, S and V to be supplied by those routines. FSUMSQ is the

residual sum of squares F(x), and S and V contain the singular values and right singular vectors respectively in the singular value decomposition of J. S and V are returned directly by the comprehensive routines E04FCF(*), E04GBF(*), E04GDF(*) and E04HEF(*), but are returned as part of the workspace parameter W from the easy-to-use routines E04FDF, E04GCF, E04GEF(*) and E04HFF(*). In the case of E04FDF, S starts at W(NS), where

NS=6*N+2*M+M*N+1+max(1,N*(N-1)/2)

and in the cases of the remaining easy-to-use routines, S starts at W(NS), where

NS=7*N+2*M+M*N+N*(N+1)/2+1+max(1,N*(N-1)/2)

The parameter V starts immediately following the elements of S, so that V starts at W(NV), where

NV=NS+N.

For all the easy-to-use routines the parameter LV must be supplied as N. Thus a call to E04YCF following E04FDF can be illustrated as

```
      CALL E04FDF (M, N, X, FSUMSQ, IW, LIW, W, LW, IFAIL)
      NS = 6*N + 2*M + M*N + 1 + MAX((1,(N*(N-1))/2)
      NV = NS + N
      CALL E04YCF (JOB, M, N, FSUMSQ, W(NS), W(NV),
     *             N, CJ, WORK, IFAIL)
```

where the parameters M, N, FSUMSQ and the $(n+n^2)$ elements W(NS), WS(NS+1),..., W(NV+N*N-1) must not be altered between the calls to E04FDF and E04YCF. The above illustration also holds for a call to E04YCF following a call to one of E04GCF, E04GEF(*),

E04HFF(*) except that NS must be computed as

NS = 7*N + 2*M + M*N + (N*(N+1))/2 + 1 + MAX((1,N*(N-1))/2)

4. References

[1] Bard Y (1974) Nonlinear Parameter Estimation. Academic Press.

[2] Wolberg J R (1967) Prediction Analysis. Van Nostrand.

5. Parameters

1: JOB -- INTEGER Input
On entry: which elements of C are returned as follows:
JOB = -1
The n by n symmetric matrix C is returned.

JOB = 0
The diagonal elements of C are returned.

JOB > 0
The elements of column JOB of C are returned.
Constraint: -1 <= JOB <= N.

2: M -- INTEGER Input
On entry: the number m of observations (residuals $f_i(x)$).
Constraint: M >= N.

3: N -- INTEGER Input
On entry: the number n of variables (x_j). Constraint: 1 <= N <= M.

4: FSUMSQ -- DOUBLE PRECISION Input

On entry: the sum of squares of the residuals, F(x), at the

solution x, as returned by the nonlinear least-squares routine. Constraint: FSUMSQ >= 0.0.

5: S(N) -- DOUBLE PRECISION array Input
On entry: the n singular values of the Jacobian as returned

by the nonlinear least-squares routine. See Section 3 for information on supplying S following one of the easy-to-use routines.

6: V(LV,N) -- DOUBLE PRECISION array Input/Output
On entry: the n by n right-hand orthogonal matrix (the right singular vectors) of J as returned by the nonlinear least-squares routine. See Section 3 for information on supplying V following one of the easy-to-use routines. On exit: when JOB >= 0 then V is unchanged.

When JOB = -1 then the leading n by n part of V is overwritten by the n by n matrix C. When E04YCF is called with JOB = -1 following an easy-to-use routine this means that C is returned, column by column, in the n^2 elements of W given by W(NV),W(NV+1),...,W(NV+N^2 -1). (See Section 3 for the definition of NV).

7: LV -- INTEGER Input
On entry:
the first dimension of the array V as declared in the (sub)program from which E04YCF is called.
When V is passed in the workspace parameter W following one of the easy-to-use least-square routines, LV must be the value N.

8: CJ(N) -- DOUBLE PRECISION array Output
On exit: with JOB = 0, CJ returns the n diagonal elements of C.

With JOB = j>0, CJ returns the n elements of the jth column of C.

When JOB = -1, CJ is not referenced.

9: WORK(N) -- DOUBLE PRECISION array Workspace
When JOB = -1 or 0 then WORK is used as internal workspace.

When JOB > 0, WORK is not referenced.

10: IFAIL -- INTEGER Input/Output
On entry: IFAIL must be set to 0, -1 or 1. Users who are unfamiliar with this parameter should refer to the Essential Introduction for details.

```
     On exit: IFAIL = 0 unless the routine detects an error or
     gives a warning (see Section 6).

     For this routine, because the values of output parameters
     may be useful even if IFAIL /=0 on exit, users are
     recommended to set IFAIL to -1 before entry. It is then
     essential to test the value of IFAIL on exit. To suppress
     the output of an error message when soft failure occurs, set
     IFAIL to 1.

6. Error Indicators and Warnings

Errors or warnings specified by the routine:

IFAIL= 1
     On entry JOB < -1,

     or       JOB > N,

     or       N < 1,

     or       M < N,

     or       FSUMSQ < 0.0.

IFAIL= 2
     The singular values are all zero, so that at the solution
     the Jacobian matrix J has rank 0.

IFAIL> 2
     At the solution the Jacobian matrix contains linear, or near
     linear, dependencies amongst its columns. In this case the
     required elements of C have still been computed based upon J
     having an assumed rank given by (IFAIL-2). The rank is
     computed by regarding singular values SV(j) that are not
     larger than 10*(epsilon)*SV(1) as zero, where (epsilon) is
     the machine precision (see X02AJF(*)). Users who expect near
     linear dependencies at the solution and are happy with this
     tolerance in determining rank should call E04YCF with IFAIL
     = 1 in order to prevent termination by P01ABF(*). It is then
     essential to test the value of IFAIL on exit from E04YCF.

IFAILOverflow
     If overflow occurs then either an element of C is very
     large, or the singular values or singular vectors have been
```

```
     incorrectly supplied.

7. Accuracy

The computed elements of C will be the exact covariances
corresponding to a closely neighbouring Jacobian matrix J.

8. Further Comments

When JOB = -1 the time taken by the routine is approximately
                 3
proportional to n . When JOB >= 0 the time taken by the routine
                                  2
is approximately proportional to n .

9. Example

To estimate the variance-covariance matrix C for the least-
squares estimates of x , x  and x  in the model
                      1   2      3

                                 t
                                  1
                       y=x + ---------
                          1  x t +x t
                              2 2  3 3

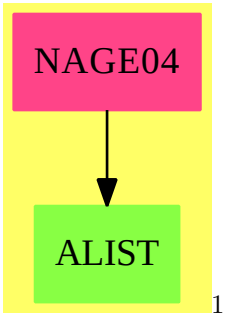
using the 15 sets of data given in the following table:

                        y    t   t    t
                             1   2    3
                      0.14  1.0 15.0 1.0
                      0.18  2.0 14.0 2.0
                      0.22  3.0 13.0 3.0
                      0.25  4.0 12.0 4.0
                      0.29  5.0 11.0 5.0
                      0.32  6.0 10.0 6.0
                      0.35  7.0  9.0 7.0
                      0.39  8.0  8.0 8.0
                      0.37  9.0  7.0 7.0
                      0.58 10.0  6.0 6.0
                      0.73 11.0  5.0 5.0
                      0.96 12.0  4.0 4.0
                      1.34 13.0  3.0 3.0
                      2.10 14.0  2.0 2.0
                      4.39 15.0  1.0 1.0
```

```
The program uses (0.5,1.0,1.5) as the initial guess at the
position of the minimum and computes the least-squares solution
using E04FDF. See the routine document E04FDF for further
information.

The example program is not reproduced here. The source code for
all example programs is distributed with the NAG Foundation
Library software and should be available on-line.
```

15.18 NagOptimisationPackage



1.00

Exports:
e04dgf e04fdf e04gcf e04jaf e04mbf
e04naf e04ucf e04ycf

⟨*package NAGE04 NagOptimisationPackage*⟩≡
```
)abbrev package NAGE04 NagOptimisationPackage
++ Author: Godfrey Nolan and Mike Dewar
++ Date Created: Jan 1994
++ Date Last Updated: Thu May 12 17:45:09 1994
++ Description:
++ This package uses the NAG Library to perform optimization.
++ An optimization problem involves minimizing a function (called
++ the objective function) of several variables, possibly subject to
++ restrictions on the values of the variables defined by a set of
++ constraint functions. The routines in the NAG Foundation Library
++ are concerned with function minimization only, since the problem
++ of maximizing a given function can be transformed into a
++ minimization problem simply by multiplying the function by -1.
++ See \downlink{Manual Page}{manpageXXe04}.
NagOptimisationPackage(): Exports == Implementation where
  S ==> Symbol
  FOP ==> FortranOutputStackPackage

  Exports ==> with
    e04dgf : (Integer,DoubleFloat,DoubleFloat,Integer,_
        DoubleFloat,Boolean,DoubleFloat,DoubleFloat,Integer,Integer,Integer,Integer,Matrix I
     ++ e04dgf(n,es,fu,it,lin,list,ma,op,pr,sta,sto,ve,x,ifail,objfun)
     ++ minimizes an unconstrained nonlinear function of several
     ++ variables using a pre-conditioned, limited memory quasi-Newton
     ++ conjugate gradient method. First derivatives are required. The
     ++ routine is intended for use on large scale problems.
     ++ See \downlink{Manual Page}{manpageXXe04dgf}.
    e04fdf : (Integer,Integer,Integer,Integer,_
        Matrix DoubleFloat,Integer,Union(fn:FileName,fp:Asp50(LSFUN1))) -> Result
```

```
 ++ e04fdf(m,n,liw,lw,x,ifail,lsfun1)
 ++ is an easy-to-use algorithm for finding an unconstrained
 ++ minimum of a sum of squares of m nonlinear functions in n
 ++ variables (m>=n). No derivatives are required.
 ++ See \downlink{Manual Page}{manpageXXe04fdf}.
e04gcf : (Integer,Integer,Integer,Integer,_
    Matrix DoubleFloat,Integer,Union(fn:FileName,fp:Asp19(LSFUN2))) -> Result
 ++ e04gcf(m,n,liw,lw,x,ifail,lsfun2)
 ++ is an easy-to-use quasi-Newton algorithm for finding an
 ++ unconstrained minimum of m nonlinear
 ++ functions in n variables (m>=n). First derivatives are required.
 ++ See \downlink{Manual Page}{manpageXXe04gcf}.
e04jaf : (Integer,Integer,Integer,Integer,_
    Matrix DoubleFloat,Matrix DoubleFloat,Matrix DoubleFloat,Integer,Union(fn
 ++ e04jaf(n,ibound,liw,lw,bl,bu,x,ifail,funct1)
 ++ is an easy-to-use quasi-Newton algorithm for finding a
 ++ minimum of a function F(x ,x ,...,x ), subject to fixed upper and
 ++                          1  2      n
 ++ lower bounds of the independent variables x ,x ,...,x , using
 ++                                            1  2      n
 ++ function values only.
 ++ See \downlink{Manual Page}{manpageXXe04jaf}.
e04mbf : (Integer,Integer,Integer,Integer,_
    Integer,Integer,Matrix DoubleFloat,Matrix DoubleFloat,Matrix DoubleFloat,
 ++ e04mbf(itmax,msglvl,n,nclin,nctotl,nrowa,a,bl,bu,cvec,linobj,liwork,lwork
 ++ is an easy-to-use routine for solving linear programming
 ++ problems, or for finding a feasible point for such problems. It
 ++ is not intended for large sparse problems.
 ++ See \downlink{Manual Page}{manpageXXe04mbf}.
e04naf : (Integer,Integer,Integer,Integer,_
    Integer,Integer,Integer,Integer,DoubleFloat,Matrix DoubleFloat,Matrix Dou
 ++ e04naf(itmax,msglvl,n,nclin,nctotl,nrowa,nrowh,ncolh,bigbnd,a,bl,bu,cvec,
 ++ is a comprehensive
 ++ programming (QP) or linear programming (LP) problems. It is not
 ++ intended for large sparse problems.
 ++ See \downlink{Manual Page}{manpageXXe04naf}.
e04ucf : (Integer,Integer,Integer,Integer,_
    Integer,Integer,Matrix DoubleFloat,Matrix DoubleFloat,Matrix DoubleFloat,
 ++ e04ucf(n,nclin,ncnln,nrowa,nrowj,nrowr,a,bl,bu,liwork,lwork,sta,cra,der,f
 ++ is designed to minimize an arbitrary smooth function
 ++ subject to constraints on the
 ++ variables, linear constraints.
 ++ (E04UCF  may be used for unconstrained, bound-constrained and
 ++ linearly constrained optimization.) The user must provide
 ++ subroutines that define the objective and constraint functions
 ++ and as many of their first partial derivatives as possible.
```

```
    ++ Unspecified derivatives are approximated by finite differences.
    ++ All matrices are treated as dense, and hence E04UCF is not
    ++ intended for large sparse problems.
    ++ See \downlink{Manual Page}{manpageXXe04ucf}.
  e04ycf : (Integer,Integer,Integer,DoubleFloat,_
      Matrix DoubleFloat,Integer,Matrix DoubleFloat,Integer) -> Result
    ++ e04ycf(job,m,n,fsumsq,s,lv,v,ifail)
    ++ returns estimates of elements of the variance
    ++ matrix of the estimated regression coefficients for a nonlinear
    ++ least squares problem. The estimates are derived from the
    ++ Jacobian of the function f(x) at the solution.
    ++ See \downlink{Manual Page}{manpageXXe04ycf}.
Implementation ==> add

  import Lisp
  import DoubleFloat
  import Any
  import Record
  import Integer
  import Matrix DoubleFloat
  import Boolean
  import NAGLinkSupportPackage
  import FortranPackage
  import Union(fn:FileName,fp:Asp49(OBJFUN))
  import AnyFunctions1(Integer)
  import AnyFunctions1(DoubleFloat)
  import AnyFunctions1(Boolean)
  import AnyFunctions1(Matrix DoubleFloat)
  import AnyFunctions1(Matrix Integer)


  e04dgf(nArg:Integer,esArg:DoubleFloat,fuArg:DoubleFloat,_
      itArg:Integer,linArg:DoubleFloat,listArg:Boolean,_
      maArg:DoubleFloat,opArg:DoubleFloat,prArg:Integer,_
      staArg:Integer,stoArg:Integer,veArg:Integer,_
      xArg:Matrix DoubleFloat,ifailArg:Integer,objfunArg:Union(fn:FileName,fp:Asp49(OBJFUN
      pushFortranOutputStack(objfunFilename := aspFilename "objfun")$FOP
      if objfunArg case fn
                then outputAsFortran(objfunArg.fn)
                else outputAsFortran(objfunArg.fp)
      popFortranOutputStack()$FOP
      [(invokeNagman([objfunFilename]$Lisp,_
      "e04dgf",_
      ["n"::S,"es"::S,"fu"::S,"it"::S,"lin"::S_
      ,"list"::S,"ma"::S,"op"::S,"pr"::S,"sta"::S_
      ,"sto"::S,"ve"::S,"iter"::S,"objf"::S,"ifail"::S_
```

```
    ,"objfun"::S,"objgrd"::S,"x"::S,"iwork"::S,"work"::S,"iuser"::S_
    ,"user"::S]$Lisp,_
    ["iter"::S,"objf"::S,"objgrd"::S,"iwork"::S,"work"::S,"iuser"::S,"user"::
    [["double"::S,"es"::S,"fu"::S,"lin"::S,"ma"::S_
    ,"op"::S,"objf"::S,["objgrd"::S,"n"::S]$Lisp,["x"::S,"n"::S]$Lisp,["work"
    ,"objfun"::S]$Lisp_
    ,["integer"::S,"n"::S,"it"::S,"pr"::S,"sta"::S_
    ,"sto"::S,"ve"::S,"iter"::S,"ifail"::S,["iwork"::S,["+"::S,"n"::S,1$Lisp]
    ,["logical"::S,"list"::S]$Lisp_
    ]$Lisp,_
    ["iter"::S,"objf"::S,"objgrd"::S,"x"::S,"ifail"::S]$Lisp,_
    [([nArg::Any,esArg::Any,fuArg::Any,itArg::Any,linArg::Any,listArg::Any,ma
    @List Any]$Lisp)$Lisp)_
    pretend List (Record(key:Symbol,entry:Any))]$Result

e04fdf(mArg:Integer,nArg:Integer,liwArg:Integer,_
    lwArg:Integer,xArg:Matrix DoubleFloat,ifailArg:Integer,_
    lsfun1Arg:Union(fn:FileName,fp:Asp50(LSFUN1))): Result ==
    pushFortranOutputStack(lsfun1Filename := aspFilename "lsfun1")$FOP
    if lsfun1Arg case fn
              then outputAsFortran(lsfun1Arg.fn)
              else outputAsFortran(lsfun1Arg.fp)
    popFortranOutputStack()$FOP
    [(invokeNagman([lsfun1Filename]$Lisp,_
    "e04fdf",_
    ["m"::S,"n"::S,"liw"::S,"lw"::S,"fsumsq"::S_
    ,"ifail"::S,"lsfun1"::S,"w"::S,"x"::S,"iw"::S]$Lisp,_
    ["fsumsq"::S,"w"::S,"iw"::S,"lsfun1"::S]$Lisp,_
    [["double"::S,"fsumsq"::S,["w"::S,"lw"::S]$Lisp_
    ,["x"::S,"n"::S]$Lisp,"lsfun1"::S]$Lisp_
    ,["integer"::S,"m"::S,"n"::S,"liw"::S,"lw"::S_
    ,"ifail"::S,["iw"::S,"liw"::S]$Lisp]$Lisp_
    ]$Lisp,_
    ["fsumsq"::S,"w"::S,"x"::S,"ifail"::S]$Lisp,_
    [([mArg::Any,nArg::Any,liwArg::Any,lwArg::Any,ifailArg::Any,xArg::Any ])_
    @List Any]$Lisp)$Lisp)_
    pretend List (Record(key:Symbol,entry:Any))]$Result

e04gcf(mArg:Integer,nArg:Integer,liwArg:Integer,_
    lwArg:Integer,xArg:Matrix DoubleFloat,ifailArg:Integer,_
    lsfun2Arg:Union(fn:FileName,fp:Asp19(LSFUN2))): Result ==
    pushFortranOutputStack(lsfun2Filename := aspFilename "lsfun2")$FOP
    if lsfun2Arg case fn
              then outputAsFortran(lsfun2Arg.fn)
              else outputAsFortran(lsfun2Arg.fp)
    popFortranOutputStack()$FOP
```

```
    [(invokeNagman([lsfun2Filename]$Lisp,_
    "e04gcf",_
    ["m"::S,"n"::S,"liw"::S,"lw"::S,"fsumsq"::S_
    ,"ifail"::S,"lsfun2"::S,"w"::S,"x"::S,"iw"::S]$Lisp,_
    ["fsumsq"::S,"w"::S,"iw"::S,"lsfun2"::S]$Lisp,_
    [["double"::S,"fsumsq"::S,["w"::S,"lw"::S]$Lisp_
    ,["x"::S,"n"::S]$Lisp,"lsfun2"::S]$Lisp_
    ,["integer"::S,"m"::S,"n"::S,"liw"::S,"lw"::S_
    ,"ifail"::S,["iw"::S,"liw"::S]$Lisp]$Lisp_
    ]$Lisp,_
    ["fsumsq"::S,"w"::S,"x"::S,"ifail"::S]$Lisp,_
    [([mArg::Any,nArg::Any,liwArg::Any,lwArg::Any,ifailArg::Any,xArg::Any ])_
    @List Any]$Lisp)$Lisp)_
    pretend List (Record(key:Symbol,entry:Any))]$Result

e04jaf(nArg:Integer,iboundArg:Integer,liwArg:Integer,_
    lwArg:Integer,blArg:Matrix DoubleFloat,buArg:Matrix DoubleFloat,_
    xArg:Matrix DoubleFloat,ifailArg:Integer,funct1Arg:Union(fn:FileName,fp:Asp24(FUNCT1
    pushFortranOutputStack(funct1Filename := aspFilename "funct1")$FOP
    if funct1Arg case fn
              then outputAsFortran(funct1Arg.fn)
              else outputAsFortran(funct1Arg.fp)
    popFortranOutputStack()$FOP
    [(invokeNagman([funct1Filename]$Lisp,_
    "e04jaf",_
    ["n"::S,"ibound"::S,"liw"::S,"lw"::S,"f"::S_
    ,"ifail"::S,"funct1"::S,"bl"::S,"bu"::S,"x"::S,"iw"::S,"w"::S_
    ]$Lisp,_
    ["f"::S,"iw"::S,"w"::S,"funct1"::S]$Lisp,_
    [["double"::S,"f"::S,["bl"::S,"n"::S]$Lisp_
    ,["bu"::S,"n"::S]$Lisp,["x"::S,"n"::S]$Lisp,["w"::S,"lw"::S]$Lisp,"funct1"::S]$Lisp_
    ,["integer"::S,"n"::S,"ibound"::S,"liw"::S_
    ,"lw"::S,"ifail"::S,["iw"::S,"liw"::S]$Lisp]$Lisp_
    ]$Lisp,_
    ["f"::S,"bl"::S,"bu"::S,"x"::S,"ifail"::S]$Lisp,_
    [([nArg::Any,iboundArg::Any,liwArg::Any,lwArg::Any,ifailArg::Any,blArg::Any,buArg::A
    @List Any]$Lisp)$Lisp)_
    pretend List (Record(key:Symbol,entry:Any))]$Result

e04mbf(itmaxArg:Integer,msglvlArg:Integer,nArg:Integer,_
    nclinArg:Integer,nctotlArg:Integer,nrowaArg:Integer,_
    aArg:Matrix DoubleFloat,blArg:Matrix DoubleFloat,buArg:Matrix DoubleFloat,_
    cvecArg:Matrix DoubleFloat,linobjArg:Boolean,liworkArg:Integer,_
    lworkArg:Integer,xArg:Matrix DoubleFloat,ifailArg:Integer): Result ==
    [(invokeNagman(NIL$Lisp,_
    "e04mbf",_
```

```
        ["itmax"::S,"msglvl"::S,"n"::S,"nclin"::S,"nctotl"::S_
        ,"nrowa"::S,"linobj"::S,"liwork"::S,"lwork"::S,"objlp"::S_
        ,"ifail"::S,"a"::S,"bl"::S,"bu"::S,"cvec"::S,"istate"::S_
        ,"clamda"::S,"x"::S,"iwork"::S,"work"::S]$Lisp,_
        ["istate"::S,"objlp"::S,"clamda"::S,"iwork"::S,"work"::S]$Lisp,_
        [["double"::S,["a"::S,"nrowa"::S,"n"::S]$Lisp_
        ,["bl"::S,"nctotl"::S]$Lisp,["bu"::S,"nctotl"::S]$Lisp,["cvec"::S,"n"::S]
        ,["x"::S,"n"::S]$Lisp,["work"::S,"lwork"::S]$Lisp]$Lisp_
        ,["integer"::S,"itmax"::S,"msglvl"::S,"n"::S_
        ,"nclin"::S,"nctotl"::S,"nrowa"::S,"liwork"::S,"lwork"::S,["istate"::S,"n
        ]$Lisp_
        ,["logical"::S,"linobj"::S]$Lisp_
        ]$Lisp,_
        ["istate"::S,"objlp"::S,"clamda"::S,"x"::S,"ifail"::S]$Lisp,_
        [([itmaxArg::Any,msglvlArg::Any,nArg::Any,nclinArg::Any,nctotlArg::Any,nr
        @List Any]$Lisp)$Lisp)_
        pretend List (Record(key:Symbol,entry:Any))]$Result

    e04naf(itmaxArg:Integer,msglvlArg:Integer,nArg:Integer,_
        nclinArg:Integer,nctotlArg:Integer,nrowaArg:Integer,_
        nrowhArg:Integer,ncolhArg:Integer,bigbndArg:DoubleFloat,_
        aArg:Matrix DoubleFloat,blArg:Matrix DoubleFloat,buArg:Matrix DoubleFloat
        cvecArg:Matrix DoubleFloat,featolArg:Matrix DoubleFloat,hessArg:Matrix Do
        coldArg:Boolean,lppArg:Boolean,orthogArg:Boolean,_
        liworkArg:Integer,lworkArg:Integer,xArg:Matrix DoubleFloat,_
        istateArg:Matrix Integer,ifailArg:Integer,qphessArg:Union(fn:FileName,fp:
        pushFortranOutputStack(qphessFilename := aspFilename "qphess")$FOP
        if qphessArg case fn
                  then outputAsFortran(qphessArg.fn)
                  else outputAsFortran(qphessArg.fp)
        popFortranOutputStack()$FOP
        [(invokeNagman([qphessFilename]$Lisp,_
        "e04naf",_
        ["itmax"::S,"msglvl"::S,"n"::S,"nclin"::S,"nctotl"::S_
        ,"nrowa"::S,"nrowh"::S,"ncolh"::S,"bigbnd"::S,"cold"::S_
        ,"lpp"::S,"orthog"::S,"liwork"::S,"lwork"::S,"iter"::S_
        ,"obj"::S,"ifail"::S,"qphess"::S,"a"::S,"bl"::S,"bu"::S,"cvec"::S,"featol
        ,"hess"::S,"clamda"::S,"x"::S,"istate"::S,"iwork"::S_
        ,"work"::S]$Lisp,_
        ["iter"::S,"obj"::S,"clamda"::S,"iwork"::S,"work"::S,"qphess"::S]$Lisp,_
        [["double"::S,"bigbnd"::S,["a"::S,"nrowa"::S,"n"::S]$Lisp_
        ,["bl"::S,"nctotl"::S]$Lisp,["bu"::S,"nctotl"::S]$Lisp,["cvec"::S,"n"::S]
        ,["hess"::S,"nrowh"::S,"ncolh"::S]$Lisp,"obj"::S,["clamda"::S,"nctotl"::S
        ,"qphess"::S]$Lisp_
        ,["integer"::S,"itmax"::S,"msglvl"::S,"n"::S_
        ,"nclin"::S,"nctotl"::S,"nrowa"::S,"nrowh"::S,"ncolh"::S,"liwork"::S,"lwo
```

```
        ,"ifail"::S,["iwork"::S,"liwork"::S]$Lisp]$Lisp_
        ,["logical"::S,"cold"::S,"lpp"::S,"orthog"::S]$Lisp_
        ]$Lisp,_
        ["iter"::S,"obj"::S,"clamda"::S,"x"::S,"istate"::S,"ifail"::S]$Lisp,_
        [([itmaxArg::Any,msglvlArg::Any,nArg::Any,nclinArg::Any,nctotlArg::Any,nrowaArg::Any
        @List Any]$Lisp)$Lisp)_
        pretend List (Record(key:Symbol,entry:Any))]$Result

    e04ucf(nArg:Integer,nclinArg:Integer,ncnlnArg:Integer,_
        nrowaArg:Integer,nrowjArg:Integer,nrowrArg:Integer,_
        aArg:Matrix DoubleFloat,blArg:Matrix DoubleFloat,buArg:Matrix DoubleFloat,_
        liworkArg:Integer,lworkArg:Integer,staArg:Boolean,_
        craArg:DoubleFloat,derArg:Integer,feaArg:DoubleFloat,_
        funArg:DoubleFloat,hesArg:Boolean,infbArg:DoubleFloat,_
        infsArg:DoubleFloat,linfArg:DoubleFloat,lintArg:DoubleFloat,_
        listArg:Boolean,majiArg:Integer,majpArg:Integer,_
        miniArg:Integer,minpArg:Integer,monArg:Integer,_
        nonfArg:DoubleFloat,optArg:DoubleFloat,steArg:DoubleFloat,_
        staoArg:Integer,stacArg:Integer,stooArg:Integer,_
        stocArg:Integer,veArg:Integer,istateArg:Matrix Integer,_
        cjacArg:Matrix DoubleFloat,clamdaArg:Matrix DoubleFloat,rArg:Matrix DoubleFloat,_
        xArg:Matrix DoubleFloat,ifailArg:Integer,confunArg:Union(fn:FileName,fp:Asp55(CONFUN
        objfunArg:Union(fn:FileName,fp:Asp49(OBJFUN))): Result ==
        pushFortranOutputStack(confunFilename := aspFilename "confun")$FOP
        if confunArg case fn
                  then outputAsFortran(confunArg.fn)
                  else outputAsFortran(confunArg.fp)
        popFortranOutputStack()$FOP
        pushFortranOutputStack(objfunFilename := aspFilename "objfun")$FOP
        if objfunArg case fn
                  then outputAsFortran(objfunArg.fn)
                  else outputAsFortran(objfunArg.fp)
        popFortranOutputStack()$FOP
        [(invokeNagman([confunFilename,objfunFilename]$Lisp,_
        "e04ucf",_
        ["n"::S,"nclin"::S,"ncnln"::S,"nrowa"::S,"nrowj"::S_
        ,"nrowr"::S,"liwork"::S,"lwork"::S,"sta"::S,"cra"::S_
        ,"der"::S,"fea"::S,"fun"::S,"hes"::S,"infb"::S_
        ,"infs"::S,"linf"::S,"lint"::S,"list"::S,"maji"::S_
        ,"majp"::S,"mini"::S,"minp"::S,"mon"::S,"nonf"::S_
        ,"opt"::S,"ste"::S,"stao"::S,"stac"::S,"stoo"::S_
        ,"stoc"::S,"ve"::S,"iter"::S,"objf"::S,"ifail"::S_
        ,"confun"::S,"objfun"::S,"a"::S,"bl"::S,"bu"::S,"c"::S,"objgrd"::S_
        ,"istate"::S,"cjac"::S,"clamda"::S,"r"::S,"x"::S_
        ,"iwork"::S,"work"::S,"iuser"::S,"user"::S]$Lisp,_
        ["iter"::S,"c"::S,"objf"::S,"objgrd"::S,"iwork"::S,"work"::S,"iuser"::S,"user"::S,"c
```

```
        [["double"::S,["a"::S,"nrowa"::S,"n"::S]$Lisp_
        ,["bl"::S,["+"::S,["+"::S,"nclin"::S,"ncnln"::S]$Lisp,"n"::S]$Lisp]$Lisp,
        ,"cra"::S,"fea"::S,"fun"::S,"infb"::S,"infs"::S,"linf"::S,"lint"::S,"nonf
        ,["c"::S,"ncnln"::S]$Lisp,"objf"::S,["objgrd"::S,"n"::S]$Lisp,["cjac"::S,
        ,["r"::S,"nrowr"::S,"n"::S]$Lisp,["x"::S,"n"::S]$Lisp,["work"::S,"lwork":
        ,["user"::S,1$Lisp]$Lisp,"confun"::S,"objfun"::S]$Lisp_
        ,["integer"::S,"n"::S,"nclin"::S,"ncnln"::S_
        ,"nrowa"::S,"nrowj"::S,"nrowr"::S,"liwork"::S,"lwork"::S,"der"::S,"maji":
        ,"stac"::S,"stoo"::S,"stoc"::S,"ve"::S,"iter"::S,["istate"::S,["+"::S,["+
        ,"ifail"::S,["iwork"::S,"liwork"::S]$Lisp,["iuser"::S,1$Lisp]$Lisp]$Lisp_
        ,["logical"::S,"sta"::S,"hes"::S,"list"::S]$Lisp_
        ]$Lisp,_
        ["iter"::S,"c"::S,"objf"::S,"objgrd"::S,"istate"::S,"cjac"::S,"clamda"::S
        [([nArg::Any,nclinArg::Any,ncnlnArg::Any,nrowaArg::Any,nrowjArg::Any,nrow
        @List Any]$Lisp)$Lisp)_
        pretend List (Record(key:Symbol,entry:Any))]$Result

    e04ycf(jobArg:Integer,mArg:Integer,nArg:Integer,_
        fsumsqArg:DoubleFloat,sArg:Matrix DoubleFloat,lvArg:Integer,_
        vArg:Matrix DoubleFloat,ifailArg:Integer): Result ==
        [(invokeNagman(NIL$Lisp,_
        "e04ycf",_
        ["job"::S,"m"::S,"n"::S,"fsumsq"::S,"lv"::S_
        ,"ifail"::S,"s"::S,"cj"::S,"v"::S,"work"::S]$Lisp,_
        ["cj"::S,"work"::S]$Lisp,_
        [["double"::S,"fsumsq"::S,["s"::S,"n"::S]$Lisp_
        ,["cj"::S,"n"::S]$Lisp,["v"::S,"lv"::S,"n"::S]$Lisp,["work"::S,"n"::S]$Li
        ,["integer"::S,"job"::S,"m"::S,"n"::S,"lv"::S_
        ,"ifail"::S]$Lisp_
        ]$Lisp,_
        ["cj"::S,"v"::S,"ifail"::S]$Lisp,_
        [([jobArg::Any,mArg::Any,nArg::Any,fsumsqArg::Any,lvArg::Any,ifailArg::An
        @List Any]$Lisp)$Lisp)_
        pretend List (Record(key:Symbol,entry:Any))]$Result
```

⟨*NAGE04.dotabb*⟩≡

```
"NAGE04" [color="#FF4488",href="bookvol10.4.pdf#nameddest=NAGE04"]
"ALIST" [color="#88FF44",href="bookvol10.3.pdf#nameddest=ALIST"]
"NAGE04" -> "ALIST"
```

15.19 package NAGD02 NagOrdinaryDifferentialEquationsPackage

⟨*NagOrdinaryDifferentialEquationsPackage.help*⟩≡

```
D02(3NAG)          Foundation Library (12/10/92)          D02(3NAG)

     D02 -- Ordinary Differential Equations        Introduction -- D02
                              Chapter D02
                    Ordinary Differential Equations

     1. Scope of the Chapter

     This chapter is concerned with the numerical solution of ordinary
     differential equations. There are two main types of problem,
     those in which all boundary conditions are specified at one point
     (initial-value problems), and those in which the boundary
     conditions are distributed between two or more points (boundary-
     value problems and eigenvalue problems). Routines are available
     for initial-value problems, two-point boundary-value problems and
     Sturm-Liouville eigenvalue problems.

     2. Background to the Problems

     For most of the routines in this chapter a system of ordinary
     differential equations must be written in the form

                          y'  = f (x,y ,y ,...,y ),
                            1    1    1  2      n

                          y'  = f (x,y ,y ,...,y ),
                            2    2    1  2      n

                            .. ...

                          y'  = f (x,y ,y ,...y ),
                            n    n    1  2     n

     that is the system must be given in first-order form. The n
     dependent variables (also, the solution) y ,y ,...,y  are
                                              1  2      n
     functions of the independent variable x, and the differential
     equations give expressions for the first derivatives y' =dy /dx
                                                           i   i
     in terms of x and y ,y ,...,y . For a system of n first-order
```

```
                    1  2      n
```
equations, n associated boundary conditions are usually required to define the solution.

A more general system may contain derivatives of higher order, but such systems can almost always be reduced to the first-order form by introducing new variables. For example, suppose we have the third-order equation

```
                                  2
                    z'''+zz''+k(l-z' )=0.
```

We write y =z, y =z', y =z'', and the third order equation may
```
          1     2      3
```
then be written as the system of first-order equations

```
                    y'  = y
                      1    2

                    y'  = y
                      2    3
                                  2
                    y'  = -y y -k(l-y ).
                      3     1 3      2
```

For this system n = 3 and we require 3 boundary conditions in order to define the solution. These conditions must specify values of the dependent variables at certain points. For example, we have an initial-value problem if the conditions are:

```
    y   =  0        at x=0
     1

    y   =  0        at x=0
     2

    y   =  0.1      at x=0.
     3
```

These conditions would enable us to integrate the equations numerically from the point x=0 to some specified end-point. We have a boundary-value problem if the conditions are:

```
    y   =  0        at x=0
     1
```

```
y   =  0        at x=0
 2

y   =  1        at x=10.
 2
```

These conditions would be sufficient to define a solution in the range 0<=x<=10, but the problem could not be solved by direct integration (see Section 2.2). More general boundary conditions are permitted in the boundary-value case.

2.1. Initial-value Problems

To solve first-order systems, initial values of the dependent variables y_i, for i=1,2,...,n must be supplied at a given point, a. Also a point, b, at which the values of the dependent variables are required, must be specified. The numerical solution is then obtained by a step-by-step calculation which approximates values of the variables y_i, for i=1,2,...,n at finite intervals over the required range [a,b]. The routines in this chapter adjust the step length automatically to meet specified accuracy tolerances. Although the accuracy tests used are reliable over each step individually, in general an accuracy requirement cannot be guaranteed over a long range. For many problems there may be no serious accumulation of error, but for unstable systems small perturbations of the solution will often lead to rapid divergence of the calculated values from the true values. A simple check for stability is to carry out trial calculations with different tolerances; if the results differ appreciably the system is probably unstable. Over a short range, the difficulty may possibly be overcome by taking sufficiently small tolerances, but over a long range it may be better to try to reformulate the problem.

A special class of initial-value problems are those for which the solutions contain rapidly decaying transient terms. Such problems are called stiff; an alternative way of describing them is to say that certain eigenvalues of the Jacobian matrix (ddf_i/ddy_j) have large negative real parts when compared to others. These problems require special methods for efficient numerical solution; the methods designed for non-stiff problems when applied to stiff problems tend to be very slow, because they need small step

lengths to avoid numerical instability. A full discussion is given in Hall and Watt [6] and a discussion of the methods for stiff problems is given in Berzins, Brankin and Gladwell [1].

2.2. Boundary-value Problems

A full discussion of the design of the methods and codes for boundary-value problems is given in Gladwell [4]. In general, a system of nonlinear differential equations with boundary conditions given at two or more points cannot be guaranteed to have a solution. The solution has to be determined iteratively (if it exists). Finite-difference equations are set up on a mesh of points and estimated values for the solution at the grid points are chosen. Using these estimated values as starting values a Newton iteration is used to solve the finite-difference equations. The accuracy of the solution is then improved by deferred corrections or the addition of points to the mesh or a combination of both. Good initial estimates of the solution may be required in some cases but results may be difficult to compute when the solution varies very rapidly over short ranges. A discussion is given in Chapters 9 and 11 of Gladwell and Sayers [5] and Chapter 4 of Childs et al [2].

2.3. Eigenvalue Problems

Sturm-Liouville problems of the form

(p(x)y')'+q(x,(lambda))y=0

with appropriate boundary conditions given at two points, can be solved by a Scaled Pruefer method. In this method the differential equation is transformed to another which can be solved for a specified eigenvalue by a shooting method. A discussion is given in Chapter 11 of Gladwell and Sayers [5] and a complete description is given in Pryce [7].

2.6. References

[1] Berzins M, Brankin R W and Gladwell I (1987) Design of the Stiff Integrators in the NAG Library. Technical Report. TR14/87 NAG.

[2] (1979) Codes for Boundary-value Problems in Ordinary Differential Equations. Lecture Notes in Computer Science. (ed Childs B, Scott M, Daniel J W, Denman E and Nelson P) 76 Springer-Verlag.

[3] Gladwell I (1979) Initial Value Routines in the NAG Library. ACM Trans Math Softw. 5 386--400.

[4] Gladwell I (1987) The NAG Library Boundary Value Codes. Numerical Analysis Report. 134 Manchester University.

[5] (1980) Computational Techniques for Ordinary Differential Equations. (Gladwell I and Sayers D K) Academic Press.

[6] Hall G and Watt J M (eds) (1976) Modern Numerical Methods for Ordinary Differential Equations. Clarendon Press.

[7] Pryce J D (1986) Error Estimation for Phase-function Shooting Methods for Sturm-Liouville Problems. J. Num. Anal. 6 103--123.

3. Recommendations on Choice and Use of Routines

There are no routines which deal directly with COMPLEX equations. These may however be transformed to larger systems of real equations of the required form. Split each equation into its real and imaginary parts and solve for the real and imaginary parts of each component of the solution. Whilst this process doubles the size of the system and may not always be appropriate it does make available for use the full range of routines provided presently.

3.1. Initial-value Problems

For simple first-order problems with low accuracy requirements, that is problems on a short range of integration, with derivative functions f_i which are inexpensive to calculate and where only a few correct figures are required, the best routines to use are likely to be the Runge-Kutta-Merson (RK) routines, D02BBF and D02BHF. For larger problems, over long ranges or with high accuracy requirements the variable-order, variable-step Adams routine D02CJF should usually be preferred. For stiff equations, that is those with rapidly decaying transient solutions, the Backward Differentiation Formula (BDF) variable-order, variable-step routine D02EJF should be used.

There are four routines for initial-value problems, two of which use the Runge-Kutta-Merson method:

D02BBF integrates a system of first order ordinary

differential equations over a range with intermediate output and a choice of error control

D02BHF integrates a system of first order ordinary differential equations with a choice of error control until a position is determined where a function of the solution is zero.

one uses an Adams method:

D02CJF combines the functionality of D02BBF and D02BHF

and one uses a BDF method:

D02EJF combines the functionality of D02BBF and D02BHF.

3.2. Boundary-value Problems

D02GAF may be used for simple boundary-value problems with assigned boundary values. The user may find that convergence is difficult to achieve using D02GAF since only specifying the unknown boundary values and the position of the finite-difference mesh is permitted. In such cases the user may use D02RAF which permits specification of an initial estimate for the solution at all mesh points and allows the calculation to be influenced in other ways too. D02RAF is designed to solve a general nonlinear two-point boundary value problem with nonlinear boundary conditions.

A routine, D02GBF, is also supplied specifically for the general linear two-point boundary-value problem written in a standard

The user is advised to use interpolation routines from the E01 Chapter to obtain solution values at points not on the final mesh.

3.3. Eigenvalue Problems

There is one general purpose routine for eigenvalue problems, D02KEF. It may be used to solve regular or singular second-order Sturm-Liouville problems on a finite or infinite range. Discontinous coefficient functions can be treated and eigenfunctions can be computed.

```
Chapter D02

Ordinary Differential Equations

D02BBF  ODEs, IVP, Runge-Kutta-Merson method, over a range,
        intermediate output

D02BHF  ODEs, IVP, Runge-Kutta-Merson method, until function of
        solution is zero

D02CJF  ODEs, IVP, Adams method, until function of solution is
        zero, intermediate output

D02EJF  ODEs, stiff IVP, BDF method, until function of solution
        is zero, intermediate output

D02GAF  ODEs, boundary value problem, finite difference technique
        with deferred correction, simple nonlinear problem

D02GBF  ODEs, boundary value problem, finite difference technique
        with deferred correction, general linear problem

D02KEF  2nd order Sturm-Liouville problem, regular/singular
        system, finite/infinite range, eigenvalue and
        eigenfunction, user-specified break-points

D02RAF  ODEs, general nonlinear boundary value problem, finite
        difference technique with deferred correction,
        continuation facility
```

%%%

```
D02BBF(3NAG)                D02BBF                D02BBF(3NAG)

D02 -- Ordinary Differential Equations                    D02BBF
        D02BBF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for
your implementation to check implementation-dependent details.
The symbol (*) after a NAG routine name denotes a routine that is
not included in the Foundation Library.

Note for users via the AXIOM system: the interface to this routine
has been enhanced for use with AXIOM and is slightly different to
that offered in the standard version of the Foundation Library.
```

1. Purpose

D02BBF integrates a system of first-order ordinary differential equations over an interval with suitable initial conditions, using a Runge-Kutta-Merson method, and returns the solution at points specified by the user.

2. Specification

```
      SUBROUTINE D02BBF (X, XEND, M, N, Y, TOL, IRELAB, RESULT,
     1                   FCN, OUTPUT, W, IFAIL)
      INTEGER           M, N, IRELAB, IFAIL
      DOUBLE PRECISION X, XEND, Y(N), TOL, W(N,7), RESULT(M,N)
      EXTERNAL          FCN, OUTPUT
```

3. Description

The routine integrates a system of ordinary differential equations

$$y_i' = f_i(x, y_1, y_2, \ldots, y_n) \quad i = 1, 2, \ldots, n$$

from x = X to x = XEND using a Merson form of the Runge-Kutta method. The system is defined by a subroutine FCN supplied by the user, which evaluates f_i in terms of x and $y_1, y_2, \ldots, y_n$, and the values of $y_1, y_2, \ldots, y_n$ must be given at x = X.

The solution is returned via the user-supplied routine OUTPUT at a set of points specified by the user. This solution is obtained by quintic Hermite interpolation on solution values produced by the Runge-Kutta method.

The accuracy of the integration and, indirectly, the interpolation is controlled by the parameters TOL and IRELAB.

For a description of Runge-Kutta methods and their practical implementation see Hall and Watt [1].

4. References

[1] Hall G and Watt J M (eds) (1976) Modern Numerical Methods for Ordinary Differential Equations. Clarendon Press.

5. Parameters

1: X -- DOUBLE PRECISION Input/Output
On entry: X must be set to the initial value of the independent variable x. On exit: XEND, unless an error has occurred, when it contains the value of x at the error.

2: XEND -- DOUBLE PRECISION Input
On entry: the final value of the independent variable. If XEND < X on entry, integration will proceed in a negative direction.

3: M -- INTEGER Input
On entry: the first dimension of the array RESULT. This will usually be equal to the number of points at which the solution is required.
Constraint: M > 0.

4: N -- INTEGER Input
On entry: the number of differential equations.
Constraint: N > 0.

5: Y(N) -- DOUBLE PRECISION array Input/Output
On entry: the initial values of the solution $y_1, y_2, \ldots, y_n$.
On exit: the computed values of the solution at the final value of X.

6: TOL -- DOUBLE PRECISION Input/Output
On entry: TOL must be set to a positive tolerance for controlling the error in the integration.

D02BBF has been designed so that, for most problems, a reduction in TOL leads to an approximately proportional reduction in the error in the solution at XEND. The relation between changes in TOL and the error at intermediate output points is less clear, but for TOL small enough the error at intermediate output points should also be approximately proportional to TOL. However, the actual relation between TOL and the accuracy achieved cannot be guaranteed. The user is strongly recommended to call D02BBF with more than one value for TOL and to compare the results obtained to estimate their accuracy. In the absence of any prior knowledge, the user might compare the results obtained by calling D02BBF with TOL=10.0^{-p} and TOL=10.0^{-p-1} if p correct

decimal digits in the solution are required. Constraint: TOL > 0.0. On exit: normally unchanged. However if the range X to XEND is so short that a small change in TOL is unlikely to make any change in the computed solution then, on return, TOL has its sign changed. This should be treated as a warning that the computed solution is likely to be more accurate than would be produced by using the same value of TOL on a longer range of integration.

7: IRELAB -- INTEGER Input
On entry: IRELAB determines the type of error control. At each step in the numerical solution an estimate of the local error, EST, is made. For the current step to be accepted the following condition must be satisfied:
IRELAB = 0

$$EST = 10.0 \leq TOL * \max\{1.0, |y_1|, |y_2|, \ldots, |y_n|\};$$

IRELAB = 1

$$EST \leq TOL;$$

IRELAB = 2

$$EST \leq TOL * \max\{(epsilon), |y_1|, |y_2|, \ldots, |y_n|\}, \text{ where}$$

(epsilon) is machine precision.
If the appropriate condition is not satisfied, the step size is reduced and the solution is recomputed on the current step.

If the user wishes to measure the error in the computed solution in terms of the number of correct decimal places, then IRELAB should be given the value 1 on entry, whereas if the error requirement is in terms of the number of correct significant digits, then IRELAB should be given the value 2. Where there is no preference in the choice of error test IRELAB = 0 will result in a mixed error test. Constraint: 0 <= IRELAB <= 2.

8: RESULT(M,N) -- DOUBLE PRECISION array Output
On exit: the computed values of the solution at the points given by OUTPUT.

9: FCN -- SUBROUTINE, supplied by the user.
External Procedure
FCN must evaluate the functions f_i (i.e., the derivatives

```
y' ) for given values of its arguments x,y ,...,y .
  i                                        1      n

Its specification is:

       SUBROUTINE FCN (X, Y, F)
       DOUBLE PRECISION X, Y(n), F(n)

where n is the actual value of N in the call of D02BBF.

 1:  X -- DOUBLE PRECISION                                Input
     On entry: the value of the argument x.

 2:  Y(*) -- DOUBLE PRECISION array                       Input
     On entry: the value of the argument y , for
                                          i
     i=1,2,...,n.

 3:  F(*) -- DOUBLE PRECISION array                      Output
     On exit: the value of f , for i=1,2,...,n.
                           i
FCN must be declared as EXTERNAL in the (sub)program
from which D02BBF is called. Parameters denoted as
Input must not be changed by this procedure.

10:  OUTPUT -- SUBROUTINE, supplied by the user.
                                          External Procedure
     OUTPUT allows the user to have access to intermediate values
     of the computed solution at successive points specified by the
     user. These solution values may be returned to the user via
     the array RESULT if desired (this is a non-standard feature
     added for use with the AXIOM system).  OUTPUT is initially
     called by D02BBF with XSOL = X (the initial value of x). The
     user must reset XSOL to the next point where OUTPUT is to be
     called, and so on at each call to OUTPUT. If, after a call
     to OUTPUT, the reset point XSOL is beyond XEND, D02BBF will
     integrate to XEND with no further calls to OUTPUT; if a call
     to OUTPUT is required at the point XSOL = XEND, then XSOL
     must be given precisely the value XEND.

     Its specification is:

          SUBROUTINE OUTPUT(XSOL,Y,COUNT,M,N,RESULT)
          DOUBLE PRECISION Y(N),RESULT(M,N),XSOL
          INTEGER M,N,COUNT
```

```
     1:  XSOL -- DOUBLE PRECISION                    Input/Output
         On entry: the current value of the independent
         variable x. On exit: the next value of x at which
         OUTPUT is to be called.

     2:  Y(N) -- DOUBLE PRECISION array                     Input
         On entry: the computed solution at the point XSOL.

     3:  COUNT -- INTEGER                            Input/Output
         On entry: Zero if OUTPUT has not been called before, or
         the previous value of COUNT.
         On exit: A new value of COUNT: this can be used to keep
         track of the number of times OUTPUT has been called.

     4:  M -- INTEGER                                       Input
         On entry: The first dimension of RESULT.

     5:  N -- INTEGER                                       Input
         On entry: The dimension of Y.

     6:  RESULT(M,N) -- DOUBLE PRECISION array      Input/Output
         On entry: the previous contents of RESULT.
         On exit: RESULT may be used to return the values of the
         intermediate solutions to the user.

    OUTPUT must be declared as EXTERNAL in the (sub)program
    from which D02BBF is called. Parameters denoted as
    Input must not be changed by this procedure.

11:  W(N,7) -- DOUBLE PRECISION array                  Workspace

12:  IFAIL -- INTEGER                                  Input/Output
    On entry: IFAIL must be set to 0, -1 or 1. For users not
    familiar with this parameter (described in the Essential
    Introduction) the recommended value is 0.

    On exit: IFAIL = 0 unless the routine detects an error (see
    Section 6).

6. Error Indicators and Warnings

Errors detected by the routine:

IFAIL= 1
    On entry TOL <= 0,
```

```
     or       N <= 0,

     or       IRELAB /= 0, 1 or 2.

IFAIL= 2
     With the given value of TOL, no further progress can be made
     across the integration range from the current point x = X,
     or the dependence of the error on TOL would be lost if
     further progress across the integration range were attempted
     (see Section 8 for a discussion of this error exit). The
     components Y(1),Y(2),...,Y(n) contain the computed values of
     the solution at the current point x = X.

IFAIL= 3
     TOL is too small for the routine to take an initial step
     (see Section 8). X and Y(1),Y(2),...,Y(n) retain their
     initial values.

IFAIL= 4
     X = XEND and XSOL /= X after the initial call to OUTPUT.

IFAIL= 5
     A value of XSOL returned by OUTPUT lies behind the previous
     value of XSOL in the direction of integration.

IFAIL= 6
     A serious error has occurred in an internal call to
     D02PAF(*). Check all subroutine calls and array dimensions.
     Seek expert help.

IFAIL= 7
     A serious error has occurred in an internal call to
     D02XAF(*). Check all subroutine calls and array dimensions.
     Seek expert help.

7. Accuracy

The accuracy depends on TOL, on the mathematical properties of
the differential system, on the length of the range of
integration and on the method. It can be controlled by varying
TOL but the approximate proportionality of the error to TOL holds
only for a restricted range of values of TOL. For TOL too large,
the underlying theory may break down and the result of varying
TOL may be unpredictable. For TOL too small, rounding errors may
affect the solution significantly and an error exit with IFAIL =
2 or IFAIL = 3 is possible.
```

At the intermediate output points the same remarks apply. For large values of TOL it is possible that the errors at some intermediate output points may be much larger than at XEND. In any case, it must not be expected that the error will have the same size at all output points. At any point, it is a combination of the errors arising from the integration of the differential equation and the interpolation. The effect of combining these errors will vary, though in most cases the integration error will dominate.

The user who requires a more reliable estimate of the accuracy achieved than can be obtained by varying TOL, is recommended to call D02BDF(*) where both the solution and a global error estimate are computed.

8. Further Comments

The time taken by the routine depends on the complexity and mathematical properties of the system of differential equations defined by FCN, on the range, the tolerance and the number of calls to OUTPUT. There is also an overhead of the form a+b*n where a and b are machine-dependent computing times.

If the routine fails with IFAIL = 3, then it can be called again with a larger value of TOL (if this has not already been tried). If the accuracy requested is really needed and cannot be obtained with this routine, the system may be very stiff (see below) or so badly scaled that it cannot be solved to the required accuracy.

If the routine fails with IFAIL = 2, it is probable that it has been called with a value of TOL which is so small that the solution cannot be obtained on the range X to XEND. This can happen for well-behaved systems and very small values of TOL. The user should, however, consider whether there is a more fundamental difficulty. For example:

(a) in the region of a singularity (infinite value) of the solution, the routine will usually stop with IFAIL = 2, unless overflow occurs first. If overflow occurs using D02BBF, D02PAF(*) can be used instead to trap the increasing solution before overflow occurs. In any case, numerical integration cannot be continued through a singularity, and analytic treatment should be considered;

(b) for 'stiff' equations, where the solution contains rapidly

decaying components, the routine will use very small steps in x (internally to D02BBF) to preserve stability. This will usually exhibit itself by making the computing time excessively long, or occasionally by an exit with IFAIL = 2. Merson's method is not efficient in such cases, and the user should try using D02EBF(*) (Backward Differentiation Formula). To determine whether a problem is stiff, D02BDF(*) may be used.

For well-behaved systems with no difficulties such as stiffness or singularities, the Merson method should work well for low accuracy calculations (three or four figures). For higher accuracy calculations or where FCN is costly to evaluate, Merson's method may not be appropriate and a computationally less expensive method may be D02CBF(*) which uses an Adams method.

Users with problems for which D02BBF is not sufficiently general should consider using D02PAF(*) with D02XAF(*). D02PAF(*) is a more general Merson routine with many facilities including more general error control options and several criteria for interrupting the calculations. D02XAF(*) interpolates on values produced by D02PAF(*).

9. Example

To integrate the following equations (for a projectile)

$$y' = \tan(\phi)$$

$$v' = \frac{-0.032\tan(\phi)}{v} - \frac{0.02v}{\cos(\phi)}$$

$$(\phi)' = \frac{-0.032}{v^2}$$

over an interval X = 0.0 to XEND = 8.0, starting with values y=0.0, v=0.5 and (phi)=(pi)/5 and printing the solution at steps of 1.0. We write y=Y(1), v=Y(2) and (phi)=Y(3), and we set TOL=1.0E-4 and TOL=1.0E-5 in turn so that we may compare the solutions. The value of (pi) is obtained by using X01AAF(*).

Note the careful construction of routine OUT to ensure that the

```
value of XEND is printed.

The example program is not reproduced here. The source code for
all example programs is distributed with the NAG Foundation
Library software and should be available on-line.
```

%%%

```
D02BHF(3NAG)      Foundation Library (12/10/92)      D02BHF(3NAG)

     D02 -- Ordinary Differential Equations                    D02BHF
            D02BHF -- NAG Foundation Library Routine Document

     Note: Before using this routine, please read the Users' Note for
     your implementation to check implementation-dependent details.
     The symbol (*) after a NAG routine name denotes a routine that is
     not included in the Foundation Library.

     1. Purpose

     D02BHF integrates a system of first-order ordinary differential
     equations over an interval with suitable initial conditions,
     using a Runge-Kutta-Merson method, until a user-specified
     function of the solution is zero.

     2. Specification

           SUBROUTINE D02BHF (X, XEND, N, Y, TOL, IRELAB, HMAX, FCN,
          1                   G, W, IFAIL)
           INTEGER          N, IRELAB, IFAIL
           DOUBLE PRECISION X, XEND, Y(N), TOL, HMAX, G, W(N,7)
           EXTERNAL         FCN, G

     3. Description

     The routine advances the solution of a system of ordinary
     differential equations

                    y' =f (x,y ,y ,...,y ),  i=1,2,...,n,
                      i  i    1  2      n

     from x = X towards x = XEND using a Merson form of the Runge-
     Kutta method. The system is defined by a subroutine FCN supplied
     by the user, which evaluates f  in terms of x and y ,y ,...,y
                                   i                     1  2      n
     (see Section 5), and the values of y ,y ,...,y  must be given at
```

1 2 n
x = X.

As the integration proceeds, a check is made on the function g(x,y) specified by the user, to determine an interval where it changes sign. The position of this sign change is then determined accurately by interpolating for the solution and its derivative. It is assumed that g(x,y) is a continuous function of the variables, so that a solution of g(x,y) = 0 can be determined by searching for a change in sign in g(x,y).

The accuracy of the integration and, indirectly, of the determination of the position where g(x,y) = 0, is controlled by the parameter TOL.

For a description of Runge-Kutta methods and their practical implementation see Hall and Watt [1].

4. References

[1] Hall G and Watt J M (eds) (1976) Modern Numerical Methods for Ordinary Differential Equations. Clarendon Press.

5. Parameters

1: X -- DOUBLE PRECISION Input/Output
On entry: X must be set to the initial value of the independent variable x. On exit: the point where g(x,y) = 0. 0 unless an error has occurred, when it contains the value of x at the error. In particular, if g(x,y)/=0.0 anywhere on the range X to XEND, it will contain XEND on exit.

2: XEND -- DOUBLE PRECISION Input
On entry: the final value of the independent variable x.

If XEND < X on entry, integration proceeds in a negative direction.

3: N -- INTEGER Input
On entry: the number of differential equations, n.
Constraint: N > 0.

4: Y(N) -- DOUBLE PRECISION array Input/Output
On entry: the initial values of the solution y ,y ,...,y .
1 2 n
On exit: the computed values of the solution at the final

point x = X.

5: TOL -- DOUBLE PRECISION Input/Output
On entry: TOL must be set to a positive tolerance for controlling the error in the integration and in the determination of the position where g(x,y) = 0.0.

D02BHF has been designed so that, for most problems, a reduction in TOL leads to an approximately proportional reduction in the error in the solution obtained in the integration. The relation between changes in TOL and the error in the determination of the position where g(x,y) = 0.0 is less clear, but for TOL small enough the error should be approximately proportional to TOL. However, the actual relation between TOL and the accuracy cannot be guaranteed. The user is strongly recommended to call D02BHF with more than one value for TOL and to compare the results obtained to estimate their accuracy. In the absence of any prior knowledge the user might compare results obtained by calling D02BHF with TOL=10.0^{-p} and TOL=10.0^{-p-1} if p correct decimal digits in the solution are required. Constraint: TOL > 0.0.

On exit: normally unchanged. However if the range from x = X to the position where g(x,y) = 0.0 (or to the final value of x if an error occurs) is so short that a small change in TOL is unlikely to make any change in the computed solution, then TOL is returned with its sign changed. To check results returned with TOL < 0.0, D02BHF should be called again with a positive value of TOL whose magnitude is considerably smaller than that of the previous call.

6: IRELAB -- INTEGER Input
On entry: IRELAB determines the type of error control. At each step in the numerical solution an estimate of the local error, EST, is made. For the current step to be accepted the following condition must be satisfied:
IRELAB = 0

$$\mathrm{EST} \le \mathrm{TOL} * \max\{1.0, |y_1|, |y_2|, \ldots, |y_n|\};$$

IRELAB = 1

$$\mathrm{EST} \le \mathrm{TOL};$$

IRELAB = 2

$$\mathrm{EST} \le \mathrm{TOL} * \max\{(\mathrm{epsilon}), |y_1|, |y_2|, \ldots, |y_n|\},$$

```
                                        1    2        n

      where (epsilon) is machine precision.
    If the appropriate condition is not satisfied, the step size
    is reduced and the solution recomputed on the current step.

    If the user wishes to measure the error in the computed
    solution in terms of the number of correct decimal places,
    then IRELAB should be given the value 1 on entry, whereas if
    the error requirement is in terms of the number of correct
    significant digits, then IRELAB should be given the value 2.
    Where there is no preference in the choice of error test,
    IRELAB = 0 will result in a mixed error test. It should be
    borne in mind that the computed solution will be used in
    evaluating g(x,y). Constraint: 0 <= IRELAB <= 2.

7:  HMAX -- DOUBLE PRECISION                                  Input
    On entry: if HMAX = 0.0, no special action is taken.

    If HMAX /= 0.0, a check is made for a change in sign of
    g(x,y) at steps not greater than |HMAX|. This facility
    should be used if there is any chance of 'missing' the
    change in sign by checking too infrequently. For example, if
    two changes of sign of g(x,y) are expected within a distance
    h, say, of each other, then a suitable value for HMAX might
    be HMAX = h/2. If only one change of sign in g(x,y) is
    expected on the range X to XEND, then the choice HMAX = 0.0
    is most appropriate.

8:  FCN -- SUBROUTINE, supplied by the user.
                                                 External Procedure
    FCN must evaluate the functions f  (i.e., the derivatives
                                    i
    y' ) for given values of its arguments x,y ,...,y .
      i                                        1      n

    Its specification is:

          SUBROUTINE FCN (X, Y, F)
          DOUBLE PRECISION X, Y(n), F(n)

     1:  X -- DOUBLE PRECISION                                Input
         On entry: the value of the argument x.

     2:  Y(*) -- DOUBLE PRECISION array                       Input
         On entry: the value of the argument y , for
```

```
                                                    i
     i=1,2,...,n.

      3:  F(*) -- DOUBLE PRECISION array                    Output
          On exit: the value of f , for i=1,2,...,n.
                                  i
     FCN must be declared as EXTERNAL in the (sub)program
     from which D02BHF is called. Parameters denoted as
     Input must not be changed by this procedure.

      9:  G -- DOUBLE PRECISION FUNCTION, supplied by the user.
                                              External Procedure
     G must evaluate the function g(x,y) at a specified point.

     Its specification is:

            DOUBLE PRECISION FUNCTION G (X, Y)
            DOUBLE PRECISION X, Y(n)
     where n is the actual value of N in the call of D02BHF.

      1:  X -- DOUBLE PRECISION                             Input
          On entry: the value of the independent variable x.

      2:  Y(*) -- DOUBLE PRECISION array                    Input
          On entry: the value of y , for i=1,2,...,n.
                                   i
     G must be declared as EXTERNAL in the (sub)program from
     which D02BHF is called. Parameters denoted as Input
     must not be changed by this procedure.

     10:  W(N,7) -- DOUBLE PRECISION array                 Workspace

     11:  IFAIL -- INTEGER                              Input/Output
          On entry: IFAIL must be set to 0, -1 or 1. For users not
          familiar with this parameter (described in the Essential
          Introduction) the recommended value is 0.

          On exit: IFAIL = 0 unless the routine detects an error (see
          Section 6).

     6. Error Indicators and Warnings

     Errors detected by the routine:

     IFAIL= 1
          On entry TOL <= 0.0,
```

```
     or       N <= 0,

     or       IRELAB /= 0, 1 or 2.

IFAIL= 2
     With the given value of TOL, no further progress can be made
     across the integration range from the current point x = X,
     or dependence of the error on TOL would be lost if further
     progress across the integration range were attempted (see
     Section 8 for a discussion of this error exit). The
     components Y(1),Y(2),...,Y(n) contain the computed values of
     the solution at the current point x = X. No point at which
     g(x,y) changes sign has been located up to the point x = X.

IFAIL= 3
     TOL is too small for the routine to take an initial step
     (see Section 8). X and Y(1),Y(2),...,Y(n) retain their
     initial values.

IFAIL= 4
     At no point in the range X to XEND did the function g(x,y)
     change sign. It is assumed that g(x,y) = 0.0 has no
     solution.

IFAIL= 5
     A serious error has occurred in an internal call to
     C05AZF(*). Check all subroutine calls and array dimensions.
     Seek expert help.

IFAIL= 6
     A serious error has occurred in an internal call to
     D02PAF(*). Check all subroutine calls and array dimensions.
     Seek expert help.

IFAIL= 7
     A serious error has occurred in an internal call to
     D02XAF(*). Check all subroutine calls and array dimensions.
     Seek expert help.

7. Accuracy

The accuracy depends on TOL, on the mathematical properties of
the differential system, on the position where g(x,y) = 0.0 and
on the method. It can be controlled by varying TOL but the
approximate proportionality of the error to TOL holds only for a
```

```
restricted range of values of TOL. For TOL too large, the
underlying theory may break down and the result of varying TOL
may be unpredictable. For TOL too small, rounding error may
affect the solution significantly and an error exit with IFAIL =
2 or IFAIL = 3 is possible.

The accuracy may also be restricted by the properties of g(x,y).
The user should try to code G without introducing any unnecessary
cancellation errors.

8. Further Comments

The time taken by the routine depends on the complexity and
mathematical properties of the system of differential equations
defined by FCN, the complexity of G, on the range, the position
of the solution and the tolerance. There is also an overhead of
the form a+b*n where a and b are machine-dependent computing
times.

For some problems it is possible that D02BHF will return IFAIL =
4 because of inaccuracy of the computed values Y, leading to
inaccuracy in the computed values of g(x,y) used in the search
for the solution of g(x,y) = 0.0. This difficulty can be overcome
by reducing TOL sufficiently, and if necessary, by choosing HMAX
sufficiently small. If possible, the user should choose XEND well
beyond the expected point where g(x,y) = 0.0; for example make
|XEND-X| about 50 larger than the expected range. As a simple
check, if, with XEND fixed, a change in TOL does not lead to a
significant change in Y at XEND, then inaccuracy is not a likely
source of error.

If the routine fails with IFAIL = 3, then it could be called
again with a larger value of TOL if this has not already been
tried. If the accuracy requested is really needed and cannot be
obtained with this routine, the system may be very stiff (see
below) or so badly scaled that it cannot be solved to the
required accuracy.

If the routine fails with IFAIL = 2, it is likely that it has
been called with a value of TOL which is so small that a solution
cannot be obtained on the range X to XEND. This can happen for
well-behaved systems and very small values of TOL. The user
should, however, consider whether there is a more fundamental
difficulty. For example:

(a)   in the region of a singularity (infinite value) of the
```

solution, the routine will usually stop with IFAIL = 2, unless overflow occurs first. If overflow occurs using D02BHF, D02PAF(*) can be used instead to trap the increasing solution, before overflow occurs. In any case, numerical integration cannot be continued through a singularity, and analytical treatment should be considered;

(b) for 'stiff' equations, where the solution contains rapidly decaying components, the routine will compute in very small steps in x (internally to D02BHF) to preserve stability. This will usually exhibit itself by making the computing time excessively long, or occasionally by an exit with IFAIL = 2. Merson's method is not efficient in such cases, and the user should try D02EHF(*) which uses a Backward Differentiation Formula method. To determine whether a problem is stiff, D02BDF(*) may be used.

For well-behaved systems with no difficulties such as stiffness or singularities, the Merson method should work well for low accuracy calculations (three or four figures). For high accuracy calculations or where FCN is costly to evaluate, Merson's method may not be appropriate and a computationally less expensive method may be D02CHF(*) which uses an Adams method.

For problems for which D02BHF is not sufficiently general, the user should consider D02PAF(*). D02PAF(*) is a more general Merson routine with many facilities including more general error control options and several criteria for interrupting the calculations. D02PAF(*) can be combined with the rootfinder C05AZF(*) and the interpolation routine D02XAF(*) to solve equations involving $y_1, y_2, \ldots, y_n$ and their derivatives.

D02BHF can also be used to solve an equation involving x, $y_1, y_2, \ldots, y_n$ and the derivatives of $y_1, y_2, \ldots, y_n$. For example in Section 9, D02BHF is used to find a value of X > 0.0 where Y(1) = 0.0. It could instead be used to find a turning-point of y_1 by replacing the function g(x,y) in the program by:

```
      DOUBLE PRECISION FUNCTION G(X,Y)
      DOUBLE PRECISION X,Y(3),F(3)
      CALL FCN(X,Y,F)
      G = F(1)
      RETURN
```

```
      END
```

This routine is only intended to locate the first zero of g(x,y). If later zeros are required, users are strongly advised to construct their own more general root finding routines as discussed above.

9. Example

To find the value X > 0.0 at which y=0.0, where y, v, (phi) are defined by

$$y' = \tan(\phi)$$

$$v' = \frac{-0.032\tan(\phi)}{v} - \frac{0.02v}{\cos(\phi)}$$

$$(\phi)' = -\frac{0.032}{v^2}$$

and where at X = 0.0 we are given y=0.5, v=0.5 and (phi)=(pi)/5. We write y=Y(1), v=Y(2) and (phi)=Y(3) and we set TOL=1.0E-4 and TOL=1.0E-5 in turn so that we can compare the solutions. We expect the solution X ~= 7.3 and so we set XEND = 10.0 to avoid determining the solution of y=0.0 too near the end of the range of integration. The value of (pi) is obtained by using X01AAF(*).

The example program is not reproduced here. The source code for all example programs is distributed with the NAG Foundation Library software and should be available on-line.

%%%

D02CJF(3NAG) D02CJF D02CJF(3NAG)

D02 -- Ordinary Differential Equations D02CJF
D02CJF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for your implementation to check implementation-dependent details. The symbol (*) after a NAG routine name denotes a routine that is not included in the Foundation Library.

Note for users via the AXIOM system: the interface to this routine has been enhanced for use with AXIOM and is slightly different to that offered in the standard version of the Foundation Library.

1. Purpose

D02CJF integrates a system of first-order ordinary differential equations over a range with suitable initial conditions, using a variable-order, variable-step Adams method until a user-specified function, if supplied, of the solution is zero, and returns the solution at points specified by the user, if desired.

2. Specification

```
      SUBROUTINE D02CJF (X, XEND, M, N, Y, FCN, TOL, RELABS,
     1                   RESULT, OUTPUT, G, W, IFAIL)
      INTEGER           M, N, IFAIL
      DOUBLE PRECISION X, XEND, Y(N), TOL, G, W(28+21*N), RESULT(M,N)
      CHARACTER*1       RELABS
      EXTERNAL          FCN, OUTPUT, G
```

3. Description

The routine advances the solution of a system of ordinary differential equations

$$y_i' = f_i(x, y_1, y_2, \ldots, y_n), \quad i = 1, 2, \ldots, n,$$

from x = X to x = XEND using a variable-order, variable-step Adams method. The system is defined by a subroutine FCN supplied by the user, which evaluates f_i in terms of x and $y_1, y_2, \ldots, y_n$. The initial values of $y_1, y_2, \ldots, y_n$ must be given at x = X.

The solution is returned via the user-supplied routine OUTPUT at points specified by the user, if desired: this solution is obtained by C^1 interpolation on solution values produced by the method. As the integration proceeds a check can be made on the user-specified function g(x,y) to determine an interval where it changes sign. The position of this sign change is then determined accurately by C^1 interpolation to the solution. It is assumed

that g(x,y) is a continuous function of the variables, so that a solution of g(x,y)=0.0 can be determined by searching for a change in sign in g(x,y). The accuracy of the integration, the interpolation and, indirectly, of the determination of the position where g(x,y)=0.0, is controlled by the parameters TOL and RELABS.

For a description of Adams methods and their practical implementation see Hall and Watt [1].

4. References

[1] Hall G and Watt J M (eds) (1976) Modern Numerical Methods for Ordinary Differential Equations. Clarendon Press.

5. Parameters

1: X -- DOUBLE PRECISION Input/Output
On entry: the initial value of the independent variable x. Constraint: X /= XEND. On exit: if g is supplied by the user, it contains the point where g(x,y)=0.0, unless g(x,y)/=0.0 anywhere on the range X to XEND, in which case, X will contain XEND. If g is not supplied by the user it contains XEND, unless an error has occurred, when it contains the value of x at the error.

2: XEND -- DOUBLE PRECISION Input
On entry: the final value of the independent variable. If XEND < X, integration proceeds in the negative direction.
Constraint: XEND /= X.

3: M -- INTEGER Input
On entry: the first dimension of the array RESULT. This will usually be equal to the number of points at which the solution is required.
Constraint: M > 0.

4: N -- INTEGER Input
On entry: the number of differential equations.
Constraint: N >= 1.

5: Y(N) -- DOUBLE PRECISION array Input/Output
On entry: the initial values of the solution $y_1, y_2, \ldots, y_n$ at x = X. On exit: the computed values of the solution at the final point x = X.

6: FCN -- SUBROUTINE, supplied by the user.
External Procedure
FCN must evaluate the functions f_i (i.e., the derivatives y_i') for given values of their arguments $x, y_1, y_2, \ldots, y_n$.

Its specification is:

```
SUBROUTINE FCN (X, Y, F)
DOUBLE PRECISION X, Y(n), F(n)
```

where n is the actual value of N in the call of D02CJF.

1: X -- DOUBLE PRECISION Input
On entry: the value of the independent variable x.

2: Y(*) -- DOUBLE PRECISION array Input
On entry: the value of the variable y_i, for $i=1,2,\ldots,n$.

3: F(*) -- DOUBLE PRECISION array Output
On exit: the value of f_i, for $i=1,2,\ldots,n$.

FCN must be declared as EXTERNAL in the (sub)program from which D02CJF is called. Parameters denoted as Input must not be changed by this procedure.

7: TOL -- DOUBLE PRECISION Input
On entry: a positive tolerance for controlling the error in the integration. Hence TOL affects the determination of the position where g(x,y)=0.0, if g is supplied.

D02CJF has been designed so that, for most problems, a reduction in TOL leads to an approximately proportional reduction in the error in the solution. However, the actual relation between TOL and the accuracy achieved cannot be guaranteed. The user is strongly recommended to call D02CJF with more than one value for TOL and to compare the results obtained to estimate their accuracy. In the absence of any prior knowledge, the user might compare the results obtained by calling D02CJF with $TOL=10.0^{-p}$ and $TOL=10.0^{-p-1}$ where p correct decimal digits are required in the solution.
Constraint: TOL > 0.0.

8: RELABS -- CHARACTER*1 Input
On entry: the type of error control. At each step in the numerical solution an estimate of the local error, EST, is made. For the current step to be accepted the following condition must be satisfied:

$$EST = \sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(e_i/((\tau)_r\,|y_i|+(\tau)_a)\right)^2} \leq 1.0$$

where $(\tau)_r$ and $(\tau)_a$ are defined by

RELABS	$(\tau)_r$	$(\tau)_a$
'M'	TOL	TOL
'A'	0.0	TOL
'R'	TOL	(epsilon)
'D'	TOL	TOL

where (epsilon) is a small machine-dependent number and e_i is an estimate of the local error at y_i, computed internally. If the appropriate condition is not satisfied, the step size is reduced and the solution is recomputed on the current step. If the user wishes to measure the error in the computed solution in terms of the number of correct decimal places, then RELABS should be set to 'A' on entry, whereas if the error requirement is in terms of the number of correct significant digits, then RELABS should be set to 'R'. If the user prefers a mixed error test, then RELABS should be set to 'M', otherwise if the user has no preference, RELABS should be set to the default 'D'. Note that in this case 'D' is taken to be 'M'. Constraint: RELABS = 'M', 'A', 'R', 'D'.

9: RESULT(M,N) -- DOUBLE PRECISION array Output
On exit: the computed values of the solution at the points given by OUTPUT.

10: OUTPUT -- SUBROUTINE, supplied by the user.

```
                                                    External Procedure
OUTPUT allows the user to have access to intermediate values
of the computed solution at successive points specified by the
user. These solution values may be returned to the user via
the array RESULT if desired (this is a non-standard feature
added for use with the AXIOM system).  OUTPUT is initially
called by D02CJF with XSOL = X (the initial value of x). The
user must reset XSOL to the next point where OUTPUT is to be
called, and so on at each call to OUTPUT. If, after a call
to OUTPUT, the reset point XSOL is beyond XEND, D02CJF will
integrate to XEND with no further calls to OUTPUT; if a call
to OUTPUT is required at the point XSOL = XEND, then XSOL
must be given precisely the value XEND.

Its specification is:

     SUBROUTINE OUTPUT(XSOL,Y,COUNT,M,N,RESULT)
     DOUBLE PRECISION Y(N),RESULT(M,N),XSOL
     INTEGER M,N,COUNT

 1:  XSOL -- DOUBLE PRECISION                        Input/Output
     On entry: the current value of the independent
     variable x. On exit: the next value of x at which
     OUTPUT is to be called.

 2:  Y(N) -- DOUBLE PRECISION array                        Input
     On entry: the computed solution at the point XSOL.

 3:  COUNT -- INTEGER                                Input/Output
     On entry: Zero if OUTPUT has not been called before, or
     the previous value of COUNT.
     On exit: A new value of COUNT: this can be used to keep
     track of the number of times OUTPUT has been called.

 4:  M -- INTEGER                                          Input
     On entry: The first dimension of RESULT.

 5:  N -- INTEGER                                          Input
     On entry: The dimension of Y.

 6:  RESULT(M,N) -- DOUBLE PRECISION array       Input/Output
     On entry: the previous contents of RESULT.
     On exit: RESULT may be used to return the values of the
     intermediate solutions to the user.

OUTPUT must be declared as EXTERNAL in the (sub)program
```

from which D02CJF is called. Parameters denoted as Input must not be changed by this procedure.

11: G -- DOUBLE PRECISION FUNCTION, supplied by the user. External Procedure

G must evaluate the function g(x,y) for specified values x,y . It specifies the function g for which the first position x where g(x,y)=0 is to be found.

If the user does not require the root finding option, the actual argument G mustbe the dummy routine D02CJW. (D02CJW is included in the NAG Foundation Library and so need not be supplied by the user).

Its specification is:

```
DOUBLE PRECISION FUNCTION G (X, Y)
DOUBLE PRECISION X, Y(n)
```

where n is the actual value of N in the call of D02CJF.

1: X -- DOUBLE PRECISION Input

On entry: the value of the independent variable x.

2: Y(*) -- DOUBLE PRECISION array Input

On entry: the value of the variable y_i, for $i=1,2,\ldots,n$.

G must be declared as EXTERNAL in the (sub)program from which D02CJF is called. Parameters denoted as Input must not be changed by this procedure.

12: W(28+21*N) -- DOUBLE PRECISION array Workspace

13: IFAIL -- INTEGER Input/Output

On entry: IFAIL must be set to 0, -1 or 1. For users not familiar with this parameter (described in the Essential Introduction) the recommended value is 0.

On exit: IFAIL = 0 unless the routine detects an error (see Section 6).

6. Error Indicators and Warnings

Errors detected by the routine:

If on entry IFAIL = 0 or -1, explanatory error messages are

```
output on the current error message unit (as defined by X04AAF).

IFAIL= 1
     On entry TOL <= 0.0,

     or       N <= 0,

     or       RELABS /= 'M', 'A', 'R' or 'D'.

     or       X = XEND.

IFAIL= 2
     With the given value of TOL, no further progress can be made
     across the integration range from the current point x = X.
     (See Section 8 for a discussion of this error exit.) The
     components Y(1),Y(2),...,Y(N) contain the computed values of
     the solution at the current point x = X. If the user has
     supplied g, then no point at which g(x,y) changes sign has
     been located up to the point x = X.

IFAIL= 3
     TOL is too small for D02CJF to take an initial step. X and Y
     (1),Y(2),...,Y(N) retain their initial values.

IFAIL= 4
     XSOL has not been reset or XSOL lies behind X in the
     direction of integration, after the initial call to OUTPUT,
     if the OUTPUT option was selected.

IFAIL= 5
     A value of XSOL returned by OUTPUT has not been reset or
     lies behind the last value of XSOL in the direction of
     integration, if the OUTPUT option was selected.

IFAIL= 6
     At no point in the range X to XEND did the function g(x,y)
     change sign, if g was supplied. It is assumed that g(x,y)=0
     has no solution.

IFAIL= 7
     A serious error has occurred in an internal call. Check all
     subroutine calls and array sizes. Seek expert help.

7. Accuracy

The accuracy of the computation of the solution vector Y may be
```

controlled by varying the local error tolerance TOL. In general, a decrease in local error tolerance should lead to an increase in accuracy. Users are advised to choose RELABS = 'M' unless they have a good reason for a different choice.

If the problem is a root-finding one, then the accuracy of the root determined will depend on the properties of g(x,y). The user should try to code G without introducing any unnecessary cancellation errors.

8. Further Comments

If more than one root is required then D02QFF(*) should be used.

If the routine fails with IFAIL = 3, then it can be called again with a larger value of TOL if this has not already been tried. If the accuracy requested is really needed and cannot be obtained with this routine, the system may be very stiff (see below) or so badly scaled that it cannot be solved to the required accuracy.

If the routine fails with IFAIL = 2, it is probable that it has been called with a value of TOL which is so small that a solution cannot be obtained on the range X to XEND. This can happen for well-behaved systems and very small values of TOL. The user should, however, consider whether there is a more fundamental difficulty. For example:

(a) in the region of a singularity (infinite value) of the solution, the routine will usually stop with IFAIL = 2, unless overflow occurs first. Numerical integration cannot be continued through a singularity, and analytic treatment should be considered;

(b) for 'stiff' equations where the solution contains rapidly decaying components, the routine will use very small steps in x (internally to D02CJF) to preserve stability. This will exhibit itself by making the computing time excessively long, or occasionally by an exit with IFAIL = 2. Adams methods are not efficient in such cases, and the user should try D02EJF.

9. Example

We illustrate the solution of four different problems. In each case the differential system (for a projectile) is

$$y' = \tan(\phi)$$

$$v' = \frac{-0.032\tan(\phi)}{v} - \frac{0.02v}{\cos(\phi)}$$

$$(\phi)' = \frac{-0.032}{v^2}$$

over an interval X = 0.0 to XEND = 10.0 starting with values y=0.5, v=0.5 and (phi)=(pi)/5. We solve each of the following problems with local error tolerances 1.0E-4 and 1.0E-5.

(i) To integrate to x=10.0 producing output at intervals of 2.0 until a point is encountered where y=0.0.

(ii) As (i) but with no intermediate output.

(iii) As (i) but with no termination on a root-finding condition.

(iv) As (i) but with no intermediate output and no root-finding termination condition.

The example program is not reproduced here. The source code for all example programs is distributed with the NAG Foundation Library software and should be available on-line.

%%%

D02EJF(3NAG) D02EJF D02EJF(3NAG)

D02 -- Ordinary Differential Equations D02EJF
D02EJF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for your implementation to check implementation-dependent details. The symbol (*) after a NAG routine name denotes a routine that is not included in the Foundation Library.

Note for users via the AXIOM system: the interface to this routine has been enhanced for use with AXIOM and is slightly different to that offered in the standard version of the Foundation Library.

1. Purpose

D02EJF integrates a stiff system of first-order ordinary differential equations over an interval with suitable initial conditions, using a variable-order, variable-step method implementing the Backward Differentiation Formulae (BDF), until a user-specified function, if supplied, of the solution is zero, and returns the solution at points specified by the user, if desired.

2. Specification

```
      SUBROUTINE D02EJF (X, XEND, M, N, Y, FCN, PEDERV, TOL,
     1                   RELABS, OUTPUT, G, W, IW, RESULT, IFAIL)
      INTEGER          M, N, IW, IFAIL
      DOUBLE PRECISION X, XEND, Y(N), TOL, G, W(IW), RESULT(M,N)
      CHARACTER*1      RELABS
      EXTERNAL         FCN, PEDERV, OUTPUT, G
```

3. Description

The routine advances the solution of a system of ordinary differential equations

$$y_i' = f_i(x, y_1, y_2, \ldots, y_n), \quad i = 1, 2, \ldots, n,$$

from x = X to x = XEND using a variable-order, variable-step method implementing the BDF. The system is defined by a subroutine FCN supplied by the user, which evaluates f_i in terms of x and $y_1, y_2, \ldots, y_n$ (see Section 5). The initial values of $y_1, y_2, \ldots, y_n$ must be given at x = X.

The solution is returned via the user-supplied routine OUTPUT at points specified by the user, if desired: this solution is obtained by C^1 interpolation on solution values produced by the method. As the integration proceeds a check can be made on the user-specified function g(x,y) to determine an interval where it changes sign. The position of this sign change is then determined accurately by C^1 interpolation to the solution. It is assumed that g(x,y) is a continuous function of the variables, so that a solution of g(x,y) = 0.0 can be determined by searching for a

```
change in sign in g(x,y). The accuracy of the integration, the
interpolation and, indirectly, of the determination of the
position where g(x,y) = 0.0, is controlled by the parameters TOL
and RELABS. The Jacobian of the system y'=f(x,y) may be supplied
in routine PEDERV, if it is available.

For a description of BDF and their practical implementation see
Hall and Watt [1].

4. References

[1]   Hall G and Watt J M (eds) (1976) Modern Numerical Methods
      for Ordinary Differential Equations. Clarendon Press.

5. Parameters

 1:  X -- DOUBLE PRECISION                              Input/Output
     On entry: the initial value of the independent variable x.
     Constraint: X /= XEND On exit: if G is supplied by the user,
     X contains the point where g(x,y) = 0.0, unless g(x,y) /= 0.
     0 anywhere on the range X to XEND, in which case, X will
     contain XEND. If G is not supplied X contains XEND, unless
     an error has occured, when it contains the value of x at the
     error.

 2:  XEND -- DOUBLE PRECISION                                 Input
     On entry: the final value of the independent variable. If
     XEND < X, integration proceeds in the negative direction.
     Constraint: XEND /= X.

 3:  M -- INTEGER                                             Input
     On entry: the first dimension of the array RESULT.  This
     will usually be equal to the number of points at which the
     solution is required.
     Constraint: M > 0.

 4:  N -- INTEGER                                             Input
     On entry: the number of differential equations, n.
     Constraint: N >= 1.

 5:  Y(N) -- DOUBLE PRECISION array                     Input/Output
     On entry: the initial values of the solution y ,y ,...,y
                                                   1  2      n
     at x = X. On exit: the computed values of the solution at
     the final point x = X.
```

6: FCN -- SUBROUTINE, supplied by the user.
External Procedure
FCN must evaluate the functions f (i.e., the derivatives
i
y') for given values of their arguments x,y ,y ,...,y .
i 1 2 n

Its specification is:

```
SUBROUTINE FCN (X, Y, F)
DOUBLE PRECISION X, Y(n), F(n)
```

where n is the actual value of N in the call of D02EJF.

1: X -- DOUBLE PRECISION Input
On entry: the value of the independent variable x.

2: Y(*) -- DOUBLE PRECISION array Input
On entry: the value of the variable y , for
i
i=1,2,...,n.

3: F(*) -- DOUBLE PRECISION array Output
On exit: the value of f , for i=1,2,...,n.
i
FCN must be declared as EXTERNAL in the (sub)program
from which D02EJF is called. Parameters denoted as
Input must not be changed by this procedure.

7: PEDERV -- SUBROUTINE, supplied by the user.
External Procedure
PEDERV must evaluate the Jacobian of the system (that is,
ddf
i
the partial derivatives ----) for given values of the
ddy
j
variables x,y ,y ,...,y .
1 2 n

Its specification is:

```
SUBROUTINE PEDERV (X, Y, PW)
DOUBLE PRECISION X, Y(n), PW(n,n)
```

where n is the actual value of N in the call of D02EJF.

1: X -- DOUBLE PRECISION Input

On entry: the value of the independent variable x.

2: Y(*) -- DOUBLE PRECISION array Input
On entry: the value of the variable y_i, for $i=1,2,\ldots,n$.

3: PW(n,*) -- DOUBLE PRECISION array Output
On exit: the value of $\frac{ddf_i}{ddy_j}$, for i,j=1,2,...,n.

If the user does not wish to supply the Jacobian, the actual argument PEDERV must be the dummy routine D02EJY . (D02EJY is included in the NAG Foundation Library and so need not be supplied by the user. The name may be implementation dependent: see the User's Note for your implementation for details).
PEDERV must be declared as EXTERNAL in the (sub)program from which D02EJF is called. Parameters denoted as Input must not be changed by this procedure.

8: TOL -- DOUBLE PRECISION Input/Output
On entry: TOL must be set to a positive tolerance for controlling the error in the integration. Hence TOL affects the determination of the position where g(x,y) = 0.0, if G is supplied.

D02EJF has been designed so that, for most problems, a reduction in TOL leads to an approximately proportional reduction in the error in the solution. However, the actual relation between TOL and the accuracy achieved cannot be guaranteed. The user is strongly recommended to call D02EJF with more than one value for TOL and to compare the results obtained to estimate their accuracy. In the absence of any prior knowledge, the user might compare the results obtained by calling D02EJF with $TOL=10^{-p}$ and $TOL=10^{-p-1}$ if p correct decimal digits are required in the solution. Constraint: TOL > 0.0. On exit: normally unchanged. However if the range X to XEND is so short that a small change in TOL is unlikely to make any change in the computed solution, then, on return, TOL has its sign changed.

9: RELABS -- CHARACTER*1 Input

On entry: the type of error control. At each step in the numerical solution an estimate of the local error, EST, is made. For the current step to be accepted the following condition must be satisfied:

$$EST = \sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(\frac{e_i}{(\tau)_r\,|y_i| + (\tau)_a}\right)^2} <= 1.0$$

where $(\tau)_r$ and $(\tau)_a$ are defined by

RELABS	$(\tau)_r$	$(\tau)_a$
'M'	TOL	TOL
'A'	0.0	TOL
'R'	TOL	(epsilon)
'D'	TOL	(epsilon)

where (epsilon) is a small machine-dependent number and e_i is an estimate of the local error at y_i, computed internally. If the appropriate condition is not satisfied, the step size is reduced and the solution is recomputed on the current step. If the user wishes to measure the error in the computed solution in terms of the number of correct decimal places, then RELABS should be set to 'A' on entry, whereas if the error requirement is in terms of the number of correct significant digits, then RELABS should be set to 'R'. If the user prefers a mixed error test, then RELABS should be set to 'M', otherwise if the user has no preference, RELABS should be set to the default 'D'. Note that in this case 'D' is taken to be 'R'. Constraint: RELABS = 'A', 'M', 'R' or 'D'.

10: OUTPUT -- SUBROUTINE, supplied by the user.
External Procedure
OUTPUT allows the user to have access to intermediate values of the computed solution at successive points specified by the user. These solution values may be returned to the user via the array RESULT if desired (this is a non-standard feature added for use with the AXIOM system). OUTPUT is initially

called by D02EJF with XSOL = X (the initial value of x). The user must reset XSOL to the next point where OUTPUT is to be called, and so on at each call to OUTPUT. If, after a call to OUTPUT, the reset point XSOL is beyond XEND, D02EJF will integrate to XEND with no further calls to OUTPUT; if a call to OUTPUT is required at the point XSOL = XEND, then XSOL must be given precisely the value XEND.

Its specification is:

```
SUBROUTINE OUTPUT(XSOL,Y,COUNT,M,N,RESULT)
DOUBLE PRECISION Y(N),RESULT(M,N),XSOL
INTEGER M,N,COUNT
```

1: XSOL -- DOUBLE PRECISION Input/Output
On entry: the current value of the independent variable x. On exit: the next value of x at which OUTPUT is to be called.

2: Y(N) -- DOUBLE PRECISION array Input
On entry: the computed solution at the point XSOL.

3: COUNT -- INTEGER Input/Output
On entry: Zero if OUTPUT has not been called before, or the previous value of COUNT.
On exit: A new value of COUNT: this can be used to keep track of the number of times OUTPUT has been called.

4: M -- INTEGER Input
On entry: The first dimension of RESULT.

5: N -- INTEGER Input
On entry: The dimension of Y.

6: RESULT(M,N) -- DOUBLE PRECISION array Input/Output
On entry: the previous contents of RESULT.
On exit: RESULT may be used to return the values of the intermediate solutions to the user.

OUTPUT must be declared as EXTERNAL in the (sub)program from which D02EJF is called. Parameters denoted as Input must not be changed by this procedure.

11: G -- DOUBLE PRECISION FUNCTION, supplied by the user.
External Procedure
G must evaluate the function g(x,y) for specified values x,y

```
. It specifies the function g for which the first position x
where g(x,y) = 0 is to be found.

Its specification is:

       DOUBLE PRECISION FUNCTION G (X, Y)
       DOUBLE PRECISION X, Y(n)
where n is the actual value of N in the call of D02EJF.

 1:  X -- DOUBLE PRECISION                                  Input
     On entry: the value of the independent variable x.

 2:  Y(*) -- DOUBLE PRECISION array                         Input
     On entry: the value of the variable y , for
                                          i
     i=1,2,...,n.
If the user does not require the root finding option,
the actual argument G must be the dummy routine D02EJW.
(D02EJW is included in the NAG Foundation Library and
so need not be supplied by the user).
G must be declared as EXTERNAL in the (sub)program from
which D02EJF is called. Parameters denoted as Input
must not be changed by this procedure.

12:  W(IW) -- DOUBLE PRECISION array                      Workspace

13:  IW -- INTEGER                                           Input
     On entry:
     the dimension of the array W as declared in the (sub)program
     from which D02EJF is called.
     Constraint: IW>=(12+N)*N+50.

14:  RESULT(M,N) -- DOUBLE PRECISION array                  Output
     On exit: the computed values of the solution at the points
     given by OUTPUT.

15:  IFAIL -- INTEGER                                 Input/Output
     On entry: IFAIL must be set to 0, -1 or 1. For users not
     familiar with this parameter (described in the Essential
     Introduction) the recommended value is 0.

     On exit: IFAIL = 0 unless the routine detects an error (see
     Section 6).

6. Error Indicators and Warnings
```

```
Errors detected by the routine:

If on entry IFAIL = 0 or -1, explanatory error messages are
output on the current error message unit (as defined by X04AAF).

IFAIL= 1
     On entry TOL <= 0.0,

     or       X = XEND,

     or       N <= 0,

     or       RELABS /= 'M', 'A', 'R', 'D'.

     or       IW<(12+N)*N+50.

IFAIL= 2
     With the given value of TOL, no further progress can be made
     across the integration range from the current point x = X.
     (See Section 5 for a discussion of this error test.) The
     components Y(1),Y(2),...,Y(n) contain the computed values of
     the solution at the current point x = X. If the user has
     supplied G, then no point at which g(x,y) changes sign has
     been located up to the point x = X.

IFAIL= 3
     TOL is too small for D02EJF to take an initial step. X and Y
     (1),Y(2),...,Y(n) retain their initial values.

IFAIL= 4
     XSOL lies behind X in the direction of integration, after
     the initial call to OUTPUT, if the OUTPUT option was
     selected.

IFAIL= 5
     A value of XSOL returned by OUTPUT lies behind the last
     value of XSOL in the direction of integration, if the OUTPUT
     option was selected.

IFAIL= 6
     At no point in the range X to XEND did the function g(x,y)
     change sign, if G was supplied. It is assumed that g(x,y) =
     0 has no solution.

IFAIL= 7
     A serious error has occurred in an internal call to
```

C05AZF(*). Check all subroutine calls and array dimensions. Seek expert help.

IFAIL= 8
A serious error has occurred in an internal call to D02XKF(*). Check all subroutine calls and array dimensions. Seek expert help.

IFAIL= 9
A serious error has occurred in an internal call to D02NMF(*). Check all subroutine calls and array dimensions. Seek expert help.

7. Accuracy

The accuracy of the computation of the solution vector Y may be controlled by varying the local error tolerance TOL. In general, a decrease in local error tolerance should lead to an increase in accuracy. Users are advised to choose RELABS = 'R' unless they have a good reason for a different choice. It is particularly appropriate if the solution decays.

If the problem is a root-finding one, then the accuracy of the root determined will depend strongly on $\frac{dg}{dx}$ and $\frac{dg}{dy_i}$, for i=1,2,...,n. Large values for these quantities may imply large errors in the root.

8. Further Comments

If more than one root is required, then to determine the second and later roots D02EJF may be called again starting a short distance past the previously determined roots. Alternatively the user may construct his own root finding code using D02QDF(*) (or the routines of the subchapter D02M-D02N), D02XKF(*) and C05AZF(*).

If it is easy to code, the user should supply the routine PEDERV. However, it is important to be aware that if PEDERV is coded incorrectly, a very inefficient integration may result and possibly even a failure to complete the integration (IFAIL = 2).

9. Example

We illustrate the solution of five different problems. In each case the differential system is the well-known stiff Robertson problem.

$$
\begin{aligned}
a' &= -0.04a - 10^{4}\,bc \\
b' &= 0.04a - 10^{4}\,bc - 3*10^{7}\,b^{2} \\
c' &= 3*10^{7}\,b^{2}
\end{aligned}
$$

with initial conditions a=1.0, b=c=0.0 at x=0.0. We solve each of the following problems with local error tolerances 1.0E-3 and 1.0 E-4.

(i) To integrate to x=10.0 producing output at intervals of 2.0 until a point is encountered where a=0.9. The Jacobian is calculated numerically.

(ii) As (i) but with the Jacobian calculated analytically.

(iii) As (i) but with no intermediate output.

(iv) As (i) but with no termination on a root-finding condition.

(v) Integrating the equations as in (i) but with no intermediate output and no root-finding termination condition.

The example program is not reproduced here. The source code for all example programs is distributed with the NAG Foundation Library software and should be available on-line.

%%%

D02GAF(3NAG) Foundation Library (12/10/92) D02GAF(3NAG)

D02 -- Ordinary Differential Equations D02GAF
D02GAF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for your implementation to check implementation-dependent details. The symbol (*) after a NAG routine name denotes a routine that is not included in the Foundation Library.

1. Purpose

D02GAF solves the two-point boundary-value problem with assigned boundary values for a system of ordinary differential equations, using a deferred correction technique and a Newton iteration.

2. Specification

```
      SUBROUTINE D02GAF (U, V, N, A, B, TOL, FCN, MNP, X, Y, NP,
     1                   W, LW, IW, LIW, IFAIL)
      INTEGER           N, MNP, NP, LW, IW(LIW), LIW, IFAIL
      DOUBLE PRECISION U(N,2), V(N,2), A, B, TOL, X(MNP), Y
     1                 (N,MNP), W(LW)
      EXTERNAL          FCN
```

3. Description

D02GAF solves a two-point boundary-value problem for a system of n differential equations in the interval [a,b]. The system is written in the form

$$y_i' = f_i(x, y_1, y_2, \ldots, y_n), \quad i = 1,2,\ldots,n \qquad (1)$$

and the derivatives are evaluated by a subroutine FCN supplied by the user. Initially, n boundary values of the variables y_i must be specified (assigned), some at a and some at b. The user also supplies estimates of the remaining n boundary values and all the boundary values are used in constructing an initial approximation to the solution. This approximate solution is corrected by a finite-difference technique with deferred correction allied with a Newton iteration to solve the finite-difference equations. The technique used is described fully in Pereyra [1]. The Newton iteration requires a Jacobian matrix $\frac{ddf_i}{ddy_j}$ and this is calculated by numerical differentiation using an algorithm described in Curtis et al [2].

The user supplies an absolute error tolerance and may also supply an initial mesh for the construction of the finite-difference equations (alternatively a default mesh is used). The algorithm constructs a solution on a mesh defined by adding points to the initial mesh. This solution is chosen so that the error is

everywhere less than the user's tolerance and so that the error is approximately equidistributed on the final mesh. The solution is returned on this final mesh.

If the solution is required at a few specific points then these should be included in the initial mesh. If on the other hand the solution is required at several specific points then the user should use the interpolation routines provided in Chapter E01 if these points do not themselves form a convenient mesh.

4. References

[1] Pereyra V (1979) PASVA3: An Adaptive Finite-Difference Fortran Program for First Order Nonlinear, Ordinary Boundary Problems. Codes for Boundary Value Problems in Ordinary Differential Equations. Lecture Notes in Computer Science. (ed B Childs, M Scott, J W Daniel, E Denman and P Nelson) 76 Springer-Verlag.

[2] Curtis A R, Powell M J D and Reid J K (1974) On the Estimation of Sparse Jacobian Matrices. J. Inst. Maths Applics. 13 117--119.

5. Parameters

1: U(N,2) -- DOUBLE PRECISION array Input
On entry: U(i,1) must be set to the known (assigned) or estimated values of y_i at a and U(i,2) must be set to the known or estimated values of y_i at b, for i=1,2,...,n.

2: V(N,2) -- DOUBLE PRECISION array Input
On entry: V(i,j) must be set to 0.0 if U(i,j) is a known (assigned) value and to 1.0 if U(i,j) is an estimated value, i=1,2,...,n; j=1,2. Constraint: precisely N of the V(i,j) must be set to 0.0, i.e., precisely N of the U(i,j) must be known values, and these must not be all at a or all at b.

3: N -- INTEGER Input
On entry: the number of equations. Constraint: N >= 2.

4: A -- DOUBLE PRECISION Input
On entry: the left-hand boundary point, a.

5: B -- DOUBLE PRECISION Input

On entry: the right-hand boundary point, b. Constraint: B > A.

6: TOL -- DOUBLE PRECISION Input
On entry: a positive absolute error tolerance. If
$$a=x_1<x_2<\ldots<x_{NP}=b$$
is the final mesh, $z(x_j)$ is the jth component of the approximate solution at x_i, and $y_j(x)$ is the jth component of the true solution of equation (1) (see Section 3) and the boundary conditions, then, except in extreme cases, it is expected that
$$|z_j(x_i)-y_j(x_i)|\leq TOL,\ i=1,2,\ldots,NP;j=1,2,\ldots,n \qquad (2)$$
Constraint: TOL > 0.0.

7: FCN -- SUBROUTINE, supplied by the user.
External Procedure
FCN must evaluate the functions f_i (i.e., the derivatives y_i') at the general point x_i.

Its specification is:

```
SUBROUTINE FCN (X, Y, F)
DOUBLE PRECISION X, Y(n), F(n)
```

where n is the actual value of N in the call of D02GAF.

1: X -- DOUBLE PRECISION Input
On entry: the value of the argument x.

2: Y(*) -- DOUBLE PRECISION array Input
On entry: the value of the argument y_i, for $i=1,2,\ldots,n$.

3: F(*) -- DOUBLE PRECISION array Output
On exit: the values of f_i, for $i=1,2,\ldots,n$.

FCN must be declared as EXTERNAL in the (sub)program from which D02GAF is called. Parameters denoted as Input must not be changed by this procedure.

```
 8:  MNP -- INTEGER                                              Input
     On entry: the maximum permitted number of mesh points.
     Constraint: MNP >= 32.

 9:  X(MNP) -- DOUBLE PRECISION array                  Input/Output
     On entry: if NP >= 4 (see NP below), the first NP elements
     must define an initial mesh. Otherwise the elements of X
     need not be set. Constraint:
               A=X(1)<X(2)<...<X(NP)=B for NP>=4                (3)
     On exit: X(1),X(2),...,X(NP) define the final mesh (with
     the returned value of NP) satisfying the relation (3).

10:  Y(N,MNP) -- DOUBLE PRECISION array                        Output
     On exit: the approximate solution z (x ) satisfying (2), on
                                        j  i
     the final mesh, that is
             Y(j,i)=z (x ) , i=1,2,...,NP;j=1,2,...,n,
                     j  i
     where NP is the number of points in the final mesh.

     The remaining columns of Y are not used.

11:  NP -- INTEGER                                    Input/Output
     On entry: determines whether a default or user-supplied
     mesh is used. If NP = 0, a default value of 4 for NP and a
     corresponding equispaced mesh X(1),X(2),...,X(NP) are used.
     If NP >= 4, then the user must define an initial mesh using
     the array X as described. Constraint: NP = 0 or 4 <= NP <=
     MNP. On exit: the number of points in the final (returned)
     mesh.

12:  W(LW) -- DOUBLE PRECISION array                        Workspace

13:  LW -- INTEGER                                              Input
     On entry: the length of the array W as declared in the
                                                  2         2
     calling (sub)program. Constraint: LW>=MNP*(3N +6N+2)+4N +4N

14:  IW(LIW) -- INTEGER array                               Workspace

15:  LIW -- INTEGER                                             Input
     On entry: the length of the array IW as declared in the
                                                    2
     calling (sub)program. Constraint: LIW>=MNP*(2N+1)+N +4N+2.

16:  IFAIL -- INTEGER                                 Input/Output
```

For this routine, the normal use of IFAIL is extended to control the printing of error and warning messages as well as specifying hard or soft failure (see the Essential Introduction).

Before entry, IFAIL must be set to a value with the decimal expansion cba, where each of the decimal digits c, b and a must have a value of 0 or 1.
a=0 specifies hard failure, otherwise soft failure;

b=0 suppresses error messages, otherwise error messages will be printed (see Section 6);

c=0 suppresses warning messages, otherwise warning messages will be printed (see Section 6).
The recommended value for inexperienced users is 110 (i.e., hard failure with all messages printed).

Unless the routine detects an error (see Section 6), IFAIL contains 0 on exit.

6. Error Indicators and Warnings

Errors detected by the routine:

If on entry IFAIL = 0 or -1, explanatory error messages are output on the current error message unit (as defined by X04AAF).

IFAIL= 1
One or more of the parameters N, TOL, NP, MNP, LW or LIW has been incorrectly set, or B <= A, or the condition (3) on X is not satisfied, or the number of known boundary values (specified by V) is not N.

IFAIL= 2
The Newton iteration has failed to converge. This could be due to there being too few points in the initial mesh or to the initial approximate solution being too inaccurate. If this latter reason is suspected the user should use subroutine D02RAF instead. If the warning 'Jacobian matrix is singular' is printed this could be due to specifying zero estimated boundary values and these should be varied. This warning could also be printed in the unlikely event of the Jacobian matrix being calculated inaccurately. If the user cannot make changes to prevent the warning then subroutine D02RAF should be used.

IFAIL= 3
The Newton iteration has reached round-off level. It could be, however, that the answer returned is satisfactory. This error might occur if too much accuracy is requested.

IFAIL= 4
A finer mesh is required for the accuracy requested; that is MNP is not large enough.

IFAIL= 5
A serious error has occurred in a call to D02GAF. Check all array subscripts and subroutine parameter lists in calls to D02GAF. Seek expert help.

7. Accuracy

The solution returned by the routine will be accurate to the user's tolerance as defined by the relation (2) except in extreme circumstances. If too many points are specified in the initial mesh, the solution may be more accurate than requested and the error may not be approximately equidistributed.

8. Further Comments

The time taken by the routine depends on the difficulty of the problem, the number of mesh points used (and the number of different meshes used), the number of Newton iterations and the number of deferred corrections.

The user is strongly recommended to set IFAIL to obtain self-explanatory error messages, and also monitoring information about the course of the computation. The user may select the channel numbers on which this output is to appear by calls of X04AAF (for error messages) or X04ABF (for monitoring information) - see Section 9 for an example. Otherwise the default channel numbers will be used, as specified in the implementation document.

A common cause of convergence problems in the Newton iteration is the user specifying too few points in the initial mesh. Although the routine adds points to the mesh to improve accuracy it is unable to do so until the solution on the initial mesh has been calculated in the Newton iteration.

If the user specifies zero known and estimated boundary values, the routine constructs a zero initial approximation and in many

cases the Jacobian is singular when evaluated for this approximation, leading to the breakdown of the Newton iteration.

The user may be unable to provide a sufficiently good choice of initial mesh and estimated boundary values, and hence the Newton iteration may never converge. In this case the continuation facility provided in D02RAF is recommended.

In the case where the user wishes to solve a sequence of similar problems, the final mesh from solving one case is strongly recommended as the initial mesh for the next.

9. Example

We solve the differential equation

$$y'''=-yy''-(\mathrm{beta})(1-y'^{2})$$

with boundary conditions

$$y(0)=y'(0)=0,$$

$$y'(10)=1$$

for (beta)=0.0 and (beta)=0.2 to an accuracy specified by TOL=1.0 E-3. We solve first the simpler problem with (beta)=0.0 using an equispaced mesh of 26 points and then we solve the problem with (beta)=0.2 using the final mesh from the first problem.

Note the call to X04ABF prior to the call to D02GAF.

The example program is not reproduced here. The source code for all example programs is distributed with the NAG Foundation Library software and should be available on-line.

%%%

D02GBF(3NAG) Foundation Library (12/10/92) D02GBF(3NAG)

D02 -- Ordinary Differential Equations D02GBF
D02GBF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for your implementation to check implementation-dependent details. The symbol (*) after a NAG routine name denotes a routine that is

not included in the Foundation Library.

1. Purpose

D02GBF solves a general linear two-point boundary value problem for a system of ordinary differential equations using a deferred correction technique.

2. Specification

```
      SUBROUTINE D02GBF (A, B, N, TOL, FCNF, FCNG, C, D, GAM,
     1                   MNP, X, Y, NP, W, LW, IW, LIW, IFAIL)
      INTEGER           N, MNP, NP, LW, IW(LIW), LIW, IFAIL
      DOUBLE PRECISION A, B, TOL, C(N,N), D(N,N), GAM(N), X(MNP),
     1                 Y(N,MNP), W(LW)
      EXTERNAL          FCNF, FCNG
```

3. Description

D02GBF solves the linear two-point boundary value problem for a system of n ordinary differential equations in the interval

[a,b]. The system is written in the form

$$y'=F(x)y+g(x) \qquad (1)$$

and the boundary conditions are written in the form

$$Cy(a)+Dy(b)=(gamma) \qquad (2)$$

Here F(x), C and D are n by n matrices, and g(x) and (gamma) are n-component vectors. The approximate solution to (1) and (2) is found using a finite-difference method with deferred correction. The algorithm is a specialisation of that used in subroutine D02RAF which solves a nonlinear version of (1) and (2). The nonlinear version of the algorithm is described fully in Pereyra [1].

The user supplies an absolute error tolerance and may also supply an initial mesh for the construction of the finite-difference equations (alternatively a default mesh is used). The algorithm constructs a solution on a mesh defined by adding points to the initial mesh. This solution is chosen so that the error is everywhere less than the user's tolerance and so that the error is approximately equidistributed on the final mesh. The solution is returned on this final mesh.

If the solution is required at a few specific points then these should be included in the initial mesh. If, on the other hand, the solution is required at several specific points, then the user should use the interpolation routines provided in Chapter E01 if these points do not themselves form a convenient mesh.

4. References

[1] Pereyra V (1979) PASVA3: An Adaptive Finite-Difference Fortran Program for First Order Nonlinear, Ordinary Boundary Problems. Codes for Boundary Value Problems in Ordinary Differential Equations. Lecture Notes in Computer Science. (ed B Childs, M Scott, J W Daniel, E Denman and P Nelson) 76 Springer-Verlag.

5. Parameters

1: A -- DOUBLE PRECISION Input
On entry: the left-hand boundary point, a.

2: B -- DOUBLE PRECISION Input
On entry: the right-hand boundary point, b. Constraint: B > A.

3: N -- INTEGER Input
On entry: the number of equations; that is n is the order of system (1). Constraint: N >= 2.

4: TOL -- DOUBLE PRECISION Input
On entry: a positive absolute error tolerance. If

$$a=x_1<x_2<\ldots<x_{NP}=b$$

is the final mesh, z(x) is the approximate solution from D02GBF and y(x) is the true solution of equations (1) and (2) then, except in extreme cases, it is expected that

$$||z-y||\leq TOL \qquad (3)$$

where

$$||u||=\max_{1\leq i\leq N}\ \max_{1\leq j\leq NP}|u_i(x_j)|.$$

Constraint: TOL > 0.0.

5: FCNF -- SUBROUTINE, supplied by the user.
External Procedure
FCNF must evaluate the matrix F(x) in (1) at a general point x.

```
    Its specification is:

           SUBROUTINE FCN (X, F)
           DOUBLE PRECISION X, F(n,n)
    where n is the actual value of N in the call of D02GBF.

     1:  X -- DOUBLE PRECISION                                Input
         On entry: the value of the independent variable x.

     2:  F(n,n) -- DOUBLE PRECISION array                     Output
         On exit: the (i,j)th element of the matrix F(x), for
         i,j=1,2,...,n. (See Section 9 for an example.)
    FCN must be declared as EXTERNAL in the (sub)program
    from which D02GBF is called. Parameters denoted as
    Input must not be changed by this procedure.

6:  FCNG -- SUBROUTINE, supplied by the user.
                                              External Procedure
    FCNG must evaluate the vector g(x) in (1) at a general point
    x.

    Its specification is:

           SUBROUTINE FCNG (X, G)
           DOUBLE PRECISION X, G(n)
    where n is the actual value of N in the call of D02GBF.

     1:  X -- DOUBLE PRECISION                                Input
         On entry: the value of the independent variable x.

     2:  G(*) -- DOUBLE PRECISION array                       Output
         On exit: the ith element of the vector g(x), for
         i=1,2,...,n. (See Section Section 9 for an example.)
    FCNG must be declared as EXTERNAL in the (sub)program
    from which D02GBF is called. Parameters denoted as
    Input must not be changed by this procedure.

7:  C(N,N) -- DOUBLE PRECISION array                  Input/Output

8:  D(N,N) -- DOUBLE PRECISION array                  Input/Output

9:  GAM(N) -- DOUBLE PRECISION array                  Input/Output
    On entry: the arrays C and D must be set to the matrices C
    and D in (2). GAM must be set to the vector (gamma) in (2).
    On exit: the rows of C and D and the components of GAM are
```

re-ordered so that the boundary conditions are in the order:
(i) conditions on y(a) only;

(ii) condition involving y(a) and y(b); and

(iii) conditions on y(b) only.

The routine will be slightly more efficient if the arrays C, D and GAM are ordered in this way before entry, and in this event they will be unchanged on exit.

Note that the problems (1) and (2) must be of boundary value type, that is neither C nor D may be identically zero. Note also that the rank of the matrix [C,D] must be n for the problem to be properly posed. Any violation of these conditions will lead to an error exit.

10: MNP -- INTEGER Input
On entry: the maximum permitted number of mesh points.
Constraint: MNP >= 32.

11: X(MNP) -- DOUBLE PRECISION array Input/Output
On entry: if NP >= 4 (see NP below), the first NP elements must define an initial mesh. Otherwise the elements of x need not be set. Constraint:
A=X(1)<X(2)<...<X(NP)=B, for NP>=4. (4)
On exit: X(1),X(2),...,X(NP) define the final mesh (with the returned value of NP) satisfying the relation (4).

12: Y(N,MNP) -- DOUBLE PRECISION array Output
On exit: the approximate solution z(x) satisfying (3), on the final mesh, that is
Y(j,i)=z (x) , i=1,2,...,NP;j=1,2,...,n
j i
where NP is the number of points in the final mesh.

The remaining columns of Y are not used.

13: NP -- INTEGER Input/Output
On entry: determines whether a default mesh or user-supplied mesh is used. If NP = 0, a default value of 4 for NP and a corresponding equispaced mesh X(1),X(2),...,X(NP) are used. If NP >= 4, then the user must define an initial mesh X as in (4) above. On exit: the number of points in the final (returned) mesh.

```
14:  W(LW) -- DOUBLE PRECISION array                        Workspace

15:  LW -- INTEGER                                              Input
     On entry: the length of the array W, Constraint:
                 2         2
     LW>=MNP*(3N +5N+2)+3N +5N.

16:  IW(LIW) -- INTEGER array                               Workspace

17:  LIW -- INTEGER                                             Input
     On entry: the length of the array IW. Constraint:
     LIW>=MNP*(2N+1)+N.

18:  IFAIL -- INTEGER                                    Input/Output
     For this routine, the normal use of IFAIL is extended to
     control the printing of error and warning messages as well
     as specifying hard or soft failure (see the Essential
     Introduction).

     Before entry, IFAIL must be set to a value with the decimal
     expansion cba, where each of the decimal digits c, b and a
     must have a value of 0 or 1.
     a=0   specifies hard failure, otherwise soft failure;

     b=0   suppresses error messages, otherwise error messages
           will be printed (see Section 6);

     c=0   suppresses warning messages, otherwise warning
           messages will be printed (see Section 6).
     The recommended value for inexperienced users is 110 (i.e.,
     hard failure with all messages printed).

     Unless the routine detects an error (see Section 6), IFAIL
     contains 0 on exit.

6. Error Indicators and Warnings

Errors detected by the routine:

For each error, an explanatory error message is output on the
current error message unit (as defined by X04AAF), unless
suppressed by the value of IFAIL on entry.

IFAIL= 1
     One or more of the parameters N, TOL, NP, MNP, LW or LIW is
     incorrectly set, B <= A or the condition (4) on X is not
```

```
     satisfied.

IFAIL= 2
     There are three possible reasons for this error exit to be
     taken:
       (i)  one of the matrices C or D is identically zero (that
            is the problem is of initial value and not boundary
            value type). In this case, IW(1) = 0 on exit;

       (ii) a row of C and the corresponding row of D are
            identically zero (that is the boundary conditions are
            rank deficient). In this case, on exit IW(1) contains
            the index of the first such row encountered; and

       (iii more than n of the columns of the n by 2n matrix [C,D
       )    ] are identically zero (that is the boundary
            conditions are rank deficient). In this case, on exit
            IW(1) contains minus the number of non-identically
            zero columns.

IFAIL= 3
     The routine has failed to find a solution to the specified
     accuracy. There are a variety of possible reasons including:
       (i)   the boundary conditions are rank deficient, which may
             be indicated by the message that the Jacobian is
             singular. However this is an unlikely explanation for
             the error exit as all rank deficient boundary
             conditions should lead instead to error exits with
             either IFAIL = 2 or IFAIL = 5; see also (iv) below;

       (ii)  not enough mesh points are permitted in order to
             attain the required accuracy. This is indicated by NP
             = MNP on return from a call to D02GBF. This
             difficulty may be aggravated by a poor initial choice
             of mesh points;

       (iii) the accuracy requested cannot be attained on the
             computer being used; and

       (iv)  an unlikely combination of values of F(x) has led to
             a singular Jacobian. The error should not persist if
             more mesh points are allowed.

IFAIL= 4
     A serious error has occurred in a call to D02GBF. Check all
     array subscripts and subroutine parameter lists in calls to
```

D02GBF. Seek expert help.

IFAIL= 5
There are two possible reasons for this error exit which occurs when checking the rank of the boundary conditions by reduction to a row echelon form:
(i) at least one row of the n by 2n matrix [C,D] is a linear combination of the other rows and hence the boundary conditions are rank deficient. The index of the first such row encountered is given by IW(1) on exit; and

(ii) as (i) but the rank deficiency implied by this error exit has only been determined up to a numerical tolerance. Minus the index of the first such row encountered is given by IW(1) on exit.
In case (ii) above there is some doubt as to the rank deficiency of the boundary conditions. However even if the boundary conditions are not rank deficient they are not posed in a suitable form for use with this routine.

For example, if

$$C=\begin{pmatrix}1 & 0\\ 1 & \text{(epsilon)}\end{pmatrix},\quad D=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix},\quad \text{(gamma)}=\begin{pmatrix}\text{((gamma)}_1\text{)}\\ \text{((gamma)}_2\text{)}\end{pmatrix}$$

and (epsilon) is small enough, this error exit is likely to be taken. A better form for the boundary conditions in this case would be

$$C=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\quad D=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix},\quad \text{(gamma)}=\begin{pmatrix}\text{((gamma)}_1\\ -1\\ \text{((epsilon)}_2\text{ ((gamma)}_1\text{ -(gamma)}_2\text{))}\end{pmatrix}$$

7. Accuracy

The solution returned by the routine will be accurate to the user's tolerance as defined by the relation (3) except in extreme circumstances. If too many points are specified in the initial mesh, the solution may be more accurate than requested and the error may not be approximately equidistributed.

8. Further Comments

The time taken by the routine depends on the difficulty of the

problem, the number of mesh points (and meshes) used and the number of deferred corrections.

The user is strongly recommended to set IFAIL to obtain self-explanatory error messages, and also monitoring information about the course of the computation. The user may select the channel numbers on which this output is to appear by calls of X04AAF (for error messages) or X04ABF (for monitoring information) - see Section 9 for an example. Otherwise the default channel numbers will be used, as specified in the implementation document.

In the case where the user wishes to solve a sequence of similar problems, the use of the final mesh from one case is strongly recommended as the initial mesh for the next.

9. Example

We solve the problem (written as a first order system)

(epsilon)y''+y'=0

with boundary conditions

y(0)=0, y(1)=1

for the cases (epsilon)=10^{-1} and (epsilon)=10^{-2} using the default initial mesh in the first case, and the final mesh of the first case as initial mesh for the second (more difficult) case. We give the solution and the error at each mesh point to illustrate the accuracy of the method given the accuracy request TOL=1.0E-3.

Note the call to X04ABF prior to the call to D02GBF.

The example program is not reproduced here. The source code for all example programs is distributed with the NAG Foundation Library software and should be available on-line.

%%%

D02 -- Ordinary Differential Equations D02KEF
D02KEF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for your implementation to check implementation-dependent details. The symbol (*) after a NAG routine name denotes a routine that is

not included in the Foundation Library.

1. Purpose

D02KEF finds a specified eigenvalue of a regular singular second-order Sturm-Liouville system on a finite or infinite range, using a Pruefer transformation and a shooting method. It also reports values of the eigenfunction and its derivatives. Provision is made for discontinuities in the coefficient functions or their derivatives.

2. Specification

```
      SUBROUTINE D02KEF (XPOINT, M, MATCH, COEFFN, BDYVAL, K,
     1                   TOL, ELAM, DELAM, HMAX, MAXIT, MAXFUN,
     2                   MONIT, REPORT, IFAIL)
      INTEGER          M, MATCH, K, MAXIT, MAXFUN, IFAIL
      DOUBLE PRECISION XPOINT(M), TOL, ELAM, DELAM, HMAX(2,M)
      EXTERNAL         COEFFN, BDYVAL, MONIT, REPORT
```

3. Description

D02KEF has essentially the same purpose as D02KDF(*) with minor modifications to enable values of the eigenfunction to be obtained after convergence to the eigenvalue has been achieved.

~~~~~~~~

It first finds a specified eigenvalue (lambda) of a Sturm-Liouville system defined by a self-adjoint differential equation of the second-order

(p(x)y')'+q(x;(lambda))y=0, a<x<b

together with the appropriate boundary conditions at the two (finite or infinite) end-points a and b. The functions p and q, which are real-valued, must be defined by a subroutine COEFFN. The boundary conditions must be defined by a subroutine BDYVAL, and, in the case of a singularity at a or b, take the form of an asymptotic formula for the solution near the relevant end-point.

~~~~~~~~

When the final estimate (lambda)=(lambda) of the eigenvalue has been found, the routine integrates the differential equation once more with that value of (lambda), and with initial conditions chosen so that the integral

```
                  b
                  /    2    ddq
              S= |y(x)  ----------(x;(lambda))dx
                  /      dd(lambda)
                  a
```

is approximately one. When q(x;(lambda)) is of the form (lambda)w(x)+q(x), which is the most common case, S represents the square of the norm of y induced by the inner product

```
                        b
                        /
                 <f,g>= |f(x)g(x)w(x)dx,
                        /
                        a
```

with respect to which the eigenfunctions are mutually orthogonal. This normalisation of y is only approximate, but experience shows that S generally differs from unity by only one or two per cent.

During this final integration the REPORT routine supplied by the user is called at each integration mesh point x. Sufficient information is returned to permit the user to compute y(x) and y'(x) for printing or plotting. For reasons described in Section 8.2, D02KEF passes across to REPORT, not y and y', but the Pru efer variables (beta), (phi) and (rho) on which the numerical method is based. Their relationship to y and y' is given by the equations

```
         ______   ( (rho))   ( (phi))
p(x)y'=\/(beta)exp( -----)cos( -----);
                  (   2  )   (   2  )

           1       ( (rho))   ( (phi))
     y= --------exp( -----)sin( -----).
          ______   (   2  )   (   2  )
        \/(beta)
```

For the theoretical basis of the numerical method to be valid, the following conditions should hold on the coefficient functions:

(a) p(x) must be non-zero and of one sign throughout the interval (a,b); and,

```
          ddq
(b)    ---------- must be of one sign throughout (a,b) for all
        dd(lambda)
      relevant values of (lambda), and must not be identically
      zero as x varies, for any (lambda).

Points of discontinuity in the functions p and q or their
derivatives are allowed, and should be included as 'break-points'
in the array XPOINT.

A good account of the theory of Sturm-Liouville systems, with
some description of Pruefer transformations, is given in Birkhoff
and Rota [4], Chapter X. An introduction for the user of Pruefer
transformations for the numerical solution of eigenvalue problems
arising from physics and chemistry is Bailey [2].

The scaled Pruefer method is fairly recent, and is described in a
short note by Pryce [6] and in some detail in the technical
report [5].

4. References
```


```
[1]   Abramowitz M and Stegun I A (1968) Handbook of Mathematical
      Functions. Dover Publications.

[2]   Bailey P B (1966) Sturm-Liouville Eigenvalues via a Phase
      Function. SIAM J. Appl. Math . 14 242--249.

[3]   Banks D O and Kurowski I (1968) Computation of Eigenvalues
      of Singular Sturm-Liouville Systems. Math. Computing. 22
      304--310.

[4]   Birkhoff G and Rota G C (1962) Ordinary Differential
      Equations. Ginn & Co., Boston and New York.

[5]   Pryce J D (1981) Two codes for Sturm-Liouville problems.
      Technical Report CS-81-01. Dept of Computer Science, Bristol
      University .

[6]   Pryce J D and Hargrave B A (1977) The Scale Pruefer Method
      for one-parameter and multi-parameter eigenvalue problems in
      ODEs. Inst. Math. Appl., Numerical Analysis Newsletter. 1(3)
```


```
5. Parameters

 1:  XPOINT(M) -- DOUBLE PRECISION array                    Input
```

```
On entry: the points where the boundary conditions computed
by BDYVAL are to be imposed, and also any break-points,
i.e., XPOINT(1) to XPOINT(m) must contain values x ,...,x
                                                  1      m
such that
                  x <=x <x <...<x   <=x
                   1   2  3      m-1   m
with the following meanings:
(a)   x  and x  are the left and right end-points, a and b,
       1      m
      of the domain of definition of the Sturm-Liouville
      system if these are finite. If either a or b is
      infinite, the corresponding value x  or x  may be a
                                        1     m
      more-or-less arbitrarily 'large' number of appropriate
      sign.

(b)   x  and x    are the Boundary Matching Points (BMP's),
       2      m-1
      that is the points at which the left and right
      boundary conditions computed in BDYVAL are imposed.

      If the left-hand end-point is a regular point then the
      user should set x =x (=a), while if it is a singular
                      2  1
      point the user must set x >x . Similarly x   =x (=b)
                               2  1             m-1  m
      if the right-hand end-point is regular, and x   <x  if
                                                   m-1  m
      it is singular.

(c)   The remaining m-4 points x ,...,x   , if any, define
                                3      m-2
      'break-points' which divide the interval [x ,x   ]
                                                 2  m-1
      into m-3 sub-intervals
                i =[x ,x ],...,i   =[x   ,x   ]
                 1   2  3       m-3   m-2  m-1
      Numerical integration of the differential equation is
      stopped and restarted at each break-point. In simple
      cases no break-points are needed. However if p(x) or
      q(x;(lambda)) are given by different formulae in
      different parts of the range, then integration is more
      efficient if the range is broken up by break-points in
      the appropriate way. Similarly points where any jumps
      occur in p(x) or q(x;(lambda)), or in their
```

derivatives up to the fifth order, should appear as break-points.
Constraint: X(1) <= X(2) < ... < X(M-1) <= X(M).

2: M -- INTEGER Input
On entry: the number of points in the array XPOINT.
Constraint: M >= 4.

3: MATCH -- INTEGER Input/Output
On entry: MATCH must be set to the index of the 'break-point ' to be used as the matching point (see Section 8.3). If MATCH is set to a value outside the range [2,m-1] then a default value is taken, corresponding to the break-point nearest the centre of the interval [XPOINT(2),XPOINT(m-1)].
On exit: the index of the break-point actually used as the matching point.

4: COEFFN -- SUBROUTINE, supplied by the user.
External Procedure
COEFFN must compute the values of the coefficient functions p(x) and q(x;(lambda)) for given values of x and (lambda). Section 3 states conditions which p and q must satisfy.

Its specification is:

```
SUBROUTINE COEFFN (P, Q, DQDL, X, ELAM, JINT)
DOUBLE PRECISION P, Q, DQDL, X, ELAM
INTEGER          JINT
```

1: P -- DOUBLE PRECISION Output
On exit: the value of p(x) for the current value of x.

2: Q -- DOUBLE PRECISION Output
On exit: the value of q(x;(lambda)) for the current value of x and the current trial value of (lambda).

3: DQDL -- DOUBLE PRECISION Output
On exit: the value of $\frac{\mathrm{ddq}}{\mathrm{dd(lambda)}}(x;(\mathrm{lambda}))$ for the current value of x and the current trial value of (lambda). However DQDL is only used in error estimation and an approximation (say to within 20%) will suffice.

4: X -- DOUBLE PRECISION Input
On entry: the current value of x.

5: ELAM -- DOUBLE PRECISION Input
On entry: the current trial value of the eigenvalue parameter (lambda).

6: JINT -- INTEGER Input
On entry: the index j of the sub-interval i_j (see specification of XPOINT) in which x lies.

See Section 8.4 and Section 9 for examples.

COEFFN must be declared as EXTERNAL in the (sub)program from which D02KEF is called. Parameters denoted as Input must not be changed by this procedure.

5: BDYVAL -- SUBROUTINE, supplied by the user.
External Procedure

BDYVAL must define the boundary conditions. For each end-point, BDYVAL must return (in YL or YR) values of y(x) and p(x)y'(x) which are consistent with the boundary conditions at the end-points; only the ratio of the values matters. Here x is a given point (XL or XR) equal to, or close to, the end-point.

For a regular end-point (a, say), x=a; and a boundary condition of the form

$$c_1\, y(a)+c_2\, y'(a)=0$$

can be handled by returning constant values in YL, e.g. $YL(1)=c_2$ and $YL(2)=-c_1$ p(a).

For a singular end-point however, YL(1) and YL(2) will in general be functions of XL and ELAM, and YR(1) and YR(2) functions of XR and ELAM, usually derived analytically from a power-series or asymptotic expansion. Examples are given in Section 8.5 Section 9.

Its specification is:

```
SUBROUTINE BDYVAL (XL, XR, ELAM, YL, YR))
DOUBLE PRECISION XL, XR, ELAM, YL(3), YR(3)
```

1: XL -- DOUBLE PRECISION Input
On entry: if a is a regular end-point of the system (so that a=x =x), then XL contains a. If a is a singular

```
          1  2
    point (so that a<=x <x ), then XL contains a point x
                       1  2
    such that x <x<=x ).
               1     2
```

2: XR -- DOUBLE PRECISION Input
On entry: if b is a regular end-point of the system (so that x =x =b), then XR contains b. If b is a singular point (so that x <x <=b), then XR contains a point x such that x <=x<x .

3: ELAM -- DOUBLE PRECISION Input
On entry: the current trial value of (lambda).

4: YL(3) -- DOUBLE PRECISION array Output
On exit: YL(1) and YL(2) should contain values of y(x) and p(x)y'(x) respectively (not both zero) which are consistent with the boundary condition at the left-hand end-point, given by x = XL. YL(3) should not be set.

5: YR(3) -- DOUBLE PRECISION array Output
On exit: YR(1) and YR(2) should contain values of y(x) and p(x)y'(x) respectively (not both zero) which are consistent with the boundary condition at the right-hand end-point, given by x = XR. YR(3) should not be set.

BDYVAL must be declared as EXTERNAL in the (sub)program from which D02KEF is called. Parameters denoted as Input must not be changed by this procedure.

6: K -- INTEGER Input
On entry: the index k of the required eigenvalue when the eigenvalues are ordered
(lambda) <(lambda) <(lambda) <...<(lambda) <....
0 1 2 k
Constraint: K >= 0.

7: TOL -- DOUBLE PRECISION Input
On entry: the tolerance parameter which determines the accuracy of the computed eigenvalue. The error estimate held in DELAM on exit satisfies the mixed absolute/relative error

test

DELAM<=TOL*max(1.0,|ELAM|) (*)

where ELAM is the final estimate of the eigenvalue. DELAM is usually somewhat smaller than the right-hand side of (*) but not several orders of magnitude smaller. Constraint: TOL > 0.0.

8: ELAM -- DOUBLE PRECISION Input/Output

On entry: an initial estimate of the eigenvalue (lambda). On exit: the final computed estimate, whether or not an error occurred.

9: DELAM -- DOUBLE PRECISION Input/Output

On entry: an indication of the scale of the problem in the (lambda)-direction. DELAM holds the initial 'search step' (positive or negative). Its value is not critical but the first two trial evaluations are made at ELAM and ELAM + DELAM, so the routine will work most efficiently if the eigenvalue lies between these values. A reasonable choice (if a closer bound is not known) is half the distance between adjacent eigenvalues in the neighbourhood of the one sought. In practice, there will often be a problem, similar to the one in hand but with known eigenvalues, which will help one to choose initial values for ELAM and DELAM.

If DELAM = 0.0 on entry, it is given the default value of 0.25*max(1.0,|ELAM|). On exit: with IFAIL = 0, DELAM holds an estimate of the absolute error in the computed

eigenvalue, that is |(lambda)-ELAM|~=DELAM. (In Section 8.2 we discuss the assumptions under which this is true.) The true error is rarely more than twice, or less than a tenth, of the estimated error.

With IFAIL /= 0, DELAM may hold an estimate of the error, or its initial value, depending on the value of IFAIL. See Section 6 for further details.

10: HMAX(2,M) -- DOUBLE PRECISION array Input/Output

On entry: HMAX(1,j) a maximum step size to be used by the differential equation code in the jth sub-interval i_j (as described in the specification of parameter XPOINT), for j=1,2,...,m-3. If it is zero the routine generates a maximum step size internally.

It is recommended that HMAX(1,j) be set to zero unless the coefficient functions p and q have features (such as a narrow peak) within the jth sub-interval that could be ' missed' if a long step were taken. In such a case HMAX(1,j) should be set to about half the distance over which the feature should be observed. Too small a value will increase the computing time for the routine. See Section 8 for further suggestions.

The rest of the array is used as workspace. On exit: HMAX(1,m-1) and HMAX(1,m) contain the sensitivity coefficients (sigma) ,(sigma) , described in Section 8.6. Other entries
l r
contain diagnostic output in case of an error (see Section 6).

11: MAXIT -- INTEGER Input/Output
On entry: a bound on n , the number of root-finding
r
iterations allowed, that is the number of trial values of (lambda) that are used. If MAXIT <= 0, no such bound is assumed. (See also under MAXFUN.) Suggested value: MAXIT = 0. On exit: MAXIT will have been decreased by the number of iterations actually performed, whether or not it was positive on entry.

12: MAXFUN -- INTEGER Input
On entry: a bound on n , the number of calls to COEFFN made
f
in any one root-finding iteration. If MAXFUN <= 0, no such bound is assumed. Suggested value: MAXFUN = 0.

MAXFUN and MAXIT may be used to limit the computational cost of a call to D02KEF, which is roughly proportional to n *n .
r f

13: MONIT -- SUBROUTINE, supplied by the user.
External Procedure
MONIT is called by D02KEF at the end of each root-finding iteration and allows the user to monitor the course of the computation by printing out the parameters (see Section 8 for an example).
If no monitoring is required, the dummy subroutine D02KAY may be used. (D02KAY is included in the NAG Foundation Library).

Its specification is:

```
      SUBROUTINE MONIT (MAXIT, IFLAG, ELAM, FINFO)
      INTEGER          MAXIT, IFLAG
      DOUBLE PRECISION ELAM, FINFO(15)
```

1: MAXIT -- INTEGER Input
On entry: the current value of the parameter MAXIT of D02KEF; this is decreased by one at each iteration.

2: IFLAG -- INTEGER Input
On entry: IFLAG describes what phase the computation is in, as follows:
IFLAG < 0
an error occurred in the computation of the 'miss-distance' at this iteration;

an error exit from D02KEF with IFAIL =-IFLAG will follow.

IFLAG = 1
the routine is trying to bracket the eigenvalue (lambda).

IFLAG = 2
the routine is converging to the eigenvalue (lambda) (having already bracketed it).

3: ELAM -- DOUBLE PRECISION Input
On entry: the current trial value of (lambda).

4: FINFO(15) -- DOUBLE PRECISION array Input
On entry: information about the behaviour of the shooting method, and diagnostic information in the case of errors. It should not normally be printed in full if no error has occurred (that is, if IFLAG > 0), though the first few components may be of interest to the user. In case of an error (IFLAG < 0) all the components of FINFO should be printed. The contents of FINFO are as follows:

FINFO(1): the current value of the 'miss-distance' or 'residual' function f((lambda)) on which the shooting

```
method is based. FINFO(1) is set to zero if IFLAG < 0.

FINFO(2): an estimate of the quantity dd(lambda)
defined as follows. Consider the perturbation in the
miss-distance f((lambda)) that would result if the
local error, in the solution of the differential
equation, were always positive and equal to its maximum
permitted value. Then dd(lambda) is the perturbation in
(lambda) that would have the same effect on f((lambda))
. Thus, at the zero of f((lambda)),|dd(lambda)| is an
approximate bound on the perturbation of the zero (that
is the eigenvalue) caused by errors in numerical
solution. If dd(lambda) is very large then it is
possible that there has been a programming error in
COEFFN such that q is independent of (lambda). If this
is the case, an error exit with IFAIL = 5 should
follow. FINFO(2) is set to zero if IFLAG < 0.

FINFO(3): the number of internal iterations, using the
same value of (lambda) and tighter accuracy tolerances,
needed to bring the accuracy (that is the value of
dd(lambda)) to an acceptable value. Its value should
normally be 1.0, and should almost never exceed 2.0.

FINFO(4): the number of calls to COEFFN at this
iteration.

FINFO(5): the number of successful steps taken by the
internal differential equation solver at this
iteration. A step is successful if it is used to
advance the integration (cf. COUT(8) in specification
of D02PAF(*)).

FINFO(6): the number of unsuccessful steps used by the
internal integrator at this iteration (cf. COUT(9) in
specification of D02PAF(*)).

FINFO(7): the number of successful steps at the maximum
step size taken by the internal integrator at this
iteration (cf. COUT(3) in specification of D02PAF(*)).

FINFO(8): is not used.

FINFO(9) to FINFO(15): set to zero, unless IFLAG < 0 in
which case they hold the following values describing
the point of failure:
```

FINFO(9): contains the index of the sub-interval where failure occurred, in the range 1 to m-3. In case of an error in BDYVAL, it is set to 0 or m-2 depending on whether the left or right boundary condition caused the error.

FINFO(10): the value f the independent variable x, the point at which error occurred. In case of an error in BDYVAL, it is set to the value of XL or XR as appropriate (see the specification of BDYVAL).

FINFO(11), FINFO(12), FINFO(13): the current values of the Pruefer dependent variables (beta), (phi) and (rho) respectively. These are set to zero in case of an error in BDYVAL.

FINFO(14): the local-error tolerance being used by the internal integrator at the point of failure. This is set to zero in the case of an error in BDYVAL.

FINFO(15): the last integration mesh point. This is set to zero in the case of an error in BDYVAL.

MONIT must be declared as EXTERNAL in the (sub)program from which D02KEF is called. Parameters denoted as Input must not be changed by this procedure.

14: REPORT -- SUBROUTINE, supplied by the user.
External Procedure

This routine provides the means by which the user may compute the eigenfunction y(x) and its derivative at each integration mesh point x. (See Section 8 for an example).

Its specification is:

```
SUBROUTINE REPORT (X, V, JINT)
INTEGER          JINT
DOUBLE PRECISION X, V(3)
```

1: X -- DOUBLE PRECISION Input
On entry: the current value of the independent variable x. See Section 8.3 for the order in which values of x are supplied.

2: V(3) -- DOUBLE PRECISION array Input
On entry: V(1), V(2), V(3) hold the current values of

```
          the Pruefer variables (beta), (phi), (rho)
          respectively.

      3:  JINT -- INTEGER                                    Input
          On entry: JINT indicates the sub-interval between
          break-points in which X lies exactly as for the routine
          COEFFN, except that at the extreme left end-point (when
          x = XPOINT(2)) JINT is set to 0 and at the extreme
          right end-point (when x=x =XPOINT(m-1)) JINT is set to
                                    r
          m-2.
     REPORT must be declared as EXTERNAL in the (sub)program
     from which D02KEF is called. Parameters denoted as
     Input must not be changed by this procedure.

15:  IFAIL -- INTEGER                                  Input/Output
     On entry: IFAIL must be set to 0, -1 or 1. For users not
     familiar with this parameter (described in the Essential
     Introduction) the recommended value is 0.

     On exit: IFAIL = 0 unless the routine detects an error (see
     Section 6).

6. Error Indicators and Warnings

Errors detected by the routine:

IFAIL= 1
     A parameter error. All parameters (except IFAIL) are left
     unchanged. The reason for the error is shown by the value of
     HMAX(2,1) as follows:
     HMAX(2,1) = 1: M < 4;

     HMAX(2,1) = 2: K < 0;

     HMAX(2,1) = 3: TOL <= 0.0;

     HMAX(2,1) = 4: XPOINT(1) to XPOINT(m) are not in ascending
                    order.

                    HMAX(2,2) gives the position i in XPOINT
                    where this was detected.

IFAIL= 2
     At some call to BDYVAL, invalid values were returned, that
     is, either YL(1) = YL(2) = 0.0, or YR(1) = YR(2) = 0.0 (a
```

programming error in BDYVAL). See the last call of MONIT for details.

This error exit will also occur if p(x) is zero at the point where the boundary condition is imposed. Probably BDYVAL was called with XL equal to a singular end-point a or with XR equal to a singular end-point b.

IFAIL= 3
At some point between XL and XR the value of p(x) computed by COEFFN became zero or changed sign. See the last call of MONIT for details.

IFAIL= 4
MAXIT > 0 on entry, and after MAXIT iterations the eigenvalue had not been found to the required accuracy.

IFAIL= 5
The 'bracketing' phase (with parameter IFLAG of MONIT equal to 1) failed to bracket the eigenvalue within ten iterations. This is caused by an error in formulating the problem (for example, q is independent of (lambda)), or by very poor initial estimates of ELAM, DELAM.

On exit ELAM and ELAM + DELAM give the end-points of the interval within which no eigenvalue was located by the routine.

IFAIL= 6
MAXFUN > 0 on entry, and the last iteration was terminated because more than MAXFUN calls to COEFFN were used. See the last call of MONIT for details.

IFAIL= 7
To obtain the desired accuracy the local error tolerance was set so small at the start of some sub-interval that the differential equation solver could not choose an initial step size large enough to make significant progress. See the last call of MONIT for diagnostics.

IFAIL= 8
At some point inside a sub-interval the step size in the differenital equation solver was reduced to a value too small to make significant progress (for the same reasons as with IFAIL = 7). This could be due to pathological behaviour of p(x) and q(x;(lambda)) or to an unreasonable accuracy

```
     requirement or to the current value of (lambda) making the
     equations 'stiff'. See the last call of MONIT for details.

IFAIL= 9
     TOL is too small for the problem being solved and the
     machine precision is being used. The final value of ELAM
     should be a very good approximation to the eigenvalue.

IFAIL= 10
     C05AZF(*), called by D02KEF, has terminated with the error
     exit corresponding to a pole of the residual function
     f((lambda)). This error exit should not occur, but if it
     does, try solving the problem again with a smaller value for
     TOL.

IFAIL= 11
     A serious error has occurred in an internal call to D02KDY.
     Check all subroutine calls and array dimensions. Seek expert
     help.

IFAIL= 12
     A serious error has occurred in an internal call to
     C05AZF(*). Check all subroutine calls and array dimensions.
     Seek expert help.

     HMAX(2,1) holds the failure exit number from the routine
     where the failure occurred. In the case of a failure in
     C05AZF(*), HMAX(2,2) holds the value of parameter IND of
     C05AZF(*).

Note: error exits with IFAIL = 2, 3, 6, 7, 8, 11 are caused by
being unable to set up or solve the differential equation at some
iteration, and will be immediately preceded by a call of MONIT
giving diagnostic information. For other errors, diagnostic
information is contained in HMAX(2,j), for j=1,2,...,m, where
appropriate.

7. Accuracy

See the discussion in Section 8.2.

8. Further Comments

8.1. Timing

The time taken by the routine depends on the complexity of the
```

coefficient functions, whether they or their derivatives are rapidly changing, the tolerance demanded, and how many iterations are needed to obtain convergence. The amount of work per iteration is roughly doubled when TOL is divided by 16. To make the most economical use of the routine, one should try to obtain good initial values for ELAM and DELAM, and, where appropriate, good asymptotic formulae. The boundary matching points should not be set unnecessarily close to singular points. The extra time needed to compute the eigenfunction is principally the cost of one additional integration once the eigenvalue has been found.

8.2. General Description of the Algorithm

A shooting method, for differential equation problems containing unknown parameters, relies on the construction of a 'miss-distance function', which for given trial values of the parameters measures how far the conditions of the problem are from being met. The problem is then reduced to one of finding the values of the parameters for which the miss-distance function is zero, that is to a root-finding process. Shooting methods differ mainly in how the miss-distance is defined.

This routine defines a miss-distance f((lambda)) based on the rotation around the origin of the point P(x)=(p(x)y'(x),y(x)) in the Phase Plane as the solution proceeds from a to b. The boundary-conditions define the ray (i.e., two-sided line through the origin) on which p(x) should start, and the ray on which it should finish. The eigenvalue index k defines the total number of half-turns it should make. Numerical solution is actually done by matching point x=c. Then f((lambda)) is taken as the angle between the rays to the two resulting points P (c) and P (c). A
                                                a   b
relative scaling of the py' and y axes, based on the behaviour of the coefficient functions p and q, is used to improve the numerical behaviour.

Please see figure in printed Reference Manual

The resulting function f((lambda)) is monotonic over -
                                                ddq
infty<(lambda)<infty, increasing if ----------->0 and decreasing
                                              dd(lambda)
       ddq
if ----------<0, with a unique zero at the desired eigenvalue

```
    dd(lambda)
~~~~~~~~
(lambda). The routine measures f((lambda)) in units of a half-
turn. This means that as (lambda) increases, f((lambda)) varies
by about 1 as each eigenvalue is passed. (This feature implies
that the values of f((lambda)) at successive iterations -
especially in the early stages of the iterative process - can be
used with suitable extrapolation or interpolation to help the
choice of initial estimates for eigenvalues near to the one
currently being found.)

The routine actually computes a value for f((lambda)) with
errors, arising from the local errors of the differential
equation code and from the asymptotic formulae provided by the
user if singular points are involved. However, the error estimate
output in DELAM is usually fairly realistic, in that the actual
       ~~~~~~~~
error |(lambda)-ELAM| is within an order of magnitude of DELAM.

We pass the values of (beta), (phi), (rho) across through REPORT
rather than converting them to values of y, y' inside D02KEF, for
the following reasons. First, there may be cases where auxiliary
quantities can be more accurately computed from the Pruefer
variables than from y and y'. Second, in singular problems on an
infinite interval y and y' may underflow towards the end of the
range, whereas the Pruefer variables remain well-behaved. Third,
with high-order eigenvalues (and therefore highly oscillatory
eigenfunctions) the eigenfunction may have a complete oscillation
(or more than one oscillation) between two mesh points, so that
values of y and y' at mesh points give a very poor representation
of the curve. The probable behaviour of the Pruefer variables in
this case is that (beta) and (rho) vary slowly whilst (phi)
increases quickly: for all three Pruefer variables linear
interpolation between the values at adjacent mesh points is
probably sufficiently accurate to yield acceptable intermediate
values of (beta), (phi), (rho) (and hence of y,y') for graphical
purposes.

Similar considerations apply to the exponentially decaying 'tails
Here (phi) has approximately constant value whilst (rho)
increases rapidly in the direction of integration, though the
step length is generally fairly small over such a range.

If the solution is output through REPORT at x-values which are
too widely spaced, the step length can be controlled by choosing
HMAX suitably, or, preferably, by reducing TOL. Both these
```

choices will lead to more accurate eigenvalues and eigenfunctions but at some computational cost.

8.3. The Position of the Shooting Matching Point c

This point is always one of the values x_i in array XPOINT. It may be specified using the parameter MATCH. The default value is chosen to be the value of that x_i, $2<=i<=m-1$, that lies closest to the middle of the interval $[x_2, x_{m-1}]$. If there is a tie, the rightmost candidate is chosen. In particular if there are no break-points then $c=x_{m-1}$ $(=x_3)$ - that is the shooting is from left to right in this case. A break-point may be inserted purely to move c to an interior point of the interval, even though the form of the equations does not require it. This often speeds up convergence especially with singular problems.

Note that the shooting method used by the code integrates first from the left-hand end x_l, then from the right-hand end x_r, to meet at the matching point c in the middle. This will of course be reflected in printed or graphical output. The diagram shows a possible sequence of nine mesh points $(tau)_1$ through $(tau)_9$ in the order in which they appear, assuming there are just two sub-intervals (so m=5).

Figure 1
Please see figure in printed Reference Manual

Since the shooting method usually fails to match up the two 'legs p(x)y' or both, at the matching point c. The code in fact 'shares large jump does not imply an inaccurate eigenvalue, but implies either

(a) a badly chosen matching point: if q(x;(lambda)) has a ' humped' shape, c should be chosen near the maximum value of q, especially if q is negative at the ends of the interval.

(b) An inherently ill-conditioned problem, typically one where another eigenvalue is pathologically close to the one being sought. In this case it is extremely difficult to obtain an

accurate eigenfunction.

In Section 9 below, we find the 11th eigenvalue and corresponding eigenfunction of the equation

$$y'' + ((\lambda) - x - 2/x^2)y = 0 \quad \text{on} \quad 0 < x < \infty$$

the boundary conditions being that y should remain bounded as x tends to 0 and x tends to infty. The coding of this problem is discussed in detail in Section 8.5.

The choice of matching point c is open. If we choose c=30.0 as in the D02KDF(*) example program we find that the exponentially increasing component of the solution dominates and we get extremely inaccurate values for the eigenfunction (though the eigenvalue is determined accurately). The values of the eigenfunction calculated with c=30.0 are given schematically in Figure 2.

Figure 2
Please see figure in printed Reference Manual

If we choose c as the maximum of the hump in q(x;(lambda)) (see (a) above) we instead obtain the accurate results given in Figure 3.

Figure 3
Please see figure in printed Reference Manual

8.4. Examples of Coding the COEFFN Routine

Coding COEFFN is straightforward except when break-points are needed. The examples below show:

(a) a simple case,

(b) a case in which discontinuities in the coefficient functions or their derivatives necessitate break-points, and

(c) a case where break-points together with the HMAX parameter are an efficient way to deal with a coefficient function that is well-behaved except over one short interval.

Example A

The modified Bessel equation

$$x(xy')'+((\lambda)x^2-(\nu)^2)y=0$$

Assuming the interval of solution does not contain the origin, dividing through by x, we have p(x)=x, $q(x;(\lambda))=(\lambda)x-(\nu)^2/x$. The code could be

```
      SUBROUTINE COEFFN(P,Q,DQDL,X,ELAM,JINT)
      P = X
      Q = ELAM*X + NU*NU/X
      DQDL = X
      RETURN
      END
```

where NU (standing for (nu)) is a real variable that might be defined in a DATA statement, or might be in user-declared COMMON so that its value could be set in the main program.

Example B

The Schroedinger equation

$$y''+((\lambda)+q(x))y=0$$

where $q(x)=\begin{cases} x^2-10 & (|x|\le 4), \\ 6/|x| & (|x|>4), \end{cases}$

over some interval 'approximating to (-infty,infty)', say [-20, 20]. Here we need break-points at +- 4, forming three sub-intervals $i_1=[-20,-4]$, $i_2=[-4,4]$, $i_3=[4,20]$. The code could be

```
      SUBROUTINE COEFFN(P,Q,DQDL,X,ELAM,JINT)
      IF (JINT.EQ.2) THEN
        Q = ELAM + X*X - 10.0E0
      ELSE
```

```
        Q = ELAM + 6.0E0/ABS(X)
      ENDIF
      P = 1.0E0
      DQDL = 1.0E0
      RETURN
      END
```

The array XPOINT would contain the values x , -20.0, -4.0, +4.0,
                                           1
+20.0, x and m would be 6. The choice of appropriate values for
        6
x and x depends on the form of the asymptotic formula computed
 1   6
by BDYVAL and the technique is discussed in the next subsection.

Example C

$$y''+(\lambda)(1-2e^{-100x^2})y=0, \text{ over } -10<=x<=10$$

Here q(x;(lambda)) is nearly constant over the range except for a sharp inverted spike over approximately -0.1<=x<=0.1. There is a danger that the routine will build up to a large step size and ' step over' the spike without noticing it. By using break-points - say at +- 0.5 - one can restrict the step size near the spike without impairing the efficiency elsewhere.

The code for COEFFN could be

```
      SUBROUTINE COEFFN(P,Q,DQDL,X,ELAM,JINT)
      P = 1.0E0
      DQDL = 1.0E0 - 2.0E0*EXP(-100.0E0*X*X)
      Q = ELAM*DQDL
      RETURN
      END
```

XPOINT might contain -0.0, -10.0, -0.5, 0.5, 10.0, 10.0 (assuming +- 10 are regular points) and m would be 6. HMAX(1,j), j=1,2,3 might contain 0.0, 0.1 and 0.0.

8.5. Examples of Boundary Conditions at Singular Points

Quoting from Bailey [2] page 243: 'Usually... the differential equation has two essentially different types of solution near a singular point, and the boundary condition there merely serves to distinguish one kind from the other. This is the case in all the standard examples of mathematical physics.'

In most cases the behaviour of the ratio p(x)y'/y near the point is quite different for the two types of solution. Essentially what the user provides through his BDYVAL routine is an approximation to this ratio, valid as x tends to the singular point (SP).

The user must decide (a) how accurate to make this approximation or asymptotic formula, for example how many terms of a series to use, and (b) where to place the boundary matching point (BMP) at which the numerical solution of the differential equation takes over from the asymptotic formula. Taking the BMP closer to the SP will generally improve the accuracy of the asymptotic formula, but will make the computation more expensive as the Pruefer differential equations generally become progressively more ill-behaved as the SP is approached. The user is strongly recommended to experiment with placing the BMPs. In many singular problems quite crude asymptotic formulae will do. To help the user avoid needlessly accurate formulae, D02KEF outputs two 'sensitivity coefficients' (sigma) ,(sigma) which estimate how much the
                    l   r
errors at the BMP's affect the computed eigenvalue. They are described in detail below, see Section 8.6.

Example of coding BDYVAL:

The example below illustrates typical situations:

$$y''+\left((\lambda)-x-\frac{2}{2}\right)y=0, \text{ for } 0<x<\infty$$

the boundary conditions being that y should remain bounded as x tends to 0 and x tends to infty.

At the end x=0 there is one solution that behaves like x^{2} and another that behaves like x^{-1}. For the first of these solutions

p(x)y'/y is asymptotically 2/x while for the second it is
asymptotically -1/x. Thus the desired ratio is specified by
setting

YL(1)=x and YL(2)=2.0.

At the end x=infty the equation behaves like Airy's equation
shifted through (lambda), i.e., like y''-ty=0 where t=x-(lambda),
so again there are two types of solution. The solution we require
behaves as

$$\exp\left(-\frac{2}{3}t^{3/2}\right)/\sqrt{4}t$$

and the other as

$$\exp\left(+\frac{2}{3}t^{3/2}\right)/\sqrt{4}t$$

once, the desired solution has $p(x)y'/y \sim -\sqrt{t}$ so that we could set
YR(1) = 1.0 and YR(2)=$-\sqrt{x-(\text{lambda})}$. The complete subroutine
might thus be

```
      SUBROUTINE BDYVAL(XL,XR,ELAM,YL,YR)
      real XL, XR, ELAM, YL(3), YR(3)
      YL(1) = XL
      YL(2) = 2.0E0
      YR(1) = 1.0E0
      YR(2) = -SQRT(XR - ELAM)
      RETURN
      END
```

Clearly for this problem it is essential that any value given by
D02KEF to XR is well to the right of the value of ELAM, so that
the user must vary the right-hand BMP with the eigenvalue index k
k
function Ai(x), so there is no problem in estimating ELAM.

More accurate asymptotic formulae are easily found - near x=0 by
the standard Frobenius method, and near x=infty by using standard

asymptotics for Ai(x), Ai'(x) (see [1], p. 448). For example, by the Frobenius method the solution near x=0 has the expansion

$$y = x^2 (c_0 + c_1 x + c_2 x^2 + \ldots)$$

with

$$c_0 = 1, c_1 = 0, c_2 = -\frac{(\lambda)}{10}, c_3 = \frac{1}{18}, \ldots, c_n = \frac{c_{n-3} - (\lambda) c_{n-2}}{n(n+3)}$$

This yields

$$\frac{p(x)y'}{y} = \frac{2 - \frac{2}{5}(\lambda) x^2 + \ldots}{x\left(1 - \frac{(\lambda)}{10} x^2 + \ldots\right)}.$$

8.6. The Sensitivity Parameters $(\sigma)_l$ and $(\sigma)_r$

The sensitivity parameters $(\sigma)_l$, $(\sigma)_r$ (held in HMAX(1,m-1) and HMAX(1,m) on output) estimate the effect of errors in the boundary conditions. For sufficiently small errors $(\Delta)y$, $(\Delta)py'$ in y and py' respectively, the relations

$$(\Delta)(\lambda)\tilde{}=(y.(\Delta)py'-py'.(\Delta)y)\ (\sigma)_l \qquad l \quad l$$

$$(\Delta)(\lambda)\tilde{}=(y.(\Delta)py'-py'.(\Delta)y)\ (\sigma)_r \qquad r \quad r$$

are satisfied where the subscripts l, r denote errors committed at left- and right-hand BMP's respectively, and $(\Delta)(\lambda)$ denotes the consequent error in the computed eigenvalue.

8.7. Missed Zeros

This is a pitfall to beware of at a singular point. If the BMP is

chosen so far from the SP that a zero of the desired
eigenfunction lies in between them, then the routine will fail to
number of zeros of its eigenfunction, the result will be that:

(a) The wrong eigenvalue will be computed for the given index k
- in fact some (lambda) will be found where k'>=1.
k+k'

(b) The same index k can cause convergence to any of several
eigenvalues depending on the initial values of ELAM and
DELAM.

It is up to the user to take suitable precautions - for instance
by varying the position of the BMP's in the light of his
knowledge of the asymptotic behaviour of the eigenfunction at
different eigenvalues.

9. Example

To find the 11th eigenvalue and eigenfunction of the example of
Section 8.5, using the simple asymptotic formulae for the
boundary conditions.

Comparison of the results from this example program with the
corresponding results from D02KDF(*) example program shows that
similar output is produced from the routine MONIT, followed by
the eigenfunction values from REPORT, and then a further line of
information from MONIT (corresponding to the integration to find
the eigenfunction). Final information is printed within the
example program exactly as with D02KDF(*).

3 _
Note the discrepancy at the matching point c(=\/4 , the maximum
of q(x;(lambda)), in this case) between the solutions obtained by
integrations from left and right end-points.

The example program is not reproduced here. The source code for
all example programs is distributed with the NAG Foundation
Library software and should be available on-line.

%%%

D02 -- Ordinary Differential Equations D02RAF
D02RAF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for

your implementation to check implementation-dependent details. The symbol (*) after a NAG routine name denotes a routine that is not included in the Foundation Library.

1. Purpose

D02RAF solves the two-point boundary-value problem with general boundary conditions for a system of ordinary differential equations, using a deferred correction technique and Newton iteration.

2. Specification

```
 SUBROUTINE D02RAF (N, MNP, NP, NUMBEG, NUMMIX, TOL, INIT,
1                   X, Y, IY, ABT, FCN, G, IJAC, JACOBF,
2                   JACOBG, DELEPS, JACEPS, JACGEP, WORK,
3                   LWORK, IWORK, LIWORK, IFAIL)
 INTEGER          N, MNP, NP, NUMBEG, NUMMIX, INIT, IY,
1                 IJAC, LWORK, IWORK(LIWORK), LIWORK, IFAIL
 DOUBLE PRECISION TOL, X(MNP), Y(IY,MNP), ABT(N), DELEPS,
1                 WORK(LWORK)
 EXTERNAL         FCN, G, JACOBF, JACOBG, JACEPS, JACGEP
```

3. Description

D02RAF solves a two-point boundary-value problem for a system of n ordinary differential equations in the interval (a,b) with b>a. The system is written in the form

$$y_i' = f_i(x, y_1, y_2, \ldots, y_n), \quad i = 1, 2, \ldots, n \tag{1}$$

and the derivatives f_i are evaluated by a subroutine FCN supplied by the user. With the differential equations (1) must be given a system of n (nonlinear) boundary conditions

$$g_i(y(a), y(b)) = 0, \quad i = 1, 2, \ldots, n$$

where

$$y(x) = [y_1(x), y_2(x), \ldots, y_n(x)]^T. \tag{2}$$

The functions g are evaluated by a subroutine G supplied by the
               i
user. The solution is computed using a finite-difference technique with deferred correction allied to a Newton iteration to solve the finite-difference equations. The technique used is described fully in Pereyra [1].

The user must supply an absolute error tolerance and may also supply an initial mesh for the finite-difference equations and an initial approximate solution (alternatively a default mesh and approximation are used). The approximate solution is corrected using Newton iteration and deferred correction. Then, additional points are added to the mesh and the solution is recomputed with the aim of making the error everywhere less than the user's tolerance and of approximately equidistributing the error on the final mesh. The solution is returned on this final mesh.

If the solution is required at a few specific points then these should be included in the initial mesh. If, on the other hand, the solution is required at several specific points then the user should use the interpolation routines provided in Chapter E01 if these points do not themselves form a convenient mesh.

The Newton iteration requires Jacobian matrices

$$\begin{pmatrix} \partial f_i \\ \hline \partial y_j \end{pmatrix}, \begin{pmatrix} \partial g_i \\ \hline \partial y(a)_j \end{pmatrix} \text{ and } \begin{pmatrix} \partial g_i \\ \hline \partial y(b)_j \end{pmatrix}.$$

These may be supplied by the user through subroutines JACOBF for
$\begin{pmatrix} \partial f_i \\ \hline \partial y_j \end{pmatrix}$
and JACOBG for the others. Alternatively the Jacobians
$\begin{pmatrix} \partial f_i \\ \hline \partial y_j \end{pmatrix}$
may be calculated by numerical differentiation using the algorithm described in Curtis et al [2].

For problems of the type (1) and (2) for which it is difficult to determine an initial approximation from which the Newton iteration will converge, a continuation facility is provided. The user must set up a family of problems

$$y' = f(x, y, (\text{epsilon})), \quad g(y(a), y(b), (\text{epsilon})) = 0, \qquad (3)$$

                    T
where f=[f ,f ,...,f] etc, and where (epsilon) is a
         1 2   n
continuation parameter. The choice (epsilon)=0 must give a
problem (3) which is easy to solve and (epsilon)=1 must define
the problem whose solution is actually required. The routine
solves a sequence of problems with (epsilon) values

        0=(epsilon) <(epsilon) <...<(epsilon) =1.   (4)
                   1   2   p

The number p and the values (epsilon) are chosen by the routine
                                    i
so that each problem can be solved using the solution of its
                                                        ddf
predecessor as a starting approximation. Jacobians -----------
                                                    dd(epsilon)
        ddg
and ----------- are required and they may be supplied by the
     dd(epsilon)
user via routines JACEPS and JACGEP respectively or may be
computed by numerical differentiation.

4. References

[1] Pereyra V (1979) PASVA3: An Adaptive Finite-Difference
     Fortran Program for First Order Nonlinear, Ordinary Boundary
     Problems. Codes for Boundary Value Problems in Ordinary
     Differential Equations. Lecture Notes in Computer Science.
     (ed B Childs, M Scott, J W Daniel, E Denman and P Nelson) 76
     Springer-Verlag.

[2] Curtis A R, Powell M J D and Reid J K (1974) On the
     Estimation of Sparse Jacobian Matrices. J. Inst. Maths
     Applics. 13 117--119.

5. Parameters

 1: N -- INTEGER   Input
     On entry: the number of differential equations, n.
     Constraint: N > 0.

 2: MNP -- INTEGER   Input
     On entry: MNP must be set to the maximum permitted number
     of points in the finite-difference mesh. If LWORK or LIWORK

(see below) is too small then internally MNP will be replaced by the maximum permitted by these values. (A warning message will be output if on entry IFAIL is set to obtain monitoring information.) Constraint: MNP >= 32.

3: NP -- INTEGER Input/Output
On entry: NP must be set to the number of points to be used in the initial mesh. Constraint: 4 <= NP <= MNP. On exit: the number of points in the final mesh.

4: NUMBEG -- INTEGER Input
On entry: the number of left-hand boundary conditions (that is the number involving y(a) only). Constraint: 0 <= NUMBEG < N.

5: NUMMIX -- INTEGER Input
On entry: the number of coupled boundary conditions (that is the number involving both y(a) and y(b)). Constraint: 0 <= NUMMIX <= N - NUMBEG.

6: TOL -- DOUBLE PRECISION Input
On entry: a positive absolute error tolerance. If
$$a=x_1<x_2<\ldots<x_{NP}=b$$
is the final mesh, $z_j(x_i)$ is the jth component of the approximate solution at x_i, and $y_j(x_i)$ is the jth component of the true solution of (1) and (2), then, except in extreme circumstances, it is expected that
$$|z_j(x_i)-y_j(x_i)|\leq TOL,\ i=1,2,\ldots,NP;\ j=1,2,\ldots,n. \qquad (5)$$
Constraint: TOL > 0.0.

7: INIT -- INTEGER Input
On entry: indicates whether the user wishes to supply an initial mesh and approximate solution (INIT /= 0) or whether default values are to be used, (INIT = 0).

8: X(MNP) -- DOUBLE PRECISION array Input/Output
On entry: the user must set X(1) = a and X(NP) = b. If INIT = 0 on entry a default equispaced mesh will be used, otherwise the user must specify a mesh by setting $X(i)=x_i$, for i = 2,3,...NP-1. Constraints:
X(1) < X(NP), if INIT = 0,

X(1) < X(2) <... < X(NP), if INIT /= 0.
On exit: X(1),X(2),...,X(NP) define the final mesh (with the returned value of NP) and X(1) = a and X(NP) = b.

9: Y(IY,MNP) -- DOUBLE PRECISION array Input/Output
On entry: if INIT = 0, then Y need not be set.

If INIT /= 0, then the array Y must contain an initial approximation to the solution such that Y(j,i) contains an approximation to

$$y_j(x_i),\ i=1,2,\ldots,NP;\ j=1,2,\ldots,n.$$

On exit: the approximate solution $z_j(x_i)$ satisfying (5) on the final mesh, that is

$$Y(j,i)=z_j(x_i),\ i=1,2,\ldots,NP;\ j=1,2,\ldots,n,$$

where NP is the number of points in the final mesh. If an error has occurred then Y contains the latest approximation to the solution. The remaining columns of Y are not used.

10: IY -- INTEGER Input
On entry:
the first dimension of the array Y as declared in the (sub)program from which D02RAF is called.
Constraint: IY >= N.

11: ABT(N) -- DOUBLE PRECISION array Output
On exit: ABT(i), for i=1,2,...,n, holds the largest estimated error (in magnitude) of the ith component of the solution over all mesh points.

12: FCN -- SUBROUTINE, supplied by the user.
External Procedure
FCN must evaluate the functions f_i (i.e., the derivatives y_i') at a general point x for a given value of (epsilon), the continuation parameter (see Section 3).

Its specification is:

```
SUBROUTINE FCN (X, EPS, Y, F, N)
INTEGER          N
DOUBLE PRECISION X, EPS, Y(N), F(N)
```

```
 1:  X -- DOUBLE PRECISION                                  Input
     On entry: the value of the argument x.

 2:  EPS -- DOUBLE PRECISION                                Input
     On entry: the value of the continuation parameter,
     (epsilon). This is 1 if continuation is not being used.

 3:  Y(N) -- DOUBLE PRECISION array                         Input
     On entry: the value of the argument y , for
                                          i
     i=1,2,...,n.

 4:  F(N) -- DOUBLE PRECISION array                        Output
     On exit: the values of f , for i=1,2,...,n.
                            i

 5:  N -- INTEGER                                           Input
     On entry: the number of equations.
FCN must be declared as EXTERNAL in the (sub)program
from which D02RAF is called. Parameters denoted as
Input must not be changed by this procedure.

13:  G -- SUBROUTINE, supplied by the user.
                                       External Procedure
G must evaluate the boundary conditions in equation (3) and
place them in the array BC.

Its specification is:

       SUBROUTINE G (EPS, YA, YB, BC, N)
       INTEGER          N
       DOUBLE PRECISION EPS, YA(N), YB(N), BC(N)

 1:  EPS -- DOUBLE PRECISION                                Input
     On entry: the value of the continuation parameter,
     (epsilon). This is 1 if continuation is not being used.

 2:  YA(N) -- DOUBLE PRECISION array                        Input
     On entry: the value y (a), for i=1,2,...,n.
                          i

 3:  YB(N) -- DOUBLE PRECISION array                        Input
     On entry: the value y (b), for i=1,2,...,n.
                          i
```

```
 4:  BC(N) -- DOUBLE PRECISION array                    Output
     On exit: the values g (y(a),y(b),(epsilon)), for
                           i
     i=1,2,...,n. These must be ordered as follows:
       (i)   first, the conditions involving only y(a) (see
             NUMBEG description above);

       (ii)  next, the NUMMIX coupled conditions involving
             both y(a) and y(b) (see NUMMIX description
             above); and,

       (iii) finally, the conditions involving only y(b) (N-
             NUMBEG-NUMMIX).

 5:  N -- INTEGER                                       Input
     On entry: the number of equations, n.
G must be declared as EXTERNAL in the (sub)program from
which D02RAF is called. Parameters denoted as Input
must not be changed by this procedure.

14:  IJAC -- INTEGER                                    Input
     On entry: indicates whether or not the user is supplying
     Jacobian evaluation routines. If IJAC /= 0 then the user
     must supply routines JACOBF and JACOBG and also, when
     continuation is used, routines JACEPS and JACGEP. If IJAC =
     0 numerical differentiation is used to calculate the
     Jacobian and the routines D02GAZ, D02GAY, D02GAZ and D02GAX
                                      ( ddf )
                                      (    i)
     JACOBF must evaluate the Jacobian ( ----) for i,j=1,2,...,n,
                                      ( ddy )
                                      (    j)
     given x and y , for j=1,2,...,n.
                 j

     Its specification is:

       SUBROUTINE JACOBF (X, EPS, Y, F, N)
       INTEGER          N
       DOUBLE PRECISION X, EPS, Y(N), F(N,N)

 1:  X -- DOUBLE PRECISION                              Input
     On entry: the value of the argument x.

 2:  EPS -- DOUBLE PRECISION                            Input
     On entry: the value of the continuation parameter
```

```
     (epsilon). This is 1 if continuation is not being used.

  3: Y(N) -- DOUBLE PRECISION array                    Input
     On entry: the value of the argument y , for
                                          i
     i=1,2,...,n.

  4: F(N,N) -- DOUBLE PRECISION array                  Output
                                                      ddf
                                                        i
     On exit: F(i,j) must be set to the value of      ----,
                                                      ddy
                                                        j
     evaluated at the point (x,y), for i,j=1,2,...,n.

  5: N -- INTEGER                                      Input
     On entry: the number of equations, n.
JACOBF must be declared as EXTERNAL in the (sub)program
from which D02RAF is called. Parameters denoted as
Input must not be changed by this procedure.

16: JACOBG -- SUBROUTINE, supplied by the user.
                                          External Procedure
                                    (  ddg   )     (  ddg   )
                                    (     i  )     (     i  )
JACOBG must evaluate the Jacobians  ( -------) and ( -------)
                                    ( ddy (a))     ( ddy (b))
                                    (    j   )     (    j   )
The ordering of the rows of AJ and BJ must correspond to
the ordering of the boundary conditions described in the
specification of subroutine G above.

Its specification is:

       SUBROUTINE JACOBG (EPS, YA, YB, AJ, BJ, N)
       INTEGER          N
       DOUBLE PRECISION EPS, YA(N), YB(N), AJ(N,N), BJ
      1                 (N,N)

  1: EPS -- DOUBLE PRECISION                           Input
     On entry: the value of the continuation parameter,
     (epsilon). This is 1 if continuation is not being used.

  2: YA(N) -- DOUBLE PRECISION array                   Input
     On entry: the value y (a), for i=1,2,...,n.
                          i
```

```
 3:  YB(N) -- DOUBLE PRECISION array                    Input
     On entry: the value y (b), for i=1,2,...,n.
                          i

 4:  AJ(N,N) -- DOUBLE PRECISION array                  Output
                                                      ddg
                                                         i
     On exit: AJ(i,j) must be set to the value        -------,
                                                     ddy (a)
                                                        j
     for i,j=1,2,...,n.

 5:  BJ(N,N) -- DOUBLE PRECISION array                  Output
                                                      ddg
                                                         i
     On exit: BJ(i,j) must be set to the value        -------,
                                                     ddy (b)
                                                        j
     for i,j=1,2...,n.

 6:  N -- INTEGER                                       Input
     On entry: the number of equations, n.
JACOBG must be declared as EXTERNAL in the (sub)program
from which D02RAF is called. Parameters denoted as
Input must not be changed by this procedure.

17:  DELEPS -- DOUBLE PRECISION                      Input/Output
     On entry: DELEPS must be given a value which specifies
     whether continuation is required. If DELEPS <= 0.0 or DELEPS
     >= 1.0 then it is assumed that continuation is not required.
     If 0.0 < DELEPS < 1.0 then it is assumed that continuation
     is required unless DELEPS < square root of machine precision
     when an error exit is taken. DELEPS is used as the increment
     (epsilon) -(epsilon)  (see (4)) and the choice DELEPS = 0.1
              2          1
     is recommended. On exit: an overestimate of the increment
     (epsilon) -(epsilon)    (in fact the value of the increment
              p          p-1
     which would have been tried if the restriction (epsilon) =1
                                                              p
     had not been imposed). If continuation was not requested
     then DELEPS = 0.0.

     If continuation is not requested then the parameters JACEPS
     and JACGEP may be replaced by dummy actual parameters in the
```

call to D02RAF. (D02GAZ and D02GAX respectively may be used as the dummy parameters.)

18: JACEPS -- SUBROUTINE, supplied by the user.
External Procedure

JACEPS must evaluate the derivative $\frac{\partial f_i}{\partial \text{dd}(\text{epsilon})}$ given x and y if continuation is being used.

Its specification is:

```
SUBROUTINE JACEPS (X, EPS, Y, F, N)
INTEGER          N
DOUBLE PRECISION X, EPS, Y(N), F(N)
```

1: X -- DOUBLE PRECISION Input
On entry: the value of the argument x.

2: EPS -- DOUBLE PRECISION Input
On entry: the value of the continuation parameter, (epsilon).

3: Y(N) -- DOUBLE PRECISION array Input
On entry: the solution values y_i at the point x, for i=1,2,...,n.

4: F(N) -- DOUBLE PRECISION array Output

On exit: F(i) must contain the value $\frac{\partial f_i}{\partial \text{dd}(\text{epsilon})}$ at the point (x,y), for i=1,2,...,n.

5: N -- INTEGER Input
On entry: the number of equations, n.

JACEPS must be declared as EXTERNAL in the (sub)program from which D02RAF is called. Parameters denoted as Input must not be changed by this procedure.

19: JACGEP -- SUBROUTINE, supplied by the user.
External Procedure

JACGEP must evaluate the derivatives ---------- if dd(epsilon) continuation is being used.

Its specification is:

```
SUBROUTINE JACGEP (EPS, YA, YB, BCEP, N)
INTEGER          N
DOUBLE PRECISION EPS, YA(N), YB(N), BCEP(N)
```

1: EPS -- DOUBLE PRECISION Input
On entry: the value of the continuation parameter, (epsilon).

2: YA(N) -- DOUBLE PRECISION array Input
On entry: the value of $y_i(a)$, for i=1,2,...,n.

3: YB(N) -- DOUBLE PRECISION array Input
On entry: the value of $y_i(b)$, for i=1,2,...,n.

4: BCEP(N) -- DOUBLE PRECISION array Output
On exit: BCEP(i) must contain the value of $\frac{dg_i}{d(\epsilon)}$, for i=1,2,...,n.

5: N -- INTEGER Input
On entry: the number of equations, n.

JACGEP must be declared as EXTERNAL in the (sub)program from which D02RAF is called. Parameters denoted as Input must not be changed by this procedure.

20: WORK(LWORK) -- DOUBLE PRECISION array Workspace

21: LWORK -- INTEGER Input
On entry:
the dimension of the array WORK as declared in the (sub)program from which D02RAF is called.
Constraint: LWORK>=MNP*(3N^2+6N+2)+4N^2+3N.

22: IWORK(LIWORK) -- INTEGER array Workspace

23: LIWORK -- INTEGER Input
On entry:
the dimension of the array IWORK as declared in the (sub)program from which D02RAF is called.
Constraints:
LIWORK>=MNP*(2*N+1)+N, if IJAC /= 0,

$$\text{LIWORK} \geq \text{MNP}*(2*\text{N}+1)+\text{N} + 4*\text{N}+2, \text{ if IJAC} = 0.$$

24: IFAIL -- INTEGER Input/Output
For this routine, the normal use of IFAIL is extended to control the printing of error and warning messages as well as specifying hard or soft failure (see the Essential Introduction).

Before entry, IFAIL must be set to a value with the decimal expansion cba, where each of the decimal digits c, b and a must have a value of 0 or 1.
a=0 specifies hard failure, otherwise soft failure;

b=0 suppresses error messages, otherwise error messages will be printed (see Section 6);

c=0 suppresses warning messages, otherwise warning messages will be printed (see Section 6).
The recommended value for inexperienced users is 110 (i.e., hard failure with all messages printed).

Unless the routine detects an error (see Section 6), IFAIL contains 0 on exit.

6. Error Indicators and Warnings

Errors detected by the routine:

For each error, an explanatory error message is output on the current error message unit (as defined by X04AAF), unless suppressed by the value of IFAIL on entry.

IFAIL= 1
One or more of the parameters N, MNP, NP, NUMBEG, NUMMIX, TOL, DELEPS, LWORK or LIWORK has been incorrectly set, or X(1) >= X(NP) or the mesh points X(i) are not in strictly ascending order.

IFAIL= 2
A finer mesh is required for the accuracy requested; that is MNP is not large enough. This error exit normally occurs when the problem being solved is difficult (for example, there is a boundary layer) and high accuracy is requested. A poor initial choice of mesh points will make this error exit more likely.

IFAIL= 3
The Newton iteration has failed to converge. There are several possible causes for this error:
(i) faulty coding in one of the Jacobian calculation routines;

(ii) if IJAC = 0 then inaccurate Jacobians may have been calculated numerically (this is a very unlikely cause); or,

(iii) a poor initial mesh or initial approximate solution has been selected either by the user or by default or there are not enough points in the initial mesh. Possibly, the user should try the continuation facility.

IFAIL= 4
The Newton iteration has reached round-off error level. It could be however that the answer returned is satisfactory. The error is likely to occur if too high an accuracy is requested.

IFAIL= 5
The Jacobian calculated by JACOBG (or the equivalent matrix calculated by numerical differentiation) is singular. This may occur due to faulty coding of JACOBG or, in some circumstances, to a zero initial choice of approximate solution (such as is chosen when INIT = 0).

IFAIL= 6
There is no dependence on (epsilon) when continuation is being used. This can be due to faulty coding of JACEPS or JACGEP or, in some circumstances, to a zero initial choice of approximate solution (such as is chosen when INIT = 0).

IFAIL= 7
DELEPS is required to be less than machine precision for continuation to proceed. It is likely that either the

```
     problem (3) has no solution for some value near the current
     value of (epsilon) (see the advisory print out from D02RAF)
     or that the problem is so difficult that even with
     continuation it is unlikely to be solved using this routine.
     If the latter cause is suspected then using more mesh points
     initially may help.

IFAIL= 8
     Indicates that a serious error has occurred in a call to
     D02RAF. Check all array subscripts and subroutine parameter
     lists in calls to D02RAF. Seek expert help.

IFAIL= 9
     Indicates that a serious error has occurred in a call to
     D02RAR. Check all array subscripts and subroutine parameter
     lists in calls to D02RAF. Seek expert help.

7. Accuracy

The solution returned by the routine will be accurate to the
user's tolerance as defined by the relation (5) except in extreme
circumstances. The final error estimate over the whole mesh for
each component is given in the array ABT. If too many points are
specified in the initial mesh, the solution may be more accurate
than requested and the error may not be approximately
equidistributed.

8. Further Comments

There are too many factors present to quantify the timing. The
time taken by the routine is negligible only on very simple
problems.

The user is strongly recommended to set IFAIL to obtain self-
explanatory error messages, and also monitoring information about
the course of the computation.

In the case where the user wishes to solve a sequence of similar
problems, the use of the final mesh and solution from one case as
the initial mesh is strongly recommended for the next.

9. Example

We solve the differential equation
```

```
                    y'''=-yy''-2(epsilon)(1-y' )

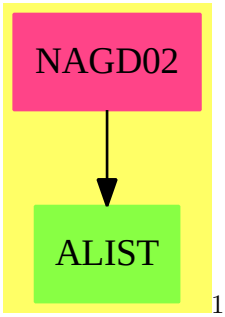
with (epsilon)=1 and boundary conditions

                       y(0)=y'(0)=0,  y'(10)=1

to an accuracy specified by TOL=1.0E-4. The continuation facility
is used with the continuation parameter (epsilon) introduced as
in the differential equation above and with DELEPS = 0.1
initially. (The continuation facility is not needed for this
problem and is used here for illustration.)

The example program is not reproduced here. The source code for
all example programs is distributed with the NAG Foundation
Library software and should be available on-line.
```

15.20 NagOrdinaryDifferentialEquationsPackage



1.00

Exports:

d02bbf	d02bhf	d02cjf	d02ejf	d02gaf
d02gbf	d02kef	d02kef	d02raf	

⟨*package NAGD02 NagOrdinaryDifferentialEquationsPackage*⟩≡

```
)abbrev package NAGD02 NagOrdinaryDifferentialEquationsPackage
++ Author: Godfrey Nolan and Mike Dewar
++ Date Created: Jan 1994
++ Date Last Updated: Mon Jun 20 17:56:33 1994
++ Description:
++ This package uses the NAG Library to calculate the numerical solution of
++ ordinary differential equations. There are two main types of problem,
++ those in which all boundary conditions are specified at one point
++ (initial-value problems), and those in which the boundary
++ conditions are distributed between two or more points (boundary-
++ value problems and eigenvalue problems). Routines are available
++ for initial-value problems, two-point boundary-value problems and
++ Sturm-Liouville eigenvalue problems.
++ See \downlink{Manual Page}{manpageXXd02}.
NagOrdinaryDifferentialEquationsPackage(): Exports == Implementation where
  S ==> Symbol
  FOP ==> FortranOutputStackPackage

  Exports ==> with
    d02bbf : (DoubleFloat,Integer,Integer,Integer,_
        DoubleFloat,Matrix DoubleFloat,DoubleFloat,Integer,Union(fn:FileName,fp:Asp7(FCN)),U
     ++ d02bbf(xend,m,n,irelab,x,y,tol,ifail,fcn,output)
     ++ integrates a system of first-order ordinary differential
     ++ equations over an interval with suitable initial conditions,
     ++ using a Runge-Kutta-Merson method, and returns the solution at
     ++ points specified by the user.
     ++ See \downlink{Manual Page}{manpageXXd02bbf}.
    d02bhf : (DoubleFloat,Integer,Integer,DoubleFloat,_
        DoubleFloat,Matrix DoubleFloat,DoubleFloat,Integer,Union(fn:FileName,fp:Asp9(G)),Uni
```

```
 ++ d02bhf(xend,n,irelab,hmax,x,y,tol,ifail,g,fcn)
 ++ integrates a system of first-order ordinary differential
 ++ equations over an interval with suitable initial conditions,
 ++ using a Runge-Kutta-Merson method, until a user-specified
 ++ function of the solution is zero.
 ++ See \downlink{Manual Page}{manpageXXd02bhf}.
d02cjf : (DoubleFloat,Integer,Integer,DoubleFloat,_
    String,DoubleFloat,Matrix DoubleFloat,Integer,Union(fn:FileName,fp:Asp9(G
 ++ d02cjf(xend,m,n,tol,relabs,x,y,ifail,g,fcn,output)
 ++ integrates a system of first-order ordinary differential
 ++ equations over a range with suitable initial conditions, using a
 ++ variable-order, variable-step Adams method until a user-specified
 ++ function, if supplied, of the solution is zero, and returns the
 ++ solution at points specified by the user, if desired.
 ++ See \downlink{Manual Page}{manpageXXd02cjf}.
d02ejf : (DoubleFloat,Integer,Integer,String,_
    Integer,DoubleFloat,Matrix DoubleFloat,DoubleFloat,Integer,Union(fn:FileN
 ++ d02ejf(xend,m,n,relabs,iw,x,y,tol,ifail,g,fcn,pederv,output)
 ++ integrates a stiff system of first-order ordinary
 ++ differential equations over an interval with suitable initial
 ++ conditions, using a variable-order, variable-step method
 ++ implementing the Backward Differentiation Formulae (BDF), until a
 ++ user-specified function, if supplied, of the solution is zero,
 ++ and returns the solution at points specified by the user, if
 ++ desired.
 ++ See \downlink{Manual Page}{manpageXXd02ejf}.
d02gaf : (Matrix DoubleFloat,Matrix DoubleFloat,Integer,DoubleFloat,_
    DoubleFloat,DoubleFloat,Integer,Integer,Integer,Matrix DoubleFloat,Intege
 ++ d02gaf(u,v,n,a,b,tol,mnp,lw,liw,x,np,ifail,fcn)
 ++ solves the two-point boundary-value problem with assigned
 ++ boundary values for a system of ordinary differential equations,
 ++ using a deferred correction technique and a Newton iteration.
 ++ See \downlink{Manual Page}{manpageXXd02gaf}.
d02gbf : (DoubleFloat,DoubleFloat,Integer,DoubleFloat,_
    Integer,Integer,Integer,Matrix DoubleFloat,Matrix DoubleFloat,Matrix Doub
 ++ d02gbf(a,b,n,tol,mnp,lw,liw,c,d,gam,x,np,ifail,fcnf,fcng)
 ++ solves a general linear two-point boundary value problem
 ++ for a system of ordinary differential equations using a deferred
 ++ correction technique.
 ++ See \downlink{Manual Page}{manpageXXd02gbf}.
d02kef : (Matrix DoubleFloat,Integer,Integer,DoubleFloat,_
    Integer,Integer,DoubleFloat,DoubleFloat,Matrix DoubleFloat,Integer,Intege
 ++ d02kef(xpoint,m,k,tol,maxfun,match,elam,delam,hmax,maxit,ifail,coeffn,bdy
 ++ finds a specified eigenvalue of a regular singular second-
 ++ order Sturm-Liouville system on a finite or infinite range, using
 ++ a Pruefer transformation and a shooting method. It also reports
```

```
    ++ values of the eigenfunction and its derivatives. Provision is
    ++ made for discontinuities in the coefficient functions or their
    ++ derivatives.
    ++ See \downlink{Manual Page}{manpageXXd02kef}.
    ++ ASP domains Asp12 and Asp33 are used to supply default
    ++ subroutines for the MONIT and REPORT arguments via their \axiomOp{outputAsFortran} c
   d02kef : (Matrix DoubleFloat,Integer,Integer,DoubleFloat,_
       Integer,Integer,DoubleFloat,DoubleFloat,Matrix DoubleFloat,Integer,Integer,Union(fn
    ++ d02kef(xpoint,m,k,tol,maxfun,match,elam,delam,hmax,maxit,ifail,coeffn,bdyval,monit,r
    ++ finds a specified eigenvalue of a regular singular second-
    ++ order Sturm-Liouville system on a finite or infinite range, using
    ++ a Pruefer transformation and a shooting method. It also reports
    ++ values of the eigenfunction and its derivatives. Provision is
    ++ made for discontinuities in the coefficient functions or their
    ++ derivatives.
    ++ See \downlink{Manual Page}{manpageXXd02kef}.
    ++ Files \spad{monit} and \spad{report} will be used to define the subroutines for the
    ++ MONIT and REPORT arguments.
    ++ See \downlink{Manual Page}{manpageXXd02gbf}.
   d02raf : (Integer,Integer,Integer,Integer,_
       DoubleFloat,Integer,Integer,Integer,Integer,Integer,Integer,Matrix DoubleFloat,Matri
    ++ d02raf(n,mnp,numbeg,nummix,tol,init,iy,ijac,lwork,liwork,np,x,y,deleps,ifail,fcn,g)
    ++ solves the two-point boundary-value problem with general
    ++ boundary conditions for a system of ordinary differential
    ++ equations, using a deferred correction technique and Newton
    ++ iteration.
    ++ See \downlink{Manual Page}{manpageXXd02raf}.
  Implementation ==> add

   import Lisp
   import DoubleFloat
   import Any
   import Record
   import Integer
   import Matrix DoubleFloat
   import Boolean
   import NAGLinkSupportPackage
   import FortranPackage
   import Union(fn:FileName,fp:Asp7(FCN))
   import Union(fn:FileName,fp:Asp8(OUTPUT))
   import AnyFunctions1(DoubleFloat)
   import AnyFunctions1(Integer)
   import AnyFunctions1(String)
   import AnyFunctions1(Matrix DoubleFloat)
```

```
d02bbf(xendArg:DoubleFloat,mArg:Integer,nArg:Integer,_
    irelabArg:Integer,xArg:DoubleFloat,yArg:Matrix DoubleFloat,_
    tolArg:DoubleFloat,ifailArg:Integer,fcnArg:Union(fn:FileName,fp:Asp7(FCN)
    outputArg:Union(fn:FileName,fp:Asp8(OUTPUT))): Result ==
    pushFortranOutputStack(fcnFilename := aspFilename "fcn")$FOP
    if fcnArg case fn
              then outputAsFortran(fcnArg.fn)
              else outputAsFortran(fcnArg.fp)
    popFortranOutputStack()$FOP
    pushFortranOutputStack(outputFilename := aspFilename "output")$FOP
    if outputArg case fn
              then outputAsFortran(outputArg.fn)
              else outputAsFortran(outputArg.fp)
    popFortranOutputStack()$FOP
    [(invokeNagman([fcnFilename, outputFilename]$Lisp,_
    "d02bbf",_
    ["xend"::S,"m"::S,"n"::S,"irelab"::S,"x"::S_
    ,"tol"::S,"ifail"::S,"fcn"::S,"output"::S,"result"::S,"y"::S,"w"::S]$Lisp
    ["result"::S,"w"::S,"fcn"::S,"output"::S]$Lisp,_
    [["double"::S,"xend"::S,["result"::S,"m"::S,"n"::S]$Lisp_
    ,"x"::S,["y"::S,"n"::S]$Lisp,"tol"::S,["w"::S,"n"::S,7$Lisp]$Lisp,"fcn"::
    ,["integer"::S,"m"::S,"n"::S,"irelab"::S,"ifail"::S_
    ]$Lisp_
    ]$Lisp,_
    ["result"::S,"x"::S,"y"::S,"tol"::S,"ifail"::S]$Lisp,_
    [([xendArg::Any,mArg::Any,nArg::Any,irelabArg::Any,xArg::Any,tolArg::Any,
    @List Any]$Lisp)$Lisp)_
    pretend List (Record(key:Symbol,entry:Any))]$Result

d02bhf(xendArg:DoubleFloat,nArg:Integer,irelabArg:Integer,_
    hmaxArg:DoubleFloat,xArg:DoubleFloat,yArg:Matrix DoubleFloat,_
    tolArg:DoubleFloat,ifailArg:Integer,gArg:Union(fn:FileName,fp:Asp9(G)),_
    fcnArg:Union(fn:FileName,fp:Asp7(FCN))): Result ==
    pushFortranOutputStack(gFilename := aspFilename "g")$FOP
    if gArg case fn
              then outputAsFortran(gArg.fn)
              else outputAsFortran(gArg.fp)
    popFortranOutputStack()$FOP
    pushFortranOutputStack(fcnFilename := aspFilename "fcn")$FOP
    if fcnArg case fn
              then outputAsFortran(fcnArg.fn)
              else outputAsFortran(fcnArg.fp)
    popFortranOutputStack()$FOP
    [(invokeNagman([gFilename,fcnFilename]$Lisp,_
    "d02bhf",_
    ["xend"::S,"n"::S,"irelab"::S,"hmax"::S,"x"::S_
```

```
    ,"tol"::S,"ifail"::S,"g"::S,"fcn"::S,"y"::S,"w"::S]$Lisp,_
    ["w"::S,"g"::S,"fcn"::S]$Lisp,_
    [["double"::S,"xend"::S,"hmax"::S,"x"::S,["y"::S,"n"::S]$Lisp_
    ,"tol"::S,["w"::S,"n"::S,7$Lisp]$Lisp,"g"::S,"fcn"::S]$Lisp_
    ,["integer"::S,"n"::S,"irelab"::S,"ifail"::S_
    ]$Lisp_
    ]$Lisp,_
    ["x"::S,"y"::S,"tol"::S,"ifail"::S]$Lisp,_
    [([xendArg::Any,nArg::Any,irelabArg::Any,hmaxArg::Any,xArg::Any,tolArg::Any,ifailArg
    @List Any]$Lisp)$Lisp)_
    pretend List (Record(key:Symbol,entry:Any))]$Result

d02cjf(xendArg:DoubleFloat,mArg:Integer,nArg:Integer,_
    tolArg:DoubleFloat,relabsArg:String,xArg:DoubleFloat,_
    yArg:Matrix DoubleFloat,ifailArg:Integer,gArg:Union(fn:FileName,fp:Asp9(G)),_
    fcnArg:Union(fn:FileName,fp:Asp7(FCN)),outputArg:Union(fn:FileName,fp:Asp8(OUTPUT))
    pushFortranOutputStack(gFilename := aspFilename "g")$FOP
    if gArg case fn
              then outputAsFortran(gArg.fn)
              else outputAsFortran(gArg.fp)
    popFortranOutputStack()$FOP
    pushFortranOutputStack(fcnFilename := aspFilename "fcn")$FOP
    if fcnArg case fn
              then outputAsFortran(fcnArg.fn)
              else outputAsFortran(fcnArg.fp)
    popFortranOutputStack()$FOP
    pushFortranOutputStack(outputFilename := aspFilename "output")$FOP
    if outputArg case fn
              then outputAsFortran(outputArg.fn)
              else outputAsFortran(outputArg.fp)
    popFortranOutputStack()$FOP
    [(invokeNagman([gFilename,fcnFilename,outputFilename]$Lisp,_
    "d02cjf",_
    ["xend"::S,"m"::S,"n"::S,"tol"::S,"relabs"::S_
    ,"x"::S,"ifail"::S,"g"::S,"fcn"::S,"output"::S_
    ,"result"::S,"y"::S,"w"::S]$Lisp,_
    ["result"::S,"w"::S,"g"::S,"fcn"::S,"output"::S]$Lisp,_
    [["double"::S,"xend"::S,"tol"::S,["result"::S,"m"::S,"n"::S]$Lisp_
    ,"x"::S,["y"::S,"n"::S]$Lisp,["w"::S,["+"::S,["*"::S,21$Lisp,"n"::S]$Lisp,28$Lisp]$I
    ,"fcn"::S,"output"::S]$Lisp_
    ,["integer"::S,"m"::S,"n"::S,"ifail"::S]$Lisp_
    ,["character"::S,"relabs"::S]$Lisp_
    ]$Lisp,_
    ["result"::S,"x"::S,"y"::S,"ifail"::S]$Lisp,_
    [([xendArg::Any,mArg::Any,nArg::Any,tolArg::Any,relabsArg::Any,xArg::Any,ifailArg::A
    @List Any]$Lisp)$Lisp)_
```

```
    pretend List (Record(key:Symbol,entry:Any))]$Result

d02ejf(xendArg:DoubleFloat,mArg:Integer,nArg:Integer,_
    relabsArg:String,iwArg:Integer,xArg:DoubleFloat,_
    yArg:Matrix DoubleFloat,tolArg:DoubleFloat,ifailArg:Integer,_
    gArg:Union(fn:FileName,fp:Asp9(G)),fcnArg:Union(fn:FileName,fp:Asp7(FCN))
    outputArg:Union(fn:FileName,fp:Asp8(OUTPUT))): Result ==
    pushFortranOutputStack(gFilename := aspFilename "g")$FOP
    if gArg case fn
              then outputAsFortran(gArg.fn)
              else outputAsFortran(gArg.fp)
    popFortranOutputStack()$FOP
    pushFortranOutputStack(fcnFilename := aspFilename "fcn")$FOP
    if fcnArg case fn
              then outputAsFortran(fcnArg.fn)
              else outputAsFortran(fcnArg.fp)
    popFortranOutputStack()$FOP
    pushFortranOutputStack(pedervFilename := aspFilename "pederv")$FOP
    if pedervArg case fn
              then outputAsFortran(pedervArg.fn)
              else outputAsFortran(pedervArg.fp)
    popFortranOutputStack()$FOP
    pushFortranOutputStack(outputFilename := aspFilename "output")$FOP
    if outputArg case fn
              then outputAsFortran(outputArg.fn)
              else outputAsFortran(outputArg.fp)
    popFortranOutputStack()$FOP
    [(invokeNagman([gFilename,fcnFilename,pedervFilename,outputFilename]$Lisp
    "d02ejf",_
    ["xend"::S,"m"::S,"n"::S,"relabs"::S,"iw"::S_
    ,"x"::S,"tol"::S,"ifail"::S,"g"::S,"fcn"::S_
    ,"pederv"::S,"output"::S,"result"::S,"y"::S,"w"::S]$Lisp,_
    ["result"::S,"w"::S,"g"::S,"fcn"::S,"pederv"::S,"output"::S]$Lisp,_
    [["double"::S,"xend"::S,["result"::S,"m"::S,"n"::S]$Lisp_
    ,"x"::S,["y"::S,"n"::S]$Lisp,"tol"::S,["w"::S,"iw"::S]$Lisp,"g"::S,"fcn":
    ,["integer"::S,"m"::S,"n"::S,"iw"::S,"ifail"::S_
    ]$Lisp_
    ,["character"::S,"relabs"::S]$Lisp_
    ]$Lisp,_
    ["result"::S,"x"::S,"y"::S,"tol"::S,"ifail"::S]$Lisp,_
    [([xendArg::Any,mArg::Any,nArg::Any,relabsArg::Any,iwArg::Any,xArg::Any,t
    @List Any]$Lisp)$Lisp)_
    pretend List (Record(key:Symbol,entry:Any))]$Result

d02gaf(uArg:Matrix DoubleFloat,vArg:Matrix DoubleFloat,nArg:Integer,_
    aArg:DoubleFloat,bArg:DoubleFloat,tolArg:DoubleFloat,_
```

```
    mnpArg:Integer,lwArg:Integer,liwArg:Integer,_
    xArg:Matrix DoubleFloat,npArg:Integer,ifailArg:Integer,_
    fcnArg:Union(fn:FileName,fp:Asp7(FCN))): Result ==
    pushFortranOutputStack(fcnFilename := aspFilename "fcn")$FOP
    if fcnArg case fn
              then outputAsFortran(fcnArg.fn)
              else outputAsFortran(fcnArg.fp)
    popFortranOutputStack()$FOP
    [(invokeNagman([fcnFilename]$Lisp,_
    "d02gaf",_
    ["n"::S,"a"::S,"b"::S,"tol"::S,"mnp"::S_
    ,"lw"::S,"liw"::S,"np"::S,"ifail"::S,"fcn"::S_
    ,"u"::S,"v"::S,"y"::S,"x"::S,"w"::S_
    ,"iw"::S]$Lisp,_
    ["y"::S,"w"::S,"iw"::S,"fcn"::S]$Lisp,_
    [["double"::S,["u"::S,"n"::S,2$Lisp]$Lisp,["v"::S,"n"::S,2$Lisp]$Lisp_
    ,"a"::S,"b"::S,"tol"::S,["y"::S,"n"::S,"mnp"::S]$Lisp,["x"::S,"mnp"::S]$Lisp,["w"::S
    ,"fcn"::S]$Lisp_
    ,["integer"::S,"n"::S,"mnp"::S,"lw"::S,"liw"::S_
    ,"np"::S,"ifail"::S,["iw"::S,"liw"::S]$Lisp]$Lisp_
    ]$Lisp,_
    ["y"::S,"x"::S,"np"::S,"ifail"::S]$Lisp,_
    [([nArg::Any,aArg::Any,bArg::Any,tolArg::Any,mnpArg::Any,lwArg::Any,liwArg::Any,npAr
    @List Any]$Lisp)$Lisp)_
    pretend List (Record(key:Symbol,entry:Any))]$Result

d02gbf(aArg:DoubleFloat,bArg:DoubleFloat,nArg:Integer,_
    tolArg:DoubleFloat,mnpArg:Integer,lwArg:Integer,_
    liwArg:Integer,cArg:Matrix DoubleFloat,dArg:Matrix DoubleFloat,_
    gamArg:Matrix DoubleFloat,xArg:Matrix DoubleFloat,npArg:Integer,_
    ifailArg:Integer,fcnfArg:Union(fn:FileName,fp:Asp77(FCNF)),fcngArg:Union(fn:FileName
    pushFortranOutputStack(fcnfFilename := aspFilename "fcnf")$FOP
    if fcnfArg case fn
              then outputAsFortran(fcnfArg.fn)
              else outputAsFortran(fcnfArg.fp)
    popFortranOutputStack()$FOP
    pushFortranOutputStack(fcngFilename := aspFilename "fcng")$FOP
    if fcngArg case fn
              then outputAsFortran(fcngArg.fn)
              else outputAsFortran(fcngArg.fp)
    popFortranOutputStack()$FOP
    [(invokeNagman([fcnfFilename,fcngFilename]$Lisp,_
    "d02gbf",_
    ["a"::S,"b"::S,"n"::S,"tol"::S,"mnp"::S_
    ,"lw"::S,"liw"::S,"np"::S,"ifail"::S,"fcnf"::S_
    ,"fcng"::S,"y"::S,"c"::S,"d"::S,"gam"::S,"x"::S_
```

```
    ,"w"::S,"iw"::S]$Lisp,_
    ["y"::S,"w"::S,"iw"::S,"fcnf"::S,"fcng"::S]$Lisp,_
    [["double"::S,"a"::S,"b"::S,"tol"::S,["y"::S,"n"::S,"mnp"::S]$Lisp_
    ,["c"::S,"n"::S,"n"::S]$Lisp,["d"::S,"n"::S,"n"::S]$Lisp,["gam"::S,"n"::S
    ,["w"::S,"lw"::S]$Lisp,"fcnf"::S,"fcng"::S]$Lisp_
    ,["integer"::S,"n"::S,"mnp"::S,"lw"::S,"liw"::S_
    ,"np"::S,"ifail"::S,["iw"::S,"liw"::S]$Lisp]$Lisp_
    ]$Lisp,_
    ["y"::S,"c"::S,"d"::S,"gam"::S,"x"::S,"np"::S,"ifail"::S]$Lisp,_
    [([aArg::Any,bArg::Any,nArg::Any,tolArg::Any,mnpArg::Any,lwArg::Any,liwAr
    @List Any]$Lisp)$Lisp)_
    pretend List (Record(key:Symbol,entry:Any))]$Result

d02kef(xpointArg:Matrix DoubleFloat,mArg:Integer,kArg:Integer,_
    tolArg:DoubleFloat,maxfunArg:Integer,matchArg:Integer,_
    elamArg:DoubleFloat,delamArg:DoubleFloat,hmaxArg:Matrix DoubleFloat,_
    maxitArg:Integer,ifailArg:Integer,coeffnArg:Union(fn:FileName,fp:Asp10(CO
    bdyvalArg:Union(fn:FileName,fp:Asp80(BDYVAL))): Result ==
    pushFortranOutputStack(coeffnFilename := aspFilename "coeffn")$FOP
    if coeffnArg case fn
              then outputAsFortran(coeffnArg.fn)
              else outputAsFortran(coeffnArg.fp)
    popFortranOutputStack()$FOP
    pushFortranOutputStack(bdyvalFilename := aspFilename "bdyval")$FOP
    if bdyvalArg case fn
              then outputAsFortran(bdyvalArg.fn)
              else outputAsFortran(bdyvalArg.fp)
    popFortranOutputStack()$FOP
    pushFortranOutputStack(monitFilename := aspFilename "monit")$FOP
    outputAsFortran()$Asp12(MONIT)
    popFortranOutputStack()$FOP
    pushFortranOutputStack(reportFilename := aspFilename "report")$FOP
    outputAsFortran()$Asp33(REPORT)
    popFortranOutputStack()$FOP
    [(invokeNagman([coeffnFilename,bdyvalFilename,monitFilename,reportFilenam
    "d02kef",_
    ["m"::S,"k"::S,"tol"::S,"maxfun"::S,"match"::S_
    ,"elam"::S,"delam"::S,"maxit"::S,"ifail"::S,"coeffn"::S_
    ,"bdyval"::S,"monit"::S,"report"::S,"xpoint"::S,"hmax"::S]$Lisp,_
    ["coeffn"::S,"bdyval"::S,"monit"::S,"report"::S]$Lisp,_
    [["double"::S,["xpoint"::S,"m"::S]$Lisp,"tol"::S_
    ,"elam"::S,"delam"::S,["hmax"::S,2$Lisp,"m"::S]$Lisp,"coeffn"::S,"bdyval"
    ,["integer"::S,"m"::S,"k"::S,"maxfun"::S,"match"::S_
    ,"maxit"::S,"ifail"::S]$Lisp_
    ]$Lisp,_
    ["match"::S,"elam"::S,"delam"::S,"hmax"::S,"maxit"::S,"ifail"::S]$Lisp,_
```

```
    [([mArg::Any,kArg::Any,tolArg::Any,maxfunArg::Any,matchArg::Any,elamArg::Any,delamAr
    @List Any]$Lisp)$Lisp)_
    pretend List (Record(key:Symbol,entry:Any))]$Result

d02kef(xpointArg:Matrix DoubleFloat,mArg:Integer,kArg:Integer,_
    tolArg:DoubleFloat,maxfunArg:Integer,matchArg:Integer,_
    elamArg:DoubleFloat,delamArg:DoubleFloat,hmaxArg:Matrix DoubleFloat,_
    maxitArg:Integer,ifailArg:Integer,coeffnArg:Union(fn:FileName,fp:Asp10(COEFFN)),_
    bdyvalArg:Union(fn:FileName,fp:Asp80(BDYVAL)),monitArg:FileName,reportArg:FileName)
    pushFortranOutputStack(coeffnFilename := aspFilename "coeffn")$FOP
    if coeffnArg case fn
              then outputAsFortran(coeffnArg.fn)
              else outputAsFortran(coeffnArg.fp)
    popFortranOutputStack()$FOP
    pushFortranOutputStack(bdyvalFilename := aspFilename "bdyval")$FOP
    if bdyvalArg case fn
              then outputAsFortran(bdyvalArg.fn)
              else outputAsFortran(bdyvalArg.fp)
    popFortranOutputStack()$FOP
    pushFortranOutputStack(monitFilename := aspFilename "monit")$FOP
    outputAsFortran(monitArg)
    popFortranOutputStack()$FOP
    pushFortranOutputStack(reportFilename := aspFilename "report")$FOP
    outputAsFortran(reportArg)
    popFortranOutputStack()$FOP
    [(invokeNagman([coeffnFilename,bdyvalFilename,monitFilename,reportFilename]$Lisp,_
    "d02kef",_
    ["m"::S,"k"::S,"tol"::S,"maxfun"::S,"match"::S_
    ,"elam"::S,"delam"::S,"maxit"::S,"ifail"::S,"coeffn"::S_
    ,"bdyval"::S,"monit"::S,"report"::S,"xpoint"::S,"hmax"::S]$Lisp,_
    ["coeffn"::S,"bdyval"::S,"monit"::S,"report"::S]$Lisp,_
    [["double"::S,["xpoint"::S,"m"::S]$Lisp,"tol"::S_
    ,"elam"::S,"delam"::S,["hmax"::S,2$Lisp,"m"::S]$Lisp,"coeffn"::S,"bdyval"::S,"monit"
    ,["integer"::S,"m"::S,"k"::S,"maxfun"::S,"match"::S_
    ,"maxit"::S,"ifail"::S]$Lisp_
    ]$Lisp,_
    ["match"::S,"elam"::S,"delam"::S,"hmax"::S,"maxit"::S,"ifail"::S]$Lisp,_
    [([mArg::Any,kArg::Any,tolArg::Any,maxfunArg::Any,matchArg::Any,elamArg::Any,delamAr
    @List Any]$Lisp)$Lisp)_
    pretend List (Record(key:Symbol,entry:Any))]$Result

d02raf(nArg:Integer,mnpArg:Integer,numbegArg:Integer,_
    nummixArg:Integer,tolArg:DoubleFloat,initArg:Integer,_
    iyArg:Integer,ijacArg:Integer,lworkArg:Integer,_
    liworkArg:Integer,npArg:Integer,xArg:Matrix DoubleFloat,_
    yArg:Matrix DoubleFloat,delepsArg:DoubleFloat,ifailArg:Integer,_
```

```
fcnArg:Union(fn:FileName,fp:Asp41(FCN,JACOBF,JACEPS)),gArg:Union(fn:FileN
pushFortranOutputStack(fcnFilename := aspFilename "fcn")$FOP
if fcnArg case fn
          then outputAsFortran(fcnArg.fn)
          else outputAsFortran(fcnArg.fp)
popFortranOutputStack()$FOP
pushFortranOutputStack(gFilename := aspFilename "g")$FOP
if gArg case fn
          then outputAsFortran(gArg.fn)
          else outputAsFortran(gArg.fp)
popFortranOutputStack()$FOP
[(invokeNagman([fcnFilename,gFilename]$Lisp,_
"d02raf",_
["n"::S,"mnp"::S,"numbeg"::S,"nummix"::S,"tol"::S_
,"init"::S,"iy"::S,"ijac"::S,"lwork"::S,"liwork"::S_
,"np"::S,"deleps"::S,"ifail"::S,"fcn"::S,"g"::S_
,"abt"::S,"x"::S,"y"::S,"work"::S,"iwork"::S_
]$Lisp,_
["abt"::S,"work"::S,"iwork"::S,"fcn"::S,"g"::S]$Lisp,_
[["double"::S,"tol"::S,["abt"::S,"n"::S]$Lisp_
,["x"::S,"mnp"::S]$Lisp,["y"::S,"iy"::S,"mnp"::S]$Lisp,"deleps"::S,["work
,["integer"::S,"n"::S,"mnp"::S,"numbeg"::S_
,"nummix"::S,"init"::S,"iy"::S,"ijac"::S,"lwork"::S,"liwork"::S,"np"::S,"
]$Lisp,_
["abt"::S,"np"::S,"x"::S,"y"::S,"deleps"::S,"ifail"::S]$Lisp,_
[([nArg::Any,mnpArg::Any,numbegArg::Any,nummixArg::Any,tolArg::Any,initAr
@List Any]$Lisp)$Lisp)_
pretend List (Record(key:Symbol,entry:Any))]$Result
```

⟨*NAGD02.dotabb*⟩≡

```
"NAGD02" [color="#FF4488",href="bookvol10.4.pdf#nameddest=NAGD02"]
"ALIST" [color="#88FF44",href="bookvol10.3.pdf#nameddest=ALIST"]
"NAGD02" -> "ALIST"
```

15.21 package NAGD03 NagPartialDifferentialEquationsPackage

⟨*NagPartialDifferentialEquationsPackage.help*⟩≡

```
D03 -- Partial Differential Equations          Introduction -- D03
                            Chapter D03
                  Partial Differential Equations

1. Scope of the Chapter

This chapter is concerned with the solution of partial
differential equations.

2. Background to the Problems

The definition of a partial differential equation problem
includes not only the equation itself but also the domain of
interest and appropriate subsidiary conditions. Indeed, partial
differential equations are usually classified as elliptic,
hyperbolic or parabolic according to the form of the equation and
the form of the subsidiary conditions which must be assigned to
produce a well-posed problem. Ultimately it is hoped that this
chapter will contain routines for the solution of equations of
each of these types together with automatic mesh generation
routines and other utility routines particular to the solution of
partial differential equations. The routines in this chapter will
often call upon routines from the Linear Algebra Chapter F04 --
Simultaneous Linear Equations.

The classification of partial differential equations is easily
described in the case of linear equations of the second order in
two independent variables, i.e., equations of the form

              au  +2bu  +cu  +du +eu +fu+g=0,                (1)
                xx    xy   yy   x   y

where a, b, c, d, e, f and g are functions of x and y only.
Equation (1) is called elliptic, hyperbolic or parabolic
                2
according as ac-b  is positive, negative or zero. Useful
definitions of the concepts of elliptic, hyperbolic and parabolic
character can also be given for differential equations in more
than two independent variables, for systems and for nonlinear
differential equations.
```

For elliptic equations, of which Laplace's equation

$$u_{xx} + u_{yy} = 0 \tag{2}$$

is the simplest example of second order, the subsidiary conditions take the form of boundary conditions, i.e., conditions which provide information about the solution at all points of a closed boundary. For example, if equation (2) holds in a plane domain D bounded by a contour C, a solution u may be sought subject to the condition

$$u = f \quad \text{on } C, \tag{3}$$

where f is a given function. The condition (3) is known as a Dirichlet boundary condition. Equally common is the Neumann boundary condition

$$u' = g \quad \text{on } C, \tag{4}$$

which is one form of a more general condition

$$u' + fu = g \quad \text{on } C, \tag{5}$$

where u' denotes the derivative of u normal to the contour C and f and g are given functions. Provided that f and g satisfy certain restrictions, condition (5) yields a well-posed boundary value problem for Laplace's equation. In the case of the Neumann problem, one further piece of information, e.g. the value of u at a particular point, is necessary for uniqueness of the solution. Boundary conditions similar to the above are applicable to more general second order elliptic equations, whilst two such conditions are required for equations of fourth order.

For hyperbolic equations, the wave equation

$$u_{tt} - u_{xx} = 0 \tag{6}$$

is the simplest example of second order. It is equivalent to a first order system

$$u_t - v_x = 0, \quad v_t - u_x = 0. \tag{7}$$

The subsidiary conditions may take the form of initial conditions, i.e., conditions which provide information about the solution at points on a suitable open boundary. For example, if equation (6) is satisfied for t>0, a solution u may be sought such that

$$u(x,0)=f(x), \quad u_t(x,0)=g(x), \tag{8}$$

where f and g are given functions. This is an example of an initial value problem, sometimes known as Cauchy's problem.

For parabolic equations, of which the heat conduction equation

$$u_t - u_{xx} = 0 \tag{9}$$

is the simplest example, the subsidiary conditions always include some of initial type and may also include some of boundary type. For example, if equation (9) is satisfied for t>0 and 0<x<1, a solution u may be sought such that

$$u(x,0)=f(x), \quad 0<x<1, \tag{10}$$

and

$$u(0,t)=0, \quad u(1,t)=1, \quad t>0. \tag{11}$$

This is an example of a mixed initial/boundary value problem.

For all types of partial differential equations, finite difference methods (Mitchell and Griffiths [5]) and finite element methods (Wait and Mitchell [9]) are the most common means of solution and such methods obviously feature prominently either in this chapter or in the companion NAG Finite Element Library. Many of the utility routines in this chapter are concerned with the solution of the large sparse systems of equations which arise from the finite difference and finite element methods.

Alternative methods of solution are often suitable for special classes of problems. For example, the method of characteristics is the most common for hyperbolic equations involving time and one space dimension (Smith [7]). The method of lines (see Mikhlin and Smolitsky [4]) may be used to reduce a parabolic equation to a (stiff) system of ordinary differential equations, which may be solved by means of routines from Chapter D02 -- Ordinary

Differential Equations. Similarly, integral equation or boundary element methods (Jaswon and Symm [3]) are frequently used for elliptic equations. Typically, in the latter case, the solution of a boundary value problem is represented in terms of certain boundary functions by an integral expression which satisfies the differential equation throughout the relevant domain. The boundary functions are obtained by applying the given boundary conditions to this representation. Implementation of this method necessitates discretisation of only the boundary of the domain, the dimensionality of the problem thus being effectively reduced by one. The boundary conditions yield a full system of simultaneous equations, as opposed to the sparse systems yielded by the finite difference and finite element methods, but the full system is usually of much lower order. Solution of this system yields the boundary functions, from which the solution of the problem may be obtained, by quadrature, as and where required.

2.1. References

[1] Ames W F (1977) Nonlinear Partial Differential Equations in Engineering. Academic Press (2nd Edition).

[2] Berzins M (1990) Developments in the NAG Library Software for Parabolic Equations. Scientific Software Systems. (ed J C Mason and M G Cox) Chapman and Hall. 59--72.

[3] Jaswon M A and Symm G T (1977) Integral Equation Methods in Potential Theory and Elastostatics. Academic Press.

[4] Mikhlin S G and Smolitsky K L (1967) Approximate Methods for the Solution of Differential and Integral Equations. Elsevier.

[5] Mitchell A R and Griffiths D F (1980) The Finite Difference Method in Partial Differential Equations. Wiley.

[6] Richtmyer R D and Morton K W (1967) Difference Methods for Initial-value Problems. Interscience (2nd Edition).

[7] Smith G D (1985) Numerical Solution of Partial Differential Equations: Finite Difference Methods. Oxford University Press (3rd Edition).

[8] Swarztrauber P N and Sweet R A (1979) Efficient Fortran Subprograms for the Solution of Separable Elliptic Partial Differential Equations. ACM Trans. Math. Softw. 5 352--364.

[9] Wait R and Mitchell A R (1985) Finite Element Analysis and Application. Wiley.

3. Recommendations on Choice and Use of Routines

The choice of routine will depend first of all upon the type of partial differential equation to be solved. At present no special allowances are made for problems with boundary singularities such as may arise at corners of domains or at points where boundary conditions change. For such problems results should be treated with caution.

Users may wish to construct their own partial differential equation solution software for problems not solvable by the routines described in Sections 3.1 to 3.4 below. In such cases users can employ appropriate routines from the Linear Algebra Chapters to solve the resulting linear systems; see Section 3.5 for further details.

3.1. Elliptic Equations

The routine D03EDF solves a system of seven-point difference equations in a rectangular grid (in two dimensions), using the multigrid iterative method. The equations are supplied by the user, and the seven-point form allows cross-derivative terms to be represented (see Mitchell and Griffiths [5]). The method is particularly efficient for large systems of equations with diagonal dominance.

The routine D03EEF discretises a second-order equation on a two-dimensional rectangular region using finite differences and a seven-point molecule. The routine allows for cross-derivative terms, Dirichlet, Neumann or mixed boundary conditions, and either central or upwind differences. The resulting seven-diagonal difference equations are in a form suitable for passing directly to the multigrid routine D03EDF, although other solution methods could easily be used.

The routine D03FAF, based on the routine HW3CRT from FISHPACK (Swarztrauber and Sweet [8]), solves the Helmholtz equation in a three-dimensional cuboidal region, with any combination of Dirichlet, Neumann or periodic boundary conditions. The method used is based on the fast Fourier transform algorithm, and is likely to be particularly efficient on vector-processing machines.

3.2. Hyperbolic Equations

There are no routines available yet for the solution of these equations.

3.3. Parabolic Equations

There are no routines available yet for the solution of these equations.

But problems in two space dimensions plus time may be treated as a succession of elliptic equations [1], [6] using appropriate D03E- routines or one may use codes from the NAG Finite Element Library.

3.4. Utility Routines

There are no utility routines available yet, but routines are available in the Linear Algebra Chapters for the direct and iterative solution of linear equations. Here we point to some of the routines that may be of use in solving the linear systems that arise from finite difference or finite element approximations to partial differential equation solutions. Chapters F01 and F04 should be consulted for further information and for the routine documents. Decision trees for the solution of linear systems are given in Section 3.5 of the F04 Chapter Introduction.

The following routines allow the direct solution of symmetric positive-definite systems:

Band	F04ACF
Variable band (skyline)	F01MCF and F04MCF
Tridiagonal	F04FAF
Sparse	F01MAF* and F04MAF

(* the parameter DROPTL should be set to zero for F01MAF to give a direct solution)

and the following routines allow the iterative solution of symmetric positive-definite systems:

```
    Sparse (incomplete F01MAF and F04MBF
    Cholesky)

    Sparse (conjugate  F04MBF
    gradient)

The latter routine above allows the user to supply a pre-
conditioner and also allows the solution of indefinite symmetric
systems.

The following routines allow the direct solution of unsymmetric
systems:

    Band               F01LBF and F04LDF

    Almost block-      F01LHF and F04LHF
    diagonal

    Tridiagonal        F01LEF and F04LEF or F04EAF

    Sparse             F01BRF (and F01BSF) and F04AXF

and the following routine allows the iterative solution of
unsymmetric systems:

    Sparse             F04QAF

The above routine allows the user to supply a pre-conditioner and
also allows the solution of least-squares systems.

3.5. Index

Elliptic equations
     equations on rectangular grid (seven-point 2-D        D03EDF
     molecule)
     discretisation on rectangular grid (seven-point 2-D   D03EEF
     molecule)
     Helmholtz's equation in three dimensions              D03FAF



D03 -- Partial Differential Equations            Contents -- D03
Chapter D03

Partial Differential Equations
```

```
D03EDF  Elliptic PDE, solution of finite difference equations by
        a multigrid technique

D03EEF  Discretize a 2nd order elliptic PDE on a rectangle

D03FAF  Elliptic PDE, Helmholtz equation, 3-D Cartesian co-
        ordinates
```

%%%

```
D03 -- Partial Differential Equations                          D03EDF
        D03EDF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for
your implementation to check implementation-dependent details.
The symbol (*) after a NAG routine name denotes a routine that is
not included in the Foundation Library.

1. Purpose

D03EDF solves seven-diagonal systems of linear equations which
arise from the discretization of an elliptic partial differential
equation on a rectangular region. This routine uses a multigrid
technique.

2. Specification

       SUBROUTINE D03EDF (NGX, NGY, LDA, A, RHS, UB, MAXIT, ACC,
      1                   US, U, IOUT, NUMIT, IFAIL)
       INTEGER          NGX, NGY, LDA, MAXIT, IOUT, NUMIT, IFAIL
       DOUBLE PRECISION A(LDA,7), RHS(LDA), UB(NGX*NGY), ACC, US
      1                 (LDA), U(LDA)

3. Description

D03EDF solves, by multigrid iteration, the seven-point scheme

 6            7
A   u       +A   u
 i,j i-1,j+1  i,j i,j+1

  3           4        5
+A   u      +A   u   +A   u
  i,j i-1,j   i,j ij   i,j i+1,j
```

$$+A^{1}_{i,j}\,u_{i,j-1} + A^{2}_{i,j}\,u_{i+1,j-1} = f_{ij}\,,$$

$$i=1,2,\ldots,n_x\ ;\ j=1,2,\ldots,n_y\ ,$$

which arises from the discretization of an elliptic partial differential equation of the form

$$\alpha(x,y)U_{xx} + \beta(x,y)U_{xy} + \gamma(x,y)U_{yy} + \delta(x,y)U_x + \epsilon(x,y)U_y + \phi(x,y)U = \psi(x,y)$$

and its boundary conditions, defined on a rectangular region. This we write in matrix form as

$$Au=f$$

The algorithm is described in separate reports by Wesseling [2], [3] and McCarthy [1].

Systems of linear equations, matching the seven-point stencil defined above, are solved by a multigrid iteration. An initial estimate of the solution must be provided by the user. A zero guess may be supplied if no better approximation is available.

A 'smoother' based on incomplete Crout decomposition is used to eliminate the high frequency components of the error. A restriction operator is then used to map the system on to a sequence of coarser grids. The errors are then smoothed and prolongated (mapped onto successively finer grids). When the finest cycle is reached, the approximation to the solution is corrected. The cycle is repeated for MAXIT iterations or until the required accuracy, ACC, is reached.

D03EDF will automatically determine the number l of possible coarse grids, 'levels' of the multigrid scheme, for a particular problem. In other words, D03EDF determines the maximum integer l so that n_x and n_y can be expressed in the form

$$2^{l-1}\qquad 2^{l-1}$$

$$n_x = m_2 + 1, \quad n_y = n_2 + 1, \quad \text{with } m \geq 2 \text{ and } n \geq 2.$$

It should be noted that the rate of convergence improves significantly with the number of levels used (see McCarthy [1]), so that n_x and n_y should be carefully chosen so that $n_x - 1$ and $n_y - 1$ have factors of the form 2^l, with l as large as possible. For good convergence the integer l should be at least 2.

D03EDF has been found to be robust in application, but being an iterative method the problem of divergence can arise. For a strictly diagonally dominant matrix A

$$|A_{ij}| > \sum_{k \neq j} |A_{ik}|$$

no such problem is foreseen. The diagonal dominance of A is not a necessary condition, but should this condition be strongly violated then divergence may occur. The quickest test is to try the routine.

4. References

[1] McCarthy G J (1983) Investigation into the Multigrid Code MGD1. Report AERE-R 10889. Harwell.

[2] Wesseling P (1982) MGD1 - A Robust and Efficient Multigrid Method. Multigrid Methods. Lecture Notes in Mathematics. 960 Springer-Verlag. 614--630.

[3] Wesseling P (1982) Theoretical Aspects of a Multigrid Method. SIAM J. Sci. Statist. Comput. 3 387--407.

5. Parameters

1: NGX -- INTEGER Input
On entry: the number of interior grid points in the x-direction, n_x. NGX-1 should preferably be divisible by as high a power of 2 as possible. Constraint: NGX >= 3.

2: NGY -- INTEGER Input
On entry: the number of interior grid points in the y-direction, n_y. NGY-1 should preferably be divisible by as high a power of 2 as possible. Constraint: NGY >= 3.

3: LDA -- INTEGER Input
On entry: the first dimension of the array A as declared in the (sub)program from which D03EDF is called, which must also be a lower bound for the dimensions of the arrays RHS, US and U. It is always sufficient to set LDA>=(4*(NGX+1)*(NGY+1))/3, but slightly smaller values may be permitted, depending on the values of NGX and NGY. If on entry, LDA is too small, an error message gives the minimum permitted value. (LDA must be large enough to allow space for the coarse-grid approximations).

4: A(LDA,7) -- DOUBLE PRECISION array Input/Output
On entry: A(i+(j-1)*NGX,k) must be set to A_{ij}^{k}, for i = 1,2,...,NGX; j = 1,2,...,NGY and k = 1,2,...,7. On exit: A is overwritten.

5: RHS(LDA) -- DOUBLE PRECISION array Input/Output
On entry: RHS(i+(j-1)*NGX) must be set to f_{ij}, for i = 1,2,...,NGX; j = 1,2,...,NGY. On exit: the first NGX*NGY elements are unchanged and the rest of the array is used as workspace.

6: UB(NGX*NGY) -- DOUBLE PRECISION array Input/Output
On entry: UB(i+(j-1)*NGX) must be set to the initial estimate for the solution u_{ij}. On exit: the corresponding component of the residual r=f-Au.

7: MAXIT -- INTEGER Input
On entry: the maximum permitted number of multigrid iterations. If MAXIT = 0, no multigrid iterations are performed, but the coarse-grid approximations and incomplete Crout decompositions are computed, and may be output if IOUT is set accordingly. Constraint: MAXIT >= 0.

8: ACC -- DOUBLE PRECISION Input

On entry: the required tolerance for convergence of the residual 2-norm:

$$||r||_2 = \sqrt{\sum_{k=1}^{NGX*NGY} (r_k)^2}$$

where r=f-Au and u is the computed solution. Note that the norm is not scaled by the number of equations. The routine will stop after fewer than MAXIT iterations if the residual 2-norm is less than the specified tolerance. (If MAXIT > 0, at least one iteration is always performed.)

If on entry ACC = 0.0, then the machine precision is used as a default value for the tolerance; if ACC > 0.0, but ACC is less than the machine precision, then the routine will stop when the residual 2-norm is less than the machine precision and IFAIL will be set to 4. Constraint: ACC >= 0.0.

9: US(LDA) -- DOUBLE PRECISION array Output
On exit: the residual 2-norm, stored in element US(1).

10: U(LDA) -- DOUBLE PRECISION array Output
On exit: the computed solution u_{ij} is returned in U(i+(j-1)*NGX), for i = 1,2,...,NGX; j = 1,2,...,NGY.

11: IOUT -- INTEGER Input
On entry: controls the output of printed information to the advisory message unit as returned by X04ABF:
IOUT = 0
No output.

IOUT = 1
The solution u_{ij}, for i = 1,2,...,NGX; j = 1,2,...,NGY.

IOUT = 2
The residual 2-norm after each iteration, with the reduction factor over the previous iteration.

IOUT = 3
As for IOUT = 1 and IOUT = 2.

```
      IOUT = 4
            As for IOUT = 3, plus the final residual (as returned
            in UB).

      IOUT = 5
            As for IOUT = 4, plus the initial elements of A and
            RHS.

      IOUT = 6
            As for IOUT = 5, plus the Galerkin coarse grid
            approximations.

      IOUT = 7
            As for IOUT = 6, plus the incomplete Crout
            decompositions.

      IOUT = 8
            As for IOUT = 7, plus the residual after each
            iteration.
      The elements A(p,k), the Galerkin coarse grid approximations
      and the incomplete Crout decompositions are output in the
      format:
           Y-index = j

           X-index = i A(p,1) A(p,2) A(p,3) A(p,4) A(p,5) A(p,6)
           A(p,7)

           where p=i+(j-1)*NGX, i = 1,2,...,NGX and j = 1,2,...
           ,NGY.
      The vectors U(p), UB(p), RHS(p) are output in matrix form
      with NGY rows and NGX columns. Where NGX > 10, the NGX
      values for a given j-value are produced in rows of 10.
      Values of IOUT > 4 may yield considerable amounts of output.
      Constraint: 0 <= IOUT <= 8.

12:   NUMIT -- INTEGER                                      Output
      On exit: the number of iterations performed.

13:   IFAIL -- INTEGER                                  Input/Output
      On entry: IFAIL must be set to 0, -1 or 1. For users not
      familiar with this parameter (described in the Essential
      Introduction) the recommended value is 0.

      On exit: IFAIL = 0 unless the routine detects an error (see
      Section 6).
```

```
6. Error Indicators and Warnings

Errors detected by the routine:

If on entry IFAIL = 0 or -1, explanatory error messages are
output on the current error message unit (as defined by X04AAF).

IFAIL= 1
     On entry NGX < 3,

     or       NGY < 3,

     or       LDA is too small,

     or       ACC < 0.0,

     or       MAXIT < 0,

     or       IOUT < 0,

     or       IOUT > 8.

IFAIL= 2
     MAXIT iterations have been performed with the residual 2-
     norm decreasing at each iteration but the residual 2-norm
     has not been reduced to less than the specified tolerance
     (see ACC). Examine the progress of the iteration by setting
     IOUT >= 2.

IFAIL= 3
     As for IFAIL = 2, except that at one or more iterations the
     residual 2-norm did not decrease. It is likely that the
     method fails to converge for the given matrix A.

IFAIL= 4
     On entry ACC is less than the machine precision. The routine
     terminated because the residual norm is less than the
     machine precision.

7. Accuracy

See ACC (Section 5).

8. Further Comments
```

The rate of convergence of this routine is strongly dependent upon the number of levels, l, in the multigrid scheme, and thus the choice of NGX and NGY is very important. The user is advised to experiment with different values of NGX and NGY to see the effect they have on the rate of convergence; for example, using a
                        6
value such as NGX = 65 (=2 +1) followed by NGX = 64 (for which l = 1).

9. Example

The program solves the elliptic partial differential equation

$$U_{xx} - (\alpha)U_{xy} + U_{yy} = -4, \quad (\alpha) = 1.7$$

on the unit square 0<=x,y<=1, with boundary conditions

$$U = 0 \text{ on } \begin{cases} x=0, & (0<=y<=1) \\ y=0, & (0<=x<=1) \\ y=1, & (0<=x<=1) \end{cases} \quad U=1 \text{ on } x=1, \quad 0<=y<=1.$$

For the equation to be elliptic, (alpha) must be less than 2.

The equation is discretized on a square grid with mesh spacing h in both directions using the following approximations:

Please see figure in printed Reference Manual

$$U_{xx} \tilde{=} \frac{1}{h^2}(U_E - 2U_O + U_W)$$

$$U_{yy} \tilde{=} \frac{1}{h^2}(U_N - 2U_O + U_S)$$

$$U_{xy} \tilde{=} \frac{1}{2h^2}(U_N - U_{NW} + U_E - 2U_O + U_W - U_{SE} + U_S).$$

Thus the following equations are solved:

$$\frac{1}{2}\alpha u_{i-1,j+1} + \left(1-\frac{1}{2}\alpha\right)u_{i,j+1}$$

$$+\left(1-\frac{1}{2}\alpha\right)u_{i-1,j} + (-4+\alpha)u_{ij} + \left(1-\frac{1}{2}\alpha\right)u_{i+1,j}$$

$$+\left(1-\frac{1}{2}\alpha\right)u_{i,j-1} + \frac{1}{2}\alpha u_{i+1,j-1}$$

$$= -4h^2$$

The example program is not reproduced here. The source code for all example programs is distributed with the NAG Foundation Library software and should be available on-line.

%%%

D03 -- Partial Differential Equations D03EEF

D03EEF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for your implementation to check implementation-dependent details. The symbol (*) after a NAG routine name denotes a routine that is not included in the Foundation Library.

1. Purpose

D03EEF discretizes a second order elliptic partial differential equation (PDE) on a rectangular region.

2. Specification

```
      SUBROUTINE D03EEF (XMIN, XMAX, YMIN, YMAX, PDEF, BNDY,
     1                   NGX, NGY, LDA, A, RHS, SCHEME, IFAIL)
      INTEGER           NGX, NGY, LDA, IFAIL
      DOUBLE PRECISION XMIN, XMAX, YMIN, YMAX, A(LDA,7), RHS(LDA)
      CHARACTER*1       SCHEME
      EXTERNAL          PDEF, BNDY
```

3. Description

D03EEF discretizes a second order linear elliptic partial differential equation of the form

$$(\alpha)(x,y)\frac{\partial^2 U}{\partial x^2}+(\beta)(x,y)\frac{\partial^2 U}{\partial x \partial y}+(\gamma)(x,y)\frac{\partial^2 U}{\partial y^2}+(\delta)(x,y)\frac{\partial U}{\partial x}+(\epsilon)(x,y)\frac{\partial U}{\partial x}+(\phi)(x,y)U=(\psi)(x,y) \quad (1)$$

on a rectangular region

$$x_A \le x \le x_B$$

$$y_A \le y \le y_B$$

subject to boundary conditions of the form

$$a(x,y)U+b(x,y)\frac{\partial U}{\partial n}=c(x,y)$$

where $\frac{\partial U}{\partial n}$ denotes the outward pointing normal derivative on the boundary. Equation (1) is said to be elliptic if

$$4(\alpha)(x,y)(\gamma)(x,y) \ge ((\beta)(x,y))^2$$

for all points in the rectangular region. The linear equations produced are in a form suitable for passing directly to the multigrid routine D03EDF.

The equation is discretized on a rectangular grid, with n_x grid

points in the x-direction and n grid points in the y-direction.
                               y
The grid spacing used is therefore

$$h_x = (x_B - x_A)/(n_x - 1)$$

$$h_y = (y_B - y_A)/(n_y - 1)$$

and the co-ordinates of the grid points (x_i, y_j) are

$$x_i = x_A + (i-1)h_x, \quad i=1,2,\ldots,n_x,$$

$$y_j = y_A + (j-1)h_y, \quad j=1,2,\ldots,n_y.$$

At each grid point (x_i, y_j) six neighbouring grid points are used
to approximate the partial differential equation, so that the
equation is discretized on the following seven-point stencil:

Please see figure in printed Reference Manual

For convenience the approximation u_{ij} to the exact solution
$U(x_i, y_j)$ is denoted by u_0, and the neighbouring approximations
are labelled according to points of the compass as shown. Where
numerical labels for the seven points are required, these are
also shown above.

The following approximations are used for the second derivatives:

$$\frac{d^2U}{dx^2} \tilde{=} \frac{1}{h_x^2}(u_E - 2u_0 + u_W)$$

$$\frac{d^2}{dx^2}$$

```
     dd U   1
     ----~= --(u -2u +u )
       2     2  N   O  S
     dd y   h
             y

        2
      dd U    1
     ------~= -----(u -u  +u -2u +u -u  +u ).
     ddxddy 2h h   N  NW  E   O  W  SE  S
              x y
```

Two possible schemes may be used to approximate the first derivatives:

Central Differences

```
     ddU     1
     ---~=  ---(u -u )
     ddx    2h   W  E
              x

     ddU     1
     ---~=  ---(u -u )
     ddy    2h   N  S
              y
```

Upwind Differences

```
     ddU    1
     ---~=  --(u -u )if (delta)(x,y)>0
     ddx    h   E  O
             x

     ddU    1
     ---~=  --(u -u )if (delta)(x,y)<0
     ddx    h   O  W
             x

     ddU    1
     ---~=  --(u -u )if (epsilon)(x,y)>0
     ddy    h   N  O
             y

     ddU    1
     ---~=  --(u -u )if (epsilon)(x,y)<0.
```

```
     ddy    h   O  S
             y
```

Central differences are more accurate than upwind differences, but upwind differences may lead to a more diagonally dominant matrix for those problems where the coefficients of the first derivatives are significantly larger than the coefficients of the second derivatives.

The approximations used for the first derivatives may be written in a more compact form as follows:

```
     ddU     1
     ---~=  ---((k -1)u -2k u +(k +1)u )
     ddx    2h (  x    W   x O   x    E)
              x

     ddU     1
     ---~=  ---((k -1)u -2k u +(k +1)u )
     ddy    2h (  y    S   y O   y    N)
              y
```

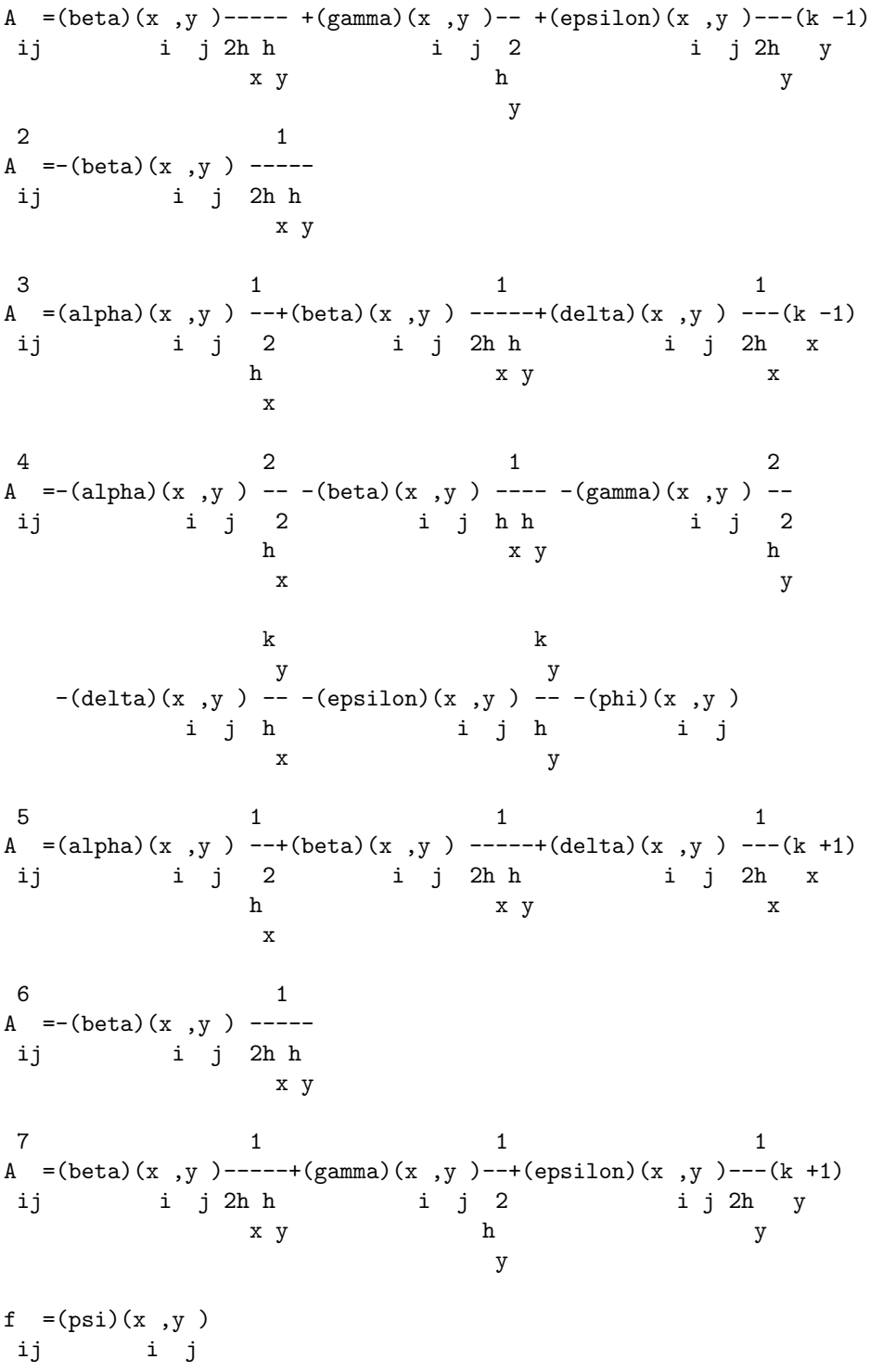
where k =sign (delta) and k =sign (epsilon) for upwind
       x   y
differences, and k =k =0 for central differences.
                  x y

At all points in the rectangular domain, including the boundary, the coefficients in the partial differential equation are evaluated by calling the user-supplied subroutine PDEF, and applying the approximations. This leads to a seven-diagonal system of linear equations of the form:

```
    6            7
   A  u        +A  u
    ij i-1,j+1   ij i,j+1
     3          4       5
   +A  u      +A  u   +A  u
     ij i-1,j   ij ij   ij i+1,j
       1          2
     +A  u      +A  u        =f  , i=1,2,...,n ;j=1,2,...,n ,
       ij i,j-1   ij i+1,j-1   ij             x            y
```

where the coefficients are given by

```
 1                1                  1                   1
```


A =(beta)(x ,y)----- +(gamma)(x ,y)-- +(epsilon)(x ,y)---(k -1)
ij i j 2h h i j 2 i j 2h y
x y h y
2 1
A =-(beta)(x ,y) -----
ij i j 2h h
x y
3 1 1 1
A =(alpha)(x ,y) --+(beta)(x ,y) -----+(delta)(x ,y) ---(k -1)
ij i j 2 i j 2h h i j 2h x
h x y x
x
4 2 1 2
A =-(alpha)(x ,y) -- -(beta)(x ,y) ---- -(gamma)(x ,y) --
ij i j 2 i j h h i j 2
h x y h
x y
k k
y y
-(delta)(x ,y) -- -(epsilon)(x ,y) -- -(phi)(x ,y)
i j h i j h i j
x y
5 1 1 1
A =(alpha)(x ,y) --+(beta)(x ,y) -----+(delta)(x ,y) ---(k +1)
ij i j 2 i j 2h h i j 2h x
h x y x
x
6 1
A =-(beta)(x ,y) -----
ij i j 2h h
x y
7 1 1 1
A =(beta)(x ,y)-----+(gamma)(x ,y)--+(epsilon)(x ,y)---(k +1)
ij i j 2h h i j 2 i j 2h y
x y h y
y
f =(psi)(x ,y)
ij i j

These equations then have to be modified to take account of the boundary conditions. These may be Dirichlet (where the solution is given), Neumann (where the derivative of the solution is given), or mixed (where a linear combination of solution and derivative is given).

If the boundary conditions are Dirichlet, there are an infinity of possible equations which may be applied:

$$(\mu)u_{ij} = (\mu)f_{ij}, \quad (\mu)/=0. \tag{2}$$

If D03EDF is used to solve the discretized equations, it turns out that the choice of (mu) can have a dramatic effect on the rate of convergence, and the obvious choice (mu)=1 is not the best. Some choices may even cause the multigrid method to fail altogether. In practice it has been found that a value of the same order as the other diagonal elements of the matrix is best, and the following value has been found to work well in practice:

$$(\mu)_{ij} = \min\left(-\left\{\frac{2}{h}+\frac{2}{x}\right\}, \left\{\frac{2}{h}+\frac{2}{y}\right\}, A_{ij}\right).$$

If the boundary conditions are either mixed or Neumann (i.e., B /= 0 on return from the user-supplied subroutine BNDY), then one of the points in the seven-point stencil lies outside the domain. In this case the normal derivative in the boundary conditions is used to eliminate the 'fictitious' point, $u_{outside}$:

$$\frac{\partial U}{\partial n} \tilde{=} \frac{1}{2h}(u_{outside} - u_{inside}). \tag{3}$$

It should be noted that if the boundary conditions are Neumann and (phi)(x,y)==0, then there is no unique solution. The routine returns with IFAIL = 5 in this case, and the seven-diagonal matrix is singular.

The four corners are treated separately. The user-supplied subroutine BNDY is called twice, once along each of the edges meeting at the corner. If both boundary conditions at this point

are Dirichlet and the prescribed solution values agree, then this value is used in an equation of the form (2). If the prescribed solution is discontinuous at the corner, then the average of the two values is used. If one boundary condition is Dirichlet and the other is mixed, then the value prescribed by the Dirichlet condition is used in an equation of the form given above. Finally, if both conditions are mixed or Neumann, then two ' fictitious' points are eliminated using two equations of the form (3).

It is possible that equations for which the solution is known at all points on the boundary, have coefficients which are not defined on the boundary. Since this routine calls the user-supplied subroutine PDEF at all points in the domain, including boundary points, arithmetic errors may occur in the user's routine PDEF which this routine cannot trap. If the user has an equation with Dirichlet boundary conditions (i.e., B = 0 at all points on the boundary), but with PDE coefficients which are singular on the boundary, then D03EDF could be called directly only using interior grid points with the user's own discretization.

After the equations have been set up as described above, they are checked for diagonal dominance. That is to say,

$$|A_{ij}^{4}| > \sum_{k/=4} |A_{ij}^{k}|, \quad i=1,2,\ldots,n_x ; \; j=1,2,\ldots,n_y .$$

If this condition is not satisfied then the routine returns with IFAIL = 6. The multigrid routine D03EDF may still converge in this case, but if the coefficients of the first derivatives in the partial differential equation are large compared with the coefficients of the second derivative, the user should consider using upwind differences (SCHEME = 'U').

Since this routine is designed primarily for use with D03EDF, this document should be read in conjunction with the document for that routine.

4. References

[1] Wesseling P (1982) MGD1 - A Robust and Efficient Multigrid Method. Multigrid Methods. Lecture Notes in Mathematics. 960 Springer-Verlag. 614--630.

5. Parameters

1: XMIN -- DOUBLE PRECISION Input

2: XMAX -- DOUBLE PRECISION Input
On entry: the lower and upper x co-ordinates of the rectangular region respectively, x_A and x_B. Constraint: XMIN < XMAX.

3: YMIN -- DOUBLE PRECISION Input

4: YMAX -- DOUBLE PRECISION Input
On entry: the lower and upper y co-ordinates of the rectangular region respectively, y_A and y_B. Constraint: YMIN < YMAX.

5: PDEF -- SUBROUTINE, supplied by the user. External Procedure
PDEF must evaluate the functions (alpha)(x,y), (beta)(x,y), (gamma)(x,y), (delta)(x,y), (epsilon)(x,y), (phi)(x,y) and (psi)(x,y) which define the equation at a general point (x,y).

Its specification is:

```
      SUBROUTINE PDEF (X, Y, ALPHA, BETA, GAMMA,
     1                 DELTA, EPSLON, PHI, PSI)
      DOUBLE PRECISION X, Y, ALPHA, BETA, GAMMA, DELTA,
     1                 EPSLON, PHI, PSI
```

1: X -- DOUBLE PRECISION Input

2: Y -- DOUBLE PRECISION Input
On entry: the x and y co-ordinates of the point at which the coefficients of the partial differential equation are to be evaluated. 8

3: ALPHA -- DOUBLE PRECISION Output

4: BETA -- DOUBLE PRECISION Output

5: GAMMA -- DOUBLE PRECISION Output

```
 6:  DELTA -- DOUBLE PRECISION                              Output

 7:  EPSLON -- DOUBLE PRECISION                             Output

 8:  PHI -- DOUBLE PRECISION                                Output

 9:  PSI -- DOUBLE PRECISION                                Output
     On exit: ALPHA, BETA, GAMMA, DELTA, EPSLON, PHI and PSI
     must be set to the values of (alpha)(x,y), (beta)(x,y),
     (gamma)(x,y), (delta)(x,y), (epsilon)(x,y), (phi)(x,y)
     and (psi)(x,y) respectively at the point specified by X
     and Y.
PDEF must be declared as EXTERNAL in the (sub)program
from which D03EEF is called. Parameters denoted as
Input must not be changed by this procedure.

6:  BNDY -- SUBROUTINE, supplied by the user.
                                              External Procedure
BNDY must evaluate the functions a(x,y), b(x,y), and c(x,y)
involved in the boundary conditions.

Its specification is:

       SUBROUTINE BNDY (X, Y, A, B, C, IBND)
       INTEGER          IBND
       DOUBLE PRECISION X, Y, A, B, C

 1:  X -- DOUBLE PRECISION                                   Input

 2:  Y -- DOUBLE PRECISION                                   Input
     On entry: the x and y co-ordinates of the point at
     which the boundary conditions are to be evaluated.

 3:  A -- DOUBLE PRECISION                                  Output

 4:  B -- DOUBLE PRECISION                                  Output

 5:  C -- DOUBLE PRECISION                                  Output
     On exit: A, B and C must be set to the values of the
     functions appearing in the boundary conditions.

 6:  IBND -- INTEGER                                         Input
     On entry: specifies on which boundary the point (X,Y)
     lies. IBND = 0, 1, 2 or 3 according as the point lies
     on the bottom, right, top or left boundary.
BNDY must be declared as EXTERNAL in the (sub)program
```

from which D03EEF is called. Parameters denoted as
Input must not be changed by this procedure.

7: NGX -- INTEGER Input

8: NGY -- INTEGER Input
On entry: the number of interior grid points in the x- and y
-directions respectively, n and n . If the seven-diagonal
x y
equations are to be solved by D03EDF, then NGX-1 and NGY-1
should preferably be divisible by as high a power of 2 as
possible. Constraint: NGX >= 3, NGY >= 3.

9: LDA -- INTEGER Input
On entry:
the first dimension of the array A as declared in the
(sub)program from which D03EEF is called.
Constraint: if only the seven-diagonal equations are
required, then LDA >= NGX*NGY. If a call to this routine is
to be followed by a call to D03EDF to solve the seven-
diagonal linear equations, LDA >= (4*(NGX+1)*(NGY+1))/3.

Note: this routine only checks the former condition. D03EDF,
if called, will check the latter condition.

10: A(LDA,7) -- DOUBLE PRECISION array Output
On exit: A(i,j), for i=1,2,...,NGX*NGY; j = 1,2,...,7,
contains the seven-diagonal linear equations produced by the
discretization described above. If LDA > NGX*NGY, the
remaining elements are not referenced by the routine, but if
LDA >= (4*(NGX+1)*(NGY+1))/3 then the array A can be passed
directly to D03EDF, where these elements are used as
workspace.

11: RHS(LDA) -- DOUBLE PRECISION array Output
On exit: the first NGX*NGY elements contain the right-hand
sides of the seven-diagonal linear equations produced by the
discretization described above. If LDA > NGX*NGY, the
remaining elements are not referenced by the routine, but if
LDA >= (4*(NGY+1)*(NGY+1))/3 then the array RHS can be
passed directly to D03EDF, where these elements are used as
workspace.

12: SCHEME -- CHARACTER*1 Input
On entry: the type of approximation to be used for the first
derivatives which occur in the partial differential

```
     equation.

     If SCHEME = 'C', then central differences are used.

     If SCHEME = 'U', then upwind differences are used.
     Constraint: SCHEME = 'C' or 'U'.

     Note: generally speaking, if at least one of the
     coefficients multiplying the first derivatives (DELTA or
     EPSLON as returned by PDEF) are large compared with the
     coefficients multiplying the second derivatives, then upwind
     differences may be more appropriate. Upwind differences are
     less accurate than central differences, but may result in
     more rapid convergence for strongly convective equations.
     The easiest test is to try both schemes.

13:  IFAIL -- INTEGER                                  Input/Output
     On entry: IFAIL must be set to 0, -1 or 1. Users who are
     unfamiliar with this parameter should refer to the Essential
     Introduction for details.

     On exit: IFAIL = 0 unless the routine detects an error or
     gives a warning (see Section 6).

     For this routine, because the values of output parameters
     may be useful even if IFAIL /=0 on exit, users are
     recommended to set IFAIL to -1 before entry. It is then
     essential to test the value of IFAIL on exit.

6. Error Indicators and Warnings

Errors or warnings specified by the routine:

If on entry IFAIL = 0 or -1, explanatory error messages are
output on the current error message unit (as defined by X04AAF).

IFAIL= 1
     On entry XMIN >= XMAX,

     or       YMIN >= YMAX,

     or       NGX < 3,

     or       NGY < 3,

     or       LDA < NGX*NGY,
```

```
     or       SCHEME is not one of 'C' or 'U'.

IFAIL= 2
     At some point on the boundary there is a derivative in the
     boundary conditions (B /= 0 on return from a BNDY) and there
                                                              2
                                                             dd U
     is a non-zero coefficient of the mixed derivative       ------
                                                            ddxddy
     (BETA /= 0 on return from PDEF).

IFAIL= 3
     A null boundary has been specified, i.e., at some point both
     A and B are zero on return from a call to BNDY.

IFAIL= 4
                                                               2
     The equation is not elliptic, i.e., 4*ALPHA*GAMMA<BETA  after
     a call to PDEF. The discretization has been completed, but
     the convergence of D03EDF cannot be guaranteed.

IFAIL= 5
     The boundary conditions are purely Neumann (only the
     derivative is specified) and there is, in general, no unique
     solution.

IFAIL= 6
     The equations were not diagonally dominant. (See Section 3).

7. Accuracy

Not applicable.

8. Further Comments

If this routine is used as a pre-processor to the multigrid
routine D03EDF it should be noted that the rate of convergence of
that routine is strongly dependent upon the number of levels in
the multigrid scheme, and thus the choice of NGX and NGY is very
important.

9. Example

The program solves the elliptic partial differential equation
```

```
                          2     2
                         dd U  dd U   { ddU  ddU}
                         ----+ ----+50{ ---+ ---}=f(x,y)
                            2     2   { ddx  ddy}
                         ddx   ddy
```

on the unit square 0<=x, y<=1, with boundary conditions

```
      ddU
      --- given on x=0 and y=0,
      ddn

     U given on x=1 and y=1.
```

The function f(x,y) and the exact form of the boundary conditions are derived from the exact solution U(x,y)=sinxsiny.

The equation is first solved using central differences. Since the coefficients of the first derivatives are large, the linear equations are not diagonally dominated, and convergence is slow. The equation is solved a second time with upwind differences, showing that convergence is more rapid, but the solution is less accurate.

The example program is not reproduced here. The source code for all example programs is distributed with the NAG Foundation Library software and should be available on-line.

%%%

D03 -- Partial Differential Equations D03FAF
D03FAF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for your implementation to check implementation-dependent details. The symbol (*) after a NAG routine name denotes a routine that is not included in the Foundation Library.

1. Purpose

D03FAF solves the Helmholtz equation in Cartesian co-ordinates in three dimensions using the standard seven-point finite difference approximation. This routine is designed to be particularly efficient on vector processors.

2. Specification

```
 SUBROUTINE D03FAF (XS, XF, L, LBDCND, BDXS, BDXF, YS, YF,
1                   M, MBDCND, BDYS, BDYF, ZS, ZF, N,
2                   NBDCND, BDZS, BDZF, LAMBDA, LDIMF,
3                   MDIMF, F, PERTRB, W, LWRK, IFAIL)
 INTEGER           L, LBDCND, M, MBDCND, N, NBDCND, LDIMF,
1                  MDIMF, LWRK, IFAIL
 DOUBLE PRECISION XS, XF, BDXS(MDIMF,N+1), BDXF(MDIMF,N+1),
1                  YS, YF, BDYS(LDIMF,N+1), BDYF(LDIMF,N+1),
2                  ZS, ZF, BDZS(LDIMF,M+1), BDZF(LDIMF,M+1),
3                  LAMBDA, F(LDIMF,MDIMF,N+1), PERTRB, W
4                  (LWRK)
```

3. Description

D03FAF solves the three-dimensional Helmholtz equation in cartesian co-ordinates:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + (\lambda)u = f(x,y,z)$$

This subroutine forms the system of linear equations resulting from the standard seven-point finite difference equations, and then solves the system using a method based on the fast Fourier transform (FFT) described by Swarztrauber [1]. This subroutine is based on the routine HW3CRT from FISHPACK (see Swarztrauber and Sweet [2]).

More precisely, the routine replaces all the second derivatives by second-order central difference approximations, resulting in a block tridiagonal system of linear equations. The equations are modified to allow for the prescribed boundary conditions. Either the solution or the derivative of the solution may be specified on any of the boundaries, or the solution may be specified to be periodic in any of the three dimensions. By taking the discrete Fourier transform in the x- and y-directions, the equations are reduced to sets of tridiagonal systems of equations. The Fourier transforms required are computed using the multiple FFT routines found in Chapter C06 of the NAG Fortran Library.

4. References

[1] Swarztrauber P N (1984) Fast Poisson Solvers. Studies in

```
      Numerical Analysis. (ed G H Golub) Mathematical Association
      of America.

[2]   Swarztrauber P N and Sweet R A (1979) Efficient Fortran
      Subprograms for the Solution of Separable Elliptic Partial
      Differential Equations. ACM Trans. Math. Softw. 5 352--364.

5. Parameters

 1:  XS -- DOUBLE PRECISION                                    Input
     On entry: the lower bound of the range of x, i.e., XS <= x
     <= XF. Constraint: XS < XF.

 2:  XF -- DOUBLE PRECISION                                    Input
     On entry: the upper bound of the range of x, i.e., XS <= x
     <= XF. Constraint: XS < XF.

 3:  L -- INTEGER                                              Input
     On entry: the number of panels into which the interval
     (XS,XF) is subdivided. Hence, there will be L+1 grid points
     in the x-direction given by x =XS+(i-1)*(delta)x, for
                                  i
     i=1,2,...,L+1, where (delta)x=(XF-XS)/L is the panel width.
     Constraint: L >= 5.

 4:  LBDCND -- INTEGER                                         Input
     On entry: indicates the type of boundary conditions at x =
     XS and x = XF.
     LBDCND = 0
           if the solution is periodic in x, i.e.,
           u(XS,y,z)=u(XF,y,z).

     LBDCND = 1
           if the solution is specified at x = XS and x = XF.

     LBDCND = 2
           if the solution is specified at x = XS and the
           derivative of the solution with respect to x is
           specified at x = XF.

     LBDCND = 3
           if the derivative of the solution with respect to x is
           specified at x = XS and x = XF.

     LBDCND = 4
           if the derivative of the solution with respect to x is
```

```
      specified at x = XS and the solution is specified at x
      = XF.
 Constraint: 0 <= LBDCND <= 4.

 5:  BDXS(MDIMF,N+1) -- DOUBLE PRECISION array              Input
     On entry: the values of the derivative of the solution with
     respect to x at x = XS. When LBDCND = 3 or 4, BDXS
     (j,k)=u (XS,y ,z ), for j=1,2,...,M+1; k=1,2,...,N+1.
             x     j  k

     When LBDCND has any other value, BDXS is not referenced.

 6:  BDXF(MDIMF,N+1) -- DOUBLE PRECISION array              Input
     On entry: the values of the derivative of the solution with
     respect to x at x = XF. When LBDCND = 2 or 3, BDXF(j,k) =
     u (XF,y ,z ), for j=1,2,...,M+1; k=1,2,...,N+1.
      x     i  k

     When LBDCND has any other value, BDXF is not referenced.

 7:  YS -- DOUBLE PRECISION                                 Input
     On entry: the lower bound of the range of y, i.e., YS <= y
     <= YF. Constraint: YS < YF.

 8:  YF -- DOUBLE PRECISION                                 Input
     On entry: the upper bound of the range of y, i.e., YS <= y
     <= YF. Constraint: YS < YF.

 9:  M -- INTEGER                                           Input
     On entry: the number of panels into which the interval
     (YS,YF) is subdivided. Hence, there will be M+1 grid points
     in the y-direction given by y  = YS + (j-1)*(delta)y for
                                  j
     j=1,2,...,M+1, where (delta)y=(YF-YS)/M is the panel width.
     Constraint: M >= 5.

10:  MBDCND -- INTEGER                                      Input
     On entry: indicates the type of boundary conditions at y =
     YS and y = YF.
     MBDCND = 0
           if the solution is periodic in y, i.e.,
           u(x,YF,z)=u(x,YS,z).

     MBDCND = 1
           if the solution is specified at y = YS and y = YF.
```

```
      MBDCND = 2
            if the solution is specified at y = YS and the
            derivative of the solution with respect to y is
            specified at y = YF.

      MBDCND = 3
            if the derivative of the solution with respect to y is
            specified at y = YS and y = YF.

      MBDCND = 4
            if the derivative of the solution with respect to y is
            specified at y = YS and the solution is specified at y
            = YF.
       Constraint: 0 <= MBDCND <= 4.

11:   BDYS(LDIMF,N+1) -- DOUBLE PRECISION array              Input
      On entry: the values of the derivative of the solution with
      respect to y at y = YS. When MBDCND = 3 or 4, BDYS
      (i,k)=u (x ,y ,z ), for i=1,2,...,L+1; k=1,2,...,N+1.
              y  i  s  k

      When MBDCND has any other value, BDYS is not referenced.

12:   BDYF(LDIMF,N+1) -- DOUBLE PRECISION array              Input
      On entry: the values of the derivative of the solution with
      respect to y at y = YF. When MBDCND = 2 or 3, BDYF
      (i,k)=u (x ,YF,z ), for i=1,2,...,L+1; k=1,2,...,N+1.
              y  i     k

      When MBDCND has any other value, BDYF is not referenced.

13:   ZS -- DOUBLE PRECISION                                 Input
      On entry: the lower bound of the range of z, i.e., ZS <= z
      <= ZF. Constraint: ZS < ZF.

14:   ZF -- DOUBLE PRECISION                                 Input
      On entry: the upper bound of the range of z, i.e., ZS <= z
      <= ZF. Constraint: ZS < ZF.

15:   N -- INTEGER                                          Input
      On entry: the number of panels into which the interval
      (ZS,ZF) is subdivided. Hence, there will be N+1 grid points
      in the z-direction given by z =ZS+(k-1)*(delta)z, for
                                    k
      k=1,2,...,N+1, where (delta)z=(ZF-ZS)/N is the panel width.
      Constraint: N >= 5.
```

```
16:  NBDCND -- INTEGER                                          Input
     On entry: specifies the type of boundary conditions at z =
     ZS and z = ZF.
     NBDCND = 0
           if the solution is periodic in z, i.e.,
           u(x,y,ZF)=u(x,y,ZS).

     NBDCND = 1
           if the solution is specified at z = ZS and z = ZF.

     NBDCND = 2
           if the solution is specified at z = ZS and the
           derivative of the solution with respect to z is
           specified at z = ZF.

     NBDCND = 3
           if the derivative of the solution with respect to z is
           specified at z = ZS and z = ZF.

     NBDCND = 4
           if the derivative of the solution with respect to z is
           specified at z = ZS and the solution is specified at z
           = ZF.
      Constraint: 0 <= NBDCND <= 4.

17:  BDZS(LDIMF,M+1) -- DOUBLE PRECISION array                  Input
     On entry: the values of the derivative of the solution with
     respect to z at z = ZS. When NBDCND = 3 or 4, BDZS
     (i,j)=u (x ,y ,ZS)=u(x,y,z), for i=1,2,...,L+1;
             z  i  j
     j=1,2,...,M+1.

     When NBDCND has any other value, BDZS is not referenced.

18:  BDZF(LDIMF,M+1) -- DOUBLE PRECISION array                  Input
     On entry: the values of the derivative of the solution with
     respect to z at z = ZF. When NBDCND = 2 or 3, BDZF
     (i,j)=u (x ,y ,ZF)=u(x,y,z), for i=1,2,...,L+1;
             z  i  j
     j=1,2,...,M+1.

     When NBDCND has any other value, BDZF is not referenced.

19:  LAMBDA -- DOUBLE PRECISION                                 Input
     On entry: the constant (lambda) in the Helmholtz equation.
```

For certain positive values of (lambda) a solution to the differential equation may not exist, and close to these values the solution of the discretized problem will be extremely ill-conditioned. If (lambda)>0, then D03FAF will set IFAIL to 3, but will still attempt to find a solution. However, since in general the values of (lambda) for which no solution exists cannot be predicted a priori, the user is advised to treat any results computed with (lambda)>0 with great caution.

20: LDIMF -- INTEGER Input
On entry:
the first dimension of the arrays F, BDYS, BDYF, BDZS and BDZF as declared in the (sub)program from which D03FAF is called.
Constraint: LDIMF >= L + 1.

21: MDIMF -- INTEGER Input
On entry: the second dimension of the array F and
the first dimension of the arrays BDXS and BDXF as declared in the (sub)program from which D03FAF is called.
Constraint: MDIMF >= M + 1.

22: F(LDIMF,MDIMF,N+1) -- DOUBLE PRECISION array Input/Output
On entry: the values of the right-side of the Helmholtz equation and boundary values (if any).

F(i,j,k)=f(x ,y ,z) i=2,3,...,L, j=2,3,...,M and k
            i j k
=2,3,...,N.

On the boundaries F is defined by

LBDCND	F(1,j,k)	F(L+1,j,k)	
0	f(XS,y ,z)	f(XS,y ,z)	
	j k	j k	
1	u(XS,y ,z)	u(XF,y ,z)	
	j k	j k	
2	u(XS,y ,z)	f(XF,y ,z)	j=1,2,...,M+1
	j k	j k	
3	f(XS,y ,z)	f(XF,y ,z)	k=1,2,...,N+1
	j k	j k	

```
4         f(XS,y ,z )u(XF,y ,z )
                j  k      j  k



MBDCND    F(i,1,k)  F(
                    i,M+1,k)

0         f(x ,YS,z )f(x ,YS,z )
             i     k   i     k

1         u(x ,YS,z )u(x ,YF,z )
             i     k   i     k

2         u(x ,YS,z )f(x ,YF,z )  i=1,2,...,L+1
             i     k   i     k

3         f(x ,YS,z )f(x ,YF,z )  k=1,2,...,N+1
             i     k   i     k

4         f(x ,YS,z )u(x ,YF,z )
             i     k   i     k



NBDCND    F(i,j,1)  F(i,j,N+1
                    )

0         f(x ,y ,ZS)f(x ,y ,ZS)
             i  j      i  j

1         u(x ,y ,ZS)u(x ,y ,ZF)
             i  j      i  j

2         u(x ,y ,ZS)f(x ,y ,ZF)  i=1,2,...,L+1
             i  j      i  j

3         f(x ,y ,ZS)f(x ,y ,ZF)  j=1,2,...,M+1
             i  j      i  j

4         f(x ,y ,ZS)u(x ,y ,ZF)
             i  j      i  j
Note: if the table calls for both the solution u and the
right-hand side f on a boundary, then the solution must be
specified. On exit: F contains the solution u(i,j,k) of the
finite difference approximation for the grid point (
```

x ,y ,z) for i=1,2,...,L+1, j=1,2,...,M+1 and k=1,2,...,N+1.
 i j k

23: PERTRB -- DOUBLE PRECISION Output
On exit: PERTRB = 0, unless a solution to Poisson's equation ((lambda)=0) is required with a combination of periodic or derivative boundary conditions (LBDCND, MBDCND and NBDCND = 0 or 3). In this case a solution may not exist. PERTRB is a constant, calculated and subtracted from the array F, which ensures that a solution exists. D03FAF then computes this solution, which is a least-squares solution to the original approximation. This solution is not unique and is unnormalised. The value of PERTRB should be small compared to the right-hand side F, otherwise a solution has been obtained to an essentially different problem. This comparison should always be made to insure that a meaningful solution has been obtained.

24: W(LWRK) -- DOUBLE PRECISION array Workspace

25: LWRK -- INTEGER Input
On entry:
the dimension of the array W as declared in the (sub)program from which D03FAF is called.
LRWK>=2*(N+1)*max(L,M)+3*L+3*M+4*N+6 is an upper bound on the required size of W. If LWRK is too small, the routine exits with IFAIL = 2, and if on entry IFAIL = 0 or IFAIL = -1, a message is output giving the exact value of LWRK required to solve the current problem.

26: IFAIL -- INTEGER Input/Output
On entry: IFAIL must be set to 0, -1 or 1. Users who are unfamiliar with this parameter should refer to the Essential Introduction for details.

On exit: IFAIL = 0 unless the routine detects an error or gives a warning (see Section 6).

For this routine, because the values of output parameters may be useful even if IFAIL /=0 on exit, users are recommended to set IFAIL to -1 before entry. It is then essential to test the value of IFAIL on exit.

6. Error Indicators and Warnings

Errors or warnings specified by the routine:

```
If on entry IFAIL = 0 or -1, explanatory error messages are
output on the current error message unit (as defined by X04AAF).

IFAIL= 1
     On entry XS >= XF,

     or       L < 5,

     or       LBDCND < 0,

     or       LBDCND > 4,

     or       YS >= YF,

     or       M < 5,

     or       MBDCND < 0,

     or       MBDCND > 4,

     or       ZS >= ZF,

     or       N < 5,

     or       NBDCND < 0,

     or       NBDCND > 4,

     or       LDIMF < L + 1 > 0,

     or       MDIMF < M + 1.

IFAIL= 2
     On entry LWRK is too small.

IFAIL= 3
     On entry (lambda) > 0.

7. Accuracy

None.

8. Further Comments

The execution time is roughly proportional to
```

L*M*N*(log L+log M+5), but also depends on input parameters
          2   2
LBDCND and MBDCND.

9. Example

The example solves the Helmholz equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}+(\text{lambda})u=f(x,y,z)$$

$$\text{for } (x,y,z) \text{ is in } [0,1]*[0,2(\text{pi})]*[0,\frac{(\text{pi})}{2}] \text{ where } (\text{lambda})=-2, \text{ and}$$

f(x,y,z) is derived from the exact solution

$$u(x,y,z)=x^4 \sin(y)\cos(z).$$

The equation is subject to the following boundary conditions,
again derived from the exact solution given above.

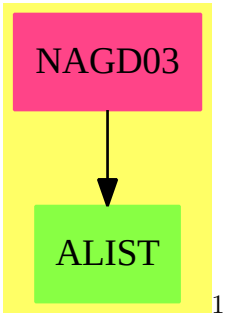
u(0,y,z) and u(1,y,z) are prescribed (i.e., LBDCND = 1).

u(x,0,z)=u(x,2(pi),z) (i.e., MBDCND = 0).

$$u(x,y,0) \text{ and } u_x(x,y,\frac{(\text{pi})}{2}) \text{ are prescribed (i.e. NBDCND = 2).}$$

The example program is not reproduced here. The source code for
all example programs is distributed with the NAG Foundation
Library software and should be available on-line.

15.22 NagPartialDifferentialEquationsPackage



1.00

Exports:
d03edf d03eef d03faf

⟨*package NAGD03 NagPartialDifferentialEquationsPackage*⟩≡

```
)abbrev package NAGD03 NagPartialDifferentialEquationsPackage
++ Author: Godfrey Nolan and Mike Dewar
++ Date Created: Jan 1994
++ Date Last Updated: Thu May 12 17:44:51 1994
++ Description:
++ This package uses the NAG Library to solve partial
++ differential equations.
++ See \downlink{Manual Page}{manpageXXd03}.
NagPartialDifferentialEquationsPackage(): Exports == Implementation where
  S ==> Symbol
  FOP ==> FortranOutputStackPackage

  Exports ==> with
    d03edf : (Integer,Integer,Integer,Integer,_
        DoubleFloat,Integer,Matrix DoubleFloat,Matrix DoubleFloat,Matrix DoubleFl
     ++ d03edf(ngx,ngy,lda,maxit,acc,iout,a,rhs,ub,ifail)
     ++ solves seven-diagonal systems of linear equations which
     ++ arise from the discretization of an elliptic partial differential
     ++ equation on a rectangular region. This routine uses a multigrid
     ++ technique.
     ++ See \downlink{Manual Page}{manpageXXd03edf}.
    d03eef : (DoubleFloat,DoubleFloat,DoubleFloat,DoubleFloat,_
        Integer,Integer,Integer,String,Integer,Union(fn:FileName,fp:Asp73(PDEF)),
     ++ d03eef(xmin,xmax,ymin,ymax,ngx,ngy,lda,scheme,ifail,pdef,bndy)
     ++ discretizes a second order elliptic partial differential
     ++ equation (PDE) on a rectangular region.
     ++ See \downlink{Manual Page}{manpageXXd03eef}.
    d03faf : (DoubleFloat,DoubleFloat,Integer,Integer,_
        Matrix DoubleFloat,Matrix DoubleFloat,DoubleFloat,DoubleFloat,Integer,Int
     ++ d03faf(xs,xf,l,lbdcnd,bdxs,bdxf,ys,yf,m,mbdcnd,bdys,bdyf,zs,zf,n,nbdcnd,b
```

```
    ++ solves the Helmholtz equation in Cartesian co-ordinates in
    ++ three dimensions using the standard seven-point finite difference
    ++ approximation. This routine is designed to be particularly
    ++ efficient on vector processors.
    ++ See \downlink{Manual Page}{manpageXXd03faf}.
  Implementation ==> add

    import Lisp
    import DoubleFloat
    import Any
    import Record
    import Integer
    import Matrix DoubleFloat
    import Boolean
    import NAGLinkSupportPackage
    import AnyFunctions1(Integer)
    import AnyFunctions1(String)
    import AnyFunctions1(DoubleFloat)
    import AnyFunctions1(Matrix DoubleFloat)
    import AnyFunctions1(ThreeDimensionalMatrix DoubleFloat)
    import FortranPackage
    import Union(fn:FileName,fp:Asp73(PDEF))
    import Union(fn:FileName,fp:Asp74(BNDY))




    d03edf(ngxArg:Integer,ngyArg:Integer,ldaArg:Integer,_
        maxitArg:Integer,accArg:DoubleFloat,ioutArg:Integer,_
        aArg:Matrix DoubleFloat,rhsArg:Matrix DoubleFloat,ubArg:Matrix DoubleFloat,_
        ifailArg:Integer): Result ==
        [(invokeNagman(NIL$Lisp,_
        "d03edf",_
        ["ngx"::S,"ngy"::S,"lda"::S,"maxit"::S,"acc"::S_
        ,"iout"::S,"numit"::S,"ifail"::S,"us"::S,"u"::S,"a"::S,"rhs"::S,"ub"::S_
        ]$Lisp,_
        ["us"::S,"u"::S,"numit"::S]$Lisp,_
        [["double"::S,"acc"::S,["us"::S,"lda"::S]$Lisp_
        ,["u"::S,"lda"::S]$Lisp,["a"::S,"lda"::S,7$Lisp]$Lisp,["rhs"::S,"lda"::S]$Lisp,["ub"
        ]$Lisp_
        ,["integer"::S,"ngx"::S,"ngy"::S,"lda"::S,"maxit"::S_
        ,"iout"::S,"numit"::S,"ifail"::S]$Lisp_
        ]$Lisp,_
        ["us"::S,"u"::S,"numit"::S,"a"::S,"rhs"::S,"ub"::S,"ifail"::S]$Lisp,_
        [([ngxArg::Any,ngyArg::Any,ldaArg::Any,maxitArg::Any,accArg::Any,ioutArg::Any,ifailA
        @List Any]$Lisp)$Lisp)_
```

```
    pretend List (Record(key:Symbol,entry:Any))]$Result

d03eef(xminArg:DoubleFloat,xmaxArg:DoubleFloat,yminArg:DoubleFloat,_
    ymaxArg:DoubleFloat,ngxArg:Integer,ngyArg:Integer,_
    ldaArg:Integer,schemeArg:String,ifailArg:Integer,_
    pdefArg:Union(fn:FileName,fp:Asp73(PDEF)),bndyArg:Union(fn:FileName,fp:As
    pushFortranOutputStack(pdefFilename := aspFilename "pdef")$FOP
    if pdefArg case fn
              then outputAsFortran(pdefArg.fn)
              else outputAsFortran(pdefArg.fp)
    popFortranOutputStack()$FOP
    pushFortranOutputStack(bndyFilename := aspFilename "bndy")$FOP
    if bndyArg case fn
              then outputAsFortran(bndyArg.fn)
              else outputAsFortran(bndyArg.fp)
    popFortranOutputStack()$FOP
    [(invokeNagman([pdefFilename,bndyFilename]$Lisp,_
    "d03eef",_
    ["xmin"::S,"xmax"::S,"ymin"::S,"ymax"::S,"ngx"::S_
    ,"ngy"::S,"lda"::S,"scheme"::S,"ifail"::S,"pdef"::S_
    ,"bndy"::S,"a"::S,"rhs"::S]$Lisp,_
    ["a"::S,"rhs"::S,"pdef"::S,"bndy"::S]$Lisp,_
    [["double"::S,"xmin"::S,"xmax"::S,"ymin"::S_
    ,"ymax"::S,["a"::S,"lda"::S,7$Lisp]$Lisp,["rhs"::S,"lda"::S]$Lisp,"pdef":
    ,["integer"::S,"ngx"::S,"ngy"::S,"lda"::S,"ifail"::S_
    ]$Lisp_
    ,["character"::S,"scheme"::S]$Lisp_
    ]$Lisp,_
    ["a"::S,"rhs"::S,"ifail"::S]$Lisp,_
    [([xminArg::Any,xmaxArg::Any,yminArg::Any,ymaxArg::Any,ngxArg::Any,ngyArg
    @List Any]$Lisp)$Lisp)_
    pretend List (Record(key:Symbol,entry:Any))]$Result

d03faf(xsArg:DoubleFloat,xfArg:DoubleFloat,lArg:Integer,_
    lbdcndArg:Integer,bdxsArg:Matrix DoubleFloat,bdxfArg:Matrix DoubleFloat,_
    ysArg:DoubleFloat,yfArg:DoubleFloat,mArg:Integer,_
    mbdcndArg:Integer,bdysArg:Matrix DoubleFloat,bdyfArg:Matrix DoubleFloat,_
    zsArg:DoubleFloat,zfArg:DoubleFloat,nArg:Integer,_
    nbdcndArg:Integer,bdzsArg:Matrix DoubleFloat,bdzfArg:Matrix DoubleFloat,_
    lambdaArg:DoubleFloat,ldimfArg:Integer,mdimfArg:Integer,_
    lwrkArg:Integer,fArg:ThreeDimensionalMatrix DoubleFloat,ifailArg:Integer)
    [(invokeNagman(NIL$Lisp,_
    "d03faf",_
    ["xs"::S,"xf"::S,"l"::S,"lbdcnd"::S,"ys"::S_
    ,"yf"::S,"m"::S,"mbdcnd"::S,"zs"::S,"zf"::S_
    ,"n"::S,"nbdcnd"::S,"lambda"::S,"ldimf"::S,"mdimf"::S_
```

```
,"lwrk"::S,"pertrb"::S,"ifail"::S,"bdxs"::S,"bdxf"::S,"bdys"::S,"bdyf"::S,"bdzs"::S_
,"bdzf"::S,"f"::S,"w"::S]$Lisp,_
["pertrb"::S,"w"::S]$Lisp,_
[["double"::S,"xs"::S,"xf"::S,["bdxs"::S,"mdimf"::S,["+"::S,"n"::S,1$Lisp]$Lisp]$Lis
,["bdxf"::S,"mdimf"::S,["+"::S,"n"::S,1$Lisp]$Lisp]$Lisp,"ys"::S,"yf"::S,["bdys"::S
,["bdyf"::S,"ldimf"::S,["+"::S,"n"::S,1$Lisp]$Lisp]$Lisp,"zs"::S_
,"zf"::S,["bdzs"::S,"ldimf"::S,["+"::S,"m"::S,1$Lisp]$Lisp]$Lisp,["bdzf"::S,"ldimf"
,"lambda"::S,"pertrb"::S,["f"::S,"ldimf"::S,"mdimf"::S,["+"::S,"n"::S,1$Lisp]$Lisp]$
,["integer"::S,"l"::S,"lbdcnd"::S,"m"::S,"mbdcnd"::S_
,"n"::S,"nbdcnd"::S,"ldimf"::S,"mdimf"::S,"lwrk"::S,"ifail"::S]$Lisp_
]$Lisp,_
["pertrb"::S,"f"::S,"ifail"::S]$Lisp,_
[([xsArg::Any,xfArg::Any,lArg::Any,lbdcndArg::Any,ysArg::Any,yfArg::Any,mArg::Any,mb
@List Any]$Lisp)$Lisp)_
pretend List (Record(key:Symbol,entry:Any))]$Result
```

⟨*NAGD03.dotabb*⟩≡

```
"NAGD03" [color="#FF4488",href="bookvol10.4.pdf#nameddest=NAGD03"]
"ALIST" [color="#88FF44",href="bookvol10.3.pdf#nameddest=ALIST"]
"NAGD03" -> "ALIST"
```

15.23 package NAGC02 NagPolynomialRootsPackage

⟨*NagPolynomialRootsPackage.help*⟩≡

```
C02(3NAG)          Foundation Library (12/10/92)          C02(3NAG)

     C02 -- Zeros of Polynomials                    Introduction -- C02
                                Chapter C02
                          Zeros of Polynomials

     1. Scope of the Chapter

     This chapter is concerned with computing the zeros of a
     polynomial with real or complex coefficients.

     2. Background to the Problems

     Let f(z) be a polynomial of degree n with complex coefficients
     a :
      i

                          n    n-1    n-2
              f(z)==a z +a z   +a z   +...+a   z+a ,  a /=0.
                     0    1      2          n-1   n    0

     A complex number z  is called a zero of f(z) (or equivalently a
                       1
     root of the equation f(z)=0), if:

                                  f(z )=0.
                                     1

     If z  is a zero, then f(z) can be divided by a factor (z-z ):
         1                                                     1

                           f(z)=(z-z )f (z)                          (1)
                                    1  1

     where f (z) is a polynomial of degree n-1. By the Fundamental
            1
     Theorem of Algebra, a polynomial f(z) always has a zero, and so
     the process of dividing out factors (z-z ) can be continued until
                                            i
     we have a complete factorization of f(z)
```

$$f(z)==a_0(z-z_1)(z-z_2)\ldots(z-z_n).$$

Here the complex numbers $z_1, z_2, \ldots, z_n$ are the zeros of f(z); they may not all be distinct, so it is sometimes more convenient to write:

$$f(z)==a_0(z-z_1)^{m_1}(z-z_2)^{m_2}\ldots(z-z_k)^{m_k},\quad k<=n,$$

with distinct zeros $z_1, z_2, \ldots, z_k$ and multiplicities $m_i >=1$. If $m_i =1$, z_i is called a single zero, if $m_i >1$, z_i is called a multiple or repeated zero; a multiple zero is also a zero of the derivative of f(z).

If the coefficients of f(z) are all real, then the zeros of f(z) are either real or else occur as pairs of conjugate complex numbers x+iy and x-iy. A pair of complex conjugate zeros are the zeros of a quadratic factor of f(z), (z^2+rz+s), with real coefficients r and s.

Mathematicians are accustomed to thinking of polynomials as pleasantly simple functions to work with. However the problem of numerically computing the zeros of an arbitrary polynomial is far from simple. A great variety of algorithms have been proposed, of which a number have been widely used in practice; for a fairly comprehensive survey, see Householder [1]. All general algorithms are iterative. Most converge to one zero at a time; the corresponding factor can then be divided out as in equation (1) above - this process is called deflation or, loosely, dividing out the zero - and the algorithm can be applied again to the polynomial $f_1(z)$. A pair of complex conjugate zeros can be divided out together - this corresponds to dividing f(z) by a quadratic factor.

Whatever the theoretical basis of the algorithm, a number of practical problems arise: for a thorough discussion of some of them see Peters and Wilkinson [2] and Wilkinson [3]. The most

elementary point is that, even if z_1 is mathematically an exact zero of f(z), because of the fundamental limitations of computer arithmetic the computed value of $f(z_1)$ will not necessarily be exactly 0.0. In practice there is usually a small region of values of z about the exact zero at which the computed value of f(z) becomes swamped by rounding errors. Moreover in many algorithms this inaccuracy in the computed value of f(z) results in a similar inaccuracy in the computed step from one iterate to the next. This limits the precision with which any zero can be computed. Deflation is another potential cause of trouble, since, in the notation of equation (1), the computed coefficients of $f_1(z)$ will not be completely accurate, especially if z_1 is not an exact zero of f(z); so the zeros of the computed $f_1(z)$ will deviate from the zeros of f(z).

A zero is called ill-conditioned if it is sensitive to small changes in the coefficients of the polynomial. An ill-conditioned zero is likewise sensitive to the computational inaccuracies just mentioned. Conversely a zero is called well-conditioned if it is comparatively insensitive to such perturbations. Roughly speaking a zero which is well separated from other zeros is well-conditioned, while zeros which are close together are ill-conditioned, but in talking about 'closeness' the decisive factor is not the absolute distance between neighbouring zeros but their ratio: if the ratio is close to 1 the zeros are ill-conditioned. In particular, multiple zeros are ill-conditioned. A multiple zero is usually split into a cluster of zeros by perturbations in the polynomial or computational inaccuracies.

2.1. References

[1] Householder A S (1970) The Numerical Treatment of a Single Nonlinear Equation. McGraw-Hill.

[2] Peters G and Wilkinson J H (1971) Practical Problems Arising in the Solution of Polynomial Equations. J. Inst. Maths Applics. 8 16--35.

[3] Wilkinson J H (1963) Rounding Errors in Algebraic Processes, Chapter 2. HMSO.

3. Recommendations on Choice and Use of Routines

```
3.1. Discussion

Two routines are available: C02AFF for polynomials with complex
coefficients and C02AGF for polynomials with real coefficients.

C02AFF and C02AGF both use a variant of Laguerre's Method due to
BT Smith to calculate each zero until the degree of the deflated
polynomial is less than 3, whereupon the remaining zeros are
obtained using the 'standard' closed formulae for a quadratic or
linear equation.

The accuracy of the roots will depend on how ill-conditioned they
are. Peters and Wilkinson [2] describe techniques for estimating
the errors in the zeros after they have been computed.

3.2. Index

 Zeros of a complex polynomial                              C02AFF
 Zeros of a real polynomial                                 C02AGF


C02 -- Zeros of Polynomials                      Contents -- C02
Chapter C02

Zeros of Polynomials

C02AFF  All zeros of complex polynomial, modified Laguerre method

C02AGF  All zeros of real polynomial, modified Laguerre method
```

%%%

```
C02AFF(3NAG)      Foundation Library (12/10/92)      C02AFF(3NAG)

C02 -- Zeros of Polynomials                                  C02AFF
        C02AFF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for
your implementation to check implementation-dependent details.
The symbol (*) after a NAG routine name denotes a routine that is
not included in the Foundation Library.

1. Purpose

C02AFF finds all the roots of a complex polynomial equation,
```

using a variant of Laguerre's Method.

2. Specification

```
      SUBROUTINE C02AFF (A, N, SCALE, Z, W, IFAIL)
      INTEGER          N, IFAIL
      DOUBLE PRECISION A(2,N+1), Z(2,N), W(4*(N+1))
      LOGICAL          SCALE
```

3. Description

The routine attempts to find all the roots of the nth degree complex polynomial equation

$$P(z)=a_0 z^n + a_1 z^{n-1} + a_2 z^{n-2} + \ldots + a_{n-1} z + a_n = 0.$$

The roots are located using a modified form of Laguerre's Method, originally proposed by Smith [2].

The method of Laguerre [3] can be described by the iterative scheme

$$L(z_k) = z_{k+1} - z_k = \frac{-n P(z_k)}{P'(z_k) \pm \sqrt{H(z_k)}},$$

where $H(z_k)=(n-1)[(n-1)(P'(z_k))^2 - nP(z_k)P''(z_k)]$, and z_0 is specified.

The sign in the denominator is chosen so that the modulus of the Laguerre step at z_k, viz. $|L(z_k)|$, is as small as possible. The method can be shown to be cubically convergent for isolated roots (real or complex) and linearly convergent for multiple roots. The routine generates a sequence of iterates $z_1, z_2, z_3, \ldots$, such that $|P(z_{k+1})| < |P(z_k)|$ and ensures that $z_{k+1} + L(z_{k+1})$ 'roughly' lies inside a circular region of radius $|F|$ about z_{k+1} known to

contain a zero of P(z); that is, |L(z k)|<=|F|, where F denotes the Fejer bound (see Marden [1]) at the point z k+1. Following Smith [2], F is taken to be min(B,1.445*n*R), where B is an upper bound for the magnitude of the smallest zero given by

$$B=1.0001*\min(\backslash/n*L(z_k),|r_1|,|a_n/a_0|^{1/n}),$$

r_1 is the zero X of smaller magnitude of the quadratic equation

$$2(P''(z_k)/(2*n*(n-1)))X^2+2(P'(z_k)/n)X+P(z_k)=0$$

and the Cauchy lower bound R for the smallest zero is computed (using Newton's Method) as the positive root of the polynomial equation

$$|a_0|z^n+|a_1|z^{n-1}+|a_2|z^{n-2}+\ldots+|a_{n-1}|z-|a_n|=0.$$

Starting from the origin, successive iterates are generated according to the rule $z_{k+1}=z_k+L(z_k)$ for k = 1,2,3,... and $L(z_k)$ is 'adjusted' so that $|P(z_{k+1})|<|P(z_k)|$ and $|L(z_k)|<=|F|$. The iterative procedure terminates if $P(z_{k+1})$ is smaller in absolute value than the bound on the rounding error in $P(z_{k+1})$ and the current iterate $z=z_{k+1}$ is taken to be a zero of P(z). The deflated polynomial $\tilde{P}(z)=P(z)/(z-z_{p\,k+1})$ of degree n-1 is then formed, and the above procedure is repeated on the deflated polynomial until n<3, whereupon the remaining roots are obtained via the 'standard' closed formulae for a linear (n = 1) or quadratic (n = 2) equation.

To obtain the roots of a quadratic polynomial, C02AHF(*) can be used.

4. References

[1] Marden M (1966) Geometry of Polynomials. Mathematical Surveys. 3 Am. Math. Soc., Providence, RI.

[2] Smith B T (1967) ZERPOL: A Zero Finding Algorithm for Polynomials Using Laguerre's Method. Technical Report. Department of Computer Science, University of Toronto, Canada.

[3] Wilkinson J H (1965) The Algebraic Eigenvalue Problem. Clarendon Press.

5. Parameters

1: A(2,N+1) -- DOUBLE PRECISION array Input
On entry: if A is declared with bounds (2,0:N), then A(1,i) and A(2,i) must contain the real and imaginary parts of a_i (i.e., the coefficient of z^{n-i}), for i=0,1,...,n.
Constraint: A(1,0) /= 0.0 or A(2,0) /= 0.0.

2: N -- INTEGER Input
On entry: the degree of the polynomial, n. Constraint: N >= 1.

3: SCALE -- LOGICAL Input
On entry: indicates whether or not the polynomial is to be scaled. See Section 8 for advice on when it may be preferable to set SCALE = .FALSE. and for a description of the scaling strategy. Suggested value: SCALE = .TRUE..

4: Z(2,N) -- DOUBLE PRECISION array Output
On exit: the real and imaginary parts of the roots are stored in Z(1,i) and Z(2,i) respectively, for i=1,2,...,n.

5: W(4*(N+1)) -- DOUBLE PRECISION array Workspace

6: IFAIL -- INTEGER Input/Output
On entry: IFAIL must be set to 0, -1 or 1. For users not familiar with this parameter (described in the Essential Introduction) the recommended value is 0.

```
     On exit: IFAIL = 0 unless the routine detects an error (see
     Section 6).

6. Error Indicators and Warnings

Errors detected by the routine:

If on entry IFAIL = 0 or -1, explanatory error messages are
output on the current error message unit (as defined by X04AAF).

IFAIL= 1
     On entry A(1,0) = 0.0 and A(2,0) = 0.0,

     or       N < 1.

IFAIL= 2
     The iterative procedure has failed to converge. This error
     is very unlikely to occur. If it does, please contact NAG
     immediately, as some basic assumption for the arithmetic has
     been violated. See also Section 8.

IFAIL= 3
     Either overflow or underflow prevents the evaluation of P(z)
     near some of its zeros. This error is very unlikely to
     occur. If it does, please contact NAG immediately. See also
     Section 8.

7. Accuracy

All roots are evaluated as accurately as possible, but because of
the inherent nature of the problem complete accuracy cannot be
guaranteed.

8. Further Comments

If SCALE = .TRUE., then a scaling factor for the coefficients is
chosen as a power of the base B of the machine so that the
                                                        EMAX-P
largest coefficient in magnitude approaches THRESH = B      .
Users should note that no scaling is performed if the largest
coefficient in magnitude exceeds THRESH, even if SCALE = .TRUE..
(For definition of B, EMAX and P see the Chapter Introduction
X02.)

However, with SCALE = .TRUE., overflow may be encountered when
```

the input coefficients a ,a ,a ,...,a vary widely in magnitude,
                       0 1 2   n
particularly on those machines for which B^(4*P) overflows. In such cases, SCALE should be set to .FALSE. and the coefficients scaled so that the largest coefficient in magnitude does not exceed B^(EMAX-2*P).

Even so, the scaling strategy used in C02AFF is sometimes insufficient to avoid overflow and/or underflow conditions. In such cases, the user is recommended to scale the independent variable (z) so that the disparity between the largest and smallest coefficient in magnitude is reduced. That is, use the routine to locate the zeros of the polynomial d*P(cz) for some suitable values of c and d. For example, if the original polynomial was P(z)=2^-100 + 2^100 z^20 + 2^-10, then choosing c=2^100 and d=2^20, for instance, would yield the scaled polynomial i+z^20, which is well-behaved relative to overflow and underflow and has zeros which are 2^10 times those of P(z).

If the routine fails with IFAIL = 2 or 3, then the real and imaginary parts of any roots obtained before the failure occurred are stored in Z in the reverse order in which they were found. Let n_R denote the number of roots found before the failure occurred. Then Z(1,n) and Z(2,n) contain the real and imaginary parts of the 1st root found, Z(1,n-1) and Z(2,n-1) contain the real and imaginary parts of the 2nd root found, ..., Z(1,n_R) and Z(2,n_R) contain the real and imaginary parts of the n_R th root found. After the failure has occurred, the remaining 2*(n-n_R) elements of Z contain a large negative number (equal to

-1/(X02AMF().\/2)).

9. Example

To find the roots of the polynomial a z^5 +a z^4 +a z^3 +a z^2 +a z+a =0,

```
                                          0    1    2    3    4   5
where a =(5.0+6.0i), a =(30.0+20.0i), a =-(0.2+6.0i),
       0              1                 2
a =(50.0+100000.0i), a =-(2.0-40.0i) and a =(10.0+1.0i).
 3                    4                   5
```

The example program is not reproduced here. The source code for all example programs is distributed with the NAG Foundation Library software and should be available on-line.

%%%

C02AGF(3NAG) Foundation Library (12/10/92) C02AGF(3NAG)

C02 -- Zeros of Polynomials C02AGF

C02AGF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for your implementation to check implementation-dependent details. The symbol (*) after a NAG routine name denotes a routine that is not included in the Foundation Library.

1. Purpose

C02AGF finds all the roots of a real polynomial equation, using a variant of Laguerre's Method.

2. Specification

```
SUBROUTINE C02AGF (A, N, SCALE, Z, W, IFAIL)
INTEGER          N, IFAIL
DOUBLE PRECISION A(N+1), Z(2,N), W(2*(N+1))
LOGICAL          SCALE
```

3. Description

The routine attempts to find all the roots of the nth degree real polynomial equation

$$P(z)=a_0 z^n + a_1 z^{n-1} + a_2 z^{n-2} + \ldots + a_{n-1} z + a_n = 0.$$

The roots are located using a modified form of Laguerre's Method, originally proposed by Smith [2].

The method of Laguerre [3] can be described by the iterative scheme

$$L(z_k)=z_{k+1}-z_k=\frac{-n*P(z_k)}{P'(z_k)+\sqrt{H(z_k)}},$$

where $H(z_k)=(n-1)*[(n-1)*(P'(z_k))^2-n*P(z_k)P''(z_k)]$, and z_0 is specified.

The sign in the denominator is chosen so that the modulus of the Laguerre step at z_k, viz. $|L(z_k)|$, is as small as possible. The method can be shown to be cubically convergent for isolated roots (real or complex) and linearly convergent for multiple roots. The routine generates a sequence of iterates $z_1, z_2, z_3, \ldots$, such that $|P(z_{k+1})|<|P(z_k)|$ and ensures that $z_{k+1}+L(z_{k+1})$ 'roughly' lies inside a circular region of radius $|F|$ about z_{k+1} known to contain a zero of $P(z)$; that is, $|L(z_{k+1})|<=|F|$, where F denotes the Fejer bound (see Marden [1]) at the point z_k. Following Smith [2], F is taken to be min(B,1.445*n*R), where B is an upper bound for the magnitude of the smallest zero given by

$$B=1.0001*\min(\sqrt{n*L(z_k)},|r_1|,|a_0/a_n|^{1/n}),$$

r_1 is the zero X of smaller magnitude of the quadratic equation

$$2(P''(z_k)/(2*n*(n-1)))X^2+2(P'(z_k)/n)X+P(z_k)=0$$

and the Cauchy lower bound R for the smallest zero is computed (using Newton's Method) as the positive root of the polynomial

equation

$$|a_0|z^n + |a_1|z^{n-1} + |a_2|z^{n-2} + \ldots + |a_{n-1}|z - |a_n| = 0.$$

Starting from the origin, successive iterates are generated according to the rule $z_{k+1} = z_k + L(z_k)$ for k=1,2,3,... and $L(z_k)$ is the iterative procedure terminates if $P(z_{k+1})$ is smaller in absolute value than the bound on the rounding error in $P(z_{k+1})$ and the current iterate $z_{k+1} = z_{k-1}$ is taken to be a zero of P(z) (as is its conjugate z_p if z_p is complex). The deflated polynomial $\tilde{P}(z) = P(z)/(z - z_p)$ of degree n-1 if z_p is real $\tilde{P}(z) = P(z)/((z - z_p)(z - \bar{z}_p))$ of degree n-2 if z_p is complex) is then formed, and the above procedure is repeated on the deflated polynomial until n<3, whereupon the remaining roots are obtained via the 'standard' closed formulae for a linear (n = 1) or quadratic (n = 2) equation.

To obtain the roots of a quadratic polynomial, C02AJF(*) can be used.

4. References

[1] Marden M (1966) Geometry of Polynomials. Mathematical Surveys. 3 Am. Math. Soc., Providence, RI.

[2] Smith B T (1967) ZERPOL: A Zero Finding Algorithm for Polynomials Using Laguerre's Method. Technical Report. Department of Computer Science, University of Toronto, Canada.

[3] Wilkinson J H (1965) The Algebraic Eigenvalue Problem. Clarendon Press.

5. Parameters

1: A(N+1) -- DOUBLE PRECISION array Input
On entry: if A is declared with bounds (0:N), then A(i) must contain a z_i^{n-i} (i.e., the coefficient of z_i), for i=0,1,...,n. Constraint: A(0) /= 0.0.

2: N -- INTEGER Input
On entry: the degree of the polynomial, n. Constraint: N >= 1.

3: SCALE -- LOGICAL Input
On entry: indicates whether or not the polynomial is to be scaled. See Section 8 for advice on when it may be preferable to set SCALE = .FALSE. and for a description of the scaling strategy. Suggested value: SCALE = .TRUE..

4: Z(2,N) -- DOUBLE PRECISION array Output
On exit: the real and imaginary parts of the roots are stored in Z(1,i) and Z(2,i) respectively, for i=1,2,...,n. Complex conjugate pairs of roots are stored in consecutive pairs of elements of Z; that is, Z(1,i+1) = Z(1,i) and Z(2,i+1)=-Z(2,i).

5: W(2*(N+1)) -- DOUBLE PRECISION array Workspace

6: IFAIL -- INTEGER Input/Output
On entry: IFAIL must be set to 0, -1 or 1. For users not familiar with this parameter (described in the Essential Introduction) the recommended value is 0.
On exit: IFAIL = 0 unless the routine detects an error (see Section 6).

6. Error Indicators and Warnings

Errors detected by the routine:

If on entry IFAIL = 0 or -1, explanatory error messages are output on the current error message unit (as defined by X04AAF).

IFAIL= 1
On entry A(0) = 0.0,

```
     or       N < 1.

IFAIL= 2
     The iterative procedure has failed to converge. This error
     is very unlikely to occur. If it does, please contact NAG
     immediately, as some basic assumption for the arithmetic has
     been violated. See also Section 8.

IFAIL= 3
     Either overflow or underflow prevents the evaluation of P(z)
     near some of its zeros. This error is very unlikely to
     occur. If it does, please contact NAG immediately. See also
     Section 8.

7. Accuracy

All roots are evaluated as accurately as possible, but because of
the inherent nature of the problem complete accuracy cannot be
guaranteed.

8. Further Comments

If SCALE = .TRUE., then a scaling factor for the coefficients is
chosen as a power of the base B of the machine so that the
                                                        EMAX-P
largest coefficient in magnitude approaches THRESH = B      .
Users should note that no scaling is performed if the largest
coefficient in magnitude exceeds THRESH, even if SCALE = .TRUE..
(For definition of B, EMAX and P see the Chapter Introduction
X02.)

However, with SCALE = .TRUE., overflow may be encountered when
the input coefficients a ,a ,...,a  vary widely in magnitude,
                        0  1  2      n
                                             (4*P)
particularly on those machines for which B      overflows. In
such cases, SCALE should be set to .FALSE. and the coefficients
scaled so that the largest coefficient in magnitude does not
        (EMAX-2*P)
exceed B          .

Even so, the scaling strategy used in C02AGF is sometimes
insufficient to avoid overflow and/or underflow conditions. In
such cases, the user is recommended to scale the independent
variable (z) so that the disparity between the largest and
smallest coefficient in magnitude is reduced. That is, use the
```

routine to locate the zeros of the polynomial d*P(cz) for some suitable values of c and d. For example, if the original polynomial was $P(z)=2^{-100}+2^{100}z^{20}$, then choosing $c=2^{-10}$ and $d=2^{100}$, for instance, would yield the scaled polynomial $1+z^{20}$, which is well-behaved relative to overflow and underflow and has zeros which are 2^{10} times those of P(z).

If the routine fails with IFAIL = 2 or 3, then the real and imaginary parts of any roots obtained before the failure occurred are stored in Z in the reverse order in which they were found. Let n_R denote the number of roots found before the failure occurred. Then Z(1,n) and Z(2,n) contain the real and imaginary parts of the 1st root found, Z(1,n-1) and Z(2,n-1) contain the real and imaginary parts of the 2nd root found, ..., Z(1,n_R) and Z(2,n_R) contain the real and imaginary parts of the n_R th root found. After the failure has occurred, the remaining $2*(n-n_R)$ elements of Z contain a large negative number (equal to

-1/(X02AMF().\/2)).

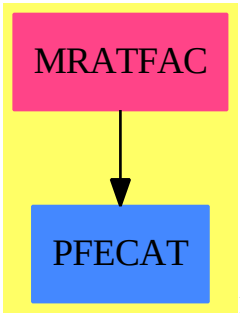
9. Example

To find the roots of the 5th degree polynomial

$$z^5+2z^4+3z^3+4z^2+5z+6=0.$$

The example program is not reproduced here. The source code for all example programs is distributed with the NAG Foundation Library software and should be available on-line.

15.24 NagPolynomialRootsPackage



1.00

Exports:
c02aff c02agf

⟨*package NAGC02 NagPolynomialRootsPackage*⟩≡
```
)abbrev package NAGC02 NagPolynomialRootsPackage
++ Author: Godfrey Nolan and Mike Dewar
++ Date Created: Jan 1994
++ Date Last Updated: Thu May 12 17:44:27 1994
++ Description:
++ This package uses the NAG Library to compute the zeros of a
++ polynomial with real or complex coefficients.
++ See \downlink{Manual Page}{manpageXXc02}.

NagPolynomialRootsPackage(): Exports == Implementation where
  S ==> Symbol
  FOP ==> FortranOutputStackPackage

  Exports ==> with
    c02aff : (Matrix DoubleFloat,Integer,Boolean,Integer) -> Result
     ++ c02aff(a,n,scale,ifail)
     ++ finds all the roots of a complex polynomial equation,
     ++ using a variant of Laguerre's Method.
     ++ See \downlink{Manual Page}{manpageXXc02aff}.
    c02agf : (Matrix DoubleFloat,Integer,Boolean,Integer) -> Result
     ++ c02agf(a,n,scale,ifail)
     ++ finds all the roots of a real polynomial equation, using a
     ++ variant of Laguerre's Method.
     ++ See \downlink{Manual Page}{manpageXXc02agf}.
  Implementation ==> add

    import Lisp
    import DoubleFloat
    import Matrix DoubleFloat
    import Any
```

```
import Record
import Integer
import Boolean
import NAGLinkSupportPackage
import AnyFunctions1(Matrix DoubleFloat)
import AnyFunctions1(Integer)
import AnyFunctions1(Boolean)


c02aff(aArg:Matrix DoubleFloat,nArg:Integer,scaleArg:Boolean,_
    ifailArg:Integer): Result ==
    [(invokeNagman(NIL$Lisp,_
    "c02aff",_
    ["n"::S,"scale"::S,"ifail"::S,"a"::S,"z"::S,"w"::S]$Lisp,_
    ["z"::S,"w"::S]$Lisp,_
    [["double"::S,["a"::S,2$Lisp,["+"::S,"n"::S,1$Lisp]$Lisp]$Lisp_
    ,["z"::S,2$Lisp,"n"::S]$Lisp,["w"::S,["*"::S,["+"::S,"n"::S,1$Lisp]$Lisp,
    ,["integer"::S,"n"::S,"ifail"::S]$Lisp_
    ,["logical"::S,"scale"::S]$Lisp_
    ]$Lisp,_
    ["z"::S,"ifail"::S]$Lisp,_
    [([nArg::Any,scaleArg::Any,ifailArg::Any,aArg::Any ])_
    @List Any]$Lisp)$Lisp)_
    pretend List (Record(key:Symbol,entry:Any))]$Result



c02agf(aArg:Matrix DoubleFloat,nArg:Integer,scaleArg:Boolean,_
    ifailArg:Integer): Result ==
    [(invokeNagman(NIL$Lisp,_
    "c02agf",_
    ["n"::S,"scale"::S,"ifail"::S,"a"::S,"z"::S,"w"::S]$Lisp,_
    ["z"::S,"w"::S]$Lisp,_
    [["double"::S,["a"::S,["+"::S,"n"::S,1$Lisp]$Lisp]$Lisp_
    ,["z"::S,2$Lisp,"n"::S]$Lisp,["w"::S,["*"::S,["+"::S,"n"::S,1$Lisp]$Lisp,
    ,["integer"::S,"n"::S,"ifail"::S]$Lisp_
    ,["logical"::S,"scale"::S]$Lisp_
    ]$Lisp,_
    ["z"::S,"ifail"::S]$Lisp,_
    [([nArg::Any,scaleArg::Any,ifailArg::Any,aArg::Any ])_
    @List Any]$Lisp)$Lisp)_
    pretend List (Record(key:Symbol,entry:Any))]$Result
```

⟨*NAGC02.dotabb*⟩≡
```
"NAGC02" [color="#FF4488",href="bookvol10.4.pdf#nameddest=NAGC02"]
"ALIST" [color="#88FF44",href="bookvol10.3.pdf#nameddest=ALIST"]
"NAGC02" -> "ALIST"
```

15.25 package NAGC05 NagRootFindingPackage

⟨*NagRootFindingPackage.help*⟩≡

```
C05(3NAG)          Foundation Library (12/10/92)          C05(3NAG)

     C05 -- Roots of One or More Transcendental Equations
                                                  Introduction -- C05
                              Chapter C05
                 Roots of One or More Transcendental Equations

     1. Scope of the Chapter

     This chapter is concerned with the calculation of real zeros of
     continuous real functions of one or more variables. (Complex
     equations must be expressed in terms of the equivalent larger
     system of real equations.)

     2. Background to the Problems

     The chapter divides naturally into two parts.

     2.1. A Single Equation

     The first deals with the real zeros of a real function of a
     single variable f(x).

     At present, there is only one routine with a simple calling
     sequence. This routine assumes that the user can determine an
     initial interval [a,b] within which the desired zero lies, that
     is f(a)*f(b)<0, and outside which all other zeros lie. The
     routine then systematically subdivides the interval to produce a
     final interval containing the zero. This final interval has a
     length bounded by the user's specified error requirements; the
     end of the interval where the function has smallest magnitude is
     returned as the zero. This routine is guaranteed to converge to a
     simple zero of the function. (Here we define a simple zero as a
     zero corresponding to a sign-change of the function.) The
     algorithm used is due to Bus and Dekker.

     2.2. Systems of Equations

     The routines in the second part of this chapter are designed to
     solve a set of nonlinear equations in n unknowns
```

$$f_i(x)=0, \quad i=1,2,\ldots,n, \quad x=(x_1,x_2,\ldots,x_n)^T \qquad (1)$$

where T stands for transpose.

It is assumed that the functions are continuous and differentiable so that the matrix of first partial derivatives of the functions, the Jacobian matrix $J_{ij}(x)=\partial f_i/\partial x_j$ evaluated at the point x, exists, though it may not be possible to calculate it directly.

The functions f_i must be independent, otherwise there will be an infinity of solutions and the methods will fail. However, even when the functions are independent the solutions may not be unique. Since the methods are iterative, an initial guess at the solution has to be supplied, and the solution located will usually be the one closest to this initial guess.

2.3. References

[1] Gill P E and Murray W (1976) Algorithms for the Solution of the Nonlinear Least-squares Problem. NAC 71 National Physical Laboratory.

[2] More J J, Garbow B S and Hillstrom K E (1974) User Guide for Minpack-1. ANL-80-74 Argonne National Laboratory.

[3] Ortega J M and Rheinboldt W C (1970) Iterative Solution of Nonlinear Equations in Several Variables. Academic Press.

[4] Rabinowitz P (1970) Numerical Methods for Nonlinear Algebraic Equations. Gordon and Breach.

3. Recommendations on Choice and Use of Routines

3.1. Zeros of Functions of One Variable

There is only one routine (C05ADF) for solving a single nonlinear equation. This routine is designed for solving problems where the function f(x) whose zero is to be calculated, can be coded as a user-supplied routine.

C05ADF may only be used when the user can supply an interval [a,b] containing the zero, that is f(a)*f(b)<0.

3.2. Solution of Sets of Nonlinear Equations

The solution of a set of nonlinear equations

$$f_i(x_1, x_2, \ldots, x_n) = 0, \quad i = 1, 2, \ldots, n \tag{2}$$

can be regarded as a special case of the problem of finding a minimum of a sum of squares

$$s(x) = \sum_{i=1}^{m} [f_i(x_1, x_2, \ldots, x_n)]^2 \quad (m \geq n). \tag{3}$$

So the routines in Chapter E04 of the Library are relevant as well as the special nonlinear equations routines.

There are two routines (C05NBF and C05PBF) for solving a set of nonlinear equations. These routines require the f_i (and possibly their derivatives) to be calculated in user-supplied routines. These should be set up carefully so the Library routines can work as efficiently as possible.

The main decision which has to be made by the user is whether to supply the derivatives $\frac{\partial f_i}{\partial x_j}$. It is advisable to do so if possible, since the results obtained by algorithms which use derivatives are generally more reliable than those obtained by algorithms which do not use derivatives.

C05PBF requires the user to provide the derivatives, whilst C05NBF does not. C05NBF and C05PBF are easy-to-use routines. A routine, C05ZAF, is provided for use in conjunction with C05PBF to check the user-provided derivatives for consistency with the functions themselves. The user is strongly advised to make use of this routine whenever C05PBF is used.

Firstly, the calculation of the functions and their derivatives should be ordered so that cancellation errors are avoided. This is particularly important in a routine that uses these quantities to build up estimates of higher derivatives.

Secondly, scaling of the variables has a considerable effect on the efficiency of a routine. The problem should be designed so that the elements of x are of similar magnitude. The same comment applies to the functions, all the f_i should be of comparable size.

The accuracy is usually determined by the accuracy parameters of the routines, but the following points may be useful:

(i) Greater accuracy in the solution may be requested by choosing smaller input values for the accuracy parameters. However, if unreasonable accuracy is demanded, rounding errors may become important and cause a failure.

(ii) Some idea of the accuracies of the x_i may be obtained by monitoring the progress of the routine to see how many figures remain unchanged during the last few iterations.

(iii) An approximation to the error in the solution x, given by e where e is the solution to the set of linear equations

$$J(x)e=-f(x)$$

where $f(x)=(f_1(x),f_2(x),\ldots,f_n(x))^T$ (see Chapter F04).

(iv) If the functions $f_i(x)$ are changed by small amounts $(\text{epsilon})_i$, for i=1,2,...,n, then the corresponding change in the solution x is given approximately by (sigma), where (sigma) is the solution of the set of linear equations

$$J(x)(\text{sigma})=-(\text{epsilon}),$$ (see Chapter F04).

Thus one can estimate the sensitivity of x to any uncertainties in the specification of $f_i(x)$, for

```
                                                    i
      i=1,2,...,n.

3.3. Index

Zeros of functions of one variable:
     Bus and Dekker algorithm                                C05ADF
Zeros of functions of several variables:
     easy-to-use                                             C05NBF
     easy-to-use, derivatives required                       C05PBF
Checking Routine:
     Checks user-supplied Jacobian                           C05ZAF


C05 -- Roots of One or More Transcendental Equations
                                                    Contents -- C05
Chapter C05

Roots of One or More Transcendental Equations

C05ADF  Zero of continuous function in given interval, Bus and
        Dekker algorithm

C05NBF  Solution of system of nonlinear equations using function
        values only

C05PBF  Solution of system of nonlinear equations using 1st
        derivatives

C05ZAF  Check user's routine for calculating 1st derivatives
```

%%%

```
C05ADF(3NAG)      Foundation Library (12/10/92)      C05ADF(3NAG)

C05 -- Roots of One or More Transcendental Equations       C05ADF
        C05ADF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for
your implementation to check implementation-dependent details.
The symbol (*) after a NAG routine name denotes a routine that is
not included in the Foundation Library.

1. Purpose

C05ADF locates a zero of a continuous function in a given
```

interval by a combination of the methods of linear interpolation, extrapolation and bisection.

2. Specification

```
SUBROUTINE C05ADF (A, B, EPS, ETA, F, X, IFAIL)
INTEGER          IFAIL
DOUBLE PRECISION A, B, EPS, ETA, F, X
EXTERNAL         F
```

3. Description

The routine attempts to obtain an approximation to a simple zero of the function f(x) given an initial interval [a,b] such that f(a)*f(b)<=0. The zero is found by calls to C05AZF(*) whose specification should be consulted for details of the method used.

The approximation x to the zero (alpha) is determined so that one or both of the following criteria are satisfied:

(i) |x-(alpha)|<EPS,

(ii) |f(x)|<ETA.

4. References

None.

5. Parameters

1: A -- DOUBLE PRECISION Input
On entry: the lower bound of the interval, a.

2: B -- DOUBLE PRECISION Input
On entry: the upper bound of the interval, b. Constraint: B /= A.

3: EPS -- DOUBLE PRECISION Input
On entry: the absolute tolerance to which the zero is required (see Section 3). Constraint: EPS > 0.0.

4: ETA -- DOUBLE PRECISION Input
On entry: a value such that if |f(x)|<ETA, x is accepted as the zero. ETA may be specified as 0.0 (see Section 7).

5: F -- DOUBLE PRECISION FUNCTION, supplied by the user.

External Procedure

F must evaluate the function f whose zero is to be determined.

Its specification is:

```
DOUBLE PRECISION FUNCTION F (XX)
DOUBLE PRECISION XX
```

1: XX -- DOUBLE PRECISION Input
On entry: the point at which the function must be evaluated.

F must be declared as EXTERNAL in the (sub)program from which C05ADF is called. Parameters denoted as Input must not be changed by this procedure.

6: X -- DOUBLE PRECISION Output
On exit: the approximation to the zero.

7: IFAIL -- INTEGER Input/Output
Before entry, IFAIL must be assigned a value. For users not familiar with this parameter (described in the Essential Introduction) the recommended value is 0.

Unless the routine detects an error (see Section 6), IFAIL contains 0 on exit.

6. Error Indicators and Warnings

Errors detected by the routine:

IFAIL= 1
On entry EPS <= 0.0,

or A = B,

or F(A)*F(B)>0.0.

IFAIL= 2
Too much accuracy has been requested in the computation, that is, EPS is too small for the computer being used. The final value of X is an accurate approximation to the zero.

IFAIL= 3
A change in sign of f(x) has been determined as occurring near the point defined by the final value of X. However,

```
          there is some evidence that this sign-change corresponds to
          a pole of f(x).

     IFAIL= 4
          Indicates that a serious error has occurred in C05AZF(*).
          Check all routine calls. Seek expert help.

     7. Accuracy

     This depends on the value of EPS and ETA. If full machine
     accuracy is required, they may be set very small, resulting in an
     error exit with IFAIL = 2, although this may involve more
     iterations than a lesser accuracy. The user is recommended to set
     ETA = 0.0 and to use EPS to control the accuracy, unless he has
     considerable knowledge of the size of f(x) for values of x near
     the zero.

     8. Further Comments

     The time taken by the routine depends primarily on the time spent
     evaluating F (see Section 5).

     If it is important to determine an interval of length less than
     EPS containing the zero, or if the function F is expensive to
     evaluate and the number of calls to F is to be restricted, then
     use of C05AZF(*) is recommended. Use of C05AZF(*) is also
     recommended when the structure of the problem to be solved does
     not permit a simple function F to be written: the reverse
     communication facilities of C05AZF(*) are more flexible than the
     direct communication of F required by C05ADF.

     9. Example

                                                          -x
     The example program below calculates the zero of e  -x within the
     interval [0,1] to approximately 5 decimal places.

     The example program is not reproduced here. The source code for
     all example programs is distributed with the NAG Foundation
     Library software and should be available on-line.

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

C05NBF(3NAG)      Foundation Library (12/10/92)      C05NBF(3NAG)

     C05 -- Roots of One or More Transcendental Equations       C05NBF
```

C05NBF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for your implementation to check implementation-dependent details. The symbol (*) after a NAG routine name denotes a routine that is not included in the Foundation Library.

1. Purpose

C05NBF is an easy-to-use routine to find a solution of a system of nonlinear equations by a modification of the Powell hybrid method.

2. Specification

```
SUBROUTINE C05NBF (FCN, N, X, FVEC, XTOL, WA, LWA, IFAIL)
INTEGER          N, LWA, IFAIL
DOUBLE PRECISION X(N), FVEC(N), XTOL, WA(LWA)
EXTERNAL         FCN
```

3. Description

The system of equations is defined as:

$$f_i(x_1, x_2, \ldots, x_n) = 0, \text{ for } i = 1, 2, \ldots, n.$$

C05NBF is based upon the MINPACK routine HYBRD1 (More et al [1]). It chooses the correction at each step as a convex combination of the Newton and scaled gradient directions. Under reasonable conditions this guarantees global convergence for starting points far from the solution and a fast rate of convergence. The Jacobian is updated by the rank-1 method of Broyden. At the starting point the Jacobian is approximated by forward differences, but these are not used again until the rank-1 method fails to produce satisfactory progress. For more details see Powell [2].

4. References

[1] More J J, Garbow B S and Hillstrom K E User Guide for MINPACK-1. Technical Report ANL-80-74. Argonne National Laboratory.

[2] Powell M J D (1970) A Hybrid Method for Nonlinear Algebraic Equations. Numerical Methods for Nonlinear Algebraic

Equations. (ed P Rabinowitz) Gordon and Breach.

5. Parameters

1: FCN -- SUBROUTINE, supplied by the user. External Procedure
FCN must return the values of the functions f_i at a point x.

Its specification is:

```
SUBROUTINE FCN (N, X, FVEC, IFLAG)
INTEGER          N, IFLAG
DOUBLE PRECISION X(N), FVEC(N)
```

1: N -- INTEGER Input
On entry: the number of equations, n.

2: X(N) -- DOUBLE PRECISION array Input
On entry: the components of the point x at which the functions must be evaluated.

3: FVEC(N) -- DOUBLE PRECISION array Output
On exit: the function values $f_i(x)$ (unless IFLAG is set to a negative value by FCN).

4: IFLAG -- INTEGER Input/Output
On entry: IFLAG > 0. On exit: in general, IFLAG should not be reset by FCN. If, however, the user wishes to terminate execution (perhaps because some illegal point X has been reached), then IFLAG should be set to a negative integer. This value will be returned through IFAIL.

FCN must be declared as EXTERNAL in the (sub)program from which C05NBF is called. Parameters denoted as Input must not be changed by this procedure.

2: N -- INTEGER Input
On entry: the number of equations, n. Constraint: N > 0.

3: X(N) -- DOUBLE PRECISION array Input/Output
On entry: an initial guess at the solution vector. On exit: the final estimate of the solution vector.

4: FVEC(N) -- DOUBLE PRECISION array Output

```
     On exit: the function values at the final point, X.

 5:  XTOL -- DOUBLE PRECISION                                    Input
     On entry: the accuracy in X to which the solution is
     required. Suggested value: the square root of the machine
     precision. Constraint: XTOL >= 0.0.

 6:  WA(LWA) -- DOUBLE PRECISION array                        Workspace

 7:  LWA -- INTEGER                                              Input
     On entry: the dimension of the array WA. Constraint:
     LWA>=N*(3*N+13)/2.

 8:  IFAIL -- INTEGER                                      Input/Output
     On entry: IFAIL must be set to 0, -1 or 1. For users not
     familiar with this parameter (described in the Essential
     Introduction) the recommended value is 0.

     On exit: IFAIL = 0 unless the routine detects an error (see
     Section 6).

6. Error Indicators and Warnings

Errors detected by the routine:

If on entry IFAIL = 0 or -1, explanatory error messages are
output on the current error message unit (as defined by X04AAF).

IFAIL< 0
     The user has set IFLAG negative in FCN. The value of IFAIL
     will be the same as the user's setting of IFLAG.

IFAIL= 1
     On entry N <= 0,

     or       XTOL < 0.0,

     or       LWA<N*(3*N+13)/2.

IFAIL= 2
     There have been at least 200*(N+1) evaluations of FCN.
     Consider restarting the calculation from the final point
     held in X.

IFAIL= 3
     No further improvement in the approximate solution X is
```

possible; XTOL is too small.

IFAIL= 4
The iteration is not making good progress. This failure exit may indicate that the system does not have a zero, or that the solution is very close to the origin (see Section 7). Otherwise, rerunning C05NBF from a different starting point may avoid the region of difficulty.

7. Accuracy

$\hat{}$
If x is the true solution, C05NBF tries to ensure that

$$||x-\hat{x}|| \leq XTOL * ||x||.$$

If this condition is satisfied with XTOL=10^{-k}, then the larger components of x have k significant decimal digits. There is a danger that the smaller components of x may have large relative errors, but the fast rate of convergence of C05NBF usually avoids this possibility.

If XTOL is less than machine precision, and the above test is satisfied with the machine precision in place of XTOL, then the routine exits with IFAIL = 3.

Note: this convergence test is based purely on relative error, and may not indicate convergence if the solution is very close to the origin.

The test assumes that the functions are reasonably well behaved. If this condition is not satisfied, then C05NBF may incorrectly indicate convergence. The validity of the answer can be checked, for example, by rerunning C05NBF with a tighter tolerance.

8. Further Comments

The time required by C05NBF to solve a given problem depends on n , the behaviour of the functions, the accuracy requested and the starting point. The number of arithmetic operations executed by C05NBF to process each call of FCN is about $11.5*n^2$. Unless FCN can be evaluated quickly, the timing of C05NBF will be strongly influenced by the time spent in FCN.

Ideally the problem should be scaled so that at the solution the function values are of comparable magnitude.

9. Example

To determine the values $x_1, \ldots, x_9$ which satisfy the tridiagonal equations:

$$(3-2x_1)x_1 - 2x_2 = -1$$

$$-x_{i-1} + (3-2x_i)x_i - 2x_{i+1} = -1, \quad i = 2,3,\ldots,8$$

$$-x_8 + (3-2x_9)x_9 = -1.$$

The example program is not reproduced here. The source code for all example programs is distributed with the NAG Foundation Library software and should be available on-line.

%%%

C05PBF(3NAG) Foundation Library (12/10/92) C05PBF(3NAG)

C05 -- Roots of One or More Transcendental Equations C05PBF
C05PBF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for your implementation to check implementation-dependent details. The symbol (*) after a NAG routine name denotes a routine that is not included in the Foundation Library.

1. Purpose

C05PBF is an easy-to-use routine to find a solution of a system of nonlinear equations by a modification of the Powell hybrid method. The user must provide the Jacobian.

2. Specification

```
      SUBROUTINE C05PBF (FCN, N, X, FVEC, FJAC, LDFJAC, XTOL,
     1                   WA, LWA, IFAIL)
      INTEGER          N, LDFJAC, LWA, IFAIL
```

```
      DOUBLE PRECISION X(N), FVEC(N), FJAC(LDFJAC,N), XTOL, WA
     1                 (LWA)
      EXTERNAL         FCN
```

3. Description

The system of equations is defined as:

$$f_i(x_1, x_2, \ldots, x_n) = 0, \quad i = 1, 2, \ldots, n.$$

C05PBF is based upon the MINPACK routine HYBRJ1 (More et al [1]). It chooses the correction at each step as a convex combination of the Newton and scaled gradient directions. Under reasonable conditions this guarantees global convergence for starting points far from the solution and a fast rate of convergence. The Jacobian is updated by the rank-1 method of Broyden. At the starting point the Jacobian is calculated, but it is not recalculated until the rank-1 method fails to produce satisfactory progress. For more details see Powell [2].

4. References

[1] More J J, Garbow B S and Hillstrom K E User Guide for MINPACK-1. Technical Report ANL-80-74. Argonne National Laboratory.

[2] Powell M J D (1970) A Hybrid Method for Nonlinear Algebraic Equations. Numerical Methods for Nonlinear Algebraic Equations. (ed P Rabinowitz) Gordon and Breach.

5. Parameters

1: FCN -- SUBROUTINE, supplied by the user.
External Procedure

Depending upon the value of IFLAG, FCN must either return the values of the functions f_i at a point x or return the Jacobian at x.

Its specification is:

```
      SUBROUTINE FCN (N, X, FVEC, FJAC, LDFJAC, IFLAG)
      INTEGER          N, LDFJAC, IFLAG
      DOUBLE PRECISION X(N), FVEC(N), FJAC(LDFJAC,N)
```

1: N -- INTEGER Input
On entry: the number of equations, n.

2: X(N) -- DOUBLE PRECISION array Input
On entry: the components of the point x at which the functions or the Jacobian must be evaluated.

3: FVEC(N) -- DOUBLE PRECISION array Output
On exit: if IFLAG = 1 on entry, FVEC must contain the function values $f_i(x)$ (unless IFLAG is set to a negative value by FCN). If IFLAG = 2 on entry, FVEC must not be changed.

4: FJAC(LDFJAC,N) -- DOUBLE PRECISION array Output
On exit: if IFLAG = 2 on entry, FJAC(i,j) must contain the value of $\frac{\mathrm{d}f_i}{\mathrm{d}x_j}$ at the point x, for i,j=1,2,...,n (unless IFLAG is set to a negative value by FCN).

If IFLAG = 1 on entry, FJAC must not be changed.

5: LDFJAC -- INTEGER Input
On entry: the first dimension of FJAC.

6: IFLAG -- INTEGER Input/Output
On entry: IFLAG = 1 or 2:
if IFLAG = 1, FVEC is to be updated;

if IFLAG = 2, FJAC is to be updated.
On exit: in general, IFLAG should not be reset by FCN. If, however, the user wishes to terminate execution (perhaps because some illegal point x has been reached) then IFLAG should be set to a negative integer. This value will be returned through IFAIL.
FCN must be declared as EXTERNAL in the (sub)program from which C05PBF is called. Parameters denoted as Input must not be changed by this procedure.

2: N -- INTEGER Input
On entry: the number of equations, n. Constraint: N > 0.

3: X(N) -- DOUBLE PRECISION array Input/Output

```
     On entry: an initial guess at the solution vector. On
     exit: the final estimate of the solution vector.

 4:  FVEC(N) -- DOUBLE PRECISION array                        Output
     On exit: the function values at the final point, X.

 5:  FJAC(LDFJAC,N) -- DOUBLE PRECISION array                 Output
     On exit: the orthogonal matrix Q produced by the QR
     factorization of the final approximate Jacobian.

 6:  LDFJAC -- INTEGER                                         Input
     On entry:
     the first dimension of the array FJAC as declared in the
     (sub)program from which C05PBF is called.
     Constraint: LDFJAC >= N.

 7:  XTOL -- DOUBLE PRECISION                                  Input
     On entry: the accuracy in X to which the solution is
     required. Suggested value: the square root of the machine
     precision. Constraint: XTOL >= 0.0.

 8:  WA(LWA) -- DOUBLE PRECISION array                      Workspace

 9:  LWA -- INTEGER                                            Input
     On entry: the dimension of the array WA. Constraint:
     LWA>=N*(N+13)/2.

10:  IFAIL -- INTEGER                                   Input/Output
     On entry: IFAIL must be set to 0, -1 or 1. For users not
     familiar with this parameter (described in the Essential
     Introduction) the recommended value is 0.

     On exit: IFAIL = 0 unless the routine detects an error (see
     Section 6).

6. Error Indicators and Warnings

Errors detected by the routine:

If on entry IFAIL = 0 or -1, explanatory error messages are
output on the current error message unit (as defined by X04AAF).

IFAIL< 0
     A negative value of IFAIL indicates an exit from C05PBF
     because the user has set IFLAG negative in FCN. The value of
     IFAIL will be the same as the user's setting of IFLAG.
```

IFAIL= 1
     On entry N <= 0,

     or   LDFJAC < N,

     or   XTOL < 0.0,

     or   LWA<N*(N+13)/2.

IFAIL= 2
     There have been 100*(N+1) evaluations of the functions.
     Consider restarting the calculation from the final point
     held in X.

IFAIL= 3
     No further improvement in the approximate solution X is
     possible; XTOL is too small.

IFAIL= 4
     The iteration is not making good progress. This failure exit
     may indicate that the system does not have a zero or that
     the solution is very close to the origin (see Section 7).
     Otherwise, rerunning C05PBF from a different starting point
     may avoid the region of difficulty.

7. Accuracy

   ^
If x is the true solution, C05PBF tries to ensure that

                        ^   ^
                  ||x-x|| <=XTOL*||x|| .
                        2   2

                                                -k
If this condition is satisfied with XTOL=10   , then the larger
components of x have k significant decimal digits. There is a
danger that the smaller components of x may have large relative
errors, but the fast rate of convergence of C05PBF usually avoids
the possibility.

If XTOL is less than machine precision and the above test is
satisfied with the machine precision in place of XTOL, then the
routine exits with IFAIL = 3.

Note: this convergence test is based purely on relative error, and may not indicate convergence if the solution is very close to the origin.

The test assumes that the functions and Jacobian are coded consistently and that the functions are reasonably well behaved. If these conditions are not satisfied then C05PBF may incorrectly indicate convergence. The coding of the Jacobian can be checked using C05ZAF. If the Jacobian is coded correctly, then the validity of the answer can be checked by rerunning C05PBF with a tighter tolerance.

8. Further Comments

The time required by C05PBF to solve a given problem depends on n , the behaviour of the functions, the accuracy requested and the starting point. The number of arithmetic operations executed by
                    2
C05PBF is about 11.5*n to process each evaluation of the
                     3
functions and about 1.3*n to process each evaluation of the Jacobian. Unless FCN can be evaluated quickly, the timing of C05PBF will be strongly influenced by the time spent in FCN.

Ideally the problem should be scaled so that, at the solution, the function values are of comparable magnitude.

9. Example

To determine the values $x_1, \ldots, x_9$ which satisfy the tridiagonal equations:

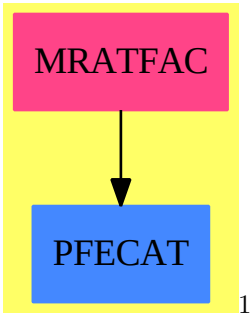
$$(3-2x_1)x_1 - 2x_2 = -1$$

$$-x_{i-1} + (3-2x_i)x_i - 2x_{i+1} = -1, \quad i=2,3,\ldots,8.$$

$$-x_8 + (3-2x_9)x_9 = -1.$$

The example program is not reproduced here. The source code for all example programs is distributed with the NAG Foundation Library software and should be available on-line.

15.26 NagRootFindingPackage



1.00

Exports:
c05adf c05nbf c05pbf

⟨*package NAGC05 NagRootFindingPackage*⟩≡
```
)abbrev package NAGC05 NagRootFindingPackage
++ Author: Godfrey Nolan and Mike Dewar
++ Date Created: Jan 1994
++ Date Last Updated: Thu May 12 17:44:28 1994
++ Description:
++ This package uses the NAG Library to calculate real zeros of
++ continuous real functions of one or more variables. (Complex
++ equations must be expressed in terms of the equivalent larger
++ system of real equations.)
++ See \downlink{Manual Page}{manpageXXc05}.

NagRootFindingPackage(): Exports == Implementation where
  S ==> Symbol
  FOP ==> FortranOutputStackPackage

  Exports ==> with
    c05adf : (DoubleFloat,DoubleFloat,DoubleFloat,DoubleFloat,_
        Integer,Union(fn:FileName,fp:Asp1(F))) -> Result
     ++ c05adf(a,b,eps,eta,ifail,f)
     ++ locates a zero of a continuous function in a given
     ++ interval by a combination of the methods of linear interpolation,
     ++ extrapolation and bisection.
     ++ See \downlink{Manual Page}{manpageXXc05adf}.
    c05nbf : (Integer,Integer,Matrix DoubleFloat,DoubleFloat,_
        Integer,Union(fn:FileName,fp:Asp6(FCN))) -> Result
     ++ c05nbf(n,lwa,x,xtol,ifail,fcn)
     ++ is an easy-to-use routine to find a solution of a system
     ++ of nonlinear equations by a modification of the Powell hybrid
     ++ method.
     ++ See \downlink{Manual Page}{manpageXXc05nbf}.
```

```
    c05pbf : (Integer,Integer,Integer,Matrix DoubleFloat,_
        DoubleFloat,Integer,Union(fn:FileName,fp:Asp35(FCN))) -> Result
     ++ c05pbf(n,ldfjac,lwa,x,xtol,ifail,fcn)
     ++ is an easy-to-use routine to find a solution of a system
     ++ of nonlinear equations by a modification of the Powell hybrid
     ++ method. The user must provide the Jacobian.
     ++ See \downlink{Manual Page}{manpageXXc05pbf}.
  Implementation ==> add

    import Lisp
    import DoubleFloat
    import Any
    import Record
    import Integer
    import Matrix DoubleFloat
    import Boolean
    import NAGLinkSupportPackage
    import FortranPackage
    import Union(fn:FileName,fp:Asp1(F))
    import AnyFunctions1(DoubleFloat)
    import AnyFunctions1(Matrix DoubleFloat)
    import AnyFunctions1(Integer)


    c05adf(aArg:DoubleFloat,bArg:DoubleFloat,epsArg:DoubleFloat,_
        etaArg:DoubleFloat,ifailArg:Integer,fArg:Union(fn:FileName,fp:Asp1(F))): Result ==
        pushFortranOutputStack(fFilename := aspFilename "f")$FOP
        if fArg case fn
                  then outputAsFortran(fArg.fn)
                  else outputAsFortran(fArg.fp)
        popFortranOutputStack()$FOP
        [(invokeNagman([fFilename]$Lisp,_
        "c05adf",_
        ["a"::S,"b"::S,"eps"::S,"eta"::S,"x"::S_
        ,"ifail"::S,"f"::S]$Lisp,_
        ["x"::S,"f"::S]$Lisp,_
        [["double"::S,"a"::S,"b"::S,"eps"::S,"eta"::S_
        ,"x"::S,"f"::S]$Lisp_
        ,["integer"::S,"ifail"::S]$Lisp_
        ]$Lisp,_
        ["x"::S,"ifail"::S]$Lisp,_
        [([aArg::Any,bArg::Any,epsArg::Any,etaArg::Any,ifailArg::Any ])_
        @List Any]$Lisp)$Lisp)_
        pretend List (Record(key:Symbol,entry:Any))]$Result

    c05nbf(nArg:Integer,lwaArg:Integer,xArg:Matrix DoubleFloat,_
```

```
        xtolArg:DoubleFloat,ifailArg:Integer,fcnArg:Union(fn:FileName,fp:Asp6(FCN
        pushFortranOutputStack(fcnFilename := aspFilename "fcn")$FOP
        if fcnArg case fn
                  then outputAsFortran(fcnArg.fn)
                  else outputAsFortran(fcnArg.fp)
        popFortranOutputStack()$FOP
        [(invokeNagman([fcnFilename]$Lisp,_
        "c05nbf",_
        ["n"::S,"lwa"::S,"xtol"::S,"ifail"::S,"fcn"::S_
        ,"fvec"::S,"x"::S,"wa"::S]$Lisp,_
        ["fvec"::S,"wa"::S,"fcn"::S]$Lisp,_
        [["double"::S,["fvec"::S,"n"::S]$Lisp,["x"::S,"n"::S]$Lisp_
        ,"xtol"::S,["wa"::S,"lwa"::S]$Lisp,"fcn"::S]$Lisp_
        ,["integer"::S,"n"::S,"lwa"::S,"ifail"::S]$Lisp_
        ]$Lisp,_
        ["fvec"::S,"x"::S,"xtol"::S,"ifail"::S]$Lisp,_
        [([nArg::Any,lwaArg::Any,xtolArg::Any,ifailArg::Any,xArg::Any ])_
        @List Any]$Lisp)$Lisp)_
        pretend List (Record(key:Symbol,entry:Any))]$Result

    c05pbf(nArg:Integer,ldfjacArg:Integer,lwaArg:Integer,_
        xArg:Matrix DoubleFloat,xtolArg:DoubleFloat,ifailArg:Integer,_
        fcnArg:Union(fn:FileName,fp:Asp35(FCN))): Result ==
        pushFortranOutputStack(fcnFilename := aspFilename "fcn")$FOP
        if fcnArg case fn
                  then outputAsFortran(fcnArg.fn)
                  else outputAsFortran(fcnArg.fp)
        popFortranOutputStack()$FOP
        [(invokeNagman([fcnFilename]$Lisp,_
        "c05pbf",_
        ["n"::S,"ldfjac"::S,"lwa"::S,"xtol"::S,"ifail"::S_
        ,"fcn"::S,"fvec"::S,"fjac"::S,"x"::S,"wa"::S]$Lisp,_
        ["fvec"::S,"fjac"::S,"wa"::S,"fcn"::S]$Lisp,_
        [["double"::S,["fvec"::S,"n"::S]$Lisp,["fjac"::S,"ldfjac"::S,"n"::S]$Lisp
        ,["x"::S,"n"::S]$Lisp,"xtol"::S,["wa"::S,"lwa"::S]$Lisp,"fcn"::S]$Lisp_
        ,["integer"::S,"n"::S,"ldfjac"::S,"lwa"::S_
        ,"ifail"::S]$Lisp_
        ]$Lisp,_
        ["fvec"::S,"fjac"::S,"x"::S,"xtol"::S,"ifail"::S]$Lisp,_
        [([nArg::Any,ldfjacArg::Any,lwaArg::Any,xtolArg::Any,ifailArg::Any,xArg::
        @List Any]$Lisp)$Lisp)_
        pretend List (Record(key:Symbol,entry:Any))]$Result
```

⟨*NAGC05.dotabb*⟩≡

```
"NAGC05" [color="#FF4488",href="bookvol10.4.pdf#nameddest=NAGC05"]
"ALIST" [color="#88FF44",href="bookvol10.3.pdf#nameddest=ALIST"]
"NAGC05" -> "ALIST"
```

15.27 package NAGC06 NagSeriesSummationPackage

⟨*NagSeriesSummationPackage.help*⟩≡

```
C06(3NAG)          Foundation Library (12/10/92)          C06(3NAG)


C06 -- Summation of Series                    Introduction -- C06
                          Chapter C06
                      Summation of Series

1. Scope of the Chapter

This chapter is concerned with calculating the discrete Fourier
transform of a sequence of real or complex data values, and
applying it to calculate convolutions and correlations.

2. Background to the Problems

2.1. Discrete Fourier Transforms

2.1.1.  Complex transforms

Most of the routines in this chapter calculate the finite
discrete Fourier transform  (DFT) of a sequence of n complex
numbers z , for j=0,1,...,n-1. The transform is defined by:
         j

                        n-1
                ^    1  --      (   2(pi)jk)
                z = --- >  z exp(-i -------)                  (1)
                 k      --  j   (      n   )
                    \/n j=0

for k=0,1,...,n-1. Note that equation (1) makes sense for all
                                  ^
integral k and with this extension z  is periodic with period n,
                                   k
     ^  ^                        ^   ^
i.e. z =z    , and in particular z  =z   .
      k  k+-n                    -k  n-k

                           ^                                  ^
```

If we write $z_j = x_j + iy_j$ and $z_k = a_k + ib_k$, then the definition of z_k may be written in terms of sines and cosines as:

$$a_k = \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} \left(x_j \cos\left(\frac{2\pi jk}{n}\right) + y_j \sin\left(\frac{2\pi jk}{n}\right) \right)$$

$$b_k = \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} \left(y_j \cos\left(\frac{2\pi jk}{n}\right) - x_j \sin\left(\frac{2\pi jk}{n}\right) \right).$$

The original data values z_j may conversely be recovered from the transform $\hat{z}_k$ by an inverse discrete Fourier transform:

$$z_j = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \hat{z}_k \exp\left(+i \frac{2\pi jk}{n}\right) \qquad (2)$$

for $j = 0, 1, \ldots, n-1$. If we take the complex conjugate of (2), we find that the sequence $\hat{z}_k$ is the DFT of the sequence $\overline{z}_j$. Hence the inverse DFT of the sequence z_k may be obtained by: taking the complex conjugates of the z_k; performing a DFT; and taking the complex conjugates of the result.

Notes: definitions of the discrete Fourier transform vary. Sometimes (2) is used as the definition of the DFT, and (1) as

the definition of the inverse. Also the scale-factor of 1/\/n may be omitted in the definition of the DFT, and replaced by 1/n in the definition of the inverse.

2.1.2. Real transforms

If the original sequence is purely real valued, i.e. z =x , then
                                                    j j

$$\hat{z}_k = a_k + ib_k = \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} x_j \exp\left(-i\,\frac{2\pi jk}{n}\right)$$

and $\hat{z}_{n-k}$ is the complex conjugate of $\hat{z}_k$. Thus the DFT of a real sequence is a particular type of complex sequence, called a Hermitian sequence, or half-complex or conjugate symmetric with the properties:

$$a_{n-k} = a_k \quad b_{n-k} = -b_k \quad b_0 = 0 \text{ and, if n is even, } b_{n/2} = 0.$$

Thus a Hermitian sequence of n complex data values can be represented by only n, rather than 2n, independent real values. This can obviously lead to economies in storage, the following scheme being used in this chapter: the real parts a_k for $0 \le k \le n/2$ are stored in normal order in the first n/2+1 locations of an array X of length n; the corresponding non-zero imaginary parts are stored in reverse order in the remaining locations of X. In other words, if X is declared with bounds (0:n-1) in the user's (sub)program, the real and imaginary parts of $\hat{z}_k$ are stored as follows:

	if n=2s	if n=2s-1
X(0)	a_0	a_0
X(1)	a_1	a_1

```
    X(2)       a          a
                2          2

    .          .          .

    .          .          .

    .          .          .

    X(s-1)     a          a
                s-1        s-1

    X(s)       a          b
                s          s-1

    X(s+1)     b          b
                s-1        s-2

    .          .          .

    .          .          .

    .          .          .

    X(n-2)     b          b
                2          2

    X(n-1)     b          b
                1          1


                 (     n/2-1                                      )
               1 (     --   (     ( 2(pi)jk)      ( 2(pi)jk))     )
Hence   x = ---(a +2 >    (a cos( -------)-b sin( -------))+a   )
         j     ( 0   --   ( k   (    n   )  k   (    n   ))  n/2)
             \/n(     k=0                                        )
```

where $a_{n/2} = 0$ if n is odd.

2.1.3. Fourier integral transforms

The usual application of the discrete Fourier transform is that of obtaining an approximation of the Fourier integral transform

```
                    +infty
                    /
          F(s)= |     f(t)exp(-i2(pi)st)dt
                    /
                    -infty
```

when f(t) is negligible outside some region (0,c). Dividing the region into n equal intervals we have

```
                      n-1
                    c --
          F(s)~= - > f exp(-i2(pi)sjc/n)
                    n --  j
                      j=0
```

and so

```
                      n-1
                    c --
           F ~= - > f exp(-i2(pi)jk/n)
            k   n --  j
                  j=0
```

for k=0,1,...,n-1, where f =f(jc/n) and F =F(k/c).
                          j   k

Hence the discrete Fourier transform gives an approximation to the Fourier integral transform in the region s=0 to s=n/c.

If the function f(t) is defined over some more general interval (a,b), then the integral transform can still be approximated by the discrete transform provided a shift is applied to move the point a to the origin.

2.1.4. Convolutions and correlations

One of the most important applications of the discrete Fourier transform is to the computation of the discrete convolution or correlation of two vectors x and y defined (as in Brigham [1]) by:

```
                    n-1
                    --
     convolution: z = >  x y
                   k  --  j k-j
```

$$\text{correlation: } w_k = \sum_{j=0}^{n-1} x_j y_{k+j}$$

(Here x and y are assumed to be periodic with period n.)

Under certain circumstances (see Brigham [1]) these can be used as approximations to the convolution or correlation integrals defined by:

$$z(s) = \int_{-\infty}^{+\infty} x(t)y(s-t)\,dt$$

and

$$w(s) = \int_{-\infty}^{+\infty} x(t)y(s+t)\,dt, \quad -\infty<s<+\infty.$$

For more general advice on the use of Fourier transforms, see Hamming [2]; more detailed information on the fast Fourier transform algorithm can be found in Van Loan [3] and Brigham [1].

2.2. References

[1] Brigham E O (1973) The Fast Fourier Transform. Prentice-Hall.

[2] Hamming R W (1962) Numerical Methods for Scientists and Engineers. McGraw-Hill.

[3] Van Loan C (1992) Computational Frameworks for the Fast Fourier Transform. SIAM Philadelphia.

3. Recommendations on Choice and Use of Routines

3.1. One-dimensional Fourier Transforms

The choice of routine is determined first of all by whether the data values constitute a real, Hermitian or general complex sequence. It is wasteful of time and storage to use an inappropriate routine.

Two groups, each of three routines, are provided

	Group 1	Group 2
Real sequences	C06EAF	C06FPF
Hermitian sequences	C06EBF	C06FQF
General complex sequences	C06ECF	C06FRF

Group 1 routines each compute a single transform of length n, without requiring any extra working storage. The Group 1 routines impose some restrictions on the value of n, namely that no prime factor of n may exceed 19 and the total number of prime factors (including repetitions) may not exceed 20 (though the latter
6
restriction only becomes relevant when $n>10$).

Group 2 routines are designed to perform several transforms in a single call, all with the same value of n. They do however require more working storage. Even on scalar processors, they may be somewhat faster than repeated calls to Group 1 routines because of reduced overheads and because they pre-compute and store the required values of trigonometric functions. Group 2 routines impose no practical restrictions on the value of n; however the fast Fourier transform algorithm ceases to be 'fast' if applied to values of n which cannot be expressed as a product of small prime factors. All the above routines are particularly efficient if the only prime factors of n are 2, 3 or 5.

If extensive use is to be made of these routines, users who are concerned about efficiency are advised to conduct their own timing tests.

To compute inverse discrete Fourier transforms the above routines should be used in conjunction with the utility routines C06GBF, C06GCF and C06GQF which form the complex conjugate of a Hermitian or general sequence of complex data values.

3.2. Multi-dimensional Fourier Transforms

C06FUF computes a 2-dimensional discrete Fourier transform of a 2-dimensional sequence of complex data values. This is defined by

$$\hat{z}_{k_1 k_2} = \frac{1}{n_1 n_2} \sum_{j_1=0}^{n_1-1} \sum_{j_2=0}^{n_2-1} z_{j_1 j_2} \exp\left(-i \frac{2\pi j_1 k_1}{n_1}\right) \exp\left(-i \frac{2\pi j_2 k_2}{n_2}\right).$$

3.3. Convolution and Correlation

C06EKF computes either the discrete convolution or the discrete correlation of two real vectors.

3.4. Index

Summation of Series

```
C06EAF  Single 1-D real discrete Fourier transform, no extra
        workspace

C06EBF  Single 1-D Hermitian discrete Fourier transform, no extra
        workspace

C06ECF  Single 1-D complex discrete Fourier transform, no extra
        workspace

C06EKF  Circular convolution or correlation of two real vectors,
        no extra workspace

C06FPF  Multiple 1-D real discrete Fourier transforms

C06FQF  Multiple 1-D Hermitian discrete Fourier transforms

C06FRF  Multiple 1-D complex discrete Fourier transforms

C06FUF  2-D complex discrete Fourier transform

C06GBF  Complex conjugate of Hermitian sequence

C06GCF  Complex conjugate of complex sequence

C06GQF  Complex conjugate of multiple Hermitian sequences

C06GSF  Convert Hermitian sequences to general complex sequences
```

%%

```
C06EAF(3NAG)      Foundation Library (12/10/92)      C06EAF(3NAG)

C06 -- Summation of Series                                   C06EAF
        C06EAF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for
your implementation to check implementation-dependent details.
The symbol (*) after a NAG routine name denotes a routine that is
not included in the Foundation Library.

1. Purpose

C06EAF calculates the discrete Fourier transform of a sequence of
n real data values. (No extra workspace required.)

2. Specification
```

```
      SUBROUTINE C06EAF (X, N, IFAIL)
      INTEGER          N, IFAIL
      DOUBLE PRECISION X(N)
```

3. Description

Given a sequence of n real data values x_j, for j=0,1,...,n-1, this routine calculates their discrete Fourier transform defined by:

$$\hat{z}_k = \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} x_j \exp\left(-i \frac{2\pi jk}{n}\right), \quad k=0,1,\ldots,n-1.$$

(Note the scale factor of $\frac{1}{\sqrt{n}}$ in this definition.) The transformed values $\hat{z}_k$ are complex, but they form a Hermitian sequence (i.e., $\hat{z}_{n-k}$ is the complex conjugate of $\hat{z}_k$), so they are completely determined by n real numbers (see also the Chapter Introduction).

To compute the inverse discrete Fourier transform defined by:

$$\hat{w}_k = \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} x_j \exp\left(+i \frac{2\pi jk}{n}\right),$$

this routine should be followed by a call of C06GBF to form the complex conjugates of the $\hat{z}_k$.

The routine uses the fast Fourier transform (FFT) algorithm

(Brigham [1]). There are some restrictions on the value of n (see Section 5).

4. References

[1] Brigham E O (1973) The Fast Fourier Transform. Prentice-Hall.

5. Parameters

1: X(N) -- DOUBLE PRECISION array Input/Output
On entry: if X is declared with bounds (0:N-1) in the (sub) program from which C06EAF is called, then X(j) must contain x_j, for j=0,1,...,n-1. On exit: the discrete Fourier transform stored in Hermitian form. If the components of the transform $\hat{z}_k$ are written as $a_k + ib_k$, and if X is declared with bounds (0:N-1) in the (sub)program from which C06EAF is called, then for 0<=k<=n/2, a_k is contained in X(k), and for 1<=k<=(n-1)/2, b_k is contained in X(n-k). (See also Section 2.1.2 of the Chapter Introduction, and the Example Program.)

2: N -- INTEGER Input
On entry: the number of data values, n. The largest prime factor of N must not exceed 19, and the total number of prime factors of N, counting repetitions, must not exceed 20. Constraint: N > 1.

3: IFAIL -- INTEGER Input/Output
On entry: IFAIL must be set to 0, -1 or 1. For users not familiar with this parameter (described in the Essential Introduction) the recommended value is 0.

On exit: IFAIL = 0 unless the routine detects an error (see Section 6).

6. Error Indicators and Warnings

Errors detected by the routine:

IFAIL= 1
At least one of the prime factors of N is greater than 19.

```
IFAIL= 2
     N has more than 20 prime factors.

IFAIL= 3
     N <= 1.

7. Accuracy

Some indication of accuracy can be obtained by performing a
subsequent inverse transform and comparing the results with the
original sequence (in exact arithmetic they would be identical).

8. Further Comments

The time taken by the routine is approximately proportional to
n*logn, but also depends on the factorization of n. The routine
is somewhat faster than average if the only prime factors of n
are 2, 3 or 5; and fastest of all if n is a power of 2.

On the other hand, the routine is particularly slow if n has
several unpaired prime factors, i.e., if the 'square-free' part
of n has several factors. For such values of n, routine C06FAF(*)
(which requires an additional n elements of workspace) is
considerably faster.

9. Example

This program reads in a sequence of real data values, and prints
their discrete Fourier transform (as computed by C06EAF), after
expanding it from Hermitian form into a full complex sequence.

It then performs an inverse transform using C06GBF and C06EBF,
and prints the sequence so obtained alongside the original data
values.

The example program is not reproduced here. The source code for
all example programs is distributed with the NAG Foundation
Library software and should be available on-line.
```

%%%

```
C06EBF(3NAG)      Foundation Library (12/10/92)      C06EBF(3NAG)

C06 -- Summation of Series                                  C06EBF
        C06EBF -- NAG Foundation Library Routine Document
```

Note: Before using this routine, please read the Users' Note for your implementation to check implementation-dependent details. The symbol (*) after a NAG routine name denotes a routine that is not included in the Foundation Library.

1. Purpose

C06EBF calculates the discrete Fourier transform of a Hermitian sequence of n complex data values. (No extra workspace required.)

2. Specification

```
SUBROUTINE C06EBF (X, N, IFAIL)
INTEGER          N, IFAIL
DOUBLE PRECISION X(N)
```

3. Description

Given a Hermitian sequence of n complex data values z_j (i.e., a sequence such that z_0 is real and z_{n-j} is the complex conjugate of z_j, for j=1,2,...,n-1) this routine calculates their discrete Fourier transform defined by:

$$\hat{x}_k = \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} z_j \exp\left(-i\,\frac{2\pi jk}{n}\right), \quad k=0,1,\ldots,n-1.$$

(Note the scale factor of $\frac{1}{\sqrt{n}}$ in this definition.) The transformed values $\hat{x}_k$ are purely real (see also the the Chapter Introduction).

To compute the inverse discrete Fourier transform defined by:

$$n-1$$

$$\hat{y}_k = \frac{1}{\sqrt{n}} \sum_{j=0}^{n} z_j \exp\left(+i\,\frac{2\pi jk}{n}\right),$$

this routine should be preceded by a call of C06GBF to form the complex conjugates of the z_j.

The routine uses the fast Fourier transform (FFT) algorithm (Brigham [1]). There are some restrictions on the value of n (see Section 5).

4. References

[1] Brigham E O (1973) The Fast Fourier Transform. Prentice-Hall.

5. Parameters

1: X(N) -- DOUBLE PRECISION array Input/Output
On entry: the sequence to be transformed stored in Hermitian form. If the data values z_j are written as $x_j + iy_j$, and if X is declared with bounds (0:N-1) in the subroutine from which C06EBF is called, then for $0 \le j \le n/2$, x_j is contained in X(j), and for $1 \le j \le (n-1)/2$, y_j is contained in X(n-j). (See also Section 2.1.2 of the Chapter Introduction and the Example Program.) On exit: the components of the discrete Fourier transform $\hat{x}_k$. If X is declared with bounds (0:N-1) in the (sub)program from which C06EBF is called, then $\hat{x}_k$ is stored in X(k), for k=0,1,...,n-1.

2: N -- INTEGER Input
On entry: the number of data values, n. The largest prime factor of N must not exceed 19, and the total number of prime factors of N, counting repetitions, must not exceed 20. Constraint: N > 1.

3: IFAIL -- INTEGER Input/Output

```
     On entry: IFAIL must be set to 0, -1 or 1. For users not
     familiar with this parameter (described in the Essential
     Introduction) the recommended value is 0.

     On exit: IFAIL = 0 unless the routine detects an error (see
     Section 6).

6. Error Indicators and Warnings

Errors detected by the routine:

IFAIL= 1
     At least one of the prime factors of N is greater than 19.

IFAIL= 2
     N has more than 20 prime factors.

IFAIL= 3
     N <= 1.

7. Accuracy

Some indication of accuracy can be obtained by performing a
subsequent inverse transform and comparing the results with the
original sequence (in exact arithmetic they would be identical).

8. Further Comments

The time taken by the routine is approximately proportional to
n*logn, but also depends on the factorization of n. The routine
is somewhat faster than average if the only prime factors of n
are 2, 3 or 5; and fastest of all if n is a power of 2.

On the other hand, the routine is particularly slow if n has
several unpaired prime factors, i.e., if the 'square-free' part
of n has several factors. For such values of n, routine C06FBF(*)
(which requires an additional n elements of workspace) is
considerably faster.

9. Example

This program reads in a sequence of real data values which is
assumed to be a Hermitian sequence of complex data values stored
in Hermitian form. The input sequence is expanded into a full
complex sequence and printed alongside the original sequence. The
discrete Fourier transform (as computed by C06EBF) is printed
```

```
out.

The program then performs an inverse transform using C06EAF and
C06GBF, and prints the sequence so obtained alongside the
original data values.

The example program is not reproduced here. The source code for
all example programs is distributed with the NAG Foundation
Library software and should be available on-line.
```

%%%

```
C06ECF(3NAG)      Foundation Library (12/10/92)      C06ECF(3NAG)

C06 -- Summation of Series                                   C06ECF
        C06ECF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for
your implementation to check implementation-dependent details.
The symbol (*) after a NAG routine name denotes a routine that is
not included in the Foundation Library.

1. Purpose

C06ECF calculates the discrete Fourier transform of a sequence of
n complex data values. (No extra workspace required.)

2. Specification

       SUBROUTINE C06ECF (X, Y, N, IFAIL)
       INTEGER          N, IFAIL
       DOUBLE PRECISION X(N), Y(N)

3. Description

Given a sequence of n complex data values z , for j=0,1,...,n-1,
                                           j
this routine calculates their discrete Fourier transform defined
by:

                n-1
          ^    1  --       (   2(pi)jk)
          z = --- >  z *exp(-i -------), k=0,1,...,n-1.
           k      --  j    (      n   )
              \/n j=0
```

(Note the scale factor of $\frac{1}{\sqrt{n}}$ in this definition.)

To compute the inverse discrete Fourier transform defined by:

$$\hat{w}_k = \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} z_j \times \exp\left(+i\,\frac{2(\pi)jk}{n}\right),$$

this routine should be preceded and followed by calls of C06GCF to form the complex conjugates of the z_j and the $\hat{z}_k$.

The routine uses the fast Fourier transform (FFT) algorithm (Brigham [1]). There are some restrictions on the value of n (see Section 5).

4. References

[1] Brigham E O (1973) The Fast Fourier Transform. Prentice-Hall.

5. Parameters

1: X(N) -- DOUBLE PRECISION array Input/Output
On entry: if X is declared with bounds (0:N-1) in the (sub) program from which C06ECF is called, then X(j) must contain x_j, the real part of z_j, for j=0,1,...,n-1. On exit: the real parts a_k of the components of the discrete Fourier transform. If X is declared with bounds (0:N-1) in the (sub) program from which C06ECF is called, then a_k is contained in X(k), for k=0,1,...,n-1.

2: Y(N) -- DOUBLE PRECISION array Input/Output
On entry: if Y is declared with bounds (0:N-1) in the (sub) program from which C06ECF is called, then Y(j) must contain y_j, the imaginary part of z_j, for j=0,1,...,n-1. On exit:

```
      j                          j
     the imaginary parts b  of the components of the discrete
                          k
     Fourier transform. If Y is declared with bounds (0:N-1) in
     the (sub)program from which C06ECF is called, then b  is
                                                         k
     contained in Y(k), for k=0,1,...,n-1.

 3:  N -- INTEGER                                          Input
     On entry: the number of data values, n. The largest prime
     factor of N must not exceed 19, and the total number of
     prime factors of N, counting repetitions, must not exceed
     20. Constraint: N > 1.

 4:  IFAIL -- INTEGER                                Input/Output
     On entry: IFAIL must be set to 0, -1 or 1. For users not
     familiar with this parameter (described in the Essential
     Introduction) the recommended value is 0.

     On exit: IFAIL = 0 unless the routine detects an error (see
     Section 6).

6. Error Indicators and Warnings

Errors detected by the routine:

IFAIL= 1
     At least one of the prime factors of N is greater than 19.

IFAIL= 2
     N has more than 20 prime factors.

IFAIL= 3
     N <= 1.

7. Accuracy

Some indication of accuracy can be obtained by performing a
subsequent inverse transform and comparing the results with the
original sequence (in exact arithmetic they would be identical).

8. Further Comments

The time taken by the routine is approximately proportional to
n*logn, but also depends on the factorization of n. The routine
is somewhat faster than average if the only prime factors of n
```

```
are 2, 3 or 5; and fastest of all if n is a power of 2.

On the other hand, the routine is particularly slow if n has
several unpaired prime factors, i.e., if the 'square-free' part
of n has several factors. For such values of n, routine C06FCF(*)
(which requires an additional n real elements of workspace) is
considerably faster.

9. Example

This program reads in a sequence of complex data values and
prints their discrete Fourier transform.

It then performs an inverse transform using C06GCF and C06ECF,
and prints the sequence so obtained alongside the original data
values.

The example program is not reproduced here. The source code for
all example programs is distributed with the NAG Foundation
Library software and should be available on-line.
```

%%%

```
     C06EKF(3NAG)      Foundation Library (12/10/92)      C06EKF(3NAG)

     C06 -- Summation of Series                                  C06EKF
            C06EKF -- NAG Foundation Library Routine Document

     Note: Before using this routine, please read the Users' Note for
     your implementation to check implementation-dependent details.
     The symbol (*) after a NAG routine name denotes a routine that is
     not included in the Foundation Library.

     1. Purpose

     C06EKF calculates the circular convolution or correlation of two
     real vectors of period n. No extra workspace is required.

     2. Specification

            SUBROUTINE C06EKF (JOB, X, Y, N, IFAIL)
            INTEGER          JOB, N, IFAIL
            DOUBLE PRECISION X(N), Y(N)

     3. Description
```

This routine computes:

if JOB =1, the discrete convolution of x and y, defined by:

$$z_k = \sum_{j=0}^{n-1} x_j y_{k-j} = \sum_{j=0}^{n-1} x_{k-j} y_j ;$$

if JOB =2, the discrete correlation of x and y defined by:

$$w_k = \sum_{j=0}^{n-1} x_j y_{k+j} .$$

Here x and y are real vectors, assumed to be periodic, with period n, i.e., $x_j = x_{j+n} = x_{j+2n} = \ldots$; z and w are then also periodic with period n.

Note: this usage of the terms 'convolution' and 'correlation' is taken from Brigham [1]. The term 'convolution' is sometimes used to denote both these computations.

If $\hat{x}$, $\hat{y}$, $\hat{z}$ and $\hat{w}$ are the discrete Fourier transforms of these sequences,

$$\text{i.e., } \hat{x}_k = \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} x_j \exp\left(-i \frac{2\pi jk}{n}\right), \text{ etc,}$$

$$\text{then } \hat{z}_k = \sqrt{n}\, \hat{x}_k \hat{y}_k$$

$$\text{and } \hat{w}_k = \sqrt{n}\, \hat{x}_k \hat{y}_k$$

(the bar denoting complex conjugate).

This routine calls the same auxiliary routines as C06EAF and C06EBF to compute discrete Fourier transforms, and there are some restrictions on the value of n.

4. References

[1] Brigham E O (1973) The Fast Fourier Transform. Prentice-Hall.

5. Parameters

1: JOB -- INTEGER Input
On entry: the computation to be performed:

$$\text{if JOB = 1, } z = \sum_{j=0}^{n-1} x_j\, y_{k-j} \quad \text{(convolution);}$$

$$\text{if JOB = 2, } w = \sum_{j=0}^{n-1} x_j\, y_{k+j} \quad \text{(correlation).}$$

Constraint: JOB = 1 or 2.

2: X(N) -- DOUBLE PRECISION array Input/Output
On entry: the elements of one period of the vector x. If X is declared with bounds (0:N-1) in the (sub)program from which C06EKF is called, then X(j) must contain x_j, for j=0,1,...,n-1. On exit: the corresponding elements of the discrete convolution or correlation.

3: Y(N) -- DOUBLE PRECISION array Input/Output
On entry: the elements of one period of the vector y. If Y is declared with bounds (0:N-1) in the (sub)program from which C06EKF is called, then Y(j) must contain y_j, for j=0,1,...,n-1. On exit: the discrete Fourier transform of the convolution or correlation returned in the array X; the transform is stored in Hermitian form, exactly as described in the document C06EAF.

```
 4:  N -- INTEGER                                              Input
     On entry: the number of values, n, in one period of the
     vectors X and Y. The largest prime factor of N must not
     exceed 19, and the total number of prime factors of N,
     counting repetitions, must not exceed 20. Constraint: N > 1.

 5:  IFAIL -- INTEGER                                  Input/Output
     On entry: IFAIL must be set to 0, -1 or 1. For users not
     familiar with this parameter (described in the Essential
     Introduction) the recommended value is 0.

     On exit: IFAIL = 0 unless the routine detects an error (see
     Section 6).

6. Error Indicators and Warnings

Errors detected by the routine:

IFAIL= 1
     At least one of the prime factors of N is greater than 19.

IFAIL= 2
     N has more than 20 prime factors.

IFAIL= 3
     N <= 1.

IFAIL= 4
     JOB /= 1 or 2.

7. Accuracy

The results should be accurate to within a small multiple of the
machine precision.

8. Further Comments

The time taken by the routine is approximately proportional to
n*logn, but also depends on the factorization of n. The routine
is faster than average if the only prime factors are 2, 3 or 5;
and fastest of all if n is a power of 2.

The routine is particularly slow if n has several unpaired prime
factors, i.e., if the 'square free' part of n has several
factors. For such values of n, routine C06FKF(*) is considerably
```

faster (but requires an additional workspace of n elements).

9. Example

This program reads in the elements of one period of two real vectors x and y and prints their discrete convolution and correlation (as computed by C06EKF). In realistic computations the number of data values would be much larger.

The example program is not reproduced here. The source code for all example programs is distributed with the NAG Foundation Library software and should be available on-line.

%%%

C06FPF(3NAG) Foundation Library (12/10/92) C06FPF(3NAG)

C06 -- Summation of Series C06FPF
C06FPF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for your implementation to check implementation-dependent details. The symbol (*) after a NAG routine name denotes a routine that is not included in the Foundation Library.

1. Purpose

C06FPF computes the discrete Fourier transforms of m sequences, each containing n real data values. This routine is designed to be particularly efficient on vector processors.

2. Specification

```
SUBROUTINE C06FPF (M, N, X, INIT, TRIG, WORK, IFAIL)
INTEGER          M, N, IFAIL
DOUBLE PRECISION X(M*N), TRIG(2*N), WORK(M*N)
CHARACTER*1      INIT
```

3. Description

Given m sequences of n real data values x_j^p, for $j=0,1,\ldots,n-1$; $p=1,2,\ldots,m$, this routine simultaneously calculates the Fourier transforms of all the sequences defined by:

$$\hat{z}_k^p = \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} x_j^p \exp\left(-i\,\frac{2\pi jk}{n}\right), \quad k=0,1,\ldots,n-1;\ p=1,2,\ldots,m.$$

(Note the scale factor $\frac{1}{\sqrt{n}}$ in this definition.)

The transformed values $\hat{z}_k^p$ are complex, but for each value of p the $\hat{z}_k^p$ form a Hermitian sequence (i.e., $\hat{z}_{n-k}^p$ is the complex conjugate of $\hat{z}_k^p$), so they are completely determined by mn real numbers (see also the Chapter Introduction).

The discrete Fourier transform is sometimes defined using a positive sign in the exponential term:

$$\hat{z}_k^p = \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} x_j^p \exp\left(+i\,\frac{2\pi jk}{n}\right).$$

To compute this form, this routine should be followed by a call to C06GQF to form the complex conjugates of the $\hat{z}_k^p$.

The routine uses a variant of the fast Fourier transform (FFT) algorithm (Brigham [1]) known as the Stockham self-sorting algorithm, which is described in Temperton [2]. Special coding is provided for the factors 2, 3, 4, 5 and 6. This routine is designed to be particularly efficient on vector processors, and it becomes especially fast as M, the number of transforms to be computed in parallel, increases.

4. References

[1] Brigham E O (1973) The Fast Fourier Transform. Prentice-Hall.

[2] Temperton C (1983) Fast Mixed-Radix Real Fourier Transforms. J. Comput. Phys. 52 340--350.

5. Parameters

1: M -- INTEGER Input
On entry: the number of sequences to be transformed, m.
Constraint: M >= 1.

2: N -- INTEGER Input
On entry: the number of real values in each sequence, n.
Constraint: N >= 1.

3: X(M,N) -- DOUBLE PRECISION array Input/Output
On entry: the data must be stored in X as if in a two-dimensional array of dimension (1:M,0:N-1); each of the m sequences is stored in a row of the array. In other words, if the data values of the pth sequence to be transformed are denoted by x_j^p, for j=0,1,...,n-1, then the mn elements of the array X must contain the values
$$x_0^1, x_0^2, \ldots, x_0^m, x_1^1, x_1^2, \ldots, x_1^m, \ldots, x_{n-1}^1, x_{n-1}^2, \ldots, x_{n-1}^m .$$
On exit: the m discrete Fourier transforms stored as if in a two-dimensional array of dimension (1:M,0:N-1). Each of the m transforms is stored in a row of the array in Hermitian form, overwriting the corresponding original sequence. If the n components of the discrete Fourier transform $\hat{z}_k^p$ are written as $a_k^p + ib_k^p$, then for $0 \le k \le n/2$, a_k^p is contained in X(p,k), and for $1 \le k \le (n-1)/2$, b_k^p is contained in X(p,n-k). (See also Section 2.1.2 of the Chapter Introduction.)

4: INIT -- CHARACTER*1 Input
On entry: if the trigonometric coefficients required to compute the transforms are to be calculated by the routine

```
     and stored in the array TRIG, then INIT must be set equal to
     'I' (Initial call).

     If INIT contains 'S' (Subsequent call), then the routine
     assumes that trigonometric coefficients for the specified
     value of n are supplied in the array TRIG, having been
     calculated in a previous call to one of C06FPF, C06FQF or
     C06FRF.

     If INIT contains 'R' (Restart then the routine assumes that
     trigonometric coefficients for the particular value of n are
     supplied in the array TRIG, but does not check that C06FPF,
     C06FQF or C06FRF have previously been called. This option
     allows the TRIG array to be stored in an external file, read
     in and re-used without the need for a call with INIT equal
     to 'I'. The routine carries out a simple test to check that
     the current value of n is consistent with the array TRIG.
     Constraint: INIT = 'I', 'S' or 'R'.

 5:  TRIG(2*N) -- DOUBLE PRECISION array              Input/Output
     On entry: if INIT = 'S' or 'R', TRIG must contain the
     required coefficients calculated in a previous call of the
     routine. Otherwise TRIG need not be set. On exit: TRIG
     contains the required coefficients (computed by the routine
     if INIT = 'I').

 6:  WORK(M*N) -- DOUBLE PRECISION array                 Workspace

 7:  IFAIL -- INTEGER                                 Input/Output
     On entry: IFAIL must be set to 0, -1 or 1. For users not
     familiar with this parameter (described in the Essential
     Introduction) the recommended value is 0.

     On exit: IFAIL = 0 unless the routine detects an error (see
     Section 6).


6. Error Indicators and Warnings

Errors detected by the routine:

If on entry IFAIL = 0 or -1, explanatory error messages are
output on the current error message unit (as defined by X04AAF).

IFAIL= 1
     On entry M < 1.
```

```
IFAIL= 2
     N < 1.

IFAIL= 3
     INIT is not one of 'I', 'S' or 'R'.

IFAIL= 4
     INIT = 'S', but none of C06FPF, C06FQF or C06FRF has
     previously been called.

IFAIL= 5
     INIT = 'S' or 'R', but the array TRIG and the current value
     of N are inconsistent.

7. Accuracy

Some indication of accuracy can be obtained by performing a
subsequent inverse transform and comparing the results with the
original sequence (in exact arithmetic they would be identical).

8. Further Comments

The time taken by the routine is approximately proportional to
nm*logn, but also depends on the factors of n. The routine is
fastest if the only prime factors of n are 2, 3 and 5, and is
particularly slow if n is a large prime, or has large prime
factors.

9. Example

This program reads in sequences of real data values and prints
their discrete Fourier transforms (as computed by C06FPF). The
Fourier transforms are expanded into full complex form using
C06GSF and printed. Inverse transforms are then calculated by
calling C06GQF followed by C06FQF showing that the original
sequences are restored.

The example program is not reproduced here. The source code for
all example programs is distributed with the NAG Foundation
Library software and should be available on-line.

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

     C06FQF(3NAG)      Foundation Library (12/10/92)      C06FQF(3NAG)
```

C06 -- Summation of Series C06FQF
C06FQF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for your implementation to check implementation-dependent details. The symbol (*) after a NAG routine name denotes a routine that is not included in the Foundation Library.

1. Purpose

C06FQF computes the discrete Fourier transforms of m Hermitian sequences, each containing n complex data values. This routine is designed to be particularly efficient on vector processors.

2. Specification

```
SUBROUTINE C06FQF (M, N, X, INIT, TRIG, WORK, IFAIL)
INTEGER          M, N, IFAIL
DOUBLE PRECISION X(M*N), TRIG(2*N), WORK(M*N)
CHARACTER*1      INIT
```

3. Description

Given m Hermitian sequences of n complex data values z_j^p, for $j=0,1,\ldots,n-1$; $p=1,2,\ldots,m$, this routine simultaneously calculates the Fourier transforms of all the sequences defined by:

$$\hat{x}_k^p = \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} z_j^p \exp\left(-i\frac{2\pi jk}{n}\right), \quad k=0,1,\ldots,n-1;\ p=1,2,\ldots,m.$$

(Note the scale factor $\frac{1}{\sqrt{n}}$ in this definition.)

The transformed values are purely real (see also the Chapter Introduction).

The discrete Fourier transform is sometimes defined using a

positive sign in the exponential term

$$\hat{x}_k = \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} \hat{z}_j \exp\left(+i\,\frac{2\pi jk}{n}\right).$$

To compute this form, this routine should be preceded by a call to C06GQF to form the complex conjugates of the $\hat{z}_j$.

The routine uses a variant of the fast Fourier transform (FFT) algorithm (Brigham [1]) known as the Stockham self-sorting algorithm, which is described in Temperton [2]. Special code is included for the factors 2, 3, 4, 5 and 6. This routine is designed to be particularly efficient on vector processors, and it becomes especially fast as m, the number of transforms to be computed in parallel, increases.

4. References

[1] Brigham E O (1973) The Fast Fourier Transform. Prentice-Hall.

[2] Temperton C (1983) Fast Mixed-Radix Real Fourier Transforms. J. Comput. Phys. 52 340--350.

5. Parameters

1: M -- INTEGER Input
On entry: the number of sequences to be transformed, m.
Constraint: M >= 1.

2: N -- INTEGER Input
On entry: the number of data values in each sequence, n.
Constraint: N >= 1.

3: X(M,N) -- DOUBLE PRECISION array Input/Output
On entry: the data must be stored in X as if in a two-dimensional array of dimension (1:M,0:N-1); each of the m sequences is stored in a row of the array in Hermitian form.
If the n data values $\hat{z}_j$ are written as $x_j + iy_j$, then for

p
0<=j<=n/2, x is contained in X(p,j), and for 1<=j<=(n-1)/2,
           j
 p
y is contained in X(p,n-j). (See also Section 2.1.2 of the
 j
Chapter Introduction.) On exit: the components of the m
discrete Fourier transforms, stored as if in a two-
dimensional array of dimension (1:M,0:N-1). Each of the m
transforms is stored as a row of the array, overwriting the
corresponding original sequence. If the n components of the
                                          ^p
discrete Fourier transform are denoted by x , for
                                           k
k=0,1,...,n-1, then the mn elements of the array X contain
the values
    ^1 ^2   ^m ^1 ^2   ^m   ^1   ^2   ^m
    x ,x ,...,x , x ,x ,..., x ,...,x   ,x   ,...,x   .
     0 0   0   1 1   1   n-1 n-1   n-1

4: INIT -- CHARACTER*1 Input
On entry: if the trigonometric coefficients required to compute the transforms are to be calculated by the routine and stored in the array TRIG, then INIT must be set equal to 'I' (Initial call).

If INIT contains 'S' (Subsequent call), then the routine assumes that trigonometric coefficients for the specified value of n are supplied in the array TRIG, having been calculated in a previous call to one of C06FPF, C06FQF or C06FRF.

If INIT contains 'R' (Restart), then the routine assumes that trigonometric coefficients for the particular value of N are supplied in the array TRIG, but does not check that C06FPF, C06FQF or C06FRF have previously been called. This option allows the TRIG array to be stored in an external file, read in and re-used without the need for a call with INIT equal to 'I'. The routine carries out a simple test to check that the current value of n is compatible with the array TRIG. Constraint: INIT = 'I', 'S' or 'R'.

5: TRIG(2*N) -- DOUBLE PRECISION array Input/Output
On entry: if INIT = 'S' or 'R', TRIG must contain the required coefficients calculated in a previous call of the routine. Otherwise TRIG need not be set. On exit: TRIG

```
     contains the required coefficients (computed by the routine
     if INIT = 'I').

 6:  WORK(M*N) -- DOUBLE PRECISION array                Workspace

 7:  IFAIL -- INTEGER                                   Input/Output
     On entry: IFAIL must be set to 0, -1 or 1. For users not
     familiar with this parameter (described in the Essential
     Introduction) the recommended value is 0.

     On exit: IFAIL = 0 unless the routine detects an error (see
     Section 6).

6. Error Indicators and Warnings

Errors detected by the routine:

If on entry IFAIL = 0 or -1, explanatory error messages are
output on the current error message unit (as defined by X04AAF).

IFAIL= 1
     On entry M < 1.

IFAIL= 2
     On entry N < 1.

IFAIL= 3
     On entry INIT is not one of 'I', 'S' or 'R'.

IFAIL= 4
     On entry INIT = 'S', but none of C06FPF, C06FQF and C06FRF
     has previously been called.

IFAIL= 5
     On entry INIT = 'S' or 'R', but the array TRIG and the
     current value of n are inconsistent.

7. Accuracy

Some indication of accuracy can be obtained by performing a
subsequent inverse transform and comparing the results with the
original sequence (in exact arithmetic they would be identical).

8. Further Comments

The time taken by the routine is approximately proportional to
```

nm*logn, but also depends on the factors of n. The routine is fastest if the only prime factors of n are 2, 3 and 5, and is particularly slow if n is a large prime, or has large prime factors.

9. Example

This program reads in sequences of real data values which are assumed to be Hermitian sequences of complex data stored in Hermitian form. The sequences are expanded into full complex form using C06GSF and printed. The discrete Fourier transforms are then computed (using C06FQF) and printed out. Inverse transforms are then calculated by calling C06FPF followed by C06GQF showing that the original sequences are restored.

The example program is not reproduced here. The source code for all example programs is distributed with the NAG Foundation Library software and should be available on-line.

%%%

C06FRF(3NAG) Foundation Library (12/10/92) C06FRF(3NAG)

C06 -- Summation of Series C06FRF
C06FRF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for your implementation to check implementation-dependent details. The symbol (*) after a NAG routine name denotes a routine that is not included in the Foundation Library.

1. Purpose

C06FRF computes the discrete Fourier transforms of m sequences, each containing n complex data values. This routine is designed to be particularly efficient on vector processors.

2. Specification

```
SUBROUTINE C06FRF (M, N, X, Y, INIT, TRIG, WORK, IFAIL)
INTEGER          M, N, IFAIL
DOUBLE PRECISION X(M*N), Y(M*N), TRIG(2*N), WORK(2*M*N)
CHARACTER*1      INIT
```

3. Description

Given m sequences of n complex data values z_j^p, for j=0,1,...,n-1; p=1,2,...,m, this routine simultaneously calculates the Fourier transforms of all the sequences defined by:

$$\hat{z}_k^p = \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} z_j^p \times \exp\left(-i\,\frac{2\pi jk}{n}\right), \quad k=0,1,\ldots,n-1;\ p=1,2,\ldots,m.$$

(Note the scale factor $\frac{1}{\sqrt{n}}$ in this definition.)

The discrete Fourier transform is sometimes defined using a positive sign in the exponential term

$$\hat{z}_k^p = \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} z_j^p \times \exp\left(+i\,\frac{2\pi jk}{n}\right).$$

To compute this form, this routine should be preceded and followed by a call of C06GCF to form the complex conjugates of z_j^p and the $\hat{z}_k^p$.

The routine uses a variant of the fast Fourier transform (FFT) algorithm (Brigham [1]) known as the Stockham self-sorting algorithm, which is described in Temperton [2]. Special code is provided for the factors 2, 3, 4, 5 and 6. This routine is designed to be particularly efficient on vector processors, and it becomes especially fast as m, the number of transforms to be computed in parallel, increases.

4. References

[1] Brigham E O (1973) The Fast Fourier Transform. Prentice-Hall.

[2] Temperton C (1983) Self-sorting Mixed-radix Fast Fourier Transforms. J. Comput. Phys. 52 1--23.

5. Parameters

1: M -- INTEGER Input
On entry: the number of sequences to be transformed, m.
Constraint: M >= 1.

2: N -- INTEGER Input
On entry: the number of complex values in each sequence, n.
Constraint: N >= 1.

3: X(M,N) -- DOUBLE PRECISION array Input/Output

4: Y(M,N) -- DOUBLE PRECISION array Input/Output
On entry: the real and imaginary parts of the complex data must be stored in X and Y respectively as if in a two-dimensional array of dimension (1:M,0:N-1); each of the m sequences is stored in a row of each array. In other words, if the real parts of the pth sequence to be transformed are denoted by x_j^p, for j=0,1,...,n-1, then the mn elements of the array X must contain the values

$$x_0^1, x_0^2, \ldots, x_0^m, x_1^1, x_1^2, \ldots, x_1^m, \ldots, x_{n-1}^1, x_{n-1}^2, \ldots, x_{n-1}^m .$$

On exit: X and Y are overwritten by the real and imaginary parts of the complex transforms.

5: INIT -- CHARACTER*1 Input
On entry: if the trigonometric coefficients required to compute the transforms are to be calculated by the routine and stored in the array TRIG, then INIT must be set equal to 'I' (Initial call).

If INIT contains 'S' (Subsequent call), then the routine assumes that trigonometric coefficients for the specified value of n are supplied in the array TRIG, having been calculated in a previous call to one of C06FPF, C06FQF or C06FRF.

If INIT contains 'R' (Restart) then the routine assumes that trigonometric coefficients for the particular value of n are supplied in the array TRIG, but does not check that C06FPF,

```
     C06FQF or C06FRF have previously been called. This option
     allows the TRIG array to be stored in an external file, read
     in and re-used without the need for a call with INIT equal
     to 'I'. The routine carries out a simple test to check that
     the current value of n is compatible with the array TRIG.
     Constraint: INIT = 'I', 'S' or 'R'.

 6:  TRIG(2*N) -- DOUBLE PRECISION array              Input/Output
     On entry: if INIT = 'S' or 'R', TRIG must contain the
     required coefficients calculated in a previous call of the
     routine. Otherwise TRIG need not be set. On exit: TRIG
     contains the required coefficients (computed by the routine
     if INIT = 'I').

 7:  WORK(2*M*N) -- DOUBLE PRECISION array               Workspace

 8:  IFAIL -- INTEGER                                   Input/Output
     On entry: IFAIL must be set to 0, -1 or 1. For users not
     familiar with this parameter (described in the Essential
     Introduction) the recommended value is 0.

     On exit: IFAIL = 0 unless the routine detects an error (see
     Section 6).

6. Error Indicators and Warnings

Errors detected by the routine:

If on entry IFAIL = 0 or -1, explanatory error messages are
output on the current error message unit (as defined by X04AAF).

IFAIL= 1
     On entry M < 1.

IFAIL= 2
     On entry N < 1.

IFAIL= 3
     On entry INIT is not one of 'I', 'S' or 'R'.

IFAIL= 4
     On entry INIT = 'S', but none of C06FPF, C06FQF and C06FRF
     has previously been called.

IFAIL= 5
     On entry INIT = 'S' or 'R', but the array TRIG and the
```

```
     current value of n are inconsistent.

7. Accuracy

Some indication of accuracy can be obtained by performing a
subsequent inverse transform and comparing the results with the
original sequence (in exact arithmetic they would be identical).

8. Further Comments

The time taken by the routine is approximately proportional to
nm*logn, but also depends on the factors of n. The routine is
fastest if the only prime factors of n are 2, 3 and 5, and is
particularly slow if n is a large prime, or has large prime
factors.

9. Example

This program reads in sequences of complex data values and prints
their discrete Fourier transforms (as computed by C06FRF).
Inverse transforms are then calculated using C06GCF and C06FRF
and printed out, showing that the original sequences are
restored.

The example program is not reproduced here. The source code for
all example programs is distributed with the NAG Foundation
Library software and should be available on-line.
```

%%%

```
C06FUF(3NAG)      Foundation Library (12/10/92)      C06FUF(3NAG)

C06 -- Summation of Series                                  C06FUF
        C06FUF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for
your implementation to check implementation-dependent details.
The symbol (*) after a NAG routine name denotes a routine that is
not included in the Foundation Library.

1. Purpose

C06FUF computes the two-dimensional discrete Fourier transform of
a bivariate sequence of complex data values. This routine is
designed to be particularly efficient on vector processors.
```

2. Specification

```
 SUBROUTINE C06FUF (M, N, X, Y, INIT, TRIGM, TRIGN, WORK,
1                   IFAIL)
 INTEGER          M, N, IFAIL
 DOUBLE PRECISION X(M*N), Y(M*N), TRIGM(2*M), TRIGN(2*N),
1                 WORK(2*M*N)
 CHARACTER*1      INIT
```

3. Description

This routine computes the two-dimensional discrete Fourier transform of a bivariate sequence of complex data values $z_{j_1 j_2}$, where $j_1 = 0,1,\ldots,m-1$, $j_2 = 0,1,\ldots,n-1$.

The discrete Fourier transform is here defined by:

$$\hat{z}_{k_1 k_2} = \frac{1}{\sqrt{mn}} \sum_{j_1=0}^{m-1} \sum_{j_2=0}^{n-1} z_{j_1 j_2} \exp\left(-2\pi i \left(\frac{j_1 k_1}{m} + \frac{j_2 k_2}{n}\right)\right),$$

where $k_1 = 0,1,\ldots,m-1$, $k_2 = 0,1,\ldots,n-1$.

(Note the scale factor of $\frac{1}{\sqrt{mn}}$ in this definition.)

To compute the inverse discrete Fourier transform, defined with exp(+2(pi)i(...)) in the above formula instead of exp(-2(pi)i(...)), this routine should be preceded and followed by calls of C06GCF to form the complex conjugates of the data values and the transform.

This routine calls C06FRF to perform multiple one-dimensional discrete Fourier transforms by the fast Fourier transform (FFT) algorithm in Brigham [1]. It is designed to be particularly efficient on vector processors.

4. References

[1] Brigham E O (1973) The Fast Fourier Transform. Prentice-Hall.

[2] Temperton C (1983) Self-sorting Mixed-radix Fast Fourier Transforms. J. Comput. Phys. 52 1--23.

5. Parameters

1: M -- INTEGER Input
On entry: the number of rows, m, of the arrays X and Y.
Constraint: M >= 1.

2: N -- INTEGER Input
On entry: the number of columns, n, of the arrays X and Y.
Constraint: N >= 1.

3: X(M,N) -- DOUBLE PRECISION array Input/Output

4: Y(M,N) -- DOUBLE PRECISION array Input/Output
On entry: the real and imaginary parts of the complex data values must be stored in arrays X and Y respectively. If X and Y are regarded as two-dimensional arrays of dimension (0:M-1,0:N-1), then X(j ,j) and Y(j ,j) must contain the
1 2 1 2
real and imaginary parts of $z_{j_1 j_2}$. On exit: the real and imaginary parts respectively of the corresponding elements of the computed transform.

5: INIT -- CHARACTER*1 Input
On entry: if the trigonometric coefficients required to compute the transforms are to be calculated by the routine and stored in the arrays TRIGM and TRIGN, then INIT must be set equal to 'I', (Initial call).

If INIT contains 'S', (Subsequent call), then the routine assumes that trigonometric coefficients for the specified values of m and n are supplied in the arrays TRIGM and TRIGN, having been calculated in a previous call to the routine.

If INIT contains 'R', (Restart), then the routine assumes

that trigonometric coefficients for the particular values of m and n are supplied in the arrays TRIGM and TRIGN, but does not check that the routine has previously been called. This option allows the TRIGM and TRIGN arrays to be stored in an external file, read in and re-used without the need for a call with INIT equal to 'I'. The routine carries out a simple test to check that the current values of m and n are compatible with the arrays TRIGM and TRIGN. Constraint: INIT = 'I', 'S' or 'R'.

6: TRIGM(2*M) -- DOUBLE PRECISION array Input/Output

7: TRIGN(2*N) -- DOUBLE PRECISION array Input/Output
On entry: if INIT = 'S' or 'R',TRIGM and TRIGN must contain the required coefficients calculated in a previous call of the routine. Otherwise TRIGM and TRIGN need not be set.

If m=n the same array may be supplied for TRIGM and TRIGN.
On exit: TRIGM and TRIGN contain the required coefficients (computed by the routine if INIT = 'I').

8: WORK(2*M*N) -- DOUBLE PRECISION array Workspace

9: IFAIL -- INTEGER Input/Output
On entry: IFAIL must be set to 0, -1 or 1. For users not familiar with this parameter (described in the Essential Introduction) the recommended value is 0.

On exit: IFAIL = 0 unless the routine detects an error (see Section 6).

6. Error Indicators and Warnings

Errors detected by the routine:

If on entry IFAIL = 0 or -1, explanatory error messages are output on the current error message unit (as defined by X04AAF).

IFAIL= 1
On entry M < 1.

IFAIL= 2
On entry N < 1.

IFAIL= 3
On entry INIT is not one of 'I', 'S' or 'R'.

```
IFAIL= 4
     On entry INIT = 'S', but C06FUF has not previously been
     called.

IFAIL= 5
     On entry INIT = 'S' or 'R', but at least one of the arrays
     TRIGM and TRIGN is inconsistent with the current value of M
     or N.

7. Accuracy

Some indication of accuracy can be obtained by performing a
subsequent inverse transform and comparing the results with the
original sequence (in exact arithmetic they would be identical).

8. Further Comments

The time taken by the routine is approximately proportional to
mn*log(mn), but also depends on the factorization of the
individual dimensions m and n. The routine is somewhat faster
than average if their only prime factors are 2, 3 or 5; and
fastest of all if they are powers of 2.

9. Example

This program reads in a bivariate sequence of complex data values
and prints the two-dimensional Fourier transform. It then
performs an inverse transform and prints the sequence so
obtained, which may be compared to the original data values.

The example program is not reproduced here. The source code for
all example programs is distributed with the NAG Foundation
Library software and should be available on-line.
```

%%%

```
C06GBF(3NAG)      Foundation Library (12/10/92)      C06GBF(3NAG)

C06 -- Summation of Series                                  C06GBF
        C06GBF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for
your implementation to check implementation-dependent details.
The symbol (*) after a NAG routine name denotes a routine that is
not included in the Foundation Library.
```

1. Purpose

C06GBF forms the complex conjugate of a Hermitian sequence of n data values.

2. Specification

```
SUBROUTINE C06GBF (X, N, IFAIL)
INTEGER          N, IFAIL
DOUBLE PRECISION X(N)
```

3. Description

This is a utility routine for use in conjunction with C06EAF, C06EBF, C06FAF(*) or C06FBF(*) to calculate inverse discrete Fourier transforms (see the Chapter Introduction).

4. References

None.

5. Parameters

1: X(N) -- DOUBLE PRECISION array Input/Output
On entry: if the data values z_j are written as $x_j + iy_j$ and if X is declared with bounds (0:N-1) in the (sub)program from which C06GBF is called, then for $0 \le j \le n/2$, X(j) must contain x_j (=x_{n-j}), while for $n/2 < j \le n-1$, X(j) must contain $-y_j$ (=y_{n-j}). In other words, X must contain the Hermitian sequence in Hermitian form. (See also Section 2.1.2 of the Chapter Introduction). On exit: the imaginary parts y_j are negated. The real parts x_j are not referenced.

2: N -- INTEGER Input
On entry: the number of data values, n. Constraint: N >= 1.

3: IFAIL -- INTEGER Input/Output
On entry: IFAIL must be set to 0, -1 or 1. For users not familiar with this parameter (described in the Essential Introduction) the recommended value is 0.

```
          On exit: IFAIL = 0 unless the routine detects an error (see
          Section 6).

     6. Error Indicators and Warnings

     Errors detected by the routine:

     IFAIL= 1
          N < 1.

     7. Accuracy

     Exact.

     8. Further Comments

     The time taken by the routine is negligible.

     9. Example

     This program reads in a sequence of real data values, calls
     C06EAF followed by C06GBF to compute their inverse discrete
     Fourier transform, and prints this after expanding it from
     Hermitian form into a full complex sequence.

     The example program is not reproduced here. The source code for
     all example programs is distributed with the NAG Foundation
     Library software and should be available on-line.
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
C06GCF(3NAG)      Foundation Library (12/10/92)      C06GCF(3NAG)

     C06 -- Summation of Series                                C06GCF
             C06GCF -- NAG Foundation Library Routine Document

     Note: Before using this routine, please read the Users' Note for
     your implementation to check implementation-dependent details.
     The symbol (*) after a NAG routine name denotes a routine that is
     not included in the Foundation Library.

     1. Purpose

     C06GCF forms the complex conjugate of a sequence of n data
     values.
```

2. Specification

```
SUBROUTINE C06GCF (Y, N, IFAIL)
INTEGER          N, IFAIL
DOUBLE PRECISION Y(N)
```

3. Description

This is a utility routine for use in conjunction with C06ECF or C06FCF(*) to calculate inverse discrete Fourier transforms (see the Chapter Introduction).

4. References

None.

5. Parameters

1: Y(N) -- DOUBLE PRECISION array Input/Output
On entry: if Y is declared with bounds (0:N-1) in the (sub) program which C06GCF is called, then Y(j) must contain the imaginary part of the jth data value, for 0<=j<=n-1. On exit: these values are negated.

2: N -- INTEGER Input
On entry: the number of data values, n. Constraint: N >= 1.

3: IFAIL -- INTEGER Input/Output
On entry: IFAIL must be set to 0, -1 or 1. For users not familiar with this parameter (described in the Essential Introduction) the recommended value is 0.

On exit: IFAIL = 0 unless the routine detects an error (see Section 6).

6. Error Indicators and Warnings

Errors detected by the routine:

IFAIL= 1
N < 1.

7. Accuracy

Exact.

8. Further Comments

The time taken by the routine is negligible.

9. Example

This program reads in a sequence of complex data values and prints their inverse discrete Fourier transform as computed by calling C06GCF, followed by C06ECF and C06GCF again.

The example program is not reproduced here. The source code for all example programs is distributed with the NAG Foundation Library software and should be available on-line.

%%%

C06GQF(3NAG) Foundation Library (12/10/92) C06GQF(3NAG)

C06 -- Summation of Series C06GQF
C06GQF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for your implementation to check implementation-dependent details. The symbol (*) after a NAG routine name denotes a routine that is not included in the Foundation Library.

1. Purpose

C06GQF forms the complex conjugates of m Hermitian sequences, each containing n data values.

2. Specification

```
SUBROUTINE C06GQF (M, N, X, IFAIL)
INTEGER          M, N, IFAIL
DOUBLE PRECISION X(M*N)
```

3. Description

This is a utility routine for use in conjunction with C06FPF and C06FQF to calculate inverse discrete Fourier transforms (see the Chapter Introduction).

4. References

None.

5. Parameters

1: M -- INTEGER Input
On entry: the number of Hermitian sequences to be conjugated, m. Constraint: M >= 1.

2: N -- INTEGER Input
On entry: the number of data values in each Hermitian sequence, n. Constraint: N >= 1.

3: X(M,N) -- DOUBLE PRECISION array Input/Output
On entry: the data must be stored in array X as if in a two-dimensional array of dimension (1:M,0:N-1); each of the m sequences is stored in a row of the array in Hermitian form. If the n data values z_j^p are written as $x_j^p + iy_j^p$, then for $0 \le j \le n/2$, x_j^p is contained in X(p,j), and for $1 \le j \le (n-1)/2$, y_j^p is contained in X(p,n-j). (See also Section 2.1.2 of the Chapter Introduction.) On exit: the imaginary parts y_j^p are negated. The real parts x_j^p are not referenced.

4: IFAIL -- INTEGER Input/Output
On entry: IFAIL must be set to 0, -1 or 1. For users not familiar with this parameter (described in the Essential Introduction) the recommended value is 0.

On exit: IFAIL = 0 unless the routine detects an error (see Section 6).

6. Error Indicators and Warnings

Errors detected by the routine:

If on entry IFAIL = 0 or -1, explanatory error messages are output on the current error message unit (as defined by X04AAF).

IFAIL= 1

```
          On entry M < 1.

     IFAIL= 2
          On entry N < 1.

     7. Accuracy

     Exact.

     8. Further Comments

     None.

     9. Example

     This program reads in sequences of real data values which are
     assumed to be Hermitian sequences of complex data stored in
     Hermitian form. The sequences are expanded into full complex form
     using C06GSF and printed. The sequences are then conjugated
     (using C06GQF) and the conjugated sequences are expanded into
     complex form using C06GSF and printed out.

     The example program is not reproduced here. The source code for
     all example programs is distributed with the NAG Foundation
     Library software and should be available on-line.
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
C06GSF(3NAG)      Foundation Library (12/10/92)      C06GSF(3NAG)

     C06 -- Summation of Series                                C06GSF
            C06GSF -- NAG Foundation Library Routine Document

     Note: Before using this routine, please read the Users' Note for
     your implementation to check implementation-dependent details.
     The symbol (*) after a NAG routine name denotes a routine that is
     not included in the Foundation Library.

     1. Purpose

     C06GSF takes m Hermitian sequences, each containing n data
     values, and forms the real and imaginary parts of the m
     corresponding complex sequences.

     2. Specification
```

```
      SUBROUTINE C06GSF (M, N, X, U, V, IFAIL)
      INTEGER          M, N, IFAIL
      DOUBLE PRECISION X(M*N), U(M*N), V(M*N)
```

3. Description

This is a utility routine for use in conjunction with C06FPF and C06FQF (see the Chapter Introduction).

4. References

None.

5. Parameters

1: M -- INTEGER Input
On entry: the number of Hermitian sequences, m, to be converted into complex form. Constraint: M >= 1.

2: N -- INTEGER Input
On entry: the number of data values, n, in each sequence. Constraint: N >= 1.

3: X(M,N) -- DOUBLE PRECISION array Input
On entry: the data must be stored in X as if in a two-dimensional array of dimension (1:M,0:N-1); each of the m sequences is stored in a row of the array in Hermitian form.
If the n data values z_j^p are written as $x_j^p + iy_j^p$, then for $0 \le j \le n/2$, x_j^p is contained in X(p,j), and for $1 \le j \le (n-1)/2$, y_j^p is contained in X(p,n-j). (See also Section 2.1.2 of the Chapter Introduction.)

4: U(M,N) -- DOUBLE PRECISION array Output

5: V(M,N) -- DOUBLE PRECISION array Output
On exit: the real and imaginary parts of the m sequences of length n, are stored in U and V respectively, as if in two-dimensional arrays of dimension (1:M,0:N-1); each of the m sequences is stored as if in a row of each array. In other words, if the real parts of the pth sequence are denoted by

```
 p
x , for j=0,1,...,n-1 then the mn elements of the array U
 j
contain the values
      1  2      m   1  2      m       1    2        m
     x ,x ,...,x , x ,x ,...,x ,..., x   ,x   ,...,x
      0  0      0   1  1      1       n-1  n-1      n-1
```

6: IFAIL -- INTEGER Input/Output
On entry: IFAIL must be set to 0, -1 or 1. For users not familiar with this parameter (described in the Essential Introduction) the recommended value is 0.

On exit: IFAIL = 0 unless the routine detects an error (see Section 6).

6. Error Indicators and Warnings

Errors detected by the routine:

If on entry IFAIL = 0 or -1, explanatory error messages are output on the current error message unit (as defined by X04AAF).

IFAIL= 1
On entry M < 1.

IFAIL= 2
On entry N < 1.

7. Accuracy

Exact.

8. Further Comments

None.

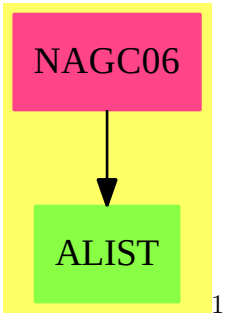
9. Example

This program reads in sequences of real data values which are assumed to be Hermitian sequences of complex data stored in Hermitian form. The sequences are then expanded into full complex form using C06GSF and printed.

The example program is not reproduced here. The source code for all example programs is distributed with the NAG Foundation

```
Library software and should be available on-line.
```

15.28 NagSeriesSummationPackage



1.00

Exports:

c06eaf	c06ebf	c06ecf	c06ekf	c06fpf
c06fqf	c06frf	c06fuf	c06gbf	c06gcf
c06gqf	c06gsf			

⟨*package NAGC06 NagSeriesSummationPackage*⟩≡

```
)abbrev package NAGC06 NagSeriesSummationPackage
++ Author: Godfrey Nolan and Mike Dewar
++ Date Created: Jan 1994
++ Date Last Updated: Thu May 12 17:44:30 1994
++ Description:
++ This package uses the NAG Library to calculate the discrete Fourier
++ transform of a sequence of real or complex data values, and
++ applies it to calculate convolutions and correlations.
++ See \downlink{Manual Page}{manpageXXc06}.

NagSeriesSummationPackage(): Exports == Implementation where
  S ==> Symbol
  FOP ==> FortranOutputStackPackage

  Exports ==> with
    c06eaf : (Integer,Matrix DoubleFloat,Integer) -> Result
     ++ c06eaf(n,x,ifail)
     ++ calculates the discrete Fourier transform of a sequence of
     ++ n real data values. (No extra workspace required.)
     ++ See \downlink{Manual Page}{manpageXXc06eaf}.
    c06ebf : (Integer,Matrix DoubleFloat,Integer) -> Result
     ++ c06ebf(n,x,ifail)
     ++ calculates the discrete Fourier transform of a Hermitian
     ++ sequence of n complex data values. (No extra workspace required.)
     ++ See \downlink{Manual Page}{manpageXXc06ebf}.
    c06ecf : (Integer,Matrix DoubleFloat,Matrix DoubleFloat,Integer) -> Result
     ++ c06ecf(n,x,y,ifail)
     ++ calculates the discrete Fourier transform of a sequence of
```

```
 ++ n complex data values. (No extra workspace required.)
 ++ See \downlink{Manual Page}{manpageXXc06ecf}.
c06ekf : (Integer,Integer,Matrix DoubleFloat,Matrix DoubleFloat,_
    Integer) -> Result
 ++ c06ekf(job,n,x,y,ifail)
 ++ calculates the circular convolution of two
 ++ real vectors of period n. No extra workspace is required.
 ++ See \downlink{Manual Page}{manpageXXc06ekf}.
c06fpf : (Integer,Integer,String,Matrix DoubleFloat,_
    Matrix DoubleFloat,Integer) -> Result
 ++ c06fpf(m,n,init,x,trig,ifail)
 ++ computes the discrete Fourier transforms of m sequences,
 ++ each containing n real data values. This routine is designed to
 ++ be particularly efficient on vector processors.
 ++ See \downlink{Manual Page}{manpageXXc06fpf}.
c06fqf : (Integer,Integer,String,Matrix DoubleFloat,_
    Matrix DoubleFloat,Integer) -> Result
 ++ c06fqf(m,n,init,x,trig,ifail)
 ++ computes the discrete Fourier transforms of m Hermitian
 ++ sequences, each containing n complex data values. This routine is
 ++ designed to be particularly efficient on vector processors.
 ++ See \downlink{Manual Page}{manpageXXc06fqf}.
c06frf : (Integer,Integer,String,Matrix DoubleFloat,_
    Matrix DoubleFloat,Matrix DoubleFloat,Integer) -> Result
 ++ c06frf(m,n,init,x,y,trig,ifail)
 ++ computes the discrete Fourier transforms of m sequences,
 ++ each containing n complex data values. This routine is designed
 ++ to be particularly efficient on vector processors.
 ++ See \downlink{Manual Page}{manpageXXc06frf}.
c06fuf : (Integer,Integer,String,Matrix DoubleFloat,_
    Matrix DoubleFloat,Matrix DoubleFloat,Matrix DoubleFloat,Integer) -> Resu
 ++ c06fuf(m,n,init,x,y,trigm,trign,ifail)
 ++ computes the two-dimensional discrete Fourier transform of
 ++ a bivariate sequence of complex data values. This routine is
 ++ designed to be particularly efficient on vector processors.
 ++ See \downlink{Manual Page}{manpageXXc06fuf}.
c06gbf : (Integer,Matrix DoubleFloat,Integer) -> Result
 ++ c06gbf(n,x,ifail)
 ++ forms the complex conjugate of n
 ++ data values.
 ++ See \downlink{Manual Page}{manpageXXc06gbf}.
c06gcf : (Integer,Matrix DoubleFloat,Integer) -> Result
 ++ c06gcf(n,y,ifail)
 ++ forms the complex conjugate of a sequence of n data
 ++ values.
 ++ See \downlink{Manual Page}{manpageXXc06gcf}.
```

```
    c06gqf : (Integer,Integer,Matrix DoubleFloat,Integer) -> Result
     ++ c06gqf(m,n,x,ifail)
     ++ forms the complex conjugates,
     ++ each containing n data values.
     ++ See \downlink{Manual Page}{manpageXXc06gqf}.
    c06gsf : (Integer,Integer,Matrix DoubleFloat,Integer) -> Result
     ++ c06gsf(m,n,x,ifail)
     ++ takes m Hermitian sequences, each containing n data
     ++ values, and forms the real and imaginary parts of the m
     ++ corresponding complex sequences.
     ++ See \downlink{Manual Page}{manpageXXc06gsf}.
  Implementation ==> add

    import Lisp
    import DoubleFloat
    import Any
    import Record
    import Integer
    import Matrix DoubleFloat
    import Boolean
    import NAGLinkSupportPackage
    import AnyFunctions1(Integer)
    import AnyFunctions1(String)
    import AnyFunctions1(Matrix DoubleFloat)


    c06eaf(nArg:Integer,xArg:Matrix DoubleFloat,ifailArg:Integer): Result ==
        [(invokeNagman(NIL$Lisp,_
        "c06eaf",_
        ["n"::S,"ifail"::S,"x"::S]$Lisp,_
        []$Lisp,_
        [["double"::S,["x"::S,"n"::S]$Lisp]$Lisp_
        ,["integer"::S,"n"::S,"ifail"::S]$Lisp_
        ]$Lisp,_
        ["x"::S,"ifail"::S]$Lisp,_
        [([nArg::Any,ifailArg::Any,xArg::Any ])_
        @List Any]$Lisp)$Lisp)_
        pretend List (Record(key:Symbol,entry:Any))]$Result


    c06ebf(nArg:Integer,xArg:Matrix DoubleFloat,ifailArg:Integer): Result ==
        [(invokeNagman(NIL$Lisp,_
        "c06ebf",_
        ["n"::S,"ifail"::S,"x"::S]$Lisp,_
        []$Lisp,_
        [["double"::S,["x"::S,"n"::S]$Lisp]$Lisp_
```

```
    ,["integer"::S,"n"::S,"ifail"::S]$Lisp_
    ]$Lisp,_
    ["x"::S,"ifail"::S]$Lisp,_
    [([nArg::Any,ifailArg::Any,xArg::Any ])_
    @List Any]$Lisp)$Lisp)_
    pretend List (Record(key:Symbol,entry:Any))]$Result

c06ecf(nArg:Integer,xArg:Matrix DoubleFloat,yArg:Matrix DoubleFloat,_
    ifailArg:Integer): Result ==
    [(invokeNagman(NIL$Lisp,_
    "c06ecf",_
    ["n"::S,"ifail"::S,"x"::S,"y"::S]$Lisp,_
    []$Lisp,_
    [["double"::S,["x"::S,"n"::S]$Lisp,["y"::S,"n"::S]$Lisp_
    ]$Lisp_
    ,["integer"::S,"n"::S,"ifail"::S]$Lisp_
    ]$Lisp,_
    ["x"::S,"y"::S,"ifail"::S]$Lisp,_
    [([nArg::Any,ifailArg::Any,xArg::Any,yArg::Any ])_
    @List Any]$Lisp)$Lisp)_
    pretend List (Record(key:Symbol,entry:Any))]$Result

c06ekf(jobArg:Integer,nArg:Integer,xArg:Matrix DoubleFloat,_
    yArg:Matrix DoubleFloat,ifailArg:Integer): Result ==
    [(invokeNagman(NIL$Lisp,_
    "c06ekf",_
    ["job"::S,"n"::S,"ifail"::S,"x"::S,"y"::S]$Lisp,_
    []$Lisp,_
    [["double"::S,["x"::S,"n"::S]$Lisp,["y"::S,"n"::S]$Lisp_
    ]$Lisp_
    ,["integer"::S,"job"::S,"n"::S,"ifail"::S]$Lisp_
    ]$Lisp,_
    ["x"::S,"y"::S,"ifail"::S]$Lisp,_
    [([jobArg::Any,nArg::Any,ifailArg::Any,xArg::Any,yArg::Any ])_
    @List Any]$Lisp)$Lisp)_
    pretend List (Record(key:Symbol,entry:Any))]$Result

c06fpf(mArg:Integer,nArg:Integer,initArg:String,_
    xArg:Matrix DoubleFloat,trigArg:Matrix DoubleFloat,ifailArg:Integer): Res
    [(invokeNagman(NIL$Lisp,_
    "c06fpf",_
    ["m"::S,"n"::S,"init"::S,"ifail"::S,"x"::S,"trig"::S,"work"::S]$Lisp,_
    ["work"::S]$Lisp,_
    [["double"::S,["x"::S,["*"::S,"m"::S,"n"::S]$Lisp]$Lisp_
    ,["trig"::S,["*"::S,2$Lisp,"n"::S]$Lisp]$Lisp,["work"::S,["*"::S,"m"::S,"
    ,["integer"::S,"m"::S,"n"::S,"ifail"::S]$Lisp_
```

```
    ,["character"::S,"init"::S]$Lisp_
    ]$Lisp,_
    ["x"::S,"trig"::S,"ifail"::S]$Lisp,_
    [([mArg::Any,nArg::Any,initArg::Any,ifailArg::Any,xArg::Any,trigArg::Any ])_
    @List Any]$Lisp)$Lisp)_
    pretend List (Record(key:Symbol,entry:Any))]$Result

c06fqf(mArg:Integer,nArg:Integer,initArg:String,_
    xArg:Matrix DoubleFloat,trigArg:Matrix DoubleFloat,ifailArg:Integer): Result ==
    [(invokeNagman(NIL$Lisp,_
    "c06fqf",_
    ["m"::S,"n"::S,"init"::S,"ifail"::S,"x"::S,"trig"::S,"work"::S]$Lisp,_
    ["work"::S]$Lisp,_
    [["double"::S,["x"::S,["*"::S,"m"::S,"n"::S]$Lisp]$Lisp_
    ,["trig"::S,["*"::S,2$Lisp,"n"::S]$Lisp]$Lisp,["work"::S,["*"::S,"m"::S,"n"::S]$Lisp
    ,["integer"::S,"m"::S,"n"::S,"ifail"::S]$Lisp_
    ,["character"::S,"init"::S]$Lisp_
    ]$Lisp,_
    ["x"::S,"trig"::S,"ifail"::S]$Lisp,_
    [([mArg::Any,nArg::Any,initArg::Any,ifailArg::Any,xArg::Any,trigArg::Any ])_
    @List Any]$Lisp)$Lisp)_
    pretend List (Record(key:Symbol,entry:Any))]$Result

c06frf(mArg:Integer,nArg:Integer,initArg:String,_
    xArg:Matrix DoubleFloat,yArg:Matrix DoubleFloat,trigArg:Matrix DoubleFloat,_
    ifailArg:Integer): Result ==
    [(invokeNagman(NIL$Lisp,_
    "c06frf",_
    ["m"::S,"n"::S,"init"::S,"ifail"::S,"x"::S,"y"::S,"trig"::S,"work"::S]$Lisp,_
    ["work"::S]$Lisp,_
    [["double"::S,["x"::S,["*"::S,"m"::S,"n"::S]$Lisp]$Lisp_
    ,["y"::S,["*"::S,"m"::S,"n"::S]$Lisp]$Lisp,["trig"::S,["*"::S,2$Lisp,"n"::S]$Lisp]$
    ]$Lisp_
    ,["integer"::S,"m"::S,"n"::S,"ifail"::S]$Lisp_
    ,["character"::S,"init"::S]$Lisp_
    ]$Lisp,_
    ["x"::S,"y"::S,"trig"::S,"ifail"::S]$Lisp,_
    [([mArg::Any,nArg::Any,initArg::Any,ifailArg::Any,xArg::Any,yArg::Any,trigArg::Any
    @List Any]$Lisp)$Lisp)_
    pretend List (Record(key:Symbol,entry:Any))]$Result

c06fuf(mArg:Integer,nArg:Integer,initArg:String,_
    xArg:Matrix DoubleFloat,yArg:Matrix DoubleFloat,trigmArg:Matrix DoubleFloat,_
    trignArg:Matrix DoubleFloat,ifailArg:Integer): Result ==
    [(invokeNagman(NIL$Lisp,_
    "c06fuf",_
```

```
    ["m"::S,"n"::S,"init"::S,"ifail"::S,"x"::S,"y"::S,"trigm"::S,"trign"::S,"
    ]$Lisp,_
    ["work"::S]$Lisp,_
    [["double"::S,["x"::S,["*"::S,"m"::S,"n"::S]$Lisp]$Lisp_
    ,["y"::S,["*"::S,"m"::S,"n"::S]$Lisp]$Lisp,["trigm"::S,["*"::S,2$Lisp,"m"
    ,["work"::S,["*"::S,["*"::S,2$Lisp,"m"::S]$Lisp,"n"::S]$Lisp]$Lisp]$Lisp_
    ,["integer"::S,"m"::S,"n"::S,"ifail"::S]$Lisp_
    ,["character"::S,"init"::S]$Lisp_
    ]$Lisp,_
    ["x"::S,"y"::S,"trigm"::S,"trign"::S,"ifail"::S]$Lisp,_
    [([mArg::Any,nArg::Any,initArg::Any,ifailArg::Any,xArg::Any,yArg::Any,tri
    @List Any]$Lisp)$Lisp)_
    pretend List (Record(key:Symbol,entry:Any))]$Result

c06gbf(nArg:Integer,xArg:Matrix DoubleFloat,ifailArg:Integer): Result ==
    [(invokeNagman(NIL$Lisp,_
    "c06gbf",_
    ["n"::S,"ifail"::S,"x"::S]$Lisp,_
    []$Lisp,_
    [["double"::S,["x"::S,"n"::S]$Lisp]$Lisp_
    ,["integer"::S,"n"::S,"ifail"::S]$Lisp_
    ]$Lisp,_
    ["x"::S,"ifail"::S]$Lisp,_
    [([nArg::Any,ifailArg::Any,xArg::Any ])_
    @List Any]$Lisp)$Lisp)_
    pretend List (Record(key:Symbol,entry:Any))]$Result

c06gcf(nArg:Integer,yArg:Matrix DoubleFloat,ifailArg:Integer): Result ==
    [(invokeNagman(NIL$Lisp,_
    "c06gcf",_
    ["n"::S,"ifail"::S,"y"::S]$Lisp,_
    []$Lisp,_
    [["double"::S,["y"::S,"n"::S]$Lisp]$Lisp_
    ,["integer"::S,"n"::S,"ifail"::S]$Lisp_
    ]$Lisp,_
    ["y"::S,"ifail"::S]$Lisp,_
    [([nArg::Any,ifailArg::Any,yArg::Any ])_
    @List Any]$Lisp)$Lisp)_
    pretend List (Record(key:Symbol,entry:Any))]$Result

c06gqf(mArg:Integer,nArg:Integer,xArg:Matrix DoubleFloat,_
    ifailArg:Integer): Result ==
    [(invokeNagman(NIL$Lisp,_
    "c06gqf",_
    ["m"::S,"n"::S,"ifail"::S,"x"::S]$Lisp,_
    []$Lisp,_
```

```
        [["double"::S,["x"::S,["*"::S,"m"::S,"n"::S]$Lisp]$Lisp_
        ]$Lisp_
        ,["integer"::S,"m"::S,"n"::S,"ifail"::S]$Lisp_
        ]$Lisp,_
        ["x"::S,"ifail"::S]$Lisp,_
        [([mArg::Any,nArg::Any,ifailArg::Any,xArg::Any ])_
        @List Any]$Lisp)$Lisp)_
        pretend List (Record(key:Symbol,entry:Any))]$Result

    c06gsf(mArg:Integer,nArg:Integer,xArg:Matrix DoubleFloat,_
        ifailArg:Integer): Result ==
        [(invokeNagman(NIL$Lisp,_
        "c06gsf",_
        ["m"::S,"n"::S,"ifail"::S,"x"::S,"u"::S,"v"::S]$Lisp,_
        ["u"::S,"v"::S]$Lisp,_
        [["double"::S,["x"::S,["*"::S,"m"::S,"n"::S]$Lisp]$Lisp_
        ,["u"::S,["*"::S,"m"::S,"n"::S]$Lisp]$Lisp,["v"::S,["*"::S,"m"::S,"n"::S]$Lisp]$Lisp
        ,["integer"::S,"m"::S,"n"::S,"ifail"::S]$Lisp_
        ]$Lisp,_
        ["u"::S,"v"::S,"ifail"::S]$Lisp,_
        [([mArg::Any,nArg::Any,ifailArg::Any,xArg::Any ])_
        @List Any]$Lisp)$Lisp)_
        pretend List (Record(key:Symbol,entry:Any))]$Result
```

⟨*NAGC06.dotabb*⟩≡

```
"NAGC06" [color="#FF4488",href="bookvol10.4.pdf#nameddest=NAGC06"]
"ALIST" [color="#88FF44",href="bookvol10.3.pdf#nameddest=ALIST"]
"NAGC06" -> "ALIST"
```

15.29 package NAGS NagSpecialFunctionsPackage

⟨*NagSpecialFunctionsPackage.help*⟩≡

```
S -- Approximations of Special Functions        Introduction -- S
                              Chapter S
                 Approximations of Special Functions

1. Scope of the Chapter

This chapter is concerned with the provision of some commonly
occurring physical and mathematical functions.

2. Background to the Problems

The majority of the routines in this chapter approximate real-
valued functions of a single real argument, and the techniques
involved are described in Section 2.1. In addition the chapter
contains routines for elliptic integrals (see Section 2.2),
Bessel and Airy functions of a complex argument (see Section 2.3)
, and exponential of a complex argument.

2.1. Functions of a Single Real Argument

Most of the routines for functions of a single real argument have
been based on truncated Chebyshev expansions. This method of
approximation was adopted as a compromise between the conflicting
requirements of efficiency and ease of implementation on many
different machine ranges. For details of the reasons behind this
choice and the production and testing procedures followed in
constructing this chapter see Schonfelder [7].

Basically if the function to be approximated is f(x), then for
xis in  [a,b] an approximation of the form


                                  --'
                      f(x)=g(x) >  C T (t)
                                  --  r r
                                  r=0


            --'
is used, ( >  denotes, according to the usual convention, a
            --
```

summation in which the first term is halved), where g(x) is some suitable auxiliary function which extracts any singularities, asymptotes and, if possible, zeros of the function in the range in question and t=t(x) is a mapping of the general range [a,b] to the specific range [-1,+1] required by the Chebyshev polynomials, $T_r(t)$. For a detailed description of the properties of the Chebyshev polynomials see Clenshaw [5] and Fox and Parker [6].

The essential property of these polynomials for the purposes of function approximation is that $T_n(t)$ oscillates between +-1 and it takes its extreme values n+1 times in the interval [-1,+1]. Therefore, provided the coefficients C_r decrease in magnitude sufficiently rapidly the error made by truncating the Chebyshev expansion after n terms is approximately given by

$$E(t)\tilde{=}C_n T_n(t)$$

That is the error oscillates between +-C_n and takes its extreme value n+1 times in the interval in question. Now this is just the condition that the approximation be a mini-max representation, one which minimizes the maximum error. By suitable choice of the interval, [a,b], the auxiliary function, g(x), and the mapping of the independent variable, t(x), it is almost always possible to obtain a Chebyshev expansion with rapid convergence and hence truncations that provide near mini-max polynomial approximations to the required function. The difference between the true mini-max polynomial and the truncated Chebyshev expansion is seldom sufficiently great to be of significance.

The evaluation of the Chebyshev expansions follows one of two methods. The first and most efficient, and hence most commonly used, works with the equivalent simple polynomial. The second method, which is used on the few occasions when the first method proves to be unstable, is based directly on the truncated Chebyshev series and uses backward recursion to evaluate the sum. For the first method, a suitably truncated Chebyshev expansion (truncation is chosen so that the error is less than the machine precision) is converted to the equivalent simple polynomial. That is we evaluate the set of coefficients b_r such that

$$y(t)= \sum_{r=0}^{n-1} b_r t^r = \sum_{r=0}^{n-1} C_r T_r(t).$$

The polynomial can then be evaluated by the efficient Horner's method of nested multiplications,

$$y(t)=(b_0 +t(b_1 +t(b_2 +\ldots t(b_{n-2} +tb_{n-1}))\ldots).$$

This method of evaluation results in efficient routines but for some expansions there is considerable loss of accuracy due to cancellation effects. In these cases the second method is used. It is well known that if

$$b_{n-1} = C_{n-1}$$

$$b_{n-2} = 2tb_{n-1} + C_{n-2}$$

$$b_j = 2tb_{j+1} - b_{j+2} + C_j , \quad j=n-3,n-4,\ldots,0$$

then

$$\sum_{r=0}^{n-1}{}' C_r T_r(t) = \frac{1}{2}(b_0 - b_2)$$

and this is always stable. This method is most efficiently implemented by using three variables cyclically and explicitly constructing the recursion.

That is,

$$(\alpha) = C_{n-1}$$

$$(\beta) = 2t(\alpha)+C_{n-2}$$

$$(\gamma) = 2t(\beta)-(\alpha)+C_{n-3}$$

```
(alpha) = 2t(gamma)-(beta)+C
                              n-4
(beta) = 2t(alpha)-(gamma)+C
                              n-5
  ...
  ...
(alpha) = 2t(gamma)-(beta)+C   (say)
                              2
(beta) = 2t(alpha)-(gamma)+C
                              1
                         1
y(t) = t(beta)-(alpha)+ -C
                         2 0
```

The auxiliary functions used are normally functions compounded of simple polynomial (usually linear) factors extracting zeros, and the primary compiler-provided functions, sin, cos, ln, exp, sqrt, which extract singularities and/or asymptotes or in some cases basic oscillatory behaviour, leaving a smooth well-behaved function to be approximated by the Chebyshev expansion which can therefore be rapidly convergent.

The mappings of [a,b] to [-1,+1] used, range from simple linear mappings to the case when b is infinite and considerable improvement in convergence can be obtained by use of a bilinear form of mapping. Another common form of mapping is used when the function is even, that is it involves only even powers in its expansion. In this case an approximation over the whole interval
[-a,a] can be provided using a mapping $t=2(-)^{(x)2}_{(a)} -1$. This embodies the evenness property but the expansion in t involves all powers and hence removes the necessity of working with an expansion with half its coefficients zero.

For many of the routines an analysis of the error in principle is given, viz, if E and (nabla) are the absolute errors in function and argument and (epsilon) and (delta) are the corresponding relative errors, then

$$E\tilde{}=|f'(x)|(nabla)$$

$$E\tilde{}=|xf'(x)|(delta)$$

$$(epsilon)\tilde{}=\left|\frac{xf'(x)}{\quad}\right|(delta)$$

```
|  f(x) |
```

If we ignore errors that arise in the argument of the function by propagation of data errors etc and consider only those errors that result from the fact that a real number is being represented in the computer in floating-point form with finite precision, then (delta) is bounded and this bound is independent of the magnitude of x; e.g. on an 11-digit machine

$$|(\text{delta})| \leq 10^{-11} .$$

(This of course implies that the absolute error (nabla)=x(delta) is also bounded but the bound is now dependent on x). However because of this the last two relations above are probably of more interest. If possible the relative error propagation is discussed; that is the behaviour of the error amplification factor |xf'(x)/f(x)| is described, but in some cases, such as near zeros of the function which cannot be extracted explicitly, absolute error in the result is the quantity of significance and here the factor |xf'(x)| is described. In general, testing of the functions has shown that their error behaviour follows fairly well these theoretical error behaviours. In regions, where the error amplification factors are less than or of the order of one, the errors are slightly larger than the above predictions. The errors are here limited largely by the finite precision of arithmetic in the machine but (epsilon) is normally no more than a few times greater than the bound on (delta). In regions where the amplification factors are large, order of ten or greater, the theoretical analysis gives a good measure of the accuracy obtainable.

It should be noted that the definitions and notations used for the functions in this chapter are all taken from Abramowitz and Stegun [1]. Users are strongly recommended to consult this book for details before using the routines in this chapter.

2.2. Approximations to Elliptic Integrals

The functions provided here are symmetrised variants of the classic elliptic integrals. These alternative definitions have been suggested by Carlson (see [2], [3] and [4]) and he also developed the basic algorithms used in this chapter.

The standard integral of the first kind is represented by

$$R_F(x,y,z)=\frac{1}{2}\int_0^{\infty}\frac{dt}{\sqrt{(t+x)(t+y)(t+z)}}$$

where x,y,z>=0 and at most one may be equal to zero.

The normalisation factor, $\frac{1}{2}$, is chosen so as to make

$$R_F(x,x,x)=1/\sqrt{x}.$$

If any two of the variables are equal, R_F degenerates into the second function

$$R_C(x,y)=R_F(x,y,y)=\frac{1}{2}\int_0^{\infty}\frac{dt}{\sqrt{t+x}(t+y)}$$

where the argument restrictions are now x>=0 and y/=0.

This function is related to the logarithm or inverse hyperbolic functions if 0<y<x, and to the inverse circular functions if 0<=x<=y.

The integrals of the second kind are defined by

$$R_D(x,y,z)=\frac{3}{2}\int_0^{\infty}\frac{dt}{\sqrt{(t+x)(t+y)(t+z)^3}}$$

with z>0, x>=0 and y>=0 but only one of x or y may be zero.

The function is a degenerate special case of the integral of the third kind

$$R_J(x,y,z,\rho) = \frac{3}{2}\int_0^{\infty} \frac{dt}{\left(\sqrt{(t+x)(t+y)(t+z)}\right)(t+\rho)}$$

with $\rho \neq 0$, $x,y,z \geq 0$ with at most one equality holding. Thus $R_D(x,y,z)=R_J(x,y,z,z)$. The normalisation of both these functions is chosen so that

$$R_D(x,x,x)=R_J(x,x,x,x)=1/(x\sqrt{x})$$

The algorithms used for all these functions are based on duplication theorems. These allow a recursion system to be established which constructs a new set of arguments from the old using a combination of arithmetic and geometric means. The value of the function at the original arguments can then be simply related to the value at the new arguments. These recursive reductions are used until the arguments differ from the mean by an amount small enough for a Taylor series about the mean to give sufficient accuracy when retaining terms of order less than six. Each step of the recurrences reduces the difference from the mean by a factor of four, and as the truncation error is of order six, the truncation error goes like $(4096)^{-n}$, where n is the number of iterations.

The above forms can be related to the more traditional canonical forms (see Abramowitz and Stegun [1], 17.2).

If we write $q=\cos^2(\phi)$, $r=1-m.\sin^2(\phi)$, $s=1+n.\sin^2(\phi)$, where $0<\phi \leq \frac{1}{2}\pi$, we have: the elliptic integral of the first kind:

```
sin(phi)      -1/2      1-/2
/           2         2
```

$$F((\phi)|m) = \int_0^{\phi} (1-t)^{-1/2} (1-mt)^{-1/2}\, dt = \sin(\phi).R_F(q,r,1);$$

the elliptic integral of the second kind:

$$E((\phi)|m) = \int_0^{\sin(\phi)} (1-t^2)^{-1/2} (1-mt^2)^{1/2}\, dt = \sin(\phi).R_F(q,r,1) - \frac{m}{3}\sin^3(\phi).R_D(q,r,1)$$

the elliptic integral of the third kind:

$$(Pi)(n;(\phi)|m) = \int_0^{\sin(\phi)} (1-t^2)^{-1/2} (1-mt^2)^{-1/2} (1+nt^2)^{-1}\, dt = \sin(\phi).R_F(q,r,1) - \frac{n}{3}\sin^3(\phi).R_J(q,r,1,s)$$

Also the complete elliptic integral of the first kind:

$$K(m) = \int_0^{\pi/2} (1-m.\sin^2(\theta))^{-1/2}\, d(\theta) = R_F(0,1-m,1);$$

the complete elliptic integral of the second kind:

$$E(m) = \int_0^{\pi/2} (1-m.\sin^2(\theta))^{1/2}\, d(\theta) = R_F(0,1-m,1) - \frac{m}{3}R_D(0,1-m,1).$$

2.3. Bessel and Airy Functions of a Complex Argument

The routines for Bessel and Airy functions of a real argument are based on Chebyshev expansions, as described in Section 2.1. The routines for functions of a complex argument, however, use different methods. These routines relate all functions to the modified Bessel functions I (z) and K (z) computed in the
(nu) (nu)
right-half complex plane, including their analytic continuations. I and K are computed by different methods according to
(nu) (nu)
the values of zand (nu). The methods include power series, asymptotic expansions and Wronskian evaluations. The relations between functions are based on well known formulae (see Abramowitz and Stegun [1]).

2.4. References

[1] Abramowitz M and Stegun I A (1968) Handbook of Mathematical Functions. Dover Publications.

[2] Carlson B C (1977) Special Functions of Applied Mathematics. Academic Press.

[3] Carlson B C (1965) On Computing Elliptic Integrals and Functions. J Math Phys. 44 36--51.

[4] Carlson B C (1977) Elliptic Integrals of the First Kind. SIAM J Math Anal. 8 231--242.

[5] Clenshaw C W (1962) Mathematical Tables. Chebyshev Series for Mathematical Functions. HMSO.

[6] Fox L and Parker I B (1968) Chebyshev Polynomials in Numerical Analysis. Oxford University Press.

[7] Schonfelder J L (1976) The Production of Special Function Routines for a Multi-Machine Library. Software Practice and Experience. 6(1)

3. Recommendations on Choice and Use of Routines

3.1. Elliptic Integrals

IMPORTANT ADVICE: users who encounter elliptic integrals in the course of their work are strongly recommended to look at transforming their analysis directly to one of the Carlson forms, rather than the traditional canonical Legendre forms. In general,

the extra symmetry of the Carlson forms is likely to simplify the analysis, and these symmetric forms are much more stable to calculate.

The routine S21BAF for R_C is largely included as an auxiliary to the other routines for elliptic integrals. This integral essentially calculates elementary functions, e.g.

$$\ln x=(x-1).R_C\left(\left(\frac{1+x}{2}\right)^2,x\right),x>0;$$

$$\arcsin x=x.R_C\left(1-x^2,1\right),|x|\leq 1;$$

$$\operatorname{arcsinh} x=x.R_C\left(1+x^2,1\right),\ \text{etc}$$

In general this method of calculating these elementary functions is not recommended as there are usually much more efficient specific routines available in the Library. However, S21BAF may be used, for example, to compute lnx/(x-1) when x is close to 1, without the loss of significant figures that occurs when lnx and x-1 are computed separately.

3.2. Bessel and Airy Functions

For computing the Bessel functions $J_{(nu)}(x)$, $Y_{(nu)}(x)$, $I_{(nu)}(x)$ and $K_{(nu)}(x)$ where x is real and (nu)=0 or 1, special routines are provided, which are much faster than the more general routines that allow a complex argument and arbitrary real (nu)>=0 functions and their derivatives Ai(x), Bi(x), Ai'(x), Bi'(x) for a real argument which are much faster than the routines for complex arguments.

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```
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```

```
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%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

S01 -- Approximations of Special Functions                S01EAF
        S01EAF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for
your implementation to check implementation-dependent details.
The symbol (*) after a NAG routine name denotes a routine that is
not included in the Foundation Library.
```

1. Purpose

S01EAF evaluates the exponential function e^z, for complex z.

2. Specification

```
COMPLEX(KIND(1.0D0)) FUNCTION S01EAF (Z, IFAIL)
INTEGER                              IFAIL
COMPLEX(KIND(1.0D0))                 Z
```

3. Description

This routine evaluates the exponential function e^z, taking care to avoid machine overflow, and giving a warning if the result cannot be computed to more than half precision. The function is evaluated as $e^z = e^x(\cos y + i \sin y)$, where x and y are the real and imaginary parts respectively of z.

Since cos y and sin y are less than or equal to 1 in magnitude, it is possible that e^x may overflow although $e^x \cos y$ or $e^x \sin y$ does not. In this case the alternative formula $\text{sign}(\cos y) e^{x+\ln|\cos y|}$ is used for the real part of the result, and $\text{sign}(\sin y) e^{x+\ln|\sin y|}$ for the imaginary part. If either part of the result still overflows, a warning is returned through parameter IFAIL.

If Im z is too large, precision may be lost in the evaluation of sin y and cos y. Again, a warning is returned through IFAIL.

4. References

None.

5. Parameters

1: Z -- COMPLEX(KIND(1.0D0)) Input
On entry: the argument z of the function.

2: IFAIL -- INTEGER Input/Output
On entry: IFAIL must be set to 0, -1 or 1. For users not

```
     familiar with this parameter (described in the Essential
     Introduction) the recommended value is 0.

     On exit: IFAIL = 0 unless the routine detects an error (see
     Section 6).

6. Error Indicators and Warnings

Errors detected by the routine:

IFAIL= 1
     The real part of the result overflows, and is set to the
     largest safe number with the correct sign. The imaginary
     part of the result is meaningful.

IFAIL= 2
     The imaginary part of the result overflows, and is set to
     the largest safe number with the correct sign. The real part
     of the result is meaningful.

IFAIL= 3
     Both real and imaginary parts of the result overflow, and
     are set to the largest safe number with the correct signs.

IFAIL= 4
     The computed result is accurate to less than half precision,
     due to the size of Im z.

IFAIL= 5
     The computed result has no precision, due to the size of Im
     z, and is set to zero.

7. Accuracy

Accuracy is limited in general only by the accuracy of the
Fortran intrinsic functions in the computation of siny, cosy and
 x
e , where x=Re z, y=Im z. As y gets larger, precision will
probably be lost due to argument reduction in the evaluation of
the sine and cosine functions, until the warning error IFAIL = 4


occurs when y gets larger than \/1/(epsilon), where (epsilon) is
the machine precision. Note that on some machines, the intrinsic
functions SIN and COS will not operate on arguments larger than
```

about \/1/(epsilon), and so IFAIL can never return as 4.

In the comparatively rare event that the result is computed by
the formulae sign(cosy)e$^{x+\ln|\cos y|}$ and sign(siny)e$^{x+\ln|\sin y|}$, a
further small loss of accuracy may be expected due to rounding
errors in the logarithmic function.

8. Further Comments

None.

9. Example

The example program reads values of the argument z from a file,
evaluates the function at each value of z and prints the results.

The example program is not reproduced here. The source code for
all example programs is distributed with the NAG Foundation
Library software and should be available on-line.

%%%

S13 -- Approximations of Special Functions S13AAF
S13AAF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for
your implementation to check implementation-dependent details.
The symbol (*) after a NAG routine name denotes a routine that is
not included in the Foundation Library.

1. Purpose

S13AAF returns the value of the exponential integral $E_1(x)$, via
the routine name.

2. Specification

```
DOUBLE PRECISION FUNCTION S13AAF (X, IFAIL)
INTEGER          IFAIL
DOUBLE PRECISION X
```

3. Description

The routine calculates an approximate value for

$$E_1(x) = \int_x^{\infty} \frac{e^{-u}}{u}\,du \ , \quad x>0.$$

For 0<x<=4, the approximation is based on the Chebyshev expansion

$$E_1(x) = y(t) - \ln x = \sum_r {}' a_r T_r(t) - \ln x \ , \quad \text{where } t = \frac{-x-1}{2}.$$

For x>4,

$$E_1(x) = \frac{e^{-x}}{x} y(t) = \frac{e^{-x}}{x} \sum_r {}' a_r T_r(t),$$

$$\text{where } t = -1.0 + 14.5/(x+3.25) = \frac{11.25-x}{3.25+x}.$$

In both cases, -1<=t<=+1.

To guard against producing underflows, if x>x_hi the result is set directly to zero. For the value x_hi see the Users' Note for your implementation.

4. References

[1] Abramowitz M and Stegun I A (1965) Handbook of Mathematical Functions. Dover Publications. Ch. 26.

5. Parameters

1: X -- DOUBLE PRECISION Input
On entry: the argument x of the function. Constraint: X > 0.0.

2: IFAIL -- INTEGER Input/Output
Before entry, IFAIL must be assigned a value. For users not familiar with this parameter (described in the Essential Introduction) the recommended value is 0.

Unless the routine detects an error (see Section 6), IFAIL contains 0 on exit.

6. Error Indicators and Warnings

Errors detected by the routine:

IFAIL= 1
The routine has been called with an argument less than or equal to zero for which the function is not defined. The result returned is zero.

7. Accuracy

If (delta) and (epsilon) are the relative errors in argument and result respectively, then in principle,

$$|\text{(epsilon)}| \tilde{=} \left| \frac{e^{-x}}{E_1(x)} * \text{(delta)} \right|$$

so the relative error in the argument is amplified in the result by at least a factor $e^{-x}/E_1(x)$. The equality should hold if (delta) is greater than the machine precision ((delta) due to data errors etc) but if (delta) is simply a result of round-off in the machine representation, it is possible that an extra figure may be lost in internal calculation and round-off.

The behaviour of this amplification factor is shown in Figure 1.

Figure 1
Please see figure in printed Reference Manual

It should be noted that, for small x, the amplification factor tends to zero and eventually the error in the result will be limited by machine precision.

For large x,

(epsilon)~x(delta)=(Delta),

the absolute error in the argument.

8. Further Comments

None.

9. Example

The example program reads values of the argument x from a file, evaluates the function at each value of x and prints the results.

The example program is not reproduced here. The source code for all example programs is distributed with the NAG Foundation Library software and should be available on-line.

%%%

S13 -- Approximations of Special Functions S13ACF
S13ACF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for your implementation to check implementation-dependent details. The symbol (*) after a NAG routine name denotes a routine that is not included in the Foundation Library.

1. Purpose

S13ACF returns the value of the cosine integral

$$Ci(x)=(\gamma)+\ln x+\int_0^x \frac{\cos u-1}{u}\,du, \quad x>0$$

via the routine name, where (γ) denotes Euler's constant.

2. Specification

```
DOUBLE PRECISION FUNCTION S13ACF (X, IFAIL)
INTEGER          IFAIL
```

```
      DOUBLE PRECISION X
```

3. Description

The routine calculates an approximate value for Ci(x).

For 0<x<=16 it is based on the Chebyshev expansion

$$Ci(x)=\ln x+\sum_{r=0}^{--'} a_r T_r(t),\quad t=2\left(\frac{x}{16}\right)^2-1.$$

For 16<x<x$_{hi}$ where the value of x$_{hi}$ is given in the Users' Note for your implementation,

$$Ci(x)=\frac{f(x)\sin x}{x}-\frac{g(x)\cos x}{x^2}$$

$$\text{where } f(x)=\sum_{r=0}^{--'} f_r T_r(t) \text{ and } g(x)=\sum_{r=0}^{--'} g_r T_r(t),\quad t=2\left(\frac{16}{x}\right)^2-1.$$

For x>=x$_{hi}$, Ci(x)=0 to within the accuracy possible (see Section 7).

4. References

[1] Abramowitz M and Stegun I A (1968) Handbook of Mathematical Functions. Dover Publications.

5. Parameters

1: X -- DOUBLE PRECISION Input
On entry: the argument x of the function. Constraint: X > 0.0.

2: IFAIL -- INTEGER Input/Output
On entry: IFAIL must be set to 0, -1 or 1. For users not familiar with this parameter (described in the Essential Introduction) the recommended value is 0.

On exit: IFAIL = 0 unless the routine detects an error (see Section 6).

6. Error Indicators and Warnings

Errors detected by the routine:

IFAIL= 1
The routine has been called with an argument less than or equal to zero for which the function is not defined. The result returned is zero.

7. Accuracy

If E and (epsilon) are the absolute and relative errors in the result and (delta) is the relative error in the argument then in principle these are related by

$$|E| \tilde{=} |(\text{delta})\cos x| \text{ and } |(\text{epsilon})| \tilde{=} \frac{|(\text{delta})\cos x|}{|\text{Ci}(x)|}.$$

That is accuracy will be limited by machine precision near the origin and near the zeros of cosx, but near the zeros of Ci(x) only absolute accuracy can be maintained.

The behaviour of this amplification is shown in Figure 1.

Figure 1
Please see figure in printed Reference Manual

For large values of x, $\text{Ci}(x) \tilde{} \frac{\sin x}{x}$ therefore
(epsilon)~(delta)xcotx and since (delta) is limited by the finite precision of the machine it becomes impossible to return results which have any relative accuracy. That is, when x>=1/(delta) we have that |Ci(x)|<=1/x~E and hence is not significantly different from zero.

Hence x_{hi} is chosen such that for values of $x \geq x_{hi}$, Ci(x) in principle would have values less than the machine precision and so is essentially zero.

```
8. Further Comments

For details of the time taken by the routine see the Users' Note
for your implementation.

9. Example

The example program reads values of the argument x from a file,
evaluates the function at each value of x and prints the results.

The example program is not reproduced here. The source code for
all example programs is distributed with the NAG Foundation
Library software and should be available on-line.
```

%%

```
S13 -- Approximations of Special Functions                  S13ADF
        S13ADF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for
your implementation to check implementation-dependent details.
The symbol (*) after a NAG routine name denotes a routine that is
not included in the Foundation Library.

1. Purpose

S13ADF returns the value of the sine integral

                                x
                                / sinu
                      Si(x)= | ----du,
                                /  u
                                0

via the routine name.

2. Specification

       DOUBLE PRECISION FUNCTION S13ADF (X, IFAIL)
       INTEGER          IFAIL
       DOUBLE PRECISION X

3. Description

The routine calculates an approximate value for Si(x).
```

For |x|<=16.0 it is based on the Chebyshev expansion

$$Si(x)=x \sum_{r=0}^{--'} a_r T_r(t), \quad t=2\left(\frac{x}{16}\right)^2 - 1.$$

For $16<|x|<x_{hi}$, where x_{hi} is an implementation dependent number,

$$Si(x)=sign(x)\left\{\frac{(pi)}{2} - \frac{f(x)\cos x}{x} - \frac{g(x)\sin x}{x^2}\right\}$$

where $f(x)= \sum_{r=0}^{--'} f_r T_r(t)$ and $g(x)= \sum_{r=0}^{--'} g_r T_r(t)$, $t=2\left(\frac{16}{x}\right)^2 - 1$.

For $|x|>=x_{hi}$, $Si(x)= \frac{1}{2}(pi)signx$ to within machine precision.

4. References

[1] Abramowitz M and Stegun I A (1968) Handbook of Mathematical Functions. Dover Publications.

5. Parameters

1: X -- DOUBLE PRECISION Input
On entry: the argument x of the function.

2: IFAIL -- INTEGER Input/Output
On entry: IFAIL must be set to 0, -1 or 1. For users not familiar with this parameter (described in the Essential Introduction) the recommended value is 0.

On exit: IFAIL = 0 unless the routine detects an error (see Section 6).

6. Error Indicators and Warnings

There are no failure exits from this routine. The parameter IFAIL

has been included for consistency with other routines in this chapter.

7. Accuracy

If (delta) and (epsilon) are the relative errors in the argument and result, respectively, then in principle

$$|(\text{epsilon})| \tilde{=} \left| \frac{(\text{delta})\sin x}{\text{Si}(x)} \right| .$$

The equality may hold if (delta) is greater than the machine precision ((delta) due to data errors etc) but if (delta) is simply due to round-off in the machine representation, then since the factor relating (delta) to (epsilon) is always less than one, the accuracy will be limited by machine precision.

8. Further Comments

For details of the time taken by the routine see the Users' Note for your implementation.

9. Example

The example program reads values of the argument x from a file, evaluates the function at each value of x and prints the results.

The example program is not reproduced here. The source code for all example programs is distributed with the NAG Foundation Library software and should be available on-line.

%%%

S14 -- Approximations of Special Functions S14AAF
S14AAF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for your implementation to check implementation-dependent details. The symbol (*) after a NAG routine name denotes a routine that is not included in the Foundation Library.

1. Purpose

S14AAF returns the value of the Gamma function (Gamma)(x), via the routine name.

2. Specification

```
DOUBLE PRECISION FUNCTION S14AAF (X, IFAIL)
INTEGER          IFAIL
DOUBLE PRECISION X
```

3. Description

This routine evaluates an approximation to the Gamma function (Gamma)(x). The routine is based on the Chebyshev expansion:

$$(\text{Gamma})(1+u) = \sum_{r=0}'' a_r T_r(t), \quad \text{where } 0 \le u < 1,\ t = 2u-1,$$

and uses the property (Gamma)(1+x)=x(Gamma)(x). If x=N+1+u where N is integral and 0<=u<1 then it follows that:

for N>0 $(\text{Gamma})(x) = (x-1)(x-2)\ldots(x-N)(\text{Gamma})(1+u)$,

for N=0 $(\text{Gamma})(x) = (\text{Gamma})(1+u)$,

for N<0 $$(\text{Gamma})(x) = \frac{(\text{Gamma})(1+u)}{x(x+1)(x+2)\ldots(x-N-1)}.$$

There are four possible failures for this routine:

(i) if x is too large, there is a danger of overflow since (Gamma)(x) could become too large to be represented in the machine;

(ii) if x is too large and negative, there is a danger of underflow;

(iii) if x is equal to a negative integer, (Gamma)(x) would overflow since it has poles at such points;

(iv) if x is too near zero, there is again the danger of overflow on some machines. For small x, $(\text{Gamma})(x) \tilde{=} \frac{1}{x}$, and on some machines there exists a range of non-zero but small values of x for which 1/x is larger than the greatest

```
     representable value.

4. References

[1]   Abramowitz M and Stegun I A (1968) Handbook of Mathematical
      Functions. Dover Publications.

5. Parameters

 1:  X -- DOUBLE PRECISION                                    Input
     On entry: the argument x of the function. Constraint: X must
     not be a negative integer.

 2:  IFAIL -- INTEGER                                  Input/Output
     On entry: IFAIL must be set to 0, -1 or 1. For users not
     familiar with this parameter (described in the Essential
     Introduction) the recommended value is 0.

     On exit: IFAIL = 0 unless the routine detects an error (see
     Section 6).

6. Error Indicators and Warnings

Errors detected by the routine:

IFAIL= 1
     The argument is too large. On soft failure the routine
     returns the approximate value of (Gamma)(x) at the nearest
     valid argument.

IFAIL= 2
     The argument is too large and negative. On soft failure the
     routine returns zero.

IFAIL= 3
     The argument is too close to zero. On soft failure the
     routine returns the approximate value of (Gamma)(x) at the
     nearest valid argument.

IFAIL= 4
     The argument is a negative integer, at which value
     (Gamma)(x) is infinite. On soft failure the routine returns
     a large positive value.

7. Accuracy
```

Let (delta) and (epsilon) be the relative errors in the argument and the result respectively. If (delta) is somewhat larger than the machine precision (i.e., is due to data errors etc), then (epsilon) and (delta) are approximately related by:

(epsilon)~=|x(Psi)(x)|(delta)

(provided (epsilon) is also greater than the representation error). Here (Psi)(x) is the digamma function $\frac{(\text{Gamma})'(x)}{(\text{Gamma})(x)}$.
Figure 1 shows the behaviour of the error amplification factor |x(Psi)(x)|.

Figure 1
Please see figure in printed Reference Manual

If (delta) is of the same order as machine precision, then rounding errors could make (epsilon) slightly larger than the above relation predicts.

There is clearly a severe, but unavoidable, loss of accuracy for arguments close to the poles of (Gamma)(x) at negative integers. However relative accuracy is preserved near the pole at x=0 right up to the point of failure arising from the danger of overflow.

Also accuracy will necessarily be lost as x becomes large since in this region

(epsilon)~=(delta)xlnx.

However since (Gamma)(x) increases rapidly with x, the routine must fail due to the danger of overflow before this loss of accuracy is too great. (e.g. for x=20, the amplification factor ~= 60.)

8. Further Comments

For details of the time taken by the routine see the Users' Note for your implementation.

9. Example

The example program reads values of the argument x from a file, evaluates the function at each value of x and prints the results.

```
The example program is not reproduced here. The source code for
all example programs is distributed with the NAG Foundation
Library software and should be available on-line.

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

S14 -- Approximations of Special Functions                    S14ABF
        S14ABF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for
your implementation to check implementation-dependent details.
The symbol (*) after a NAG routine name denotes a routine that is
not included in the Foundation Library.

1. Purpose

S14ABF returns a value for the logarithm of the Gamma function,
ln(Gamma)(x), via the routine name.

2. Specification

       DOUBLE PRECISION FUNCTION S14ABF (X, IFAIL)
       INTEGER          IFAIL
       DOUBLE PRECISION X

3. Description

This routine evaluates an approximation to ln(Gamma)(x). It is
based on two Chebyshev expansions.

For 0<x<=x     , ln(Gamma)(x)=-lnx to within machine accuracy.
          small

For x     <x<=15.0, the recursive relation
     small
(Gamma)(1+x)=x(Gamma)(x) is used to reduce the calculation to one
involving (Gamma)(1+u), 0<=u<1 which is evaluated as:

                                --'
               (Gamma)(1+u)= >   a T (t),  t=2u-1.
                             --   r r
                             r=0

Once (Gamma)(x) has been calculated, the required result is
produced by taking the logarithm.
```

For 15.0<x<=x ,
big

$$\ln(\Gamma)(x)=(x-\tfrac{1}{2})\ln x-x+\tfrac{1}{2}\ln 2\pi+y(x)/x$$

$$\text{where } y(x)=\sum_{r=0}^{15} b_r T_r(t),\ t=2\left(\frac{x}{15}\right)^2-1.$$

For x <x<=x the term y(x)/x is negligible and so its
big vbig
calculation is omitted.

For x>x there is a danger of setting overflow so the routine
vbig
must fail.

For x<=0.0 the function is not defined and the routine fails.

Note: x is calculated so that if x<x , (Gamma)(x)=1/x to
small small
within machine accuracy. x is calculated so that if x>x ,
big big

$$\ln(\Gamma)(x)=(x-\tfrac{1}{2})\ln x-x+\tfrac{1}{2}\ln 2\pi$$

to within machine accuracy. x is calculated so that
vbig
ln(Gamma)(x) is close to the value returned by X02ALF(*).
vbig

4. References

[1] Abramowitz M and Stegun I A (1968) Handbook of Mathematical
Functions. Dover Publications.

5. Parameters

1: X -- DOUBLE PRECISION Input
On entry: the argument x of the function. Constraint: X > 0.

```
     0.

 2:  IFAIL -- INTEGER                                    Input/Output
     On entry: IFAIL must be set to 0, -1 or 1. For users not
     familiar with this parameter (described in the Essential
     Introduction) the recommended value is 0.

     On exit: IFAIL = 0 unless the routine detects an error (see
     Section 6).

6. Error Indicators and Warnings

Errors detected by the routine:

IFAIL= 1
     X <= 0.0, the function is undefined. On soft failure, the
     routine returns zero.

IFAIL= 2
     X is too large, the function would overflow. On soft
     failure, the routine returns the value of the function at
     the largest permissible argument.

7. Accuracy

Let (delta) and (epsilon) be the relative errors in the argument
and result respectively, and E be the absolute error in the
result.

If (delta) is somewhat larger than the relative machine
precision, then

                                          |  x*(Psi)(x) |
  E~=|x*(Psi)(x)|(delta) and (epsilon)~=| ------------|(delta)
                                          | ln(Gamma)(x)|

                                          (Gamma)'(x)
where (Psi)(x) is the digamma function    -----------. Figure 1 and
                                          (Gamma)(x)
Figure 2 show the behaviour of these error amplification factors.


                          Figure 1
        Please see figure in printed Reference Manual

                          Figure 2
```

Please see figure in printed Reference Manual

These show that relative error can be controlled, since except near x=1 or 2 relative error is attenuated by the function or at least is not greatly amplified.

```
                          (    1 )
For large x, (epsilon)~=(1+ ---)(delta) and for small x,
                          (   lnx)
              1
(epsilon)~= ---(delta).
            lnx
```

The function ln(Gamma)(x) has zeros at x=1 and 2 and hence relative accuracy is not maintainable near those points. However absolute accuracy can still be provided near those zeros as is shown above.

If however, (delta) is of the order of the machine precision, then rounding errors in the routine's internal arithmetic may result in errors which are slightly larger than those predicted by the equalities. It should be noted that even in areas where strong attenuation of errors is predicted the relative precision is bounded by the effective machine precision.

8. Further Comments

For details of the time taken by the routine see the Users' Note for your implementation.

9. Example

The example program reads values of the argument x from a file, evaluates the function at each value of x and prints the results.

The example program is not reproduced here. The source code for all example programs is distributed with the NAG Foundation Library software and should be available on-line.

%%%

S14 -- Approximations of Special Functions S14BAF
S14BAF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for your implementation to check implementation-dependent details.

The symbol (*) after a NAG routine name denotes a routine that is not included in the Foundation Library.

1. Purpose

S14BAF computes values for the incomplete gamma functions P(a,x) and Q(a,x).

2. Specification

```
SUBROUTINE S14BAF (A, X, TOL, P, Q, IFAIL)
INTEGER          IFAIL
DOUBLE PRECISION A, X, TOL, P, Q
```

3. Description

This subroutine evaluates the incomplete gamma functions in the normalised form

$$P(a,x) = \frac{1}{(\Gamma)(a)} \int_0^x t^{a-1} e^{-t}\, dt,$$

$$Q(a,x) = \frac{1}{(\Gamma)(a)} \int_x^{\infty} t^{a-1} e^{-t}\, dt,$$

with x>=0 and a>0, to a user-specified accuracy. With this normalisation, P(a,x)+Q(a,x)=1.

Several methods are used to evaluate the functions depending on the arguments a and x, the methods including Taylor expansion for P(a,x), Legendre's continued fraction for Q(a,x), and power series for Q(a,x). When both a and x are large, and a~=x, the uniform asymptotic expansion of Temme [3] is employed for greater efficiency - specifically, this expansion is used when a>=20 and 0.7a<=x<=1.4a.

Once either of P or Q is computed, the other is obtained by subtraction from 1. In order to avoid loss of relative precision in this subtraction, the smaller of P and Q is computed first.

This routine is derived from subroutine GAM in Gautschi [2].

4. References

[1] Gautschi W (1979) A Computational Procedure for Incomplete Gamma Functions. ACM Trans. Math. Softw. 5 466--481.

[2] Gautschi W (1979) Algorithm 542: Incomplete Gamma Functions. ACM Trans. Math. Softw. 5 482--489.

[3] Temme N M (1987) On the Computation of the Incomplete Gamma Functions for Large Values of the Parameters. Algorithms for Approximation. (ed J C Mason and M G Cox) Oxford University Press.

5. Parameters

1: A -- DOUBLE PRECISION Input
On entry: the argument a of the functions. Constraint: A > 0.0.

2: X -- DOUBLE PRECISION Input
On entry: the argument x of the functions. Constraint: X >= 0.0.

3: TOL -- DOUBLE PRECISION Input
On entry: the relative accuracy required by the user in the results. If S14BAF is entered with TOL greater than 1.0 or less than machine precision, then the value of machine precision is used instead.

4: P -- DOUBLE PRECISION Output

5: Q -- DOUBLE PRECISION Output
On exit: the values of the functions P(a,x) and Q(a,x) respectively.

6: IFAIL -- INTEGER Input/Output
On entry: IFAIL must be set to 0, -1 or 1. For users not familiar with this parameter (described in the Essential Introduction) the recommended value is 0.

On exit: IFAIL = 0 unless the routine detects an error (see Section 6).

6. Error Indicators and Warnings

```
Errors detected by the routine:

IFAIL= 1
     On entry A <= 0.0.

IFAIL= 2
     On entry X < 0.0.

IFAIL= 3
     Convergence of the Taylor series or Legendre continued
     fraction fails within 600 iterations. This error is
     extremely unlikely to occur; if it does, contact NAG.

7. Accuracy

There are rare occasions when the relative accuracy attained is
somewhat less than that specified by parameter TOL. However, the
error should never exceed more than one or two decimal places.
Note also that there is a limit of 18 decimal places on the
achievable accuracy, because constants in the routine are given
to this precision.

8. Further Comments

The time taken for a call of S14BAF depends on the precision
requested through TOL, and also varies slightly with the input
arguments a and x.

9. Example

The example program reads values of the argument a and x from a
file, evaluates the function and prints the results.

The example program is not reproduced here. The source code for
all example programs is distributed with the NAG Foundation
Library software and should be available on-line.
```

%%%

```
S15 -- Approximations of Special Functions                  S15ADF
        S15ADF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for
your implementation to check implementation-dependent details.
The symbol (*) after a NAG routine name denotes a routine that is
```

not included in the Foundation Library.

1. Purpose

S15ADF returns the value of the complementary error function, erfcx, via the routine name.

2. Specification

```
DOUBLE PRECISION FUNCTION S15ADF (X, IFAIL)
INTEGER          IFAIL
DOUBLE PRECISION X
```

3. Description

The routine calculates an approximate value for the complement of the error function

$$\operatorname{erfc} x = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-u^2}\, du = 1 - \operatorname{erf} x.$$

For x>=0, it is based on the Chebyshev expansion

$$\operatorname{erfc} x = e^{-x^2} y(x),$$

$$\text{where } y(x) = \sum_{r=0}^{\prime\prime} a_r T_r(t) \text{ and } t=(x-3.75)/(x+3.75),\ -1 \le t \le +1.$$

For $x \ge x_{hi}$, where there is a danger of setting underflow, the result is returned as zero.

For x<0, $\operatorname{erfc} x = 2 - e^{-x^2} y(|x|)$.

For x<x_low <0, the result is returned as 2.0 which is correct to

within machine precision. The values of x_{hi} and x_{low} are given in the Users' Note for your implementation.

4. References

[1] Abramowitz M and Stegun I A (1968) Handbook of Mathematical Functions. Dover Publications.

5. Parameters

1: X -- DOUBLE PRECISION Input
On entry: the argument x of the function.

2: IFAIL -- INTEGER Input/Output
On entry: IFAIL must be set to 0, -1 or 1. For users not familiar with this parameter (described in the Essential Introduction) the recommended value is 0.

On exit: IFAIL = 0 unless the routine detects an error (see Section 6).

6. Error Indicators and Warnings

There are no failure exits from this routine. The parameter IFAIL has been included for consistency with other routines in this chapter.

7. Accuracy

If (delta) and (epsilon) are relative errors in the argument and result, respectively, then in principle

$$|\text{(epsilon)}| \tilde{=} \left| \frac{2xe^{-x^2}}{\sqrt{\pi}\,\mathrm{erfc}\,x} \right| |\text{(delta)}|.$$

That is, the relative error in the argument, x, is amplified by a factor $\frac{2xe^{-x^2}}{}$ in the result.

\/(pi)erfc x

The behaviour of this factor is shown in Figure 1.

Figure 1

Please see figure in printed Reference Manual

It should be noted that near x=0 this factor behaves as $\frac{2x}{\sqrt{\pi}}$ and hence the accuracy is largely determined by the machine precision. Also for large negative x, where the factor is xe^{-x^2} ~ ------, accuracy is mainly limited by machine precision.

$\sqrt{\pi}$

However, for large positive x, the factor becomes ~$2x^2$ and to an extent relative accuracy is necessarily lost. The absolute accuracy E is given by

$$E \tilde{=} \frac{2xe^{-x^2}}{\sqrt{\pi}}(\delta)$$

so absolute accuracy is guaranteed for all x.

8. Further Comments

For details of the time taken by the routine see the Users' Note for your implementation.

9. Example

```
The example program reads values of the argument x from a file,
evaluates the function at each value of x and prints the results.

The example program is not reproduced here. The source code for
all example programs is distributed with the NAG Foundation
Library software and should be available on-line.
```

%%%

```
S15 -- Approximations of Special Functions                  S15AEF
        S15AEF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for
your implementation to check implementation-dependent details.
The symbol (*) after a NAG routine name denotes a routine that is
not included in the Foundation Library.

1. Purpose

S15AEF returns the value of the error function erfx, via the
routine name.

2. Specification

       DOUBLE PRECISION FUNCTION S15AEF (X, IFAIL)
       INTEGER          IFAIL
       DOUBLE PRECISION X

3. Description

Evaluates the error function,

                                  2
                             x -t
                 2    /
     erf x =  ------ |e   dt.
                     /
              \/(pi) 0

For |x|<=2,
                 --'                    1 2
      erf x = x >   a T (t), where t= -x -1.
                 --   r r               2
                 r=0
```

For 2<|x|<x ,
          hi

$$\text{erf } x = \text{signx}\left\{1 - \frac{e^{-x^2}}{\sqrt{\pi}\,|x|} \sum_{r=0}^{\infty} {}'b_r T_r(t)\right\}, \text{ where } t = \frac{x-7}{x+3}.$$

For |x|>=x ,
         hi
      erf x = signx.

x   is the value above which erf x=+-1 within machine precision.
 hi
Its value is given in the Users' Note for your implementation.

4. References

[1] Abramowitz M and Stegun I A (1968) Handbook of Mathematical Functions. Dover Publications.

5. Parameters

1: X -- DOUBLE PRECISION Input
On entry: the argument x of the function.

2: IFAIL -- INTEGER Input/Output
On entry: IFAIL must be set to 0, -1 or 1. For users not familiar with this parameter (described in the Essential Introduction) the recommended value is 0.

On exit: IFAIL = 0 unless the routine detects an error (see Section 6).

6. Error Indicators and Warnings

There are no failure exits from this routine. The parameter IFAIL has been included for consistency with other routines in this chapter.

7. Accuracy

On a machine with approximately 11 significant figures the routine agrees with available tables to 10 figures and consistency checking with S15ADF of the relation

erf x+erfc x=1.0

shows errors in at worst the 11th figure.

8. Further Comments

None.

9. Example

The example program reads values of the argument x from a file, evaluates the function at each value of x and prints the results.

The example program is not reproduced here. The source code for all example programs is distributed with the NAG Foundation Library software and should be available on-line.

%%%

S17 -- Approximations of Special Functions S17ACF
S17ACF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for your implementation to check implementation-dependent details. The symbol (*) after a NAG routine name denotes a routine that is not included in the Foundation Library.

1. Purpose

S17ACF returns the value of the Bessel Function $Y_0(x)$, via the routine name.

2. Specification

```
DOUBLE PRECISION FUNCTION S17ACF (X, IFAIL)
INTEGER          IFAIL
DOUBLE PRECISION X
```

3. Description

This routine evaluates an approximation to the Bessel Function of the second kind $Y_0(x)$.

Note: Y (x) is undefined for x<=0 and the routine will fail for
       0
such arguments.

The routine is based on four Chebyshev expansions:

For 0<x<=8,

$$Y_0(x) = \frac{2}{(\pi)} \ln x \sum_{r=0}^{\prime\prime} a_r T_r(t) + \sum_{r=0}^{\prime\prime} b_r T_r(t), \quad \text{with } t = 2\left(\frac{x}{8}\right)^2 - 1,$$

For x>8,

$$Y_0(x) = \sqrt{\frac{2}{(\pi)x}}\left\{P_0(x)\sin\left(x - \frac{(\pi)}{4}\right) + Q_0(x)\cos\left(x - \frac{(\pi)}{4}\right)\right\}$$

where $P_0(x) = \sum_{r=0}^{\prime\prime} c_r T_r(t)$,

and $Q_0(x) = \frac{8}{x} \sum_{r=0}^{\prime\prime} d_r T_r(t)$, with $t = 2\left(\frac{8}{x}\right)^2 - 1$.

For x near zero, $Y_0(x) \tilde{=} \frac{2}{(\pi)}\left(\ln\left(\frac{x}{2}\right) + (\gamma)\right)$, where (γ) denotes Euler's constant. This approximation is used when x is sufficiently small for the result to be correct to machine precision.

For very large x, it becomes impossible to provide results with any reasonable accuracy (see Section 7), hence the routine fails. Such arguments contain insufficient information to determine the

phase of oscillation of $Y_0(x)$; only the amplitude, $\sqrt{\frac{2}{(\pi)x}}$, can be

```
determined and this is returned on soft failure. The range for
which this occurs is roughly related to the machine precision:
the routine will fail if x>~1/machine precision (see the Users'
Note for your implementation for details).

4. References

[1]   Abramowitz M and Stegun I A (1968) Handbook of Mathematical
      Functions. Dover Publications.

[2]   Clenshaw C W (1962) Mathematical Tables. Chebyshev Series
      for Mathematical Functions. HMSO.

5. Parameters

 1:  X -- DOUBLE PRECISION                                    Input
     On entry: the argument x of the function. Constraint: X > 0.
     0.

 2:  IFAIL -- INTEGER                                  Input/Output
     On entry: IFAIL must be set to 0, -1 or 1. For users not
     familiar with this parameter (described in the Essential
     Introduction) the recommended value is 0.

     On exit: IFAIL = 0 unless the routine detects an error (see
     Section 6).

6. Error Indicators and Warnings

Errors detected by the routine:

IFAIL= 1
     X is too large. On soft failure the routine returns the
     amplitude of the Y  oscillation, \/2/(pi)x.
                      0

IFAIL= 2
     X <= 0.0, Y  is undefined. On soft failure the routine
                0
     returns zero.

7. Accuracy

Let (delta) be the relative error in the argument and E be the
absolute error in the result. (Since Y (x) oscillates about zero,
                                       0
```

absolute error and not relative error is significant, except for very small x.)

If (delta) is somewhat larger than the machine representation error (e.g. if (delta) is due to data errors etc), then E and (delta) are approximately related by

$$E \tilde{=} |xY_1(x)|(\delta)$$

(provided E is also within machine bounds). Figure 1 displays the behaviour of the amplification factor $|xY_1(x)|$.

Figure 1
Please see figure in printed Reference Manual

However, if (delta) is of the same order as the machine representation errors, then rounding errors could make E slightly larger than the above relation predicts.

For very small x, the errors are essentially independent of (delta) and the routine should provide relative accuracy bounded by the machine precision.

For very large x, the above relation ceases to apply. In this region, $Y_0(x) \tilde{=} \sqrt{\frac{2}{\pi x}}\sin\left(x - \frac{\pi}{4}\right)$. The amplitude can be calculated with reasonable accuracy for all x, but $\sin\left(x - \frac{\pi}{4}\right)$ cannot. If $x - \frac{\pi}{4}$ is written as $2N\pi+\theta$ where N is an integer and $0 \le \theta < 2\pi$, then $\sin\left(x - \frac{\pi}{4}\right)$ is determined by θ only. If $x > \tilde{} \delta^{-1}$, θ cannot be determined with any accuracy at all. Thus if x is greater than, or of the order of the inverse of machine precision, it is impossible to calculate the phase of $Y_0(x)$ and the routine must fail.

```
8. Further Comments

For details of the time taken by the routine see the Users' Note
for your implementation.

9. Example

The example program reads values of the argument x from a file,
evaluates the function at each value of x and prints the results.

The example program is not reproduced here. The source code for
all example programs is distributed with the NAG Foundation
Library software and should be available on-line.
```

%%%

```
S17 -- Approximations of Special Functions                  S17ADF
        S17ADF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for
your implementation to check implementation-dependent details.
The symbol (*) after a NAG routine name denotes a routine that is
not included in the Foundation Library.

1. Purpose

S17ADF returns the value of the Bessel Function Y (x), via the
                                                  1
routine name.

2. Specification

       DOUBLE PRECISION FUNCTION S17ADF (X, IFAIL)
       INTEGER          IFAIL
       DOUBLE PRECISION X

3. Description

This routine evaluates an approximation to the Bessel Function of
the second kind Y (x).
                1

Note: Y (x) is undefined for x<=0 and the routine will fail for
       1
such arguments.
```

The routine is based on four Chebyshev expansions:

For 0<x<=8,

$$Y_1(x) = \frac{2}{(\pi)}\ln x - \sum_{r=0}^{8}{}' a_r T_r(t) - \frac{2}{(\pi)}x + \sum_{r=0}^{8}{}' b_r T_r(t), \text{ with } t = 2\left(\frac{x}{8}\right)^2 - 1;$$

For x>8,

$$Y_1(x) = \sqrt{\frac{2}{(\pi)x}}\left\{P_1(x)\sin\left(x - \frac{(\pi)}{4}\right) + Q_1(x)\cos\left(x - \frac{(\pi)}{4}\right)\right\}$$

$$\text{where } P_1(x) = \sum_{r=0}^{8}{}' c_r T_r(t),$$

$$\text{and } Q_1(x) = \frac{1}{x}\sum_{r=0}^{8}{}' d_r T_r(t), \text{ with } t = 2\left(\frac{8}{x}\right)^2 - 1.$$

For x near zero, $Y_1(x) \tilde{=} -\frac{2}{(\pi)x}$. This approximation is used when x is sufficiently small for the result to be correct to machine precision. For extremely small x, there is a danger of overflow in calculating $-\frac{2}{(\pi)x}$ and for such arguments the routine will fail.

For very large x, it becomes impossible to provide results with any reasonable accuracy (see Section 7), hence the routine fails. Such arguments contain insufficient information to determine the

phase of oscillation of Y (x), only the amplitude, $\frac{1}{\sqrt{\frac{2}{(\pi)x}}}$, can be determined and this is returned on soft failure. The range for which this occurs is roughly related to machine precision; the routine will fail if x>~1/machine precision (see the Users' Note for your implementation for details).

4. References

[1] Abramowitz M and Stegun I A (1968) Handbook of Mathematical Functions. Dover Publications.

[2] Clenshaw C W (1962) Mathematical Tables. Chebyshev Series for Mathematical Functions. HMSO.

5. Parameters

1: X -- DOUBLE PRECISION Input
On entry: the argument x of the function. Constraint: X > 0.0.

2: IFAIL -- INTEGER Input/Output
On entry: IFAIL must be set to 0, -1 or 1. For users not familiar with this parameter (described in the Essential Introduction) the recommended value is 0.

On exit: IFAIL = 0 unless the routine detects an error (see Section 6).

6. Error Indicators and Warnings

Errors detected by the routine:

IFAIL= 1
X is too large. On soft failure the routine returns the

amplitude of the Y_1 oscillation, $\frac{1}{\sqrt{\frac{2}{(\pi)x}}}$.

IFAIL= 2
$X \leq 0.0$, Y_1 is undefined. On soft failure the routine

returns zero.

IFAIL= 3
X is too close to zero, there is a danger of overflow. On soft failure, the routine returns the value of $Y_1(x)$ at the smallest valid argument.

7. Accuracy

Let (delta) be the relative error in the argument and E be the absolute error in the result. (Since $Y_1(x)$ oscillates about zero, absolute error and not relative error is significant, except for very small x.)

If (delta) is somewhat larger than the machine precision (e.g. if (delta) is due to data errors etc), then E and (delta) are approximately related by:

$$E \tilde{=} |x Y_0(x) - Y_1(x)| (\text{delta})$$

(provided E is also within machine bounds). Figure 1 displays the behaviour of the amplification factor $|x Y_0(x) - Y_1(x)|$.

Figure 1
Please see figure in printed Reference Manual

However, if (delta) is of the same order as machine precision, then rounding errors could make E slightly larger than the above relation predicts.

For very small x, absolute error becomes large, but the relative error in the result is of the same order as (delta).

For very large x, the above relation ceases to apply. In this region, $Y_1(x) \tilde{=} \frac{2(\pi)}{x}\sin\left(x - \frac{3(\pi)}{4}\right)$. The amplitude $\frac{2(\pi)}{x}$ can be calculated with reasonable accuracy for all x, but $\sin\left(x - \frac{3(\pi)}{4}\right)$

cannot. If x- ----- is written as 2N(pi)+(theta) where N is an
4
(3(pi))
integer and 0<=(theta)<2(pi), then sin(x- -----) is determined by
(4)
-1
(theta) only. If x>(delta) , (theta) cannot be determined with any accuracy at all. Thus if x is greater than, or of the order of, the inverse of the machine precision, it is impossible to calculate the phase of Y (x) and the routine must fail.
1

8. Further Comments

For details of the time taken by the routine see the Users' Note for your implementation.

9. Example

The example program reads values of the argument x from a file, evaluates the function at each value of x and prints the results.

The example program is not reproduced here. The source code for all example programs is distributed with the NAG Foundation Library software and should be available on-line.

%%%

S17 -- Approximations of Special Functions S17AEF
S17AEF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for your implementation to check implementation-dependent details. The symbol (*) after a NAG routine name denotes a routine that is not included in the Foundation Library.

1. Purpose

S17AEF returns the value of the Bessel Function J (x), via the
0
routine name.

2. Specification

```
DOUBLE PRECISION FUNCTION S17AEF (X, IFAIL)
INTEGER          IFAIL
```

```
       DOUBLE PRECISION X

3. Description

This routine evaluates an approximation to the Bessel Function of
the first kind J (x).
               0

Note: J (-x)=J (x), so the approximation need only consider x>=0.
       0      0

The routine is based on three Chebyshev expansions:

For 0<x<=8,                              2
             --'                    ( x)
      J (x)= >   a T (t), with t=2( -) -1.
       0     --   r r              ( 8)
             r=0

For x>8,


               /   2  {        (   (pi))         (   (pi))}
      J (x)=  /  -----{P (x)cos(x- ----)-Q (x)sin(x- ----)}
       0    \/   (pi)x{ 0      (    4  )  0      (    4  )}

                  --'
      where P (x) =  >   b T (t),
             0       --   r r
                     r=0
                                                 2
                   8 --'                    ( 8)
      and Q (x) =  - >   c T (t), with t=2( -) -1.
           0       x --   r r              ( x)
                     r=0

For x near zero, J (x)~=1. This approximation is used when x is
                  0
sufficiently small for the result to be correct to machine
precision.

For very large x, it becomes impossible to provide results with
any reasonable accuracy (see Section 7), hence the routine fails.
Such arguments contain insufficient information to determine the
```

phase of oscillation of $J_0(x)$; only the amplitude, $\sqrt{\frac{2}{(\pi)|x|}}$, can be determined and this is returned on soft failure. The range for which this occurs is roughly related to the machine precision; the routine will fail if |x|>~1/machine precision (see the Users' Note for your implementation).

4. References

[1] Abramowitz M and Stegun I A (1968) Handbook of Mathematical Functions. Dover Publications.

[2] Clenshaw C W (1962) Mathematical Tables. Chebyshev Series for Mathematical Functions. HMSO.

5. Parameters

1: X -- DOUBLE PRECISION Input
On entry: the argument x of the function.

2: IFAIL -- INTEGER Input/Output
On entry: IFAIL must be set to 0, -1 or 1. For users not familiar with this parameter (described in the Essential Introduction) the recommended value is 0.

On exit: IFAIL = 0 unless the routine detects an error (see Section 6).

6. Error Indicators and Warnings

Errors detected by the routine:

IFAIL= 1
X is too large. On soft failure the routine returns the

amplitude of the J_0 oscillation, $\sqrt{\frac{2}{(\pi)|x|}}$.

7. Accuracy

Let (delta) be the relative error in the argument and E be the absolute error in the result. (Since $J_0(x)$ oscillates about zero,

absolute error and not relative error is significant.)

If (delta) is somewhat larger than the machine precision (e.g. if (delta) is due to data errors etc), then E and (delta) are approximately related by:

$$E\tilde{}=|xJ_1(x)|(delta)$$

(provided E is also within machine bounds). Figure 1 displays the behaviour of the amplification factor $|xJ_1(x)|$.

Figure 1
Please see figure in printed Reference Manual

However, if (delta) is of the same order as machine precision, then rounding errors could make E slightly larger than the above relation predicts.

For very large x, the above relation ceases to apply. In this region, $J_0(x)\tilde{}=\sqrt{\frac{2}{\pi|x|}}\cos(x-\frac{\pi}{4})$. The amplitude $\sqrt{\frac{2}{\pi|x|}}$ can be calculated with reasonable accuracy for all x but $\cos(x-\frac{\pi}{4})$ cannot. If $x-\frac{\pi}{4}$ is written as $2N\pi+(theta)$ where N is an integer and $0 \le (theta) < 2\pi$, then $\cos(x-\frac{\pi}{4})$ is determined by (theta) only. If $x>\tilde{}(delta)^{-1}$, (theta) cannot be determined with any accuracy at all. Thus if x is greater than, or of the order of, the inverse of the machine precision, it is impossible to calculate the phase of $J_0(x)$ and the routine must fail.

8. Further Comments

For details of the time taken by the routine see the Users' Note for your implementation.

9. Example

The example program reads values of the argument x from a file, evaluates the function at each value of x and prints the results.

The example program is not reproduced here. The source code for all example programs is distributed with the NAG Foundation Library software and should be available on-line.

%%%

S17 -- Approximations of Special Functions S17AFF
S17AFF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for your implementation to check implementation-dependent details. The symbol (*) after a NAG routine name denotes a routine that is not included in the Foundation Library.

1. Purpose

S17AFF returns the value of the Bessel Function $J_1(x)$, via the routine name.

2. Specification

```
DOUBLE PRECISION FUNCTION S17AFF (X, IFAIL)
INTEGER          IFAIL
DOUBLE PRECISION X
```

3. Description

This routine evaluates an approximation to the Bessel Function of the first kind $J_1(x)$.

Note: $J_1(-x) = -J_1(x)$, so the approximation need only consider $x \geq 0$

The routine is based on three Chebyshev expansions:

For 0<x<=8,

$$J_1(x) = \frac{x}{8} \sum_{r=0}^{--'} a_r T_r(t), \text{ with } t = 2\left(\frac{x}{8}\right)^2 - 1.$$

For x>8,

$$J_1(x) = \sqrt{\frac{2}{(\pi)x}}\left\{P_1(x)\cos\left(x - \frac{3(\pi)}{4}\right) - Q_1(x)\sin\left(x - \frac{3(\pi)}{4}\right)\right\}$$

where

$$P_1(x) = \sum_{r=0}^{--'} b_r T_r(t),$$

and

$$Q_1(x) = \frac{8}{x} \sum_{r=0}^{--'} c_r T_r(t), \text{ with } t = 2\left(\frac{8}{x}\right)^2 - 1.$$

For x near zero, $J_1(x) \tilde{=} \frac{x}{2}$. This approximation is used when x is sufficiently small for the result to be correct to machine precision.

For very large x, it becomes impossible to provide results with any reasonable accuracy (see Section 7), hence the routine fails. Such arguments contain insufficient information to determine the phase of oscillation of $J_1(x)$; only the amplitude, $\sqrt{\frac{2}{(\pi)|x|}}$, can be determined and this is returned on soft failure. The range for which this occurs is roughly related to the machine precision; the routine will fail if $|x| > \tilde{} 1/$machine precision (see the Users' Note for your implementation for details).

4. References

[1] Abramowitz M and Stegun I A (1968) Handbook of Mathematical Functions. Dover Publications.

[2] Clenshaw C W (1962) Mathematical Tables. Chebyshev Series for Mathematical Functions. HMSO.

5. Parameters

1: X -- DOUBLE PRECISION Input
On entry: the argument x of the function.

2: IFAIL -- INTEGER Input/Output
On entry: IFAIL must be set to 0, -1 or 1. For users not familiar with this parameter (described in the Essential Introduction) the recommended value is 0.

On exit: IFAIL = 0 unless the routine detects an error (see Section 6).

6. Error Indicators and Warnings

Errors detected by the routine:

IFAIL= 1
X is too large. On soft failure the routine returns the

$$\text{amplitude of the } J_1 \text{ oscillation, } \sqrt{\frac{2}{\pi|x|}}.$$

7. Accuracy

Let (delta) be the relative error in the argument and E be the absolute error in the result. (Since $J_1(x)$ oscillates about zero, absolute error and not relative error is significant.)

If (delta) is somewhat larger than machine precision (e.g. if (delta) is due to data errors etc), then E and (delta) are approximately related by:

$$E \tilde{=} |xJ_0(x)-J_1(x)|(\text{delta})$$

(provided E is also within machine bounds). Figure 1 displays the behaviour of the amplification factor $|xJ_0(x)-J_1(x)|$.

Figure 1
Please see figure in printed Reference Manual

However, if (delta) is of the same order as machine precision, then rounding errors could make E slightly larger than the above relation predicts.

For very large x, the above relation ceases to apply. In this

region, $J_1(x) \tilde{=} \sqrt{\frac{2}{\pi|x|}}\cos\left(x - \frac{3\pi}{4}\right)$. The amplitude

$\sqrt{\frac{2}{\pi|x|}}$ can be calculated with reasonable accuracy for all x but $\cos\left(x - \frac{3\pi}{4}\right)$ cannot. If $x - \frac{3\pi}{4}$ is written as $2N\pi+\theta$ where N is an integer and $0 \le \theta < 2\pi$, then $\cos\left(x - \frac{3\pi}{4}\right)$ is determined by θ only. If $x > \tilde{}(\delta)^{-1}$, θ cannot be determined with any accuracy at all. Thus if x is greater than, or of the order of, machine precision, it is impossible to calculate the phase of $J_1(x)$ and the routine must fail.

8. Further Comments

For details of the time taken by the routine see the Users' Note for your implementation.

9. Example

The example program reads values of the argument x from a file, evaluates the function at each value of x and prints the results.

The example program is not reproduced here. The source code for all example programs is distributed with the NAG Foundation Library software and should be available on-line.

%%

S17 -- Approximations of Special Functions S17AGF
S17AGF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for your implementation to check implementation-dependent details. The symbol (*) after a NAG routine name denotes a routine that is not included in the Foundation Library.

1. Purpose

S17AGF returns a value for the Airy function, Ai(x), via the routine name.

2. Specification

```
DOUBLE PRECISION FUNCTION S17AGF (X, IFAIL)
INTEGER          IFAIL
DOUBLE PRECISION X
```

3. Description

This routine evaluates an approximation to the Airy function, Ai(x). It is based on a number of Chebyshev expansions:

For x<-5,

$$Ai(x) = \frac{a(t)\sin z - b(t)\cos z}{(-x)^{1/4}}$$

where $z = \frac{(pi)}{4} + \frac{2}{3}\sqrt{-x^3}$, and a(t) and b(t) are expansions in the variable $t = -2\left(\frac{5}{x}\right)^3 - 1$.

For -5<=x<=0,

$$Ai(x)=f(t)-xg(t),$$

where f and g are expansions in $t=-2\left(\frac{x}{5}\right)^3-1$.

For 0<x<4.5,

$$Ai(x)=e^{-3x/2}y(t),$$

where y is an expansion in t=4x/9-1.

For 4.5<=x<9,

$$Ai(x)=e^{-5x/2}u(t),$$

where u is an expansion in t=4x/9-3.

For x>=9,

$$Ai(x)=\frac{e^{-z}v(t)}{x^{1/4}},$$

where $z=-\frac{2}{3}\sqrt{x}^3$ and v is an expansion in $t=2\left(\frac{18}{z}\right)-1$.

For |x|< the machine precision, the result is set directly to Ai(0). This both saves time and guards against underflow in intermediate calculations.

For large negative arguments, it becomes impossible to calculate the phase of the oscillatory function with any precision and so the routine must fail. This occurs if $x<-\left(\frac{3}{2(\text{epsilon})}\right)^{2/3}$, where (epsilon) is the machine precision.

For large positive arguments, where Ai decays in an essentially exponential manner, there is a danger of underflow so the routine must fail.

4. References

[1] Abramowitz M and Stegun I A (1968) Handbook of Mathematical Functions. Dover Publications.

5. Parameters

1: X -- DOUBLE PRECISION Input
On entry: the argument x of the function.

2: IFAIL -- INTEGER Input/Output
On entry: IFAIL must be set to 0, -1 or 1. For users not familiar with this parameter (described in the Essential Introduction) the recommended value is 0.
On exit: IFAIL = 0 unless the routine detects an error (see Section 6).

6. Error Indicators and Warnings

Errors detected by the routine:

IFAIL= 1
X is too large and positive. On soft failure, the routine returns zero.

IFAIL= 2
X is too large and negative. On soft failure, the routine returns zero.

7. Accuracy

For negative arguments the function is oscillatory and hence absolute error is the appropriate measure. In the positive region the function is essentially exponential-like and here relative error is appropriate. The absolute error, E, and the relative error, (epsilon), are related in principle to the relative error in the argument, (delta), by

$$E \tilde{=} |xAi'(x)|(\delta), \quad \varepsilon \tilde{=} \left|\frac{xAi'(x)}{Ai(x)}\right|(\delta).$$

In practice, approximate equality is the best that can be expected. When (delta), (epsilon) or E is of the order of the machine precision, the errors in the result will be somewhat larger.

For small x, errors are strongly damped by the function and hence will be bounded by the machine precision.

For moderate negative x, the error behaviour is oscillatory but the amplitude of the error grows like

$$\text{amplitude} \sim \frac{\left(\frac{E}{\text{(delta)}}\right)^{\sim} |x|^{5/4}}{\sqrt{\pi}}$$

However the phase error will be growing roughly like $\sim \sqrt{3} |x|^{2/3}$

and hence all accuracy will be lost for large negative arguments due to the impossibility of calculating sin and cos to any

$$\text{accuracy if } \sqrt{3} |x|^{2/3} > \frac{1}{\text{(delta)}}.$$

For large positive arguments, the relative error amplification is considerable:

$$\frac{\text{(epsilon)}}{\text{(delta)}} \sim \sqrt{3} x .$$

This means a loss of roughly two decimal places accuracy for arguments in the region of 20. However very large arguments are not possible due to the danger of setting underflow and so the errors are limited in practice.

8. Further Comments

None.

9. Example

The example program reads values of the argument x from a file, evaluates the function at each value of x and prints the results.

The example program is not reproduced here. The source code for all example programs is distributed with the NAG Foundation Library software and should be available on-line.

%%%

S17 -- Approximations of Special Functions S17AHF
S17AHF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for your implementation to check implementation-dependent details. The symbol (*) after a NAG routine name denotes a routine that is not included in the Foundation Library.

1. Purpose

S17AHF returns a value of the Airy function, Bi(x), via the routine name.

2. Specification

```
DOUBLE PRECISION FUNCTION S17AHF (X, IFAIL)
INTEGER          IFAIL
DOUBLE PRECISION X
```

3. Description

This routine evaluates an approximation to the Airy function Bi(x). It is based on a number of Chebyshev expansions.

For x<-5,

$$Bi(x) = \frac{a(t)\cos z + b(t)\sin z}{(-x)^{1/4}},$$

$$\text{where } z=\frac{(\text{pi})}{4}+\sqrt[3]{-x}$$ and a(t) and b(t) are expansions in the variable $t=-2\left(\frac{5}{x}\right)^{3}-1$.

For $-5\le x\le 0$,

$$Bi(x)=\sqrt[3]{3}(f(t)+xg(t)),$$

where f and g are expansions in $t=-2\left(\frac{5}{x}\right)^{3}-1$.

For $0<x<4.5$,

$$Bi(x)=e^{11x/8}y(t),$$

where y is an expansion in $t=4x/9-1$.

For $4.5\le x\le 9$,

$$Bi(x)=e^{5x/2}v(t),$$

where v is an expansion in $t=4x/9-3$.

For $x\ge 9$,

$$Bi(x)=\frac{e^{z}u(t)}{x^{1/4}},$$

$$\text{where } z=-\frac{2}{3}\sqrt{x^{3}}$$ and u is an expansion in $t=2\left(\frac{18}{z}\right)^{3}-1$.

For |x|< the machine precision, the result is set directly to Bi(0). This both saves time and avoids possible intermediate underflows.

For large negative arguments, it becomes impossible to calculate the phase of the oscillating function with any accuracy so the routine must fail. This occurs if $x < -\left(\frac{3}{2(\text{epsilon})}\right)^{2/3}$, where (epsilon) is the machine precision.

For large positive arguments, there is a danger of causing overflow since Bi grows in an essentially exponential manner, so the routine must fail.

4. References

[1] Abramowitz M and Stegun I A (1968) Handbook of Mathematical Functions. Dover Publications.

5. Parameters

1: X -- DOUBLE PRECISION Input
On entry: the argument x of the function.

2: IFAIL -- INTEGER Input/Output
On entry: IFAIL must be set to 0, -1 or 1. For users not familiar with this parameter (described in the Essential Introduction) the recommended value is 0.

On exit: IFAIL = 0 unless the routine detects an error (see Section 6).

6. Error Indicators and Warnings

Errors detected by the routine:

IFAIL= 1
X is too large and positive. On soft failure, the routine returns zero.

IFAIL= 2
X is too large and negative. On soft failure, the routine returns zero.

7. Accuracy

For negative arguments the function is oscillatory and hence absolute error is the appropriate measure. In the positive region the function is essentially exponential-like and here relative error is appropriate. The absolute error, E, and the relative error, (epsilon), are related in principle to the relative error in the argument, (delta), by

$$E\tilde{}=|xBi'(x)|(\text{delta}),\quad (\text{epsilon})\tilde{}=\left|\frac{xBi'(x)}{Bi(x)}\right|(\text{delta}).$$

In practice, approximate equality is the best that can be expected. When (delta), (epsilon) or E is of the order of the machine precision, the errors in the result will be somewhat larger.

For small x, errors are strongly damped and hence will be bounded essentially by the machine precision.

For moderate to large negative x, the error behaviour is clearly oscillatory but the amplitude of the error grows like amplitude

$$\left(\frac{E}{(\text{delta})}\right)\tilde{}\ \frac{|x|^{5/4}}{\sqrt{\pi}}.$$

However the phase error will be growing roughly as $\frac{2}{3}\sqrt{|x|^3}$ and hence all accuracy will be lost for large negative arguments.

This is due to the impossibility of calculating sin and cos to

$$\text{any accuracy if } \frac{2}{3}\sqrt{|x|^3} > \frac{1}{(\text{delta})}.$$

For large positive arguments, the relative error amplification is considerable:

```
                                  (epsilon)   / 3
                                  ---------~\/ x .
                                   (delta)
```

This means a loss of roughly two decimal places accuracy for arguments in the region of 20. However very large arguments are not possible due to the danger of causing overflow and errors are therefore limited in practice.

8. Further Comments

For details of the time taken by the routine see the Users' Note for your implementation.

9. Example

The example program reads values of the argument x from a file, evaluates the function at each value of x and prints the results.

The example program is not reproduced here. The source code for all example programs is distributed with the NAG Foundation Library software and should be available on-line.

%%%

S17 -- Approximations of Special Functions S17AJF
S17AJF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for your implementation to check implementation-dependent details. The symbol (*) after a NAG routine name denotes a routine that is not included in the Foundation Library.

1. Purpose

S17AJF returns a value of the derivative of the Airy function Ai(x), via the routine name.

2. Specification

```
DOUBLE PRECISION FUNCTION S17AJF (X, IFAIL)
INTEGER          IFAIL
DOUBLE PRECISION X
```

3. Description

This routine evaluates an approximation to the derivative of the Airy function Ai(x). It is based on a number of Chebyshev expansions.

For x<-5,

$$Ai'(x)=\sqrt{-x}\left[a(t)\cos z+\frac{b(t)}{(\zeta eta)}\sin z\right],$$

where $z=\frac{\pi}{4}+(\zeta eta)$, $(\zeta eta)=-\sqrt{-x^3}$ and a(t) and b(t) are expansions in variable $t=-2\left(\frac{5}{x}\right)^3-1$.

For -5<=x<=0,

$$Ai'(x)=x^2 f(t)-g(t),$$

where f and g are expansions in $t=-2\left(\frac{5}{x}\right)^3-1$.

For 0<x<4.5,

$$Ai'(x)=e^{-11x/8}y(t),$$

where y(t) is an expansion in $t=4\left(\frac{x}{9}\right)-1$.

For 4.5<=x<9,

$$Ai'(x)=e^{-5x/2}v(t),$$

$\left(\frac{x}{}\right)$

where v(t) is an expansion in t=4(-)-3.
(9)

For x>=9,

$$Ai'(x)=\sqrt{-x}e^{-z}u(t),$$

where $z=-\frac{2}{3}\sqrt{x^3}$ and u(t) is an expansion in $t=2(\frac{--}{z})-1$. (18)

For |x|< the square of the machine precision, the result is set directly to Ai'(0). This both saves time and avoids possible intermediate underflows.

For large negative arguments, it becomes impossible to calculate a result for the oscillating function with any accuracy and so

$$\text{the routine must fail. This occurs for } x<-\left(\frac{\sqrt{(pi)}}{\text{(epsilon)}}\right)^{4/7}, \text{ where}$$

(epsilon) is the machine precision.

For large positive arguments, where Ai' decays in an essentially exponential manner, there is a danger of underflow so the routine must fail.

4. References

[1] Abramowitz M and Stegun I A (1968) Handbook of Mathematical Functions. Dover Publications.

5. Parameters

1: X -- DOUBLE PRECISION Input
On entry: the argument x of the function.

2: IFAIL -- INTEGER Input/Output
On entry: IFAIL must be set to 0, -1 or 1. For users not familiar with this parameter (described in the Essential Introduction) the recommended value is 0.

On exit: IFAIL = 0 unless the routine detects an error (see

Section 6).

6. Error Indicators and Warnings

Errors detected by the routine:

IFAIL= 1
X is too large and positive. On soft failure, the routine returns zero.

IFAIL= 2
X is too large and negative. On soft failure, the routine returns zero.

7. Accuracy

For negative arguments the function is oscillatory and hence absolute error is the appropriate measure. In the positive region the function is essentially exponential in character and here relative error is needed. The absolute error, E, and the relative error, (epsilon), are related in principle to the relative error in the argument, (delta), by

$$E \tilde{=} |x\ \mathrm{Ai}(x)|(\mathrm{delta}) \qquad (\mathrm{epsilon}) \tilde{=} \left| \frac{x\ \mathrm{Ai}(x)}{\mathrm{Ai}'(x)} \right| (\mathrm{delta}).$$

In practice, approximate equality is the best that can be expected. When (delta), (epsilon) or E is of the order of the machine precision, the errors in the result will be somewhat larger.

For small x, positive or negative, errors are strongly attenuated by the function and hence will be roughly bounded by the machine precision.

For moderate to large negative x, the error, like the function, is oscillatory; however the amplitude of the error grows like

$$\frac{|x|^{7/4}}{\sqrt{\pi}}.$$

Therefore it becomes impossible to calculate the function with

$$\text{any accuracy if } |x| > \frac{7/4\ \sqrt{\pi}}{\delta}.$$

For large positive x, the relative error amplification is considerable:

$$\frac{\varepsilon}{\delta} \approx \frac{3}{\sqrt{x}}.$$

However, very large arguments are not possible due to the danger of underflow. Thus in practice error amplification is limited.

8. Further Comments

None.

9. Example

The example program reads values of the argument x from a file, evaluates the function at each value of x and prints the results.

The example program is not reproduced here. The source code for all example programs is distributed with the NAG Foundation Library software and should be available on-line.

%%%

S17 -- Approximations of Special Functions S17AKF
S17AKF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for your implementation to check implementation-dependent details. The symbol (*) after a NAG routine name denotes a routine that is not included in the Foundation Library.

1. Purpose

S17AKF returns a value for the derivative of the Airy function

Bi(x), via the routine name.

2. Specification

```
DOUBLE PRECISION FUNCTION S17AKF (X, IFAIL)
INTEGER          IFAIL
DOUBLE PRECISION X
```

3. Description

This routine calculates an approximate value for the derivative of the Airy function Bi(x). It is based on a number of Chebyshev expansions.

For x<-5,

$$Bi'(x)=\sqrt{-x}\left[-a(t)\sin z+\frac{b(t)}{(\mathrm{zeta})}\cos z\right],$$

$$\text{where } z=\frac{(\mathrm{pi})}{4}+(\mathrm{zeta}),(\mathrm{zeta})=\left(-\sqrt{-x}\right)^{3}/3 \text{ and } a(t) \text{ and } b(t) \text{ are}$$

expansions in the variable $t=-2\left(\frac{5}{x}\right)^{3}-1$.

For -5<=x<=0,

$$Bi'(x)=\sqrt{3}\left(x^{2} f(t)+g(t)\right),$$

where f and g are expansions in $t=-2\left(\frac{x}{5}\right)^{3}-1$.

For 0<x<4.5,

$$Bi'(x)=e^{3x/2} y(t),$$

where y(t) is an expansion in t=4x/9-1.

For 4.5<=x<9,

$$Bi'(x)=e^{21x/8}u(t),$$

where u(t) is an expansion in t=4x/9-3.

For x>=9,

$$Bi'(x)=\sqrt{x}e^{4}v(t),$$

where $z=-\frac{2}{3}x^{3/2}$ and v(t) is an expansion in $t=2\left(\frac{18}{z}\right)-1$.

For |x|< the square of the machine precision, the result is set directly to Bi'(0). This saves time and avoids possible underflows in calculation.

For large negative arguments, it becomes impossible to calculate a result for the oscillating function with any accuracy so the routine must fail. This occurs for $x<-\left(\frac{\sqrt{(pi)}}{(epsilon)}\right)^{4/7}$, where (epsilon) is the machine precision.

For large positive arguments, where Bi' grows in an essentially exponential manner, there is a danger of overflow so the routine must fail.

4. References

[1] Abramowitz M and Stegun I A (1968) Handbook of Mathematical Functions. Dover Publications.

5. Parameters

1: X -- DOUBLE PRECISION Input
On entry: the argument x of the function.

2: IFAIL -- INTEGER Input/Output

On entry: IFAIL must be set to 0, -1 or 1. For users not familiar with this parameter (described in the Essential Introduction) the recommended value is 0.
On exit: IFAIL = 0 unless the routine detects an error (see Section 6).

6. Error Indicators and Warnings

Errors detected by the routine:

IFAIL= 1
X is too large and positive. On soft failure the routine returns zero.

IFAIL= 2
X is too large and negative. On soft failure the routine returns zero.

7. Accuracy

For negative arguments the function is oscillatory and hence absolute error is appropriate. In the positive region the function has essentially exponential behaviour and hence relative error is needed. The absolute error, E, and the relative error (epsilon), are related in principle to the relative error in the argument (delta), by

$$E \approx |x\,\mathrm{Bi}(x)|\,(\mathrm{delta}) \qquad (\mathrm{epsilon}) \approx \left|\frac{x\,\mathrm{Bi}(x)}{\mathrm{Bi}'(x)}\right|(\mathrm{delta}).$$

In practice, approximate equality is the best that can be expected. When (delta), (epsilon) or E is of the order of the machine precision, the errors in the result will be somewhat larger.

For small x, positive or negative, errors are strongly attenuated by the function and hence will effectively be bounded by the machine precision.

For moderate to large negative x, the error is, like the function, oscillatory. However, the amplitude of the absolute error grows like $\frac{|x|^{7/4}}{}$. Therefore it becomes impossible to

\/(pi)

calculate the function with any accuracy if |x| > 7/4 -------.
(delta)

For large positive x, the relative error amplification is

considerable: (epsilon) / 3 ---------~\/ x . However, very large arguments are
(delta)
not possible due to the danger of overflow. Thus in practice the actual amplification that occurs is limited.

8. Further Comments

None.

9. Example

The example program reads values of the argument x from a file, evaluates the function at each value of x and prints the results.

The example program is not reproduced here. The source code for all example programs is distributed with the NAG Foundation Library software and should be available on-line.

%%%

S17 -- Approximations of Special Functions S17DCF
S17DCF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for your implementation to check implementation-dependent details. The symbol (*) after a NAG routine name denotes a routine that is not included in the Foundation Library.

1. Purpose

S17DCF returns a sequence of values for the Bessel functions Y (z) for complex z, non-negative (nu) and n=0,1,...,N-1,
(nu)+n

with an option for exponential scaling.

2. Specification

```
SUBROUTINE S17DCF (FNU, Z, N, SCALE, CY, NZ, CWRK, IFAIL)
INTEGER              N, NZ, IFAIL
DOUBLE PRECISION     FNU
COMPLEX(KIND(1.0D0)) Z, CY(N), CWRK(N)
CHARACTER*1          SCALE
```

3. Description

This subroutine evaluates a sequence of values for the Bessel function $Y_{(nu)}(z)$, where z is complex, $-(pi) < \arg z <= (pi)$, and (nu) is the real, non-negative order. The N-member sequence is generated for orders (nu), (nu)+1,...,(nu)+N-1. Optionally, the sequence is scaled by the factor $e^{-|\mathrm{Im}\, z|}$.

Note: although the routine may not be called with (nu) less than zero, for negative orders the formula $Y_{-(nu)}(z)=Y_{(nu)}(z)\cos((pi)(nu))+J_{(nu)}(z)\sin((pi)(nu))$ may be used (for the Bessel function $J_{(nu)}(z)$, see S17DEF).

The routine is derived from the routine CBESY in Amos [2]. It is based on the relation $Y_{(nu)}(z)= \dfrac{H^{(1)}_{(nu)}(z)-H^{(2)}_{(nu)}(z)}{2i}$, where $H^{(1)}_{(nu)}(z)$ and $H^{(2)}_{(nu)}(z)$ are the Hankel functions of the first and second kinds respectively (see S17DLF).

When N is greater than 1, extra values of $Y_{(nu)}(z)$ are computed using recurrence relations.

For very large |z| or ((nu)+N-1), argument reduction will cause total loss of accuracy, and so no computation is performed. For slightly smaller |z| or ((nu)+N-1), the computation is performed but results are accurate to less than half of machine precision.

If |z| is very small, near the machine underflow threshold, or ((nu)+N-1) is too large, there is a risk of overflow and so no computation is performed. In all the above cases, a warning is given by the routine.

4. References

[1] Abramowitz M and Stegun I A (1968) Handbook of Mathematical Functions. Dover Publications.

[2] Amos D E (1986) Algorithm 644: A Portable Package for Bessel Functions of a Complex Argument and Nonnegative Order. ACM Trans. Math. Softw. 12 265--273.

5. Parameters

1: FNU -- DOUBLE PRECISION Input
On entry: the order, (nu), of the first member of the sequence of functions. Constraint: FNU >= 0.0.

2: Z -- COMPLEX(KIND(1.0D0)) Input
On entry: the argument, z, of the functions. Constraint: Z /= (0.0, 0.0).

3: N -- INTEGER Input
On entry: the number, N, of members required in the sequence $Y_{(nu)}(z), Y_{(nu)+1}(z),\ldots,Y_{(nu)+N-1}(z)$. Constraint: N >= 1.

4: SCALE -- CHARACTER*1 Input
On entry: the scaling option.

If SCALE = 'U', the results are returned unscaled.

If SCALE = 'S', the results are returned scaled by the factor $e^{-|Imz|}$. Constraint: SCALE = 'U' or 'S'.

5: CY(N) -- COMPLEX(KIND(1.0D)) array Output
On exit: the N required function values: CY(i) contains $Y_{(nu)+i-1}(z)$, for i=1,2,...,N.

6: NZ -- INTEGER Output
On exit: the number of components of CY that are set to zero due to underflow. The positions of such components in the

```
     array CY are arbitrary.

 7:  CWRK(N) -- COMPLEX(KIND(1.0D)) array                Workspace

 8:  IFAIL -- INTEGER                                    Input/Output
     On entry: IFAIL must be set to 0, -1 or 1. For users not
     familiar with this parameter (described in the Essential
     Introduction) the recommended value is 0.

     On exit: IFAIL = 0 unless the routine detects an error (see
     Section 6).

6. Error Indicators and Warnings

Errors detected by the routine:

If on entry IFAIL = 0 or -1, explanatory error messages are
output on the current error message unit (as defined by X04AAF).

IFAIL= 1
     On entry FNU < 0.0,

     or       Z = (0.0, 0.0),

     or       N < 1,

     or       SCALE /= 'U' or 'S'.

IFAIL= 2
     No computation has been performed due to the likelihood of
     overflow, because ABS(Z) is less than a machine-dependent
     threshold value (given in the Users' Note for your
     implementation).

IFAIL= 3
     No computation has been performed due to the likelihood of
     overflow, because FNU + N - 1 is too large - how large
     depends on Z as well as the overflow threshold of the
     machine.

IFAIL= 4
     The computation has been performed, but the errors due to
     argument reduction in elementary functions make it likely
     that the results returned by S17DCF are accurate to less
     than half of machine precision. This error exit may occur if
     either ABS(Z) or FNU + N - 1 is greater than a machine-
```

dependent threshold value (given in the Users' Note for your implementation).

IFAIL= 5
No computation has been performed because the errors due to argument reduction in elementary functions mean that all precision in results returned by S17DCF would be lost. This error exit may occur if either ABS(Z) or FNU + N - 1 is greater than a machine-dependent threshold value (given in the Users' Note for your implementation).

IFAIL= 6
No results are returned because the algorithm termination condition has not been met. This may occur because the parameters supplied to S17DCF would have caused overflow or underflow.

7. Accuracy

All constants in subroutine S17DCF are given to approximately 18 digits of precision. Calling the number of digits of precision in the floating-point arithmetic being used t, then clearly the maximum number of correct digits in the results obtained is limited by p=min(t,18). Because of errors in argument reduction when computing elementary functions inside S17DCF, the actual number of correct digits is limited, in general, by p-s, where s~max(1,|log |z||,|log (nu)|) represents the number of digits
            10   10
lost due to the argument reduction. Thus the larger the values of |z| and (nu), the less the precision in the result. If S17DCF is called with N>1, then computation of function values via recurrence may lead to some further small loss of accuracy.

If function values which should nominally be identical are computed by calls to S17DCF with different base values of (nu) and different N, the computed values may not agree exactly. Empirical tests with modest values of (nu) and z have shown that the discrepancy is limited to the least significant 3-4 digits of precision.

8. Further Comments

The time taken by the routine for a call of S17DCF is approximately proportional to the value of N, plus a constant. In general it is much cheaper to call S17DCF with N greater than 1, rather than to make N separate calls to S17DCF.

```
Paradoxically, for some values of z and (nu), it is cheaper to
call S17DCF with a larger value of N than is required, and then
discard the extra function values returned. However, it is not
possible to state the precise circumstances in which this is
likely to occur. It is due to the fact that the base value used
to start recurrence may be calculated in different regions for
different N, and the costs in each region may differ greatly.

Note that if the function required is Y (x) or Y (x), i.e., (nu)
                                       0       1
= 0.0 or 1.0, where x is real and positive, and only a single
unscaled function value is required, then it may be much cheaper
to call S17ACF or S17ADF respectively.

9. Example

The example program prints a caption and then proceeds to read
sets of data from the input data stream. The first datum is a
value for the order FNU, the second is a complex value for the
argument, Z, and the third is a value for the parameter SCALE.
The program calls the routine with N = 2 to evaluate the function
for orders FNU and FNU + 1, and it prints the results. The
process is repeated until the end of the input data stream is
encountered.

The example program is not reproduced here. The source code for
all example programs is distributed with the NAG Foundation
Library software and should be available on-line.

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

S17 -- Approximations of Special Functions                  S17DEF
        S17DEF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for
your implementation to check implementation-dependent details.
The symbol (*) after a NAG routine name denotes a routine that is
not included in the Foundation Library.

1. Purpose

S17DEF returns a sequence of values for the Bessel functions
J      (z) for complex z, non-negative (nu) and n=0,1,...,N-1,
 (nu)+n
with an option for exponential scaling.
```

2. Specification

```
SUBROUTINE S17DEF (FNU, Z, N, SCALE, CY, NZ, IFAIL)
INTEGER              N, NZ, IFAIL
DOUBLE PRECISION     FNU
COMPLEX(KIND(1.0D0)) Z, CY(N)
CHARACTER*1          SCALE
```

3. Description

This subroutine evaluates a sequence of values for the Bessel function $J_{(nu)}(z)$, where z is complex, $-(pi) < \arg z <= (pi)$, and (nu) is the real, non-negative order. The N-member sequence is generated for orders (nu), (nu)+1,...,(nu)+N-1. Optionally, the sequence is scaled by the factor $e^{-|\mathrm{Im}\, z|}$.

Note: although the routine may not be called with (nu) less than zero, for negative orders the formula $J_{-(nu)}(z)=J_{(nu)}(z)\cos((pi)(nu))-Y_{(nu)}(z)\sin((pi)(nu))$ may be used (for the Bessel function $Y_{(nu)}(z)$, see S17DCF).

The routine is derived from the routine CBESJ in Amos [2]. It is based on the relations $J_{(nu)}(z)=e^{(nu)(pi)i/2} I_{(nu)}(-iz)$, $\mathrm{Im}\, z>=0.0$ and $J_{(nu)}(z)=e^{-(nu)(pi)i/2} I_{(nu)}(iz)$, $\mathrm{Im}\, z<0.0$.

The Bessel function $I_{(nu)}(z)$ is computed using a variety of techniques depending on the region under consideration.

When N is greater than 1, extra values of $J_{(nu)}(z)$ are computed using recurrence relations.

For very large |z| or ((nu)+N-1), argument reduction will cause total loss of accuracy, and so no computation is performed. For slightly smaller |z| or ((nu)+N-1), the computation is performed but results are accurate to less than half of machine precision.

If Im z is large, there is a risk of overflow and so no computation is performed. In all the above cases, a warning is given by the routine.

4. References

[1] Abramowitz M and Stegun I A (1968) Handbook of Mathematical Functions. Dover Publications.

[2] Amos D E (1986) Algorithm 644: A Portable Package for Bessel Functions of a Complex Argument and Nonnegative Order. ACM Trans. Math. Softw. 12 265--273.

5. Parameters

1: FNU -- DOUBLE PRECISION Input
On entry: the order, (nu), of the first member of the sequence of functions. Constraint: FNU >= 0.0.

2: Z -- COMPLEX(KIND(1.0D0)) Input
On entry: the argument z of the functions.

3: N -- INTEGER Input
On entry: the number, N, of members required in the sequence $J_{(nu)}(z), J_{(nu)+1}(z), \ldots, J_{(nu)+N-1}(z)$. Constraint: N >= 1.

4: SCALE -- CHARACTER*1 Input
On entry: the scaling option.
If SCALE = 'U', the results are returned unscaled.

If SCALE = 'S', the results are returned scaled by the factor $e^{-|Imz|}$.
Constraint: SCALE = 'U' or 'S'.

5: CY(N) -- COMPLEX(KIND(1.0D)) array Output
On exit: the N required function values: CY(i) contains $J_{(nu)+i-1}(z)$, for i=1,2,...,N.

6: NZ -- INTEGER Output
On exit: the number of components of CY that are set to zero due to underflow. If NZ > 0, then elements CY(N-NZ+1), CY(N-NZ+2),...,CY(N) are set to zero.

```
 7: IFAIL -- INTEGER                                    Input/Output
     On entry: IFAIL must be set to 0, -1 or 1. For users not
     familiar with this parameter (described in the Essential
     Introduction) the recommended value is 0.

     On exit: IFAIL = 0 unless the routine detects an error (see
     Section 6).

6. Error Indicators and Warnings

Errors detected by the routine:

If on entry IFAIL = 0 or -1, explanatory error messages are
output on the current error message unit (as defined by X04AAF).

IFAIL= 1
     On entry FNU < 0.0,

     or       N < 1,

     or       SCALE /= 'U' or 'S'.

IFAIL= 2
     No computation has been performed due to the likelihood of
     overflow, because Im Z is larger than a machine-dependent
     threshold value (given in the Users' Note for your
     implementation). This error exit can only occur when SCALE =
     'U'.

IFAIL= 3
     The computation has been performed, but the errors due to
     argument reduction in elementary functions make it likely
     that the results returned by S17DEF are accurate to less
     than half of machine precision. This error exit may occur if
     either ABS(Z) or FNU + N - 1 is greater than a machine-
     dependent threshold value (given in the Users' Note for
     your implementation).

IFAIL= 4
     No computation has been performed because the errors due to
     argument reduction in elementary functions mean that all
     precision in results returned by S17DEF would be lost. This
     error exit may occur when either ABS(Z) or FNU + N - 1 is
     greater than a machine-dependent threshold value (given in
     the Users' Note for your implementation).
```

```
IFAIL= 5
     No results are returned because the algorithm termination
     condition has not been met. This may occur because the
     parameters supplied to S17DEF would have caused overflow or
     underflow.

7. Accuracy

All constants in subroutine S17DEF are given to approximately 18
digits of precision. Calling the number of digits of precision in
the floating-point arithmetic being used t, then clearly the
maximum number of correct digits in the results obtained is
limited by p=min(t,18). Because of errors in argument reduction
when computing elementary functions inside S17DEF, the actual
number of correct digits is limited, in general, by p-s, where
s~max(1,|log  |(|z|),|log  (nu)|) represents the number of digits
            10           10
lost due to the argument reduction. Thus the larger the values of
|z| and (nu), the less the precision in the result. If S17DEF is
called with N>1, then computation of function values via
recurrence may lead to some further small loss of accuracy.

If function values which should nominally be identical are
computed by calls to S17DEF with different base values of (nu)
and different N, the computed values may not agree exactly.
Empirical tests with modest values of (nu) and z have shown that
the discrepancy is limited to the least significant 3-4 digits of
precision.

8. Further Comments

The time taken by the routine for a call of S17DEF is
approximately proportional to the value of N, plus a constant. In
general it is much cheaper to call S17DEF with N greater than 1,
rather than to make N separate calls to S17DEF.

Paradoxically, for some values of z and (nu), it is cheaper to
call S17DEF with a larger value of N than is required, and then
discard the extra function values returned. However, it is not
possible to state the precise circumstances in which this is
likely to occur. It is due to the fact that the base value used
to start recurrence may be calculated in different regions for
different N, and the costs in each region may differ greatly.

Note that if the function required is J (x) or J (x), i.e., (nu)
                                      0       1
```

```
= 0.0 or 1.0, where x is real and positive, and only a single
unscaled function value is required, then it may be much cheaper
to call S17AEF or S17AFF respectively.

9. Example

The example program prints a caption and then proceeds to read
sets of data from the input data stream. The first datum is a
value for the order FNU, the second is a complex value for the
argument, Z, and the third is a value for the parameter SCALE.

The program calls the routine with N = 2 to evaluate the function
for orders FNU and FNU + 1, and it prints the results. The
process is repeated until the end of the input data stream is
encountered.

The example program is not reproduced here. The source code for
all example programs is distributed with the NAG Foundation
Library software and should be available on-line.
```

%%%

```
S17 -- Approximations of Special Functions                S17DGF
        S17DGF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for
your implementation to check implementation-dependent details.
The symbol (*) after a NAG routine name denotes a routine that is
not included in the Foundation Library.

1. Purpose

S17DGF returns the value of the Airy function Ai(z) or its
derivative Ai'(z) for complex z, with an option for exponential
scaling.

2. Specification

       SUBROUTINE S17DGF (DERIV, Z, SCALE, AI, NZ, IFAIL)
       INTEGER              NZ, IFAIL
       COMPLEX(KIND(1.0D0)) Z, AI
       CHARACTER*1          DERIV, SCALE

3. Description

This subroutine returns a value for the Airy function Ai(z) or
```

its derivative Ai'(z), where z is complex, -(pi) < arg z <= (pi).

Optionally, the value is scaled by the factor $e^{2z\sqrt{z}/3}$.

The routine is derived from the routine CAIRY in Amos [2]. It is

based on the relations $Ai(z) = \frac{\sqrt{z}K_{1/3}(w)}{(pi)\sqrt{3}}$, and $Ai'(z) = -\frac{zK_{2/3}(w)}{(pi)/3}$,

where $K_{(nu)}$ is the modified Bessel function and $w=2z\sqrt{z}/3$.

For very large |z|, argument reduction will cause total loss of accuracy, and so no computation is performed. For slightly smaller |z|, the computation is performed but results are accurate to less than half of machine precision. If Re w is too large, and the unscaled function is required, there is a risk of overflow and so no computation is performed. In all the above cases, a warning is given by the routine.

4. References

[1] Abramowitz M and Stegun I A (1968) Handbook of Mathematical Functions. Dover Publications.

[2] Amos D E (1986) Algorithm 644: A Portable Package for Bessel Functions of a Complex Argument and Nonnegative Order. ACM Trans. Math. Softw. 12 265--273.

5. Parameters

1: DERIV -- CHARACTER*1 Input
On entry: specifies whether the function or its derivative is required.
If DERIV = 'F', Ai(z) is returned.

If DERIV = 'D', Ai'(z) is returned.
Constraint: DERIV = 'F' or 'D'.

```
 2:  Z -- COMPLEX(KIND(1.0D0))                                    Input
     On entry: the argument z of the function.

 3:  SCALE -- CHARACTER*1                                         Input
     On entry: the scaling option.

     If SCALE = 'U', the result is returned unscaled.

     If SCALE = 'S', the result is returned scaled by the factor


      2z\/z/3
     e       . Constraint: SCALE = 'U' or 'S'.

 4:  AI -- COMPLEX(KIND(1.0D0))                                   Output
     On exit: the required function or derivative value.

 5:  NZ -- INTEGER                                                Output
     On exit: NZ indicates whether or not AI is set to zero due
     to underflow. This can only occur when SCALE = 'U'.

     If NZ = 0, AI is not set to zero.

     If NZ = 1, AI is set to zero.

 6:  IFAIL -- INTEGER                                     Input/Output
     On entry: IFAIL must be set to 0, -1 or 1. For users not
     familiar with this parameter (described in the Essential
     Introduction) the recommended value is 0.

     On exit: IFAIL = 0 unless the routine detects an error (see
     Section 6).

6. Error Indicators and Warnings

Errors detected by the routine:

If on entry IFAIL = 0 or -1, explanatory error messages are
output on the current error message unit (as defined by X04AAF).

IFAIL= 1
     On entry DERIV /= 'F' or 'D'.

     or       SCALE /= 'U' or 'S'.
```

```
IFAIL= 2
     No computation has been performed due to the likelihood of


     overflow, because Re w is too large, where w=2Z\/Z/3 -- how
     large depends on Z and the overflow threshold of the
     machine. This error exit can only occur when SCALE = 'U'.

IFAIL= 3
     The computation has been performed, but the errors due to
     argument reduction in elementary functions make it likely
     that the result returned by S17DGF is accurate to less than
     half of machine precision. This error exit may occur if ABS
     (Z) is greater than a machine-dependent threshold value
     (given in the Users' Note for your implementation).

IFAIL= 4
     No computation has been performed because the errors due to
     argument reduction in elementary functions mean that all
     precision in the result returned by S17DGF would be lost.
     This error exit may occur if ABS(Z) is greater than a
     machine-dependent threshold value (given in the Users' Note
     for your implementation).

IFAIL= 5
     No result is returned because the algorithm termination
     condition has not been met. This may occur because the
     parameters supplied to S17DGF would have caused overflow or
     underflow.

7. Accuracy

All constants in subroutine S17DGF are given to approximately 18
digits of precision. Calling the number of digits of precision in
the floating-point arithmetic being used t, then clearly the
maximum number of correct digits in the results obtained is
limited by p=min(t,18). Because of errors in argument reduction
when computing elementary functions inside S17DGF, the actual
number of correct digits is limited, in general, by p-s, where
s~max(1,|log  |z||) represents the number of digits lost due to
           10
the argument reduction. Thus the larger the value of |z|, the
less the precision in the result.

Empirical tests with modest values of z, checking relations
between Airy functions Ai(z), Ai'(z), Bi(z) and Bi'(z), have
```

shown errors limited to the least significant 3-4 digits of precision.

8. Further Comments

Note that if the function is required to operate on a real argument only, then it may be much cheaper to call S17AGF or S17AJF.

9. Example

The example program prints a caption and then proceeds to read sets of data from the input data stream. The first datum is a value for the parameter DERIV, the second is a complex value for the argument, Z, and the third is a value for the parameter SCALE. The program calls the routine and prints the results. The process is repeated until the end of the input data stream is encountered.

The example program is not reproduced here. The source code for all example programs is distributed with the NAG Foundation Library software and should be available on-line.

%%%

S17 -- Approximations of Special Functions S17DHF
S17DHF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for your implementation to check implementation-dependent details. The symbol (*) after a NAG routine name denotes a routine that is not included in the Foundation Library.

1. Purpose

S17DHF returns the value of the Airy function Bi(z) or its derivative Bi'(z) for complex z, with an option for exponential scaling.

2. Specification

```
SUBROUTINE S17DHF (DERIV, Z, SCALE, BI, IFAIL)
INTEGER              IFAIL
COMPLEX(KIND(1.0D0)) Z, BI
CHARACTER*1          DERIV, SCALE
```

3. Description

This subroutine returns a value for the Airy function Bi(z) or its derivative Bi'(z), where z is complex, -(pi) < argz <= (pi).

Optionally, the value is scaled by the factor $e^{|\mathrm{Re}\,(2z\sqrt{z}/3)|}$.

The routine is derived from the routine CBIRY in Amos [2]. It is based on the relations $Bi(z)=\frac{\sqrt{z}}{\sqrt{3}}(I_{-1/3}(w)+I_{1/3}(w))$, and $Bi'(z)=\frac{z}{\sqrt{3}}(I_{-2/3}(w)+I_{2/3}(w))$, where $I_{(nu)}$ is the modified Bessel function and $w=2z\sqrt{z}/3$.

For very large |z|, argument reduction will cause total loss of accuracy, and so no computation is performed. For slightly smaller |z|, the computation is performed but results are accurate to less than half of machine precision. If Re z is too large, and the unscaled function is required, there is a risk of overflow and so no computation is performed. In all the above cases, a warning is given by the routine.

4. References

[1] Abramowitz M and Stegun I A (1968) Handbook of Mathematical Functions. Dover Publications.

[2] Hammersley J M and Handscomb D C (1967) Monte-Carlo Methods. Methuen.

5. Parameters

1: DERIV -- CHARACTER*1 Input
On entry: specifies whether the function or its derivative is required.

```
          If DERIV = 'F', Bi(z) is returned.

          If DERIV = 'D', Bi'(z) is returned.
     Constraint: DERIV = 'F' or 'D'.

 2:  Z -- COMPLEX(KIND(1.0D0))                                   Input
     On entry: the argument z of the function.

 3:  SCALE -- CHARACTER*1                                        Input
     On entry: the scaling option.
          If SCALE = 'U', the result is returned unscaled.

          If SCALE = 'S', the result is returned scaled by the


                          |Re(2z\/z/3)|
          factor e                    .
     Constraint: SCALE = 'U' or 'S'.

 4:  BI -- COMPLEX(KIND(1.0D0))                                  Output
     On exit: the required function or derivative value.

 5:  IFAIL -- INTEGER                                      Input/Output
     On entry: IFAIL must be set to 0, -1 or 1. For users not
     familiar with this parameter (described in the Essential
     Introduction) the recommended value is 0.

     On exit: IFAIL = 0 unless the routine detects an error (see
     Section 6).

6. Error Indicators and Warnings

Errors detected by the routine:

If on entry IFAIL = 0 or -1, explanatory error messages are
output on the current error message unit (as defined by X04AAF).

IFAIL= 1
     On entry DERIV /= 'F' or 'D'.

     or       SCALE /= 'U' or 'S'.

IFAIL= 2
     No computation has been performed due to the likelihood of
     overflow, because real(Z) is too large - how large depends
     on the overflow threshold of the machine. This error exit
```

```
     can only occur when SCALE = 'U'.

IFAIL= 3
     The computation has been performed, but the errors due to
     argument reduction in elementary functions make it likely
     that the result returned by S17DHF is accurate to less than
     half of machine precision. This error exit may occur if ABS
     (Z) is greater than a machine-dependent threshold value
     (given in the Users' Note for your implementation).

IFAIL= 4
     No computation has been performed because the errors due to
     argument reduction in elementary functions mean that all
     precision in the result returned by S17DHF would be lost.
     This error exit may occur if ABS(Z) is greater than a
     machine-dependent threshold value (given in the Users' Note
     for your implementation).

IFAIL= 5
     No result is returned because the algorithm termination
     condition has not been met. This may occur because the
     parameters supplied to S17DHF would have caused overflow or
     underflow.

7. Accuracy

All constants in subroutine S17DHF are given to approximately 18
digits of precision. Calling the number of digits of precision in
the floating-point arithmetic being used t, then clearly the
maximum number of correct digits in the results obtained is
limited by p=min(t,18). Because of errors in argument reduction
when computing elementary functions inside S17DHF, the actual
number of correct digits is limited, in general, by p-s, where
s~max(1,|log  |z||) represents the number of digits lost due to
           10
the argument reduction. Thus the larger the value of |z|, the
less the precision in the result.

Empirical tests with modest values of z, checking relations
between Airy functions Ai(z), Ai'(z), Bi(z) and Bi'(z), have
shown errors limited to the least significant 3-4 digits of
precision.

8. Further Comments

Note that if the function is required to operate on a real
```

argument only, then it may be much cheaper to call S17AHF or S17AKF.

9. Example

The example program prints a caption and then proceeds to read sets of data from the input data stream. The first datum is a value for the parameter DERIV, the second is a complex value for the argument, Z, and the third is a value for the parameter SCALE. The program calls the routine and prints the results. The process is repeated until the end of the input data stream is encountered.

The example program is not reproduced here. The source code for all example programs is distributed with the NAG Foundation Library software and should be available on-line.

%%%

S17 -- Approximations of Special Functions S17DLF
S17DLF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for your implementation to check implementation-dependent details. The symbol (*) after a NAG routine name denotes a routine that is not included in the Foundation Library.

1. Purpose

S17DLF returns a sequence of values for the Hankel functions
 (1) (2)
H (z) or H (z) for complex z, non-negative (nu) and
 (nu)+n (nu)+n
n=0,1,...,N-1, with an option for exponential scaling.

2. Specification

```
SUBROUTINE S17DLF (M, FNU, Z, N, SCALE, CY, NZ, IFAIL)
INTEGER              M, N, NZ, IFAIL
DOUBLE PRECISION     FNU
COMPLEX(KIND(1.0D0)) Z, CY(N)
CHARACTER*1          SCALE
```

3. Description

This subroutine evaluates a sequence of values for the Hankel
          (1)   (2)
function H   (z) or H   (z), where z is complex, -(pi) < argz
          (nu)   (nu)
<= (pi), and (nu) is the real, non-negative order. The N-member
sequence is generated for orders (nu), (nu)+1,...,(nu)+N-1.
                                                      -iz
Optionally, the sequence is scaled by the factor e   if the
             (1)   iz
function is H   (z) or by the factor e   if the function is
             (nu)
 (2)
H   (z).
 (nu)

Note: although the routine may not be called with (nu) less than
zero, for negative orders the formulae
 (1)   (nu)(pi)i (1)   (2)   -(nu)(pi)i (2)
H   (z)=e   H   (z), and H   (z)=e   H   (z)
 -(nu)   (nu)   -(nu)   (nu)
may be used.

The routine is derived from the routine CBESH in Amos [2]. It is
based on the relation

$$H_{(nu)}^{(m)}(z) = -e^{1\,-p(nu)}\,K_{(nu)}(ze^{-p}),$$

          (pi)   (pi)
where p=i ---- if m=1 and p=-i ---- if m=2, and the Bessel
           2   2
function K   (z) is computed in the right half-plane only.
          (nu)
Continuation of K   (z) to the left half-plane is computed in
                 (nu)
terms of the Bessel function I   (z). These functions are
                              (nu)
evaluated using a variety of different techniques, depending on
the region under consideration.

                                                    (m)
When N is greater than 1, extra values of H   (z) are computed
                                                    (nu)
using recurrence relations.

For very large |z| or ((nu)+N-1), argument reduction will cause total loss of accuracy, and so no computation is performed. For slightly smaller |z| or ((nu)+N-1), the computation is performed but results are accurate to less than half of machine precision. If |z| is very small, near the machine underflow threshold, or ((nu)+N-1) is too large, there is a risk of overflow and so no computation is performed. In all the above cases, a warning is given by the routine.

4. References

[1] Abramowitz M and Stegun I A (1968) Handbook of Mathematical Functions. Dover Publications.

[2] Amos D E (1986) Algorithm 644: A Portable Package for Bessel Functions of a Complex Argument and Nonnegative Order. ACM Trans. Math. Softw. 12 265--273.

5. Parameters

1: M -- INTEGER Input
On entry: the kind of functions required.
If M = 1, the functions are $H^{(1)}_{(nu)}(z)$.
If M = 2, the functions are $H^{(2)}_{(nu)}(z)$.
Constraint: M = 1 or 2.

2: FNU -- DOUBLE PRECISION Input
On entry: the order, (nu), of the first member of the sequence of functions. Constraint: FNU >= 0.0.

3: Z -- COMPLEX(KIND(1.0D0)) Input
On entry: the argument z of the functions. Constraint: Z /= (0.0, 0.0).

4: N -- INTEGER Input
On entry: the number, N, of members required in the sequence $H^{(M)}_{(nu)}, H^{(M)}_{(nu)+1}, \ldots, H^{(M)}_{(nu)+N-1}$. Constraint: N >= 1.

5: SCALE -- CHARACTER*1 Input

```
     On entry: the scaling option.
          If SCALE = 'U', the results are returned unscaled.

          If SCALE = 'S', the results are returned scaled by the
                  -iz                                    iz
          factor e    when M = 1, or by the factor e   when M =
          2.
     Constraint: SCALE = 'U' or 'S'.

 6:  CY(N) -- COMPLEX(KIND(1.0D)) array                    Output
     On exit: the N required function values: CY(i) contains
      (M)
     H        , for i=1,2,...,N.
      (nu)+i-1

 7:  NZ -- INTEGER                                         Output
     On exit: the number of components of CY that are set to zero
     due to underflow. If NZ > 0, then if Imz>0.0 and M = 1, or
     Imz<0.0 and M = 2, elements CY(1), CY(2),...,CY(NZ) are set
     to zero. In the complementary half-planes, NZ simply states
     the number of underflows, and not which elements they are.

 8:  IFAIL -- INTEGER                                  Input/Output
     On entry: IFAIL must be set to 0, -1 or 1. For users not
     familiar with this parameter (described in the Essential
     Introduction) the recommended value is 0.

     On exit: IFAIL = 0 unless the routine detects an error (see
     Section 6).

6. Error Indicators and Warnings

Errors detected by the routine:

If on entry IFAIL = 0 or -1, explanatory error messages are
output on the current error message unit (as defined by X04AAF).

IFAIL= 1
     On entry M /= 1 and M /= 2,

     or       FNU < 0.0,

     or       Z = (0.0, 0.0),

     or       N < 1,
```

```
     or       SCALE /= 'U' or 'S'.

IFAIL= 2
     No computation has been performed due to the likelihood of
     overflow, because ABS(Z) is less than a machine-dependent
     threshold value (given in the Users' Note for your
     implementation).

IFAIL= 3
     No computation has been performed due to the likelihood of
     overflow, because FNU + N - 1 is too large - how large
     depends on Z and the overflow threshold of the machine.

IFAIL= 4
     The computation has been performed, but the errors due to
     argument reduction in elementary functions make it likely
     that the results returned by S17DLF are accurate to less
     than half of machine precision. This error exit may occur if
     either ABS(Z) or FNU + N - 1 is greater than a machine-
     dependent threshold value (given in the Users' Note for
     your implementation).

IFAIL= 5
     No computation has been performed because the errors due to
     argument reduction in elementary functions mean that all
     precision in results returned by S17DLF would be lost. This
     error exit may occur when either of ABS(Z) or FNU + N - 1 is
     greater than a machine-dependent threshold value (given in
     the Users' Note for your implementation).

IFAIL= 6
     No results are returned because the algorithm termination
     condition has not been met. This may occur because the
     parameters supplied to S17DLF would have caused overflow or
     underflow.

7. Accuracy

All constants in subroutine S17DLF are given to approximately 18
digits of precision. Calling the number of digits of precision in
the floating-point arithmetic being used t, then clearly the
maximum number of correct digits in the results obtained is
limited by p=min(t,18). Because of errors in argument reduction
when computing elementary functions inside S17DLF, the actual
number of correct digits is limited, in general, by p-s, where
s~max(1,|log  |z||,|log  (nu)|) represents the number of digits
```

```
            10         10
lost due to the argument reduction. Thus the larger the values of
|z| and (nu), the less the precision in the result. If S17DLF is
called with N>1, then computation of function values via
recurrence may lead to some further small loss of accuracy.

If function values which should nominally be identical are
computed by calls to S17DLF with different base values of (nu)
and different N, the computed values may not agree exactly.
Empirical tests with modest values of (nu) and z have shown that
the discrepancy is limited to the least significant 3-4 digits of
precision.

8. Further Comments

The time taken by the routine for a call of S17DLF is
approximately proportional to the value of N, plus a constant. In
general it is much cheaper to call S17DLF with N greater than 1,
rather than to make N separate calls to S17DLF.

Paradoxically, for some values of z and (nu), it is cheaper to
call S17DLF with a larger value of N than is required, and then
discard the extra function values returned. However, it is not
possible to state the precise circumstances in which this is
likely to occur. It is due to the fact that the base value used
to start recurrence may be calculated in different regions for
different N, and the costs in each region may differ greatly.

9. Example

The example program prints a caption and then proceeds to read
sets of data from the input data stream. The first datum is a
value for the kind of function, M, the second is a value for the
order FNU, the third is a complex value for the argument, Z, and
the fourth is a value for the parameter SCALE. The program calls
the routine with N = 2 to evaluate the function for orders FNU
and FNU + 1, and it prints the results. The process is repeated
until the end of the input data stream is encountered.

The example program is not reproduced here. The source code for
all example programs is distributed with the NAG Foundation
Library software and should be available on-line.

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

S18 -- Approximations of Special Functions                    S18ACF
```

S18ACF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for your implementation to check implementation-dependent details. The symbol (*) after a NAG routine name denotes a routine that is not included in the Foundation Library.

1. Purpose

S18ACF returns the value of the modified Bessel Function $K_0(x)$, via the routine name.

2. Specification

```
DOUBLE PRECISION FUNCTION S18ACF (X, IFAIL)
INTEGER          IFAIL
DOUBLE PRECISION X
```

3. Description

This routine evaluates an approximation to the modified Bessel Function of the second kind $K_0(x)$.

Note: $K_0(x)$ is undefined for $x \leq 0$ and the routine will fail for such arguments.

The routine is based on five Chebyshev expansions:

For $0 < x \leq 1$,

$$K_0(x) = -\ln x \sum_{r=0}{}' a_r T_r(t) + \sum_{r=0}{}' b_r T_r(t), \quad \text{where } t = 2x^2 - 1;$$

For $1 < x \leq 2$,

$$K_0(x) = e^{-x} \sum_{r=0}{}' c_r T_r(t), \quad \text{where } t = 2x - 3;$$

For $2 < x \leq 4$,

$$K_0(x) = e^{-x} \sum_{r=0}^{\infty} d_r T_r(t), \text{ where } t = x - 3;$$

For x>4,

$$K_0(x) = \frac{e^{-x}}{\sqrt{x}} \sum_{r=0}^{\infty} e_r T_r(t), \text{ where } t = \frac{9-x}{1+x}.$$

For x near zero, $K_0(x) \tilde{=} -(\gamma) - \ln\left(\frac{x}{2}\right)$, where (γ) denotes Euler's constant. This approximation is used when x is sufficiently small for the result to be correct to machine precision.

For large x, where there is a danger of underflow due to the smallness of K_0, the result is set exactly to zero.

4. References

[1] Abramowitz M and Stegun I A (1968) Handbook of Mathematical Functions. Dover Publications.

5. Parameters

1: X -- DOUBLE PRECISION Input
On entry: the argument x of the function. Constraint: X > 0.0.

2: IFAIL -- INTEGER Input/Output
On entry: IFAIL must be set to 0, -1 or 1. For users not familiar with this parameter (described in the Essential Introduction) the recommended value is 0.

On exit: IFAIL = 0 unless the routine detects an error (see Section 6).

6. Error Indicators and Warnings

Errors detected by the routine:

```
IFAIL= 1
     X <= 0.0, K  is undefined. On soft failure the routine
                0
     returns zero.

7. Accuracy

Let (delta) and (epsilon) be the relative errors in the argument
and result respectively.

If (delta) is somewhat larger than the machine precision (i.e.,
if (delta) is due to data errors etc), then (epsilon) and (delta)
are approximately related by:

                              | xK (x)|
                              |   1   |
                   (epsilon)~=| ------|(delta).
                              | K (x) |
                              |  0    |

Figure 1 shows the behaviour of the error amplification factor

                             | xK (x)|
                             |   1   |
                             | ------|.
                             | K (x) |
                             |  0    |

                             Figure 1
          Please see figure in printed Reference Manual

However, if (delta) is of the same order as machine precision,
then rounding errors could make (epsilon) slightly larger than
the above relation predicts.

                                                          |  1 |
For small x, the amplification factor is approximately    | ---|,
                                                          | lnx|
which implies strong attenuation of the error, but in general
(epsilon) can never be less than the machine precision.

For large x, (epsilon)~=x(delta) and we have strong amplification
of the relative error. Eventually K , which is asymptotically
                                   0
           -x
```

```
          e
given by  ---, becomes so small that it cannot be calculated


          \/x
without underflow and hence the routine will return zero. Note
that for large x the errors will be dominated by those of the
Fortran intrinsic function EXP.

8. Further Comments

For details of the time taken by the routine see the appropriate
the Users' Note for your implementation.

9. Example

The example program reads values of the argument x from a file,
evaluates the function at each value of x and prints the results.

The example program is not reproduced here. The source code for
all example programs is distributed with the NAG Foundation
Library software and should be available on-line.
```

%%%

```
S18 -- Approximations of Special Functions                  S18ADF
        S18ADF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for
your implementation to check implementation-dependent details.
The symbol (*) after a NAG routine name denotes a routine that is
not included in the Foundation Library.

1. Purpose

S18ADF returns the value of the modified Bessel Function K (x),
                                                          1
via the routine name.

2. Specification

       DOUBLE PRECISION FUNCTION S18ADF (X, IFAIL)
       INTEGER          IFAIL
       DOUBLE PRECISION X

3. Description
```

```
This routine evaluates an approximation to the modified Bessel
Function of the second kind K (x).
                              1

Note: K (x) is undefined for x<=0 and the routine will fail for
       1
such arguments.

The routine is based on five Chebyshev expansions:

For 0<x<=1,

           1      --'           --'                      2
    K (x)= -+xlnx >   a T (t)-x >   b T (t) , where t=2x -1;
     1     x      --   r r      --   r r
                  r=0           r=0

For 1<x<=2,

                      -x --'
                K (x)=e   >   c T (t) , where t=2x-3;
                 1        --   r r
                          r=0

For 2<x<=4,

                      -x --'
                K (x)=e   >   d T (t) , where t=x-3;
                 1        --   r r
                          r=0

For x>4,

                       -x
                      e   --'                    9-x
              K (x)= --- >   e T (t) , where t= ---.
               1         --   r r               1+x
                     \/x r=0

                          1
For x near zero, K (x)~= -. This approximation is used when x is
                  1      x
sufficiently small for the result to be correct to machine
precision. For very small x on some machines, it is impossible to
           1
```

```
calculate  - without overflow and the routine must fail.
          x

For large x, where there is a danger of underflow due to the
smallness of K , the result is set exactly to zero.
             1

4. References

[1]   Abramowitz M and Stegun I A (1968) Handbook of Mathematical
      Functions. Dover Publications.

5. Parameters

 1:  X -- DOUBLE PRECISION                                    Input
     On entry: the argument x of the function. Constraint: X > 0.
     0.

 2:  IFAIL -- INTEGER                                  Input/Output
     On entry: IFAIL must be set to 0, -1 or 1. For users not
     familiar with this parameter (described in the Essential
     Introduction) the recommended value is 0.

     On exit: IFAIL = 0 unless the routine detects an error (see
     Section 6).

6. Error Indicators and Warnings

Errors detected by the routine:

IFAIL= 1
     X <= 0.0, K  is undefined. On soft failure the routine
                1
     returns zero.

IFAIL= 2
     X is too small, there is a danger of overflow. On soft
     failure the routine returns approximately the largest
     representable value.

7. Accuracy

Let (delta) and (epsilon) be the relative errors in the argument
and result respectively.

If (delta) is somewhat larger than the machine precision (i.e.,
```

if (delta) is due to data errors etc), then (epsilon) and (delta) are approximately related by:

$$(\text{epsilon}) \tilde{=} \left| \frac{xK_1(x) - K_0(x)}{K_1(x)} \right| (\text{delta}).$$

Figure 1 shows the behaviour of the error amplification factor

$$\left| \frac{xK_1(x) - K_0(x)}{K_1(x)} \right|.$$

Figure 1
Please see figure in printed Reference Manual

However if (delta) is of the same order as the machine precision, then rounding errors could make (epsilon) slightly larger than the above relation predicts.

For small x, (epsilon)~=(delta) and there is no amplification of errors.

For large x, (epsilon)~=x(delta) and we have strong amplification of the relative error. Eventually K_1, which is asymptotically given by $\frac{e^{-x}}{\sqrt{x}}$, becomes so small that it cannot be calculated without underflow and hence the routine will return zero. Note that for large x the errors will be dominated by those of the Fortran intrinsic function EXP.

8. Further Comments

For details of the time taken by the routine see the Users' Note for your implementation.

9. Example

```
The example program reads values of the argument x from a file,
evaluates the function at each value of x and prints the results.

The example program is not reproduced here. The source code for
all example programs is distributed with the NAG Foundation
Library software and should be available on-line.

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

S18 -- Approximations of Special Functions                    S18AEF
        S18AEF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for
your implementation to check implementation-dependent details.
The symbol (*) after a NAG routine name denotes a routine that is
not included in the Foundation Library.

1. Purpose

S18AEF returns the value of the modified Bessel Function I (x),
                                                          0
via the routine name.

2. Specification

       DOUBLE PRECISION FUNCTION S18AEF (X, IFAIL)
       INTEGER          IFAIL
       DOUBLE PRECISION X

3. Description

This routine evaluates an approximation to the modified Bessel
Function of the first kind I (x).
                            0

Note: I (-x)=I (x), so the approximation need only consider x>=0.
       0      0

The routine is based on three Chebyshev expansions:

For 0<x<=4,

                     x --'                    ( x)
            I (x)=e  >   a T (t) , where t=2( -)-1;
             0       --   r r               ( 4)
```

$$\sum_{r=0}$$

For 4<x<=12,

$$I_0(x) = e^{\sum_{r=0}^{x} b_r T_r(t)}, \text{ where } t = \frac{x-8}{4};$$

For x>12,

$$I_0(x) = \frac{e^x}{\sqrt{x}} \sum_{r=0}^{} c_r T_r(t), \text{ where } t = 2\left(\frac{12}{x}\right) - 1.$$

For small x, $I_0(x) \tilde{=} 1$. This approximation is used when x is sufficiently small for the result to be correct to machine precision.

For large x, the routine must fail because of the danger of overflow in calculating e^x.

4. References

[1] Abramowitz M and Stegun I A (1968) Handbook of Mathematical Functions. Dover Publications.

5. Parameters

1: X -- DOUBLE PRECISION Input
On entry: the argument x of the function.

2: IFAIL -- INTEGER Input/Output
On entry: IFAIL must be set to 0, -1 or 1. For users not familiar with this parameter (described in the Essential Introduction) the recommended value is 0.

On exit: IFAIL = 0 unless the routine detects an error (see Section 6).

6. Error Indicators and Warnings

Errors detected by the routine:

IFAIL= 1
X is too large. On soft failure the routine returns the approximate value of I (x) at the nearest valid argument.
0

7. Accuracy

Let (delta) and (epsilon) be the relative errors in the argument and result respectively.

If (delta) is somewhat larger than the machine precision (i.e., if (delta) is due to data errors etc), then (epsilon) and (delta) are approximately related by:

$$(\text{epsilon})\tilde{}= \left| \frac{xI_1(x)}{I_0(x)} \right| (\text{delta}).$$

Figure 1 shows the behaviour of the error amplification factor

$$\left| \frac{xI_1(x)}{I_0(x)} \right|.$$

Figure 1
Please see figure in printed Reference Manual

However if (delta) is of the same order as machine precision, then rounding errors could make (epsilon) slightly larger than the above relation predicts.

For small x the amplification factor is approximately $\frac{x^2}{2}$, which implies strong attenuation of the error, but in general (epsilon) can never be less than the machine precision.

For large x, (epsilon)~=x(delta) and we have strong amplification of errors. However the routine must fail for quite moderate

values of x, because I (x) would overflow; hence in practice the
                      0
loss of accuracy for large x is not excessive. Note that for
large x the errors will be dominated by those of the Fortran
intrinsic function EXP.

8. Further Comments

For details of the time taken by the routine see the Users' Note
for your implementation.

9. Example

The example program reads values of the argument x from a file,
evaluates the function at each value of x and prints the results.

The example program is not reproduced here. The source code for
all example programs is distributed with the NAG Foundation
Library software and should be available on-line.

%%%

S18 -- Approximations of Special Functions S18AFF
S18AFF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for
your implementation to check implementation-dependent details.
The symbol (*) after a NAG routine name denotes a routine that is
not included in the Foundation Library.

1. Purpose

S18AFF returns a value for the modified Bessel Function I (x),
                                                         1
via the routine name.

2. Specification

```
DOUBLE PRECISION FUNCTION S18AFF (X, IFAIL)
INTEGER          IFAIL
DOUBLE PRECISION X
```

3. Description

This routine evaluates an approximation to the modified Bessel
Function of the first kind I (x).

1

Note: I (-x)=-I (x), so the approximation need only consider x>=0
       1   1

The routine is based on three Chebyshev expansions:

For 0<x<=4,

                    --'   (x)2
             I (x)=x >   a T (t) , where t=2(-) -1;
              1   --   r r   (4)
                     r=0

For 4<x<=12,

                    x --'   x-8
              I (x)=e >   b T (t) , where t= ---;
               1   --   r r   4
                       r=0

For x>12,

                   x
                  e   --'   (12)
           I (x)= --- >   c T (t) , where t=2(--)-1.
            1   --   r r   (x)
                  \/x r=0

For small x, I (x)~=x. This approximation is used when x is
              1
sufficiently small for the result to be correct to machine
precision.

For large x, the routine must fail because I (x) cannot be
                                           1
represented without overflow.

4. References

[1] Abramowitz M and Stegun I A (1968) Handbook of Mathematical
     Functions. Dover Publications.

5. Parameters

 1: X -- DOUBLE PRECISION   Input

On entry: the argument x of the function.

2: IFAIL -- INTEGER Input/Output
On entry: IFAIL must be set to 0, -1 or 1. For users not familiar with this parameter (described in the Essential Introduction) the recommended value is 0.

On exit: IFAIL = 0 unless the routine detects an error (see Section 6).

6. Error Indicators and Warnings

Errors detected by the routine:

IFAIL= 1
X is too large. On soft failure the routine returns the approximate value of I (x) at the nearest valid argument.
1

7. Accuracy

Let (delta) and (epsilon) be the relative errors in the argument and result respectively.

If (delta) is somewhat larger than the machine precision (i.e., if (delta) is due to data errors etc), then (epsilon) and (delta) are approximately related by:

$$|\epsilon| \approx \left| \frac{x I_0(x) - I_1(x)}{I_1(x)} \right| |\delta|.$$

Figure 1 shows the behaviour of the error amplification factor

$$\left| \frac{x I_0(x) - I_1(x)}{I_1(x)} \right|,$$

Figure 1
Please see figure in printed Reference Manual

However if (delta) is of the same order as machine precision,

```
then rounding errors could make (epsilon) slightly larger than
the above relation predicts.

For small x, (epsilon)~=(delta) and there is no amplification of
errors.

For large x, (epsilon)~=x(delta) and we have strong amplification
of errors. However the routine must fail for quite moderate
values of x because I (x) would overflow; hence in practice the
                      1
loss of accuracy for large x is not excessive. Note that for
large x, the errors will be dominated by those of the Fortran
intrinsic function EXP.

8. Further Comments

For details of the time taken by the routine see the Users' Note
for your implementation.

9. Example

The example program reads values of the argument x from a file,
evaluates the function at each value of x and prints the results.

The example program is not reproduced here. The source code for
all example programs is distributed with the NAG Foundation
Library software and should be available on-line.
```

%%%

```
S18 -- Approximations of Special Functions                S18DCF
        S18DCF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for
your implementation to check implementation-dependent details.
The symbol (*) after a NAG routine name denotes a routine that is
not included in the Foundation Library.

1. Purpose

S18DCF returns a sequence of values for the modified Bessel
functions K      (z) for complex z, non-negative (nu) and
          (nu)+n
n=0,1,...,N-1, with an option for exponential scaling.

2. Specification
```

```
      SUBROUTINE S18DCF (FNU, Z, N, SCALE, CY, NZ, IFAIL)
      INTEGER              N, NZ, IFAIL
      DOUBLE PRECISION     FNU
      COMPLEX(KIND(1.0D0)) Z, CY(N)
      CHARACTER*1          SCALE
```

3. Description

This subroutine evaluates a sequence of values for the modified Bessel function $K_{(nu)}(z)$, where z is complex, $-(pi) < \arg z <= (pi)$, and (nu) is the real, non-negative order. The N-member sequence is generated for orders (nu), (nu)+1,...,(nu)+N-1.

Optionally, the sequence is scaled by the factor e^{z}.

The routine is derived from the routine CBESK in Amos [2].

Note: although the routine may not be called with (nu) less than zero, for negative orders the formula $K_{-(nu)}(z)=K_{(nu)}(z)$ may be used.

When N is greater than 1, extra values of $K_{(nu)}(z)$ are computed using recurrence relations.

For very large |z| or ((nu)+N-1), argument reduction will cause total loss of accuracy, and so no computation is performed. For slightly smaller |z| or ((nu)+N-1), the computation is performed but results are accurate to less than half of machine precision. If |z| is very small, near the machine underflow threshold, or ((nu)+N-1) is too large, there is a risk of overflow and so no computation is performed. In all the above cases, a warning is given by the routine.

4. References

[1] Abramowitz M and Stegun I A (1968) Handbook of Mathematical Functions. Dover Publications.

[2] Amos D E (1986) Algorithm 644: A Portable Package for Bessel Functions of a Complex Argument and Nonnegative Order. ACM Trans. Math. Softw. 12 265--273.

5. Parameters

1: FNU -- DOUBLE PRECISION Input
On entry: the order, (nu), of the first member of the sequence of functions. Constraint: FNU >= 0.0.

2: Z -- COMPLEX(KIND(1.0D0)) Input
On entry: the argument z of the functions. Constraint: Z /= (0.0, 0.0).

3: N -- INTEGER Input
On entry: the number, N, of members required in the sequence $K_{(nu)}(z), K_{(nu)+1}(z),\ldots, K_{(nu)+N-1}(z)$. Constraint: N >= 1.

4: SCALE -- CHARACTER*1 Input
On entry: the scaling option.

If SCALE = 'U', the results are returned unscaled.

If SCALE = 'S', the results are returned scaled by the factor e^{z}. Constraint: SCALE = 'U' or 'S'.

5: CY(N) -- COMPLEX(KIND(1.0D)) array Output
On exit: the N required function values: CY(i) contains $K_{(nu)+i-1}(z)$, for i=1,2,...,N.

6: NZ -- INTEGER Output
On exit: the number of components of CY that are set to zero due to underflow. If NZ > 0 and Rez>=0.0, elements CY(1),CY(2),...,CY(NZ) are set to zero. If Rez<0.0, NZ simply states the number of underflows, and not which elements they are.

7: IFAIL -- INTEGER Input/Output
On entry: IFAIL must be set to 0, -1 or 1. For users not familiar with this parameter (described in the Essential Introduction) the recommended value is 0.

On exit: IFAIL = 0 unless the routine detects an error (see Section 6).

6. Error Indicators and Warnings

```
Errors detected by the routine:

If on entry IFAIL = 0 or -1, explanatory error messages are
output on the current error message unit (as defined by X04AAF).

IFAIL= 1
     On entry FNU < 0.0,

     or       Z = (0.0, 0.0),

     or       N < 1,

     or       SCALE /= 'U' or 'S'.

IFAIL= 2
     No computation has been performed due to the likelihood of
     overflow, because ABS(Z) is less than a machine-dependent
     threshold value (given in the Users' Note for your
     implementation).

IFAIL= 3
     No computation has been performed due to the likelihood of
     overflow, because FNU + N - 1 is too large - how large
     depends on Z and the overflow threshold of the machine.

IFAIL= 4
     The computation has been performed, but the errors due to
     argument reduction in elementary functions make it likely
     that the results returned by S18DCF are accurate to less
     than half of machine precision. This error exit may occur if
     either ABS(Z) or FNU + N - 1 is greater than a machine-
     dependent threshold value (given in the Users' Note for
     your implementation).

IFAIL= 5
     No computation has been performed because the errors due to
     argument reduction in elementary functions mean that all
     precision in results returned by S18DCF would be lost. This
     error exit may occur when either ABS(Z) or FNU + N - 1 is
     greater than a machine-dependent threshold value (given in
     the Users' Note for your implementation).

IFAIL= 6
     No results are returned because the algorithm termination
     condition has not been met. This may occur because the
     parameters supplied to S18DCF would have caused overflow or
```

underflow.

7. Accuracy

All constants in subroutine S18DCF are given to approximately 18 digits of precision. Calling the number of digits of precision in the floating-point arithmetic being used t, then clearly the maximum number of correct digits in the results obtained is limited by p=min(t,18). Because of errors in argument reduction when computing elementary functions inside S18DCF, the actual number of correct digits is limited, in general, by p-s, where s~max(1,|log |z||,|log (nu)|) represents the number of digits
            10   10
lost due to the argument reduction. Thus the larger the values of |z| and (nu), the less the precision in the result. If S18DCF is called with N>1, then computation of function values via recurrence may lead to some further small loss of accuracy.

If function values which should nominally be identical are computed by calls to S18DCF with different base values of (nu) and different N, the computed values may not agree exactly. Empirical tests with modest values of (nu) and z have shown that the discrepancy is limited to the least significant 3-4 digits of precision.

8. Further Comments

The time taken by the routine for a call of S18DCF is approximately proportional to the value of N, plus a constant. In general it is much cheaper to call S18DCF with N greater than 1, rather than to make N separate calls to S18DCF.

Paradoxically, for some values of z and (nu), it is cheaper to call S18DCF with a larger value of N than is required, and then discard the extra function values returned. However, it is not possible to state the precise circumstances in which this is likely to occur. It is due to the fact that the base value used to start recurrence may be calculated in different regions for different N, and the costs in each region may differ greatly.

Note that if the function required is K (x) or K (x), i.e.,
                                      0   1
(nu)=0.0 or 1.0, where x is real and positive, and only a single function value is required, then it may be much cheaper to call S18ACF, S18ADF, S18CCF(*) or S18CDF(*), depending on whether a

```
scaled result is required or not.

9. Example

The example program prints a caption and then proceeds to read
sets of data from the input data stream. The first datum is a
value for the order FNU, the second is a complex value for the
argument, Z, and the third is a value for the parameter SCALE.
The program calls the routine with N = 2 to evaluate the function
for orders FNU and FNU + 1, and it prints the results. The
process is repeated until the end of the input data stream is
encountered.

The example program is not reproduced here. The source code for
all example programs is distributed with the NAG Foundation
Library software and should be available on-line.
```

%%%

```
S18 -- Approximations of Special Functions                  S18DEF
        S18DEF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for
your implementation to check implementation-dependent details.
The symbol (*) after a NAG routine name denotes a routine that is
not included in the Foundation Library.

1. Purpose

S18DEF returns a sequence of values for the modified Bessel
functions I      (z) for complex z, non-negative (nu) and
           (nu)+n
n=0,1,...,N-1, with an option for exponential scaling.

2. Specification

       SUBROUTINE S18DEF (FNU, Z, N, SCALE, CY, NZ, IFAIL)
       INTEGER              N, NZ, IFAIL
       DOUBLE PRECISION     FNU
       COMPLEX(KIND(1.0D0)) Z, CY(N)
       CHARACTER*1          SCALE

3. Description

This subroutine evaluates a sequence of values for the modified
Bessel function I    (z), where z is complex, -(pi) < argz <=
```

(nu)
(pi), and (nu) is the real, non-negative order. The N-member
sequence is generated for orders (nu), (nu)+1,...,(nu)+N-1.
-|Rez|
Optionally, the sequence is scaled by the factor e .

The routine is derived from the routine CBESI in Amos [2].

Note: although the routine may not be called with (nu) less than
zero, for negative orders the formula
2
I (z)=I (z)+ ----sin((pi)(nu))K (z) may be used (for
-(nu) (nu) (pi) (nu)
the Bessel function K (z), see S18DCF).
(nu)

When N is greater than 1, extra values of I (z) are computed
(nu)
using recurrence relations.

For very large |z| or ((nu)+N-1), argument reduction will cause
total loss of accuracy, and so no computation is performed. For
slightly smaller |z| or ((nu)+N-1), the computation is performed
but results are accurate to less than half of machine precision.
If Re(z) is too large and the unscaled function is required,
there is a risk of overflow and so no computation is performed.
In all the above cases, a warning is given by the routine.

4. References

[1] Abramowitz M and Stegun I A (1968) Handbook of Mathematical
Functions. Dover Publications.

[2] Amos D E (1986) Algorithm 644: A Portable Package for Bessel
Functions of a Complex Argument and Nonnegative Order. ACM
Trans. Math. Softw. 12 265--273.

5. Parameters

1: FNU -- DOUBLE PRECISION Input
On entry: the order, (nu), of the first member of the
sequence of functions. Constraint: FNU >= 0.0.

2: Z -- COMPLEX(KIND(1.0D0)) Input
On entry: the argument z of the functions.

3: N -- INTEGER Input
On entry: the number, N, of members required in the sequence
I (z),I (z),...,I (z). Constraint: N >= 1.
(nu) (nu)+1 (nu)+N-1

4: SCALE -- CHARACTER*1 Input
On entry: the scaling option.
If SCALE = 'U', the results are returned unscaled.

If SCALE = 'S', the results are returned scaled by the
-|Rez|
factor e .
Constraint: SCALE = 'U' or 'S'.

5: CY(N) -- COMPLEX(KIND(1.0D)) array Output
On exit: the N required function values: CY(i) contains
I (z), for i=1,2,...,N.
(nu)+i-1

6: NZ -- INTEGER Output
On exit: the number of components of CY that are set to zero due to underflow.

If NZ > 0, then elements CY(N-NZ+1),CY(N-NZ+2),...,CY(N) are set to zero.

7: IFAIL -- INTEGER Input/Output
On entry: IFAIL must be set to 0, -1 or 1. For users not familiar with this parameter (described in the Essential Introduction) the recommended value is 0.

On exit: IFAIL = 0 unless the routine detects an error (see Section 6).

6. Error Indicators and Warnings

Errors detected by the routine:

If on entry IFAIL = 0 or -1, explanatory error messages are output on the current error message unit (as defined by X04AAF).

IFAIL= 1
On entry FNU < 0.0,

or N < 1,

```
     or       SCALE /= 'U' or 'S'.

IFAIL= 2
     No computation has been performed due to the likelihood of
     overflow, because real(Z) is greater than a machine-
     dependent threshold value (given in the Users' Note for
     your implementation). This error exit can only occur when
     SCALE = 'U'.

IFAIL= 3
     The computation has been performed, but the errors due to
     argument reduction in elementary functions make it likely
     that the results returned by S18DEF are accurate to less
     than half of machine precision. This error exit may occur
     when either ABS(Z) or FNU + N - 1 is greater than a machine-
     dependent threshold value (given in the Users' Note for
     your implementation).

IFAIL= 4
     No computation has been performed because the errors due to
     argument reduction in elementary functions mean that all
     precision in results returned by S18DEF would be lost. This
     error exit may occur when either ABS(Z) or FNU + N - 1 is
     greater than a machine-dependent threshold value (given in
     the Users' Note for your implementation).

IFAIL= 5
     No results are returned because the algorithm termination
     condition has not been met. This may occur because the
     parameters supplied to S18DEF would have caused overflow or
     underflow.

7. Accuracy

All constants in subroutine S18DEF are given to approximately 18
digits of precision. Calling the number of digits of precision in
the floating-point arithmetic being used t, then clearly the
maximum number of correct digits in the results obtained is
limited by p=min(t,18). Because of errors in argument reduction
when computing elementary functions inside S18DEF, the actual
number of correct digits is limited, in general, by p-s, where
s~max(1,|log  |z||,|log  (nu)|) represents the number of digits
           10         10
lost due to the argument reduction. Thus the larger the values of
|z| and (nu), the less the precision in the result. If S18DEF is
called with N>1, then computation of function values via
```

recurrence may lead to some further small loss of accuracy.

If function values which should nominally be identical are computed by calls to S18DEF with different base values of (nu) and different N, the computed values may not agree exactly. Empirical tests with modest values of (nu) and z have shown that the discrepancy is limited to the least significant 3-4 digits of precision.

8. Further Comments

The time taken by the routine for a call of S18DEF is approximately proportional to the value of N, plus a constant. In general it is much cheaper to call S18DEF with N greater than 1, rather than to make N separate calls to S18DEF.

Paradoxically, for some values of z and (nu), it is cheaper to call S18DEF with a larger value of N than is required, and then discard the extra function values returned. However, it is not possible to state the precise circumstances in which this is likely to occur. It is due to the fact that the base value used to start recurrence may be calculated in different regions for different N, and the costs in each region may differ greatly.

Note that if the function required is I (x) or I (x), i.e.,
0 1
(nu)=0.0 or 1.0, where x is real and positive, and only a single function value is required, then it may be much cheaper to call S18AEF, S18AFF, S18CEF(*) or S18CFF(*), depending on whether a scaled result is required or not.

9. Example

The example program prints a caption and then proceeds to read sets of data from the input data stream. The first datum is a value for the order FNU, the second is a complex value for the argument, Z, and the third is a value for the parameter SCALE. The program calls the routine with N = 2 to evaluate the function for orders FNU and FNU + 1, and it prints the results. The process is repeated until the end of the input data stream is encountered.

The example program is not reproduced here. The source code for all example programs is distributed with the NAG Foundation Library software and should be available on-line.

%%

S19 -- Approximations of Special Functions S19AAF
S19AAF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for your implementation to check implementation-dependent details. The symbol (*) after a NAG routine name denotes a routine that is not included in the Foundation Library.

1. Purpose

S19AAF returns a value for the Kelvin function ber x via the routine name.

2. Specification

```
DOUBLE PRECISION FUNCTION S19AAF (X, IFAIL)
INTEGER          IFAIL
DOUBLE PRECISION X
```

3. Description

This routine evaluates an approximation to the Kelvin function berx.

Note: ber(-x)=berx, so the approximation need only consider x>=0.0.

The routine is based on several Chebyshev expansions:

For 0<=x<=5,

$$\mathrm{berx} = \sum_{r=0}^{\prime\prime} a_r T_r(t) \quad \text{with } t = 2\left(\frac{x}{5}\right)^4 - 1;$$

For x>5,

$$\mathrm{berx} = \frac{e^{x/\sqrt{2}}}{\sqrt{x}}\left[\left(1+\frac{1}{x}\right)a(t)\cos(\alpha) + \frac{1}{x}b(t)\sin(\alpha)\right]$$

```
              \/2(pi)x


               -x/\/2
              e      [(   1    )           1             ]
           + --------[(1+ -c(t))sin(beta)+ -d(t)cos(beta)]
                     [(   x    )           x             ]
             \/2(pi)x

                  x   (pi)           x   (pi)
where (alpha)= ---- ----, (beta)= ---+ ----,
                    8                  8
                \/2                /2
```

and a(t), b(t), c(t), and d(t) are expansions in the variable

$$t = \frac{10}{x} - 1.$$

When x is sufficiently close to zero, the result is set directly to ber 0=1.0.

For large x, there is a danger of the result being totally inaccurate, as the error amplification factor grows in an essentially exponential manner; therefore the routine must fail.

4. References

[1] Abramowitz M and Stegun I A (1968) Handbook of Mathematical Functions. Dover Publications.

5. Parameters

1: X -- DOUBLE PRECISION Input
On entry: the argument x of the function.

2: IFAIL -- INTEGER Input/Output
On entry: IFAIL must be set to 0, -1 or 1. For users not familiar with this parameter (described in the Essential Introduction) the recommended value is 0.

On exit: IFAIL = 0 unless the routine detects an error (see Section 6).

6. Error Indicators and Warnings

Errors detected by the routine:

IFAIL= 1
On entry ABS(X) is too large for an accurate result to be returned. On soft failure, the routine returns zero.

7. Accuracy

Since the function is oscillatory, the absolute error rather than the relative error is important. Let E be the absolute error in the result and (delta) be the relative error in the argument. If (delta) is somewhat larger than the machine precision, then we have:

```
                  |  x              |
               E~=| ---(ber x+bei x)|(delta)
                  |        1     1  |
                  | \/2             |
```

(provided E is within machine bounds).

For small x the error amplification is insignificant and thus the absolute error is effectively bounded by the machine precision.

For medium and large x, the error behaviour is oscillatory and

```
                            /   x   x/\/2
its amplitude grows like   /  -----e    . Therefore it is not
                         \/   2(pi)
```

possible to calculate the function with any accuracy when

```
    x/\/2  /2(pi)
\/xe     > -------. Note that this value of x is much smaller than
           (delta)
```

the minimum value of x for which the function overflows.

8. Further Comments

For details of the time taken by the routine see the Users' Note for your implementation.

9. Example

The example program reads values of the argument x from a file, evaluates the function at each value of x and prints the results.

The example program is not reproduced here. The source code for all example programs is distributed with the NAG Foundation Library software and should be available on-line.

%%%

S19 -- Approximations of Special Functions S19ABF
S19ABF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for your implementation to check implementation-dependent details. The symbol (*) after a NAG routine name denotes a routine that is not included in the Foundation Library.

1. Purpose

S19ABF returns a value for the Kelvin function bei x via the routine name.

2. Specification

```
DOUBLE PRECISION FUNCTION S19ABF (X, IFAIL)
INTEGER          IFAIL
DOUBLE PRECISION X
```

3. Description

This routine evaluates an approximation to the Kelvin function beix.

Note: bei(-x)=beix, so the approximation need only consider x>=0.0.

The routine is based on several Chebyshev expansions:

For 0<=x<=5,

$$\text{bei } x = \frac{x^2}{4} = \sum_r a_r T_r(t), \quad \text{with} \quad t = 2\left(\frac{x}{5}\right)^4 - 1;$$

$$\sum_{r=0}$$

For x>5,

$$\text{bei } x = \frac{e^{x/\sqrt{2}}}{\sqrt{2\pi x}}\left[\left(1+\frac{1}{x}\right)\sin(\alpha) - \frac{1}{x}b(t)\cos(\alpha)\right]$$

$$+ \frac{e^{x/\sqrt{2}}}{\sqrt{2\pi x}}\left[\left(1+\frac{1}{x}\right)\cos(\beta) - \frac{1}{x}d(t)\sin(\beta)\right]$$

$$\text{where}(\alpha) = \frac{x}{\sqrt{2}} - \frac{\pi}{8}, (\beta) = \frac{x}{\sqrt{2}} + \frac{\pi}{8},$$

and a(t), b(t), c(t), and d(t) are expansions in the variable

$$t = \frac{10}{x} - 1.$$

When x is sufficiently close to zero, the result is computed as

$$\text{bei } x = \frac{x^2}{4}.$$ If this result would underflow, the result returned is beix=0.0.

For large x, there is a danger of the result being totally inaccurate, as the error amplification factor grows in an essentially exponential manner; therefore the routine must fail.

4. References

[1] Abramowitz M and Stegun I A (1968) Handbook of Mathematical Functions. Dover Publications.

5. Parameters

1: X -- DOUBLE PRECISION Input
On entry: the argument x of the function.

2: IFAIL -- INTEGER Input/Output
On entry: IFAIL must be set to 0, -1 or 1. For users not familiar with this parameter (described in the Essential Introduction) the recommended value is 0.

On exit: IFAIL = 0 unless the routine detects an error (see Section 6).

6. Error Indicators and Warnings

Errors detected by the routine:

IFAIL= 1
On entry ABS(X) is too large for an accurate result to be returned. On soft failure, the routine returns zero.

7. Accuracy

Since the function is oscillatory, the absolute error rather than the relative error is important. Let E be the absolute error in the function, and (delta) be the relative error in the argument. If (delta) is somewhat larger than the machine precision, then we have:

```
                  |  x               |
               E~=| ---(-ber x+bei x)|(delta)
                  |         1     1  |
                  | \/2              |
```

(provided E is within machine bounds).

For small x the error amplification is insignificant and thus the absolute error is effectively bounded by the machine precision.

For medium and large x, the error behaviour is oscillatory and

```
                            /   x   x\/2
its amplitude grows like   /  -----e    . Therefore it is
                         \/   2(pi)
```

```
impossible to calculate the functions with any accuracy when


    x/\/2 /2(pi)
\/xe     > -------. Note that this value of x is much smaller than
           (delta)

the minimum value of x for which the function overflows.

8. Further Comments

For details of the time taken by the routine see the Users' Note
for your implementation.

9. Example

The example program reads values of the argument x from a file,
evaluates the function at each value of x and prints the results.

The example program is not reproduced here. The source code for
all example programs is distributed with the NAG Foundation
Library software and should be available on-line.
```

%%%

```
S19 -- Approximations of Special Functions                  S19ACF
        S19ACF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for
your implementation to check implementation-dependent details.
The symbol (*) after a NAG routine name denotes a routine that is
not included in the Foundation Library.

1. Purpose

S19ACF returns a value for the Kelvin function ker x, via the
routine name.

2. Specification

       DOUBLE PRECISION FUNCTION S19ACF (X, IFAIL)
       INTEGER          IFAIL
       DOUBLE PRECISION X

3. Description
```

This routine evaluates an approximation to the Kelvin function
ker x.

Note: for x<0 the function is undefined and at x=0 it is infinite
so we need only consider x>0.

The routine is based on several Chebyshev expansions:

For 0<x<=1,

$$\ker x=-f(t)\log x+\frac{(pi)}{16}x^2 g(t)+y(t)$$

where f(t), g(t) and y(t) are expansions in the variable $t=2x^4-1$;

For 1<x<=3,

$$\ker x=\exp\left(-\frac{11}{16}x\right)q(t)$$

where q(t) is an expansion in the variable t=x-2;

For x>3,

$$\ker x=\sqrt{\frac{(pi)}{2x}}e^{-x/\sqrt{2}}\left[\left(1+\frac{1}{-c(t)}\right)\cos(beta)-\frac{1}{-d(t)}\sin(beta)\right]$$

where $(beta)=\frac{x}{8}+\frac{(pi)}{\sqrt{2}}$, and c(t) and d(t) are expansions in the variable $t=\frac{6}{x}-1$.

When x is sufficiently close to zero, the result is computed as

$$\ker x = -(\gamma) - \log\left(\frac{x}{2}\right) + \left((\pi) - \frac{x^3}{8}\right)\frac{2}{16}x$$

and when x is even closer to zero, simply as

$$\ker x = -(\gamma) - \log\left(\frac{x}{2}\right).$$

For large x, ker x is asymptotically given by $\sqrt{\frac{(\pi)}{2x}}\,e^{-x/\sqrt{2}}$ and this becomes so small that it cannot be computed without underflow and the routine fails.

4. References

[1] Abramowitz M and Stegun I A (1968) Handbook of Mathematical Functions. Dover Publications.

5. Parameters

1: X -- DOUBLE PRECISION Input
On entry: the argument x of the function. Constraint: X > 0.

2: IFAIL -- INTEGER Input/Output
On entry: IFAIL must be set to 0, -1 or 1. For users not familiar with this parameter (described in the Essential Introduction) the recommended value is 0.

On exit: IFAIL = 0 unless the routine detects an error (see Section 6).

6. Error Indicators and Warnings

Errors detected by the routine:

IFAIL= 1
On entry X is too large, the result underflows. On soft failure, the routine returns zero.

IFAIL= 2
On entry X <= 0, the function is undefined. On soft failure

the routine returns zero.

7. Accuracy

Let E be the absolute error in the result, (epsilon) be the relative error in the result and (delta) be the relative error in the argument. If (delta) is somewhat larger than the machine precision, then we have:

$$E\tilde{} = \left| \frac{x}{\sqrt{2}} \left(\ker_1 x + \mathrm{kei}_1 x \right) \right| (\delta),$$

$$(\varepsilon)\tilde{} = \left| \frac{x}{\sqrt{2}} \frac{\ker_1 x + \mathrm{kei}_1 x}{\ker x} \right| (\delta).$$

For very small x, the relative error amplification factor is approximately given by $\frac{1}{|\log x|}$, which implies a strong attenuation of relative error. However, (epsilon) in general cannot be less than the machine precision.

For small x, errors are damped by the function and hence are limited by the machine precision.

For medium and large x, the error behaviour, like the function itself, is oscillatory, and hence only the absolute accuracy for the function can be maintained. For this range of x, the

amplitude of the absolute error decays like $\sqrt{\frac{\pi}{2}} e^{-x/\sqrt{2}}$

which implies a strong attenuation of error. Eventually, ker x,

which asymptotically behaves like $\sqrt{\frac{\pi}{2x}} e^{-x/\sqrt{2}}$, becomes so

small that it cannot be calculated without causing underflow, and the routine returns zero. Note that for large x the errors are dominated by those of the Fortran intrinsic function EXP.

8. Further Comments

Underflow may occur for a few values of x close to the zeros of ker x, below the limit which causes a failure with IFAIL = 1.

9. Example

The example program reads values of the argument x from a file, evaluates the function at each value of x and prints the results.

The example program is not reproduced here. The source code for all example programs is distributed with the NAG Foundation Library software and should be available on-line.

%%%

S19 -- Approximations of Special Functions S19ADF
S19ADF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for your implementation to check implementation-dependent details. The symbol (*) after a NAG routine name denotes a routine that is not included in the Foundation Library.

1. Purpose

S19ADF returns a value for the Kelvin function keix via the routine name.

2. Specification

```
DOUBLE PRECISION FUNCTION S19ADF (X, IFAIL)
INTEGER          IFAIL
DOUBLE PRECISION X
```

3. Description

This routine evaluates an approximation to the Kelvin function keix.

Note: for x<0 the function is undefined, so we need only consider x>=0.

```
The routine is based on several Chebyshev expansions:

For 0<=x<=1,

                                  2
                         (pi)     x
                 keix=- ----f(t)+ --[-g(t)logx+v(t)]
                         4        4

                                                                  4
where f(t), g(t) and v(t) are expansions in the variable t=2x -1;

For 1<x<=3,

                                  (  9 )
                           keix=exp(- -x)u(t)
                                  (  8 )

where u(t) is an expansion in the variable t=x-2;

For x>3,


                 / (pi) -x/\/2[(   1)              1             ]
        keix=  /  ----e      [(1+ -)c(t)sin(beta)+ -d(t)cos(beta)]
             \/    2x        [(   x)              x             ]


                      x   (pi)
where (beta)= ---+ ----, and c(t) and d(t) are expansions in the
                       8
                    \/2
                   6
variable t= --1.
                   x

For x<0, the function is undefined, and hence the routine fails
and returns zero.

When x is sufficiently close to zero, the result is computed as

                                                  2
                         (pi) (             ( x)) x
                 keix=- ----+(1-(gamma)-log( -)) --
```

```
                          4   (            ( 2)) 4
```

and when x is even closer to zero simply as

```
                                 (pi)
                            keix=- ----.
                                    4
```

For large x, keix is asymptotically given by $\sqrt{\frac{\pi}{2x}}\,e^{-x/\sqrt{2}}$ and this becomes so small that it cannot be computed without underflow and the routine fails.

4. References

[1] Abramowitz M and Stegun I A (1968) Handbook of Mathematical Functions. Dover Publications.

5. Parameters

1: X -- DOUBLE PRECISION Input
On entry: the argument x of the function. Constraint: X >= 0.

2: IFAIL -- INTEGER Input/Output
On entry: IFAIL must be set to 0, -1 or 1. For users not familiar with this parameter (described in the Essential Introduction) the recommended value is 0.

On exit: IFAIL = 0 unless the routine detects an error (see Section 6).

6. Error Indicators and Warnings

Errors detected by the routine:

IFAIL= 1
On entry X is too large, the result underflows. On soft failure, the routine returns zero.

IFAIL= 2
On entry X < 0, the function is undefined. On soft failure

the routine returns zero.

7. Accuracy

Let E be the absolute error in the result, and (delta) be the relative error in the argument. If (delta) is somewhat larger than the machine representation error, then we have:

$$E \tilde{=} \left| \frac{x}{\sqrt{2}} \left(-\mathrm{ker}_1 x + \mathrm{kei}_1 x \right) \right| (\mathrm{delta}).$$

For small x, errors are attenuated by the function and hence are limited by the machine precision.

For medium and large x, the error behaviour, like the function itself, is oscillatory and hence only absolute accuracy of the function can be maintained. For this range of x, the amplitude of

the absolute error decays like $\sqrt{\frac{\pi}{2}} e^{-x/\sqrt{2}}$, which implies a

strong attenuation of error. Eventually, keix, which is

asymptotically given by $\sqrt{\frac{\pi}{2x}} e^{-x/\sqrt{2}}$, becomes so small that it

cannot be calculated without causing underflow and therefore the routine returns zero. Note that for large x, the errors are dominated by those of the Fortran intrinsic function EXP.

8. Further Comments

Underflow may occur for a few values of x close to the zeros of keix, below the limit which causes a failure with IFAIL = 1.

9. Example

The example program reads values of the argument x from a file, evaluates the function at each value of x and prints the results.

```
The example program is not reproduced here. The source code for
all example programs is distributed with the NAG Foundation
Library software and should be available on-line.

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

S20 -- Approximations of Special Functions                  S20ACF
        S20ACF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for
your implementation to check implementation-dependent details.
The symbol (*) after a NAG routine name denotes a routine that is
not included in the Foundation Library.

1. Purpose

S20ACF returns a value for the Fresnel Integral S(x), via the
routine name.

2. Specification

       DOUBLE PRECISION FUNCTION S20ACF (X, IFAIL)
       INTEGER          IFAIL
       DOUBLE PRECISION X

3. Description

This routine evaluates an approximation to the Fresnel Integral

                            x
                            /   ( (pi) 2)
                      S(x)= |sin( ----t )dt.
                            /   (  2    )
                            0

Note: S(x)=-S(-x), so the approximation need only consider x>=0.0

The routine is based on three Chebyshev expansions:

For 0<x<=3,

                     3 --'                    ( x)4
              S(x)=x  >   a T (t) , with t=2( -) -1;
                      --   r r               ( 3)
                      r=0
```

For x>3,

$$S(x)= \frac{1}{2}\frac{f(x)}{x}\cos\left(\frac{(\pi)2}{2}x\right)-\frac{g(x)}{x^3}\sin\left(\frac{(\pi)2}{2}x\right),$$

$$\text{where } f(x) = \sum_{r=0}^{} b_r T_r(t),$$

$$\text{and } g(x) = \sum_{r=0}^{} c_r T_r(t), \text{ with } t=2\left(\frac{3}{x}\right)^4-1.$$

For small x, $S(x)\tilde{=}\frac{(\pi)^3}{6}x^3$. This approximation is used when x is sufficiently small for the result to be correct to machine precision. For very small x, this approximation would underflow; the result is then set exactly to zero.

For large x, $f(x)\tilde{=}\frac{1}{(\pi)}$ and $g(x)\tilde{=}\frac{1}{(\pi)^2}$. Therefore for moderately large x, when $\frac{1}{2^3(\pi)^3x^3}$ is negligible compared with $\frac{1}{2}$, the second term in the approximation for x>3 may be dropped. For very large x, when $\frac{1}{(\pi)x}$ becomes negligible, $S(x)\tilde{=}\frac{1}{2}$. However there will be considerable difficulties in calculating $\cos\left(\frac{(\pi)2}{2}x\right)$ accurately before this final limiting value can be used. Since $\cos\left(\frac{(\pi)2}{2}x\right)$ is periodic, its value is essentially determined by the fractional part of x^2. If $x^2=N+(\theta)$ where N

is an integer and 0<=(theta)<1, then cos(----x) depends on
                                        (2   )
(theta) and on N modulo 4. By exploiting this fact, it is
possible to retain significance in the calculation of
   ((pi) 2)
cos(----x) either all the way to the very large x limit, or at
   (2   )
                                  x
least until the integer part of   - is equal to the maximum
                                  2
integer allowed on the machine.

4. References

[1] Abramowitz M and Stegun I A (1968) Handbook of Mathematical Functions. Dover Publications.

5. Parameters

1: X -- DOUBLE PRECISION Input
On entry: the argument x of the function.

2: IFAIL -- INTEGER Input/Output
On entry: IFAIL must be set to 0, -1 or 1. For users not familiar with this parameter (described in the Essential Introduction) the recommended value is 0.

On exit: IFAIL = 0 unless the routine detects an error (see Section 6).

6. Error Indicators and Warnings

There are no failure exits from this routine. The parameter IFAIL has been included for consistency with other routines in this chapter.

7. Accuracy

Let (delta) and (epsilon) be the relative errors in the argument and result respectively.

If (delta) is somewhat larger than the machine precision (i.e., if (delta) is due to data errors etc), then (epsilon) and (delta) are approximately related by:

                          | ((pi) 2)|

```
                          | xsin( ----x )|
                          |     (  2    )|
                (epsilon)~=| -------------|(delta).
                          |     S(x)     |
```

Figure 1 shows the behaviour of the error amplification factor

```
|     ( (pi) 2)|
| xsin( ----x )|
|     (  2    )|
| -------------|.
|     S(x)     |
```

Figure 1
Please see figure in printed Reference Manual

However if (delta) is of the same order as the machine precision, then rounding errors could make (epsilon) slightly larger than the above relation predicts.

For small x, (epsilon)~=3(delta) and hence there is only moderate amplification of relative error. Of course for very small x where the correct result would underflow and exact zero is returned, relative error-control is lost.

For moderately large values of x,

```
                          |     ( (pi) 2)|
             |(epsilon)|~=|2xsin( ----x )||(delta)|
                          |     (  2    )|
```

and the result will be subject to increasingly large amplification of errors. However the above relation breaks down
for large values of x (i.e., when $\frac{1}{x^2}$ is of the order of the machine precision in this region the relative error in the result is essentially bounded by $\frac{2}{(\pi)x^2}$).

Hence the effects of error amplification are limited and at worst the relative error loss should not exceed half the possible number of significant figures.

```
8. Further Comments

None.

9. Example

The example program reads values of the argument x from a file,
evaluates the function at each value of x and prints the results.

The example program is not reproduced here. The source code for
all example programs is distributed with the NAG Foundation
Library software and should be available on-line.
```

%%%

```
S20 -- Approximations of Special Functions                S20ADF
        S20ADF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for
your implementation to check implementation-dependent details.
The symbol (*) after a NAG routine name denotes a routine that is
not included in the Foundation Library.

1. Purpose

S20ADF returns a value for the Fresnel Integral C(x), via the
routine name.

2. Specification

       DOUBLE PRECISION FUNCTION S20ADF (X, IFAIL)
       INTEGER          IFAIL
       DOUBLE PRECISION X

3. Description

This routine evaluates an approximation to the Fresnel Integral

                           x
                           /   ( (pi) 2)
                     C(x)= |cos( ----t )dt.
                           /   (  2    )
                           0

Note: C(x)=-C(-x), so the approximation need only consider x>=0.0
```

The routine is based on three Chebyshev expansions:

For 0<x<=3,

$$C(x)=x\sum_{r=0}^{--'} a_r T_r(t), \text{ with } t=2\left(\frac{x}{3}\right)^4-1;$$

For x>3,

$$C(x)=\frac{1}{2}\frac{f(x)}{x}\sin\left(\frac{\pi}{2}x\right)-\frac{g(x)}{x^3}\cos\left(\frac{\pi}{2}x\right),$$

$$\text{where } f(x)=\sum_{r=0}^{--'} b_r T_r(t),$$

$$\text{and } g(x)=\sum_{r=0}^{--'} c_r T_r(t), \text{ with } t=2\left(\frac{3}{x}\right)^4-1.$$

For small x, $C(x)\tilde{}=x$. This approximation is used when x is sufficiently small for the result to be correct to machine precision.

For large x, $f(x)\tilde{}=\frac{1}{(\pi)}$ and $g(x)\tilde{}=\frac{1}{(\pi)^2}$. Therefore for moderately large x, when $\frac{1}{2\,3\,(\pi)\,x}$ is negligible compared with $\frac{1}{2}$, the second term in the approximation for x>3 may be dropped. For very large x, when $\frac{1}{(\pi)x}$ becomes negligible, $C(x)\tilde{}=\frac{1}{2}$. However there will be considerable difficulties in calculating $\sin\left(\frac{\pi}{2}x\right)$ accurately before this final limiting value can be

```
   (  2    )
              ( (pi) 2)
used. Since sin( ----x ) is periodic, its value is essentially
              (  2    )
                                      2      2
determined by the fractional part of x . If x =N+(theta), where N
                                        ( (pi) 2)
is an integer and 0<=(theta)<1, then sin( ----x ) depends on
                                        (  2    )
(theta) and on N modulo 4. By exploiting this fact, it is
possible to retain some significance in the calculation of
   ( (pi) 2)
sin( ----x ) either all the way to the very large x limit, or at
   (  2    )
                                  x
least until the integer part of   - is equal to the maximum
                                  2
integer allowed on the machine.

4. References

[1]   Abramowitz M and Stegun I A (1968) Handbook of Mathematical
      Functions. Dover Publications.

5. Parameters

 1:  X -- DOUBLE PRECISION                                    Input
     On entry: the argument x of the function.

 2:  IFAIL -- INTEGER                                 Input/Output
     On entry: IFAIL must be set to 0, -1 or 1. For users not
     familiar with this parameter (described in the Essential
     Introduction) the recommended value is 0.

     On exit: IFAIL = 0 unless the routine detects an error (see
     Section 6).

6. Error Indicators and Warnings

There are no failure exits from this routine. The parameter IFAIL
has been included for consistency with other routines in this
chapter.

7. Accuracy

Let (delta) and (epsilon) be the relative errors in the argument
```

and result respectively.

If (delta) is somewhat larger than the machine precision (i.e if (delta) is due to data errors etc), then (epsilon) and (delta) are approximately related by:

```
                          |     ( (pi) 2)|
                          | xcos( ----x )|
                          |     (  2    )|
              (epsilon)~=| -------------|(delta).
                          |     C(x)     |
```

Figure 1 shows the behaviour of the error amplification factor

```
|     ( (pi) 2)|
| xcos( ----x )|
|     (  2    )|
| -------------|.
|     C(x)     |
```

Figure 1
Please see figure in printed Reference Manual

However if (delta) is of the same order as the machine precision, then rounding errors could make (epsilon) slightly larger than the above relation predicts.

For small x, (epsilon)~=(delta) and there is no amplification of relative error.

For moderately large values of x,

```
                          |     ( (pi) 2)|
            |(epsilon)|~=|2xcos( ----x )||(delta)|
                          |     (  2    )|
```

and the result will be subject to increasingly large amplification of errors. However the above relation breaks down
                                  1
for large values of x (i.e., when -- is of the order of the
                                   2
                                  x
machine precision); in this region the relative error in the
                                   2
result is essentially bounded by -----).

```
                                        (pi)x

Hence the effects of error amplification are limited and at worst
the relative error loss should not exceed half the possible
number of significant figures.

8. Further Comments

None.

9. Example

The example program reads values of the argument x from a file,
evaluates the function at each value of x and prints the results.

The example program is not reproduced here. The source code for
all example programs is distributed with the NAG Foundation
Library software and should be available on-line.
```

%%

```
S21 -- Approximations of Special Functions                S21BAF
        S21BAF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for
your implementation to check implementation-dependent details.
The symbol (*) after a NAG routine name denotes a routine that is
not included in the Foundation Library.

1. Purpose

S21BAF returns a value of an elementary integral, which occurs as
a degenerate case of an elliptic integral of the first kind, via
the routine name.

2. Specification

       DOUBLE PRECISION FUNCTION S21BAF (X, Y, IFAIL)
       INTEGER          IFAIL
       DOUBLE PRECISION X, Y

3. Description

This routine calculates an approximate value for the integral

                                    infty
```

$$R_C(x,y) = \frac{1}{2}\int_0^\infty \frac{dt}{\sqrt{t+x}(t+y)}$$

where x>=0 and y/=0.

This function, which is related to the logarithm or inverse hyperbolic functions for y<x and to inverse circular functions if x<y, arises as a degenerate form of the elliptic integral of the first kind. If y<0, the result computed is the Cauchy principal value of the integral.

The basic algorithm, which is due to Carlson [2] and [3], is to reduce the arguments recursively towards their mean by the system:

$$x_0 = x,$$

$$y_0 = y$$

$$\mu_n = (x_n + 2y_n)/3,$$

$$S_n = (y_n - x_n)/3\mu_n$$

$$\lambda_n = y_n + 2\sqrt{x_n}\sqrt{y_n}$$

$$x_{n+1} = (x_n + \lambda_n)/4,$$

$$y_{n+1} = (y_n + \lambda_n)/4.$$

The quantity $|S_n|$ for n=0,1,2,3,... decreases with increasing n, eventually $|S_n| \sim 1/4$. For small enough S_n the required function value can be approximated by the first few terms of the Taylor

series about the mean. That is

$$R_C(x,y) = \left(1 + \frac{3S_n^2}{10} + \frac{S_n^3}{7} + \frac{3S_n^4}{8} + \frac{9S_n^5}{22}\right) \Big/ \sqrt{\mu_n}\,.$$

The truncation error involved in using this approximation is bounded by $16|S_n|^6/(1-2|S_n|)$ and the recursive process is stopped when S_n is small enough for this truncation error to be negligible compared to the machine precision.

Within the domain of definition, the function value is itself representable for all representable values of its arguments. However, for values of the arguments near the extremes the above algorithm must be modified so as to avoid causing underflows or overflows in intermediate steps. In extreme regions arguments are pre-scaled away from the extremes and compensating scaling of the result is done before returning to the calling program.

4. References

[1] Abramowitz M and Stegun I A (1968) Handbook of Mathematical Functions. Dover Publications.

[2] Carlson B C (1978) Computing Elliptic Integrals by Duplication. (Preprint) Department of Physics, Iowa State University.

[3] Carlson B C (1988) A Table of Elliptic Integrals of the Third Kind. Math. Comput. 51 267--280.

5. Parameters

1: X -- DOUBLE PRECISION Input

2: Y -- DOUBLE PRECISION Input
On entry: the arguments x and y of the function, respectively. Constraint: X >= 0.0 and Y /= 0.0.

3: IFAIL -- INTEGER Input/Output
On entry: IFAIL must be set to 0, -1 or 1. For users not

```
     familiar with this parameter (described in the Essential
     Introduction) the recommended value is 0.

     On exit: IFAIL = 0 unless the routine detects an error (see
     Section 6).

6. Error Indicators and Warnings

Errors detected by the routine:

If on entry IFAIL = 0 or -1, explanatory error messages are
output on the current error message unit (as defined by X04AAF).

IFAIL= 1
     On entry X < 0.0; the function is undefined.

IFAIL= 2
     On entryY = 0.0; the function is undefined.

     On soft failure the routine returns zero.

7. Accuracy

In principle the routine is capable of producing full machine
precision. However round-off errors in internal arithmetic will
result in slight loss of accuracy. This loss should never be
excessive as the algorithm does not involve any significant
amplification of round-off error. It is reasonable to assume that
the result is accurate to within a small multiple of the machine
precision.

8. Further Comments

Users should consult the Chapter Introduction which shows the
relationship of this function to the classical definitions of the
elliptic integrals.

9. Example

This example program simply generates a small set of non-extreme
arguments which are used with the routine to produce the table of
low accuracy results.

The example program is not reproduced here. The source code for
all example programs is distributed with the NAG Foundation
Library software and should be available on-line.
```

%%

S21 -- Approximations of Special Functions S21BBF
S21BBF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for your implementation to check implementation-dependent details. The symbol (*) after a NAG routine name denotes a routine that is not included in the Foundation Library.

1. Purpose

S21BBF returns a value of the symmetrised elliptic integral of the first kind, via the routine name.

2. Specification

```
DOUBLE PRECISION FUNCTION S21BBF (X, Y, Z, IFAIL)
INTEGER          IFAIL
DOUBLE PRECISION X, Y, Z
```

3. Description

This routine calculates an approximation to the integral

$$R_F(x,y,z) = \frac{1}{2}\int_0^{\infty} \frac{dt}{\sqrt{(t+x)(t+y)(t+z)}}$$

where x, y, z>=0 and at most one is zero.

The basic algorithm, which is due to Carlson [2] and [3], is to reduce the arguments recursively towards their mean by the rule:

$x_0 = \min(x,y,z), z_0 = \max(x,y,z)$,

y_0 = remaining third intermediate value argument.

(This ordering, which is possible because of the symmetry of the function, is done for technical reasons related to the avoidance of overflow and underflow.)

```
(mu) =   (x +y +3z )/3
    n      n  n   n

X    =   (1-x )/(mu)
 n           n      n

Y    =   (1-y )/(mu)
 n           n      n

Z    =   (1-z )/(mu)
 n           n      n


(lambda) =  /x y +  /y z +  /z x
        n \/  n n /  n n /  n n

x    =   (x +(lambda) )/4
 n+1       n         n

y    =   (y +(lambda) )/4
 n+1       n         n

z    =   (z +(lambda) )/4
 n+1       n         n
```

(epsilon) =max(|X |,|Y |,|Z |) and the function may be approximated adequately by a 5th order power series:
n n n n

$$R_F(x,y,z) = \frac{1}{(\mu)\sqrt{\mu_n}}\left(1 - \frac{E_2}{10} + \frac{E_2^2}{24} - \frac{3E_2E_3}{44} + \frac{E_3}{14}\right)$$

where $E_2 = X_n Y_n + Y_n Z_n + Z_n X_n$, $E_3 = X_n Y_n Z_n$.

The truncation error involved in using this approximation is bounded by $\frac{(\epsilon_n)^6}{4(1-(\epsilon_n)^3)}$ and the recursive process

is stopped when this truncation error is negligible compared with the machine precision.

Within the domain of definition, the function value is itself representable for all representable values of its arguments. However, for values of the arguments near the extremes the above algorithm must be modified so as to avoid causing underflows or overflows in intermediate steps. In extreme regions arguments are pre-scaled away from the extremes and compensating scaling of the result is done before returning to the calling program.

4. References

[1] Abramowitz M and Stegun I A (1968) Handbook of Mathematical Functions. Dover Publications.

[2] Carlson B C (1978) Computing Elliptic Integrals by Duplication. (Preprint) Department of Physics, Iowa State University.

[3] Carlson B C (1988) A Table of Elliptic Integrals of the Third Kind. Math. Comput. 51 267--280.

5. Parameters

1: X -- DOUBLE PRECISION Input

2: Y -- DOUBLE PRECISION Input

3: Z -- DOUBLE PRECISION Input
On entry: the arguments x, y and z of the function.
Constraint: X, Y, Z >= 0.0 and only one of X, Y and Z may be zero.

4: IFAIL -- INTEGER Input/Output
On entry: IFAIL must be set to 0, -1 or 1. For users not familiar with this parameter (described in the Essential Introduction) the recommended value is 0.

On exit: IFAIL = 0 unless the routine detects an error (see Section 6).

6. Error Indicators and Warnings

Errors detected by the routine:

If on entry IFAIL = 0 or -1, explanatory error messages are output on the current error message unit (as defined by X04AAF).

IFAIL= 1
  On entry one or more of X, Y and Z is negative; the function is undefined.

IFAIL= 2
  On entry two or more of X, Y and Z are zero; the function is undefined.

  On soft failure, the routine returns zero.

7. Accuracy

In principle the routine is capable of producing full machine precision. However round-off errors in internal arithmetic will result in slight loss of accuracy. This loss should never be excessive as the algorithm does not involve any significant amplification of round-off error. It is reasonable to assume that the result is accurate to within a small multiple of the machine precision.

8. Further Comments

Users should consult the Chapter Introduction which shows the relationship of this function to the classical definitions of the elliptic integrals.

If two arguments are equal, the function reduces to the elementary integral R_C, computed by S21BAF.

9. Example

This example program simply generates a small set of non-extreme arguments which are used with the routine to produce the table of low accuracy results.

The example program is not reproduced here. The source code for all example programs is distributed with the NAG Foundation Library software and should be available on-line.

%%

S21 -- Approximations of Special Functions S21BCF

S21BCF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for your implementation to check implementation-dependent details. The symbol (*) after a NAG routine name denotes a routine that is not included in the Foundation Library.

1. Purpose

S21BCF returns a value of the symmetrised elliptic integral of the second kind, via the routine name.

2. Specification

```
DOUBLE PRECISION FUNCTION S21BCF (X, Y, Z, IFAIL)
INTEGER          IFAIL
DOUBLE PRECISION X, Y, Z
```

3. Description

This routine calculates an approximate value for the integral

$$R_D(x,y,z) = \frac{3}{2}\int_0^{\infty} \frac{dt}{\sqrt{(t+x)(t+y)(t+z)^3}}$$

where x, y>=0, at most one of x and y is zero, and z>0.

The basic algorithm, which is due to Carlson [2] and [3], is to reduce the arguments recursively towards their mean by the rule:

$$x_0 = x$$

$$y_0 = y$$

$$z_0 = z$$

$$(\mu)_n = (x_n + y_n + 3z_n)/5$$

$$X_n = (1-x_n)/(\mu)_n$$

$$Y_n = (1-y_n)/(\mu)_n$$

$$Z_n = (1-z_n)/(\mu)_n$$

$$(\lambda)_n = \sqrt{x_n y_n} + \sqrt{y_n z_n} + \sqrt{z_n x_n}$$

$$x_{n+1} = (x_n+(\lambda)_n)/4$$

$$y_{n+1} = (y_n+(\lambda)_n)/4$$

$$z_{n+1} = (z_n+(\lambda)_n)/4$$

For n sufficiently large,

$$(\epsilon)_n = \max(|X_n|,|Y_n|,|Z_n|) \sim \left(\frac{1}{4}\right)^n$$

and the function may be approximated adequately by a 5th order power series

$$R_D(x,y,z) = 3 \sum_{m=0}^{n-1} \frac{4^{-m}}{(z_m+(\lambda)_n)\sqrt{z_m}} +$$

$$\frac{4^{-n}}{\sqrt{(\mu)_n^3}}\left[1+\frac{3}{7}S_n^{(2)}+\frac{1}{3}S_n^{(3)}+\frac{3}{22}(S_n^{(2)})^2+\frac{3}{11}S_n^{(4)}+\frac{3}{13}S_n^{(2)}S_n^{(3)}+\frac{3}{13}S_n^{(5)}\right]$$

where

$$S_n^{(m)} = (X_n^m + Y_n^m + 3Z_n^m)/2m.$$

The truncation error in this expansion is bounded by

$$\frac{3(\text{epsilon})^6}{\sqrt{(1-(\text{epsilon})^3)_n}}$$

and the recursive process is terminated when

this quantity is negligible compared with the machine precision.

The routine may fail either because it has been called with arguments outside the domain of definition, or with arguments so extreme that there is an unavoidable danger of setting underflow or overflow.

Note: $R_D(x,x,x) = x^{-3/2}$, so there exists a region of extreme arguments for which the function value is not representable.

4. References

[1] Abramowitz M and Stegun I A (1968) Handbook of Mathematical Functions. Dover Publications.

[2] Carlson B C (1978) Computing Elliptic Integrals by Duplication. (Preprint) Department of Physics, Iowa State University.

[3] Carlson B C (1988) A Table of Elliptic Integrals of the Third Kind. Math. Comput. 51 267--280.

5. Parameters

1: X -- DOUBLE PRECISION Input

2: Y -- DOUBLE PRECISION Input

```
 3:  Z -- DOUBLE PRECISION                                    Input
     On entry: the arguments x, y and z of the function.
     Constraint: X, Y >= 0.0, Z > 0.0 and only one of X and Y may
     be zero.

 4:  IFAIL -- INTEGER                                  Input/Output
     On entry: IFAIL must be set to 0, -1 or 1. For users not
     familiar with this parameter (described in the Essential
     Introduction) the recommended value is 0.

     On exit: IFAIL = 0 unless the routine detects an error (see
     Section 6).

6. Error Indicators and Warnings

Errors detected by the routine:

If on entry IFAIL = 0 or -1, explanatory error messages are
output on the current error message unit (as defined by X04AAF).

IFAIL= 1
     On entry either X or Y is negative, or both X and Y are
     zero; the function is undefined.

IFAIL= 2
     On entry Z <= 0.0; the function is undefined.

IFAIL= 3
     On entry either Z is too close to zero or both X and Y are
     too close to zero: there is a danger of setting overflow.

IFAIL= 4
     On entry at least one of X, Y and Z is too large: there is a
     danger of setting underflow.

     On soft failure the routine returns zero.

7. Accuracy

In principle the routine is capable of producing full machine
precision. However round-off errors in internal arithmetic will
result in slight loss of accuracy. This loss should never be
excessive as the algorithm does not involve any significant
amplification of round-off error. It is reasonable to assume that
the result is accurate to within a small multiple of the machine
precision.
```

8. Further Comments

Users should consult the Chapter Introduction which shows the relationship of this function to the classical definitions of the elliptic integrals.

9. Example

This example program simply generates a small set of non-extreme arguments which are used with the routine to produce the table of low accuracy results.

The example program is not reproduced here. The source code for all example programs is distributed with the NAG Foundation Library software and should be available on-line.

%%%

S21 -- Approximations of Special Functions S21BDF
S21BDF -- NAG Foundation Library Routine Document

Note: Before using this routine, please read the Users' Note for your implementation to check implementation-dependent details. The symbol (*) after a NAG routine name denotes a routine that is not included in the Foundation Library.

1. Purpose

S21BDF returns a value of the symmetrised elliptic integral of the third kind, via the routine name.

2. Specification

```
DOUBLE PRECISION FUNCTION S21BDF (X, Y, Z, R, IFAIL)
INTEGER          IFAIL
DOUBLE PRECISION X, Y, Z, R
```

3. Description

This routine calculates an approximation to the integral

$$R_J(x,y,z,\rho) = \frac{3}{2}\int_0^{\infty} \frac{dt}{\ldots}$$

```
                                  0     (t+(rho))\/(t+x)(t+y)(t+z)
```

where x, y, z>=0, (rho)/=0 and at most one of x, y and z is zero.

If p<0, the result computed is the Cauchy principal value of the integral.

The basic algorithm, which is due to Carlson [2] and [3], is to reduce the arguments recursively towards their mean by the rule:

```
x    =   x
 0

y    =   y
 0

z    =   z
 0

(rho) =  (rho)
     0

(mu) =   (x +y +z +2(rho) )/5
    n      n  n  n       n

X    =   (1-x )/(mu)
 n           n      n

Y    =   (1-y )/(mu)
 n           n      n

Z    =   (1-z )/(mu)
 n           n      n

P    =   (1-(rho) )/(mu)
 n               n      n



(lambda) =   /x y +  /y z +  /z x
          n\/  n n /  n n /  n n

x    =   (x +(lambda) )/4
 n+1       n          n

y    =   (y +(lambda) )/4
```

$$
\begin{aligned}
z_{n+1} &= (z_n + \lambda_n)/4 \\
\rho_{n+1} &= (\rho_n + \lambda_n)/4 \\
\alpha_n &= \left[\rho_n \left(\frac{1}{x_n} + \frac{1}{y_n} + \frac{1}{z_n}\right) + \frac{1}{x_n y_n z_n}\right]^2 \\
\beta_n &= \rho_n (\rho_n + \lambda_n)^2
\end{aligned}
$$

For n sufficiently large,

$$
\varepsilon_n = \max(|X_n|, |Y_n|, |Z_n|, |P_n|) \sim \frac{1}{4^n}
$$

and the function may be approximated by a 5th order power series

$$
R_J(x, y, z, \rho) = 3 \sum_{m=0}^{n-1} 4^{-m} R_C(\alpha_m, \beta_m)
$$

$$
+ \frac{4^{-n}}{\sqrt{\mu_n}}\left[1 + \frac{S_n^{(3)}}{7n} + \frac{S_n^{(3)}}{3n} + \frac{(S_n^{(3)})^2}{22n} + \frac{3S_n^{(4)}}{11n} + \frac{3S_n^{(2)}S_n^{(3)}}{13n} + \frac{3S_n^{(5)}}{n} + \frac{S_n^{(5)}}{13n}\right]
$$

where $S_n^{(m)} = (X_n^m + Y_n^m + Z_n^m + 2P_n^m)/2m$.

The truncation error in this expansion is bounded by

$3\epsilon_n^6/\sqrt{(1-\epsilon_n^3)}$ and the recursion process is terminated when this quantity is negligible compared with the machine precision. The routine may fail either because it has been called with arguments outside the domain of definition or with arguments so extreme that there is an unavoidable danger of setting underflow or overflow.

Note: $R_J(x,x,x,x)=x^{-\frac{3}{2}}$, so there exists a region of extreme arguments for which the function value is not representable.

4. References

[1] Abramowitz M and Stegun I A (1968) Handbook of Mathematical Functions. Dover Publications.

[2] Carlson B C (1978) Computing Elliptic Integrals by Duplication. (Preprint) Department of Physics, Iowa State University.

[3] Carlson B C (1988) A Table of Elliptic Integrals of the Third Kind. Math. Comput. 51 267--280.

5. Parameters

1: X -- DOUBLE PRECISION Input

2: Y -- DOUBLE PRECISION Input

3: Z -- DOUBLE PRECISION Input

4: R -- DOUBLE PRECISION Input
On entry: the arguments x, y, z and (rho) of the function.
Constraint: X, Y, Z >= 0.0, R /= 0.0 and at most one of X, Y and Z may be zero.

5: IFAIL -- INTEGER Input/Output
On entry: IFAIL must be set to 0, -1 or 1. For users not familiar with this parameter (described in the Essential Introduction) the recommended value is 0.

```
     On exit: IFAIL = 0 unless the routine detects an error (see
     Section 6).

6. Error Indicators and Warnings

Errors detected by the routine:

If on entry IFAIL = 0 or -1, explanatory error messages are
output on the current error message unit (as defined by X04AAF).

IFAIL= 1
     On entry at least one of X, Y and Z is negative, or at least
     two of them are zero; the function is undefined.

IFAIL= 2
     On entry R = 0.0; the function is undefined.

IFAIL= 3
     On entry either R is too close to zero, or any two of X, Y
     and Z are too close to zero; there is a danger of setting
     overflow.

IFAIL= 4
     On entry at least one of X, Y, Z and R is too large; there
     is a danger of setting underflow.

IFAIL= 5
     An error has occurred in a call to S21BAF. Any such
     occurrence should be reported to NAG.

     On soft failure, the routine returns zero.

7. Accuracy

In principle the routine is capable of producing full machine
precision. However round-off errors in internal arithmetic will
result in slight loss of accuracy. This loss should never be
excessive as the algorithm does not involve any significant
amplification of round-off error. It is reasonable to assume that
the result is accurate to within a small multiple of the machine
precision.

8. Further Comments

Users should consult the Chapter Introduction which shows the
```

relationship of this function to the classical definitions of the elliptic integrals.

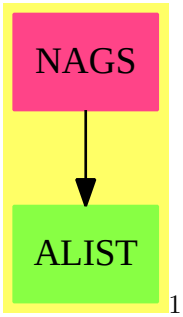
If the argument R is equal to any of the other arguments, the function reduces to the integral R_D, computed by S21BCF.

9. Example

This example program simply generates a small set of non-extreme arguments which are used with the routine to produce the table of low accuracy results.

The example program is not reproduced here. The source code for all example programs is distributed with the NAG Foundation Library software and should be available on-line.

15.30 NagSpecialFunctionsPackage



1.00

Exports:

s01eaf	s13aaf	s13acf	s13adf	s14aaf
s14abf	s14baf	s15adf	s15aef	s17acf
s17adf	s17aef	s17aff	s17agf	s17ahf
s17ajf	s17akf	s17dcf	s17def	s17dgf
s17dhf	s17dlf	s18acf	s18adf	s18aef
s18aff	s18dcf	s18def	s19aaf	s19abf
s19acf	s19adf	s20acf	s20adf	s21baf
s21bbf	s21bcf	s21bdf		

⟨*package NAGS NagSpecialFunctionsPackage*⟩≡

```
)abbrev package NAGS NagSpecialFunctionsPackage
++ Author: Godfrey Nolan and Mike Dewar
++ Date Created: Jan 1994
++ Date Last Updated: Thu May 12 17:45:44 1994
++ Description:
++ This package uses the NAG Library to compute some commonly
++ occurring physical and mathematical functions.
++ See \downlink{Manual Page}{manpageXXs}.
NagSpecialFunctionsPackage(): Exports == Implementation where
  S ==> Symbol
  FOP ==> FortranOutputStackPackage

  Exports ==> with
    s01eaf : (Complex DoubleFloat,Integer) -> Result
     ++ s01eaf(z,ifail)
     ++ S01EAF evaluates the exponential function exp(z) , for complex z.
     ++ See \downlink{Manual Page}{manpageXXs01eaf}.
    s13aaf : (DoubleFloat,Integer) -> Result
     ++ s13aaf(x,ifail)
     ++ returns the value of the exponential integral
     ++  E (x), via the routine name.
     ++   1
     ++ See \downlink{Manual Page}{manpageXXs13aaf}.
```

```
s13acf : (DoubleFloat,Integer) -> Result
 ++ s13acf(x,ifail)
 ++ returns the value of the cosine integral
 ++ See \downlink{Manual Page}{manpageXXs13acf}.
s13adf : (DoubleFloat,Integer) -> Result
 ++ s13adf(x,ifail)
 ++ returns the value of the sine integral
 ++ See \downlink{Manual Page}{manpageXXs13adf}.
s14aaf : (DoubleFloat,Integer) -> Result
 ++ s14aaf(x,ifail) returns the value of the Gamma function (Gamma)(x), via
 ++ the routine name.
 ++ See \downlink{Manual Page}{manpageXXs14aaf}.
s14abf : (DoubleFloat,Integer) -> Result
 ++ s14abf(x,ifail) returns a value for the log, ln(Gamma(x)), via
 ++ the routine name.
 ++ See \downlink{Manual Page}{manpageXXs14abf}.
s14baf : (DoubleFloat,DoubleFloat,DoubleFloat,Integer) -> Result
 ++ s14baf(a,x,tol,ifail)
 ++ computes values for the incomplete gamma functions P(a,x)
 ++ and Q(a,x).
 ++ See \downlink{Manual Page}{manpageXXs14baf}.
s15adf : (DoubleFloat,Integer) -> Result
 ++ s15adf(x,ifail)
 ++ returns the value of the complementary error function,
 ++ erfc(x), via the routine name.
 ++ See \downlink{Manual Page}{manpageXXs15adf}.
s15aef : (DoubleFloat,Integer) -> Result
 ++ s15aef(x,ifail)
 ++ returns the value of the error function erf(x), via the
 ++ routine name.
 ++ See \downlink{Manual Page}{manpageXXs15aef}.
s17acf : (DoubleFloat,Integer) -> Result
 ++ s17acf(x,ifail)
 ++ returns the value of the Bessel Function
 ++  Y (x), via the routine name.
 ++   0
 ++ See \downlink{Manual Page}{manpageXXs17acf}.
s17adf : (DoubleFloat,Integer) -> Result
 ++ s17adf(x,ifail)
 ++ returns the value of the Bessel Function
 ++  Y (x), via the routine name.
 ++   1
 ++ See \downlink{Manual Page}{manpageXXs17adf}.
s17aef : (DoubleFloat,Integer) -> Result
 ++ s17aef(x,ifail)
 ++ returns the value of the Bessel Function
```

```
 ++  J (x), via the routine name.
 ++   0
 ++ See \downlink{Manual Page}{manpageXXs17aef}.
s17aff : (DoubleFloat,Integer) -> Result
 ++ s17aff(x,ifail)
 ++ returns the value of the Bessel Function
 ++  J (x), via the routine name.
 ++   1
 ++ See \downlink{Manual Page}{manpageXXs17aff}.
s17agf : (DoubleFloat,Integer) -> Result
 ++ s17agf(x,ifail)
 ++ returns a value for the Airy function, Ai(x), via the
 ++ routine name.
 ++ See \downlink{Manual Page}{manpageXXs17agf}.
s17ahf : (DoubleFloat,Integer) -> Result
 ++ s17ahf(x,ifail)
 ++ returns a value of the Airy function, Bi(x), via the
 ++ routine name.
 ++ See \downlink{Manual Page}{manpageXXs17ahf}.
s17ajf : (DoubleFloat,Integer) -> Result
 ++ s17ajf(x,ifail)
 ++ returns a value of the derivative of the Airy function
 ++ Ai(x), via the routine name.
 ++ See \downlink{Manual Page}{manpageXXs17ajf}.
s17akf : (DoubleFloat,Integer) -> Result
 ++ s17akf(x,ifail)
 ++ returns a value for the derivative of the Airy function
 ++ Bi(x), via the routine name.
 ++ See \downlink{Manual Page}{manpageXXs17akf}.
s17dcf : (DoubleFloat,Complex DoubleFloat,Integer,String,_
    Integer) -> Result
 ++ s17dcf(fnu,z,n,scale,ifail)
 ++ returns a sequence of values for the Bessel functions
 ++  Y      (z) for complex z, non-negative (nu) and n=0,1,...,N-1,
 ++   (nu)+n
 ++ with an option for exponential scaling.
 ++ See \downlink{Manual Page}{manpageXXs17dcf}.
s17def : (DoubleFloat,Complex DoubleFloat,Integer,String,_
    Integer) -> Result
 ++ s17def(fnu,z,n,scale,ifail)
 ++ returns a sequence of values for the Bessel functions
 ++  J      (z) for complex z, non-negative (nu) and n=0,1,...,N-1,
 ++   (nu)+n
 ++ with an option for exponential scaling.
 ++ See \downlink{Manual Page}{manpageXXs17def}.
s17dgf : (String,Complex DoubleFloat,String,Integer) -> Result
```

```
 ++ s17dgf(deriv,z,scale,ifail)
 ++ returns the value of the Airy function Ai(z) or its
 ++ derivative Ai'(z) for complex z, with an option for exponential
 ++ scaling.
 ++ See \downlink{Manual Page}{manpageXXs17dgf}.
s17dhf : (String,Complex DoubleFloat,String,Integer) -> Result
 ++ s17dhf(deriv,z,scale,ifail)
 ++ returns the value of the Airy function Bi(z) or its
 ++ derivative Bi'(z) for complex z, with an option for exponential
 ++ scaling.
 ++ See \downlink{Manual Page}{manpageXXs17dhf}.
s17dlf : (Integer,DoubleFloat,Complex DoubleFloat,Integer,_
    String,Integer) -> Result
 ++ s17dlf(m,fnu,z,n,scale,ifail)
 ++ returns a sequence of values for the Hankel functions
 ++   (1)           (2)
 ++  H      (z) or H      (z) for complex z, non-negative (nu) and
 ++   (nu)+n        (nu)+n
 ++ n=0,1,...,N-1, with an option for exponential scaling.
 ++ See \downlink{Manual Page}{manpageXXs17dlf}.
s18acf : (DoubleFloat,Integer) -> Result
 ++ s18acf(x,ifail)
 ++ returns the value of the modified Bessel Function
 ++  K (x), via the routine name.
 ++   0
 ++ See \downlink{Manual Page}{manpageXXs18acf}.
s18adf : (DoubleFloat,Integer) -> Result
 ++ s18adf(x,ifail)
 ++ returns the value of the modified Bessel Function
 ++  K (x), via the routine name.
 ++   1
 ++ See \downlink{Manual Page}{manpageXXs18adf}.
s18aef : (DoubleFloat,Integer) -> Result
 ++ s18aef(x,ifail)
 ++ returns the value of the modified Bessel Function
 ++  I (x), via the routine name.
 ++   0
 ++ See \downlink{Manual Page}{manpageXXs18aef}.
s18aff : (DoubleFloat,Integer) -> Result
 ++ s18aff(x,ifail)
 ++ returns a value for the modified Bessel Function
 ++  I (x), via the routine name.
 ++   1
 ++ See \downlink{Manual Page}{manpageXXs18aff}.
s18dcf : (DoubleFloat,Complex DoubleFloat,Integer,String,_
    Integer) -> Result
```

```
 ++ s18dcf(fnu,z,n,scale,ifail)
 ++ returns a sequence of values for the modified Bessel functions
 ++  K      (z) for complex z, non-negative (nu) and
 ++   (nu)+n
 ++ n=0,1,...,N-1, with an option for exponential scaling.
 ++ See \downlink{Manual Page}{manpageXXs18dcf}.
s18def : (DoubleFloat,Complex DoubleFloat,Integer,String,_
    Integer) -> Result
 ++ s18def(fnu,z,n,scale,ifail)
 ++ returns a sequence of values for the modified Bessel functions
 ++  I      (z) for complex z, non-negative (nu) and
 ++   (nu)+n
 ++ n=0,1,...,N-1, with an option for exponential scaling.
 ++ See \downlink{Manual Page}{manpageXXs18def}.
s19aaf : (DoubleFloat,Integer) -> Result
 ++ s19aaf(x,ifail)
 ++ returns a value for the Kelvin function ber(x) via the
 ++ routine name.
 ++ See \downlink{Manual Page}{manpageXXs19aaf}.
s19abf : (DoubleFloat,Integer) -> Result
 ++ s19abf(x,ifail)
 ++ returns a value for the Kelvin function bei(x) via the
 ++ routine name.
 ++ See \downlink{Manual Page}{manpageXXs19abf}.
s19acf : (DoubleFloat,Integer) -> Result
 ++ s19acf(x,ifail)
 ++ returns a value for the Kelvin function ker(x), via the
 ++ routine name.
 ++ See \downlink{Manual Page}{manpageXXs19acf}.
s19adf : (DoubleFloat,Integer) -> Result
 ++ s19adf(x,ifail)
 ++ returns a value for the Kelvin function kei(x) via the
 ++ routine name.
 ++ See \downlink{Manual Page}{manpageXXs19adf}.
s20acf : (DoubleFloat,Integer) -> Result
 ++ s20acf(x,ifail)
 ++ returns a value for the Fresnel Integral S(x), via the
 ++ routine name.
 ++ See \downlink{Manual Page}{manpageXXs20acf}.
s20adf : (DoubleFloat,Integer) -> Result
 ++ s20adf(x,ifail)
 ++ returns a value for the Fresnel Integral C(x), via the
 ++ routine name.
 ++ See \downlink{Manual Page}{manpageXXs20adf}.
s21baf : (DoubleFloat,DoubleFloat,Integer) -> Result
 ++ s21baf(x,y,ifail)
```

```
    ++ returns a value of an elementary integral, which occurs as
    ++ a degenerate case of an elliptic integral of the first kind, via
    ++ the routine name.
    ++ See \downlink{Manual Page}{manpageXXs21baf}.
   s21bbf : (DoubleFloat,DoubleFloat,DoubleFloat,Integer) -> Result
    ++ s21bbf(x,y,z,ifail)
    ++ returns a value of the symmetrised elliptic integral of
    ++ the first kind, via the routine name.
    ++ See \downlink{Manual Page}{manpageXXs21bbf}.
   s21bcf : (DoubleFloat,DoubleFloat,DoubleFloat,Integer) -> Result
    ++ s21bcf(x,y,z,ifail)
    ++ returns a value of the symmetrised elliptic integral of
    ++ the second kind, via the routine name.
    ++ See \downlink{Manual Page}{manpageXXs21bcf}.
   s21bdf : (DoubleFloat,DoubleFloat,DoubleFloat,DoubleFloat,_
       Integer) -> Result
    ++ s21bdf(x,y,z,r,ifail)
    ++ returns a value of the symmetrised elliptic integral of
    ++ the third kind, via the routine name.
    ++ See \downlink{Manual Page}{manpageXXs21bdf}.
 Implementation ==> add

   import Lisp
   import DoubleFloat
   import Any
   import Record
   import Integer
   import Matrix DoubleFloat
   import Boolean
   import NAGLinkSupportPackage
   import AnyFunctions1(Complex DoubleFloat)
   import AnyFunctions1(Integer)
   import AnyFunctions1(DoubleFloat)
   import AnyFunctions1(String)


   s01eaf(zArg:Complex DoubleFloat,ifailArg:Integer): Result ==
       [(invokeNagman(NIL$Lisp,_
       "s01eaf",_
       ["z"::S,"ifail"::S]$Lisp,_
       []$Lisp,_
       [["integer"::S,"ifail"::S]$Lisp_
       ,["double complex"::S,"s01eafResult"::S,"z"::S]$Lisp_
       ]$Lisp,_
       ["s01eafResult"::S,"ifail"::S]$Lisp,_
       [([zArg::Any,ifailArg::Any ])_
```

```
    @List Any]$Lisp)$Lisp)_
    pretend List (Record(key:Symbol,entry:Any))]$Result

s13aaf(xArg:DoubleFloat,ifailArg:Integer): Result ==
    [(invokeNagman(NIL$Lisp,_
    "s13aaf",_
    ["x"::S,"ifail"::S]$Lisp,_
    []$Lisp,_
    [["double"::S,"s13aafResult"::S,"x"::S]$Lisp_
    ,["integer"::S,"ifail"::S]$Lisp_
    ]$Lisp,_
    ["s13aafResult"::S,"ifail"::S]$Lisp,_
    [([xArg::Any,ifailArg::Any ])_
    @List Any]$Lisp)$Lisp)_
    pretend List (Record(key:Symbol,entry:Any))]$Result

s13acf(xArg:DoubleFloat,ifailArg:Integer): Result ==
    [(invokeNagman(NIL$Lisp,_
    "s13acf",_
    ["x"::S,"ifail"::S]$Lisp,_
    []$Lisp,_
    [["double"::S,"s13acfResult"::S,"x"::S]$Lisp_
    ,["integer"::S,"ifail"::S]$Lisp_
    ]$Lisp,_
    ["s13acfResult"::S,"ifail"::S]$Lisp,_
    [([xArg::Any,ifailArg::Any ])_
    @List Any]$Lisp)$Lisp)_
    pretend List (Record(key:Symbol,entry:Any))]$Result

s13adf(xArg:DoubleFloat,ifailArg:Integer): Result ==
    [(invokeNagman(NIL$Lisp,_
    "s13adf",_
    ["x"::S,"ifail"::S]$Lisp,_
    []$Lisp,_
    [["double"::S,"s13adfResult"::S,"x"::S]$Lisp_
    ,["integer"::S,"ifail"::S]$Lisp_
    ]$Lisp,_
    ["s13adfResult"::S,"ifail"::S]$Lisp,_
    [([xArg::Any,ifailArg::Any ])_
    @List Any]$Lisp)$Lisp)_
    pretend List (Record(key:Symbol,entry:Any))]$Result

s14aaf(xArg:DoubleFloat,ifailArg:Integer): Result ==
    [(invokeNagman(NIL$Lisp,_
    "s14aaf",_
    ["x"::S,"ifail"::S]$Lisp,_
```

```
        []$Lisp,_
        [["double"::S,"s14aafResult"::S,"x"::S]$Lisp_
        ,["integer"::S,"ifail"::S]$Lisp_
        ]$Lisp,_
        ["s14aafResult"::S,"ifail"::S]$Lisp,_
        [([xArg::Any,ifailArg::Any ])_
        @List Any]$Lisp)$Lisp)_
        pretend List (Record(key:Symbol,entry:Any))]$Result

    s14abf(xArg:DoubleFloat,ifailArg:Integer): Result ==
        [(invokeNagman(NIL$Lisp,_
        "s14abf",_
        ["x"::S,"ifail"::S]$Lisp,_
        []$Lisp,_
        [["double"::S,"s14abfResult"::S,"x"::S]$Lisp_
        ,["integer"::S,"ifail"::S]$Lisp_
        ]$Lisp,_
        ["s14abfResult"::S,"ifail"::S]$Lisp,_
        [([xArg::Any,ifailArg::Any ])_
        @List Any]$Lisp)$Lisp)_
        pretend List (Record(key:Symbol,entry:Any))]$Result

    s14baf(aArg:DoubleFloat,xArg:DoubleFloat,tolArg:DoubleFloat,_
        ifailArg:Integer): Result ==
        [(invokeNagman(NIL$Lisp,_
        "s14baf",_
        ["a"::S,"x"::S,"tol"::S,"p"::S,"q"::S_
        ,"ifail"::S]$Lisp,_
        ["p"::S,"q"::S]$Lisp,_
        [["double"::S,"a"::S,"x"::S,"tol"::S,"p"::S_
        ,"q"::S]$Lisp_
        ,["integer"::S,"ifail"::S]$Lisp_
        ]$Lisp,_
        ["p"::S,"q"::S,"ifail"::S]$Lisp,_
        [([aArg::Any,xArg::Any,tolArg::Any,ifailArg::Any ])_
        @List Any]$Lisp)$Lisp)_
        pretend List (Record(key:Symbol,entry:Any))]$Result

    s15adf(xArg:DoubleFloat,ifailArg:Integer): Result ==
        [(invokeNagman(NIL$Lisp,_
        "s15adf",_
        ["x"::S,"ifail"::S]$Lisp,_
        []$Lisp,_
        [["double"::S,"s15adfResult"::S,"x"::S]$Lisp_
        ,["integer"::S,"ifail"::S]$Lisp_
        ]$Lisp,_
```

```
        ["s15adfResult"::S,"ifail"::S]$Lisp,_
        [([xArg::Any,ifailArg::Any ])_
        @List Any]$Lisp)$Lisp)_
        pretend List (Record(key:Symbol,entry:Any))]$Result

    s15aef(xArg:DoubleFloat,ifailArg:Integer): Result ==
        [(invokeNagman(NIL$Lisp,_
        "s15aef",_
        ["x"::S,"ifail"::S]$Lisp,_
        []$Lisp,_
        [["double"::S,"s15aefResult"::S,"x"::S]$Lisp_
        ,["integer"::S,"ifail"::S]$Lisp_
        ]$Lisp,_
        ["s15aefResult"::S,"ifail"::S]$Lisp,_
        [([xArg::Any,ifailArg::Any ])_
        @List Any]$Lisp)$Lisp)_
        pretend List (Record(key:Symbol,entry:Any))]$Result

    s17acf(xArg:DoubleFloat,ifailArg:Integer): Result ==
        [(invokeNagman(NIL$Lisp,_
        "s17acf",_
        ["x"::S,"ifail"::S]$Lisp,_
        []$Lisp,_
        [["double"::S,"s17acfResult"::S,"x"::S]$Lisp_
        ,["integer"::S,"ifail"::S]$Lisp_
        ]$Lisp,_
        ["s17acfResult"::S,"ifail"::S]$Lisp,_
        [([xArg::Any,ifailArg::Any ])_
        @List Any]$Lisp)$Lisp)_
        pretend List (Record(key:Symbol,entry:Any))]$Result

    s17adf(xArg:DoubleFloat,ifailArg:Integer): Result ==
        [(invokeNagman(NIL$Lisp,_
        "s17adf",_
        ["x"::S,"ifail"::S]$Lisp,_
        []$Lisp,_
        [["double"::S,"s17adfResult"::S,"x"::S]$Lisp_
        ,["integer"::S,"ifail"::S]$Lisp_
        ]$Lisp,_
        ["s17adfResult"::S,"ifail"::S]$Lisp,_
        [([xArg::Any,ifailArg::Any ])_
        @List Any]$Lisp)$Lisp)_
        pretend List (Record(key:Symbol,entry:Any))]$Result

    s17aef(xArg:DoubleFloat,ifailArg:Integer): Result ==
        [(invokeNagman(NIL$Lisp,_
```

```
        "s17aef",_
        ["x"::S,"ifail"::S]$Lisp,_
        []$Lisp,_
        [["double"::S,"s17aefResult"::S,"x"::S]$Lisp_
        ,["integer"::S,"ifail"::S]$Lisp_
        ]$Lisp,_
        ["s17aefResult"::S,"ifail"::S]$Lisp,_
        [([xArg::Any,ifailArg::Any ])_
        @List Any]$Lisp)$Lisp)_
        pretend List (Record(key:Symbol,entry:Any))]$Result

    s17aff(xArg:DoubleFloat,ifailArg:Integer): Result ==
        [(invokeNagman(NIL$Lisp,_
        "s17aff",_
        ["x"::S,"ifail"::S]$Lisp,_
        []$Lisp,_
        [["double"::S,"s17affResult"::S,"x"::S]$Lisp_
        ,["integer"::S,"ifail"::S]$Lisp_
        ]$Lisp,_
        ["s17affResult"::S,"ifail"::S]$Lisp,_
        [([xArg::Any,ifailArg::Any ])_
        @List Any]$Lisp)$Lisp)_
        pretend List (Record(key:Symbol,entry:Any))]$Result

    s17agf(xArg:DoubleFloat,ifailArg:Integer): Result ==
        [(invokeNagman(NIL$Lisp,_
        "s17agf",_
        ["x"::S,"ifail"::S]$Lisp,_
        []$Lisp,_
        [["double"::S,"s17agfResult"::S,"x"::S]$Lisp_
        ,["integer"::S,"ifail"::S]$Lisp_
        ]$Lisp,_
        ["s17agfResult"::S,"ifail"::S]$Lisp,_
        [([xArg::Any,ifailArg::Any ])_
        @List Any]$Lisp)$Lisp)_
        pretend List (Record(key:Symbol,entry:Any))]$Result

    s17ahf(xArg:DoubleFloat,ifailArg:Integer): Result ==
        [(invokeNagman(NIL$Lisp,_
        "s17ahf",_
        ["x"::S,"ifail"::S]$Lisp,_
        []$Lisp,_
        [["double"::S,"s17ahfResult"::S,"x"::S]$Lisp_
        ,["integer"::S,"ifail"::S]$Lisp_
        ]$Lisp,_
        ["s17ahfResult"::S,"ifail"::S]$Lisp,_
```

```
        [([xArg::Any,ifailArg::Any ])_
        @List Any]$Lisp)$Lisp)_
        pretend List (Record(key:Symbol,entry:Any))]$Result

    s17ajf(xArg:DoubleFloat,ifailArg:Integer): Result ==
        [(invokeNagman(NIL$Lisp,_
        "s17ajf",_
        ["x"::S,"ifail"::S]$Lisp,_
        []$Lisp,_
        [["double"::S,"s17ajfResult"::S,"x"::S]$Lisp_
        ,["integer"::S,"ifail"::S]$Lisp_
        ]$Lisp,_
        ["s17ajfResult"::S,"ifail"::S]$Lisp,_
        [([xArg::Any,ifailArg::Any ])_
        @List Any]$Lisp)$Lisp)_
        pretend List (Record(key:Symbol,entry:Any))]$Result

    s17akf(xArg:DoubleFloat,ifailArg:Integer): Result ==
        [(invokeNagman(NIL$Lisp,_
        "s17akf",_
        ["x"::S,"ifail"::S]$Lisp,_
        []$Lisp,_
        [["double"::S,"s17akfResult"::S,"x"::S]$Lisp_
        ,["integer"::S,"ifail"::S]$Lisp_
        ]$Lisp,_
        ["s17akfResult"::S,"ifail"::S]$Lisp,_
        [([xArg::Any,ifailArg::Any ])_
        @List Any]$Lisp)$Lisp)_
        pretend List (Record(key:Symbol,entry:Any))]$Result


    s17dcf(fnuArg:DoubleFloat,zArg:Complex DoubleFloat,nArg:Integer,_
        scaleArg:String,ifailArg:Integer): Result ==
        [(invokeNagman(NIL$Lisp,_
        "s17dcf",_
        ["fnu"::S,"z"::S,"n"::S,"scale"::S,"nz"::S_
        ,"ifail"::S,"cy"::S,"cwrk"::S]$Lisp,_
        ["cy"::S,"nz"::S,"cwrk"::S]$Lisp,_
        [["double"::S,"fnu"::S]$Lisp_
        ,["integer"::S,"n"::S,"nz"::S,"ifail"::S]$Lisp_
        ,["character"::S,"scale"::S]$Lisp_
        ,["double complex"::S,"z"::S,["cy"::S,"n"::S]$Lisp,["cwrk"::S,"n"::S]$Lisp]$Lisp_
        ]$Lisp,_
        ["cy"::S,"nz"::S,"ifail"::S]$Lisp,_
        [([fnuArg::Any,zArg::Any,nArg::Any,scaleArg::Any,ifailArg::Any ])_
        @List Any]$Lisp)$Lisp)_
```

```
    pretend List (Record(key:Symbol,entry:Any))]$Result

s17def(fnuArg:DoubleFloat,zArg:Complex DoubleFloat,nArg:Integer,_
    scaleArg:String,ifailArg:Integer): Result ==
    [(invokeNagman(NIL$Lisp,_
    "s17def",_
    ["fnu"::S,"z"::S,"n"::S,"scale"::S,"nz"::S_
    ,"ifail"::S,"cy"::S]$Lisp,_
    ["cy"::S,"nz"::S]$Lisp,_
    [["double"::S,"fnu"::S]$Lisp_
    ,["integer"::S,"n"::S,"nz"::S,"ifail"::S]$Lisp_
    ,["character"::S,"scale"::S]$Lisp_
    ,["double complex"::S,"z"::S,["cy"::S,"n"::S]$Lisp]$Lisp_
    ]$Lisp,_
    ["cy"::S,"nz"::S,"ifail"::S]$Lisp,_
    [([fnuArg::Any,zArg::Any,nArg::Any,scaleArg::Any,ifailArg::Any ])_
    @List Any]$Lisp)$Lisp)_
    pretend List (Record(key:Symbol,entry:Any))]$Result

s17dgf(derivArg:String,zArg:Complex DoubleFloat,scaleArg:String,_
    ifailArg:Integer): Result ==
    [(invokeNagman(NIL$Lisp,_
    "s17dgf",_
    ["deriv"::S,"z"::S,"scale"::S,"ai"::S,"nz"::S_
    ,"ifail"::S]$Lisp,_
    ["ai"::S,"nz"::S]$Lisp,_
    [["integer"::S,"nz"::S,"ifail"::S]$Lisp_
    ,["character"::S,"deriv"::S,"scale"::S]$Lisp_
    ,["double complex"::S,"z"::S,"ai"::S]$Lisp_
    ]$Lisp,_
    ["ai"::S,"nz"::S,"ifail"::S]$Lisp,_
    [([derivArg::Any,zArg::Any,scaleArg::Any,ifailArg::Any ])_
    @List Any]$Lisp)$Lisp)_
    pretend List (Record(key:Symbol,entry:Any))]$Result

s17dhf(derivArg:String,zArg:Complex DoubleFloat,scaleArg:String,_
    ifailArg:Integer): Result ==
    [(invokeNagman(NIL$Lisp,_
    "s17dhf",_
    ["deriv"::S,"z"::S,"scale"::S,"bi"::S,"ifail"::S_
    ]$Lisp,_
    ["bi"::S]$Lisp,_
    [["integer"::S,"ifail"::S]$Lisp_
    ,["character"::S,"deriv"::S,"scale"::S]$Lisp_
    ,["double complex"::S,"z"::S,"bi"::S]$Lisp_
    ]$Lisp,_
```

```
        ["bi"::S,"ifail"::S]$Lisp,_
        [([derivArg::Any,zArg::Any,scaleArg::Any,ifailArg::Any ])_
        @List Any]$Lisp)$Lisp)_
        pretend List (Record(key:Symbol,entry:Any))]$Result

    s17dlf(mArg:Integer,fnuArg:DoubleFloat,zArg:Complex DoubleFloat,_
        nArg:Integer,scaleArg:String,ifailArg:Integer): Result ==
        [(invokeNagman(NIL$Lisp,_
        "s17dlf",_
        ["m"::S,"fnu"::S,"z"::S,"n"::S,"scale"::S_
        ,"nz"::S,"ifail"::S,"cy"::S]$Lisp,_
        ["cy"::S,"nz"::S]$Lisp,_
        [["double"::S,"fnu"::S]$Lisp_
        ,["integer"::S,"m"::S,"n"::S,"nz"::S,"ifail"::S_
        ]$Lisp_
        ,["character"::S,"scale"::S]$Lisp_
        ,["double complex"::S,"z"::S,["cy"::S,"n"::S]$Lisp]$Lisp_
        ]$Lisp,_
        ["cy"::S,"nz"::S,"ifail"::S]$Lisp,_
        [([mArg::Any,fnuArg::Any,zArg::Any,nArg::Any,scaleArg::Any,ifailArg::Any ])_
        @List Any]$Lisp)$Lisp)_
        pretend List (Record(key:Symbol,entry:Any))]$Result

    s18acf(xArg:DoubleFloat,ifailArg:Integer): Result ==
        [(invokeNagman(NIL$Lisp,_
        "s18acf",_
        ["x"::S,"ifail"::S]$Lisp,_
        []$Lisp,_
        [["double"::S,"s18acfResult"::S,"x"::S]$Lisp_
        ,["integer"::S,"ifail"::S]$Lisp_
        ]$Lisp,_
        ["s18acfResult"::S,"ifail"::S]$Lisp,_
        [([xArg::Any,ifailArg::Any ])_
        @List Any]$Lisp)$Lisp)_
        pretend List (Record(key:Symbol,entry:Any))]$Result

    s18adf(xArg:DoubleFloat,ifailArg:Integer): Result ==
        [(invokeNagman(NIL$Lisp,_
        "s18adf",_
        ["x"::S,"ifail"::S]$Lisp,_
        []$Lisp,_
        [["double"::S,"s18adfResult"::S,"x"::S]$Lisp_
        ,["integer"::S,"ifail"::S]$Lisp_
        ]$Lisp,_
        ["s18adfResult"::S,"ifail"::S]$Lisp,_
        [([xArg::Any,ifailArg::Any ])_
```

```
        @List Any]$Lisp)$Lisp)_
        pretend List (Record(key:Symbol,entry:Any))]$Result

    s18aef(xArg:DoubleFloat,ifailArg:Integer): Result ==
        [(invokeNagman(NIL$Lisp,_
        "s18aef",_
        ["x"::S,"ifail"::S]$Lisp,_
        []$Lisp,_
        [["double"::S,"s18aefResult"::S,"x"::S]$Lisp_
        ,["integer"::S,"ifail"::S]$Lisp_
        ]$Lisp,_
        ["s18aefResult"::S,"ifail"::S]$Lisp,_
        [([xArg::Any,ifailArg::Any ])_
        @List Any]$Lisp)$Lisp)_
        pretend List (Record(key:Symbol,entry:Any))]$Result

    s18aff(xArg:DoubleFloat,ifailArg:Integer): Result ==
        [(invokeNagman(NIL$Lisp,_
        "s18aff",_
        ["x"::S,"ifail"::S]$Lisp,_
        []$Lisp,_
        [["double"::S,"s18affResult"::S,"x"::S]$Lisp_
        ,["integer"::S,"ifail"::S]$Lisp_
        ]$Lisp,_
        ["s18affResult"::S,"ifail"::S]$Lisp,_
        [([xArg::Any,ifailArg::Any ])_
        @List Any]$Lisp)$Lisp)_
        pretend List (Record(key:Symbol,entry:Any))]$Result

    s18dcf(fnuArg:DoubleFloat,zArg:Complex DoubleFloat,nArg:Integer,_
        scaleArg:String,ifailArg:Integer): Result ==
        [(invokeNagman(NIL$Lisp,_
        "s18dcf",_
        ["fnu"::S,"z"::S,"n"::S,"scale"::S,"nz"::S_
        ,"ifail"::S,"cy"::S]$Lisp,_
        ["cy"::S,"nz"::S]$Lisp,_
        [["double"::S,"fnu"::S]$Lisp_
        ,["integer"::S,"n"::S,"nz"::S,"ifail"::S]$Lisp_
        ,["character"::S,"scale"::S]$Lisp_
        ,["double complex"::S,"z"::S,["cy"::S,"n"::S]$Lisp]$Lisp_
        ]$Lisp,_
        ["cy"::S,"nz"::S,"ifail"::S]$Lisp,_
        [([fnuArg::Any,zArg::Any,nArg::Any,scaleArg::Any,ifailArg::Any ])_
        @List Any]$Lisp)$Lisp)_
        pretend List (Record(key:Symbol,entry:Any))]$Result
```

```
    s18def(fnuArg:DoubleFloat,zArg:Complex DoubleFloat,nArg:Integer,_
        scaleArg:String,ifailArg:Integer): Result ==
        [(invokeNagman(NIL$Lisp,_
        "s18def",_
        ["fnu"::S,"z"::S,"n"::S,"scale"::S,"nz"::S_
        ,"ifail"::S,"cy"::S]$Lisp,_
        ["cy"::S,"nz"::S]$Lisp,_
        [["double"::S,"fnu"::S]$Lisp_
        ,["integer"::S,"n"::S,"nz"::S,"ifail"::S]$Lisp_
        ,["character"::S,"scale"::S]$Lisp_
        ,["double complex"::S,"z"::S,["cy"::S,"n"::S]$Lisp]$Lisp_
        ]$Lisp,_
        ["cy"::S,"nz"::S,"ifail"::S]$Lisp,_
        [([fnuArg::Any,zArg::Any,nArg::Any,scaleArg::Any,ifailArg::Any ])_
        @List Any]$Lisp)$Lisp)_
        pretend List (Record(key:Symbol,entry:Any))]$Result

    s19aaf(xArg:DoubleFloat,ifailArg:Integer): Result ==
        [(invokeNagman(NIL$Lisp,_
        "s19aaf",_
        ["x"::S,"ifail"::S]$Lisp,_
        []$Lisp,_
        [["double"::S,"s19aafResult"::S,"x"::S]$Lisp_
        ,["integer"::S,"ifail"::S]$Lisp_
        ]$Lisp,_
        ["s19aafResult"::S,"ifail"::S]$Lisp,_
        [([xArg::Any,ifailArg::Any ])_
        @List Any]$Lisp)$Lisp)_
        pretend List (Record(key:Symbol,entry:Any))]$Result

    s19abf(xArg:DoubleFloat,ifailArg:Integer): Result ==
        [(invokeNagman(NIL$Lisp,_
        "s19abf",_
        ["x"::S,"ifail"::S]$Lisp,_
        []$Lisp,_
        [["double"::S,"s19abfResult"::S,"x"::S]$Lisp_
        ,["integer"::S,"ifail"::S]$Lisp_
        ]$Lisp,_
        ["s19abfResult"::S,"ifail"::S]$Lisp,_
        [([xArg::Any,ifailArg::Any ])_
        @List Any]$Lisp)$Lisp)_
        pretend List (Record(key:Symbol,entry:Any))]$Result

    s19acf(xArg:DoubleFloat,ifailArg:Integer): Result ==
        [(invokeNagman(NIL$Lisp,_
        "s19acf",_
```

```
        ["x"::S,"ifail"::S]$Lisp,_
        []$Lisp,_
        [["double"::S,"s19acfResult"::S,"x"::S]$Lisp_
        ,["integer"::S,"ifail"::S]$Lisp_
        ]$Lisp,_
        ["s19acfResult"::S,"ifail"::S]$Lisp,_
        [([xArg::Any,ifailArg::Any ])_
        @List Any]$Lisp)$Lisp)_
        pretend List (Record(key:Symbol,entry:Any))]$Result

    s19adf(xArg:DoubleFloat,ifailArg:Integer): Result ==
        [(invokeNagman(NIL$Lisp,_
        "s19adf",_
        ["x"::S,"ifail"::S]$Lisp,_
        []$Lisp,_
        [["double"::S,"s19adfResult"::S,"x"::S]$Lisp_
        ,["integer"::S,"ifail"::S]$Lisp_
        ]$Lisp,_
        ["s19adfResult"::S,"ifail"::S]$Lisp,_
        [([xArg::Any,ifailArg::Any ])_
        @List Any]$Lisp)$Lisp)_
        pretend List (Record(key:Symbol,entry:Any))]$Result

    s20acf(xArg:DoubleFloat,ifailArg:Integer): Result ==
        [(invokeNagman(NIL$Lisp,_
        "s20acf",_
        ["x"::S,"ifail"::S]$Lisp,_
        []$Lisp,_
        [["double"::S,"s20acfResult"::S,"x"::S]$Lisp_
        ,["integer"::S,"ifail"::S]$Lisp_
        ]$Lisp,_
        ["s20acfResult"::S,"ifail"::S]$Lisp,_
        [([xArg::Any,ifailArg::Any ])_
        @List Any]$Lisp)$Lisp)_
        pretend List (Record(key:Symbol,entry:Any))]$Result

    s20adf(xArg:DoubleFloat,ifailArg:Integer): Result ==
        [(invokeNagman(NIL$Lisp,_
        "s20adf",_
        ["x"::S,"ifail"::S]$Lisp,_
        []$Lisp,_
        [["double"::S,"s20adfResult"::S,"x"::S]$Lisp_
        ,["integer"::S,"ifail"::S]$Lisp_
        ]$Lisp,_
        ["s20adfResult"::S,"ifail"::S]$Lisp,_
        [([xArg::Any,ifailArg::Any ])_
```

```
    @List Any]$Lisp)$Lisp)_
    pretend List (Record(key:Symbol,entry:Any))]$Result

s21baf(xArg:DoubleFloat,yArg:DoubleFloat,ifailArg:Integer): Result ==
    [(invokeNagman(NIL$Lisp,_
    "s21baf",_
    ["x"::S,"y"::S,"ifail"::S]$Lisp,_
    []$Lisp,_
    [["double"::S,"s21bafResult"::S,"x"::S,"y"::S_
    ]$Lisp_
    ,["integer"::S,"ifail"::S]$Lisp_
    ]$Lisp,_
    ["s21bafResult"::S,"ifail"::S]$Lisp,_
    [([xArg::Any,yArg::Any,ifailArg::Any ])_
    @List Any]$Lisp)$Lisp)_
    pretend List (Record(key:Symbol,entry:Any))]$Result

s21bbf(xArg:DoubleFloat,yArg:DoubleFloat,zArg:DoubleFloat,_
    ifailArg:Integer): Result ==
    [(invokeNagman(NIL$Lisp,_
    "s21bbf",_
    ["x"::S,"y"::S,"z"::S,"ifail"::S]$Lisp,_
    []$Lisp,_
    [["double"::S,"s21bbfResult"::S,"x"::S,"y"::S_
    ,"z"::S]$Lisp_
    ,["integer"::S,"ifail"::S]$Lisp_
    ]$Lisp,_
    ["s21bbfResult"::S,"ifail"::S]$Lisp,_
    [([xArg::Any,yArg::Any,zArg::Any,ifailArg::Any ])_
    @List Any]$Lisp)$Lisp)_
    pretend List (Record(key:Symbol,entry:Any))]$Result

s21bcf(xArg:DoubleFloat,yArg:DoubleFloat,zArg:DoubleFloat,_
    ifailArg:Integer): Result ==
    [(invokeNagman(NIL$Lisp,_
    "s21bcf",_
    ["x"::S,"y"::S,"z"::S,"ifail"::S]$Lisp,_
    []$Lisp,_
    [["double"::S,"s21bcfResult"::S,"x"::S,"y"::S_
    ,"z"::S]$Lisp_
    ,["integer"::S,"ifail"::S]$Lisp_
    ]$Lisp,_
    ["s21bcfResult"::S,"ifail"::S]$Lisp,_
    [([xArg::Any,yArg::Any,zArg::Any,ifailArg::Any ])_
    @List Any]$Lisp)$Lisp)_
    pretend List (Record(key:Symbol,entry:Any))]$Result
```

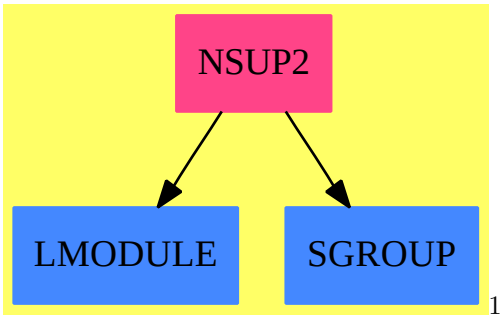
```
s21bdf(xArg:DoubleFloat,yArg:DoubleFloat,zArg:DoubleFloat,_
    rArg:DoubleFloat,ifailArg:Integer): Result ==
    [(invokeNagman(NIL$Lisp,_
    "s21bdf",_
    ["x"::S,"y"::S,"z"::S,"r"::S,"ifail"::S_
    ]$Lisp,_
    []$Lisp,_
    [["double"::S,"s21bdfResult"::S,"x"::S,"y"::S_
    ,"z"::S,"r"::S]$Lisp_
    ,["integer"::S,"ifail"::S]$Lisp_
    ]$Lisp,_
    ["s21bdfResult"::S,"ifail"::S]$Lisp,_
    [([xArg::Any,yArg::Any,zArg::Any,rArg::Any,ifailArg::Any ])_
    @List Any]$Lisp)$Lisp)_
    pretend List (Record(key:Symbol,entry:Any))]$Result
```

⟨*NAGS.dotabb*⟩≡

```
"NAGS" [color="#FF4488",href="bookvol10.4.pdf#nameddest=NAGS"]
"ALIST" [color="#88FF44",href="bookvol10.3.pdf#nameddest=ALIST"]
"NAGS" -> "ALIST"
```

15.31 package NSUP2 NewSparseUnivariatePolynomialFunctions2

15.32 NewSparseUnivariatePolynomialFunctions2



1.00

Exports:
map

⟨*package NSUP2 NewSparseUnivariatePolynomialFunctions2*⟩≡

```
)abbrev package NSUP2 NewSparseUnivariatePolynomialFunctions2
++ Author:
++ Date Created:
++ Date Last Updated:
++ Basic Functions:
++ Related Constructors:
++ Also See:
++ AMS Classifications:
++ Keywords:
++ References:
++ Description:
++ This package lifts a mapping from coefficient rings R to S to
++ a mapping from sparse univariate polynomial over R to
++ a sparse univariate polynomial over S.
++ Note that the mapping is assumed
++ to send zero to zero, since it will only be applied to the non-zero
++ coefficients of the polynomial.

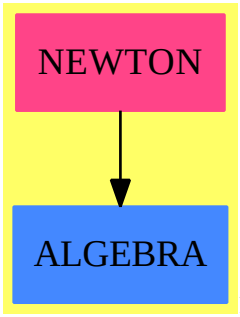
NewSparseUnivariatePolynomialFunctions2(R:Ring, S:Ring): with
  map:(R->S,NewSparseUnivariatePolynomial R) -> NewSparseUnivariatePolynomial S
    ++ \axiom{map(func, poly)} creates a new polynomial by applying func to
    ++ every non-zero coefficient of the polynomial poly.
 == add
  map(f, p) == map(f, p)$UnivariatePolynomialCategoryFunctions2(R,
           NewSparseUnivariatePolynomial R, S, NewSparseUnivariatePolynomial S)
```

⟨*NSUP2.dotabb*⟩≡

```
"NSUP2" [color="#FF4488",href="bookvol10.4.pdf#nameddest=NSUP2"]
"LMODULE" [color="#4488FF",href="bookvol10.2.pdf#nameddest=LMODULE"]
"SGROUP" [color="#4488FF",href="bookvol10.2.pdf#nameddest=SGROUP"]
"NSUP2" -> "LMODULE"
"NSUP2" -> "SGROUP"
```

15.33 package NEWTON NewtonInterpolation

15.34 NewtonInterpolation



1.00

Exports:
newton

⟨*package NEWTON NewtonInterpolation*⟩≡
```
)abbrev package NEWTON NewtonInterpolation
++ Description:
++ This package exports Newton interpolation for the special case where the
++ result is known to be in the original integral domain
++ The packages defined in this file provide fast fraction free rational
++ interpolation algorithms. (see FAMR2, FFFG, FFFGF, NEWTON)
NewtonInterpolation F: Exports == Implementation where
    F: IntegralDomain
    Exports == with

      newton: List F -> SparseUnivariatePolynomial F

      ++ \spad{newton}(l) returns the interpolating polynomial for the values
      ++ l, where the x-coordinates are assumed to be [1,2,3,...,n] and the
      ++ coefficients of the interpolating polynomial are known to be in the
      ++ domain F. I.e., it is a very streamlined version for a special case of
      ++ interpolation.

    Implementation == add

      differences(yl: List F): List F ==
        [y2-y1 for y1 in yl for y2 in rest yl]

      z: SparseUnivariatePolynomial(F) := monomial(1,1)

-- we assume x=[1,2,3,...,n]
      newtonAux(k: F, fact: F, yl: List F): SparseUnivariatePolynomial(F) ==
```

```
        if empty? rest yl
        then ((yl.1) exquo fact)::F::SparseUnivariatePolynomial(F)
        else ((yl.1) exquo fact)::F::SparseUnivariatePolynomial(F)
             + (z-k::SparseUnivariatePolynomial(F)) _
               * newtonAux(k+1$F, fact*k, differences yl)

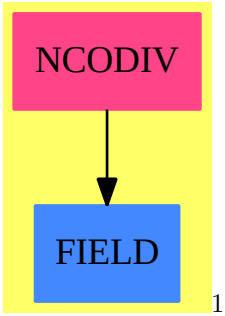

      newton yl == newtonAux(1$F, 1$F, yl)
```

⟨*NEWTON.dotabb*⟩≡

```
"NEWTON" [color="#FF4488",href="bookvol10.4.pdf#nameddest=NEWTON"]
"ALGEBRA" [color="#4488FF",href="bookvol10.2.pdf#nameddest=ALGEBRA"]
"NEWTON" -> "ALGEBRA"
```

15.35 package NCODIV NonCommutativeOperatorDivision

15.36 NonCommutativeOperatorDivision



1.00

Exports:
leftGcd leftLcm leftQuotient leftRemainder leftDivide leftExactQuotient

⟨*package NCODIV NonCommutativeOperatorDivision*⟩≡

```
)abbrev package NCODIV NonCommutativeOperatorDivision
++ Author: Jean Della Dora, Stephen M. Watt
++ Date Created: 1986
++ Date Last Updated: May 30, 1991
++ Basic Operations:
++ Related Domains: LinearOrdinaryDifferentialOperator
++ Also See:
++ AMS Classifications:
++ Keywords: gcd, lcm, division, non-commutative
++ Examples:
++ References:
++ Description:
++   This package provides a division and related operations for
++   \spadtype{MonogenicLinearOperator}s over a \spadtype{Field}.
++   Since the multiplication is in general non-commutative,
++   these operations all have left- and right-hand versions.
++   This package provides the operations based on left-division.
            -- [q,r] = leftDivide(a,b) means a=b*q+r

NonCommutativeOperatorDivision(P, F): PDcat == PDdef  where
    P: MonogenicLinearOperator(F)
    F: Field

    PDcat == with
        leftDivide:   (P, P) -> Record(quotient: P, remainder: P)
            ++ leftDivide(a,b) returns the pair \spad{[q,r]} such that
```

```
            ++ \spad{a = b*q + r} and the degree of \spad{r} is
            ++ less than the degree of \spad{b}.
            ++ This process is called ``left division''.
        leftQuotient:  (P, P) -> P
            ++ leftQuotient(a,b) computes the pair \spad{[q,r]} such that
            ++ \spad{a = b*q + r} and the degree of \spad{r} is
            ++ less than the degree of \spad{b}.
            ++ The value \spad{q} is returned.
        leftRemainder:  (P, P) -> P
            ++ leftRemainder(a,b) computes the pair \spad{[q,r]} such that
            ++ \spad{a = b*q + r} and the degree of \spad{r} is
            ++ less than the degree of \spad{b}.
            ++ The value \spad{r} is returned.
        leftExactQuotient:(P, P) -> Union(P, "failed")
            ++ leftExactQuotient(a,b) computes the value \spad{q}, if it exists,
            ++  such that \spad{a = b*q}.

        leftGcd:   (P, P) -> P
            ++ leftGcd(a,b) computes the value \spad{g} of highest degree
            ++ such that
            ++    \spad{a = aa*g}
            ++    \spad{b = bb*g}
            ++ for some values \spad{aa} and \spad{bb}.
            ++ The value \spad{g} is computed using left-division.
        leftLcm:   (P, P) -> P
            ++ leftLcm(a,b) computes the value \spad{m} of lowest degree
            ++ such that \spad{m = a*aa = b*bb} for some values
            ++ \spad{aa} and \spad{bb}.  The value \spad{m} is
            ++ computed using left-division.

    PDdef == add
        leftDivide(a, b) ==
            q: P := 0
            r: P := a
            iv:F := inv leadingCoefficient b
            while degree r >= degree b and r ^= 0 repeat
                h := monomial(iv*leadingCoefficient r,
                                (degree r - degree b)::NonNegativeInteger)$P
                r := r - b*h
                q := q + h
            [q,r]

        -- leftQuotient(a,b) is the quotient from left division, etc.
        leftQuotient(a,b)   == leftDivide(a,b).quotient
        leftRemainder(a,b)   == leftDivide(a,b).remainder
        leftExactQuotient(a,b) ==
```

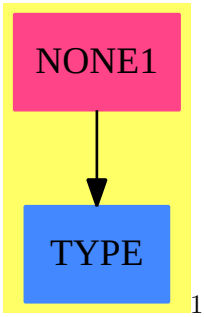
```
        qr := leftDivide(a,b)
        if qr.remainder = 0 then qr.quotient else "failed"
    -- l = leftGcd(a,b) means  a = aa*l  b = bb*l.  Uses leftDivide.
    leftGcd(a,b) ==
        a = 0 =>b
        b = 0 =>a
        while degree b > 0 repeat (a,b) := (b, leftRemainder(a,b))
        if b=0 then a else b
    -- l = leftLcm(a,b) means  l = a*aa  l = b*bb   Uses leftDivide.
    leftLcm(a,b) ==
        a = 0 =>b
        b = 0 =>a
        b0 := b
        u  := monomial(1,0)$P
        v  := 0
        while leadingCoefficient b ^= 0 repeat
            qr     := leftDivide(a,b)
            (a, b) := (b, qr.remainder)
            (u, v) := (u*qr.quotient+v, u)
        b0*u
```

⟨*NCODIV.dotabb*⟩≡

```
"NCODIV" [color="#FF4488",href="bookvol10.4.pdf#nameddest=NCODIV"]
"FIELD"  [color="#4488FF",href="bookvol10.2.pdf#nameddest=FIELD"]
"NCODIV" -> "FIELD"
```

15.37 package NONE1 NoneFunctions1

15.38 NoneFunctions1



1.00

Exports:
coerce

⟨*package NONE1 NoneFunctions1*⟩≡

```
)abbrev package NONE1 NoneFunctions1
++ Author:
++ Date Created:
++ Change History:
++ Basic Functions: coerce
++ Related Constructors: None
++ Also See:
++ AMS Classification:
++ Keywords:
++ Description:
++   \spadtype{NoneFunctions1} implements functions on \spadtype{None}.
++   It particular it includes a particulary dangerous coercion from
++   any other type to \spadtype{None}.

NoneFunctions1(S:Type): Exports == Implementation where
  Exports ==> with
    coerce: S -> None
      ++ coerce(x) changes \spad{x} into an object of type
      ++ \spadtype{None}.

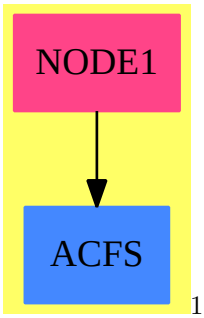
  Implementation ==> add
    coerce(s:S):None == s pretend None
```

⟨*NONE1.dotabb*⟩≡

```
"NONE1" [color="#FF4488",href="bookvol10.4.pdf#nameddest=NONE1"]
"TYPE" [color="#4488FF",href="bookvol10.2.pdf#nameddest=TYPE"]
"NONE1" -> "TYPE"
```

15.39 package NODE1 NonLinearFirstOrderODESolver

15.40 NonLinearFirstOrderODESolver



1.00

Exports:
solve

⟨*package NODE1 NonLinearFirstOrderODESolver*⟩≡

```
)abbrev package NODE1 NonLinearFirstOrderODESolver
++ Author: Manuel Bronstein
++ Date Created: 2 September 1991
++ Date Last Updated: 14 October 1994
++ Description: NonLinearFirstOrderODESolver provides a function
++ for finding closed form first integrals of nonlinear ordinary
++ differential equations of order 1.
++ Keywords: differential equation, ODE
NonLinearFirstOrderODESolver(R, F): Exports == Implementation where
  R: Join(OrderedSet, EuclideanDomain, RetractableTo Integer,
          LinearlyExplicitRingOver Integer, CharacteristicZero)
  F: Join(AlgebraicallyClosedFunctionSpace R, TranscendentalFunctionCategory,
          PrimitiveFunctionCategory)

  N   ==> NonNegativeInteger
  Q   ==> Fraction Integer
  UQ  ==> Union(Q, "failed")
  OP  ==> BasicOperator
  SY  ==> Symbol
  K   ==> Kernel F
  U   ==> Union(F, "failed")
  P   ==> SparseMultivariatePolynomial(R, K)
  REC ==> Record(coef:Q, logand:F)
  SOL ==> Record(particular: F,basis: List F)
  BER ==> Record(coef1:F, coefn:F, exponent:N)
```

```
  Exports ==> with
    solve: (F, F, OP, SY) -> U
      ++ solve(M(x,y), N(x,y), y, x) returns \spad{F(x,y)} such that
      ++ \spad{F(x,y) = c} for a constant \spad{c} is a first integral
      ++ of the equation \spad{M(x,y) dx + N(x,y) dy  = 0}, or
      ++ "failed" if no first-integral can be found.

  Implementation ==> add
    import ODEIntegration(R, F)
    import ElementaryFunctionODESolver(R, F)    -- recursive dependency!

    checkBernoulli   : (F, F, K) -> Union(BER, "failed")
    solveBernoulli   : (BER, OP, SY, F) -> Union(F, "failed")
    checkRiccati     : (F, F, K) -> Union(List F, "failed")
    solveRiccati     : (List F, OP, SY, F) -> Union(F, "failed")
    partSolRiccati   : (List F, OP, SY, F) -> Union(F, "failed")
    integratingFactor: (F, F, SY, SY) -> U

    unk    := new()$SY
    kunk:K := kernel unk

    solve(m, n, y, x) ==
-- first replace the operator y(x) by a new symbol z in m(x,y) and n(x,y)
      lk:List(K) := [retract(yx := y(x::F))@K]
      lv:List(F) := [kunk::F]
      mm := eval(m, lk, lv)
      nn := eval(n, lk, lv)
-- put over a common denominator (to balance m and n)
      d := lcm(denom mm, denom nn)::F
      mm := d * mm
      nn := d * nn
-- look for an integrating factor mu
      (u := integratingFactor(mm, nn, unk, x)) case F =>
        mu := u::F
        mm := mm * mu
        nn := nn * mu
        eval(int(mm,x) + int(nn-int(differentiate(mm,unk),x), unk),[kunk],[yx])
-- check for Bernoulli equation
      (w := checkBernoulli(m, n, k1 := first lk)) case BER =>
        solveBernoulli(w::BER, y, x, yx)
-- check for Riccati equation
      (v := checkRiccati(m, n, k1)) case List(F) =>
        solveRiccati(v::List(F), y, x, yx)
      "failed"

-- look for an integrating factor
```

```
    integratingFactor(m, n, y, x) ==
-- check first for exactness
      zero?(d := differentiate(m, y) - differentiate(n, x)) => 1
-- look for an integrating factor involving x only
      not member?(y, variables(f := d / n)) => expint(f, x)
-- look for an integrating factor involving y only
      not member?(x, variables(f := - d / m)) => expint(f, y)
-- room for more techniques later on (e.g. Prelle-Singer etc...)
      "failed"

-- check whether the equation is of the form
--    dy/dx + p(x)y + q(x)y^N = 0   with N > 1
-- i.e. whether m/n is of the form  p(x) y + q(x) y^N
-- returns [p, q, N] if the equation is in that form
    checkBernoulli(m, n, ky) ==
      r := denom(f := m / n)::F
      (not freeOf?(r, y := ky::F))
          or (d := degree(p := univariate(numer f, ky))) < 2
            or degree(pp := reductum p) ^= 1 or reductum(pp) ^= 0
              or (not freeOf?(a := (leadingCoefficient(pp)::F), y))
                or (not freeOf?(b := (leadingCoefficient(p)::F), y)) => "failed"
      [a / r, b / r, d]

-- solves the equation dy/dx + rec.coef1 y + rec.coefn y^rec.exponent = 0
-- the change of variable v = y^{1-n} transforms the above equation to
--  dv/dx + (1 - n) p v + (1 - n) q = 0
    solveBernoulli(rec, y, x, yx) ==
      n1 := 1 - rec.exponent::Integer
      deq := differentiate(yx, x) + n1 * rec.coef1 * yx + n1 * rec.coefn
      sol := solve(deq, y, x)::SOL          -- can always solve for order 1
-- if v = vp + c v0 is the general solution of the linear equation, then
-- the general first integral for the Bernoulli equation is
-- (y^{1-n} - vp) / v0  =   c   for any constant c
      (yx**n1 - sol.particular) / first(sol.basis)

-- check whether the equation is of the form
--    dy/dx + q0(x) + q1(x)y + q2(x)y^2 = 0
-- i.e. whether m/n is a quadratic polynomial in y.
-- returns the list [q0, q1, q2] if the equation is in that form
    checkRiccati(m, n, ky) ==
      q := denom(f := m / n)::F
      (not freeOf?(q, y := ky::F)) or degree(p := univariate(numer f, ky)) > 2
         or (not freeOf?(a0 := (coefficient(p, 0)::F), y))
           or (not freeOf?(a1 := (coefficient(p, 1)::F), y))
             or (not freeOf?(a2 := (coefficient(p, 2)::F), y)) => "failed"
      [a0 / q, a1 / q, a2 / q]
```

```
  -- solves the equation dy/dx + l.1 + l.2 y + l.3 y^2 = 0
      solveRiccati(l, y, x, yx) ==
  -- get first a particular solution
        (u := partSolRiccati(l, y, x, yx)) case "failed" => "failed"
  -- once a particular solution yp is known, the general solution is of the
  -- form  y = yp + 1/v  where v satisfies the linear 1st order equation
  -- v' - (l.2 + 2 l.3 yp) v = l.3
        deq := differentiate(yx, x) - (l.2 + 2 * l.3 * u::F) * yx - l.3
        gsol := solve(deq, y, x)::SOL         -- can always solve for order 1
  -- if v = vp + c v0 is the general solution of the above equation, then
  -- the general first integral for the Riccati equation is
  --  (1/(y - yp) - vp) / v0  =   c   for any constant c
        (inv(yx - u::F) - gsol.particular) / first(gsol.basis)

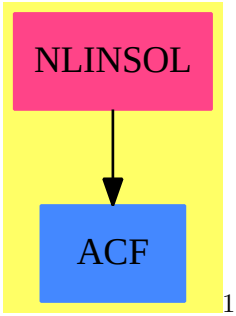
  -- looks for a particular solution of dy/dx + l.1 + l.2 y + l.3 y^2 = 0
      partSolRiccati(l, y, x, yx) ==
  -- we first do the change of variable y = z / l.3, which transforms
  -- the equation into  dz/dx + l.1 l.3 + (l.2 - l.3'/l.3) z + z^2 = 0
        q0 := l.1 * (l3 := l.3)
        q1 := l.2 - differentiate(l3, x) / l3
  -- the equation dz/dx + q0 + q1 z + z^2 = 0 is transformed by the change
  -- of variable z = w'/w into the linear equation w'' + q1 w' + q0 w = 0
        lineq := differentiate(yx, x, 2) + q1 * differentiate(yx, x) + q0 * yx
  -- should be made faster by requesting a particular nonzero solution only
        (not((gsol := solve(lineq, y, x)) case SOL))
                                or empty?(bas := (gsol::SOL).basis) => "failed"
        differentiate(first bas, x) / (l3 * first bas)
```

⟨*NODE1.dotabb*⟩≡

```
  "NODE1" [color="#FF4488",href="bookvol10.4.pdf#nameddest=NODE1"]
  "ACFS" [color="#4488FF",href="bookvol10.2.pdf#nameddest=ACFS"]
  "NODE1" -> "ACFS"
```

15.41 package NLINSOL NonLinearSolvePackage

15.42 NonLinearSolvePackage



1.00

Exports:
solve solveInField

⟨*package NLINSOL NonLinearSolvePackage*⟩≡
```
)abbrev package NLINSOL NonLinearSolvePackage
++ Author: Manuel Bronstein
++ Date Created: 31 October 1991
++ Date Last Updated: 26 June 1992
++ Description:
++ NonLinearSolvePackage is an interface to \spadtype{SystemSolvePackage}
++ that attempts to retract the coefficients of the equations before
++ solving. The solutions are given in the algebraic closure of R whenever
++ possible.

NonLinearSolvePackage(R:IntegralDomain): Exports == Implementation where
  Z   ==> Integer
  Q   ==> Fraction Z
  SY  ==> Symbol
  P   ==> Polynomial R
  F   ==> Fraction P
  EQ  ==> Equation F
  SSP ==> SystemSolvePackage
  SOL ==> RetractSolvePackage

  Exports ==> with
    solveInField:    (List P, List SY) -> List List EQ
      ++ solveInField(lp,lv) finds the solutions of the list lp of
      ++ rational functions with respect to the list of symbols lv.
    solveInField:     List P       -> List List EQ
      ++ solveInField(lp) finds the solution of the list lp of rational
```

```
      ++ functions with respect to all the symbols appearing in lp.
    solve:           (List P, List SY) -> List List EQ
      ++ solve(lp,lv) finds the solutions in the algebraic closure of R
      ++ of the list lp of
      ++ rational functions with respect to the list of symbols lv.
    solve:            List P       -> List List EQ
      ++ solve(lp) finds the solution in the algebraic closure of R
      ++ of the list lp of rational
      ++ functions with respect to all the symbols appearing in lp.

  Implementation ==> add
    solveInField l == solveInField(l, "setUnion"/[variables p for p in l])

    if R has AlgebraicallyClosedField then
      import RationalFunction(R)

      expandSol: List EQ -> List List EQ
      RIfCan   : F -> Union(R, "failed")
      addRoot  : (EQ, List List EQ) -> List List EQ
      allRoots : List P -> List List EQ
      evalSol  : (List EQ, List EQ) -> List EQ

      solve l        == solve(l, "setUnion"/[variables p for p in l])
      solve(lp, lv)  == concat([expandSol sol for sol in solveInField(lp, lv)])
      addRoot(eq, l) == [concat(eq, sol) for sol in l]
      evalSol(ls, l) == [equation(lhs eq, eval(rhs eq, l)) for eq in ls]

-- converts [p1(a1),...,pn(an)] to
-- [[a1=v1,...,an=vn]] where vi ranges over all the zeros of pi
      allRoots l ==
        empty? l => [empty()$List(EQ)]
        z := allRoots rest l
        s := mainVariable(p := first l)::SY::P::F
        concat [addRoot(equation(s, a::P::F), z) for a in zerosOf univariate p]

      expandSol l ==
        lassign := lsubs := empty()$List(EQ)
        luniv := empty()$List(P)
        for eq in l repeat
          if retractIfCan(lhs eq)@Union(SY, "failed") case SY then
            if RIfCan(rhs eq) case R then lassign := concat(eq, lassign)
                                     else lsubs := concat(eq, lsubs)
          else
            if ((u := retractIfCan(lhs eq)@Union(P, "failed")) case P) and
--               one?(# variables(u::P)) and ((r := RIfCan rhs eq) case R) then
              ((# variables(u::P)) = 1) and ((r := RIfCan rhs eq) case R) then
```

```
                luniv := concat(u::P - r::R::P, luniv)
            else return [l]
          empty? luniv => [l]
          [concat(z, concat(evalSol(lsubs,z), lassign)) for z in allRoots luniv]

        RIfCan f ==
          ((n := retractIfCan(numer f)@Union(R,"failed")) case R) and
            ((d := retractIfCan(denom f)@Union(R,"failed")) case R) => n::R / d::R
          "failed"
      else
        solve l       == solveInField l
        solve(lp, lv) == solveInField(lp, lv)

    -- 'else if' is doubtful with this compiler so all 3 conditions are explicit
      if (not(R is Q)) and (R has RetractableTo Q) then
        solveInField(lp, lv) == solveRetract(lp, lv)$SOL(Q, R)

      if (not(R is Z)) and (not(R has RetractableTo Q)) and
        (R has RetractableTo Z) then
          solveInField(lp, lv) == solveRetract(lp, lv)$SOL(Z, R)

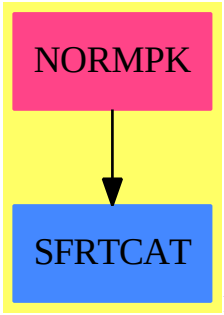
      if (not(R is Z)) and (not(R has RetractableTo Q)) and
        (not(R has RetractableTo Z)) then
          solveInField(lp, lv) == solve([p::F for p in lp]$List(F), lv)$SSP(R)
```

⟨*NLINSOL.dotabb*⟩≡

```
"NLINSOL" [color="#FF4488",href="bookvol10.4.pdf#nameddest=NLINSOL"]
"ACF" [color="#4488FF",href="bookvol10.2.pdf#nameddest=ACF"]
"NLINSOL" -> "ACF"
```

15.43 package NORMPK NormalizationPackage

15.44 NormalizationPackage



1.00

Exports:
normalizedAssociate normInvertible? normalize outputArgs recip

⟨*package NORMPK NormalizationPackage*⟩≡

```
)abbrev package NORMPK NormalizationPackage
++ Author: Marc Moreno Maza
++ Date Created: 09/23/1998
++ Date Last Updated: 12/16/1998
++ Basic Functions:
++ Related Constructors:
++ Also See:
++ AMS Classifications:
++ Keywords:
++ Description:
++ A package for computing normalized assocites of univariate polynomials
++ with coefficients in a tower of simple extensions of a field.\newline
++ References :
++  [1] D. LAZARD "A new method for solving algebraic systems of
++      positive dimension" Discr. App. Math. 33:147-160,1991
++  [2] M. MORENO MAZA and R. RIOBOO "Computations of gcd over
++      algebraic towers of simple extensions" In proceedings of AAECC11
++      Paris, 1995.
++  [3] M. MORENO MAZA "Calculs de pgcd au-dessus des tours
++      d'extensions simples et resolution des systemes d'equations
++      algebriques" These, Universite P.etM. Curie, Paris, 1997.
++ Version: 1.

NormalizationPackage(R,E,V,P,TS): Exports == Implementation where

  R : GcdDomain
```

```
E : OrderedAbelianMonoidSup
V : OrderedSet
P : RecursivePolynomialCategory(R,E,V)
TS : RegularTriangularSetCategory(R,E,V,P)
N ==> NonNegativeInteger
Z ==> Integer
B ==> Boolean
S ==> String
K ==> Fraction R
LP ==> List P
PWT ==> Record(val : P, tower : TS)

BWT ==> Record(val : Boolean, tower : TS)
LpWT ==> Record(val : (List P), tower : TS)
Split ==> List TS
--KeyGcd ==> Record(arg1: P, arg2: P, arg3: TS, arg4: B)
--EntryGcd ==> List PWT
--HGcd ==> TabulatedComputationPackage(KeyGcd, EntryGcd)
--KeyInvSet ==> Record(arg1: P, arg3: TS)
--EntryInvSet ==> List TS
--HInvSet ==> TabulatedComputationPackage(KeyInvSet, EntryInvSet)
polsetpack ==> PolynomialSetUtilitiesPackage(R,E,V,P)
regsetgcdpack ==> SquareFreeRegularTriangularSetGcdPackage(R,E,V,P,TS)

Exports ==  with

   recip: (P, TS) -> Record(num:P,den:P)
     ++ \axiom{recip(p,ts)} returns the inverse of \axiom{p} w.r.t \spad{ts}
     ++ assuming that \axiom{p} is invertible w.r.t \spad{ts}.
   normalizedAssociate: (P, TS) -> P
     ++ \axiom{normalizedAssociate(p,ts)} returns a normalized polynomial
     ++ \axiom{n} w.r.t. \spad{ts} such that \axiom{n} and \axiom{p} are
     ++ associates w.r.t \spad{ts} and assuming that \axiom{p} is invertible
     ++ w.r.t \spad{ts}.
   normalize: (P, TS) -> List PWT
     ++ \axiom{normalize(p,ts)} normalizes \axiom{p} w.r.t \spad{ts}.
   outputArgs: (S, S, P, TS) -> Void
     ++ \axiom{outputArgs(s1,s2,p,ts)}
     ++ is an internal subroutine, exported only for developement.
   normInvertible?: (P, TS) -> List BWT
     ++ \axiom{normInvertible?(p,ts)}
     ++ is an internal subroutine, exported only for developement.

Implementation == add

   if TS has SquareFreeRegularTriangularSetCategory(R,E,V,P)
```

```
then

  normInvertible?(p:P, ts:TS): List BWT ==
    stoseInvertible?_sqfreg(p,ts)$regsetgcdpack

else

  normInvertible?(p:P, ts:TS): List BWT ==
    stoseInvertible?_reg(p,ts)$regsetgcdpack

if (R has RetractableTo(Integer)) and (V has ConvertibleTo(Symbol))
then

  outputArgs(s1:S, s2: S, p:P,ts:TS): Void ==
    if not empty? s1 then output(s1, p::OutputForm)$OutputPackage
    if not empty? s1 then _
         output(s1,(convert(p)@String)::OutputForm)$OutputPackage
    output(" ")$OutputPackage
    if not empty? s2 then output(s2, ts::OutputForm)$OutputPackage
    empty? s2 => void()
    output(s2,("[")::OutputForm)$OutputPackage
    lp: List P := members(ts)
    for q in lp repeat
       output((convert(q)@String)::OutputForm)$OutputPackage
    output("]")$OutputPackage
    output(" ")$OutputPackage

else

  outputArgs(s1:S, s2: S, p:P,ts:TS): Void ==
    if not empty? s1 then output(s1, p::OutputForm)$OutputPackage
    output(" ")$OutputPackage
    if not empty? s2 then output(s2, ts::OutputForm)$OutputPackage
    output(" ")$OutputPackage

recip(p:P,ts:TS): Record(num:P, den:P) ==
-- ASSUME p is invertible w.r.t. ts
-- ASSUME mvar(p) is algebraic w.r.t. ts
  v := mvar(p)
  ts_v := select(ts,v)::P
  if mdeg(p) < mdeg(ts_v)
    then
      hesrg: Record (gcd : P, coef2 : P)  := _
                halfExtendedSubResultantGcd2(ts_v,p)$P
      d: P :=  hesrg.gcd; n: P := hesrg.coef2
    else
```

```
        hesrg: Record (gcd : P, coef1 : P) := _
                 halfExtendedSubResultantGcd1(p,ts_v)$P
        d: P :=  hesrg.gcd; n: P := hesrg.coef1
    g := gcd(n,d)
    (n, d) := ((n exquo g)::P, (d exquo g)::P)
    remn, remd: Record(rnum:R,polnum:P,den:R)
    remn := remainder(n,ts); remd := remainder(d,ts)
    cn := remn.rnum; pn := remn.polnum; dn := remn.den
    cd := remd.rnum; pd := remd.polnum; dp := remd.den
    k: K := (cn / cd) * (dp / dn)
    pn := removeZero(pn,ts)
    pd := removeZero(pd,ts)
    [numer(k) * pn, denom(k) * pd]$Record(num:P, den:P)

  normalizedAssociate(p:P,ts:TS): P ==
  -- ASSUME p is invertible or zero w.r.t. ts
    empty? ts => p
    zero?(p) => p
    ground?(p) => 1
    zero? initiallyReduce(init(p),ts) =>
      error "in normalizedAssociate$NORMPK: bad #1"
    vp := mvar(p)
    ip: P := p
    mp: P := 1
    tp: P := 0
    while not ground?(ip) repeat
      v := mvar(ip)
      if algebraic?(v,ts)
        then
          if v = vp
            then
              ts_v := select(ts,v)::P
              ip := lastSubResultant(ip,ts_v)$P
              ip := remainder(ip,ts).polnum
              -- ip := primitivePart stronglyReduce(ip,ts)
              ip := primitivePart initiallyReduce(ip,ts)
            else
              qr := recip(ip,ts)
              ip := qr.den
              tp := qr.num * tp
              zero? ip =>
                  outputArgs("p = ", " ts = ",p,ts)
                  error _
                    "in normalizedAssociate$NORMPK: should never happen !"
        else
          tp := tail(ip) * mp + tp
```

```
            mp := mainMonomial(ip) * mp
            ip := init(ip)
        r := ip * mp + tp
        r := remainder(r,ts).polnum
        -- primitivePart stronglyReduce(r,ts)
        primitivePart initiallyReduce(r,ts)

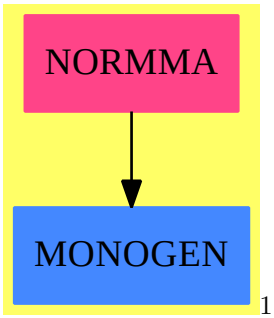
      normalize(p: P, ts: TS): List PWT ==
        zero? p => [[p,ts]$PWT]
        ground? p => [[1,ts]$PWT]
        zero? initiallyReduce(init(p),ts) =>
          error "in normalize$NORMPK: init(#1) reduces to 0 w.r.t. #2"
        --output("Entering  normalize")$OutputPackage
        --outputArgs("p = ", " ts = ",p,ts)
        --output("Calling  normInvertible?")$OutputPackage
        lbwt: List BWT := normInvertible?(p,ts)
        --output("Result is: ")$OutputPackage
        --output(lbwt::OutputForm)$OutputPackage
        lpwt: List PWT := []
        for bwt in lbwt repeat
          us := bwt.tower
          q := remainder(p,us).polnum
          q := removeZero(q,us)
          bwt.val =>
            --output("Calling  normalizedAssociate")$OutputPackage
            --outputArgs("q = ", " us = ",q,us)
            lpwt := cons([normalizedAssociate(q,us)@P,us]$PWT, lpwt)
            --output("Leaving  normalizedAssociate")$OutputPackage
          zero? q => lpwt := cons([0$P,us]$PWT, lpwt)
          lpwt := concat(normalize(q,us)@(List PWT),lpwt)
        lpwt
```

⟨*NORMPK.dotabb*⟩≡

```
"NORMPK" [color="#FF4488",href="bookvol10.4.pdf#nameddest=NORMPK"]
"SFRTCAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=SFRTCAT"]
"NORMPK" -> "SFRTCAT"
```

15.45 package NORMMA NormInMonogenicAlgebra

15.46 NormInMonogenicAlgebra



1.00

Exports:
norm

⟨*package NORMMA NormInMonogenicAlgebra*⟩≡

```
)abbrev package NORMMA NormInMonogenicAlgebra
++ Author: Manuel Bronstein
++ Date Created: 23 February 1995
++ Date Last Updated: 23 February 1995
++ Basic Functions: norm
++ Description:
++ This package implements the norm of a polynomial with coefficients
++ in a monogenic algebra (using resultants)

NormInMonogenicAlgebra(R, PolR, E, PolE): Exports == Implementation where
  R: GcdDomain
  PolR: UnivariatePolynomialCategory R
  E: MonogenicAlgebra(R, PolR)
  PolE: UnivariatePolynomialCategory E

  SUP ==> SparseUnivariatePolynomial

  Exports ==> with
    norm: PolE -> PolR
      ++ norm q returns the norm of q,
      ++ i.e. the product of all the conjugates of q.

  Implementation ==> add
    import UnivariatePolynomialCategoryFunctions2(R, PolR, PolR, SUP PolR)

    PolR2SUP: PolR -> SUP PolR
```

```
      PolR2SUP q == map(#1::PolR, q)

      defpol := PolR2SUP(definingPolynomial()$E)

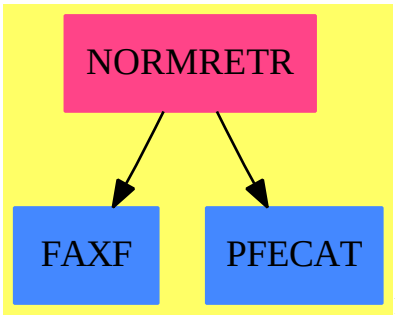
      norm q ==
        p:SUP PolR := 0
        while q ~= 0 repeat
          p := p + monomial(1,degree q)$PolR * PolR2SUP lift leadingCoefficient q
          q := reductum q
        primitivePart resultant(p, defpol)
```

⟨*NORMMA.dotabb*⟩≡

```
"NORMMA" [color="#FF4488",href="bookvol10.4.pdf#nameddest=NORMMA"]
"MONOGEN" [color="#4488FF",href="bookvol10.2.pdf#nameddest=MONOGEN"]
"NORMMA" -> "MONOGEN"
```

15.47 package NORMRETR NormRetractPackage

15.48 NormRetractPackage



1.00

Exports:
Frobenius normFactors retractIfCan

⟨*package NORMRETR NormRetractPackage*⟩≡
```
)abbrev package NORMRETR NormRetractPackage
++ Description:
++ This package \undocumented
NormRetractPackage(F, ExtF, SUEx, ExtP, n):C  == T where
  F          :   FiniteFieldCategory
  ExtF       :   FiniteAlgebraicExtensionField(F)
  SUEx       :   UnivariatePolynomialCategory ExtF
  ExtP       :   UnivariatePolynomialCategory SUEx
  n          :   PositiveInteger
  SUP       ==>  SparseUnivariatePolynomial
  R         ==>  SUP F
  P         ==>  SUP R

  C  ==> with
      normFactors : ExtP -> List ExtP
        ++ normFactors(x) \undocumented
      retractIfCan : ExtP -> Union(P, "failed")
        ++ retractIfCan(x) \undocumented
      Frobenius    : ExtP -> ExtP
        ++ Frobenius(x) \undocumented

  T  ==> add

      normFactors(p:ExtP):List ExtP ==
          facs : List ExtP := [p]
          for i in 1..n-1 repeat
```

```
            member?((p := Frobenius p), facs) => return facs
            facs := cons(p, facs)
         facs

      Frobenius(ff:ExtP):ExtP ==
         fft:ExtP:=0
         while ff^=0 repeat
           fft:=fft + monomial(map(Frobenius, leadingCoefficient ff),
                               degree ff)
           ff:=reductum ff
         fft

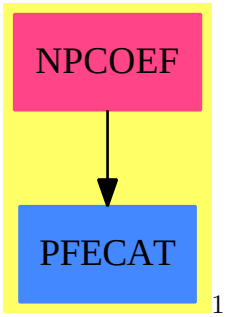
      retractIfCan(ff:ExtP):Union(P, "failed") ==
         fft:P:=0
         while ff ^= 0 repeat
           lc : SUEx := leadingCoefficient ff
           plc: SUP F := 0
           while lc ^= 0 repeat
              lclc:ExtF := leadingCoefficient lc
              (retlc := retractIfCan lclc) case "failed" => return "failed"
              plc := plc + monomial(retlc::F, degree lc)
              lc := reductum lc
           fft:=fft+monomial(plc, degree ff)
           ff:=reductum ff
         fft
```

⟨*NORMRETR.dotabb*⟩≡

```
"NORMRETR" [color="#FF4488",href="bookvol10.4.pdf#nameddest=NORMRETR"]
"FAXF" [color="#4488FF",href="bookvol10.2.pdf#nameddest=FAXF"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"NORMRETR" -> "FAXF"
"NORMRETR" -> "PFECAT"
```

15.49 package NPCOEF NPCoef

15.50 NPCoef



1.00

Exports:
listexp npcoef

⟨*package NPCOEF NPCoef*⟩≡
```
)abbrev package NPCOEF NPCoef
++ Author : P.Gianni, revised May 1990
++ Description:
++ Package for the determination of the coefficients in the lifting
++ process. Used by \spadtype{MultivariateLifting}.
++ This package will work for every euclidean domain R which has property
++ F, i.e. there exists a factor operation in \spad{R[x]}.
NPCoef(BP,E,OV,R,P) : C == T where

  OV   :   OrderedSet
  E    :   OrderedAbelianMonoidSup
  R    :   EuclideanDomain  -- with property "F"
  BP   :   UnivariatePolynomialCategory R
  P    :   PolynomialCategory(R,E,OV)

  Z       ==> Integer
  NNI     ==> NonNegativeInteger
  USP     ==> SparseUnivariatePolynomial(P)
  Term    ==> Record(expt:NNI,pcoef:P)
  Detc    ==> Record(valexp:NNI,valcoef:P,posit:NNI)
  VTerm   ==> List(Term)
  DetCoef ==> Record(deter:List(USP),dterm:List(VTerm),
                     nfacts:List(BP),nlead:List(P))
  TermC   ==> Record(coefu:P,detfacts:List(VTerm))
  TCoef   ==> List(TermC)

  C == with
```

```
      npcoef   :    (USP,List(BP),List(P))      ->   DetCoef
        ++ npcoef \undocumented
      listexp  :              BP                ->   List(NNI)
        ++ listexp \undocumented
  T == add

                 ----   Local  Functions  ----
   check      : (TermC,Vector P) -> Union(Detc,"failed")
   buildvect  : (List(VTerm),NNI) -> Vector(List(VTerm))
   buildtable : (Vector(P),List(List NNI),List P) -> TCoef
   modify : (TCoef,Detc) -> TCoef
   constructp : VTerm -> USP

   npcoef(u:USP,factlist:List(BP),leadlist:List(P)) :DetCoef ==
     detcoef:List(VTerm):=empty();detufact:List(USP):=empty()
     lexp:List(List(NNI)):=[listexp(v) for v in factlist]
     ulist :Vector(P):=vector [coefficient(u,i) for i in 0..degree u]
     tablecoef:=buildtable(ulist,lexp,leadlist)
     detcoef:=[[[ep.first,lcu]$Term]  for ep in lexp for lcu in leadlist]
     ldtcf:=detcoef
     lexp:=[ep.rest for ep in lexp]
     ndet:NNI:=#factlist
     changed:Boolean:=true
     ltochange:List(NNI):=empty()
     ltodel:List(NNI):=empty()
     while changed and ndet^=1 repeat
       changed :=false
       dt:=#tablecoef
       for i in 1..dt while ^changed repeat
         (cf:=check(tablecoef.i,ulist)) case "failed" => "next i"
         ltochange:=cons(i,ltochange)
         celtf:Detc:=cf::Detc
         tablecoef:=modify(tablecoef,celtf)
         vpos:=celtf.posit
         vexp:=celtf.valexp
         nterm:=[vexp,celtf.valcoef]$Term
         detcoef.vpos:=cons(nterm,detcoef.vpos)
         lexp.vpos:=delete(lexp.vpos,position(vexp,lexp.vpos))
         if lexp.vpos=[] then
          ltodel:=cons(vpos,ltodel)
          ndet:=(ndet-1):NNI
          detufact:=cons(constructp(detcoef.vpos),detufact)
         changed:=true
       for i in ltochange repeat tablecoef:=delete(tablecoef,i)
       ltochange:=[]
     if ndet=1 then
```

```
     uu:=u exquo */[pol for pol in detufact]
     if uu case "failed" then return
       [empty(),ldtcf,factlist,leadlist]$DetCoef
     else  detufact:=cons(uu::USP,detufact)
    else
      ltodel:=sort(#1>#2,ltodel)
      for i in ltodel repeat
        detcoef:=delete(detcoef,i)
        factlist:=delete(factlist,i)
        leadlist:=delete(leadlist,i)
    [detufact,detcoef,factlist,leadlist]$DetCoef


  check(tterm:TermC,ulist:Vector(P)) : Union(Detc,"failed") ==
    cfu:P:=1$P;doit:NNI:=0;poselt:NNI:=0;pp:Union(P,"failed")
    termlist:List(VTerm):=tterm.detfacts
    vterm:VTerm:=empty()
    #termlist=1 =>
      vterm:=termlist.first
      for elterm in vterm while doit<2 repeat
        (cu1:=elterm.pcoef)^=0 => cfu:=cu1*cfu
        doit:=doit+1
        poselt:=position(elterm,vterm):NNI
      doit=2  or (pp:=tterm.coefu exquo cfu) case "failed" => "failed"
      [vterm.poselt.expt,pp::P,poselt]$Detc
    "failed"

  buildvect(lvterm:List(VTerm),n:NNI) : Vector(List(VTerm)) ==
    vtable:Vector(List(VTerm)):=new(n,empty())
    (#lvterm)=1 =>
      for term in lvterm.first repeat vtable.(term.expt+1):=[[term]]
      vtable

    vtable:=buildvect(lvterm.rest,n)
    ntable:Vector(List(VTerm)):=new(n,empty())
    for term in lvterm.first repeat
      nexp:=term.expt
      for i in 1..n while (nexp+i)<(n+1) repeat
        ntable.(nexp+i):=append(
                            [cons(term,lvterm) for lvterm in vtable.i],
                               ntable.(nexp+i))
    ntable

  buildtable(vu:Vector(P),lvect:List(List(NNI)),leadlist:List(P)):TCoef==
    nfact:NNI:=#leadlist
    table:TCoef:=empty()
```

```
    degu:=(#vu-1)::NNI
    prelim:List(VTerm):=[[[e,0$P]$Term for e in lv] for lv in lvect]
    for i in 1..nfact repeat prelim.i.first.pcoef:=leadlist.i
    partialv:Vector(List(VTerm)):=new(nfact,empty())
    partialv:=buildvect(prelim,degu)
    for i in 1..degu repeat
      empty? partialv.i => "next i"
      table:=cons([vu.i,partialv.i]$TermC, table)
    table

  modify(tablecoef:TCoef,cfter:Detc) : TCoef ==
    cfexp:=cfter.valexp;cfcoef:=cfter.valcoef;cfpos:=cfter.posit
    lterase:List(NNI):=empty()
    for cterm in tablecoef | ^empty?(ctdet:=cterm.detfacts) repeat
      (+/[term.expt for term in ctdet.first])<cfexp => "next term"
      for celt in ctdet repeat
        if celt.cfpos.expt=cfexp then
          celt.cfpos.pcoef:=cfcoef
          if (and/[cc.pcoef ^=0 for cc in celt]) then
            k:=position(celt,ctdet):NNI
            lterase:=cons(k,lterase)
            cterm.coefu:=(cterm.coefu - */[cc.pcoef for cc in celt])
      if not empty? lterase then
        lterase:=sort(#1>#2,lterase)
        for i in lterase repeat ctdet:=delete(ctdet,i)
        cterm.detfacts:=ctdet
        lterase:=empty()
    tablecoef

  listexp(up:BP) :List(NNI) ==
    degree up=0 => [0]
    [degree up,:listexp(reductum up)]

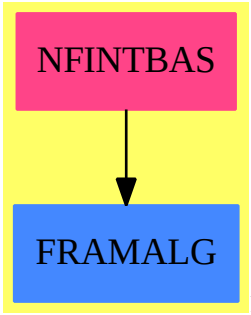
  constructp(lterm:VTerm):USP ==
    +/[monomial(term.pcoef,term.expt) for term in lterm]
```

⟨*NPCOEF.dotabb*⟩≡

```
"NPCOEF" [color="#FF4488",href="bookvol10.4.pdf#nameddest=NPCOEF"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"NPCOEF" -> "PFECAT"
```

15.51 package NFINTBAS NumberFieldIntegralBasis

15.52 NumberFieldIntegralBasis



1.00

Exports:
discriminant integralBasis localIntegralBasis

⟨*package NFINTBAS NumberFieldIntegralBasis*⟩≡

```
)abbrev package NFINTBAS NumberFieldIntegralBasis
++ Author: Victor Miller, Clifton Williamson
++ Date Created: 9 April 1990
++ Date Last Updated: 20 September 1994
++ Basic Operations: discriminant, integralBasis
++ Related Domains: IntegralBasisTools, TriangularMatrixOperations
++ Also See: FunctionFieldIntegralBasis, WildFunctionFieldIntegralBasis
++ AMS Classifications:
++ Keywords: number field, integral basis, discriminant
++ Examples:
++ References:
++ Description:
++   In this package F is a framed algebra over the integers (typically
++   \spad{F = Z[a]} for some algebraic integer a).  The package provides
++   functions to compute the integral closure of Z in the quotient
++   quotient field of F.
NumberFieldIntegralBasis(UP,F): Exports == Implementation where
  UP : UnivariatePolynomialCategory Integer
  F  : FramedAlgebra(Integer,UP)

  FR  ==> Factored Integer
  I   ==> Integer
  Mat ==> Matrix I
  NNI ==> NonNegativeInteger
  Ans ==> Record(basis: Mat, basisDen: I, basisInv:Mat,discr: I)
```

```
Exports ==> with
  discriminant: () -> Integer
    ++ \spad{discriminant()} returns the discriminant of the integral
    ++ closure of Z in the quotient field of the framed algebra F.
  integralBasis : () -> Record(basis: Mat, basisDen: I, basisInv:Mat)
    ++ \spad{integralBasis()} returns a record
    ++ \spad{[basis,basisDen,basisInv]}
    ++ containing information regarding the integral closure of Z in the
    ++ quotient field of F, where F is a framed algebra with Z-module
    ++ basis \spad{w1,w2,...,wn}.
    ++ If \spad{basis} is the matrix \spad{(aij, i = 1..n, j = 1..n)}, then
    ++ the \spad{i}th element of the integral basis is
    ++ \spad{vi = (1/basisDen) * sum(aij * wj, j = 1..n)}, i.e. the
    ++ \spad{i}th row of \spad{basis} contains the coordinates of the
    ++ \spad{i}th basis vector.  Similarly, the \spad{i}th row of the
    ++ matrix \spad{basisInv} contains the coordinates of \spad{wi} with
    ++ respect to the basis \spad{v1,...,vn}: if \spad{basisInv} is the
    ++ matrix \spad{(bij, i = 1..n, j = 1..n)}, then
    ++ \spad{wi = sum(bij * vj, j = 1..n)}.
  localIntegralBasis : I -> Record(basis: Mat, basisDen: I, basisInv:Mat)
    ++ \spad{integralBasis(p)} returns a record
    ++ \spad{[basis,basisDen,basisInv]} containing information regarding
    ++ the local integral closure of Z at the prime \spad{p} in the quotient
    ++ field of F, where F is a framed algebra with Z-module basis
    ++ \spad{w1,w2,...,wn}.
    ++ If \spad{basis} is the matrix \spad{(aij, i = 1..n, j = 1..n)}, then
    ++ the \spad{i}th element of the integral basis is
    ++ \spad{vi = (1/basisDen) * sum(aij * wj, j = 1..n)}, i.e. the
    ++ \spad{i}th row of \spad{basis} contains the coordinates of the
    ++ \spad{i}th basis vector.  Similarly, the \spad{i}th row of the
    ++ matrix \spad{basisInv} contains the coordinates of \spad{wi} with
    ++ respect to the basis \spad{v1,...,vn}: if \spad{basisInv} is the
    ++ matrix \spad{(bij, i = 1..n, j = 1..n)}, then
    ++ \spad{wi = sum(bij * vj, j = 1..n)}.

Implementation ==> add
  import IntegralBasisTools(I, UP, F)
  import ModularHermitianRowReduction(I)
  import TriangularMatrixOperations(I, Vector I, Vector I, Matrix I)

  frobMatrix              : (Mat,Mat,I,NNI) -> Mat
  wildPrimes              : (FR,I) -> List I
  tameProduct             : (FR,I) -> I
  iTameLocalIntegralBasis : (Mat,I,I) -> Ans
  iWildLocalIntegralBasis : (Mat,I,I) -> Ans
```

```
    frobMatrix(rb,rbinv,rbden,p) ==
      n := rank()$F; b := basis()$F
      v : Vector F := new(n,0)
      for i in minIndex(v)..maxIndex(v)
        for ii in minRowIndex(rb)..maxRowIndex(rb) repeat
          a : F := 0
          for j in minIndex(b)..maxIndex(b)
            for jj in minColIndex(rb)..maxColIndex(rb) repeat
              a := a + qelt(rb,ii,jj) * qelt(b,j)
          qsetelt_!(v,i,a**p)
      mat := transpose coordinates v
      ((transpose(rbinv) * mat) exquo (rbden ** p)) :: Mat

    wildPrimes(factoredDisc,n) ==
      -- returns a list of the primes <=n which divide factoredDisc to a
      -- power greater than 1
      ans : List I := empty()
      for f in factors(factoredDisc) repeat
        if f.exponent > 1 and f.factor <= n then ans := concat(f.factor,ans)
      ans

    tameProduct(factoredDisc,n) ==
      -- returns the product of the primes > n which divide factoredDisc
      -- to a power greater than 1
      ans : I := 1
      for f in factors(factoredDisc) repeat
        if f.exponent > 1 and f.factor > n then ans := f.factor * ans
      ans

    integralBasis() ==
      traceMat := traceMatrix()$F; n := rank()$F
      disc := determinant traceMat  -- discriminant of current order
      disc0 := disc                 -- this is disc(F)
      factoredDisc := factor(disc0)$IntegerFactorizationPackage(Integer)
      wilds := wildPrimes(factoredDisc,n)
      sing := tameProduct(factoredDisc,n)
      runningRb := scalarMatrix(n, 1); runningRbinv := scalarMatrix(n, 1)
      -- runningRb    = basis matrix of current order
      -- runningRbinv = inverse basis matrix of current order
      -- these are wrt the original basis for F
      runningRbden : I := 1
      -- runningRbden = denominator for current basis matrix
--      one? sing and empty? wilds => [runningRb, runningRbden, runningRbinv]
      (sing = 1) and empty? wilds => [runningRb, runningRbden, runningRbinv]
      -- id = basis matrix of the ideal (p-radical) wrt current basis
      matrixOut : Mat := scalarMatrix(n,0)
```

```
    for p in wilds repeat
      lb := iWildLocalIntegralBasis(matrixOut,disc,p)
      rb := lb.basis; rbinv := lb.basisInv; rbden := lb.basisDen
      disc := lb.discr
      -- update 'running integral basis' if newly computed
      -- local integral basis is non-trivial
      if sizeLess?(1,rbden) then
        mat := vertConcat(rbden * runningRb,runningRbden * rb)
        runningRbden := runningRbden * rbden
        runningRb := squareTop rowEchelon(mat,runningRbden)
        runningRbinv := UpTriBddDenomInv(runningRb,runningRbden)
    lb := iTameLocalIntegralBasis(traceMat,disc,sing)
    rb := lb.basis; rbinv := lb.basisInv; rbden := lb.basisDen
    disc := lb.discr
    -- update 'running integral basis' if newly computed
    -- local integral basis is non-trivial
    if sizeLess?(1,rbden) then
      mat := vertConcat(rbden * runningRb,runningRbden * rb)
      runningRbden := runningRbden * rbden
      runningRb := squareTop rowEchelon(mat,runningRbden)
      runningRbinv := UpTriBddDenomInv(runningRb,runningRbden)
    [runningRb,runningRbden,runningRbinv]

  localIntegralBasis p ==
    traceMat := traceMatrix()$F; n := rank()$F
    disc := determinant traceMat  -- discriminant of current order
    (disc exquo (p*p)) case "failed" =>
      [scalarMatrix(n, 1), 1, scalarMatrix(n, 1)]
    lb :=
      p > rank()$F =>
        iTameLocalIntegralBasis(traceMat,disc,p)
      iWildLocalIntegralBasis(scalarMatrix(n,0),disc,p)
    [lb.basis,lb.basisDen,lb.basisInv]

  iTameLocalIntegralBasis(traceMat,disc,sing) ==
    n := rank()$F; disc0 := disc
    rb := scalarMatrix(n, 1); rbinv := scalarMatrix(n, 1)
    -- rb    = basis matrix of current order
    -- rbinv = inverse basis matrix of current order
    -- these are wrt the original basis for F
    rbden : I := 1; index : I := 1; oldIndex : I := 1
    -- rbden = denominator for current basis matrix
    -- id = basis matrix of the ideal (p-radical) wrt current basis
    tfm := traceMat
    repeat
      -- compute the p-radical = p-trace-radical
```

```
        idinv := transpose squareTop rowEchelon(tfm,sing)
        -- [u1,..,un] are the coordinates of an element of the p-radical
        -- iff [u1,..,un] * idinv is in p * Z^n
        id := rowEchelon LowTriBddDenomInv(idinv, sing)
        -- id = basis matrix of the p-radical
        idinv := UpTriBddDenomInv(id, sing)
        -- id * idinv = sing * identity
        -- no need to check for inseparability in this case
        rbinv := idealiser(id * rb, rbinv * idinv, sing * rbden)
        index := diagonalProduct rbinv
        rb := rowEchelon LowTriBddDenomInv(rbinv, sing * rbden)
        g := matrixGcd(rb,sing,n)
        if sizeLess?(1,g) then rb := (rb exquo g) :: Mat
        rbden := rbden * (sing quo g)
        rbinv := UpTriBddDenomInv(rb, rbden)
        disc := disc0 quo (index * index)
        indexChange := index quo oldIndex; oldIndex := index
--        one? indexChange => return [rb, rbden, rbinv, disc]
        (indexChange = 1) => return [rb, rbden, rbinv, disc]
        tfm := ((rb * traceMat * transpose rb) exquo (rbden * rbden)) :: Mat

    iWildLocalIntegralBasis(matrixOut,disc,p) ==
      n := rank()$F; disc0 := disc
      rb := scalarMatrix(n, 1); rbinv := scalarMatrix(n, 1)
      -- rb    = basis matrix of current order
      -- rbinv = inverse basis matrix of current order
      -- these are wrt the original basis for F
      rbden : I := 1; index : I := 1; oldIndex : I := 1
      -- rbden = denominator for current basis matrix
      -- id = basis matrix of the ideal (p-radical) wrt current basis
      p2 := p * p; lp := leastPower(p::NNI,n)
      repeat
        tfm := frobMatrix(rb,rbinv,rbden,p::NNI) ** lp
        -- compute Rp = p-radical
        idinv := transpose squareTop rowEchelon(tfm, p)
        -- [u1,..,un] are the coordinates of an element of Rp
        -- iff [u1,..,un] * idinv is in p * Z^n
        id := rowEchelon LowTriBddDenomInv(idinv,p)
        -- id = basis matrix of the p-radical
        idinv := UpTriBddDenomInv(id,p)
        -- id * idinv = p * identity
        -- no need to check for inseparability in this case
        rbinv := idealiser(id * rb, rbinv * idinv, p * rbden)
        index := diagonalProduct rbinv
        rb := rowEchelon LowTriBddDenomInv(rbinv, p * rbden)
        if divideIfCan_!(rb,matrixOut,p,n) = 1
```

```
            then rb := matrixOut
            else rbden := p * rbden
          rbinv := UpTriBddDenomInv(rb, rbden)
          indexChange := index quo oldIndex; oldIndex := index
          disc := disc quo (indexChange * indexChange)
--          one? indexChange or gcd(p2,disc) ^= p2 =>
          (indexChange = 1) or gcd(p2,disc) ^= p2 =>
            return [rb, rbden, rbinv, disc]

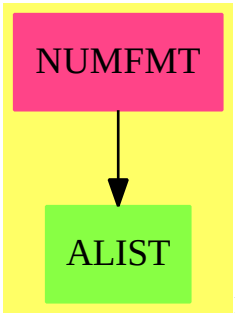
      discriminant() ==
        disc := determinant traceMatrix()$F
        intBas := integralBasis()
        rb := intBas.basis; rbden := intBas.basisDen
        index := ((rbden ** rank()$F) exquo (determinant rb)) :: Integer
        (disc exquo (index * index)) :: Integer
```

⟨*NFINTBAS.dotabb*⟩≡

```
"NFINTBAS" [color="#FF4488",href="bookvol10.4.pdf#nameddest=NFINTBAS"]
"FRAMALG" [color="#4488FF",href="bookvol10.2.pdf#nameddest=FRAMALG"]
"NFINTBAS" -> "FRAMALG"
```

15.53 package NUMFMT NumberFormats

15.54 NumberFormats



1.00

Exports:

FormatArabic	FormatRoman	ScanArabic	ScanFloatIgnoreSpaces
ScanFloatIgnoreSpacesIfCan	ScanRoman		

⟨*package NUMFMT NumberFormats*⟩≡

```
)abbrev package NUMFMT NumberFormats
++ SMW March 88
++ Keywords: string manipulation, roman numerals, format
++ Description:
++ NumberFormats provides function to format and read arabic and
++ roman numbers, to convert numbers to strings and to read
++ floating-point numbers.

NumberFormats(): NFexports == NFimplementation where
    PI ==> PositiveInteger
    I  ==> Integer
    C  ==> Character
    F  ==> Float
    S  ==> String
    V  ==> PrimitiveArray

    NFexports ==> with
        FormatArabic: PI -> S
            ++ FormatArabic(n) forms an Arabic numeral
            ++ string from an integer n.
        ScanArabic:   S -> PI
            ++ ScanArabic(s) forms an integer from an Arabic numeral string s.
        FormatRoman:  PI -> S
            ++ FormatRoman(n) forms a Roman numeral string from an integer n.
        ScanRoman:    S -> PI
            ++ ScanRoman(s) forms an integer from a Roman numeral string s.
```

```
        ScanFloatIgnoreSpaces: S -> F
            ++ ScanFloatIgnoreSpaces(s) forms a floating point number from
            ++ the string s ignoring any spaces. Error is generated if the
            ++ string is not recognised as a floating point number.
        ScanFloatIgnoreSpacesIfCan: S -> Union(F, "failed")
            ++ ScanFloatIgnoreSpacesIfCan(s) tries to form a floating point
            ++ number from the string s ignoring any spaces.


    NFimplementation ==> add
        import SExpression
        import Symbol
        replaceD: C -> C
        replaced: C -> C
        contract: S -> S
        check: S ->Boolean
        replaceD c ==
          if c = char "D" then char "E" else c
        replaced c ==
          if c = char "d" then char "E" else c
        contract s ==
          s:= map(replaceD,s)
          s:= map(replaced,s)
          ls:List S := split(s,char " ")$String
          s:= concat ls
        check s ==
          NUMBERP(READ_-FROM_-STRING(s)$Lisp)$Lisp and
           -- if there is an "E" then there must be a "."
           -- this is not caught by code above
           -- also if the exponent is v.big the above returns false
           not (any?(#1=char "E",s) and not any?(#1=char ".",s) )

--        Original interpreter function:
--        )lis (defun scanstr(x) (spadcomp::|parseFromString| x))
        sexfloat:SExpression:=convert(coerce("Float")@Symbol)$SExpression
        ScanFloatIgnoreSpaces s ==
          s := contract s
          not check s => error "Non-numeric value"
          sex := interpret(packageTran(ncParseFromString(s)$Lisp)$Lisp)$Lisp
          sCheck := car(car(sex))
          if (sCheck=sexfloat) = true then
             f := (cdr cdr sex) pretend Float
          else
             if integer?(cdr sex) = true then
                f := (cdr sex) pretend Integer
                f::F
```

```
      else
         error "Non-numeric value"

ScanFloatIgnoreSpacesIfCan s ==
  s := contract s
  not check s => "failed"
  sex := interpret(packageTran(ncParseFromString(s)$Lisp)$Lisp)$Lisp
  sCheck := car(car(sex))
  if (sCheck=sexfloat) = true then
     f := (cdr cdr sex) pretend Float
  else
     if integer?(cdr sex) = true then
        f := (cdr sex) pretend Integer
        f::F
     else
        "failed"

units:V S :=
   construct ["","I","II","III","IV","V","VI","VII","VIII","IX"]
tens :V S :=
   construct ["","X","XX","XXX","XL","L","LX","LXX","LXXX","XC"]
hunds:V S :=
   construct ["","C","CC","CCC","CD","D","DC","DCC","DCCC","CM"]
umin := minIndex units
tmin := minIndex tens
hmin := minIndex hunds
romval:V I := new(256, -1)
romval ord char(" ")$C := 0
romval ord char("I")$C := 1
romval ord char("V")$C := 5
romval ord char("X")$C := 10
romval ord char("L")$C := 50
romval ord char("C")$C := 100
romval ord char("D")$C := 500
romval ord char("M")$C := 1000
thou:C  := char "M"
plen:C  := char "("
pren:C  := char ")"
ichar:C := char "I"

FormatArabic n == STRINGIMAGE(n)$Lisp
ScanArabic   s == PARSE_-INTEGER(s)$Lisp

FormatRoman pn ==
    n := pn::Integer
    -- Units
```

```
        d := (n rem 10) + umin
        n := n quo 10
        s := units.d
        zero? n => s
        -- Tens
        d := (n rem 10) + tmin
        n := n quo 10
        s := concat(tens.d, s)
        zero? n => s
        -- Hundreds
        d := (n rem 10) + hmin
        n := n quo 10
        s := concat(hunds.d, s)
        zero? n => s
        -- Thousands
        d := n rem 10
        n := n quo 10
        s := concat(new(d::NonNegativeInteger, thou), s)
        zero? n => s
        -- Ten thousand and higher
        for i in 2.. while not zero? n repeat
            -- Coefficient of 10**(i+2)
            d := n rem 10
            n := n quo 10
            zero? d => "iterate"
            m0:String := concat(new(i,plen),concat("I",new(i,pren)))
            mm := concat([m0 for j in 1..d]$List(String))
            -- strictly speaking the blank is gratuitous
            if #s > 0 then s := concat(" ", s)
            s  := concat(mm, s)
        s

    -- ScanRoman
    --
    -- The Algorithm:
    --    Read number from right to left.  When the current
    --    numeral is lower in magnitude than the previous maximum
    --    then subtract otherwise add.
    --    Shift left and repeat until done.

    ScanRoman s ==
        s      := upperCase s
        tot: I := 0
        Max: I := 0
        i:   I := maxIndex s
        while i >= minIndex s repeat
```

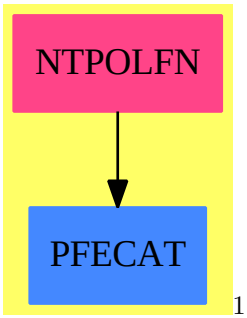
```
                -- Read a single roman digit
                c := s.i; i := i-1
                n := romval ord c
                -- (I)=1000, ((I))=10000, (((I)))=100000, etc
                if n < 0 then
                    c ^= pren =>
                       error ["Improper character in Roman numeral: ",c]
                    nprens: PI := 1
                    while c = pren and i >= minIndex s repeat
                       c := s.i; i := i-1
                       if c = pren then nprens := nprens+1
                    c ^= ichar =>
                       error "Improper Roman numeral: (x)"
                    for k in 1..nprens while i >= minIndex s repeat
                       c := s.i; i := i-1
                       c ^= plen =>
                          error "Improper Roman numeral: unbalanced ')'"
                    n := 10**(nprens + 2)
                if n < Max then
                    tot := tot - n
                else
                    tot := tot + n
                    Max := n
            tot < 0 => error ["Improper Roman numeral: ", tot]
            tot::PI
```

⟨*NUMFMT.dotabb*⟩≡

```
"NUMFMT" [color="#FF4488",href="bookvol10.4.pdf#nameddest=NUMFMT"]
"ALIST" [color="#88FF44",href="bookvol10.3.pdf#nameddest=ALIST"]
"NUMFMT" -> "ALIST"
```

15.55 package NTPOLFN NumberTheoreticPolynomialFunctions

15.56 NumberTheoreticPolynomialFunctions



1.00

Exports:
bernoulliB cyclotomic eulerE

⟨*package NTPOLFN NumberTheoreticPolynomialFunctions*⟩≡

```
)abbrev package NTPOLFN NumberTheoreticPolynomialFunctions
++ Author: Stephen M. Watt
++ Date Created:  1990
++ Date Last Updated: June 25, 1991
++ Basic Operations:
++ Related Domains:
++ Also See:
++ AMS Classifications:
++ Keywords:
++ Examples:
++ References:
++ Description:
++   This package provides polynomials as functions on a ring.

NumberTheoreticPolynomialFunctions(R: CommutativeRing): Exports == Impl where
    NNI ==> NonNegativeInteger
    RN  ==> Fraction Integer

    Exports ==> with

        cyclotomic: (NNI, R) -> R
         ++ cyclotomic(n,r) \undocumented

        if R has Algebra RN then
            bernoulliB: (NNI, R) -> R
             ++ bernoulliB(n,r) \undocumented
```

```
            eulerE:     (NNI, R) -> R
             ++ eulerE(n,r) \undocumented

    Impl ==> add

        import PolynomialNumberTheoryFunctions()

        I   ==> Integer
        SUP ==> SparseUnivariatePolynomial

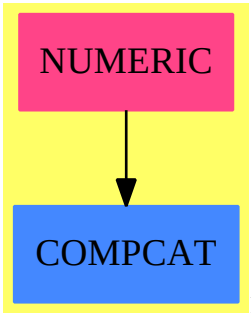
        -- This is the wrong way to evaluate the polynomial.
        cyclotomic(k, x) ==
            p: SUP(I) := cyclotomic(k)
            r: R      := 0
            while p ^= 0 repeat
                d := degree p
                c := leadingCoefficient p
                p := reductum p
                r := c*x**d + r
            r

        if R has Algebra RN then
            eulerE(k, x) ==
                p: SUP(RN) := euler(k)
                r: R       := 0
                while p ^= 0 repeat
                    d := degree p
                    c := leadingCoefficient p
                    p := reductum p
                    r := c*x**d + r
                r
            bernoulliB(k, x) ==
                p: SUP(RN) := bernoulli(k)
                r: R       := 0
                while p ^= 0 repeat
                    d := degree p
                    c := leadingCoefficient p
                    p := reductum p
                    r := c*x**d + r
                r
```

⟨*NTPOLFN.dotabb*⟩≡
```
"NTPOLFN" [color="#FF4488",href="bookvol10.4.pdf#nameddest=NTPOLFN"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"NTPOLFN" -> "PFECAT"
```

15.57 package NUMERIC Numeric

15.58 Numeric



1.00

Exports:
numeric complexNumeric complexNumericIfCan numeric numericIfCan

⟨*package NUMERIC Numeric*⟩≡

```
)abbrev package NUMERIC Numeric
++ Author: Manuel Bronstein
++ Date Created: 21 Feb 1990
++ Date Last Updated: 17 August 1995, Mike Dewar
++                    24 January 1997, Miked Dewar (added partial operators)
++ Basic Operations: numeric, complexNumeric, numericIfCan, complexNumericIfCan
++ Related Constructors:
++ Also See:
++ AMS Classifications:
++ Keywords:
++ References:
++ Description: Numeric provides real and complex numerical evaluation
++ functions for various symbolic types.

Numeric(S:ConvertibleTo Float): with
  numeric: S -> Float
    ++ numeric(x) returns a real approximation of x.
  numeric: (S, PositiveInteger) -> Float
    ++ numeric(x, n) returns a real approximation of x up to n decimal
    ++ places.
  complexNumeric: S -> Complex Float
    ++ complexNumeric(x) returns a complex approximation of x.
  complexNumeric: (S, PositiveInteger) -> Complex Float
    ++ complexNumeric(x, n) returns a complex approximation of x up
    ++ to n decimal places.
  if S has CommutativeRing then
    complexNumeric: Complex S -> Complex Float
```

```
      ++ complexNumeric(x) returns a complex approximation of x.
    complexNumeric: (Complex S, PositiveInteger) -> Complex Float
      ++ complexNumeric(x, n) returns a complex approximation of x up
      ++ to n decimal places.
    complexNumeric: Polynomial Complex S -> Complex Float
      ++ complexNumeric(x) returns a complex approximation of x.
    complexNumeric: (Polynomial Complex S, PositiveInteger) -> Complex Float
      ++ complexNumeric(x, n) returns a complex approximation of x up
      ++ to n decimal places.
  if S has Ring then
    numeric: Polynomial S -> Float
      ++ numeric(x) returns a real approximation of x.
    numeric: (Polynomial S, PositiveInteger) -> Float
      ++ numeric(x,n) returns a real approximation of x up to n decimal
      ++ places.
    complexNumeric: Polynomial S -> Complex Float
      ++ complexNumeric(x) returns a complex approximation of x.
    complexNumeric: (Polynomial S, PositiveInteger) -> Complex Float
      ++ complexNumeric(x, n) returns a complex approximation of x
      ++ up to n decimal places.
  if S has IntegralDomain then
    numeric: Fraction Polynomial S -> Float
      ++ numeric(x) returns a real approximation of x.
    numeric: (Fraction Polynomial S, PositiveInteger) -> Float
      ++ numeric(x,n) returns a real approximation of x up to n decimal
      ++ places.
    complexNumeric: Fraction Polynomial S -> Complex Float
      ++ complexNumeric(x) returns a complex approximation of x.
    complexNumeric: (Fraction Polynomial S, PositiveInteger) -> Complex Float
      ++ complexNumeric(x, n) returns a complex approximation of x
    complexNumeric: Fraction Polynomial Complex S -> Complex Float
      ++ complexNumeric(x) returns a complex approximation of x.
    complexNumeric: (Fraction Polynomial Complex S, PositiveInteger) ->
                                                              Complex Float
      ++ complexNumeric(x, n) returns a complex approximation of x
      ++ up to n decimal places.
    if S has OrderedSet then
      numeric: Expression S -> Float
        ++ numeric(x) returns a real approximation of x.
      numeric: (Expression S, PositiveInteger) -> Float
        ++ numeric(x, n) returns a real approximation of x up to n
        ++ decimal places.
      complexNumeric: Expression S -> Complex Float
        ++ complexNumeric(x) returns a complex approximation of x.
      complexNumeric: (Expression S, PositiveInteger) -> Complex Float
        ++ complexNumeric(x, n) returns a complex approximation of x
```

```
      ++ up to n decimal places.
    complexNumeric: Expression Complex S -> Complex Float
      ++ complexNumeric(x) returns a complex approximation of x.
    complexNumeric: (Expression Complex S, PositiveInteger) -> Complex Float
      ++ complexNumeric(x, n) returns a complex approximation of x
      ++ up to n decimal places.
  if S has CommutativeRing then
    complexNumericIfCan: Polynomial Complex S -> Union(Complex Float,"failed")
      ++ complexNumericIfCan(x) returns a complex approximation of x,
      ++ or "failed" if \axiom{x} is not constant.
    complexNumericIfCan: (Polynomial Complex S, PositiveInteger) -> Union(Complex
      ++ complexNumericIfCan(x, n) returns a complex approximation of x up
      ++ to n decimal places, or "failed" if \axiom{x} is not a constant.
  if S has Ring then
    numericIfCan: Polynomial S -> Union(Float,"failed")
      ++ numericIfCan(x) returns a real approximation of x,
      ++ or "failed" if \axiom{x} is not a constant.
    numericIfCan: (Polynomial S, PositiveInteger) -> Union(Float,"failed")
      ++ numericIfCan(x,n) returns a real approximation of x up to n decimal
      ++ places, or "failed" if \axiom{x} is not a constant.
    complexNumericIfCan: Polynomial S -> Union(Complex Float,"failed")
      ++ complexNumericIfCan(x) returns a complex approximation of x,
      ++ or "failed" if \axiom{x} is not a constant.
    complexNumericIfCan: (Polynomial S, PositiveInteger) -> Union(Complex Float,"
      ++ complexNumericIfCan(x, n) returns a complex approximation of x
      ++ up to n decimal places, or "failed" if \axiom{x} is not a constant.
  if S has IntegralDomain then
    numericIfCan: Fraction Polynomial S -> Union(Float,"failed")
      ++ numericIfCan(x) returns a real approximation of x,
      ++ or "failed" if \axiom{x} is not a constant.
    numericIfCan: (Fraction Polynomial S, PositiveInteger) -> Union(Float,"failed
      ++ numericIfCan(x,n) returns a real approximation of x up to n decimal
      ++ places, or "failed" if \axiom{x} is not a constant.
    complexNumericIfCan: Fraction Polynomial S -> Union(Complex Float,"failed")
      ++ complexNumericIfCan(x) returns a complex approximation of x,
      ++ or "failed" if \axiom{x} is not a constant.
    complexNumericIfCan: (Fraction Polynomial S, PositiveInteger) -> Union(Comple
      ++ complexNumericIfCan(x, n) returns a complex approximation of x,
      ++ or "failed" if \axiom{x} is not a constant.
    complexNumericIfCan: Fraction Polynomial Complex S -> Union(Complex Float,"fa
      ++ complexNumericIfCan(x) returns a complex approximation of x,
      ++ or "failed" if \axiom{x} is not a constant.
    complexNumericIfCan: (Fraction Polynomial Complex S, PositiveInteger) ->
                                                  Union(Complex Float,"failed")
      ++ complexNumericIfCan(x, n) returns a complex approximation of x
      ++ up to n decimal places, or "failed" if \axiom{x} is not a constant.
```

```
    if S has OrderedSet then
      numericIfCan: Expression S -> Union(Float,"failed")
        ++ numericIfCan(x) returns a real approximation of x,
        ++ or "failed" if \axiom{x} is not a constant.
      numericIfCan: (Expression S, PositiveInteger) -> Union(Float,"failed")
        ++ numericIfCan(x, n) returns a real approximation of x up to n
        ++ decimal places, or "failed" if \axiom{x} is not a constant.
      complexNumericIfCan: Expression S -> Union(Complex Float,"failed")
        ++ complexNumericIfCan(x) returns a complex approximation of x,
        ++ or "failed" if \axiom{x} is not a constant.
      complexNumericIfCan: (Expression S, PositiveInteger) ->
                                              Union(Complex Float,"failed")
        ++ complexNumericIfCan(x, n) returns a complex approximation of x
        ++ up to n decimal places, or "failed" if \axiom{x} is not a constant.
      complexNumericIfCan: Expression Complex S -> Union(Complex Float,"failed")
        ++ complexNumericIfCan(x) returns a complex approximation of x,
        ++ or "failed" if \axiom{x} is not a constant.
      complexNumericIfCan: (Expression Complex S, PositiveInteger) ->
                                               Union(Complex Float,"failed")
        ++ complexNumericIfCan(x, n) returns a complex approximation of x
        ++ up to n decimal places, or "failed" if \axiom{x} is not a constant.
  == add

   if S has CommutativeRing then
     complexNumericIfCan(p:Polynomial Complex S) ==
       p' : Union(Complex(S),"failed") := retractIfCan p
       p' case "failed" => "failed"
       complexNumeric(p')

     complexNumericIfCan(p:Polynomial Complex S,n:PositiveInteger) ==
       p' : Union(Complex(S),"failed") := retractIfCan p
       p' case "failed" => "failed"
       complexNumeric(p',n)

   if S has Ring then
     numericIfCan(p:Polynomial S) ==
       p' : Union(S,"failed") := retractIfCan p
       p' case "failed" => "failed"
       numeric(p')

     complexNumericIfCan(p:Polynomial S) ==
       p' : Union(S,"failed") := retractIfCan p
       p' case "failed" => "failed"
       complexNumeric(p')

     complexNumericIfCan(p:Polynomial S, n:PositiveInteger) ==
```

```
      p' : Union(S,"failed") := retractIfCan p
      p' case "failed" => "failed"
      complexNumeric(p', n)

    numericIfCan(p:Polynomial S, n:PositiveInteger) ==
      old := digits(n)$Float
      ans := numericIfCan p
      digits(old)$Float
      ans

  if S has IntegralDomain then
    numericIfCan(f:Fraction Polynomial S)==
      num := numericIfCan(numer(f))
      num case "failed" => "failed"
      den := numericIfCan(denom f)
      den case "failed" => "failed"
      num/den

    complexNumericIfCan(f:Fraction Polynomial S) ==
      num := complexNumericIfCan(numer f)
      num case "failed" => "failed"
      den := complexNumericIfCan(denom f)
      den case "failed" => "failed"
      num/den

    complexNumericIfCan(f:Fraction Polynomial S, n:PositiveInteger) ==
      num := complexNumericIfCan(numer f, n)
      num case "failed" => "failed"
      den := complexNumericIfCan(denom f, n)
      den case "failed" => "failed"
      num/den

    numericIfCan(f:Fraction Polynomial S, n:PositiveInteger) ==
      old := digits(n)$Float
      ans := numericIfCan f
      digits(old)$Float
      ans

    complexNumericIfCan(f:Fraction Polynomial Complex S) ==
      num := complexNumericIfCan(numer f)
      num case "failed" => "failed"
      den := complexNumericIfCan(denom f)
      den case "failed" => "failed"
      num/den

    complexNumericIfCan(f:Fraction Polynomial Complex S, n:PositiveInteger) ==
```

```
    num := complexNumericIfCan(numer f, n)
    num case "failed" => "failed"
    den := complexNumericIfCan(denom f, n)
    den case "failed" => "failed"
    num/den

  if S has OrderedSet then
    numericIfCan(x:Expression S) ==
      retractIfCan(map(convert, x)$ExpressionFunctions2(S, Float))

    --s2cs(u:S):Complex(S) == complex(u,0)

    complexNumericIfCan(x:Expression S) ==
       complexNumericIfCan map(coerce, x)$ExpressionFunctions2(S,Complex S)

    numericIfCan(x:Expression S, n:PositiveInteger) ==
      old := digits(n)$Float
      x' : Expression Float := map(convert, x)$ExpressionFunctions2(S, Float)
      ans : Union(Float,"failed") := retractIfCan x'
      digits(old)$Float
      ans

    complexNumericIfCan(x:Expression S, n:PositiveInteger) ==
      old := digits(n)$Float
      x' : Expression Complex S := map(coerce, x)$ExpressionFunctions2(S, Complex S)
      ans : Union(Complex Float,"failed") := complexNumericIfCan(x')
      digits(old)$Float
      ans

    if S has RealConstant then
      complexNumericIfCan(x:Expression Complex S) ==
        retractIfCan(map(convert, x)$ExpressionFunctions2(Complex S,Complex Float))

      complexNumericIfCan(x:Expression Complex S, n:PositiveInteger) ==
        old := digits(n)$Float
        x' : Expression Complex Float :=
         map(convert, x)$ExpressionFunctions2(Complex S,Complex Float)
        ans : Union(Complex Float,"failed") := retractIfCan x'
        digits(old)$Float
        ans
    else
      convert(x:Complex S):Complex(Float)==map(convert,x)$ComplexFunctions2(S,Float)

      complexNumericIfCan(x:Expression Complex S) ==
        retractIfCan(map(convert, x)$ExpressionFunctions2(Complex S,Complex Float))
```

```
      complexNumericIfCan(x:Expression Complex S, n:PositiveInteger) ==
        old := digits(n)$Float
        x' : Expression Complex Float :=
         map(convert, x)$ExpressionFunctions2(Complex S,Complex Float)
        ans : Union(Complex Float,"failed") := retractIfCan x'
        digits(old)$Float
        ans
  numeric(s:S) == convert(s)@Float

  if S has ConvertibleTo Complex Float then
    complexNumeric(s:S) == convert(s)@Complex(Float)

    complexNumeric(s:S, n:PositiveInteger) ==
      old := digits(n)$Float
      ans := complexNumeric s
      digits(old)$Float
      ans

  else
    complexNumeric(s:S) == convert(s)@Float :: Complex(Float)

    complexNumeric(s:S,n:PositiveInteger) ==
      numeric(s, n)::Complex(Float)

  if S has CommutativeRing then
    complexNumeric(p:Polynomial Complex S) ==
      p' : Union(Complex(S),"failed") := retractIfCan p
      p' case "failed" =>
        error "Cannot compute the numerical value of a non-constant polynomial"
      complexNumeric(p')

    complexNumeric(p:Polynomial Complex S,n:PositiveInteger) ==
      p' : Union(Complex(S),"failed") := retractIfCan p
      p' case "failed" =>
        error "Cannot compute the numerical value of a non-constant polynomial"
      complexNumeric(p',n)

    if S has RealConstant then
      complexNumeric(s:Complex S) == convert(s)$Complex(S)

      complexNumeric(s:Complex S, n:PositiveInteger) ==
        old := digits(n)$Float
        ans := complexNumeric s
        digits(old)$Float
        ans
```

```
    else if Complex(S) has ConvertibleTo(Complex Float) then
      complexNumeric(s:Complex S) == convert(s)@Complex(Float)

      complexNumeric(s:Complex S, n:PositiveInteger) ==
        old := digits(n)$Float
        ans := complexNumeric s
        digits(old)$Float
        ans

    else
      complexNumeric(s:Complex S) ==
        s' : Union(S,"failed") := retractIfCan s
        s' case "failed" =>
          error "Cannot compute the numerical value of a non-constant object"
        complexNumeric(s')

      complexNumeric(s:Complex S, n:PositiveInteger) ==
        s' : Union(S,"failed") := retractIfCan s
        s' case "failed" =>
          error "Cannot compute the numerical value of a non-constant object"
        old := digits(n)$Float
        ans := complexNumeric s'
        digits(old)$Float
        ans

  numeric(s:S, n:PositiveInteger) ==
    old := digits(n)$Float
    ans := numeric s
    digits(old)$Float
    ans

  if S has Ring then
    numeric(p:Polynomial S) ==
      p' : Union(S,"failed") := retractIfCan p
      p' case "failed" => error
       "Can only compute the numerical value of a constant, real-valued polynomial"
      numeric(p')

    complexNumeric(p:Polynomial S) ==
      p' : Union(S,"failed") := retractIfCan p
      p' case "failed" =>
        error "Cannot compute the numerical value of a non-constant polynomial"
      complexNumeric(p')

    complexNumeric(p:Polynomial S, n:PositiveInteger) ==
      p' : Union(S,"failed") := retractIfCan p
```

```
        p' case "failed" =>
          error "Cannot compute the numerical value of a non-constant polynomial"
        complexNumeric(p', n)

      numeric(p:Polynomial S, n:PositiveInteger) ==
        old := digits(n)$Float
        ans := numeric p
        digits(old)$Float
        ans

    if S has IntegralDomain then
      numeric(f:Fraction Polynomial S)==
          numeric(numer(f)) / numeric(denom f)

      complexNumeric(f:Fraction Polynomial S) ==
        complexNumeric(numer f)/complexNumeric(denom f)

      complexNumeric(f:Fraction Polynomial S, n:PositiveInteger) ==
        complexNumeric(numer f, n)/complexNumeric(denom f, n)

      numeric(f:Fraction Polynomial S, n:PositiveInteger) ==
        old := digits(n)$Float
        ans := numeric f
        digits(old)$Float
        ans

      complexNumeric(f:Fraction Polynomial Complex S) ==
        complexNumeric(numer f)/complexNumeric(denom f)

      complexNumeric(f:Fraction Polynomial Complex S, n:PositiveInteger) ==
        complexNumeric(numer f, n)/complexNumeric(denom f, n)

      if S has OrderedSet then
        numeric(x:Expression S) ==
          x' : Union(Float,"failed") :=
           retractIfCan(map(convert, x)$ExpressionFunctions2(S, Float))
          x' case "failed" => error
           "Can only compute the numerical value of a constant, real-valued Express
          x'

        complexNumeric(x:Expression S) ==
          x' : Union(Complex Float,"failed") := retractIfCan(
           map(complexNumeric, x)$ExpressionFunctions2(S,Complex Float))
          x' case "failed" =>
           error "Cannot compute the numerical value of a non-constant expression"
          x'
```

```
numeric(x:Expression S, n:PositiveInteger) ==
  old := digits(n)$Float
  x' : Expression Float := map(convert, x)$ExpressionFunctions2(S, Float)
  ans : Union(Float,"failed") := retractIfCan x'
  digits(old)$Float
  ans case "failed" => error
   "Can only compute the numerical value of a constant, real-valued Expression"
  ans

complexNumeric(x:Expression S, n:PositiveInteger) ==
  old := digits(n)$Float
  x' : Expression Complex Float :=
   map(complexNumeric, x)$ExpressionFunctions2(S,Complex Float)
  ans : Union(Complex Float,"failed") := retractIfCan x'
  digits(old)$Float
  ans case "failed" =>
   error "Cannot compute the numerical value of a non-constant expression"
  ans

complexNumeric(x:Expression Complex S) ==
  x' : Union(Complex Float,"failed") := retractIfCan(
   map(complexNumeric, x)$ExpressionFunctions2(Complex S,Complex Float))
  x' case "failed" =>
   error "Cannot compute the numerical value of a non-constant expression"
  x'

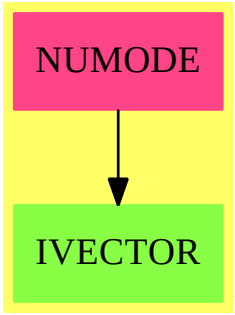
complexNumeric(x:Expression Complex S, n:PositiveInteger) ==
  old := digits(n)$Float
  x' : Expression Complex Float :=
   map(complexNumeric, x)$ExpressionFunctions2(Complex S,Complex Float)
  ans : Union(Complex Float,"failed") := retractIfCan x'
  digits(old)$Float
  ans case "failed" =>
   error "Cannot compute the numerical value of a non-constant expression"
  ans
```

⟨*NUMERIC.dotabb*⟩≡

```
"NUMERIC" [color="#FF4488",href="bookvol10.4.pdf#nameddest=NUMERIC"]
"COMPCAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=COMPCAT"]
"NUMERIC" -> "COMPCAT"
```

15.59 package NUMODE NumericalOrdinaryDifferentialEquations

15.60 NumericalOrdinaryDifferentialEquations



1.00

Exports:
rk4 rk4a rk4f rk4qc rk4qc

⟨*package NUMODE NumericalOrdinaryDifferentialEquations*⟩≡

```
)abbrev package NUMODE NumericalOrdinaryDifferentialEquations
++ Author: Yurij Baransky
++ Date Created: October 90
++ Date Last Updated: October 90
++ Basic Operations:
++ Related Constructors:
++ Also See:
++ AMS Classifications:
++ Keywords:
++ References:
++ Description:
++ This package is a suite of functions for the numerical integration of an
++ ordinary differential equation of n variables:
++
++         \center{dy/dx = f(y,x)\space{5}y is an n-vector}
++
++ \par All the routines are based on a 4-th order Runge-Kutta kernel.
++ These routines generally have as arguments:
++ n, the number of dependent variables;
++ x1, the initial point;
++ h, the step size;
++ y, a vector of initial conditions of length n which upon exit contains the sol
++ \spad{derivs}, a function which computes the right hand side of the
++ ordinary differential equation: \spad{derivs(dydx,y,x)} computes \spad{dydx},
++ a vector which contains the derivative information.
++
```

```
++ \par In order of increasing complexity:\begin{items}
++
++ \item \spad{rk4(y,n,x1,h,derivs)} advances the solution vector to
++ \spad{x1 + h} and return the values in y.
++
++ \item \spad{rk4(y,n,x1,h,derivs,t1,t2,t3,t4)} is the same as
++ \spad{rk4(y,n,x1,h,derivs)} except that you must provide 4 scratch
++ arrays t1-t4 of size n.
++
++ \item Starting with y at x1, \spad{rk4f(y,n,x1,x2,ns,derivs)}
++ uses \spad{ns} fixed
++ steps of a 4-th order Runge-Kutta integrator to advance the
++ solution vector to x2 and return the values in y.
++ Argument x2, is the final point, and
++ \spad{ns}, the number of steps to take.
++
++ \item \spad{rk4qc(y,n,x1,step,eps,yscal,derivs)} takes a 5-th order
++ Runge-Kutta step with monitoring
++ of local truncation to ensure accuracy and adjust stepsize.
++ The function takes two half steps and one full step and scales
++ the difference in solutions at the final point. If the error is
++ within \spad{eps}, the step is taken and the result is returned.
++ If the error is not within \spad{eps}, the stepsize if decreased
++ and the procedure is tried again until the desired accuracy is
++ reached. Upon input, an trial step size must be given and upon
++ return, an estimate of the next step size to use is returned as
++ well as the step size which produced the desired accuracy.
++ The scaled error is computed as
++ \center{\spad{error = MAX(ABS((y2steps(i) - y1step(i))/yscal(i)))}}
++ and this is compared against \spad{eps}. If this is greater
++ than \spad{eps}, the step size is reduced accordingly to
++ \center{\spad{hnew = 0.9 * hdid * (error/eps)**(-1/4)}}
++ If the error criterion is satisfied, then we check if the
++ step size was too fine and return a more efficient one. If
++ \spad{error > \spad{eps} * (6.0E-04)} then the next step size should be
++ \center{\spad{hnext = 0.9 * hdid * (error/\spad{eps})**(-1/5)}}
++ Otherwise \spad{hnext = 4.0 * hdid} is returned.
++ A more detailed discussion of this and related topics can be
++ found in the book "Numerical Recipies" by W.Press, B.P. Flannery,
++ S.A. Teukolsky, W.T. Vetterling published by Cambridge University Press.
++ Argument \spad{step} is a record of 3 floating point
++ numbers \spad{(try , did , next)},
++ \spad{eps} is the required accuracy,
++ \spad{yscal} is the scaling vector for the difference in solutions.
++ On input, \spad{step.try} should be the guess at a step
++ size to achieve the accuracy.
```

```
++ On output, \spad{step.did} contains the step size which achieved the
++ accuracy and \spad{step.next} is the next step size to use.
++
++ \item \spad{rk4qc(y,n,x1,step,eps,yscal,derivs,t1,t2,t3,t4,t5,t6,t7)} is the
++ same as \spad{rk4qc(y,n,x1,step,eps,yscal,derivs)} except that the user
++ must provide the 7 scratch arrays \spad{t1-t7} of size n.
++
++ \item \spad{rk4a(y,n,x1,x2,eps,h,ns,derivs)}
++ is a driver program which uses \spad{rk4qc} to integrate n ordinary
++ differential equations starting at x1 to x2, keeping the local
++ truncation error to within \spad{eps} by changing the local step size.
++ The scaling vector is defined as
++ \center{\spad{yscal(i) = abs(y(i)) + abs(h*dydx(i)) + tiny}}
++ where \spad{y(i)} is the solution at location x, \spad{dydx} is the
++ ordinary differential equation's right hand side, h is the current
++ step size and \spad{tiny} is 10 times the
++ smallest positive number representable.
++ The user must supply an estimate for a trial step size and
++ the maximum number of calls to \spad{rk4qc} to use.
++ Argument x2 is the final point,
++ \spad{eps} is local truncation,
++ \spad{ns} is the maximum number of call to \spad{rk4qc} to use.
++ \end{items}
NumericalOrdinaryDifferentialEquations(): Exports == Implementation where
  L     ==> List
  V     ==> Vector
  B     ==> Boolean
  I     ==> Integer
  E     ==> OutputForm
  NF    ==> Float
  NNI   ==> NonNegativeInteger
  VOID  ==> Void
  OFORM ==> OutputForm
  RK4STEP ==> Record(try:NF, did:NF, next:NF)

  Exports ==> with
--header definitions here
   rk4   : (V NF,I,NF,NF,  (V NF,V NF,NF) -> VOID) -> VOID
    ++ rk4(y,n,x1,h,derivs) uses a 4-th order Runge-Kutta method
    ++ to numerically integrate the ordinary differential equation
    ++ {\em dy/dx = f(y,x)} of n variables, where y is an n-vector.
    ++ Argument y is a vector of initial conditions of length n which upon exit
    ++ contains the solution at \spad{x1 + h}, n is the number of dependent
    ++ variables, x1 is the initial point, h is the step size, and
    ++ \spad{derivs} is a function which computes the right hand side of the
    ++ ordinary differential equation.
```

```
 ++ For details, see \spadtype{NumericalOrdinaryDifferentialEquations}.
rk4   : (V NF,I,NF,NF,  (V NF,V NF,NF) -> VOID
        ,V NF,V NF,V NF,V NF) -> VOID
 ++ rk4(y,n,x1,h,derivs,t1,t2,t3,t4) is the same as
 ++ \spad{rk4(y,n,x1,h,derivs)} except that you must provide 4 scratch
 ++ arrays t1-t4 of size n.
 ++ For details, see \con{NumericalOrdinaryDifferentialEquations}.
rk4a  : (V NF,I,NF,NF,NF,NF,I,(V NF,V NF,NF) -> VOID ) -> VOID
 ++ rk4a(y,n,x1,x2,eps,h,ns,derivs) is a driver function for the
 ++ numerical integration of an ordinary differential equation
 ++ {\em dy/dx = f(y,x)} of n variables, where y is an n-vector
 ++ using a 4-th order Runge-Kutta method.
 ++ For details, see \con{NumericalOrdinaryDifferentialEquations}.
rk4qc : (V NF,I,NF,RK4STEP,NF,V NF,(V NF,V NF,NF) -> VOID) -> VOID
 ++ rk4qc(y,n,x1,step,eps,yscal,derivs) is a subfunction for the
 ++ numerical integration of an ordinary differential equation
 ++ {\em dy/dx = f(y,x)} of n variables, where y is an n-vector
 ++ using a 4-th order Runge-Kutta method.
 ++ This function takes a 5-th order Runge-Kutta step with monitoring
 ++ of local truncation to ensure accuracy and adjust stepsize.
 ++ For details, see \con{NumericalOrdinaryDifferentialEquations}.
rk4qc : (V NF,I,NF,RK4STEP,NF,V NF,(V NF,V NF,NF) -> VOID
        ,V NF,V NF,V NF,V NF,V NF,V NF,V NF) -> VOID
 ++ rk4qc(y,n,x1,step,eps,yscal,derivs,t1,t2,t3,t4,t5,t6,t7) is a
 ++ subfunction for the numerical integration of an ordinary differential
 ++ equation {\em dy/dx = f(y,x)} of n variables, where y is an n-vector
 ++ using a 4-th order Runge-Kutta method.
 ++ This function takes a 5-th order Runge-Kutta step with monitoring
 ++ of local truncation to ensure accuracy and adjust stepsize.
 ++ For details, see \con{NumericalOrdinaryDifferentialEquations}.
rk4f  : (V NF,I,NF,NF,I,(V NF,V NF,NF) -> VOID ) -> VOID
 ++ rk4f(y,n,x1,x2,ns,derivs) uses a 4-th order Runge-Kutta method
 ++ to numerically integrate the ordinary differential equation
 ++ {\em dy/dx = f(y,x)} of n variables, where y is an n-vector.
 ++ Starting with y at x1, this function uses \spad{ns} fixed
 ++ steps of a 4-th order Runge-Kutta integrator to advance the
 ++ solution vector to x2 and return the values in y.
 ++ For details, see \con{NumericalOrdinaryDifferentialEquations}.

Implementation ==>  add
--some local function definitions here
rk4qclocal : (V NF,V NF,I,NF,RK4STEP,NF,V NF,(V NF,V NF,NF) -> VOID
             ,V NF,V NF,V NF,V NF,V NF,V NF) -> VOID
rk4local   : (V NF,V NF,I,NF,NF,V NF,(V NF,V NF,NF) -> VOID
             ,V NF,V NF,V NF) -> VOID
import OutputPackage
```

```
---------------------------------------------------------

   rk4a(ystart,nvar,x1,x2,eps,htry,nstep,derivs) ==
      y       : V NF := new(nvar::NNI,0.0)
      yscal   : V NF := new(nvar::NNI,1.0)
      dydx    : V NF := new(nvar::NNI,0.0)
      t1      : V NF := new(nvar::NNI,0.0)
      t2      : V NF := new(nvar::NNI,0.0)
      t3      : V NF := new(nvar::NNI,0.0)
      t4      : V NF := new(nvar::NNI,0.0)
      t5      : V NF := new(nvar::NNI,0.0)
      t6      : V NF := new(nvar::NNI,0.0)
      step    : RK4STEP := [htry,0.0,0.0]
      x       : NF   := x1
      tiny    : NF   := 10.0**(-(digits()+1)::I)
      m       : I    := nvar
      outlist : L OFORM := [x::E,x::E,x::E]
      i       : I
      iter    : I

      eps  := 1.0/eps
      for i in 1..m repeat
         y(i)  := ystart(i)
      for iter in 1..nstep repeat
--compute the derivative
         derivs(dydx,y,x)
--if overshoot, the set h accordingly
         if (x + step.try - x2) > 0.0 then
            step.try := x2 - x
--find the correct scaling
         for i in 1..m repeat
            yscal(i) := abs(y(i)) + abs(step.try * dydx(i)) + tiny
--take a quality controlled runge-kutta step
         rk4qclocal(y,dydx,nvar,x,step,eps,yscal,derivs
                   ,t1,t2,t3,t4,t5,t6)
         x         := x + step.did
--       outlist.0 := x::E
--       outlist.1 := y(0)::E
--       outlist.2 := y(1)::E
--       output(blankSeparate(outlist)::E)
--check to see if done
         if (x-x2) >= 0.0 then
            leave
--next stepsize to use
         step.try := step.next
```

```
--end nstep repeat
      if iter = (nstep+1) then
         output("ode: ERROR ")
         outlist.1 := nstep::E
         outlist.2 := " steps to small, last h = "::E
         outlist.3 := step.did::E
         output(blankSeparate(outlist))
         output(" y= ",y::E)
      for i in 1..m repeat
         ystart(i) := y(i)

------------------------------------------------------------------

   rk4qc(y,n,x,step,eps,yscal,derivs) ==
      t1 : V NF := new(n::NNI,0.0)
      t2 : V NF := new(n::NNI,0.0)
      t3 : V NF := new(n::NNI,0.0)
      t4 : V NF := new(n::NNI,0.0)
      t5 : V NF := new(n::NNI,0.0)
      t6 : V NF := new(n::NNI,0.0)
      t7 : V NF := new(n::NNI,0.0)
      derivs(t7,y,x)
      eps := 1.0/eps
      rk4qclocal(y,t7,n,x,step,eps,yscal,derivs,t1,t2,t3,t4,t5,t6)

------------------------------------------------------------

   rk4qc(y,n,x,step,eps,yscal,derivs,t1,t2,t3,t4,t5,t6,dydx) ==
      derivs(dydx,y,x)
      eps := 1.0/eps
      rk4qclocal(y,dydx,n,x,step,eps,yscal,derivs,t1,t2,t3,t4,t5,t6)

------------------------------------------------------------

   rk4qclocal(y,dydx,n,x,step,eps,yscal,derivs
             ,t1,t2,t3,ysav,dysav,ytemp) ==
      xsav   : NF := x
      h      : NF := step.try
      fcor   : NF := 1.0/15.0
      safety : NF := 0.9
      grow   : NF := -0.20
      shrink : NF := -0.25
      errcon : NF := 0.6E-04  --(this is 4/safety)**(1/grow)
      hh     : NF
      errmax : NF
      i      : I
```

```
      m      : I  := n
--
      for i in 1..m repeat
         dysav(i) := dydx(i)
         ysav(i)  := y(i)
--cut down step size till error criterion is met
      repeat
--take two little steps to get to x + h
         hh := 0.5 * h
         rk4local(ysav,dysav,n,xsav,hh,ytemp,derivs,t1,t2,t3)
         x  := xsav + hh
         derivs(dydx,ytemp,x)
         rk4local(ytemp,dydx,n,x,hh,y,derivs,t1,t2,t3)
         x  := xsav + h
--take one big step get to x + h
         rk4local(ysav,dysav,n,xsav,h,ytemp,derivs,t1,t2,t3)

--compute the maximum scaled difference
         errmax := 0.0
         for i in 1..m repeat
            ytemp(i) := y(i) - ytemp(i)
            errmax   := max(errmax,abs(ytemp(i)/yscal(i)))
--scale relative to required accuracy
         errmax := errmax * eps
--update integration stepsize
         if (errmax > 1.0) then
            h := safety * h * (errmax ** shrink)
         else
            step.did := h
            if errmax > errcon then
               step.next := safety * h * (errmax ** grow)
            else
               step.next := 4 * h
            leave
--make fifth order with 4-th order error estimate
      for i in 1..m repeat
         y(i) := y(i) + ytemp(i) * fcor

-----------------------------------------

   rk4f(y,nvar,x1,x2,nstep,derivs) ==
     yt   : V NF := new(nvar::NNI,0.0)
     dyt  : V NF := new(nvar::NNI,0.0)
     dym  : V NF := new(nvar::NNI,0.0)
     dydx : V NF := new(nvar::NNI,0.0)
     ynew : V NF := new(nvar::NNI,0.0)
```

```
      h    : NF := (x2-x1) / (nstep::NF)
      x    : NF := x1
      i    : I
      j    : I
-- start integrating
      for i in 1..nstep repeat
         derivs(dydx,y,x)
         rk4local(y,dydx,nvar,x,h,y,derivs,yt,dyt,dym)
         x := x + h

------------------------------------------------------

   rk4(y,n,x,h,derivs) ==
      t1 : V NF := new(n::NNI,0.0)
      t2 : V NF := new(n::NNI,0.0)
      t3 : V NF := new(n::NNI,0.0)
      t4 : V NF := new(n::NNI,0.0)
      derivs(t1,y,x)
      rk4local(y,t1,n,x,h,y,derivs,t2,t3,t4)

------------------------------------------------------------

   rk4(y,n,x,h,derivs,t1,t2,t3,t4) ==
      derivs(t1,y,x)
      rk4local(y,t1,n,x,h,y,derivs,t2,t3,t4)

------------------------------------------------------------

   rk4local(y,dydx,n,x,h,yout,derivs,yt,dyt,dym) ==
      hh : NF := h*0.5
      h6 : NF := h/6.0
      xh : NF := x+hh
      m  : I  := n
      i  : I
-- first step
      for i in 1..m repeat
         yt(i) := y(i) + hh*dydx(i)
-- second step
      derivs(dyt,yt,xh)
      for i in 1..m repeat
         yt(i) := y(i) + hh*dyt(i)
-- third step
      derivs(dym,yt,xh)
      for i in 1..m repeat
         yt(i)  := y(i)   + h*dym(i)
         dym(i) := dyt(i) + dym(i)
```

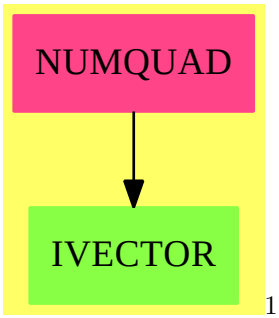
```
  -- fourth step
        derivs(dyt,yt,x+h)
        for i in 1..m repeat
           yout(i) := y(i) + h6*( dydx(i) + 2.0*dym(i) + dyt(i) )
```

⟨*NUMODE.dotabb*⟩≡

```
  "NUMODE" [color="#FF4488",href="bookvol10.4.pdf#nameddest=NUMODE"]
  "IVECTOR" [color="#88FF44",href="bookvol10.3.pdf#nameddest=IVECTOR"]
  "NUMODE" -> "IVECTOR"
```

15.61 package NUMQUAD NumericalQuadrature

15.62 NumericalQuadrature



1.00

Exports:

aromberg	asimpson	atrapezoidal	romberg	rombergo
simpson	simpsono	trapezoidal	trapezoidalo	

⟨*package NUMQUAD NumericalQuadrature*⟩≡

```
)abbrev package NUMQUAD NumericalQuadrature
++ Author: Yurij A. Baransky
++ Date Created: October 90
++ Date Last Updated: October 90
++ Basic Operations:
++ Related Constructors:
++ Also See:
++ AMS Classifications:
++ Keywords:
++ References:
++ Description:
++ This suite of routines performs numerical quadrature using
++ algorithms derived from the basic trapezoidal rule. Because
++ the error term of this rule contains only even powers of the
++ step size (for open and closed versions), fast convergence
++ can be obtained if the integrand is sufficiently smooth.
++
++ Each routine returns a Record of type TrapAns, which contains\indent{3}
++ \newline value (\spadtype{Float}):\tab{20}  estimate of the integral
++ \newline error (\spadtype{Float}):\tab{20}  estimate of the error in the computation
++ \newline totalpts (\spadtype{Integer}):\tab{20} total number of function evaluations
++ \newline success (\spadtype{Boolean}):\tab{20} if the integral was computed within the us
++ \indent{0}\indent{0}
++ To produce this estimate, each routine generates an internal
++ sequence of sub-estimates, denoted by {\em S(i)}, depending on the
```

```
++ routine, to which the various convergence criteria are applied.
++ The user must supply a relative accuracy, \spad{eps_r}, and an absolute
++ accuracy, \spad{eps_a}. Convergence is obtained when either
++ \center{\spad{ABS(S(i) - S(i-1)) < eps_r * ABS(S(i-1))}}
++ \center{or    \spad{ABS(S(i) - S(i-1)) < eps_a}}
++ are true statements.
++
++ The routines come in three families and three flavors:
++ \newline\tab{3} closed:\tab{20}romberg,\tab{30}simpson,\tab{42}trapezoidal
++ \newline\tab{3} open:  \tab{20}rombergo,\tab{30}simpsono,\tab{42}trapezoidalo
++ \newline\tab{3} adaptive closed:\tab{20}aromberg,\tab{30}asimpson,\tab{42}atra
++ \par
++ The {\em S(i)} for the trapezoidal family is the value of the
++ integral using an equally spaced absicca trapezoidal rule for
++ that level of refinement.
++ \par
++ The {\em S(i)} for the simpson family is the value of the integral
++ using an equally spaced absicca simpson rule for that level of
++ refinement.
++ \par
++ The {\em S(i)} for the romberg family is the estimate of the integral
++ using an equally spaced absicca romberg method. For
++ the \spad{i}-th level, this is an appropriate combination of all the
++ previous trapezodial estimates so that the error term starts
++ with the \spad{2*(i+1)} power only.
++ \par
++ The three families come in a closed version, where the formulas
++ include the endpoints, an open version where the formulas do not
++ include the endpoints and an adaptive version, where the user
++ is required to input the number of subintervals over which the
++ appropriate closed family integrator will apply with the usual
++ convergence parmeters for each subinterval. This is useful
++ where a large number of points are needed only in a small fraction
++ of the entire domain.
++ \par
++ Each routine takes as arguments:
++ \newline f\tab{10} integrand
++ \newline a\tab{10} starting point
++ \newline b\tab{10} ending point
++ \newline \spad{eps_r}\tab{10} relative error
++ \newline \spad{eps_a}\tab{10} absolute error
++ \newline \spad{nmin} \tab{10} refinement level when to start checking for conv
++ \newline \spad{nmax} \tab{10} maximum level of refinement
++ \par
++ The adaptive routines take as an additional parameter
++ \newline \spad{nint}\tab{10} the number of independent intervals to apply a cl
```

```
++  family integrator of the same name.
++ \par Notes:
++ \newline Closed family level i uses \spad{1 + 2**i} points.
++ \newline Open family level i uses \spad{1 + 3**i} points.
NumericalQuadrature(): Exports == Implementation where
  L        ==> List
  V        ==> Vector
  I        ==> Integer
  B        ==> Boolean
  E        ==> OutputForm
  F       ==> Float
  PI       ==> PositiveInteger
  OFORM    ==> OutputForm
  TrapAns  ==> Record(value:F, error:F, totalpts:I, success:B )

  Exports ==> with
   aromberg     : (F -> F,F,F,F,F,I,I,I) -> TrapAns
     ++ aromberg(fn,a,b,epsrel,epsabs,nmin,nmax,nint)
     ++ uses the adaptive romberg method to numerically integrate function
     ++ \spad{fn} over the closed interval from \spad{a} to \spad{b},
     ++ with relative accuracy \spad{epsrel} and absolute accuracy
     ++ \spad{epsabs}, with the refinement levels for convergence checking
     ++ vary from \spad{nmin} to \spad{nmax}, and where \spad{nint}
     ++ is the number of independent intervals to apply the integrator.
     ++ The value returned is a record containing the value of the integral,
     ++ the estimate of the error in the computation, the total number of
     ++ function evaluations, and either a boolean value which is true if
     ++ the integral was computed within the user specified error criterion.
     ++ See \spadtype{NumericalQuadrature} for details.
   asimpson     : (F -> F,F,F,F,F,I,I,I) -> TrapAns
     ++ asimpson(fn,a,b,epsrel,epsabs,nmin,nmax,nint) uses the
     ++ adaptive simpson method to numerically integrate function \spad{fn}
     ++ over the closed interval from \spad{a} to \spad{b}, with relative
     ++ accuracy \spad{epsrel} and absolute accuracy \spad{epsabs}, with the
     ++ refinement levels for convergence checking vary from \spad{nmin}
     ++ to \spad{nmax}, and where \spad{nint} is the number of independent
     ++ intervals to apply the integrator. The value returned is a record
     ++ containing the value of the integral, the estimate of the error in
     ++ the computation, the total number of function evaluations, and
     ++ either a boolean value which is true if the integral was computed
     ++ within the user specified error criterion.
     ++ See \spadtype{NumericalQuadrature} for details.
   atrapezoidal : (F -> F,F,F,F,F,I,I,I) -> TrapAns
     ++ atrapezoidal(fn,a,b,epsrel,epsabs,nmin,nmax,nint) uses the
     ++ adaptive trapezoidal method to numerically integrate function
     ++ \spad{fn} over the closed interval from \spad{a} to \spad{b}, with
```

```
  ++ relative accuracy \spad{epsrel} and absolute accuracy \spad{epsabs},
  ++ with the refinement levels for convergence checking vary from
  ++ \spad{nmin} to \spad{nmax}, and where \spad{nint} is the number
  ++ of independent intervals to apply the integrator. The value returned
  ++ is a record containing the value of the integral, the estimate of
  ++ the error in the computation, the total number of function
  ++ evaluations, and either a boolean value which is true if
  ++ the integral was computed within the user specified error criterion.
  ++ See \spadtype{NumericalQuadrature} for details.
romberg      : (F -> F,F,F,F,F,I,I) -> TrapAns
  ++ romberg(fn,a,b,epsrel,epsabs,nmin,nmax) uses the romberg
  ++ method to numerically integrate function \spadvar{fn} over the closed
  ++ interval \spad{a} to \spad{b}, with relative accuracy \spad{epsrel}
  ++ and absolute accuracy \spad{epsabs}, with the refinement levels
  ++ for convergence checking vary from \spad{nmin} to \spad{nmax}.
  ++ The value returned is a record containing the value
  ++ of the integral, the estimate of the error in the computation, the
  ++ total number of function evaluations, and either a boolean value
  ++ which is true if the integral was computed within the user specified
  ++ error criterion. See \spadtype{NumericalQuadrature} for details.
simpson      : (F -> F,F,F,F,F,I,I) -> TrapAns
  ++ simpson(fn,a,b,epsrel,epsabs,nmin,nmax) uses the simpson
  ++ method to numerically integrate function \spad{fn} over the closed
  ++ interval \spad{a} to \spad{b}, with
  ++ relative accuracy \spad{epsrel} and absolute accuracy \spad{epsabs},
  ++ with the refinement levels for convergence checking vary from
  ++ \spad{nmin} to \spad{nmax}. The value returned
  ++ is a record containing the value of the integral, the estimate of
  ++ the error in the computation, the total number of function
  ++ evaluations, and either a boolean value which is true if
  ++ the integral was computed within the user specified error criterion.
  ++ See \spadtype{NumericalQuadrature} for details.
trapezoidal  : (F -> F,F,F,F,F,I,I) -> TrapAns
  ++ trapezoidal(fn,a,b,epsrel,epsabs,nmin,nmax) uses the
  ++ trapezoidal method to numerically integrate function \spadvar{fn} over
  ++ the closed interval \spad{a} to \spad{b}, with relative accuracy
  ++ \spad{epsrel} and absolute accuracy \spad{epsabs}, with the
  ++ refinement levels for convergence checking vary
  ++ from \spad{nmin} to \spad{nmax}. The value
  ++ returned is a record containing the value of the integral, the
  ++ estimate of the error in the computation, the total number of
  ++ function evaluations, and either a boolean value which is true
  ++ if the integral was computed within the user specified error criterion.
  ++ See \spadtype{NumericalQuadrature} for details.
rombergo     : (F -> F,F,F,F,F,I,I) -> TrapAns
  ++ rombergo(fn,a,b,epsrel,epsabs,nmin,nmax) uses the romberg
```

```
      ++ method to numerically integrate function \spad{fn} over
      ++ the open interval from \spad{a} to \spad{b}, with
      ++ relative accuracy \spad{epsrel} and absolute accuracy \spad{epsabs},
      ++ with the refinement levels for convergence checking vary from
      ++ \spad{nmin} to \spad{nmax}. The value returned
      ++ is a record containing the value of the integral, the estimate of
      ++ the error in the computation, the total number of function
      ++ evaluations, and either a boolean value which is true if
      ++ the integral was computed within the user specified error criterion.
      ++ See \spadtype{NumericalQuadrature} for details.
    simpsono     : (F -> F,F,F,F,F,I,I) -> TrapAns
      ++ simpsono(fn,a,b,epsrel,epsabs,nmin,nmax) uses the
      ++ simpson method to numerically integrate function \spad{fn} over
      ++ the open interval from \spad{a} to \spad{b}, with
      ++ relative accuracy \spad{epsrel} and absolute accuracy \spad{epsabs},
      ++ with the refinement levels for convergence checking vary from
      ++ \spad{nmin} to \spad{nmax}. The value returned
      ++ is a record containing the value of the integral, the estimate of
      ++ the error in the computation, the total number of function
      ++ evaluations, and either a boolean value which is true if
      ++ the integral was computed within the user specified error criterion.
      ++ See \spadtype{NumericalQuadrature} for details.
    trapezoidalo : (F -> F,F,F,F,F,I,I) -> TrapAns
      ++ trapezoidalo(fn,a,b,epsrel,epsabs,nmin,nmax) uses the
      ++ trapezoidal method to numerically integrate function \spad{fn}
      ++ over the open interval from \spad{a} to \spad{b}, with
      ++ relative accuracy \spad{epsrel} and absolute accuracy \spad{epsabs},
      ++ with the refinement levels for convergence checking vary from
      ++ \spad{nmin} to \spad{nmax}. The value returned
      ++ is a record containing the value of the integral, the estimate of
      ++ the error in the computation, the total number of function
      ++ evaluations, and either a boolean value which is true if
      ++ the integral was computed within the user specified error criterion.
      ++ See \spadtype{NumericalQuadrature} for details.

   Implementation ==> add
    trapclosed : (F -> F,F,F,F,I) -> F
    trapopen   : (F -> F,F,F,F,I) -> F
    import OutputPackage

 -------------------------------------------------

    aromberg(func,a,b,epsrel,epsabs,nmin,nmax,nint) ==
       ans  : TrapAns
       sum  : F := 0.0
       err  : F := 0.0
```

```
      pts  : I  := 1
      done : B  := true
      hh   : F := (b-a) / nint
      x1   : F := a
      x2   : F := a + hh
      io   : L OFORM := [x1::E,x2::E]
      i    : I
      for i in 1..nint repeat
         ans := romberg(func,x1,x2,epsrel,epsabs,nmin,nmax)
         if (not ans.success) then
           io.1 := x1::E
           io.2 := x2::E
           print blankSeparate cons("accuracy not reached in interval"::E,io)
         sum  := sum + ans.value
         err  := err + abs(ans.error)
         pts  := pts + ans.totalpts-1
         done := (done and ans.success)
         x1   := x2
         x2   := x2 + hh
      return( [sum , err , pts , done] )

---------------------------------------------------

   asimpson(func,a,b,epsrel,epsabs,nmin,nmax,nint) ==
      ans  : TrapAns
      sum  : F := 0.0
      err  : F := 0.0
      pts  : I  := 1
      done : B  := true
      hh   : F := (b-a) / nint
      x1   : F := a
      x2   : F := a + hh
      io   : L OFORM := [x1::E,x2::E]
      i    : I
      for i in 1..nint repeat
         ans := simpson(func,x1,x2,epsrel,epsabs,nmin,nmax)
         if (not ans.success) then
           io.1 := x1::E
           io.2 := x2::E
           print blankSeparate cons("accuracy not reached in interval"::E,io)
         sum  := sum + ans.value
         err  := err + abs(ans.error)
         pts  := pts + ans.totalpts-1
         done := (done and ans.success)
         x1   := x2
         x2   := x2 + hh
```

```
      return( [sum , err , pts , done] )

--------------------------------------------------

   atrapezoidal(func,a,b,epsrel,epsabs,nmin,nmax,nint) ==
      ans  : TrapAns
      sum  : F := 0.0
      err  : F := 0.0
      pts  : I  := 1
      i    : I
      done : B := true
      hh   : F := (b-a) / nint
      x1   : F := a
      x2   : F := a + hh
      io   : L OFORM := [x1::E,x2::E]
      for i in 1..nint repeat
         ans := trapezoidal(func,x1,x2,epsrel,epsabs,nmin,nmax)
         if (not ans.success) then
           io.1 := x1::E
           io.2 := x2::E
           print blankSeparate cons("accuracy not reached in interval"::E,io)
         sum  := sum + ans.value
         err  := err + abs(ans.error)
         pts  := pts + ans.totalpts-1
         done := (done and ans.success)
         x1   := x2
         x2   := x2 + hh
      return( [sum , err , pts , done] )

--------------------------------------------------

   romberg(func,a,b,epsrel,epsabs,nmin,nmax) ==
      length : F := (b-a)
      delta  : F := length
      newsum : F := 0.5 * length * (func(a)+func(b))
      newest : F := 0.0
      oldsum : F := 0.0
      oldest : F := 0.0
      change : F := 0.0
      qx1    : F := newsum
      table  : V F := new((nmax+1)::PI,0.0)
      n      : I  := 1
      pts    : I  := 1
      four   : I
      j      : I
      i      : I
```

```
      if (nmin < 2) then
         output("romberg: nmin to small (nmin > 1) nmin = ",nmin::E)
         return([0.0,0.0,0,false])
      if (nmax < nmin) then
         output("romberg: nmax < nmin : nmax = ",nmax::E)
         output("                       nmin = ",nmin::E)
         return([0.0,0.0,0,false])
      if (a = b) then
         output("romberg: integration limits are equal  = ",a::E)
         return([0.0,0.0,1,true])
      if (epsrel < 0.0) then
         output("romberg: eps_r < 0.0            eps_r  = ",epsrel::E)
         return([0.0,0.0,0,false])
      if (epsabs < 0.0) then
         output("romberg: eps_a < 0.0            eps_a  = ",epsabs::E)
         return([0.0,0.0,0,false])
      for n in 1..nmax repeat
         oldsum := newsum
         newsum := trapclosed(func,a,delta,oldsum,pts)
         newest := (4.0 * newsum - oldsum) / 3.0
         four   := 4
         table(n) := newest
         for j in 2..n repeat
            i        := n+1-j
            four     := four * 4
            table(i) := table(i+1) + (table(i+1)-table(i)) / (four-1)
         if n > nmin then
            change := abs(table(1) - qx1)
            if change < abs(epsrel*qx1) then
               return( [table(1) , change , 2*pts+1 , true] )
            if change < epsabs then
               return( [table(1) , change , 2*pts+1 , true] )
         oldsum := newsum
         oldest := newest
         delta  := 0.5*delta
         pts    := 2*pts
         qx1    := table(1)
      return( [table(1) , 1.25*change , pts+1 ,false] )

---------------------------------------------------

   simpson(func,a,b,epsrel,epsabs,nmin,nmax) ==
      length : F := (b-a)
      delta  : F := length
      newsum : F := 0.5*(b-a)*(func(a)+func(b))
      newest : F := 0.0
```

```
      oldsum : F := 0.0
      oldest : F := 0.0
      change : F := 0.0
      n      : I  := 1
      pts    : I  := 1
      if (nmin < 2) then
         output("simpson: nmin to small (nmin > 1) nmin = ",nmin::E)
         return([0.0,0.0,0,false])
      if (nmax < nmin) then
         output("simpson: nmax < nmin : nmax = ",nmax::E)
         output("                       nmin = ",nmin::E)
         return([0.0,0.0,0,false])
      if (a = b) then
         output("simpson: integration limits are equal  = ",a::E)
         return([0.0,0.0,1,true])
      if (epsrel < 0.0) then
         output("simpson: eps_r < 0.0 : eps_r = ",epsrel::E)
         return([0.0,0.0,0,false])
      if (epsabs < 0.0) then
         output("simpson: eps_a < 0.0 : eps_a = ",epsabs::E)
         return([0.0,0.0,0,false])
      for n in 1..nmax repeat
         oldsum := newsum
         newsum := trapclosed(func,a,delta,oldsum,pts)
         newest := (4.0 * newsum - oldsum) / 3.0
         if n > nmin then
            change := abs(newest-oldest)
            if change < abs(epsrel*oldest) then
               return( [newest , 1.25*change , 2*pts+1 , true] )
            if change < epsabs then
               return( [newest , 1.25*change , 2*pts+1 , true] )
         oldsum := newsum
         oldest := newest
         delta  := 0.5*delta
         pts    := 2*pts
      return( [newest , 1.25*change , pts+1 ,false] )

-------------------------------------------------

   trapezoidal(func,a,b,epsrel,epsabs,nmin,nmax) ==
      length : F := (b-a)
      delta  : F := length
      newsum : F := 0.5*(b-a)*(func(a)+func(b))
      change : F := 0.0
      oldsum : F
      n      : I  := 1
```

```
      pts    : I  := 1
      if (nmin < 2) then
         output("trapezoidal: nmin to small (nmin > 1) nmin = ",nmin::E)
         return([0.0,0.0,0,false])
      if (nmax < nmin) then
         output("trapezoidal: nmax < nmin : nmax = ",nmax::E)
         output("                           nmin = ",nmin::E)
         return([0.0,0.0,0,false])
      if (a = b) then
         output("trapezoidal: integration limits are equal  = ",a::E)
         return([0.0,0.0,1,true])
      if (epsrel < 0.0) then
         output("trapezoidal: eps_r < 0.0 : eps_r = ",epsrel::E)
         return([0.0,0.0,0,false])
      if (epsabs < 0.0) then
         output("trapezoidal: eps_a < 0.0 : eps_a = ",epsabs::E)
         return([0.0,0.0,0,false])
      for n in 1..nmax repeat
         oldsum := newsum
         newsum := trapclosed(func,a,delta,oldsum,pts)
         if n > nmin then
            change := abs(newsum-oldsum)
            if change < abs(epsrel*oldsum) then
               return( [newsum , 1.25*change , 2*pts+1 , true] )
            if change < epsabs then
               return( [newsum , 1.25*change , 2*pts+1 , true] )
         delta := 0.5*delta
         pts   := 2*pts
      return( [newsum , 1.25*change , pts+1 ,false] )

---------------------------------------------------

   rombergo(func,a,b,epsrel,epsabs,nmin,nmax) ==
      length : F := (b-a)
      delta  : F := length / 3.0
      newsum : F := length * func( 0.5*(a+b) )
      newest : F := 0.0
      oldsum : F := 0.0
      oldest : F := 0.0
      change : F := 0.0
      qx1    : F := newsum
      table  : V F := new((nmax+1)::PI,0.0)
      four   : I
      j      : I
      i      : I
      n      : I  := 1
```

```
      pts    : I  := 1
      for n in 1..nmax repeat
         oldsum   := newsum
         newsum   := trapopen(func,a,delta,oldsum,pts)
         newest   := (9.0 * newsum - oldsum) / 8.0
         table(n) := newest
         nine     := 9
         output(newest::E)
         for j in 2..n repeat
            i        := n+1-j
            nine     := nine * 9
            table(i) := table(i+1) + (table(i+1)-table(i)) / (nine-1)
         if n > nmin then
            change := abs(table(1) - qx1)
            if change < abs(epsrel*qx1) then
               return( [table(1) , 1.5*change , 3*pts , true] )
            if change < epsabs then
               return( [table(1) , 1.5*change , 3*pts , true] )
         output(table::E)
         oldsum := newsum
         oldest := newest
         delta  := delta / 3.0
         pts    := 3*pts
         qx1    := table(1)
      return( [table(1) , 1.5*change , pts ,false] )

-------------------------------------------------

   simpsono(func,a,b,epsrel,epsabs,nmin,nmax) ==
      length : F := (b-a)
      delta  : F := length / 3.0
      newsum : F := length * func( 0.5*(a+b) )
      newest : F := 0.0
      oldsum : F := 0.0
      oldest : F := 0.0
      change : F := 0.0
      n      : I  := 1
      pts    : I  := 1
      for n in 1..nmax repeat
         oldsum := newsum
         newsum := trapopen(func,a,delta,oldsum,pts)
         newest := (9.0 * newsum - oldsum) / 8.0
         output(newest::E)
         if n > nmin then
            change := abs(newest - oldest)
            if change < abs(epsrel*oldest) then
```

```
               return( [newest , 1.5*change , 3*pts , true] )
            if change < epsabs then
               return( [newest , 1.5*change , 3*pts , true] )
         oldsum := newsum
         oldest := newest
         delta  := delta / 3.0
         pts    := 3*pts
      return( [newest , 1.5*change , pts ,false] )

-------------------------------------------------

   trapezoidalo(func,a,b,epsrel,epsabs,nmin,nmax) ==
      length : F := (b-a)
      delta  : F := length/3.0
      newsum : F := length*func( 0.5*(a+b) )
      change : F := 0.0
      pts    : I  := 1
      oldsum : F
      n      : I
      for n in 1..nmax repeat
         oldsum := newsum
         newsum := trapopen(func,a,delta,oldsum,pts)
         output(newsum::E)
         if n > nmin then
            change := abs(newsum-oldsum)
            if change < abs(epsrel*oldsum) then
               return([newsum , 1.5*change , 3*pts , true] )
            if change < epsabs then
               return([newsum , 1.5*change , 3*pts , true] )
         delta := delta / 3.0
         pts   := 3*pts
      return([newsum , 1.5*change , pts ,false] )

-------------------------------------------------

   trapclosed(func,start,h,oldsum,numpoints) ==
      x   : F := start + 0.5*h
      sum : F := 0.0
      i   : I
      for i in 1..numpoints repeat
          sum := sum + func(x)
          x   := x + h
      return( 0.5*(oldsum + sum*h) )

-------------------------------------------------
```

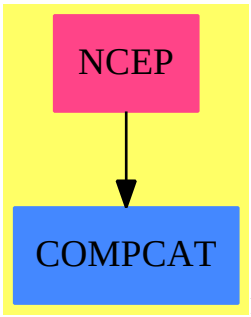
```
trapopen(func,start,del,oldsum,numpoints) ==
   ddel : F := 2.0*del
   x    : F := start + 0.5*del
   sum  : F := 0.0
   i    : I
   for i in 1..numpoints repeat
       sum := sum + func(x)
       x   := x + ddel
       sum := sum + func(x)
       x   := x + del
   return( (oldsum/3.0 + sum*del) )
```

⟨*NUMQUAD.dotabb*⟩≡

```
"NUMQUAD" [color="#FF4488",href="bookvol10.4.pdf#nameddest=NUMQUAD"]
"IVECTOR" [color="#88FF44",href="bookvol10.3.pdf#nameddest=IVECTOR"]
"NUMQUAD" -> "IVECTOR"
```

15.63 package NCEP NumericComplexEigenPackage

15.64 NumericComplexEigenPackage



1.00

Exports:
characteristicPolynomial complexEigenvalues complexEigenvectors

⟨*package NCEP NumericComplexEigenPackage*⟩≡

```
)abbrev package NCEP NumericComplexEigenPackage
++ Author: P. Gianni
++ Date Created: Summer 1990
++ Date Last Updated: Spring 1991
++ Basic Functions:
++ Related Constructors: FloatingComplexPackage
++ Also See:
++ AMS Classifications:
++ Keywords:
++ References:
++ Description:
++ This package computes explicitly eigenvalues and eigenvectors of
++ matrices with entries over the complex rational numbers.
++ The results are expressed either as complex floating numbers or as
++ complex rational numbers
++ depending on the type of the precision parameter.
NumericComplexEigenPackage(Par) : C == T
 where
   Par   :   Join(Field,OrderedRing)   -- Float or RationalNumber

   SE    ==> Symbol()
   RN    ==> Fraction Integer
   I     ==> Integer
   NF    ==> Float
   CF    ==> Complex Float
   GRN   ==> Complex RN
```

```
GI    ==> Complex Integer
PI    ==> PositiveInteger
NNI   ==> NonNegativeInteger
MRN   ==> Matrix RN

MCF        ==> Matrix CF
MGRN       ==> Matrix GRN
MCPar      ==> Matrix Complex Par
SUPGRN     ==> SparseUnivariatePolynomial GRN
outForm    ==> Record(outval:Complex Par,outmult:Integer,outvect:List MCPar)

C == with
  characteristicPolynomial :   MGRN    -> Polynomial GRN
    ++ characteristicPolynomial(m) returns the characteristic polynomial
    ++ of the matrix m expressed as polynomial
    ++ over complex rationals with a new symbol as variable.
    -- while the function in EigenPackage returns Fraction P GRN.
  characteristicPolynomial : (MGRN,SE) -> Polynomial GRN
    ++ characteristicPolynomial(m,x) returns the characteristic polynomial
    ++ of the matrix m expressed as polynomial
    ++ over Complex Rationals with variable x.
    -- while the function in EigenPackage returns Fraction P GRN.
  complexEigenvalues  :   (MGRN,Par)   ->  List Complex Par
    ++ complexEigenvalues(m,eps) computes the eigenvalues of the matrix
    ++ m to precision eps. The eigenvalues are expressed as complex
    ++ floats or complex rational numbers depending on the type of
    ++ eps (float or rational).
  complexEigenvectors    : (MGRN,Par)   ->  List(outForm)
    ++ complexEigenvectors(m,eps)  returns a list of
    ++ records each one containing
    ++ a complex eigenvalue, its algebraic multiplicity, and a list of
    ++ associated eigenvectors. All these results
    ++ are computed to precision eps and are expressed as complex floats
    ++ or complex rational numbers depending on the type of
    ++ eps (float or rational).
T == add

  import InnerNumericEigenPackage(GRN,Complex Par,Par)

  characteristicPolynomial(m:MGRN) : Polynomial GRN  ==
    x:SE:=new()$SE
    multivariate(charpol m, x)

        ----  characteristic polynomial of a matrix A ----
  characteristicPolynomial(A:MGRN,x:SE):Polynomial GRN ==
    multivariate(charpol A, x)
```

```
complexEigenvalues(m:MGRN,eps:Par) : List Complex Par  ==
  solve1(charpol m, eps)

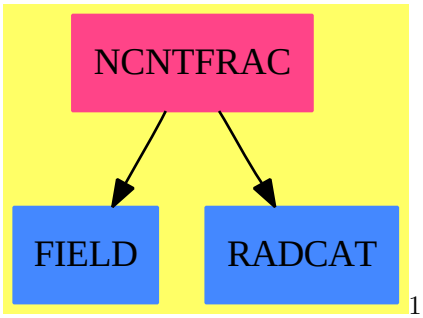
complexEigenvectors(m:MGRN,eps:Par) :List outForm ==
  innerEigenvectors(m,eps,factor$ComplexFactorization(RN,SUPGRN))
```

⟨*NCEP.dotabb*⟩≡

```
"NCEP" [color="#FF4488",href="bookvol10.4.pdf#nameddest=NCEP"]
"COMPCAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=COMPCAT"]
"NCEP" -> "COMPCAT"
```

15.65 package NCNTFRAC NumericContinuedFraction

15.66 NumericContinuedFraction



1.00

Exports:
continuedFraction

⟨*package NCNTFRAC NumericContinuedFraction*⟩≡
```
)abbrev package NCNTFRAC NumericContinuedFraction
++ Author: Clifton J. Williamson
++ Date Created: 12 April 1990
++ Change History:
++ Basic Operations: continuedFraction
++ Related Constructors: ContinuedFraction, Float
++ Also See: Fraction
++ AMS Classifications: 11J70 11A55 11K50 11Y65 30B70 40A15
++ Keywords: continued fraction
++ References:
++ Description: \spadtype{NumericContinuedFraction} provides functions
++   for converting floating point numbers to continued fractions.

NumericContinuedFraction(F): Exports == Implementation where
  F :     FloatingPointSystem
  CFC ==> ContinuedFraction Integer
  I   ==> Integer
  ST  ==> Stream I

  Exports ==> with
    continuedFraction: F -> CFC
      ++ continuedFraction(f) converts the floating point number
      ++ \spad{f} to a reduced continued fraction.

  Implementation ==> add
```

```
cfc: F -> ST
cfc(a) == delay
  aa := wholePart a
  zero?(b := a - (aa :: F)) => concat(aa,empty()$ST)
  concat(aa,cfc inv b)

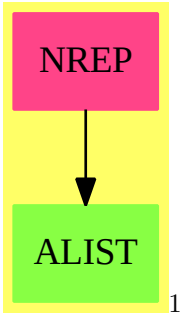
continuedFraction a ==
  aa := wholePart a
  zero?(b := a - (aa :: F)) =>
    reducedContinuedFraction(aa,empty()$ST)
  if negative? b then (aa := aa - 1; b := b + 1)
  reducedContinuedFraction(aa,cfc inv b)
```

⟨*NCNTFRAC.dotabb*⟩≡

```
"NCNTFRAC" [color="#FF4488",href="bookvol10.4.pdf#nameddest=NCNTFRAC"]
"FIELD"  [color="#4488FF",href="bookvol10.2.pdf#nameddest=FIELD"]
"RADCAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=RADCAT"]
"NCNTFRAC" -> "FIELD"
"NCNTFRAC" -> "RADCAT"
```

15.67 package NREP NumericRealEigenPackage

15.68 NumericRealEigenPackage



1.00

Exports:
characteristicPolynomial realEigenvalues realEigenvectors

⟨*package NREP NumericRealEigenPackage*⟩≡
```
)abbrev package NREP NumericRealEigenPackage
++ Author:P. Gianni
++ Date Created:Summer 1990
++ Date Last Updated:Spring 1991
++ Basic Functions:
++ Related Constructors: FloatingRealPackage
++ Also See:
++ AMS Classifications:
++ Keywords:
++ References:
++ Description:
++ This package computes explicitly eigenvalues and eigenvectors of
++ matrices with entries over the Rational Numbers.
++ The results are expressed as floating numbers or as rational numbers
++ depending on the type of the parameter Par.
NumericRealEigenPackage(Par) : C == T
 where
   Par   :   Join(Field,OrderedRing) -- Float or RationalNumber

   SE    ==> Symbol()
   RN    ==> Fraction Integer
   I     ==> Integer
   NF    ==> Float
   CF    ==> Complex Float
   GRN   ==> Complex RN
   GI    ==> Complex Integer
```

```
PI    ==> PositiveInteger
NNI   ==> NonNegativeInteger
MRN   ==> Matrix RN

MPar        ==> Matrix Par
outForm     ==> Record(outval:Par,outmult:Integer,outvect:List MPar)

C == with
  characteristicPolynomial :   MRN    -> Polynomial RN
    ++ characteristicPolynomial(m) returns the characteristic polynomial
    ++ of the matrix m expressed as polynomial
    ++ over RN with a new symbol as variable.
    -- while the function in EigenPackage returns Fraction P RN.
  characteristicPolynomial : (MRN,SE) -> Polynomial RN
    ++ characteristicPolynomial(m,x) returns the characteristic polynomial
    ++ of the matrix m expressed as polynomial
    ++ over RN with variable x.
    -- while the function in EigenPackage returns
    ++ Fraction P RN.
  realEigenvalues  :   (MRN,Par)   ->  List Par
    ++ realEigenvalues(m,eps) computes the eigenvalues of the matrix
    ++ m to precision eps. The eigenvalues are expressed as floats or
    ++ rational numbers depending on the type of eps (float or rational).
  realEigenvectors    : (MRN,Par)   ->  List(outForm)
    ++ realEigenvectors(m,eps)  returns a list of
    ++ records each one containing
    ++ a real eigenvalue, its algebraic multiplicity, and a list of
    ++ associated eigenvectors. All these results
    ++ are computed to precision eps as floats or rational
    ++ numbers depending on the type of eps .


T == add

  import InnerNumericEigenPackage(RN, Par, Par)

  characteristicPolynomial(m:MRN) : Polynomial RN ==
    x:SE:=new()$SE
    multivariate(charpol(m),x)

        ----  characteristic polynomial of a matrix A ----
  characteristicPolynomial(A:MRN,x:SE):Polynomial RN ==
    multivariate(charpol(A),x)

  realEigenvalues(m:MRN,eps:Par) : List Par  ==
    solve1(charpol m, eps)
```

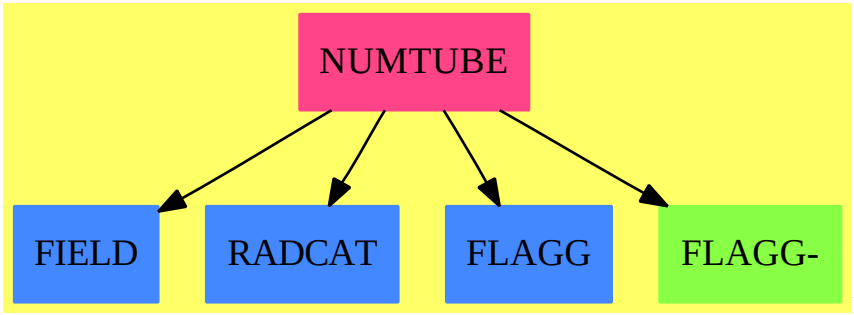
```
realEigenvectors(m:MRN,eps:Par) :List outForm ==
  innerEigenvectors(m,eps,factor$GenUFactorize(RN))
```

⟨*NREP.dotabb*⟩≡

```
"NREP" [color="#FF4488",href="bookvol10.4.pdf#nameddest=NREP"]
"ALIST" [color="#88FF44",href="bookvol10.3.pdf#nameddest=ALIST"]
"NREP" -> "ALIST"
```

15.69 package NUMTUBE NumericTubePlot

15.70 NumericTubePlot



1.00

Exports:
tube

⟨*package NUMTUBE NumericTubePlot*⟩≡
```
)abbrev package NUMTUBE NumericTubePlot
++ Author: Clifton J. Williamson
++ Date Created: Bastille Day 1989
++ Date Last Updated: 5 June 1990
++ Keywords:
++ Examples:
++ Package for constructing tubes around 3-dimensional parametric curves.
NumericTubePlot(Curve): Exports == Implementation where
  Curve : PlottableSpaceCurveCategory
  B   ==> Boolean
  I   ==> Integer
  SF  ==> DoubleFloat
  L   ==> List
  S   ==> String
  SEG ==> Segment
  Pt  ==> Point SF
  TUBE ==> TubePlot Curve
  Triad ==> Record(tang:Pt,norm:Pt,bin:Pt)

  Exports ==> with
    tube: (Curve,SF,I) -> TUBE
      ++ tube(c,r,n) creates a tube of radius r around the curve c.

  Implementation ==> add
    import TubePlotTools

    LINMAX  := convert(0.995)@SF
```

```
XHAT := point(1,0,0,0)
YHAT := point(0,1,0,0)
PREV0 := point(1,1,0,0)
PREV := PREV0

colinearity: (Pt,Pt) -> SF
colinearity(x,y) == dot(x,y)**2/(dot(x,x) * dot(y,y))

orthog: (Pt,Pt) -> Pt
orthog(x,y) ==
  if colinearity(x,y) > LINMAX then y := PREV
  if colinearity(x,y) > LINMAX then
    y := (colinearity(x,XHAT) < LINMAX => XHAT; YHAT)
  a := -dot(x,y)/dot(x,x)
  PREV := a*x + y

poTriad:(Pt,Pt,Pt) -> Triad
poTriad(pl,po,pr) ==
  -- use divided difference for t.
  t := unitVector(pr - pl)
  -- compute n as orthogonal to t in plane containing po.
  pol := pl - po
  n   := unitVector orthog(t,pol)
  [t,n,cross(t,n)]

curveTriads: L Pt -> L Triad
curveTriads l ==
  (k := #l) < 2 => error "Need at least 2 points to specify a curve"
  PREV := PREV0
  k = 2 =>
    t := unitVector(second l - first l)
    n := unitVector(t - XHAT)
    b := cross(t,n)
    triad : Triad := [t,n,b]
    [triad,triad]
  -- compute interior triads using divided differences
  midtriads : L Triad :=
    [poTriad(pl,po,pr) for pl in l for po in rest l _
           for pr in rest rest l]
  -- compute first triad using a forward difference
  x := first midtriads
  t := unitVector(second l - first l)
  n := unitVector orthog(t,x.norm)
  begtriad : Triad := [t,n,cross(t,n)]
  -- compute last triad using a backward difference
  x := last midtriads
```

```
    -- efficiency!!
    t := unitVector(l.k - l.(k-1))
    n := unitVector orthog(t,x.norm)
    endtriad : Triad := [t,n,cross(t,n)]
    concat(begtriad,concat(midtriads,endtriad))

  curveLoops: (L Pt,SF,I) -> L L Pt
  curveLoops(pts,r,nn) ==
    triads := curveTriads pts
    cosSin := cosSinInfo nn
    loops : L L Pt := nil()
    for pt in pts for triad in triads repeat
      n := triad.norm; b := triad.bin
      loops := concat(loopPoints(pt,n,b,r,cosSin),loops)
    reverse_! loops

  tube(curve,r,n) ==
    n < 3 => error "tube: n should be at least 3"
    brans := listBranches curve
    loops : L L Pt := nil()
    for bran in brans repeat
      loops := concat(loops,curveLoops(bran,r,n))
    tube(curve,loops,false)
```

⟨*NUMTUBE.dotabb*⟩≡

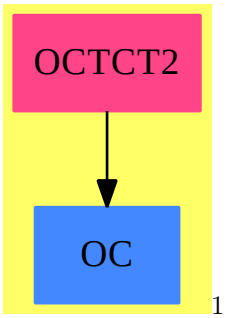
```
"NUMTUBE" [color="#FF4488",href="bookvol10.4.pdf#nameddest=NUMTUBE"]
"FIELD"  [color="#4488FF",href="bookvol10.2.pdf#nameddest=FIELD"]
"RADCAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=RADCAT"]
"FLAGG" [color="#4488FF",href="bookvol10.2.pdf#nameddest=FLAGG"]
"FLAGG-" [color="#88FF44",href="bookvol10.3.pdf#nameddest=FLAGG"]
"NUMTUBE" -> "FIELD"
"NUMTUBE" -> "RADCAT"
"NUMTUBE" -> "FLAGG"
"NUMTUBE" -> "FLAGG-"
```

Chapter 16

Chapter O

16.1 package OCTCT2 OctonionCategoryFunctions2

16.2 OctonionCategoryFunctions2



1.00

Exports:
map

⟨*package OCTCT2 OctonionCategoryFunctions2*⟩≡
```
)abbrev package OCTCT2 OctonionCategoryFunctions2
--% OctonionCategoryFunctions2
++ Author: Johannes Grabmeier
++ Date Created: 10 September 1990
++ Date Last Updated: 10 September 1990
++ Basic Operations: map
++ Related Constructors:
++ Also See:
++ AMS Classifications:
```

```
++ Keywords: octonion, non-associative algebra, Cayley-Dixon
++ References:
++ Description:
++  OctonionCategoryFunctions2 implements functions between
++  two octonion domains defined over different rings.
++  The function map is used
++  to coerce between octonion types.

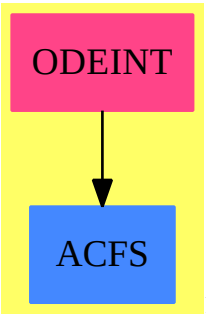
OctonionCategoryFunctions2(OR,R,OS,S) : Exports ==
  Implementation where
    R  : CommutativeRing
    S  : CommutativeRing
    OR : OctonionCategory R
    OS : OctonionCategory S
    Exports == with
      map:     (R -> S, OR) -> OS
        ++ map(f,u) maps f onto the component parts of the octonion
        ++ u.
    Implementation == add
      map(fn : R -> S, u : OR): OS ==
        octon(fn real u, fn imagi u, fn imagj u, fn imagk u,_
        fn imagE u, fn imagI u, fn imagJ u, fn imagK u)$OS
```

⟨*OCTCT2.dotabb*⟩≡

```
"OCTCT2" [color="#FF4488",href="bookvol10.4.pdf#nameddest=OCTCT2"]
"OC" [color="#4488FF",href="bookvol10.2.pdf#nameddest=OC"]
"OCTCT2" -> "OC"
```

16.3 package ODEINT ODEIntegration

16.4 ODEIntegration



1.00

Exports:
diff expint int

⟨*package ODEINT ODEIntegration*⟩≡
```
)abbrev package ODEINT ODEIntegration
++ Author: Manuel Bronstein
++ Date Created: 4 November 1991
++ Date Last Updated: 2 February 1994
++ Description:
++ \spadtype{ODEIntegration} provides an interface to the integrator.
++ This package is intended for use
++ by the differential equations solver but not at top-level.
ODEIntegration(R, F): Exports == Implementation where
  R: Join(OrderedSet, EuclideanDomain, RetractableTo Integer,
          LinearlyExplicitRingOver Integer, CharacteristicZero)
  F: Join(AlgebraicallyClosedFunctionSpace R, TranscendentalFunctionCategory,
                                              PrimitiveFunctionCategory)

  Q   ==> Fraction Integer
  UQ  ==> Union(Q, "failed")
  SY  ==> Symbol
  K   ==> Kernel F
  P   ==> SparseMultivariatePolynomial(R, K)
  REC ==> Record(coef:Q, logand:F)

  Exports ==> with
    int   : (F, SY) -> F
      ++ int(f, x) returns the integral of f with respect to x.
    expint: (F, SY) -> F
      ++ expint(f, x) returns e^{the integral of f with respect to x}.
    diff  : SY -> (F -> F)
```

```
      ++ diff(x) returns the derivation with respect to x.

  Implementation ==> add
    import FunctionSpaceIntegration(R, F)
    import ElementaryFunctionStructurePackage(R, F)

    isQ   : List F -> UQ
    isQlog: F -> Union(REC, "failed")
    mkprod: List REC -> F

    diff x == differentiate(#1, x)

-- This is the integration function to be used for quadratures
    int(f, x) ==
      (u := integrate(f, x)) case F => u::F
      first(u::List(F))

-- mkprod([q1, f1],...,[qn,fn]) returns */(fi^qi) but groups the
-- qi having the same denominator together
    mkprod l ==
      empty? l => 1
      rec := first l
      d := denom(rec.coef)
      ll := select(denom(#1.coef) = d, l)
      nthRoot(*/[r.logand ** numer(r.coef) for r in ll], d) *
        mkprod setDifference(l, ll)

-- computes exp(int(f,x)) in a non-naive way
    expint(f, x) ==
      a := int(f, x)
      (u := validExponential(tower a, a, x)) case F => u::F
      da := denom a
      l :=
        (v := isPlus(na := numer a)) case List(P) => v::List(P)
        [na]
      exponent:P := 0
      lrec:List(REC) := empty()
      for term in l repeat
        if (w := isQlog(term / da)) case REC then
          lrec := concat(w::REC, lrec)
        else
          exponent := exponent + term
      mkprod(lrec) * exp(exponent / da)

-- checks if all the elements of l are rational numbers, returns their product
    isQ l ==
```

```
        prod:Q := 1
        for x in l repeat
          (u := retractIfCan(x)@UQ) case "failed" => return "failed"
          prod := prod * u::Q
        prod

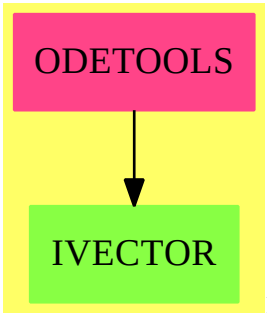
  -- checks if a non-sum expr is of the form c * log(g) for a rational number c
      isQlog f ==
        is?(f, "log"::SY) => [1, first argument(retract(f)@K)]
        (v := isTimes f) case List(F) and (#(l := v::List(F)) <= 3) =>
            l := reverse_! sort_! l
            is?(first l, "log"::SY) and ((u := isQ rest l) case Q) =>
                [u::Q, first argument(retract(first(l))@K)]
            "failed"
        "failed"
```

⟨*ODEINT.dotabb*⟩≡

```
"ODEINT" [color="#FF4488",href="bookvol10.4.pdf#nameddest=ODEINT"]
"ACFS" [color="#4488FF",href="bookvol10.2.pdf#nameddest=ACFS"]
"ODEINT" -> "ACFS"
```

16.5 package ODETOOLS ODETools

16.6 ODETools



1.00

Exports:
particularSolution variationOfParameters wronskianMatrix

⟨*package ODETOOLS ODETools*⟩≡
```
)abbrev package ODETOOLS ODETools
++ Author: Manuel Bronstein
++ Date Created: 20 March 1991
++ Date Last Updated: 2 February 1994
++ Description:
++   \spad{ODETools} provides tools for the linear ODE solver.
ODETools(F, LODO): Exports == Implementation where
  N ==> NonNegativeInteger
  L ==> List F
  V ==> Vector F
  M ==> Matrix F

  F:    Field
  LODO: LinearOrdinaryDifferentialOperatorCategory F

  Exports ==> with
    wronskianMatrix: L -> M
      ++ wronskianMatrix([f1,...,fn]) returns the \spad{n x n} matrix m
      ++ whose i^th row is \spad{[f1^(i-1),...,fn^(i-1)]}.
    wronskianMatrix: (L, N) -> M
      ++ wronskianMatrix([f1,...,fn], q, D) returns the \spad{q x n} matrix m
      ++ whose i^th row is \spad{[f1^(i-1),...,fn^(i-1)]}.
    variationOfParameters: (LODO, F, L) -> Union(V, "failed")
      ++ variationOfParameters(op, g, [f1,...,fm])
      ++ returns \spad{[u1,...,um]} such that a particular solution of the
      ++ equation \spad{op y = g} is \spad{f1 int(u1) + ... + fm int(um)}
      ++ where \spad{[f1,...,fm]} are linearly independent and \spad{op(fi)=0}.
```

```
      ++ The value "failed" is returned if \spad{m < n} and no particular
      ++ solution is found.
    particularSolution: (LODO, F, L, F -> F) -> Union(F, "failed")
      ++ particularSolution(op, g, [f1,...,fm], I) returns a particular
      ++ solution h of the equation \spad{op y = g} where \spad{[f1,...,fm]}
      ++ are linearly independent and \spad{op(fi)=0}.
      ++ The value "failed" is returned if no particular solution is found.
      ++ Note: the method of variations of parameters is used.

  Implementation ==> add
    import LinearSystemMatrixPackage(F, V, V, M)

    diff := D()$LODO

    wronskianMatrix l == wronskianMatrix(l, #l)

    wronskianMatrix(l, q) ==
      v:V := vector l
      m:M := zero(q, #v)
      for i in minRowIndex m .. maxRowIndex m repeat
        setRow_!(m, i, v)
        v := map_!(diff #1, v)
      m

    variationOfParameters(op, g, b) ==
      empty? b => "failed"
      v:V := new(n := degree op, 0)
      qsetelt_!(v, maxIndex v, g / leadingCoefficient op)
      particularSolution(wronskianMatrix(b, n), v)

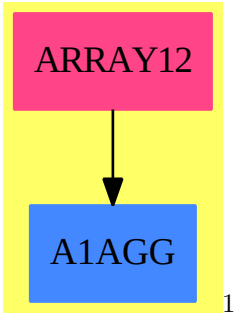
    particularSolution(op, g, b, integration) ==
      zero? g => 0
      (sol := variationOfParameters(op, g, b)) case "failed" => "failed"
      ans:F := 0
      for f in b for i in minIndex(s := sol::V) .. repeat
        ans := ans + integration(qelt(s, i)) * f
      ans
```

⟨*ODETOOLS.dotabb*⟩≡

```
"ODETOOLS" [color="#FF4488",href="bookvol10.4.pdf#nameddest=ODETOOLS"]
"IVECTOR" [color="#88FF44",href="bookvol10.3.pdf#nameddest=IVECTOR"]
"ODETOOLS" -> "IVECTOR"
```

16.7 package ARRAY12 OneDimensionalArrayFunctions2

16.8 OneDimensionalArrayFunctions2



1.00

Exports:
map reduce scan

⟨*package ARRAY12 OneDimensionalArrayFunctions2*⟩≡

```
)abbrev package ARRAY12 OneDimensionalArrayFunctions2
++ This package provides tools for operating on one-dimensional arrays
++ with unary and binary functions involving different underlying types
OneDimensionalArrayFunctions2(A, B): Exports == Implementation where
  A, B: Type

  VA ==> OneDimensionalArray A
  VB ==> OneDimensionalArray B
  O2 ==> FiniteLinearAggregateFunctions2(A, VA, B, VB)

  Exports ==> with
   scan   : ((A, B) -> B, VA, B) -> VB
    ++ scan(f,a,r) successively applies
    ++ \spad{reduce(f,x,r)} to more and more leading sub-arrays
    ++ x of one-dimensional array \spad{a}.
    ++ More precisely, if \spad{a} is \spad{[a1,a2,...]}, then
    ++ \spad{scan(f,a,r)} returns
    ++ \spad{[reduce(f,[a1],r),reduce(f,[a1,a2],r),...]}.
    ++
    ++X T1:=OneDimensionalArrayFunctions2(Integer,Integer)
    ++X adder(a:Integer,b:Integer):Integer == a+b
    ++X scan(adder,[i for i in 1..10],0)$T1

   reduce : ((A, B) -> B, VA, B) -> B
    ++ reduce(f,a,r) applies function f to each
    ++ successive element of the
```

```
      ++ one-dimensional array \spad{a} and an accumulant initialized to r.
      ++ For example, \spad{reduce(_+$Integer,[1,2,3],0)}
      ++ does \spad{3+(2+(1+0))}. Note: third argument r
      ++ may be regarded as the identity element for the function f.
      ++
      ++X T1:=OneDimensionalArrayFunctions2(Integer,Integer)
      ++X adder(a:Integer,b:Integer):Integer == a+b
      ++X reduce(adder,[i for i in 1..10],0)$T1

     map : (A -> B, VA) -> VB
      ++ map(f,a) applies function f to each member of one-dimensional array
      ++ \spad{a} resulting in a new one-dimensional array over a
      ++ possibly different underlying domain.
      ++
      ++X T1:=OneDimensionalArrayFunctions2(Integer,Integer)
      ++X map(x+->x+2,[i for i in 1..10])$T1

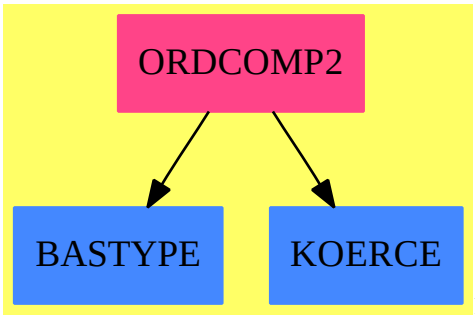
    Implementation ==> add
      map(f, v)       == map(f, v)$O2
      scan(f, v, b)   == scan(f, v, b)$O2
      reduce(f, v, b) == reduce(f, v, b)$O2
```

⟨*ARRAY12.dotabb*⟩≡

```
"ARRAY12" [color="#FF4488",href="bookvol10.4.pdf#nameddest=ARRAY12"]
"A1AGG" [color="#4488FF",href="bookvol10.2.pdf#nameddest=A1AGG"]
"ARRAY12" -> "A1AGG"
```

16.9 package ONECOMP2 OnePointCompletionFunctions2

16.10 OnePointCompletionFunctions2



1.00

Exports:
map

⟨*package ONECOMP2 OnePointCompletionFunctions2*⟩≡

```
)abbrev package ONECOMP2 OnePointCompletionFunctions2
++ Lifting of maps to one-point completions
++ Author: Manuel Bronstein
++ Description: Lifting of maps to one-point completions.
++ Date Created: 4 Oct 1989
++ Date Last Updated: 4 Oct 1989
OnePointCompletionFunctions2(R, S): Exports == Implementation where
  R, S: SetCategory

  OPR ==> OnePointCompletion R
  OPS ==> OnePointCompletion S

  Exports ==> with
    map: (R -> S, OPR) -> OPS
      ++ map(f, r) lifts f and applies it to r, assuming that
      ++ f(infinity) = infinity.
    map: (R -> S, OPR, OPS) -> OPS
      ++ map(f, r, i) lifts f and applies it to r, assuming that
      ++ f(infinity) = i.

  Implementation ==> add
    map(f, r) == map(f, r, infinity())

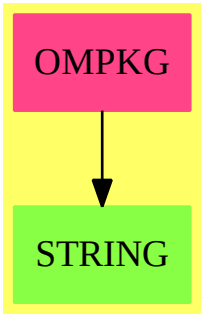
    map(f, r, i) ==
      (u := retractIfCan r) case R => (f(u::R))::OPS
      i
```

⟨*ONECOMP2.dotabb*⟩≡

```
"ONECOMP2" [color="#FF4488",href="bookvol10.4.pdf#nameddest=ONECOMP2"]
"BASTYPE" [color="#4488FF",href="bookvol10.2.pdf#nameddest=BASTYPE"]
"KOERCE" [color="#4488FF",href="bookvol10.2.pdf#nameddest=KOERCE"]
"ONECOMP2" -> "BASTYPE"
"ONECOMP2" -> "KOERCE"
```

16.11 package OMPKG OpenMathPackage

16.12 OpenMathPackage



1.00

Exports:

OMlistCDs	OMread	OMreadFile	OMreadStr	OMsupportsCD?
OMlistSymbols	OMsupportsSymbol?	OMunhandledSymbol		

⟨*package OMPKG OpenMathPackage*⟩≡

```
)abbrev package OMPKG OpenMathPackage
++ Author: Vilya Harvey
++ Date Created:
++ Date Last Updated:
++ Basic Functions:
++ Related Constructors:
++ Also See:
++ AMS Classifications:
++ Keywords:
++ References:
++ Description: \spadtype{OpenMathPackage} provides some simple utilities
++ to make reading OpenMath objects easier.

OpenMathPackage(): with
  OMread            : OpenMathDevice -> Any
  ++ OMread(dev) reads an OpenMath object from \axiom{dev} and passes it
  ++ to AXIOM.
  OMreadFile        : String -> Any
  ++ OMreadFile(f) reads an OpenMath object from \axiom{f} and passes it
  ++ to AXIOM.
  OMreadStr         : String -> Any
  ++ OMreadStr(f) reads an OpenMath object from \axiom{f} and passes it
  ++ to AXIOM.
  OMlistCDs         : () -> List(String)
  ++ OMlistCDs() lists all the CDs supported by AXIOM.
  OMlistSymbols     : String -> List(String)
```

```
    ++ OMlistSymbols(cd) lists all the symbols in \axiom{cd}.
    OMsupportsCD?      : String -> Boolean
    ++ OMsupportsCD?(cd) returns true if AXIOM supports \axiom{cd}, false
    ++ otherwise.
    OMsupportsSymbol? : (String, String) -> Boolean
    ++ OMsupportsSymbol?(s,cd) returns true if AXIOM supports symbol \axiom{s}
    ++ from CD \axiom{cd}, false otherwise.
    OMunhandledSymbol : (String, String) -> Exit
    ++ OMunhandledSymbol(s,cd) raises an error if AXIOM reads a symbol which it
    ++ is unable to handle.  Note that this is different from an unexpected
    ++ symbol.
  == add
    import OpenMathEncoding
    import OpenMathDevice
    import String

    OMunhandledSymbol(u,v) ==
      error concat ["AXIOM is unable to process the symbol ",u," from CD ",v,"."]

    OMread(dev: OpenMathDevice): Any ==
      interpret(OM_-READ(dev)$Lisp :: InputForm)

    OMreadFile(filename: String): Any ==
      dev := OMopenFile(filename, "r", OMencodingUnknown())
      res: Any := interpret(OM_-READ(dev)$Lisp :: InputForm)
      OMclose(dev)
      res

    OMreadStr(str: String): Any ==
      strp := OM_-STRINGTOSTRINGPTR(str)$Lisp
      dev := OMopenString(strp pretend String, OMencodingUnknown())
      res: Any := interpret(OM_-READ(dev)$Lisp :: InputForm)
      OMclose(dev)
      res

    OMlistCDs(): List(String) ==
      OM_-LISTCDS()$Lisp pretend List(String)

    OMlistSymbols(cd: String): List(String) ==
      OM_-LISTSYMBOLS(cd)$Lisp pretend List(String)

    import SExpression

    OMsupportsCD?(cd: String): Boolean ==
      not null? OM_-SUPPORTSCD(cd)$Lisp
```

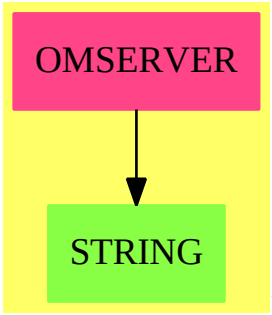
```
    OMsupportsSymbol?(cd: String, name: String): Boolean ==
      not null? OM_-SUPPORTSSYMBOL(cd, name)$Lisp
```

⟨*OMPKG.dotabb*⟩≡

```
"OMPKG" [color="#FF4488",href="bookvol10.4.pdf#nameddest=OMPKG"]
"STRING" [color="#88FF44",href="bookvol10.3.pdf#nameddest=STRING"]
"OMPKG" -> "STRING"
```

16.13 package OMSERVER OpenMathServerPackage

16.14 OpenMathServerPackage



1.00

Exports:
OMreceive OMsend OMserve

⟨*package OMSERVER OpenMathServerPackage*⟩≡
```
)abbrev package OMSERVER OpenMathServerPackage
++ Author: Vilya Harvey
++ Date Created:
++ Date Last Updated:
++ Basic Functions:
++ Related Constructors:
++ Also See:
++ AMS Classifications:
++ Keywords:
++ References:
++ Description: \spadtype{OpenMathServerPackage} provides the necessary
++ operations to run AXIOM as an OpenMath server, reading/writing objects
++ to/from a port.  Please note the facilities available here are very basic.
++ The idea is that a user calls e.g. \axiom{Omserve(4000,60)} and then
++ another process sends OpenMath objects to port 4000 and reads the result.

OpenMathServerPackage(): with
  OMreceive : OpenMathConnection -> Any
  ++ OMreceive(c) reads an OpenMath object from connection \axiom{c} and
  ++ returns the appropriate AXIOM object.
  OMsend    : (OpenMathConnection, Any) -> Void
  ++ OMsend(c,u) attempts to output \axiom{u} on \axiom{c} in OpenMath.
  OMserve   : (SingleInteger, SingleInteger) -> Void
  ++ OMserve(portnum,timeout) puts AXIOM into server mode on port number
  ++ \axiom{portnum}.  The parameter \axiom{timeout} specifies the timeout
  ++ period for the connection.
```

```
== add
 import OpenMathDevice
 import OpenMathConnection
 import OpenMathPackage
 import OpenMath



 OMreceive(conn: OpenMathConnection): Any ==
   dev: OpenMathDevice := OMconnInDevice(conn)
   OMsetEncoding(dev, OMencodingUnknown);
   OMread(dev)

 OMsend(conn: OpenMathConnection, value: Any): Void ==
   dev: OpenMathDevice := OMconnOutDevice(conn)
   OMsetEncoding(dev, OMencodingXML);
   --retractable?(value)$AnyFunctions1(Expression Integer) =>
   --  OMwrite(dev, retract(value)$AnyFunctions1(Expression Integer), true)
   retractable?(value)$AnyFunctions1(Integer) =>
     OMwrite(dev, retract(value)$AnyFunctions1(Integer), true)
   retractable?(value)$AnyFunctions1(Float) =>
     OMwrite(dev, retract(value)$AnyFunctions1(Float), true)
   retractable?(value)$AnyFunctions1(SingleInteger) =>
     OMwrite(dev, retract(value)$AnyFunctions1(SingleInteger), true)
   retractable?(value)$AnyFunctions1(DoubleFloat) =>
     OMwrite(dev, retract(value)$AnyFunctions1(DoubleFloat), true)
   retractable?(value)$AnyFunctions1(String) =>
     OMwrite(dev, retract(value)$AnyFunctions1(String), true)

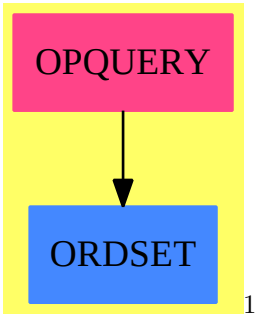
 OMserve(portNum: SingleInteger, timeout: SingleInteger): Void ==
   conn: OpenMathConnection := OMmakeConn(timeout)
   OMbindTCP(conn, portNum)
   val: Any
   while true repeat
     val := OMreceive(conn)
     OMsend(conn, val)
```

⟨*OMSERVER.dotabb*⟩≡

```
"OMSERVER" [color="#FF4488",href="bookvol10.4.pdf#nameddest=OMSERVER"]
"STRING" [color="#88FF44",href="bookvol10.3.pdf#nameddest=STRING"]
"OMSERVER" -> "STRING"
```

16.15 package OPQUERY OperationsQuery

16.16 OperationsQuery



1.00

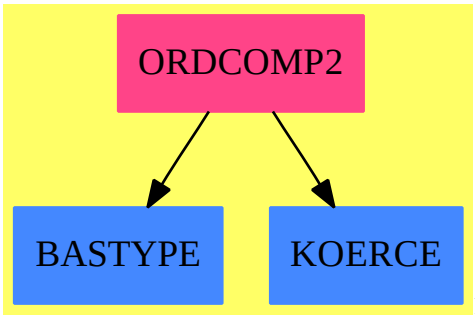
Exports:
getDatabase

⟨*package OPQUERY OperationsQuery*⟩≡
```
)abbrev package OPQUERY OperationsQuery
++ This package exports tools to create AXIOM Library information databases.
OperationsQuery(): Exports == Implementation where
  Exports == with
    getDatabase: String -> Database(IndexCard)
      ++ getDatabase("char") returns a list of appropriate entries in the
      ++ browser database.  The legal values for "char" are "o" (operations),
      ++ "k" (constructors), "d" (domains), "c" (categories) or "p" (packages).
  Implementation == add
    getDatabase(s) == getBrowseDatabase(s)$Lisp
```

⟨*OPQUERY.dotabb*⟩≡
```
"OPQUERY" [color="#FF4488",href="bookvol10.4.pdf#nameddest=OPQUERY"]
"ORDSET" [color="#4488FF",href="bookvol10.2.pdf#nameddest=ORDSET"]
"OPQUERY" -> "ORDSET"
```

16.17 package ORDCOMP2 OrderedCompletionFunctions2

16.18 OrderedCompletionFunctions2



1.00

Exports:
map

⟨*package ORDCOMP2 OrderedCompletionFunctions2*⟩≡

```
)abbrev package ORDCOMP2 OrderedCompletionFunctions2
++ Lifting of maps to ordered completions
++ Author: Manuel Bronstein
++ Description: Lifting of maps to ordered completions.
++ Date Created: 4 Oct 1989
++ Date Last Updated: 4 Oct 1989
OrderedCompletionFunctions2(R, S): Exports == Implementation where
  R, S: SetCategory

  ORR ==> OrderedCompletion R
  ORS ==> OrderedCompletion S

  Exports ==> with
    map: (R -> S, ORR) -> ORS
      ++ map(f, r) lifts f and applies it to r, assuming that
      ++ f(plusInfinity) = plusInfinity and that
      ++ f(minusInfinity) = minusInfinity.
    map: (R -> S, ORR, ORS, ORS) -> ORS
      ++ map(f, r, p, m) lifts f and applies it to r, assuming that
      ++ f(plusInfinity) = p and that f(minusInfinity) = m.

  Implementation ==> add
    map(f, r) == map(f, r, plusInfinity(), minusInfinity())

    map(f, r, p, m) ==
      zero?(n := whatInfinity r) => (f retract r)::ORS
```

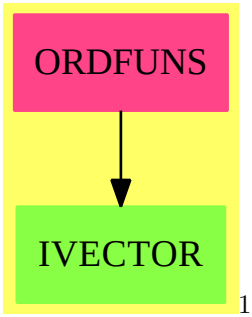
```
--      one? n => p
       (n = 1) => p
       m
```

⟨*ORDCOMP2.dotabb*⟩≡

```
"ORDCOMP2" [color="#FF4488",href="bookvol10.4.pdf#nameddest=ORDCOMP2"]
"BASTYPE" [color="#4488FF",href="bookvol10.2.pdf#nameddest=BASTYPE"]
"KOERCE" [color="#4488FF",href="bookvol10.2.pdf#nameddest=KOERCE"]
"ORDCOMP2" -> "BASTYPE"
"ORDCOMP2" -> "KOERCE"
```

16.19 package ORDFUNS OrderingFunctions

16.20 OrderingFunctions



1.00

Exports:
pureLex reverseLex totalLex

⟨*package ORDFUNS OrderingFunctions*⟩≡
```
)abbrev package ORDFUNS OrderingFunctions
++ Author: Barry Trager
++ Date Created:
++ Date Last Updated:
++ Basic Functions:
++ Related Constructors: OrderedDirectProduct
++ Also See:
++ AMS Classifications:
++ Keywords:
++ References:
++ Description:
++    This package provides ordering functions on vectors which
++ are suitable parameters for OrderedDirectProduct.

OrderingFunctions(dim,S) : T == C  where
  dim : NonNegativeInteger
  S         : OrderedAbelianMonoid
  VS       == Vector S

  T ==  with
     pureLex    :  (VS,VS)  -> Boolean
       ++ pureLex(v1,v2) return true if the vector v1 is less than the
       ++ vector v2 in the lexicographic ordering.
     totalLex   :  (VS,VS)  -> Boolean
       ++ totalLex(v1,v2) return true if the vector v1 is less than the
       ++ vector v2 in the ordering which is total degree refined by
       ++ lexicographic ordering.
```

```
    reverseLex :  (VS,VS)  -> Boolean
      ++ reverseLex(v1,v2) return true if the vector v1 is less than the
      ++ vector v2 in the ordering which is total degree refined by
      ++ the reverse lexicographic ordering.

 C == add
   n:NonNegativeInteger:=dim

-- pure lexicographical ordering
   pureLex(v1:VS,v2:VS) : Boolean ==
     for i in 1..n repeat
       if qelt(v1,i) < qelt(v2,i) then return true
       if qelt(v2,i) < qelt(v1,i) then return false
     false

-- total ordering refined with lex
   totalLex(v1:VS,v2:VS) :Boolean ==
     n1:S:=0
     n2:S:=0
     for i in 1..n repeat
       n1:= n1+qelt(v1,i)
       n2:=n2+qelt(v2,i)
     n1<n2 => true
     n2<n1 => false
     for i in 1..n repeat
       if qelt(v1,i) < qelt(v2,i) then return true
       if qelt(v2,i) < qelt(v1,i) then return false
     false

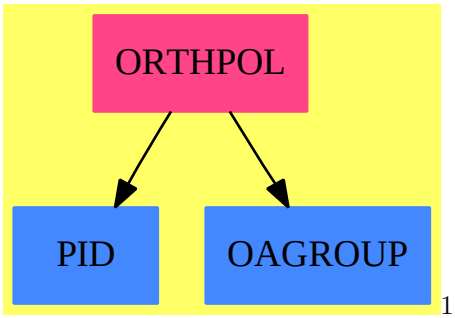
-- reverse lexicographical ordering
   reverseLex(v1:VS,v2:VS) :Boolean ==
     n1:S:=0
     n2:S:=0
     for i in 1..n repeat
       n1:= n1+qelt(v1,i)
       n2:=n2+qelt(v2,i)
     n1<n2 => true
     n2<n1 => false
     for i in reverse(1..n) repeat
       if qelt(v2,i) < qelt(v1,i) then return true
       if qelt(v1,i) < qelt(v2,i) then return false
     false
```

⟨*ORDFUNS.dotabb*⟩≡

```
"ORDFUNS" [color="#FF4488",href="bookvol10.4.pdf#nameddest=ORDFUNS"]
"IVECTOR" [color="#88FF44",href="bookvol10.3.pdf#nameddest=IVECTOR"]
"ORDFUNS" -> "IVECTOR"
```

16.21 package ORTHPOL OrthogonalPolynomialFunctions

16.22 OrthogonalPolynomialFunctions



1.00

Exports:
chebyshevT chebyshevU hermiteH laguerreL laguerreL legendreP

⟨*package ORTHPOL OrthogonalPolynomialFunctions*⟩≡

```
)abbrev package ORTHPOL OrthogonalPolynomialFunctions
++ Author: Stephen M. Watt
++ Date Created:  1990
++ Date Last Updated: June 25, 1991
++ Basic Operations:
++ Related Domains:
++ Also See:
++ AMS Classifications:
++ Keywords:
++ Examples:
++ References:
++ Description:
++   This package provides orthogonal polynomials as functions on a ring.

OrthogonalPolynomialFunctions(R: CommutativeRing): Exports == Impl where
    NNI ==> NonNegativeInteger
    RN  ==> Fraction Integer

    Exports ==> with

        chebyshevT: (NNI, R) -> R
           ++ chebyshevT(n,x) is the n-th Chebyshev polynomial of the first
           ++ kind, \spad{T[n](x)}.  These are defined by
           ++ \spad{(1-t*x)/(1-2*t*x+t**2) = sum(T[n](x) *t**n, n = 0..)}.

        chebyshevU: (NNI, R) -> R
```

```
        ++ chebyshevU(n,x) is the n-th Chebyshev polynomial of the second
        ++ kind, \spad{U[n](x)}. These are defined by
        ++ \spad{1/(1-2*t*x+t**2) = sum(T[n](x) *t**n, n = 0..)}.

    hermiteH:   (NNI, R) -> R
        ++ hermiteH(n,x) is the n-th Hermite polynomial, \spad{H[n](x)}.
        ++ These are defined by
        ++ \spad{exp(2*t*x-t**2) = sum(H[n](x)*t**n/n!, n = 0..)}.

    laguerreL:  (NNI, R) -> R
        ++ laguerreL(n,x) is the n-th Laguerre polynomial, \spad{L[n](x)}.
        ++ These are defined by
        ++ \spad{exp(-t*x/(1-t))/(1-t) = sum(L[n](x)*t**n/n!, n = 0..)}.

    laguerreL:  (NNI, NNI, R) -> R
        ++ laguerreL(m,n,x) is the associated Laguerre polynomial,
        ++ \spad{L<m>[n](x)}.  This is the m-th derivative of \spad{L[n](x)}.

    if R has Algebra RN then
        legendreP:  (NNI, R) -> R
            ++ legendreP(n,x) is the n-th Legendre polynomial,
            ++ \spad{P[n](x)}.  These are defined by
            ++ \spad{1/sqrt(1-2*x*t+t**2) = sum(P[n](x)*t**n, n = 0..)}.

Impl ==> add
    p0, p1: R
    cx:     Integer

    import IntegerCombinatoricFunctions()

    laguerreL(n, x) ==
        n = 0 => 1
        (p1, p0) := (-x + 1, 1)
        for i in 1..n-1 repeat
            (p1, p0) := ((2*i::R + 1 - x)*p1 - i**2*p0, p1)
        p1
    laguerreL(m, n, x) ==
        ni := n::Integer
        mi := m::Integer
        cx := (-1)**m * binomial(ni,ni-mi) * factorial(ni)
        p0 := 1
        p1 := cx::R
        for j in 1..ni-mi repeat
            cx := -cx*(ni-mi-j+1)
            cx := (cx exquo ((mi+j)*j))::Integer
            p0 := p0 * x
```

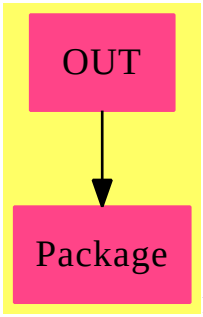
```
            p1 := p1 + cx*p0
        p1
    chebyshevT(n, x) ==
        n = 0 => 1
        (p1, p0) := (x, 1)
        for i in 1..n-1 repeat
            (p1, p0) := (2*x*p1 - p0, p1)
        p1
    chebyshevU(n, x) ==
        n = 0 => 1
        (p1, p0) := (2*x, 1)
        for i in 1..n-1 repeat
            (p1, p0) := (2*x*p1 - p0, p1)
        p1
    hermiteH(n, x) ==
        n = 0 => 1
        (p1, p0) := (2*x, 1)
        for i in 1..n-1 repeat
            (p1, p0) := (2*x*p1 - 2*i*p0, p1)
        p1
    if R has Algebra RN then
        legendreP(n, x) ==
            n = 0 => 1
            p0 := 1
            p1 := x
            for i in 1..n-1 repeat
                c: RN := 1/(i+1)
                (p1, p0) := (c*((2*i+1)*x*p1 - i*p0), p1)
            p1
```

⟨*ORTHPOL.dotabb*⟩≡

```
"ORTHPOL" [color="#FF4488",href="bookvol10.4.pdf#nameddest=ORTHPOL"]
"PID" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PID"]
"OAGROUP" [color="#4488FF",href="bookvol10.2.pdf#nameddest=OAGROUP"]
"ORTHPOL" -> "PID"
"ORTHPOL" -> "OAGROUP"
```

16.23 package OUT OutputPackage

16.24 OutputPackage



1.00

Exports:
output outputList

⟨*package OUT OutputPackage*⟩≡

```
)abbrev package OUT OutputPackage
++ Author: Stephen M. Watt
++ Date Created: February 1986
++ Date Last Updated: October 27 1995 (MCD)
++ Basic Operations: output
++ Related Constructors: OutputForm
++ Also See:
++ AMS Classifications:
++ Keywords:
++ References:
++ Description: OutPackage allows pretty-printing from programs.

OutputPackage: with
        output: String -> Void
            ++ output(s) displays the string s on the ‘‘algebra output’’
            ++ stream, as defined by \spadsyscom{set output algebra}.
        output: OutputForm -> Void
            ++ output(x) displays the output form x on the
            ++ ‘‘algebra output’’ stream, as defined by
            ++ \spadsyscom{set output algebra}.
        output: (String, OutputForm) -> Void
            ++ output(s,x) displays the string s followed by the form x
            ++ on the ‘‘algebra output’’ stream, as defined by
            ++ \spadsyscom{set output algebra}.
        outputList: (List Any) -> Void
            ++ outputList(l) displays the concatenated components of the
            ++ list l on the ‘‘algebra output’’ stream, as defined by
```

```
            ++ \spadsyscom{set output algebra}; quotes are stripped
            ++ from strings.

      == add
          --ExpressionPackage()
          E      ==> OutputForm
          putout ==> mathprint$Lisp

          s: String
          e: OutputForm
          l: List Any

          output e ==
              mathprint(e)$Lisp
              void()
          output s ==
              output(s:E)
          output(s,e) ==
              output blankSeparate [s:E, e]
          outputList(l) ==                                  -- MGR
            output hconcat
              [if retractable?(x)$AnyFunctions1(String) then
                  message(retract(x)$AnyFunctions1(String))$OutputForm
                else
                  x::OutputForm
               for x in l]
```

⟨*OUT.dotabb*⟩≡

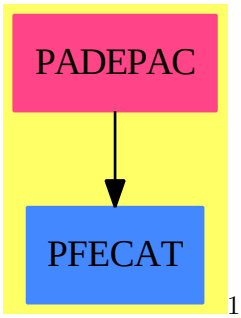
```
"OUT" [color="#FF4488",href="bookvol10.4.pdf#nameddest=OUT"]
"Package" [color="#FF4488"]
"OUT" -> "Package"
```

Chapter 17

Chapter P

17.1 package PADEPAC PadeApproximantPackage

17.2 PadeApproximantPackage



1.00

Exports:
pade

⟨*package PADEPAC PadeApproximantPackage*⟩≡
```
)abbrev package PADEPAC PadeApproximantPackage
++ This package computes reliable Pad&ea. approximants using
++ a generalized Viskovatov continued fraction algorithm.
++ Authors: Trager,Burge, Hassner & Watt.
++ Date Created: April 1987
++ Date Last Updated: 12 April 1990
++ Keywords: Pade, series
++ Examples:
++ References:
```

```
++   "Pade Approximants, Part I: Basic Theory", Baker & Graves-Morris.

PadeApproximantPackage(R: Field, x:Symbol, pt:R): Exports == Implementation where
   PS ==> UnivariateTaylorSeries(R,x,pt)
   UP ==> UnivariatePolynomial(x,R)
   QF ==> Fraction UP
   CF  ==> ContinuedFraction UP
   NNI ==> NonNegativeInteger
   Exports ==> with
     pade:   (NNI,NNI,PS,PS) -> Union(QF,"failed")
      ++ pade(nd,dd,ns,ds) computes the approximant as a quotient of polynomials
      ++ (if it exists) for arguments
      ++ nd (numerator degree of approximant),
      ++ dd (denominator degree of approximant),
      ++ ns (numerator series of function), and
      ++ ds (denominator series of function).
     pade:   (NNI,NNI,PS) -> Union(QF,"failed")
      ++ pade(nd,dd,s)
      ++ computes the quotient of polynomials
      ++ (if it exists) with numerator degree at
      ++ most nd and denominator degree at most dd
      ++ which matches the series s to order \spad{nd + dd}.

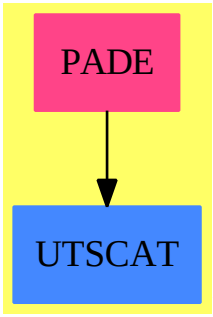
   Implementation ==> add
     n,m : NNI
     u,v : PS
     pa := PadeApproximants(R,PS,UP)
     pade(n,m,u,v) ==
       ans:=pade(n,m,u,v)$pa
       ans case "failed" => ans
       pt = 0 => ans
       num := numer(ans::QF)
       den := denom(ans::QF)
       xpt : UP := monomial(1,1)-monomial(pt,0)
       num := num(xpt)
       den := den(xpt)
       num/den
     pade(n,m,u) == pade(n,m,u,1)
```

⟨*PADEPAC.dotabb*⟩≡

```
"PADEPAC" [color="#FF4488",href="bookvol10.4.pdf#nameddest=PADEPAC"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"PADEPAC" -> "PFECAT"
```

17.3 package PADE PadeApproximants

17.4 PadeApproximants



1.00

Exports:
pade padecf

⟨*package PADE PadeApproximants*⟩≡

```
)abbrev package PADE PadeApproximants
++ This package computes reliable Pad&ea. approximants using
++ a generalized Viskovatov continued fraction algorithm.
++ Authors: Burge, Hassner & Watt.
++ Date Created: April 1987
++ Date Last Updated: 12 April 1990
++ Keywords: Pade, series
++ Examples:
++ References:
++   "Pade Approximants, Part I: Basic Theory", Baker & Graves-Morris.
PadeApproximants(R,PS,UP): Exports == Implementation where
  R:  Field -- IntegralDomain
  PS: UnivariateTaylorSeriesCategory R
  UP: UnivariatePolynomialCategory R

  NNI ==> NonNegativeInteger
  QF  ==> Fraction UP
  CF  ==> ContinuedFraction UP

  Exports ==> with
    pade:   (NNI,NNI,PS,PS) -> Union(QF,"failed")
      ++ pade(nd,dd,ns,ds)
      ++ computes the approximant as a quotient of polynomials
      ++ (if it exists) for arguments
      ++ nd (numerator degree of approximant),
      ++ dd (denominator degree of approximant),
      ++ ns (numerator series of function), and
```

```
      ++ ds (denominator series of function).
    padecf: (NNI,NNI,PS,PS) -> Union(CF, "failed")
      ++ padecf(nd,dd,ns,ds)
      ++ computes the approximant as a continued fraction of
      ++ polynomials (if it exists) for arguments
      ++ nd (numerator degree of approximant),
      ++ dd (denominator degree of approximant),
      ++ ns (numerator series of function), and
      ++ ds (denominator series of function).

  Implementation ==> add
    -- The approximant is represented as
    --   p0 + x**a1/(p1 + x**a2/(...))

    PadeRep ==> Record(ais: List UP, degs: List NNI) -- #ais= #degs
    PadeU   ==> Union(PadeRep, "failed")             -- #ais= #degs+1

    constInner(up:UP):PadeU == [[up], []]

    truncPoly(p:UP,n:NNI):UP ==
      while n < degree p repeat p := reductum p
      p

    truncSeries(s:PS,n:NNI):UP ==
      p: UP := 0
      for i in 0..n repeat p := p + monomial(coefficient(s,i),i)
      p

    -- Assumes s starts with a<n>*x**n + ... and divides out x**n.
    divOutDegree(s:PS,n:NNI):PS ==
      for i in 1..n repeat s := quoByVar s
      s

    padeNormalize: (NNI,NNI,PS,PS) -> PadeU
    padeInner:     (NNI,NNI,PS,PS) -> PadeU

    pade(l,m,gps,dps) ==
      (ad := padeNormalize(l,m,gps,dps)) case "failed" => "failed"
      plist := ad.ais; dlist := ad.degs
      approx := first(plist) :: QF
      for d in dlist for p in rest plist repeat
        approx := p::QF + (monomial(1,d)$UP :: QF)/approx
      approx

    padecf(l,m,gps,dps) ==
      (ad := padeNormalize(l,m,gps,dps)) case "failed" => "failed"
```

```
    alist := reverse(ad.ais)
    blist := [monomial(1,d)$UP for d in reverse ad.degs]
    continuedFraction(first(alist),_
                      blist::Stream UP,(rest alist) :: Stream UP)

  padeNormalize(l,m,gps,dps) ==
    zero? dps => "failed"
    zero? gps => constInner 0
    -- Normalize so numerator or denominator has constant term.
    ldeg:= min(order dps,order gps)
    if ldeg > 0 then
      dps := divOutDegree(dps,ldeg)
      gps := divOutDegree(gps,ldeg)
    padeInner(l,m,gps,dps)

  padeInner(l, m, gps, dps) ==
    zero? coefficient(gps,0) and zero? coefficient(dps,0) =>
      error "Pade' problem not normalized."
    plist: List UP := nil()
    alist: List NNI := nil()
    -- Ensure denom has constant term.
    if zero? coefficient(dps,0) then
      -- g/d = 0 + z**0/(d/g)
      (gps,dps) := (dps,gps)
      (l,m)     := (m,l)
      plist := concat(0,plist)
      alist := concat(0,alist)
    -- Ensure l >= m, maintaining coef(dps,0)^=0.
    if l < m then
      --   (a<n>*x**n + a<n+1>*x**n+1 + ...)/b
      -- = x**n/b + (a<n> + a<n+1>*x + ...)/b
      alpha := order gps
      if alpha > l then return "failed"
      gps := divOutDegree(gps, alpha)
      (l,m) := (m,(l-alpha) :: NNI)
      (gps,dps) := (dps,gps)
      plist := concat(0,plist)
      alist := concat(alpha,alist)
    degbd: NNI := l + m + 1
    g := truncSeries(gps,degbd)
    d := truncSeries(dps,degbd)
    for j in 0.. repeat
      -- Normalize d so constant coefs cancel. (B&G-M is wrong)
      d0 := coefficient(d,0)
      d := (1/d0) * d; g := (1/d0) * g
      p : UP := 0; s := g
```

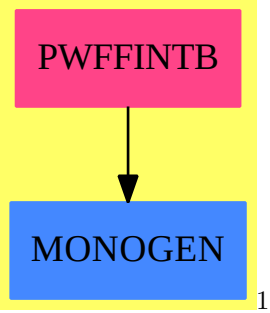
```
if l-m+1 < 0 then error "Internal pade error"
degbd := (l-m+1) :: NNI
for k in 1..degbd repeat
  pk := coefficient(s,0)
  p  := p + monomial(pk,(k-1) :: NNI)
  s  := s - pk*d
  s  := (s exquo monomial(1,1)) :: UP
plist := concat(p,plist)
s = 0 => return [plist,alist]
alpha := minimumDegree(s) + degbd
alpha > l + m => return [plist,alist]
alpha > l     => return "failed"
alist := concat(alpha,alist)
h := (s exquo monomial(1,minimumDegree s)) :: UP
degbd := (l + m - alpha) :: NNI
g := truncPoly(d,degbd)
d := truncPoly(h,degbd)
(l,m) := (m,(l-alpha) :: NNI)
```

⟨*PADE.dotabb*⟩≡

```
"PADE" [color="#FF4488",href="bookvol10.4.pdf#nameddest=PADE"]
"UTSCAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=UTSCAT"]
"PADE" -> "UTSCAT"
```

17.5 package PWFFINTB PAdicWildFunction-FieldIntegralBasis

17.6 PAdicWildFunctionFieldIntegralBasis



1.00

Exports:
reducedDiscriminant integralBasis localIntegralBasis

⟨*package PWFFINTB PAdicWildFunctionFieldIntegralBasis*⟩≡

```
)abbrev package PWFFINTB PAdicWildFunctionFieldIntegralBasis
++ Author: Clifton Williamson
++ Date Created: 5 July 1993
++ Date Last Updated: 17 August 1993
++ Basic Operations: integralBasis, localIntegralBasis
++ Related Domains: WildFunctionFieldIntegralBasis(K,R,UP,F)
++ Also See: FunctionFieldIntegralBasis
++ AMS Classifications:
++ Keywords: function field, finite field, integral basis
++ Examples:
++ References:
++ Description:
++ In this package K is a finite field, R is a ring of univariate
++ polynomials over K, and F is a monogenic algebra over R.
++ We require that F is monogenic, i.e. that \spad{F = K[x,y]/(f(x,y))},
++ because the integral basis algorithm used will factor the polynomial
++ \spad{f(x,y)}.  The package provides a function to compute the integral
++ closure of R in the quotient field of F as well as a function to compute
++ a "local integral basis" at a specific prime.

PAdicWildFunctionFieldIntegralBasis(K,R,UP,F): Exports == Implementation where
  K  : FiniteFieldCategory
  R  : UnivariatePolynomialCategory K
  UP : UnivariatePolynomialCategory R
  F  : MonogenicAlgebra(R,UP)
```

```
I        ==> Integer
L        ==> List
L2       ==> ListFunctions2
Mat      ==> Matrix R
NNI      ==> NonNegativeInteger
PI       ==> PositiveInteger
Q        ==> Fraction R
SAE      ==> SimpleAlgebraicExtension
SUP      ==> SparseUnivariatePolynomial
CDEN     ==> CommonDenominator
DDFACT   ==> DistinctDegreeFactorize
WFFINTBS ==> WildFunctionFieldIntegralBasis
Result   ==> Record(basis: Mat, basisDen: R, basisInv:Mat)
IResult  ==> Record(basis: Mat, basisDen: R, basisInv:Mat,discr: R)
IBPTOOLS ==> IntegralBasisPolynomialTools
IBACHIN  ==> ChineseRemainderToolsForIntegralBases
IRREDFFX ==> IrredPolyOverFiniteField
GHEN     ==> GeneralHenselPackage

Exports ==> with
  integralBasis : () -> Result
    ++ \spad{integralBasis()} returns a record
    ++ \spad{[basis,basisDen,basisInv] } containing information regarding
    ++ the integral closure of R in the quotient field of the framed
    ++ algebra F.  F is a framed algebra with R-module basis
    ++ \spad{w1,w2,...,wn}.
    ++ If 'basis' is the matrix \spad{(aij, i = 1..n, j = 1..n)}, then
    ++ the \spad{i}th element of the integral basis is
    ++ \spad{vi = (1/basisDen) * sum(aij * wj, j = 1..n)}, i.e. the
    ++ \spad{i}th row of 'basis' contains the coordinates of the
    ++ \spad{i}th basis vector.  Similarly, the \spad{i}th row of the
    ++ matrix 'basisInv' contains the coordinates of \spad{wi} with respect
    ++ to the basis \spad{v1,...,vn}: if 'basisInv' is the matrix
    ++ \spad{(bij, i = 1..n, j = 1..n)}, then
    ++ \spad{wi = sum(bij * vj, j = 1..n)}.
  localIntegralBasis : R -> Result
    ++ \spad{integralBasis(p)} returns a record
    ++ \spad{[basis,basisDen,basisInv] } containing information regarding
    ++ the local integral closure of R at the prime \spad{p} in the quotient
    ++ field of the framed algebra F.  F is a framed algebra with R-module
    ++ basis \spad{w1,w2,...,wn}.
    ++ If 'basis' is the matrix \spad{(aij, i = 1..n, j = 1..n)}, then
    ++ the \spad{i}th element of the local integral basis is
    ++ \spad{vi = (1/basisDen) * sum(aij * wj, j = 1..n)}, i.e. the
    ++ \spad{i}th row of 'basis' contains the coordinates of the
    ++ \spad{i}th basis vector.  Similarly, the \spad{i}th row of the
```

```
      ++ matrix 'basisInv' contains the coordinates of \spad{wi} with respect
      ++ to the basis \spad{v1,...,vn}: if 'basisInv' is the matrix
      ++ \spad{(bij, i = 1..n, j = 1..n)}, then
      ++ \spad{wi = sum(bij * vj, j = 1..n)}.
    reducedDiscriminant: UP -> R
        ++ reducedDiscriminant(up) \undocumented

  Implementation ==> add
    import IntegralBasisTools(R, UP, F)
    import GeneralHenselPackage(R,UP)
    import ModularHermitianRowReduction(R)
    import TriangularMatrixOperations(R, Vector R, Vector R, Matrix R)

    reducedDiscriminant f ==
      ff : SUP Q := mapUnivariate(#1 :: Q,f)$IBPTOOLS(R,UP,SUP UP,Q)
      ee := extendedEuclidean(ff,differentiate ff)
      cc := concat(coefficients(ee.coef1),coefficients(ee.coef2))
      cden := splitDenominator(cc)$CDEN(R,Q,L Q)
      denom := cden.den
      gg := gcd map(numer,cden.num)$L2(Q,R)
      (ans := denom exquo gg) case "failed" =>
        error "PWFFINTB: error in reduced discriminant computation"
      ans :: R

    compLocalBasis: (UP,R) -> Result
    compLocalBasis(poly,prime) ==
      -- compute a local integral basis at 'prime' for k[x,y]/(poly(x,y)).
      sae := SAE(R,UP,poly)
      localIntegralBasis(prime)$WFFINTBS(K,R,UP,sae)

    compLocalBasisOverExt: (UP,R,UP,NNI) -> Result
    compLocalBasisOverExt(poly0,prime0,irrPoly0,k) ==
      -- poly0 = irrPoly0**k (mod prime0)
      n := degree poly0; disc0 := discriminant poly0
      (disc0 exquo prime0) case "failed" =>
        [scalarMatrix(n,1), 1, scalarMatrix(n,1)]
      r := degree irrPoly0
      -- extend scalars:
      -- construct irreducible polynomial of degree r over K
      irrPoly := generateIrredPoly(r :: PI)$IRREDFFX(K)
      -- construct extension of degree r over K
      E := SAE(K,SUP K,irrPoly)
      -- lift coefficients to elements of E
      poly := mapBivariate(#1 :: E,poly0)$IBPTOOLS(K,R,UP,E)
      redDisc0 := reducedDiscriminant poly0
      redDisc := mapUnivariate(#1 :: E,redDisc0)$IBPTOOLS(K,R,UP,E)
```

```
    prime := mapUnivariate(#1 :: E,prime0)$IBPTOOLS(K,R,UP,E)
    sae := SAE(E,SUP E,prime)
    -- reduction (mod prime) of polynomial of which poly is the kth power
    redIrrPoly :=
      pp := mapBivariate(#1 :: E,irrPoly0)$IBPTOOLS(K,R,UP,E)
      mapUnivariate(reduce,pp)$IBPTOOLS(SUP E,SUP SUP E,SUP SUP SUP E,sae)
    -- factor the reduction
    factorListSAE := factors factor(redIrrPoly)$DDFACT(sae,SUP sae)
    -- list the 'primary factors' of the reduction of poly
    redFactors : List SUP sae := [(f.factor)**k for f in factorListSAE]
    -- lift these factors to elements of SUP SUP E
    primaries : List SUP SUP E :=
      [mapUnivariate(lift,ff)$IBPTOOLS(SUP E,SUP SUP E,SUP SUP SUP E,sae) _
           for ff in redFactors]
    -- lift the factors to factors modulo a suitable power of 'prime'
    deg := (1 + order(redDisc,prime) * degree(prime)) :: PI
    henselInfo := HenselLift(poly,primaries,prime,deg)$GHEN(SUP E,SUP SUP E)
    henselFactors := henselInfo.plist
    psi1 := first henselFactors
    FF := SAE(SUP E,SUP SUP E,psi1)
    factorIb := localIntegralBasis(prime)$WFFINTBS(E,SUP E,SUP SUP E,FF)
    bs := listConjugateBases(factorIb,size()$K,r)$IBACHIN(E,SUP E,SUP SUP E)
    ib := chineseRemainder(henselFactors,bs,n)$IBACHIN(E,SUP E,SUP SUP E)
    b : Matrix R :=
      bas := mapMatrixIfCan(retractIfCan,ib.basis)$IBPTOOLS(K,R,UP,E)
      bas case "failed" => error "retraction of basis failed"
      bas :: Matrix R
    bInv : Matrix R :=
      --bas := mapMatrixIfCan(ric,ib.basisInv)$IBPTOOLS(K,R,UP,E)
      bas := mapMatrixIfCan(retractIfCan,ib.basisInv)$IBPTOOLS(K,R,UP,E)
      bas case "failed" => error "retraction of basis inverse failed"
      bas :: Matrix R
    bDen : R :=
      p := mapUnivariateIfCan(retractIfCan,ib.basisDen)$IBPTOOLS(K,R,UP,E)
      p case "failed" => error "retraction of basis denominator failed"
      p :: R
    [b,bDen,bInv]

  padicLocalIntegralBasis: (UP,R,R,R) -> IResult
  padicLocalIntegralBasis(p,disc,redDisc,prime) ==
    -- polynomials in x modulo 'prime'
    sae := SAE(K,R,prime)
    -- find the factorization of 'p' modulo 'prime' and lift the
    -- prime powers to elements of UP:
    -- reduce 'p' modulo 'prime'
    reducedP := mapUnivariate(reduce,p)$IBPTOOLS(R,UP,SUP UP,sae)
```

```
    -- factor the reduced polynomial
    factorListSAE := factors factor(reducedP)$DDFACT(sae,SUP sae)
    -- if only one prime factor, perform usual integral basis computation
    (# factorListSAE) = 1 =>
      ib := localIntegralBasis(prime)$WFFINTBS(K,R,UP,F)
      index := diagonalProduct(ib.basisInv)
      [ib.basis,ib.basisDen,ib.basisInv,disc quo (index * index)]
    -- list the 'prime factors' of the reduced polynomial
    redPrimes : List SUP sae :=
      [f.factor for f in factorListSAE]
    -- lift these factors to elements of UP
    primes : List UP :=
      [mapUnivariate(lift,ff)$IBPTOOLS(R,UP,SUP UP,sae) for ff in redPrimes]
    -- list the exponents
    expons : List NNI := [((f.exponent) :: NNI) for f in factorListSAE]
    -- list the 'primary factors' of the reduced polynomial
    redPrimaries : List SUP sae :=
      [(f.factor) **((f.exponent) :: NNI) for f in factorListSAE]
    -- lift these factors to elements of UP
    primaries : List UP :=
      [mapUnivariate(lift,ff)$IBPTOOLS(R,UP,SUP UP,sae) for ff in redPrimaries]
    -- lift the factors to factors modulo a suitable power of 'prime'
    deg := (1 + order(redDisc,prime) * degree(prime)) :: PI
    henselInfo := HenselLift(p,primaries,prime,deg)
    henselFactors := henselInfo.plist
    -- compute integral bases for the factors
    factorBases : List Result := empty(); degPrime := degree prime
    for pp in primes for k in expons for qq in henselFactors repeat
      base :=
        degPp := degree pp
        degPp > 1 and gcd(degPp,degPrime) = 1 =>
          compLocalBasisOverExt(qq,prime,pp,k)
        compLocalBasis(qq,prime)
      factorBases := concat(base,factorBases)
    factorBases := reverse_! factorBases
    ib := chineseRemainder(henselFactors,factorBases,rank()$F)$IBACHIN(K,R,UP)
    index := diagonalProduct(ib.basisInv)
    [ib.basis,ib.basisDen,ib.basisInv,disc quo (index * index)]

  localIntegralBasis prime ==
    p := definingPolynomial()$F; disc := discriminant p
    --disc := determinant traceMatrix()$F
    redDisc := reducedDiscriminant p
    ib := padicLocalIntegralBasis(p,disc,redDisc,prime)
    [ib.basis,ib.basisDen,ib.basisInv]
```

```
listSquaredFactors: R -> List R
listSquaredFactors px ==
  -- returns a list of the factors of px which occur with
  -- exponent > 1
  ans : List R := empty()
  factored := factor(px)$DistinctDegreeFactorize(K,R)
  for f in factors(factored) repeat
    if f.exponent > 1 then ans := concat(f.factor,ans)
  ans

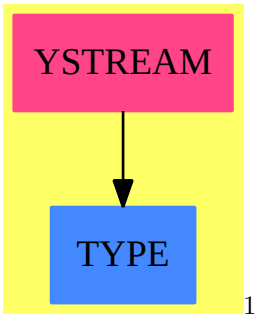
integralBasis() ==
  p := definingPolynomial()$F; disc := discriminant p; n := rank()$F
  --traceMat := traceMatrix()$F; n := rank()$F
  --disc := determinant traceMat        -- discriminant of current order
  singList := listSquaredFactors disc -- singularities of relative Spec
  redDisc := reducedDiscriminant p
  runningRb := runningRbinv := scalarMatrix(n,1)$Mat
  -- runningRb    = basis matrix of current order
  -- runningRbinv = inverse basis matrix of current order
  -- these are wrt the original basis for F
  runningRbden : R := 1
  -- runningRbden = denominator for current basis matrix
  empty? singList => [runningRb, runningRbden, runningRbinv]
  for prime in singList repeat
    lb := padicLocalIntegralBasis(p,disc,redDisc,prime)
    rb := lb.basis; rbinv := lb.basisInv; rbden := lb.basisDen
    disc := lb.discr
    mat := vertConcat(rbden * runningRb,runningRbden * rb)
    runningRbden := runningRbden * rbden
    runningRb := squareTop rowEchelon(mat,runningRbden)
    --runningRb := squareTop rowEch mat
    runningRbinv := UpTriBddDenomInv(runningRb,runningRbden)
  [runningRb, runningRbden, runningRbinv]
```

⟨*PWFFINTB.dotabb*⟩≡

```
"PWFFINTB" [color="#FF4488",href="bookvol10.4.pdf#nameddest=PWFFINTB"]
"MONOGEN" [color="#4488FF",href="bookvol10.2.pdf#nameddest=MONOGEN"]
"PWFFINTB" -> "MONOGEN"
```

17.7 package YSTREAM ParadoxicalCombinatorsForStreams

17.8 ParadoxicalCombinatorsForStreams



1.00

Exports:
Y

⟨*package YSTREAM ParadoxicalCombinatorsForStreams*⟩≡

```
)abbrev package YSTREAM ParadoxicalCombinatorsForStreams
++ Computation of fixed points of mappings on streams
++ Author: Burge, Watt (revised by Williamson)
++ Date Created: 1986
++ Date Last Updated: 21 October 1989
++ Keywords: stream, fixed point
++ Examples:
++ References:
ParadoxicalCombinatorsForStreams(A):Exports == Implementation where
  ++ This package implements fixed-point computations on streams.
  A  :   Type
  ST ==> Stream
  L  ==> List
  I  ==> Integer

  Exports ==> with
    Y: (ST A -> ST A) -> ST A
      ++ Y(f) computes a fixed point of the function f.
    Y: (L ST A -> L ST A,I) -> L ST A
      ++ Y(g,n) computes a fixed point of the function g, where g takes
      ++ a list of n streams and returns a list of n streams.

  Implementation ==> add

    Y f ==
      y : ST A := CONS(0$I,0$I)$Lisp
```

```
        j := f y
        RPLACA(y,frst j)$Lisp
        RPLACD(y,rst j)$Lisp
        y

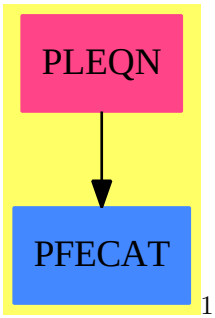
      Y(g,n) ==
        x : L ST A := [CONS(0$I,0$I)$Lisp for i in 1..n]
        j := g x
        for xi in x for ji in j repeat
          RPLACA(xi,frst ji)$Lisp
          RPLACD(xi,rst ji)$Lisp
        x
```

⟨*YSTREAM.dotabb*⟩≡

```
"YSTREAM" [color="#FF4488",href="bookvol10.4.pdf#nameddest=YSTREAM"]
"TYPE" [color="#4488FF",href="bookvol10.2.pdf#nameddest=TYPE"]
"YSTREAM" -> "TYPE"
```

17.9 package PLEQN ParametricLinearEquations

17.10 ParametricLinearEquations



1.00

Exports:

bsolve	B1solve	dmp2rfi	dmp2rfi	dmp2rfi
factorset	hasoln	inconsistent?	maxrank	minrank
minset	nextSublist	overset?	ParCond	ParCondList
pr2dmp	psolve	rdregime	redmat	redpps
regime	se2rfi	sqfree	wrregime	

```
++ This package completely solves a parametric linear system of equations
++ by decomposing the set of all parametric values for which the linear
++ system is consistent into a union of quasi-algebraic  sets (which need
++ not be irredundant, but most of the time is). Each quasi-algebraic
++ set is described by a list of polynomials that vanish on the set, and
++ a list of polynomials that vanish at no point of the set.
++ For each quasi-algebraic set, the solution of the linear system
++ is given, as a particular solution and  a basis of the homogeneous
++ system.
++ The parametric linear system should be given in matrix form, with
++ a coefficient matrix and a right hand side vector. The entries
++ of the coefficient matrix and right hand side vector should be
++ polynomials in the parametric variables, over a Euclidean domain
++ of characteristic zero.
++
++ If the system is homogeneous, the right hand side need not be given.
++ The right hand side can also be  replaced by an indeterminate vector,
++ in which case, the conditions required for consistency will also be
++ given.
++
++ The package has other facilities for saving results to external
++ files, as well as solving the system for a specified minimum rank.
++ Altogether there are 12 mode maps for psolve, as explained below.
```

```
-- modified to conform with new runtime system 06/04/90
-- updated with comments for MB, 02/16/94
-- also cleaned up some unnecessary arguments in regime routine
--
-- MB: In order to allow the rhs to be indeterminate, while working
-- mainly with the parametric variables on the lhs (less number of
-- variables), certain conversions of internal representation from
-- GR to Polynomial R and Fraction Polynomial R are done.  At the time
-- of implementation, I thought that this may be more efficient.  I
-- have not done any comparison since that requires rewriting the
-- package.  My own application needs to call this package quite often,
-- and most computations involves only polynomials in the parametric
-- variables.

-- The 12 modes of psolve are probably not all necessary.  Again, I
-- was thinking that if there are many regimes and many ranks, then
-- the output is quite big, and it may be nice to be able to save it
-- and read the results in later to continue computing rather than
-- recomputing.  Because of the combinatorial nature of the algorithm
-- (computing all subdeterminants!), it does not take a very big matrix
-- to get into many regimes.  But I now have second thoughts of this
-- design, since most of the time, the results are just intermediate,
-- passed to be further processed.  On the other hand, there is probably
-- no penalty in leaving the options as is.
```

⟨*package PLEQN ParametricLinearEquations*⟩≡

```
  )abbrev package PLEQN ParametricLinearEquations
  ++  Author: William Sit, spring 89
  ParametricLinearEquations(R,Var,Expon,GR):
              Declaration == Definition where

    R         :  Join(EuclideanDomain, CharacteristicZero)
    -- Warning: does not work if R is a field! because of Fraction R
    Var       :  Join(OrderedSet,ConvertibleTo (Symbol))
    Expon     :  OrderedAbelianMonoidSup
    GR        :  PolynomialCategory(R,Expon,Var)
    F         == Fraction R
    FILE ==> FileCategory
    FNAME ==> FileName
    GB  ==> EuclideanGroebnerBasisPackage
  --  GBINTERN ==> GroebnerInternalPackage
    I   ==> Integer
    L   ==> List
    M   ==> Matrix
    NNI ==> NonNegativeInteger
    OUT ==> OutputForm
    P   ==> Polynomial
    PI  ==> PositiveInteger
```

```
  SEG ==> Segment
  SM  ==> SquareMatrix
  S   ==> String
  V   ==> Vector
  mf    ==> MultivariateFactorize(Var,Expon,R,GR)
  rp    ==> GB(R,Expon,Var,GR)
  gb    ==> GB(R,Expon,Var,GR)
  PR    ==> P R
  GF    ==> Fraction PR
  plift ==> PolynomialCategoryLifting(Expon,Var,R,GR,GF)
  Inputmode ==> Integer
  groebner ==> euclideanGroebner
  redPol  ==> euclideanNormalForm

-- MB: The following macros are data structures to store mostly
-- intermediate results
-- Rec stores a subdeterminant with corresponding row and column indices
-- Fgb is a Groebner basis for the ideal generated by the subdeterminants
--     of a given rank.
-- Linsys specifies a linearly independent system of a given system
--     assuming a given rank, using given row and column indices
-- Linsoln stores the solution to the parametric linear system as a basis
--     and a particular solution (for a given regime)
-- Rec2 stores the rank, and a list of subdeterminants of that rank,
--     and a Groebner basis for the ideal they generate.
-- Rec3 stores a regime and the corresponding solution; the regime is
--     given by a list of equations (eqzro) and one inequation (neqzro)
--     describing the quasi-algebraic set which is the regime; the
--     additional consistency conditions due to the rhs is given by wcond.
-- Ranksolns stores a list of regimes and their solutions, and the number
--     of regimes all together.
-- Rec8 (temporary) stores a quasi-algebraic set with an indication
-- whether it is empty (sysok = false) or not (sysok = true).

-- I think psolve should be renamed parametricSolve, or even
-- parametricLinearSolve.  On the other hand, may be just solve will do.
-- Please feel free to change it to conform with system conventions.
-- Most psolve routines return a list of regimes and solutions,
-- except those that output to file when the number of regimes is
-- returned instead.
-- This version has been tested on the pc version 1.608 March 13, 1992

  Rec   ==> Record(det:GR,rows:L I,cols:L I)
  Eqns  ==> L Rec
  Fgb   ==> L GR  -- groebner basis
  Linsoln   ==> Record(partsol:V GF,basis:L V GF)
```

```
Linsys    ==> Record(mat:M GF,vec:L GF,rank:NNI,rows:L I,cols:L I)
Rec2  ==> Record(rank:NNI,eqns:Eqns,fgb:Fgb)
RankConds ==> L Rec2
Rec3  ==> Record(eqzro:L GR, neqzro:L GR,wcond:L PR, bsoln:Linsoln)
Ranksolns ==> Record(rgl:L Rec3,rgsz:I)
Rec8 ==> Record(sysok:Boolean, z0:L GR, n0:L GR)


Declaration == with
    psolve: (M GR, L GR) -> L Rec3
      ++ psolve(c,w) solves c z = w for all possible ranks
      ++ of the matrix c and given right hand side vector w
      -- this is mode 1
    psolve: (M GR, L Symbol) -> L Rec3
      ++ psolve(c,w) solves c z = w for all possible ranks
      ++ of the matrix c and indeterminate right hand side w
      -- this is mode 2
    psolve:  M GR        -> L Rec3
      ++ psolve(c) solves the homogeneous linear system
      ++ c z = 0 for all possible ranks of the matrix c
      -- this is mode 3
    psolve: (M GR, L GR, PI) -> L Rec3
      ++ psolve(c,w,k) solves c z = w for all possible ranks >= k
      ++ of the matrix c and given right hand side vector w
      -- this is mode 4
    psolve: (M GR, L Symbol, PI) -> L Rec3
      ++ psolve(c,w,k) solves c z = w for all possible ranks >= k
      ++ of the matrix c and indeterminate right hand side w
      -- this is mode 5
    psolve: (M GR,       PI) -> L Rec3
      ++ psolve(c) solves the homogeneous linear system
      ++ c z = 0 for all possible ranks >= k of the matrix c
      -- this is mode 6
    psolve: (M GR, L GR, S) -> I
      ++ psolve(c,w,s) solves c z = w for all possible ranks
      ++ of the matrix c and given right hand side vector w,
      ++ writes the results to a file named s, and returns the
      ++ number of regimes
      -- this is mode 7
    psolve: (M GR, L Symbol, S) -> I
      ++ psolve(c,w,s) solves c z = w for all possible ranks
      ++ of the matrix c and indeterminate right hand side w,
      ++ writes the results to a file named s, and returns the
      ++ number of regimes
      -- this is mode 8
    psolve: (M GR,       S) -> I
```

```
    ++ psolve(c,s) solves c z = 0 for all possible ranks
    ++ of the matrix c and given right hand side vector w,
    ++ writes the results to a file named s, and returns the
    ++ number of regimes
    -- this is mode 9
  psolve: (M GR, L GR, PI, S) -> I
    ++ psolve(c,w,k,s) solves c z = w for all possible ranks >= k
    ++ of the matrix c and given right hand side w,
    ++ writes the results to a file named s, and returns the
    ++ number of regimes
    -- this is mode 10
  psolve: (M GR, L Symbol, PI, S) -> I
    ++ psolve(c,w,k,s) solves c z = w for all possible ranks >= k
    ++ of the matrix c and indeterminate right hand side w,
    ++ writes the results to a file named s, and returns the
    ++ number of regimes
    -- this is mode 11
  psolve: (M GR,       PI, S) -> I
    ++ psolve(c,k,s) solves c z = 0 for all possible ranks >= k
    ++ of the matrix c,
    ++ writes the results to a file named s, and returns the
    ++ number of regimes
    -- this is mode 12

  wrregime   : (L Rec3, S) -> I
    ++ wrregime(l,s) writes a list of regimes to a file named s
    ++ and returns the number of regimes written
  rdregime   : S -> L Rec3
    ++ rdregime(s) reads in a list from a file with name s

    -- for internal use only --
  -- these are exported so my other packages can use them

  bsolve: (M GR, L GF, NNI, S, Inputmode) -> Ranksolns
    ++ bsolve(c, w, r, s, m) returns a list of regimes and
    ++ solutions of the system c z = w for ranks at least r;
    ++ depending on the mode m chosen, it writes the output to
    ++ a file given by the string s.
  dmp2rfi: GR -> GF
    ++ dmp2rfi(p) converts p to target domain
  dmp2rfi: M GR -> M GF
    ++ dmp2rfi(m) converts m to target domain
  dmp2rfi: L GR -> L GF
    ++ dmp2rfi(l) converts l to target domain
  se2rfi:  L Symbol -> L GF
    ++ se2rfi(l) converts l to target domain
```

```
        pr2dmp: PR -> GR
          ++ pr2dmp(p) converts p to target domain
        hasoln: (Fgb, L GR) -> Rec8
          ++ hasoln(g, l) tests whether the quasi-algebraic set
          ++ defined by p = 0 for p in g and q ^= 0 for q in l
          ++ is empty or not and returns a simplified definition
          ++ of the quasi-algebraic set
          -- this is now done in QALGSET package
        ParCondList: (M GR,NNI) -> RankConds
          ++ ParCondList(c,r) computes a list of subdeterminants of each
          ++ rank >= r of the matrix c and returns
          ++ a groebner basis for the
          ++ ideal they generate
        redpps: (Linsoln, Fgb) -> Linsoln
          ++ redpps(s,g) returns the simplified form of s after reducing
          ++ modulo a groebner basis g


    --              L O C A L   F U N C T I O N S

        B1solve: Linsys -> Linsoln
          ++ B1solve(s) solves the system (s.mat) z = s.vec
          ++ for the variables given by the column indices of s.cols
          ++ in terms of the other variables and the right hand side s.vec
          ++ by assuming that the rank is s.rank,
          ++ that the system is consistent, with the linearly
          ++ independent equations indexed by the given row indices s.rows;
          ++ the coefficients in s.mat involving parameters are treated as
          ++ polynomials.  B1solve(s) returns a particular solution to the
          ++ system and a basis of the homogeneous system (s.mat) z = 0.
        factorset: GR -> L GR
          ++ factorset(p) returns the set of irreducible factors of p.
        maxrank: RankConds -> NNI
          ++ maxrank(r) returns the maximum rank in the list r of regimes
        minrank: RankConds -> NNI
          ++ minrank(r) returns the minimum rank in the list r of regimes
        minset: L L GR -> L L GR
          ++ minset(sl) returns the sublist of sl consisting of the minimal
          ++ lists (with respect to inclusion) in the list sl of lists
        nextSublist: (I, I) -> L L I
          ++ nextSublist(n,k) returns a list of k-subsets of {1, ..., n}.
        overset?: (L GR, L L GR) -> Boolean
          ++ overset?(s,sl) returns true if s properly a sublist of a member
          ++ of sl; otherwise it returns false
        ParCond    : (M GR,NNI) -> Eqns
```

```
      ++ ParCond(m,k) returns the list of all k by k subdeterminants in
      ++ the matrix m
    redmat: (M GR, Fgb) -> M GR
      ++ redmat(m,g) returns a matrix whose entries are those of m
      ++ modulo the ideal generated by the groebner basis g
    regime: (Rec,M GR,L GF,L L GR,NNI,NNI,Inputmode) -> Rec3
      ++ regime(y,c, w, p, r, rm, m) returns a regime,
      ++ a list of polynomials specifying the consistency conditions,
      ++ a particular solution and basis representing the general
      ++ solution of the parametric linear system c z = w
      ++ on that regime. The regime returned depends on
      ++ the subdeterminant y.det and the row and column indices.
      ++ The solutions are simplified using the assumption that
      ++ the system has rank r and maximum rank rm. The list p
      ++ represents a list of list of factors of polynomials in
      ++ a groebner basis of the ideal generated by higher order
      ++ subdeterminants, and ius used for the simplification.
      ++ The mode m
      ++ distinguishes the cases when the system is homogeneous,
      ++ or the right hand side is arbitrary, or when there is no
      ++ new right hand side variables.
    sqfree: GR -> GR
      ++ sqfree(p) returns the product of square free factors of p
    inconsistent?: L GR -> Boolean
      ++ inconsistant?(pl) returns true if the system of equations
      ++ p = 0 for p in pl is inconsistent.  It is assumed
      ++ that pl is a groebner basis.
      -- this is needed because of change to
      -- EuclideanGroebnerBasisPackage
    inconsistent?: L PR -> Boolean
      ++ inconsistant?(pl) returns true if the system of equations
      ++ p = 0 for p in pl is inconsistent.  It is assumed
      ++ that pl is a groebner basis.
      -- this is needed because of change to
      -- EuclideanGroebnerBasisPackage

  Definition == add

    inconsistent?(pl:L GR):Boolean ==
      for p in pl repeat
        ground? p => return true
      false
    inconsistent?(pl:L PR):Boolean ==
      for p in pl repeat
        ground? p => return true
      false
```

```
B1solve (sys:Linsys):Linsoln ==
  i,j,i1,j1:I
  rss:L I:=sys.rows
  nss:L I:=sys.cols
  k:=sys.rank
  cmat:M GF:=sys.mat
  n:=ncols cmat
  frcols:L I:=setDifference$(L I) (expand$(SEG I) (1..n), nss)
  w:L GF:=sys.vec
  p:V GF:=new(n,0)
  pbas:L V GF:=[]
  if k ^= 0 then
    augmat:M GF:=zero(k,n+1)
    for i in rss for i1 in 1.. repeat
      for j in nss for j1 in 1.. repeat
        augmat(i1,j1):=cmat(i,j)
      for j in frcols for j1 in k+1.. repeat
        augmat(i1,j1):=-cmat(i,j)
      augmat(i1,n+1):=w.i
    augmat:=rowEchelon$(M GF) augmat
    for i in nss for i1 in 1.. repeat p.i:=augmat(i1,n+1)
    for j in frcols for j1 in k+1.. repeat
      pb:V GF:=new(n,0)
      pb.j:=1
      for i in nss for i1 in 1.. repeat
        pb.i:=augmat(i1,j1)
      pbas:=cons(pb,pbas)
  else
    for j in frcols for j1 in k+1.. repeat
      pb:V GF:=new(n,0)
      pb.j:=1
      pbas:=cons(pb,pbas)
  [p,pbas]

regime (y, coef, w, psbf, rk, rkmax, mode) ==
  i,j:I
  -- use the y.det nonzero to simplify the groebner basis
  -- of ideal generated by higher order subdeterminants
  ydetf:L GR:=factorset y.det
  yzero:L GR:=
    rk = rkmax => nil$(L GR)
    psbf:=[setDifference(x, ydetf) for x in psbf]
    groebner$gb [*/x for x in psbf]
  -- simplify coefficients by modulo ideal
  nc:M GF:=dmp2rfi redmat(coef,yzero)
```

```
        -- solve the system
        rss:L I:=y.rows;  nss:L I :=y.cols
        sys:Linsys:=[nc,w,rk,rss,nss]$Linsys
        pps:= B1solve(sys)
        pp:=pps.partsol
        frows:L I:=setDifference$(L I) (expand$(SEG I) (1..nrows coef),rss)
        wcd:L PR:= []
        -- case homogeneous rhs
        entry? (mode, [3,6,9,12]$(L I)) =>
                  [yzero, ydetf,wcd, redpps(pps, yzero)]$Rec3
        -- case arbitrary rhs, pps not reduced
        for i in frows repeat
            weqn:GF:=+/[nc(i,j)*(pp.j) for j in nss]
            wnum:PR:=numer$GF (w.i - weqn)
            wnum = 0 => "trivially satisfied"
            ground? wnum => return [yzero, ydetf,[1$PR]$(L PR),pps]$Rec3
            wcd:=cons(wnum,wcd)
        entry? (mode, [2,5,8,11]$(L I)) => [yzero, ydetf, wcd, pps]$Rec3
        -- case no new rhs variable
        if not empty? wcd then _
          yzero:=removeDuplicates append(yzero,[pr2dmp pw for pw in wcd])
        test:Rec8:=hasoln (yzero, ydetf)
        not test.sysok => [test.z0, test.n0, [1$PR]$(L PR), pps]$Rec3
        [test.z0, test.n0, [], redpps(pps, test.z0)]$Rec3

      bsolve (coeff, w, h, outname, mode) ==
        r:=nrows coeff
--      n:=ncols coeff
        r ^= #w => error "number of rows unequal on lhs and rhs"
        newfile:FNAME
        rksoln:File Rec3
        count:I:=0
        lrec3:L Rec3:=[]
        filemode:Boolean:= entry? (mode, [7,8,9,10,11,12]$(L I))
        if filemode then
          newfile:=new$FNAME  ("",outname,"regime")
          rksoln:=open$(File Rec3) newfile
        y:Rec
        k:NNI
        rrcl:RankConds:=
          entry? (mode,[1,2,3,7,8,9]$(L I)) => ParCondList (coeff,0)
          entry? (mode,[4,5,6,10,11,12]$(L I)) => ParCondList (coeff,h)
        rkmax:=maxrank rrcl
        rkmin:=minrank rrcl
        for k in rkmax-rkmin+1..1 by -1 repeat
          rk:=rrcl.k.rank
```

```
        pc:Eqns:=rrcl.k.eqns
        psb:Fgb:= (if rk=rkmax then [] else rrcl.(k+1).fgb)
        psbf:L L GR:= [factorset x for x in psb]
        psbf:= minset(psbf)
        for y in pc repeat
          rec3:Rec3:= regime (y, coeff, w, psbf, rk, rkmax, mode)
          inconsistent? rec3.wcond => "incompatible system"
          if filemode then write_!(rksoln, rec3)
          else lrec3:= cons(rec3, lrec3)
          count:=count+1
      if filemode then close_! rksoln
      [lrec3, count]$Ranksolns

    factorset y ==
      ground? y => []
      [j.factor for j in factors(factor$mf y)]

    ParCondList (mat, h) ==
      rcl: RankConds:= []
      ps: L GR:=[]
      pc:Eqns:=[]
      npc: Eqns:=[]
      psbf: Fgb:=[]
      rc: Rec
      done: Boolean := false
      r:=nrows mat
      n:=ncols mat
      maxrk:I:=min(r,n)
      k:NNI
      for k in min(r,n)..h by -1 until done repeat
        pc:= ParCond(mat,k)
        npc:=[]
        (empty? pc) and (k >= 1) => maxrk:= k - 1
        if ground? pc.1.det -- only one is sufficient (neqzro = {})
        then (npc:=pc; done:=true; ps := [1$GR])
        else
          zro:L GR:= (if k = maxrk then [] else rcl.1.fgb)
          covered:Boolean:=false
          for rc in pc until covered repeat
            p:GR:= redPol$rp (rc.det, zro)
            p = 0 => "incompatible or covered subdeterminant"
            test:=hasoln(zro, [rc.det])
--          zroideal:=ideal(zro)
--          inRadical? (p, zroideal) => "incompatible or covered"
            ^test.sysok => "incompatible or covered"
-- The next line is WRONG! cannot replace zro by test.z0
```

```
--          zro:=groebner$gb (cons(*/test.n0, test.z0))
            zro:=groebner$gb (cons(p,zro))
            npc:=cons(rc,npc)
            done:= covered:= inconsistent? zro
          ps:=zro
        pcl: Rec2:= construct(k,npc,ps)
        rcl:=cons(pcl,rcl)
      rcl

    redpps(pps, zz) ==
      pv:=pps.partsol
      r:=#pv
      pb:=pps.basis
      n:=#pb + 1
      nummat:M GR:=zero(r,n)
      denmat:M GR:=zero(r,n)
      for i in  1..r repeat
        nummat(i,1):=pr2dmp numer$GF pv.i
        denmat(i,1):=pr2dmp denom$GF pv.i
      for j in 2..n repeat
        for i in 1..r  repeat
          nummat(i,j):=pr2dmp numer$GF (pb.(j-1)).i
          denmat(i,j):=pr2dmp denom$GF (pb.(j-1)).i
      nummat:=redmat(nummat, zz)
      denmat:=redmat(denmat, zz)
      for i in 1..r repeat
        pv.i:=(dmp2rfi nummat(i,1))/(dmp2rfi denmat(i,1))
      for j in 2..n repeat
        pbj:V GF:=new(r,0)
        for i in 1..r repeat
          pbj.i:=(dmp2rfi nummat(i,j))/(dmp2rfi  denmat(i,j))
        pb.(j-1):=pbj
      [pv, pb]

    dmp2rfi (mat:M GR): M GF ==
      r:=nrows mat
      n:=ncols mat
      nmat:M GF:=zero(r,n)
      for i in 1..r repeat
        for j in 1..n repeat
          nmat(i,j):=dmp2rfi mat(i,j)
      nmat

    dmp2rfi (vl: L GR):L GF ==
      [dmp2rfi v for v in vl]
```

```
psolve (mat:M GR, w:L GR): L Rec3 ==
  bsolve(mat, dmp2rfi w, 1, "nofile", 1).rgl
psolve (mat:M GR, w:L Symbol): L Rec3 ==
  bsolve(mat,  se2rfi w, 1, "nofile", 2).rgl
psolve (mat:M GR): L Rec3 ==
  bsolve(mat, [0$GF for i in 1..nrows mat], 1, "nofile", 3).rgl

psolve (mat:M GR, w:L GR, h:PI): L Rec3 ==
  bsolve(mat, dmp2rfi w, h::NNI, "nofile", 4).rgl
psolve (mat:M GR, w:L Symbol, h:PI): L Rec3 ==
  bsolve(mat, se2rfi w, h::NNI, "nofile", 5).rgl
psolve (mat:M GR, h:PI): L Rec3 ==
  bsolve(mat, [0$GF for i in 1..nrows mat], h::NNI, "nofile", 6).rgl

psolve (mat:M GR, w:L GR, outname:S): I ==
  bsolve(mat, dmp2rfi w, 1, outname, 7).rgsz
psolve (mat:M GR, w:L Symbol, outname:S): I ==
  bsolve(mat, se2rfi w, 1, outname, 8).rgsz
psolve (mat:M GR, outname:S): I ==
  bsolve(mat, [0$GF for i in 1..nrows mat], 1, outname, 9).rgsz

nextSublist (n,k) ==
  n <= 0 => []
  k <= 0 => [ nil$(List Integer) ]
  k > n => []
  n = 1 and k = 1 => [[1]]
  mslist: L L I:=[]
  for ms in nextSublist(n-1,k-1) repeat
    mslist:=cons(append(ms,[n]),mslist)
  append(nextSublist(n-1,k), mslist)

psolve (mat:M GR, w:L GR, h:PI, outname:S): I ==
  bsolve(mat, dmp2rfi w, h::NNI, outname, 10).rgsz
psolve (mat:M GR, w:L Symbol, h:PI, outname:S): I ==
  bsolve(mat, se2rfi w, h::NNI, outname, 11).rgsz
psolve (mat:M GR, h:PI, outname:S): I ==
  bsolve(mat,[0$GF for i in 1..nrows mat],h::NNI,outname, 12).rgsz

hasoln (zro,nzro) ==
  empty? zro => [true, zro, nzro]
  zro:=groebner$gb zro
  inconsistent? zro => [false, zro, nzro]
  empty? nzro =>[true, zro, nzro]
  pnzro:GR:=redPol$rp (*/nzro, zro)
  pnzro = 0 => [false, zro, nzro]
  nzro:=factorset pnzro
```

```
      psbf:L L GR:= minset [factorset p for p in zro]
      psbf:= [setDifference(x, nzro) for x in psbf]
      entry? ([], psbf) => [false, zro, nzro]
      zro:=groebner$gb [*/x for x in psbf]
      inconsistent? zro => [false, zro, nzro]
      nzro:=[redPol$rp (p,zro) for p in nzro]
      nzro:=[p for p in nzro | ^(ground? p)]
      [true, zro, nzro]



    se2rfi w == [coerce$GF monomial$PR (1$PR, wi, 1) for wi in w]

    pr2dmp p ==
      ground? p => (ground p)::GR
      algCoerceInteractive(p,PR,GR)$(Lisp) pretend GR

    wrregime (lrec3, outname) ==
      newfile:FNAME:=new$FNAME ("",outname,"regime")
      rksoln: File Rec3:=open$(File Rec3) newfile
      count:I:=0  -- number of distinct regimes
--      rec3: Rec3
      for rec3 in lrec3 repeat
          write_!(rksoln, rec3)
          count:=count+1
      close_!(rksoln)
      count

    dmp2rfi (p:GR):GF ==
      map$plift ((convert #1)@Symbol::GF, #1::PR::GF, p)


    rdregime inname ==
      infilename:=filename$FNAME ("",inname, "regime")
      infile: File Rec3:=open$(File Rec3) (infilename, "input")
      rksoln:L Rec3:=[]
      rec3:Union(Rec3, "failed"):=readIfCan_!$(File Rec3) (infile)
      while rec3 case Rec3 repeat
        rksoln:=cons(rec3::Rec3,rksoln) -- replace : to :: for AIX
        rec3:=readIfCan_!$(File Rec3) (infile)
      close_!(infile)
      rksoln

    maxrank rcl ==
      empty? rcl => 0
      "max"/[j.rank for j in rcl]
```

```
minrank rcl ==
  empty? rcl => 0
  "min"/[j.rank for j in rcl]

minset lset ==
  empty? lset => lset
  [x for x in lset | ^(overset?(x,lset))]

sqfree p == */[j.factor for j in factors(squareFree p)]


ParCond (mat, k) ==
  k = 0 => [[1, [], []]$Rec]
  j:NNI:=k::NNI
  DetEqn :Eqns := []
  r:I:= nrows(mat)
  n:I:= ncols(mat)
  k > min(r,n) => error "k exceeds maximum possible rank "
  found:Boolean:=false
  for rss in nextSublist(r, k) until found repeat
    for nss in nextSublist(n, k) until found repeat
      matsub := mat(rss, nss) pretend SM(j, GR)
      detmat := determinant(matsub)
      if detmat ^= 0 then
        found:= (ground? detmat)
        detmat:=sqfree detmat
        neweqn:Rec:=construct(detmat,rss,nss)
        DetEqn:=cons(neweqn, DetEqn)
  found => [first DetEqn]$Eqns
  sort(degree #1.det < degree #2.det, DetEqn)



overset?(p,qlist) ==
  empty? qlist => false
  or/[(brace$(Set GR) q) <$(Set GR) (brace$(Set GR) p) _
                                              for q in qlist]

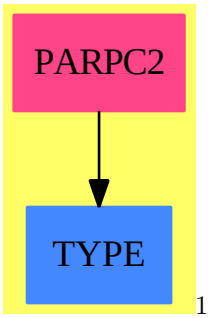

redmat (mat,psb) ==
  i,j:I
  r:=nrows(mat)
  n:=ncols(mat)
  newmat: M GR:=zero(r,n)
  for i in 1..r repeat
```

```
          for j in 1..n repeat
            p:GR:=mat(i,j)
            ground? p => newmat(i,j):=p
            newmat(i,j):=redPol$rp (p,psb)
        newmat
```

⟨*PLEQN.dotabb*⟩≡
```
"PLEQN" [color="#FF4488",href="bookvol10.4.pdf#nameddest=PLEQN"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"PLEQN" -> "PFECAT"
```

17.11 package PARPC2 ParametricPlaneCurveFunctions2

17.12 ParametricPlaneCurveFunctions2



1.00

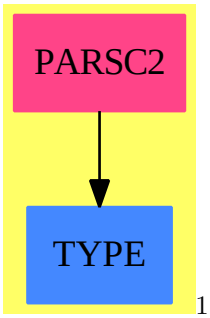
Exports:
map

⟨*package PARPC2 ParametricPlaneCurveFunctions2*⟩≡
```
)abbrev package PARPC2 ParametricPlaneCurveFunctions2
++ Description:
++ This package \undocumented
ParametricPlaneCurveFunctions2(CF1: Type, CF2:Type): with
  map: (CF1 -> CF2, ParametricPlaneCurve(CF1)) -> ParametricPlaneCurve(CF2)
       ++ map(f,x) \undocumented
 == add
  map(f, c) == curve(f coordinate(c,1), f coordinate(c, 2))
```

⟨*PARPC2.dotabb*⟩≡
```
"PARPC2" [color="#FF4488",href="bookvol10.4.pdf#nameddest=PARPC2"]
"TYPE" [color="#4488FF",href="bookvol10.2.pdf#nameddest=TYPE"]
"PARPC2" -> "TYPE"
```

17.13 package PARSC2 ParametricSpaceCurveFunctions2

17.14 ParametricSpaceCurveFunctions2



1.00

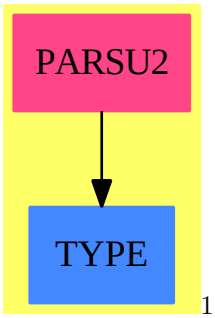
Exports:
map

⟨*package PARSC2 ParametricSpaceCurveFunctions2*⟩≡
```
)abbrev package PARSC2 ParametricSpaceCurveFunctions2
++ Description:
++ This package \undocumented
ParametricSpaceCurveFunctions2(CF1: Type, CF2:Type): with
  map: (CF1 -> CF2, ParametricSpaceCurve(CF1)) -> ParametricSpaceCurve(CF2)
       ++ map(f,x) \undocumented
 == add
  map(f, c) == curve(f coordinate(c,1), f coordinate(c,2), f coordinate(c,3))
```

⟨*PARSC2.dotabb*⟩≡
```
"PARSC2" [color="#FF4488",href="bookvol10.4.pdf#nameddest=PARSC2"]
"TYPE" [color="#4488FF",href="bookvol10.2.pdf#nameddest=TYPE"]
"PARSC2" -> "TYPE"
```

17.15 package PARSU2 ParametricSurface-Functions2

17.16 ParametricSurfaceFunctions2



1.00

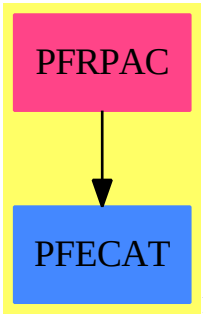
Exports:
map

⟨*package PARSU2 ParametricSurfaceFunctions2*⟩≡
```
)abbrev package PARSU2 ParametricSurfaceFunctions2
++ Description:
++ This package \undocumented
ParametricSurfaceFunctions2(CF1: Type, CF2:Type): with
  map: (CF1 -> CF2, ParametricSurface(CF1)) -> ParametricSurface(CF2)
        ++ map(f,x) \undocumented
 == add
  map(f, c) == surface(f coordinate(c,1), f coordinate(c,2), f coordinate(c,3))
```

⟨*PARSU2.dotabb*⟩≡
```
"PARSU2" [color="#FF4488",href="bookvol10.4.pdf#nameddest=PARSU2"]
"TYPE" [color="#4488FF",href="bookvol10.2.pdf#nameddest=TYPE"]
"PARSU2" -> "TYPE"
```

17.17 package PFRPAC PartialFractionPackage

17.18 PartialFractionPackage



1.00

Exports:
partialFraction

⟨*package PFRPAC PartialFractionPackage*⟩≡

```
)abbrev package PFRPAC PartialFractionPackage
++ Author: Barry M. Trager
++ Date Created: 1992
++ BasicOperations:
++ Related Constructors: PartialFraction
++ Also See:
++ AMS Classifications:
++ Keywords: partial fraction, factorization, euclidean domain
++ References:
++ Description:
++   The package \spadtype{PartialFractionPackage} gives an easier
++   to use interfact the domain \spadtype{PartialFraction}.
++   The user gives a fraction of polynomials, and a variable and
++   the package converts it to the proper datatype for the
++   \spadtype{PartialFraction} domain.

PartialFractionPackage(R): Cat == Capsule where
--  R : UniqueFactorizationDomain -- not yet supported
   R : Join(EuclideanDomain, CharacteristicZero)
   FPR ==> Fraction Polynomial R
   INDE ==> IndexedExponents Symbol
   PR ==> Polynomial R
   SUP ==> SparseUnivariatePolynomial
   Cat == with
      partialFraction: (FPR, Symbol) -> Any
         ++ partialFraction(rf, var) returns the partial fraction decomposition
         ++ of the rational function rf with respect to the variable var.
```

```
        partialFraction: (PR, Factored PR, Symbol) -> Any
           ++ partialFraction(num, facdenom, var) returns the partial fraction
           ++ decomposition of the rational function whose numerator is num and
           ++ whose factored denominator is facdenom with respect to the variable var.
     Capsule == add
        partialFraction(rf, v) ==
           df := factor(denom rf)$MultivariateFactorize(Symbol, INDE,R,PR)
           partialFraction(numer rf, df, v)

        makeSup(p:Polynomial R, v:Symbol) : SparseUnivariatePolynomial FPR ==
           up := univariate(p,v)
           map(#1::FPR,up)$UnivariatePolynomialCategoryFunctions2(PR, SUP PR, FPR, SUP FPR)

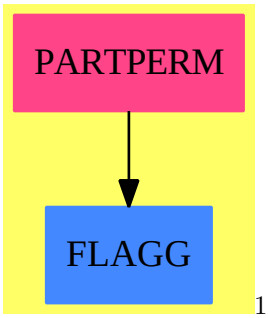
        partialFraction(p, facq, v) ==
           up := UnivariatePolynomial(v, Fraction Polynomial R)
           fup := Factored up
           ffact : List(Record(irr:up,pow:Integer))
           ffact:=[[makeSup(u.factor,v) pretend up,u.exponent]
                        for u in factors facq]
           fcont:=makeSup(unit facq,v) pretend up
           nflist:fup := fcont*(*/[primeFactor(ff.irr,ff.pow) for ff in ffact])
           pfup:=partialFraction(makeSup(p,v) pretend up, nflist)$PartialFraction(up)
           coerce(pfup)$AnyFunctions1(PartialFraction up)
```

⟨*PFRPAC.dotabb*⟩≡

```
"PFRPAC" [color="#FF4488",href="bookvol10.4.pdf#nameddest=PFRPAC"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"PFRPAC" -> "PFECAT"
```

17.19 package PARTPERM PartitionsAndPermutations

17.20 PartitionsAndPermutations



1.00

Exports:

conjugate	conjugates	partitions	permutations	sequences
shuffle	shufflein			

⟨*package PARTPERM PartitionsAndPermutations*⟩≡

```
)abbrev package PARTPERM PartitionsAndPermutations
++ Author: William H. Burge
++ Date Created: 29 October 1987
++ Date Last Updated: 3 April 1991
++ Basic Operations:
++ Related Domains:
++ Also See:
++ AMS Classifications:
++ Keywords: partition, permutation
++ References:
++ Description: PartitionsAndPermutations contains
++ functions for generating streams of integer partitions,
++ and streams of sequences of integers
++ composed from a multi-set.
PartitionsAndPermutations: Exports == Implementation where
  I   ==> Integer
  L   ==> List
  ST  ==> Stream
  ST1 ==> StreamFunctions1
  ST2 ==> StreamFunctions2
  ST3 ==> StreamFunctions3

  Exports ==> with

    partitions: (I,I,I) -> ST L I
```

```
    ++\spad{partitions(p,l,n)} is the stream of partitions
    ++ of n whose number of parts is no greater than p
    ++ and whose largest part is no greater than l.
  partitions: I -> ST L I
    ++\spad{partitions(n)} is the stream of all partitions of n.
  partitions: (I,I) -> ST L I
    ++\spad{partitions(p,l)} is the stream of all
    ++ partitions whose number of
    ++ parts and largest part are no greater than p and l.
  conjugate: L I -> L I
    ++\spad{conjugate(pt)} is the conjugate of the partition pt.
  conjugates: ST L I -> ST L I
    ++\spad{conjugates(lp)} is the stream of conjugates of a stream
    ++ of partitions lp.
  shuffle: (L I,L I) -> ST L I
    ++\spad{shuffle(l1,l2)} forms the stream of all shuffles of l1
    ++ and l2, i.e. all sequences that can be formed from
    ++ merging l1 and l2.
  shufflein: (L I,ST L I) -> ST L I
    ++\spad{shufflein(l,st)} maps shuffle(l,u) on to all
    ++ members u of st, concatenating the results.
  sequences: (L I,L I) -> ST L I
    ++\spad{sequences(l1,l2)} is the stream of all sequences that
    ++ can be composed from the multiset defined from
    ++ two lists of integers l1 and l2.
    ++ For example,the pair \spad{([1,2,4],[2,3,5])} represents
    ++ multi-set with 1 \spad{2}, 2 \spad{3}'s, and 4 \spad{5}'s.
  sequences: L I -> ST L I
    ++ \spad{sequences([l0,l1,l2,..,ln])} is the set of
    ++  all sequences formed from
    ++ \spad{l0} 0's,\spad{l1} 1's,\spad{l2} 2's,...,\spad{ln} n's.
  permutations: I -> ST L I
    ++\spad{permutations(n)} is the stream of permutations
    ++ formed from \spad{1,2,3,...,n}.

Implementation ==> add

  partitions(M,N,n) ==
    zero? n => concat(empty()$L(I),empty()$(ST L I))
    zero? M or zero? N or n < 0 => empty()
    c := map(concat(N,#1),partitions(M - 1,N,n - N))
    concat(c,partitions(M,N - 1,n))

  partitions n == partitions(n,n,n)

  partitions(M,N)==
```

```
    aaa : L ST L I := [partitions(M,N,i) for i in 0..M*N]
    concat(aaa :: ST ST L I)$ST1(L I)

  -- nogreq(n,l) is the number of elements of l that are greater or
  -- equal to n
  nogreq: (I,L I) -> I
  nogreq(n,x) == +/[1 for i in x | i >= n]

  conjugate x ==
    empty? x => empty()
    [nogreq(i,x) for i in 1..first x]

  conjugates z == map(conjugate,z)

  shuffle(x,y)==
    empty? x => concat(y,empty())$(ST L I)
    empty? y => concat(x,empty())$(ST L I)
    concat(map(concat(first x,#1),shuffle(rest x,y)),_
           map(concat(first y,#1),shuffle(x,rest y)))

  shufflein(x,yy) ==
    concat(map(shuffle(x,#1),yy)$ST2(L I,ST L I))$ST1(L I)

  -- rpt(n,m) is the list of n m's
  rpt: (I,I) -> L I
  rpt(n,m) == [m for i in 1..n]

  -- zrpt(x,y) where x is [x0,x1,x2...] and y is [y0,y1,y2...]
  -- is the stream [rpt(x0,y0),rpt(x1,y1),...]
  zrpt: (L I,L I) -> ST L I
  zrpt(x,y) == map(rpt,x :: ST I,y :: ST I)$ST3(I,I,L I)

  sequences(x,y) ==
    reduce(concat(empty()$L(I),empty()$(ST L I)),_
                  shufflein,zrpt(x,y))$ST2(L I,ST L I)

  sequences x == sequences(x,[i for i in 0..#x-1])

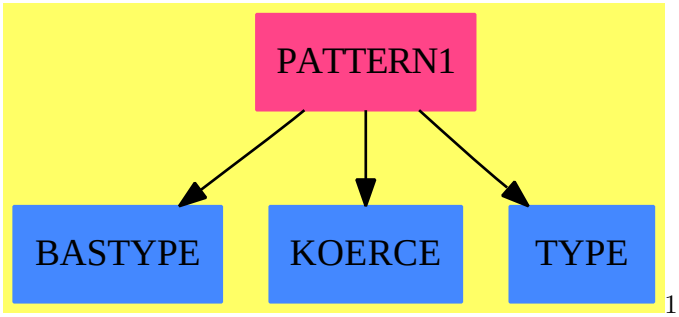
  permutations n == sequences(rpt(n,1),[i for i in 1..n])
```

⟨*PARTPERM.dotabb*⟩≡

```
"PARTPERM" [color="#FF4488",href="bookvol10.4.pdf#nameddest=PARTPERM"]
"FLAGG" [color="#4488FF",href="bookvol10.2.pdf#nameddest=FLAGG"]
"PARTPERM" -> "FLAGG"
```

17.21 package PATTERN1 PatternFunctions1

17.22 PatternFunctions1



1.00

Exports:
badValues addBadValue predicate satisfy? suchThat

⟨*package PATTERN1 PatternFunctions1*⟩≡
```
)abbrev package PATTERN1 PatternFunctions1
++ Utilities for handling patterns
++ Author: Manuel Bronstein
++ Date Created: 28 Nov 1989
++ Date Last Updated: 5 Jul 1990
++ Description: Tools for patterns;
++ Keywords: pattern, matching.
PatternFunctions1(R:SetCategory, D:Type): with
    suchThat   : (Pattern R, D -> Boolean)       -> Pattern R
      ++ suchThat(p, f) makes a copy of p and adds the predicate
      ++ f to the copy, which is returned.
    suchThat   : (Pattern R, List(D -> Boolean)) -> Pattern R
      ++ \spad{suchThat(p, [f1,...,fn])} makes a copy of p and adds the
      ++ predicate f1 and ... and fn to the copy, which is returned.
    suchThat : (Pattern R, List Symbol, List D -> Boolean)  -> Pattern R
      ++ \spad{suchThat(p, [a1,...,an], f)} returns a copy of p with
      ++ the top-level predicate set to \spad{f(a1,...,an)}.
    predicate  : Pattern R -> (D -> Boolean)
      ++ predicate(p) returns the predicate attached to p, the
      ++ constant function true if p has no predicates attached to it.
    satisfy?   : (D, Pattern R) -> Boolean
      ++ satisfy?(v, p) returns f(v) where f is the predicate
      ++ attached to p.
    satisfy?   : (List D, Pattern R) -> Boolean
      ++ \spad{satisfy?([v1,...,vn], p)} returns \spad{f(v1,...,vn)}
      ++ where f is the
      ++ top-level predicate attached to p.
```

```
    addBadValue: (Pattern R, D) -> Pattern R
      ++ addBadValue(p, v) adds v to the list of "bad values" for p;
      ++ p is not allowed to match any of its "bad values".
    badValues  : Pattern R -> List D
      ++ badValues(p) returns the list of "bad values" for p;
      ++ p is not allowed to match any of its "bad values".
  == add
    A1D ==> AnyFunctions1(D)
    A1  ==> AnyFunctions1(D -> Boolean)
    A1L ==> AnyFunctions1(List D -> Boolean)

    applyAll: (List Any, D) -> Boolean
    st      : (Pattern R, List Any) -> Pattern R

    st(p, l)          == withPredicates(p, concat(predicates p, l))
    predicate p       == applyAll(predicates p, #1)
    addBadValue(p, v) == addBadValue(p, coerce(v)$A1D)
    badValues p       == [retract(v)$A1D for v in getBadValues p]
    suchThat(p, l, f) == setTopPredicate(copy p, l, coerce(f)$A1L)
    suchThat(p:Pattern R, f:D -> Boolean) == st(p, [coerce(f)$A1])
    satisfy?(d:D, p:Pattern R)            == applyAll(predicates p, d)

    satisfy?(l:List D, p:Pattern R) ==
      empty?((rec := topPredicate p).var) => true
      retract(rec.pred)$A1L l

    applyAll(l, d) ==
      for f in l repeat
        not(retract(f)$A1 d) => return false
      true

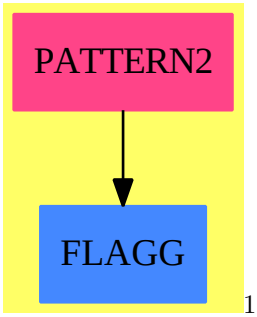
    suchThat(p:Pattern R, l:List(D -> Boolean)) ==
      st(p, [coerce(f)$A1 for f in l])
```

⟨*PATTERN1.dotabb*⟩≡

```
"PATTERN1" [color="#FF4488",href="bookvol10.4.pdf#nameddest=PATTERN1"]
"BASTYPE" [color="#4488FF",href="bookvol10.2.pdf#nameddest=BASTYPE"]
"KOERCE" [color="#4488FF",href="bookvol10.2.pdf#nameddest=KOERCE"]
"TYPE" [color="#4488FF",href="bookvol10.2.pdf#nameddest=TYPE"]
"PATTERN1" -> "BASTYPE"
"PATTERN1" -> "KOERCE"
"PATTERN1" -> "TYPE"
```

17.23 package PATTERN2 PatternFunctions2

17.24 PatternFunctions2



1.00

Exports:
map

⟨*package PATTERN2 PatternFunctions2*⟩≡

```
)abbrev package PATTERN2 PatternFunctions2
++ Lifting of maps to patterns
++ Author: Manuel Bronstein
++ Date Created: 28 Nov 1989
++ Date Last Updated: 12 Jan 1990
++ Description: Lifts maps to patterns;
++ Keywords: pattern, matching.
PatternFunctions2(R:SetCategory, S:SetCategory): with
    map: (R -> S, Pattern R) -> Pattern S
      ++ map(f, p) applies f to all the leaves of p and
      ++ returns the result as a pattern over S.
  == add
    map(f, p) ==
      (r := (retractIfCan p)@Union(R, "failed")) case R =>
        f(r::R)::Pattern(S)
      (u := isOp p) case Record(op:BasicOperator, arg:List Pattern R) =>
        ur := u::Record(op:BasicOperator, arg:List Pattern R)
        (ur.op) [map(f, x) for x in ur.arg]
      (v := isQuotient p) case Record(num:Pattern R, den:Pattern R) =>
        vr := v::Record(num:Pattern R, den:Pattern R)
        map(f, vr.num) / map(f, vr.den)
      (l := isPlus p) case List(Pattern R) =>
        reduce("+", [map(f, x) for x in l::List(Pattern R)])
      (l := isTimes p) case List(Pattern R) =>
        reduce("*", [map(f, x) for x in l::List(Pattern R)])
      (x := isPower p) case
       Record(val:Pattern R, exponent: Pattern R) =>
```

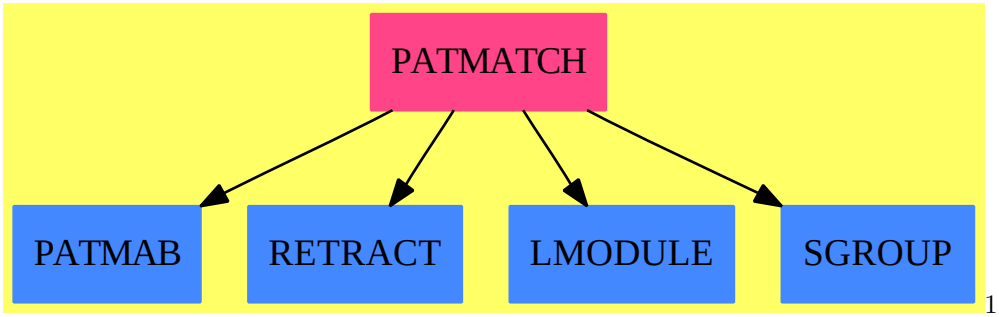
```
      xr := x::Record(val:Pattern R, exponent: Pattern R)
      map(f, xr.val) ** map(f, xr.exponent)
    (w := isExpt p) case
     Record(val:Pattern R, exponent: NonNegativeInteger) =>
      wr := w::Record(val:Pattern R, exponent: NonNegativeInteger)
      map(f, wr.val) ** wr.exponent
    sy := retract(p)@Symbol
    setPredicates(sy::Pattern(S), copy predicates p)
```

⟨*PATTERN2.dotabb*⟩≡

```
"PATTERN2" [color="#FF4488",href="bookvol10.4.pdf#nameddest=PATTERN2"]
"FLAGG" [color="#4488FF",href="bookvol10.2.pdf#nameddest=FLAGG"]
"PATTERN2" -> "FLAGG"
```

17.25 package PATMATCH PatternMatch

17.26 PatternMatch



1.00

Exports:
is? Is

⟨*package PATMATCH PatternMatch*⟩≡
```
)abbrev package PATMATCH PatternMatch
++ Top-level pattern matching functions
++ Author: Manuel Bronstein
++ Date Created: 3 Dec 1989
++ Date Last Updated: 29 Jun 1990
++ Description:
++   This package provides the top-level pattern macthing functions.
++ Keywords: pattern, matching.
PatternMatch(Base, Subject, Pat): Exports == Implementation where
  Base   : SetCategory
  Subject: PatternMatchable Base
  Pat    : ConvertibleTo Pattern Base

  Exports ==> with
    is?: (Subject, Pat)      -> Boolean
      ++ is?(expr, pat) tests if the expression expr matches
      ++ the pattern pat.
    is?: (List Subject, Pat) -> Boolean
      ++ is?([e1,...,en], pat) tests if the list of
      ++ expressions \spad{[e1,...,en]} matches
      ++ the pattern pat.
    Is : (List Subject, Pat) ->
                    PatternMatchListResult(Base, Subject, List Subject)
      ++ Is([e1,...,en], pat) matches the pattern pat on the list of
      ++ expressions \spad{[e1,...,en]} and returns the result.
    if Subject has RetractableTo(Symbol) then
      Is: (Subject, Pat) -> List Equation Subject
```

```
      ++ Is(expr, pat) matches the pattern pat on the expression
      ++ expr and returns a list of matches \spad{[v1 = e1,...,vn = en]};
      ++ returns an empty list if either expr is exactly equal to
      ++ pat or if pat does not match expr.
    else
      if Subject has Ring then
        Is: (Subject, Pat) -> List Equation Polynomial Subject
          ++ Is(expr, pat) matches the pattern pat on the expression
          ++ expr and returns a list of matches \spad{[v1 = e1,...,vn = en]};
          ++ returns an empty list if either expr is exactly equal to
          ++ pat or if pat does not match expr.
      else
        Is: (Subject, Pat) -> PatternMatchResult(Base, Subject)
          ++ Is(expr, pat) matches the pattern pat on the expression
          ++ expr and returns a match of the form \spad{[v1 = e1,...,vn = en]};
          ++ returns an empty match if expr is exactly equal to pat.
          ++ returns a \spadfun{failed} match if pat does not match expr.

  Implementation ==> add
    import PatternMatchListAggregate(Base, Subject, List Subject)

    ist: (Subject, Pat) -> PatternMatchResult(Base, Subject)

    ist(s, p)                  == patternMatch(s, convert p, new())
    is?(s:     Subject, p:Pat) == not failed? ist(s, p)
    is?(s:List Subject, p:Pat) == not failed? Is(s, p)
    Is(s:List Subject,  p:Pat) == patternMatch(s, convert p, new())

    if Subject has RetractableTo(Symbol) then
      Is(s:Subject, p:Pat):List(Equation Subject) ==
        failed?(r := ist(s, p)) => empty()
        [rec.key::Subject = rec.entry for rec in destruct r]

    else
      if Subject has Ring then
        Is(s:Subject, p:Pat):List(Equation Polynomial Subject) ==
          failed?(r := ist(s, p)) => empty()
          [rec.key::Polynomial(Subject) =$Equation(Polynomial Subject)
           rec.entry::Polynomial(Subject) for rec in destruct r]

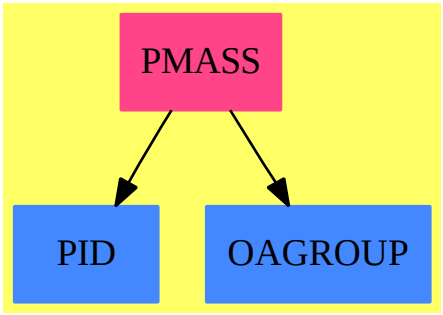
      else
        Is(s:Subject,p:Pat):PatternMatchResult(Base,Subject) == ist(s,p)
```

⟨*PATMATCH.dotabb*⟩≡

```
"PATMATCH" [color="#FF4488",href="bookvol10.4.pdf#nameddest=PATMATCH"]
"PATMAB" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PATMAB"]
"RETRACT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=RETRACT"]
"LMODULE" [color="#4488FF",href="bookvol10.2.pdf#nameddest=LMODULE"]
"SGROUP" [color="#4488FF",href="bookvol10.2.pdf#nameddest=SGROUP"]
"PATMATCH" -> "PATMAB"
"PATMATCH" -> "RETRACT"
"PATMATCH" -> "LMODULE"
"PATMATCH" -> "SGROUP"
```

17.27 package PMASS PatternMatchAssertions

17.28 PatternMatchAssertions



1.00

Exports:
assert constant multiple optional

⟨*package PMASS PatternMatchAssertions*⟩≡
```
)abbrev package PMASS PatternMatchAssertions
++ Assertions for pattern-matching
++ Author: Manuel Bronstein
++ Description: Attaching assertions to symbols for pattern matching.
++ Date Created: 21 Mar 1989
++ Date Last Updated: 23 May 1990
++ Keywords: pattern, matching.
PatternMatchAssertions(): Exports == Implementation where
  FE ==> Expression Integer

  Exports ==> with
    assert  : (Symbol, String) -> FE
      ++ assert(x, s) makes the assertion s about x.
    constant: Symbol -> FE
      ++ constant(x) tells the pattern matcher that x should
      ++ match only the symbol 'x and no other quantity.
    optional: Symbol -> FE
      ++ optional(x) tells the pattern matcher that x can match
      ++ an identity (0 in a sum, 1 in a product or exponentiation).;
    multiple: Symbol -> FE
      ++ multiple(x) tells the pattern matcher that x should
      ++ preferably match a multi-term quantity in a sum or product.
      ++ For matching on lists, multiple(x) tells the pattern matcher
      ++ that x should match a list instead of an element of a list.

  Implementation ==> add
    import FunctionSpaceAssertions(Integer, FE)
```

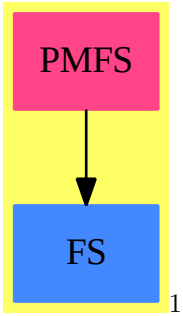
```
constant x   == constant(x::FE)
multiple x   == multiple(x::FE)
optional x   == optional(x::FE)
assert(x, s) == assert(x::FE, s)
```

⟨*PMASS.dotabb*⟩≡

```
"PMASS" [color="#FF4488",href="bookvol10.4.pdf#nameddest=PMASS"]
"PID" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PID"]
"OAGROUP" [color="#4488FF",href="bookvol10.2.pdf#nameddest=OAGROUP"]
"PMASS" -> "PID"
"PMASS" -> "OAGROUP"
```

17.29 package PMFS PatternMatchFunctionSpace

17.30 PatternMatchFunctionSpace



1.00

Exports:
patternMatch

⟨*package PMFS PatternMatchFunctionSpace*⟩≡

```
)abbrev package PMFS PatternMatchFunctionSpace
++ Pattern matching on function spaces
++ Author: Manuel Bronstein
++ Date Created: 15 Mar 1990
++ Date Last Updated: 20 June 1991
++ Description:
++  This package provides pattern matching functions on function spaces.
++ Keywords: pattern, matching, function, space.
PatternMatchFunctionSpace(S, R, F): Exports== Implementation where
  S: SetCategory
  R: Join(IntegralDomain, OrderedSet, PatternMatchable S)
  F: Join(FunctionSpace R, ConvertibleTo Pattern S, PatternMatchable S,
          RetractableTo Kernel %)  -- that one is redundant but won't
                                   -- compile without it

  N   ==> NonNegativeInteger
  K   ==> Kernel F
  PAT ==> Pattern S
  PRS ==> PatternMatchResult(S, F)
  RCP ==> Record(val:PAT, exponent:N)
  RCX ==> Record(var:K, exponent:Integer)

  Exports ==> with
    patternMatch: (F, PAT, PRS) -> PRS
      ++ patternMatch(expr, pat, res) matches the pattern pat to the
      ++ expression expr; res contains the variables of pat which
```

```
      ++ are already matched and their matches.

  Implementation ==> add
    import PatternMatchKernel(S, F)
    import PatternMatchTools(S, R, F)
    import PatternMatchPushDown(S, R, F)

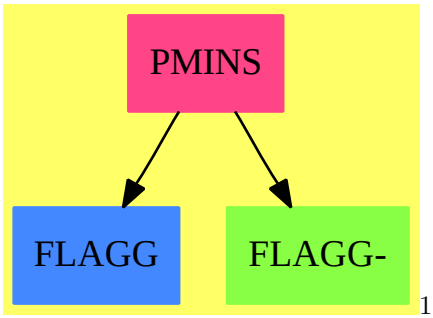
    patternMatch(x, p, l) ==
      generic? p => addMatch(p, x, l)
      (r := retractIfCan(x)@Union(R, "failed")) case R =>
        patternMatch(r::R, p, l)
      (v := retractIfCan(x)@Union(K, "failed")) case K =>
        patternMatch(v::K, p, l)
      (q := isQuotient p) case Record(num:PAT, den:PAT) =>
        uq := q::Record(num:PAT, den:PAT)
        failed?(l := patternMatch(numer(x)::F, uq.num, l)) => l
        patternMatch(denom(x)::F, uq.den, l)
      (u := isPlus p) case List(PAT) =>
        (lx := isPlus x) case List(F) =>
          patternMatch(lx::List(F), u::List(PAT), +/#1, l, patternMatch)
        (u := optpair(u::List(PAT))) case List(PAT) =>
          failed?(l := addMatch(first(u::List(PAT)), 0, l)) => failed()
          patternMatch(x, second(u::List(PAT)), l)
        failed()
      (u := isTimes p) case List(PAT) =>
        (lx := isTimes x) case List(F) =>
          patternMatchTimes(lx::List(F), u::List(PAT), l, patternMatch)
        (u := optpair(u::List(PAT))) case List(PAT) =>
          failed?(l := addMatch(first(u::List(PAT)), 1, l)) => failed()
          patternMatch(x, second(u::List(PAT)), l)
        failed()
      (uu := isPower p) case Record(val:PAT, exponent:PAT) =>
        uur := uu::Record(val:PAT, exponent: PAT)
        (ex := isExpt x) case RCX =>
          failed?(l := patternMatch((ex::RCX).exponent::Integer::F,
                                        uur.exponent, l)) => failed()
          patternMatch((ex::RCX).var, uur.val, l)
        optional?(uur.exponent) =>
          failed?(l := addMatch(uur.exponent, 1, l)) => failed()
          patternMatch(x, uur.val, l)
        failed()
      ((ep := isExpt p) case RCP) and ((ex := isExpt x) case RCX) and
           (ex::RCX).exponent = ((ep::RCP).exponent)::Integer =>
               patternMatch((ex::RCX).var, (ep::RCP).val, l)
      failed()
```

⟨*PMFS.dotabb*⟩≡

```
"PMFS" [color="#FF4488",href="bookvol10.4.pdf#nameddest=PMFS"]
"FS" [color="#4488FF",href="bookvol10.2.pdf#nameddest=FS"]
"PMFS" -> "FS"
```

17.31 package PMINS PatternMatchIntegerNumberSystem

17.32 PatternMatchIntegerNumberSystem



1.00

Exports:
patternMatch

⟨*package PMINS PatternMatchIntegerNumberSystem*⟩≡

```
)abbrev package PMINS PatternMatchIntegerNumberSystem
++ Pattern matching on integer number systems
++ Author: Manuel Bronstein
++ Date Created: 29 Nov 1989
++ Date Last Updated: 22 Mar 1990
++ Description:
++   This package provides pattern matching functions on integers.
++ Keywords: pattern, matching, integer.
PatternMatchIntegerNumberSystem(I:IntegerNumberSystem): with
  patternMatch: (I, Pattern Integer, PatternMatchResult(Integer, I)) ->
                                          PatternMatchResult(Integer, I)
    ++ patternMatch(n, pat, res) matches the pattern pat to the
    ++ integer n; res contains the variables of pat which
    ++ are already matched and their matches.
 == add
   import IntegerRoots(I)

   PAT ==> Pattern Integer
   PMR ==> PatternMatchResult(Integer, I)

   patternMatchInner     : (I, PAT, PMR) -> PMR
   patternMatchRestricted: (I, PAT, PMR, I) -> PMR
   patternMatchSumProd   :
     (I, List PAT, PMR, (I, I) -> Union(I, "failed"), I) -> PMR

   patternMatch(x, p, l) ==
```

```
    generic? p => addMatch(p, x, l)
    patternMatchInner(x, p, l)

  patternMatchRestricted(x, p, l, y) ==
    generic? p => addMatchRestricted(p, x, l, y)
    patternMatchInner(x, p, l)

  patternMatchSumProd(x, lp, l, invOp, ident) ==
    #lp = 2 =>
      p2 := last lp
      if ((r:= retractIfCan(p1 := first lp)@Union(Integer,"failed"))
                          case "failed") then (p1 := p2; p2 := first lp)
      (r := retractIfCan(p1)@Union(Integer, "failed")) case "failed" =>
                                                              failed()
      (y := invOp(x, r::Integer::I)) case "failed" => failed()
      patternMatchRestricted(y::I, p2, l, ident)
    failed()

  patternMatchInner(x, p, l) ==
    constant? p =>
      (r := retractIfCan(p)@Union(Integer, "failed")) case Integer =>
        convert(x)@Integer = r::Integer => l
        failed()
      failed()
    (u := isExpt p) case Record(val:PAT,exponent:NonNegativeInteger) =>
      ur := u::Record(val:PAT, exponent:NonNegativeInteger)
      (v := perfectNthRoot(x, ur.exponent)) case "failed" => failed()
      patternMatchRestricted(v::I, ur.val, l, 1)
    (uu := isPower p) case Record(val:PAT, exponent:PAT) =>
      uur := uu::Record(val:PAT, exponent: PAT)
      pr := perfectNthRoot x
      failed?(l := patternMatchRestricted(pr.exponent::Integer::I,
                                      uur.exponent, l,1)) => failed()
      patternMatchRestricted(pr.base, uur.val, l, 1)
    (w := isTimes p) case List(PAT) =>
      patternMatchSumProd(x, w::List(PAT), l, #1 exquo #2, 1)
    (w := isPlus p) case List(PAT) =>
     patternMatchSumProd(x,w::List(PAT),l,(#1-#2)::Union(I,"failed"),0)
    (uv := isQuotient p) case Record(num:PAT, den:PAT) =>
      uvr := uv::Record(num:PAT, den:PAT)
      (r := retractIfCan(uvr.num)@Union(Integer,"failed")) case Integer
        and (v := r::Integer::I exquo x) case I =>
          patternMatchRestricted(v::I, uvr.den, l, 1)
      (r := retractIfCan(uvr.den)@Union(Integer,"failed")) case Integer
        => patternMatch(r::Integer * x, uvr.num, l)
      failed()
```

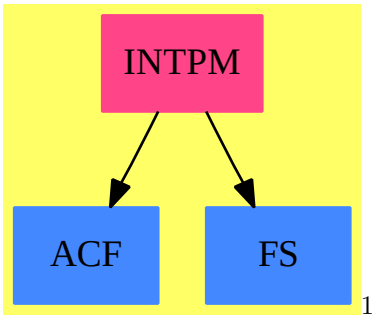
```
      failed()
```

⟨*PMINS.dotabb*⟩≡

```
"PMINS" [color="#FF4488",href="bookvol10.4.pdf#nameddest=PMINS"]
"FLAGG" [color="#4488FF",href="bookvol10.2.pdf#nameddest=FLAGG"]
"FLAGG-" [color="#88FF44",href="bookvol10.3.pdf#nameddest=FLAGG"]
"PMINS" -> "FLAGG"
"PMINS" -> "FLAGG-"
```

17.33 package INTPM PatternMatchIntegration

17.34 PatternMatchIntegration



1.00

Exports:
pmComplexintegrate pmintegrate pmintegrate splitConstant

⟨*package INTPM PatternMatchIntegration*⟩≡

```
)abbrev package INTPM PatternMatchIntegration
++ Author: Manuel Bronstein
++ Date Created: 5 May 1992
++ Date Last Updated: 27 September 1995
++ Description:
++ \spadtype{PatternMatchIntegration} provides functions that use
++ the pattern matcher to find some indefinite and definite integrals
++ involving special functions and found in the litterature.
PatternMatchIntegration(R, F): Exports == Implementation where
  R : Join(OrderedSet, RetractableTo Integer, GcdDomain,
           LinearlyExplicitRingOver Integer)
  F : Join(AlgebraicallyClosedField, TranscendentalFunctionCategory,
           FunctionSpace R)

  N   ==> NonNegativeInteger
  Z   ==> Integer
  SY  ==> Symbol
  K   ==> Kernel F
  P   ==> SparseMultivariatePolynomial(R, K)
  SUP ==> SparseUnivariatePolynomial F
  PAT ==> Pattern Z
  RES ==> PatternMatchResult(Z, F)
  OFE ==> OrderedCompletion F
  REC ==> Record(which: Z, exponent: F, coeff: F)
  ANS ==> Record(special:F, integrand:F)
  NONE ==> 0
```

```
EI   ==> 1
ERF  ==> 2
SI   ==> 3
CI   ==> 4
GAM2 ==> 5
CIO  ==> 6

Exports ==> with
  splitConstant: (F, SY) -> Record(const:F, nconst:F)
    ++ splitConstant(f, x) returns \spad{[c, g]} such that
    ++ \spad{f = c * g} and \spad{c} does not involve \spad{t}.
  if R has ConvertibleTo Pattern Integer and
     R has PatternMatchable Integer then
       if F has LiouvillianFunctionCategory then
         pmComplexintegrate: (F, SY) -> Union(ANS, "failed")
           ++ pmComplexintegrate(f, x) returns either "failed" or
           ++ \spad{[g,h]} such that
           ++ \spad{integrate(f,x) = g + integrate(h,x)}.
           ++ It only looks for special complex integrals that pmintegrate
           ++ does not return.
         pmintegrate: (F, SY) -> Union(ANS, "failed")
           ++ pmintegrate(f, x) returns either "failed" or \spad{[g,h]} such
           ++ that \spad{integrate(f,x) = g + integrate(h,x)}.
       if F has SpecialFunctionCategory then
         pmintegrate: (F, SY, OFE, OFE) -> Union(F, "failed")
           ++ pmintegrate(f, x = a..b) returns the integral of
           ++ \spad{f(x)dx} from a to b
           ++ if it can be found by the built-in pattern matching rules.

Implementation ==> add
  import PatternMatch(Z, F, F)
  import ElementaryFunctionSign(R, F)
  import FunctionSpaceAssertions(R, F)
  import TrigonometricManipulations(R, F)
  import FunctionSpaceAttachPredicates(R, F, F)

  mkalist   : RES -> AssociationList(SY, F)

  pm := new()$SY
  pmw := new pm
  pmm := new pm
  pms := new pm
  pmc := new pm
  pma := new pm
  pmb := new pm
```

```
c := optional(pmc::F)
w := suchThat(optional(pmw::F), empty? variables #1)
s := suchThat(optional(pms::F), empty? variables #1 and real? #1)
m := suchThat(optional(pmm::F),
                (retractIfCan(#1)@Union(Z,"failed") case Z) and #1 >= 0)

spi := sqrt(pi()$F)

half := 1::F / 2::F

mkalist res == construct destruct res

splitConstant(f, x) ==
  not member?(x, variables f) => [f, 1]
  (retractIfCan(f)@Union(K, "failed")) case K => [1, f]
  (u := isTimes f) case List(F) =>
    cc := nc := 1$F
    for g in u::List(F) repeat
      rec := splitConstant(g, x)
      cc  := cc * rec.const
      nc  := nc * rec.nconst
    [cc, nc]
  (u := isPlus f) case List(F) =>
    rec := splitConstant(first(u::List(F)), x)
    cc  := rec.const
    nc  := rec.nconst
    for g in rest(u::List(F)) repeat
      rec := splitConstant(g, x)
      if rec.nconst = nc then cc := cc + rec.const
      else if rec.nconst = -nc then cc := cc - rec.const
      else return [1, f]
    [cc, nc]
  if (v := isPower f) case Record(val:F, exponent:Z) then
    vv := v::Record(val:F, exponent:Z)
    (vv.exponent ^= 1) =>
      rec := splitConstant(vv.val, x)
      return [rec.const ** vv.exponent, rec.nconst ** vv.exponent]
  error "splitConstant: should not happen"

if R has ConvertibleTo Pattern Integer and
   R has PatternMatchable Integer then
     if F has LiouvillianFunctionCategory then
       import ElementaryFunctionSign(R, F)

       insqrt     : F -> F
       matchei    : (F, SY) -> REC
```

```
            matcherfei : (F, SY, Boolean) -> REC
            matchsici  : (F, SY) -> REC
            matchli    : (F, SY) -> List F
            matchli0   : (F, K, SY) -> List F
            matchdilog : (F, SY) -> List F
            matchdilog0: (F, K, SY, P, F) -> List F
            goodlilog? : (K, P) -> Boolean
            gooddilog? : (K, P, P) -> Boolean

--            goodlilog?(k, p) == is?(k, "log"::SY) and one? minimumDegree(p, k)
            goodlilog?(k, p) == is?(k, "log"::SY) and (minimumDegree(p, k) = 1)

            gooddilog?(k, p, q) ==
--              is?(k, "log"::SY) and one? degree(p, k) and zero? degree(q, k)
              is?(k, "log"::SY) and (degree(p, k) = 1) and zero? degree(q, k)

-- matches the integral to a result of the form d * erf(u) or d * ei(u)
-- returns [case, u, d]
            matcherfei(f, x, comp?) ==
              res0 := new()$RES
              pat := c * exp(pma::F)
              failed?(res := patternMatch(f, convert(pat)@PAT, res0)) =>
                comp? => [NONE, 0,0]
                matchei(f,x)
              l := mkalist res
              da := differentiate(a := l.pma, x)
              d := a * (cc := l.pmc) / da
              zero? differentiate(d, x) => [EI, a, d]
              comp? or (((u := sign a) case Z) and (u::Z) < 0) =>
                d := cc * (sa := insqrt(- a)) / da
                zero? differentiate(d, x) => [ERF, sa, - d * spi]
                [NONE, 0, 0]
              [NONE, 0, 0]

-- matches the integral to a result of the form d * ei(k * log u)
-- returns [case, k * log u, d]
            matchei(f, x) ==
              res0 := new()$RES
              a := pma::F
              pat := c * a**w / log a
              failed?(res := patternMatch(f, convert(pat)@PAT, res0)) =>
                [NONE, 0, 0]
              l := mkalist res
              da := differentiate(a := l.pma, x)
              d := (cc := l.pmc) / da
              zero? differentiate(d, x) => [EI, (1 + l.pmw) * log a, d]
```

```
              [NONE, 0, 0]

  -- matches the integral to a result of the form d * dilog(u) + int(v),
  -- returns [u,d,v] or []
            matchdilog(f, x) ==
              n := numer f
              df := (d := denom f)::F
              for k in select_!(gooddilog?(#1, n, d), variables n)$List(K) repeat
                not empty?(l := matchdilog0(f, k, x, n, df)) => return l
              empty()

  -- matches the integral to a result of the form d * dilog(a) + int(v)
  -- where k = log(a)
  -- returns [a,d,v] or []
            matchdilog0(f, k, x, p, q) ==
              zero?(da := differentiate(a := first argument k, x)) => empty()
              a1 := 1 - a
              d := coefficient(univariate(p, k), 1)::F * a1 / (q * da)
              zero? differentiate(d, x) => [a, d, f - d * da * (k::F) / a1]
              empty()

  -- matches the integral to a result of the form d * li(u) + int(v),
  -- returns [u,d,v] or []
            matchli(f, x) ==
              d := denom f
              for k in select_!(goodlilog?(#1, d), variables d)$List(K) repeat
                not empty?(l := matchli0(f, k, x)) => return l
              empty()

  -- matches the integral to a result of the form d * li(a) + int(v)
  -- where k = log(a)
  -- returns [a,d,v] or []
            matchli0(f, k, x) ==
              g := (lg := k::F) * f
              zero?(da := differentiate(a := first argument k, x)) => empty()
              zero? differentiate(d := g / da, x) => [a, d, 0]
              ug := univariate(g, k)
              (u:=retractIfCan(ug)@Union(SUP,"failed")) case "failed" => empty()
              degree(p := u::SUP) > 1 => empty()
              zero? differentiate(d := coefficient(p, 0) / da, x) =>
                [a, d, leadingCoefficient p]
              empty()

  -- matches the integral to a result of the form d * Si(u) or d * Ci(u)
  -- returns [case, u, d]
            matchsici(f, x) ==
```

```
              res0 := new()$RES
              b := pmb::F
              t := tan(a := pma::F)
              patsi := c * t / (patden := b + b * t**2)
              patci := (c - c * t**2) / patden
              patci0 := c / patden
              ci0?:Boolean
              (ci? := failed?(res := patternMatch(f, convert(patsi)@PAT, res0)))
                and (ci0?:=failed?(res:=patternMatch(f,convert(patci)@PAT,res0)))
                  and failed?(res := patternMatch(f,convert(patci0)@PAT,res0)) =>
                    [NONE, 0, 0]
              l := mkalist res
              (b := l.pmb) ^= 2 * (a := l.pma) => [NONE, 0, 0]
              db := differentiate(b, x)
              d := (cc := l.pmc) / db
              zero? differentiate(d, x) =>
                ci? =>
                   ci0? => [CI0, b, d / (2::F)]
                   [CI, b, d]
                [SI, b, d / (2::F)]
              [NONE, 0, 0]

  -- returns a simplified sqrt(y)
            insqrt y ==
              rec := froot(y, 2)$PolynomialRoots(IndexedExponents K, K, R, P, F)
  --            one?(rec.exponent) => rec.coef * rec.radicand
              ((rec.exponent) = 1) => rec.coef * rec.radicand
              rec.exponent ^=2 => error "insqrt: hould not happen"
              rec.coef * sqrt(rec.radicand)

            pmintegrate(f, x) ==
              (rc := splitConstant(f, x)).const ^= 1 =>
                (u := pmintegrate(rc.nconst, x)) case "failed" => "failed"
                rec := u::ANS
                [rc.const * rec.special, rc.const * rec.integrand]
              not empty?(l := matchli(f, x)) => [second l * li first l, third l]
              not empty?(l := matchdilog(f, x)) =>
                                            [second l * dilog first l, third l]
              cse := (rec := matcherfei(f, x, false)).which
              cse = EI   => [rec.coeff * Ei(rec.exponent), 0]
              cse = ERF  => [rec.coeff * erf(rec.exponent), 0]
              cse := (rec := matchsici(f, x)).which
              cse = SI => [rec.coeff * Si(rec.exponent), 0]
              cse = CI => [rec.coeff * Ci(rec.exponent), 0]
              cse = CI0 => [rec.coeff * Ci(rec.exponent)
                            + rec.coeff * log(rec.exponent), 0]
```

```
              "failed"

            pmComplexintegrate(f, x) ==
              (rc := splitConstant(f, x)).const ^= 1 =>
                (u := pmintegrate(rc.nconst, x)) case "failed" => "failed"
                rec := u::ANS
                [rc.const * rec.special, rc.const * rec.integrand]
              cse := (rec := matcherfei(f, x, true)).which
              cse = ERF  => [rec.coeff * erf(rec.exponent), 0]
              "failed"

          if F has SpecialFunctionCategory then
            match1    : (F, SY, F, F) -> List F
            formula1  : (F, SY, F, F) -> Union(F, "failed")

  -- tries only formula (1) of the Geddes & al, AAECC 1 (1990) paper
            formula1(f, x, t, cc) ==
              empty?(l := match1(f, x, t, cc)) => "failed"
              mw := first l
              zero?(ms := third l) or ((sgs := sign ms) case "failed")=> "failed"
              ((sgz := sign(z := (mw + 1) / ms)) case "failed") or (sgz::Z < 0)
                 => "failed"
              mmi := retract(mm := second l)@Z
              sgs * (last l) * ms**(- mmi - 1) *
                  eval(differentiate(Gamma(x::F), x, mmi::N), [kernel(x)@K], [z])

  -- returns [w, m, s, c] or []
  -- matches only formula (1) of the Geddes & al, AAECC 1 (1990) paper
            match1(f, x, t, cc) ==
              res0 := new()$RES
              pat := cc * log(t)**m * exp(-t**s)
              not failed?(res := patternMatch(f, convert(pat)@PAT, res0)) =>
                l := mkalist res
                [0, l.pmm, l.pms, l.pmc]
              pat := cc * t**w * exp(-t**s)
              not failed?(res := patternMatch(f, convert(pat)@PAT, res0)) =>
                l := mkalist res
                [l.pmw, 0, l.pms, l.pmc]
              pat := cc / t**w * exp(-t**s)
              not failed?(res := patternMatch(f, convert(pat)@PAT, res0)) =>
                l := mkalist res
                [- l.pmw, 0, l.pms, l.pmc]
              pat := cc * t**w * log(t)**m * exp(-t**s)
              not failed?(res := patternMatch(f, convert(pat)@PAT, res0)) =>
                l := mkalist res
                [l.pmw, l.pmm, l.pms, l.pmc]
```

```
              pat := cc / t**w * log(t)**m * exp(-t**s)
              not failed?(res := patternMatch(f, convert(pat)@PAT, res0)) =>
                l := mkalist res
                [- l.pmw, l.pmm, l.pms, l.pmc]
              empty()

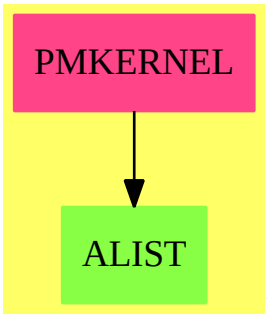
            pmintegrate(f, x, a, b) ==
--            zero? a and one? whatInfinity b =>
              zero? a and ((whatInfinity b) = 1) =>
                formula1(f, x, constant(x::F), suchThat(c, freeOf?(#1, x)))
              "failed"
```

⟨*INTPM.dotabb*⟩≡

```
"INTPM" [color="#FF4488",href="bookvol10.4.pdf#nameddest=INTPM"]
"ACF" [color="#4488FF",href="bookvol10.2.pdf#nameddest=ACF"]
"FS" [color="#4488FF",href="bookvol10.2.pdf#nameddest=FS"]
"INTPM" -> "ACF"
"INTPM" -> "FS"
```

17.35 package PMKERNEL PatternMatchKernel

17.36 PatternMatchKernel



1.00

Exports:
patternMatch

⟨*package PMKERNEL PatternMatchKernel*⟩≡

```
)abbrev package PMKERNEL PatternMatchKernel
++ Pattern matching on kernels
++ Author: Manuel Bronstein
++ Date Created: 12 Jan 1990
++ Date Last Updated: 4 May 1992
++ Description:
++   This package provides pattern matching functions on kernels.
++ Keywords: pattern, matching, kernel.
PatternMatchKernel(S, E): Exports == Implementation where
  S: SetCategory
  E: Join(OrderedSet, RetractableTo Kernel %,
          ConvertibleTo Pattern S, PatternMatchable S)

  PAT ==> Pattern S
  PRS ==> PatternMatchResult(S, E)
  POWER ==> "%power"::Symbol
  NTHRT ==> "nthRoot"::Symbol

  Exports ==> with
    patternMatch: (Kernel E, PAT, PRS) -> PRS
      ++ patternMatch(f(e1,...,en), pat, res) matches the pattern pat
      ++ to \spad{f(e1,...,en)}; res contains the variables of pat which
      ++ are already matched and their matches.

  Implementation ==> add
    patternMatchArg  : (List E, List PAT, PRS) -> PRS
```

```
patternMatchInner: (Kernel E, PAT, PRS) -> Union(PRS, "failed")

-- matches the ordered lists ls and lp.
patternMatchArg(ls, lp, l) ==
  #ls ^= #lp => failed()
  for p in lp for s in ls repeat
    generic? p and failed?(l := addMatch(p,s,l)) => return failed()
  for p in lp for s in ls repeat
    not(generic? p) and failed?(l := patternMatch(s, p, l)) =>
                                                    return failed()
  l

patternMatchInner(s, p, l) ==
  generic? p => addMatch(p, s::E, l)
  (u := isOp p) case Record(op:BasicOperator, arg: List PAT) =>
    ur := u::Record(op:BasicOperator, arg: List PAT)
    ur.op = operator s => patternMatchArg(argument s, ur.arg, l)
    failed()
  constant? p =>
    ((v := retractIfCan(p)@Union(Symbol, "failed")) case Symbol)
      and ((w := symbolIfCan s) case Symbol) and
        (v::Symbol = w::Symbol) => l
    failed()
  "failed"

if E has Monoid then
  patternMatchMonoid: (Kernel E, PAT, PRS) -> Union(PRS, "failed")
  patternMatchOpt   : (E, List PAT, PRS, E) -> PRS

  patternMatchOpt(x, lp, l, id) ==
    (u := optpair lp) case List(PAT) =>
      failed?(l := addMatch(first(u::List(PAT)), id, l)) => failed()
      patternMatch(x, second(u::List(PAT)), l)
    failed()

  patternMatchMonoid(s, p, l) ==
    (u := patternMatchInner(s, p, l)) case PRS => u::PRS
    (v := isPower p) case Record(val:PAT, exponent:PAT) =>
      vr := v::Record(val:PAT, exponent: PAT)
      is?(op := operator s, POWER) =>
        patternMatchArg(argument s, [vr.val, vr.exponent], l)
      is?(op,NTHRT) and ((r := recip(second(arg := argument s))) case E) =>
        patternMatchArg([first arg, r::E], [vr.val, vr.exponent], l)
      optional?(vr.exponent) =>
        failed?(l := addMatch(vr.exponent, 1, l)) => failed()
        patternMatch(s::E, vr.val, l)
```

```
            failed()
          (w := isTimes p) case List(PAT) =>
            patternMatchOpt(s::E, w::List(PAT), l, 1)
          "failed"

        if E has AbelianMonoid then
          patternMatch(s, p, l) ==
            (u := patternMatchMonoid(s, p, l)) case PRS => u::PRS
            (w := isPlus p) case List(PAT) =>
              patternMatchOpt(s::E, w::List(PAT), l, 0)
            failed()

        else
          patternMatch(s, p, l) ==
            (u := patternMatchMonoid(s, p, l)) case PRS => u::PRS
            failed()

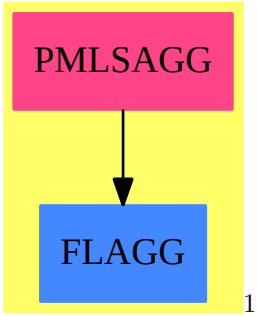
      else
        patternMatch(s, p, l) ==
          (u := patternMatchInner(s, p, l)) case PRS => u::PRS
          failed()
```

⟨*PMKERNEL.dotabb*⟩≡

```
"PMKERNEL" [color="#FF4488",href="bookvol10.4.pdf#nameddest=PMKERNEL"]
"ALIST" [color="#88FF44",href="bookvol10.3.pdf#nameddest=ALIST"]
"PMKERNEL" -> "ALIST"
```

17.37 package PMLSAGG PatternMatchListAggregate

17.38 PatternMatchListAggregate



1.00

Exports:
patternMatch

⟨*package PMLSAGG PatternMatchListAggregate*⟩≡

```
)abbrev package PMLSAGG PatternMatchListAggregate
++ Pattern matching for list aggregates
++ Author: Manuel Bronstein
++ Date Created: 4 Dec 1989
++ Date Last Updated: 29 Jun 1990
++ Description:
++   This package provides pattern matching functions on lists.
++ Keywords: pattern, matching, list.
PatternMatchListAggregate(S, R, L): Exports == Implementation where
  S: SetCategory
  R: PatternMatchable S
  L: ListAggregate R

  PLR ==> PatternMatchListResult(S, R, L)

  Exports ==> with
    patternMatch: (L, Pattern S, PLR) -> PLR
      ++ patternMatch(l, pat, res) matches the pattern pat to the
      ++ list l; res contains the variables of pat which
      ++ are already matched and their matches.

  Implementation ==> add
    match: (L, List Pattern S, PLR, Boolean) -> PLR

    patternMatch(l, p, r) ==
      (u := isList p) case "failed" => failed()
```

```
        match(l, u::List Pattern S, r, true)

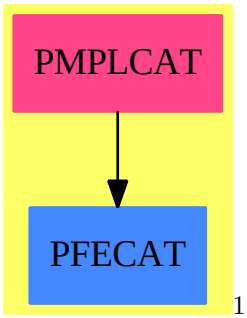
      match(l, lp, r, new?) ==
        empty? lp =>
          empty? l => r
          failed()
        multiple?(p0 := first lp) =>
          empty? rest lp =>
            if not new? then l := reverse_! l
            makeResult(atoms r, addMatchRestricted(p0,l,lists r,empty()))
          new? => match(reverse l, reverse lp, r, false)
          error "Only one multiple pattern allowed in list"
        empty? l => failed()
        failed?(r := makeResult(patternMatch(first l,p0,atoms r),lists r))
                                                                  => failed()
        match(rest l, rest lp, r, new?)
```

⟨*PMLSAGG.dotabb*⟩≡

```
"PMLSAGG" [color="#FF4488",href="bookvol10.4.pdf#nameddest=PMLSAGG"]
"FLAGG" [color="#4488FF",href="bookvol10.2.pdf#nameddest=FLAGG"]
"PMLSAGG" -> "FLAGG"
```

17.39 package PMPLCAT PatternMatchPolynomialCategory

17.40 PatternMatchPolynomialCategory



1.00

Exports:
patternMatch

⟨*package PMPLCAT PatternMatchPolynomialCategory*⟩≡

```
)abbrev package PMPLCAT PatternMatchPolynomialCategory
++ Pattern matching on polynomial objects
++ Author: Manuel Bronstein
++ Date Created: 9 Jan 1990
++ Date Last Updated: 20 June 1991
++ Description:
++   This package provides pattern matching functions on polynomials.
++ Keywords: pattern, matching, polynomial.
PatternMatchPolynomialCategory(S,E,V,R,P):Exports== Implementation where
  S: SetCategory
  E: OrderedAbelianMonoidSup
  V: OrderedSet
  R: Join(Ring, OrderedSet, PatternMatchable S)
  P: Join(PolynomialCategory(R, E, V), ConvertibleTo Pattern S)

  N   ==> NonNegativeInteger
  PAT ==> Pattern S
  PRS ==> PatternMatchResult(S, P)
  RCP ==> Record(val:PAT, exponent:N)
  RCX ==> Record(var:V, exponent:N)

  Exports ==> with
    patternMatch: (P, PAT, PRS, (V, PAT, PRS) -> PRS) -> PRS
      ++ patternMatch(p, pat, res, vmatch) matches the pattern pat to
      ++ the polynomial p. res contains the variables of pat which
      ++ are already matched and their matches; vmatch is the matching
```

```
      ++ function to use on the variables.
      -- This can be more efficient than pushing down when the variables
      -- are recursive over P (e.g. kernels)
    if V has PatternMatchable S then
      patternMatch: (P, PAT, PRS) -> PRS
        ++ patternMatch(p, pat, res) matches the pattern pat to
        ++ the polynomial p; res contains the variables of pat which
        ++ are already matched and their matches.

  Implementation ==> add
    import PatternMatchTools(S, R, P)
    import PatternMatchPushDown(S, R, P)

    if V has PatternMatchable S then
      patternMatch(x, p, l) ==
        patternMatch(x, p, l, patternMatch$PatternMatchPushDown(S,V,P))

    patternMatch(x, p, l, vmatch) ==
      generic? p => addMatch(p, x, l)
      (r := retractIfCan(x)@Union(R, "failed")) case R =>
        patternMatch(r::R, p, l)
      (v := retractIfCan(x)@Union(V, "failed")) case V =>
        vmatch(v::V, p, l)
      (u := isPlus p) case List(PAT) =>
        (lx := isPlus x) case List(P) =>
          patternMatch(lx::List(P), u::List(PAT), +/#1, l,
                                        patternMatch(#1, #2, #3, vmatch))
        (u := optpair(u::List(PAT))) case List(PAT) =>
          failed?(l := addMatch(first(u::List(PAT)), 0, l)) => failed()
          patternMatch(x, second(u::List(PAT)), l, vmatch)
        failed()
      (u := isTimes p) case List(PAT) =>
        (lx := isTimes x) case List(P) =>
          patternMatchTimes(lx::List(P), u::List(PAT), l,
                                        patternMatch(#1, #2, #3, vmatch))
        (u := optpair(u::List(PAT))) case List(PAT) =>
          failed?(l := addMatch(first(u::List(PAT)), 1, l)) => failed()
          patternMatch(x, second(u::List(PAT)), l, vmatch)
        failed()
      (uu := isPower p) case Record(val:PAT, exponent:PAT) =>
        uur := uu::Record(val:PAT, exponent: PAT)
        (ex := isExpt x) case RCX =>
          failed?(l := patternMatch((ex::RCX).exponent::Integer::P,
                                    uur.exponent, l, vmatch)) => failed()
          vmatch((ex::RCX).var, uur.val, l)
        optional?(uur.exponent) =>
```

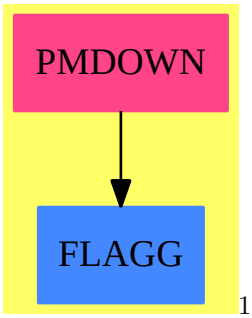
```
            failed?(l := addMatch(uur.exponent, 1, l)) => failed()
            patternMatch(x, uur.val, l, vmatch)
          failed()
        ((ep := isExpt p) case RCP) and ((ex := isExpt x) case RCX) and
             (ex::RCX).exponent = (ep::RCP).exponent =>
                 vmatch((ex::RCX).var, (ep::RCP).val, l)
        failed()
```

⟨*PMPLCAT.dotabb*⟩≡

```
"PMPLCAT" [color="#FF4488",href="bookvol10.4.pdf#nameddest=PMPLCAT"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"PMPLCAT" -> "PFECAT"
```

17.41 package PMDOWN PatternMatchPushDown

17.42 PatternMatchPushDown



1.00

Exports:
fixPredicate patternMatch

⟨*package PMDOWN PatternMatchPushDown*⟩≡

```
)abbrev package PMDOWN PatternMatchPushDown
++ Pattern matching in towers
++ Author: Manuel Bronstein
++ Date Created: 1 Dec 1989
++ Date Last Updated: 16 August 1995
++ Description:
++   This packages provides tools for matching recursively
++   in type towers.
++ Keywords: pattern, matching, quotient, field.
PatternMatchPushDown(S, A, B): Exports == Implementation where
  S: SetCategory
  A: PatternMatchable S
  B: Join(SetCategory, RetractableTo A)

  PAT ==> Pattern S
  PRA ==> PatternMatchResult(S, A)
  PRB ==> PatternMatchResult(S, B)
  REC ==> Record(pat:PAT, res:PRA)

  Exports ==> with
    fixPredicate: (B -> Boolean) -> (A -> Boolean)
      ++ fixPredicate(f) returns g defined by g(a) = f(a::B);
    patternMatch: (A, PAT, PRB)  -> PRB
      ++ patternMatch(expr, pat, res) matches the pattern pat to the
      ++ expression expr; res contains the variables of pat which
      ++ are already matched and their matches.
```

```
    ++ Note: this function handles type towers by changing the predicates
    ++ and calling the matching function provided by \spad{A}.

  Implementation ==> add
    import PatternMatchResultFunctions2(S, A, B)

    fixPred      : Any -> Union(Any, "failed")
    inA          : (PAT, PRB) -> Union(List A, "failed")
    fixPredicates: (PAT, PRB, PRA) -> Union(REC, "failed")
    fixList:(List PAT -> PAT, List PAT, PRB, PRA) -> Union(REC,"failed")

    fixPredicate f == f(#1::B)

    patternMatch(a, p, l) ==
      (u := fixPredicates(p, l, new())) case "failed" => failed()
      union(l, map(#1::B, patternMatch(a, (u::REC).pat, (u::REC).res)))

    inA(p, l) ==
      (u := getMatch(p, l)) case "failed" => empty()
      (r := retractIfCan(u::B)@Union(A, "failed")) case A => [r::A]
      "failed"

    fixList(fn, l, lb, la) ==
      ll:List(PAT) := empty()
      for x in l repeat
        (f := fixPredicates(x, lb, la)) case "failed" => return "failed"
        ll := concat((f::REC).pat, ll)
        la := (f::REC).res
      [fn ll, la]

    fixPred f ==
      (u:= retractIfCan(f)$AnyFunctions1(B -> Boolean)) case "failed" =>
                                                          "failed"
      g := fixPredicate(u::(B -> Boolean))
      coerce(g)$AnyFunctions1(A -> Boolean)

    fixPredicates(p, lb, la) ==
      (r:=retractIfCan(p)@Union(S,"failed")) case S or quoted? p =>[p,la]
      (u := isOp p) case Record(op:BasicOperator, arg:List PAT) =>
        ur := u::Record(op:BasicOperator, arg:List PAT)
        fixList((ur.op) #1, ur.arg, lb, la)
      (us := isPlus p) case List(PAT) =>
        fixList(reduce("+", #1), us::List(PAT), lb, la)
      (us := isTimes p) case List(PAT) =>
        fixList(reduce("*", #1), us::List(PAT), lb, la)
      (v := isQuotient p) case Record(num:PAT, den:PAT) =>
```

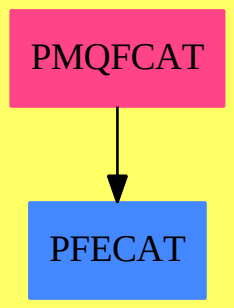
```
    vr := v::Record(num:PAT, den:PAT)
    (fn := fixPredicates(vr.num, lb, la)) case "failed" => "failed"
    la  := (fn::REC).res
    (fd := fixPredicates(vr.den, lb, la)) case "failed" => "failed"
    [(fn::REC).pat / (fd::REC).pat, (fd::REC).res]
  (w:= isExpt p) case Record(val:PAT,exponent:NonNegativeInteger) =>
    wr := w::Record(val:PAT, exponent: NonNegativeInteger)
    (f := fixPredicates(wr.val, lb, la)) case "failed" => "failed"
    [(f::REC).pat ** wr.exponent, (f::REC).res]
  (uu := isPower p) case Record(val:PAT, exponent:PAT) =>
    uur := uu::Record(val:PAT, exponent: PAT)
    (fv := fixPredicates(uur.val, lb, la)) case "failed" => "failed"
    la  := (fv::REC).res
    (fe := fixPredicates(uur.exponent, lb, la)) case "failed" =>
      "failed"
    [(fv::REC).pat ** (fe::REC).pat, (fe::REC).res]
  generic? p =>
    (ua := inA(p, lb)) case "failed" => "failed"
    lp := [if (h := fixPred g) case Any then h::Any else
                    return "failed" for g in predicates p]$List(Any)
    q := setPredicates(patternVariable(retract p, constant? p,
                                       optional? p, multiple? p), lp)
    [q, (empty?(ua::List A) => la; insertMatch(q,first(ua::List A), la))]
  error "Should not happen"
```

⟨*PMDOWN.dotabb*⟩≡

```
"PMDOWN" [color="#FF4488",href="bookvol10.4.pdf#nameddest=PMDOWN"]
"FLAGG" [color="#4488FF",href="bookvol10.2.pdf#nameddest=FLAGG"]
"PMDOWN" -> "FLAGG"
```

17.43 package PMQFCAT PatternMatchQuotientFieldCategory

17.44 PatternMatchQuotientFieldCategory



1.00

Exports:
patternMatch

⟨*package PMQFCAT PatternMatchQuotientFieldCategory*⟩≡

```
)abbrev package PMQFCAT PatternMatchQuotientFieldCategory
++ Pattern matching on quotient objects
++ Author: Manuel Bronstein
++ Date Created: 1 Dec 1989
++ Date Last Updated: 20 June 1991
++ Description:
++   This package provides pattern matching functions on quotients.
++ Keywords: pattern, matching, quotient, field.
PatternMatchQuotientFieldCategory(S,R,Q):Exports == Implementation where
  S: SetCategory
  R: Join(IntegralDomain, PatternMatchable S, ConvertibleTo Pattern S)
  Q: QuotientFieldCategory R

  PAT ==> Pattern S
  PRQ ==> PatternMatchResult(S, Q)

  Exports ==> with
    patternMatch: (Q, PAT, PRQ) -> PRQ
      ++ patternMatch(a/b, pat, res) matches the pattern pat to the
      ++ quotient a/b; res contains the variables of pat which
      ++ are already matched and their matches.

  Implementation ==> add
    import PatternMatchPushDown(S, R, Q)

    patternMatch(x, p, l) ==
```

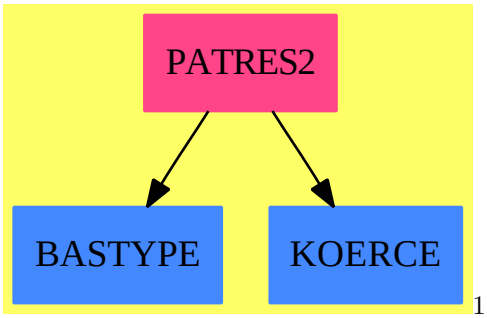
```
        generic? p => addMatch(p, x, l)
        (r := retractIfCan x)@Union(R, "failed") case R =>
          patternMatch(r::R, p, l)
        (u := isQuotient p) case Record(num:PAT, den:PAT) =>
          ur := u::Record(num:PAT, den:PAT)
          failed?(l := patternMatch(numer x, ur.num, l)) => l
          patternMatch(denom x, ur.den, l)
        failed()
```

⟨*PMQFCAT.dotabb*⟩≡

```
"PMQFCAT" [color="#FF4488",href="bookvol10.4.pdf#nameddest=PMQFCAT"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"PMQFCAT" -> "PFECAT"
```

17.45 package PATRES2 PatternMatchResultFunctions2

17.46 PatternMatchResultFunctions2



1.00

Exports:
map

⟨*package PATRES2 PatternMatchResultFunctions2*⟩≡

```
)abbrev package PATRES2 PatternMatchResultFunctions2
++ Lifts maps to pattern matching results
++ Author: Manuel Bronstein
++ Date Created: 1 Dec 1989
++ Date Last Updated: 14 Dec 1989
++ Description: Lifts maps to pattern matching results.
++ Keywords: pattern, matching.
PatternMatchResultFunctions2(R, A, B): Exports == Implementation where
  R: SetCategory
  A: SetCategory
  B: SetCategory

  Exports ==> with
    map: (A -> B, PatternMatchResult(R, A)) -> PatternMatchResult(R, B)
      ++ map(f, [(v1,a1),...,(vn,an)]) returns the matching result
      ++ [(v1,f(a1)),...,(vn,f(an))].

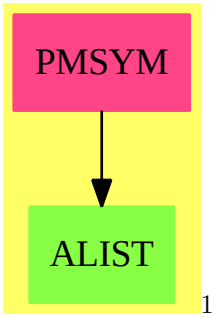
  Implementation ==> add
    map(f, r) ==
      failed? r => failed()
      construct [[rec.key, f(rec.entry)] for rec in destruct r]
```

⟨*PATRES2.dotabb*⟩≡

```
"PATRES2" [color="#FF4488",href="bookvol10.4.pdf#nameddest=PATRES2"]
"BASTYPE" [color="#4488FF",href="bookvol10.2.pdf#nameddest=BASTYPE"]
"KOERCE" [color="#4488FF",href="bookvol10.2.pdf#nameddest=KOERCE"]
"PATRES2" -> "BASTYPE"
"PATRES2" -> "KOERCE"
```

17.47 package PMSYM PatternMatchSymbol

17.48 PatternMatchSymbol



1.00

Exports:
patternMatch

⟨*package PMSYM PatternMatchSymbol*⟩≡

```
)abbrev package PMSYM PatternMatchSymbol
++ Pattern matching on symbols
++ Author: Manuel Bronstein
++ Date Created: 9 Jan 1990
++ Date Last Updated: 20 June 1991
++ Description:
++   This package provides pattern matching functions on symbols.
++ Keywords: pattern, matching, symbol.
PatternMatchSymbol(S:SetCategory): with
  patternMatch: (Symbol, Pattern S, PatternMatchResult(S, Symbol)) ->
                                          PatternMatchResult(S, Symbol)
    ++ patternMatch(expr, pat, res) matches the pattern pat to the
    ++ expression expr; res contains the variables of pat which
    ++ are already matched and their matches (necessary for recursion).
 == add
  import TopLevelPatternMatchControl

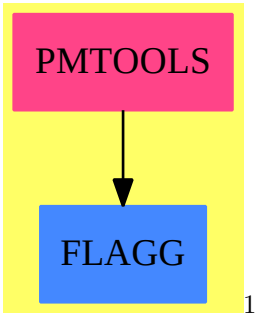
  patternMatch(s, p, l) ==
    generic? p  => addMatch(p, s, l)
    constant? p =>
      ((u := retractIfCan(p)@Union(Symbol, "failed")) case Symbol)
        and (u::Symbol) = s => l
      failed()
    failed()
```

⟨*PMSYM.dotabb*⟩≡

```
"PMSYM" [color="#FF4488",href="bookvol10.4.pdf#nameddest=PMSYM"]
"ALIST" [color="#88FF44",href="bookvol10.3.pdf#nameddest=ALIST"]
"PMSYM" -> "ALIST"
```

17.49 package PMTOOLS PatternMatchTools

17.50 PatternMatchTools



1.00

Exports:
patternMatch patternMatchTimes

⟨*package PMTOOLS PatternMatchTools*⟩≡

```
)abbrev package PMTOOLS PatternMatchTools
++ Tools for the pattern matcher
++ Author: Manuel Bronstein
++ Date Created: 13 Mar 1990
++ Date Last Updated: 4 February 1992
++ Description:
++   This package provides tools for the pattern matcher.
++ Keywords: pattern, matching, tools.
PatternMatchTools(S, R, P): Exports == Implementation where
  S: SetCategory
  R: Join(Ring, OrderedSet)
  P: Join(Ring, ConvertibleTo Pattern S, RetractableTo R)

  PAT ==> Pattern S
  PRS ==> PatternMatchResult(S, P)
  REC ==> Record(res:PRS, s:List P)
  RC  ==> Record(pat:List PAT, s:List P)

  Exports ==> with
    patternMatch: (List P, List PAT, List P -> P, PRS,
                                          (P, PAT, PRS) -> PRS) -> PRS
      ++ patternMatch(lsubj, lpat, op, res, match) matches the list
      ++ of patterns lpat to the list of subjects lsubj, allowing for
      ++ commutativity; op is the operator such that op(lpat) should
      ++ match op(lsubj) at the end, r contains the previous matches,
      ++ and match is a pattern-matching function on P.
    patternMatchTimes: (List P, List PAT, PRS,
```

```
                                                (P, PAT, PRS) -> PRS) -> PRS
      ++ patternMatchTimes(lsubj, lpat, res, match) matches the
      ++ product of patterns \spad{reduce(*,lpat)}
      ++ to the product of subjects \spad{reduce(*,lsubj)};
      ++ r contains the previous matches
      ++ and match is a pattern-matching function on P.

  Implementation ==> add
    import PatternFunctions1(S, P)

    preprocessList: (PAT, List P, PRS) -> Union(List P, "failed")
    selBestGen    : List PAT -> List PAT
    negConstant   : List P -> Union(P, "failed")
    findMatch     : (PAT, List P, PRS, P, (P, PAT, PRS) -> PRS) -> REC
    tryToMatch    : (List PAT, REC, P, (P, PAT, PRS) -> PRS) ->
                                                    Union(REC, "failed")
    filterMatchedPatterns: (List PAT, List P, PRS) -> Union(RC, "failed")

    mn1 := convert(-1::P)@Pattern(S)

    negConstant l ==
      for x in l repeat
        ((r := retractIfCan(x)@Union(R, "failed")) case R) and
          (r::R < 0) => return x
      "failed"

-- tries to match the list of patterns lp to the list of subjects rc.s
-- with rc.res being the list of existing matches.
-- updates rc with the new result and subjects still to match
    tryToMatch(lp, rc, ident, pmatch) ==
      rec:REC := [l := rc.res, ls := rc.s]
      for p in lp repeat
        rec := findMatch(p, ls, l, ident, pmatch)
        failed?(l := rec.res) => return "failed"
        ls := rec.s
      rec

-- handles -1 in the pattern list.
    patternMatchTimes(ls, lp, l, pmatch) ==
      member?(mn1, lp) =>
        (u := negConstant ls) case "failed" => failed()
        if (u::P ^= -1::P) then ls := concat(-u::P, ls)
        patternMatch(remove(u::P,ls), remove(mn1,lp), */#1, l, pmatch)
      patternMatch(ls, lp, */#1, l, pmatch)

-- finds a match for p in ls, try not to match to a "bad" value
```

```
    findMatch(p, ls, l, ident, pmatch) ==
      bad:List(P) :=
        generic? p => setIntersection(badValues p, ls)
        empty()
      l1:PRS := failed()
      for x in setDifference(ls, bad)
        while (t := x; failed?(l1 := pmatch(x, p, l))) repeat 0
      failed? l1 =>
        for x in bad
          while (t := x; failed?(l1 := pmatch(x, p, l))) repeat 0
        failed? l1 => [addMatchRestricted(p, ident, l, ident), ls]
        [l1, remove(t, ls)]
      [l1, remove(t, ls)]

  -- filters out pattern if it's generic and already matched.
    preprocessList(pattern, ls, l) ==
      generic? pattern =>
        (u := getMatch(pattern, l)) case P =>
          member?(u::P, ls) => [u::P]
          "failed"
        empty()
      empty()

  -- take out already matched generic patterns
    filterMatchedPatterns(lp, ls, l) ==
      for p in lp repeat
        (rc := preprocessList(p, ls, l)) case "failed" => return "failed"
        if not empty?(rc::List(P)) then
          lp := remove(p,  lp)
          ls := remove(first(rc::List(P)), ls)
      [lp, ls]

  -- select a generic pattern with no predicate if possible
    selBestGen l ==
      ans := empty()$List(PAT)
      for p in l | generic? p repeat
        ans := [p]
        not hasPredicate? p => return ans
      ans

  -- matches unordered lists ls and lp
    patternMatch(ls, lp, op, l, pmatch) ==
      ident := op empty()
      (rc := filterMatchedPatterns(lp, ls, l)) case "failed" => return failed()
      lp := (rc::RC).pat
      ls := (rc::RC).s
```

```
empty? lp => l
#(lpm := select(optional?, lp)) > 1 =>
  error "More than one optional pattern in sum/product"
(#ls + #lpm) < #lp => failed()
if (not empty? lpm) and (#ls + 1 = #lp) then
  lp := remove(first lpm, lp)
  failed?(l := addMatch(first lpm, ident, l)) => return l
#(lpm := select(multiple?, lp)) > 1 =>
  error "More than one expandable pattern in sum/product"
#ls > #lp and empty? lpm and empty?(lpm := selBestGen lp) =>
  failed()
if not empty? lpm then lp := remove(first lpm, lp)
-- this is the order in which we try to match predicates
-- l1 = constant patterns (i.e. 'x, or sin('x))
l1 := select(constant?, lp)
-- l2 = patterns with a predicate attached to them
l2 := select(hasPredicate? #1 and not constant? #1, lp)
-- l3 = non-generic patterns without predicates
l3 := sort_!(depth(#1) > depth(#2),
  select(not(hasPredicate? #1 or generic? #1 or constant? #1),lp))
-- l4 = generic patterns with predicates
l4 := select(generic? #1 and
                       not(hasPredicate? #1 or constant? #1), lp)
rec:REC := [l, ls]
(u := tryToMatch(l1, rec, ident, pmatch)) case "failed" =>
  failed()
(u := tryToMatch(l2, u::REC, ident, pmatch)) case "failed" =>
  failed()
(u := tryToMatch(l3, u::REC, ident, pmatch)) case "failed" =>
  failed()
rec := u::REC
(rc := filterMatchedPatterns(l4,rec.s,rec.res)) case "failed" => failed()
rec := [rec.res, (rc::RC).s]
(u := tryToMatch((rc::RC).pat,rec,ident,pmatch)) case "failed" => failed()
rec := u::REC
l := rec.res
ls := rec.s
empty? lpm =>
  empty? ls => l
  failed()
addMatch(first lpm, op ls, l)
```

⟨*PMTOOLS.dotabb*⟩≡

```
"PMTOOLS" [color="#FF4488",href="bookvol10.4.pdf#nameddest=PMTOOLS"]
"FLAGG" [color="#4488FF",href="bookvol10.2.pdf#nameddest=FLAGG"]
"PMTOOLS" -> "FLAGG"
```

17.51 package PERMAN Permanent

⟨*Permanent.input*⟩≡

```
)set break resume
)spool Permanent.output
)set message test on
)set message auto off
)clear all
--S 1 of 3
kn n ==
  r : MATRIX INT := new(n,n,1)
  for i in 1..n repeat
    r.i.i := 0
  r
--R
--R                                                                    Type: Void
--E 1

--S 2 of 3
permanent(kn(5) :: SQMATRIX(5,INT))
--R
--R   Compiling function kn with type PositiveInteger -> Matrix Integer
--R
--R   (2)  44
--R                                                       Type: PositiveInteger
--E 2

--S 3 of 3
[permanent(kn(n) :: SQMATRIX(n,INT)) for n in 1..13]
--R
--R   Cannot compile conversion for types involving local variables. In
--R      particular, could not compile the expression involving ::
--R      SQMATRIX(n,INT)
--R   AXIOM will attempt to step through and interpret the code.
--R
--R   (3)
--R   [0,1,2,9,44,265,1854,14833,133496,1334961,14684570,176214841,2290792932]
--R                                                  Type: List NonNegativeInteger
--E 3
)spool
)lisp (bye)
```

⟨*Permanent.help*⟩≡

```
==========================================================================
Permanent examples
==========================================================================

The package Permanent provides the function permanent for square
matrices.  The permanent of a square matrix can be computed in the
same way as the determinant by expansion of minors except that for the
permanent the sign for each element is 1, rather than being 1 if the
row plus column indices is positive and -1 otherwise.  This function
is much more difficult to compute efficiently than the determinant.
An example of the use of permanent is the calculation of the n-th
derangement number, defined to be the number of different possibilities
for n couples to dance but never with their own spouse.

Consider an n by n matrix with entries 0 on the diagonal and 1 elsewhere.
Think of the rows as one-half of each couple (for example, the males) and
the columns the other half.  The permanent of such a matrix gives the
desired derangement number.

  kn n ==
    r : MATRIX INT := new(n,n,1)
    for i in 1..n repeat
      r.i.i := 0
    r

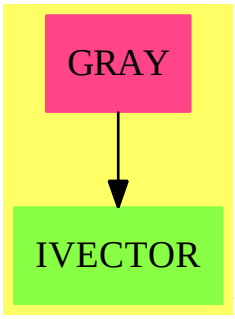
Here are some derangement numbers, which you see grow quite fast.

  permanent(kn(5) :: SQMATRIX(5,INT))

  [permanent(kn(n) :: SQMATRIX(n,INT)) for n in 1..13]

See Also:
o )show Permanent
```

17.52 Permanent



1.00

Exports:
permanent

⟨*package PERMAN Permanent*⟩≡
```
)abbrev package PERMAN Permanent
++ Authors: Johannes Grabmeier, Oswald Gschnitzer
++ Date Created: 7 August 1989
++ Date Last Updated: 23 August 1990
++ Basic Operations: permanent
++ Related Constructors: GrayCode
++ Also See: MatrixLinearAlgebraFunctions
++ AMS Classifications:
++ Keywords: permanent
++ References:
++  Henryk Minc: Evaluation of Permanents,
++    Proc. of the Edinburgh Math. Soc.(1979), 22/1 pp 27-32.
++  Nijenhuis and Wilf : Combinatorical Algorithms, Academic
++    Press, New York 1978.
++  S.G.Williamson, Combinatorics for Computer Science,
++    Computer Science Press, 1985.
++ Description:
++  Permanent implements the functions {\em permanent}, the
++  permanent for square matrices.
Permanent(n : PositiveInteger, R : Ring with commutative("*")):
 public == private where
  I  ==> Integer
  L  ==> List
  V  ==> Vector
  SM  ==> SquareMatrix(n,R)
  VECTPKG1 ==> VectorPackage1(I)
  NNI ==> NonNegativeInteger
  PI ==> PositiveInteger
  GRAY ==> GrayCode
```

```
public ==> with

  permanent:  SM  -> R
    ++ permanent(x) computes the permanent of a square matrix x.
    ++ The {\em permanent} is equivalent to
    ++ the \spadfun{determinant} except that coefficients have
    ++ no change of sign. This function
    ++ is much more difficult to compute than the
    ++ {\em determinant}. The formula used is by H.J. Ryser,
    ++ improved by [Nijenhuis and Wilf, Ch. 19].
    ++ Note: permanent(x) choose one of three algorithms, depending
    ++ on the underlying ring R and on n, the number of rows (and
    ++ columns) of x:\begin{items}
    ++ \item 1. if 2 has an inverse in R we can use the algorithm of
    ++    [Nijenhuis and Wilf, ch.19,p.158]; if 2 has no inverse,
    ++    some modifications are necessary:
    ++ \item 2. if {\em n > 6} and R is an integral domain with characteristic
    ++    different from 2 (the algorithm works if and only 2 is not a
    ++    zero-divisor of R and {\em characteristic()$R ^= 2},
    ++    but how to check that for any given R ?),
    ++    the local function {\em permanent2} is called;
    ++ \item 3. else, the local function {\em permanent3} is called
    ++    (works for all commutative rings R).
    ++ \end{items}

private ==> add

  -- local functions:

  permanent2:  SM  -> R

  permanent3:  SM  -> R

  x : SM
  a,b : R
  i,j,k,l : I

  permanent3(x) ==
    -- This algorithm is based upon the principle of inclusion-
    -- exclusion. A Gray-code is used to generate the subsets of
    -- 1,... ,n. This reduces the number of additions needed in
    -- every step.
    sgn : R := 1
    k : R
    a := 0$R
    vv : V V I := firstSubsetGray(n)$GRAY
```

```
    -- For the meaning of the elements of vv, see GRAY.
  w : V R := new(n,0$R)
  j := 1   -- Will be the number of the element changed in subset
  while j ^= (n+1) repeat  -- we sum over all subsets of (1,...,n)
    sgn := -sgn
    b := sgn
    if vv.1.j = 1 then k := -1
    else k := 1  -- was that element deleted(k=-1) or added(k=1)?
    for i in 1..(n::I) repeat
      w.i :=  w.i +$R k *$R  x(i,j)
      b := b *$R w.i
    a := a +$R b
    vv := nextSubsetGray(vv,n)$GRAY
    j := vv.2.1
  if odd?(n) then a := -a
  a


permanent(x) ==
  -- If 2 has an inverse in R, we can spare half of the calcu-
  -- lation needed in "permanent3": This is the algorithm of
  -- [Nijenhuis and Wilf, ch.19,p.158]
  n = 1 => x(1,1)
  two : R := (2:I) :: R
  half : Union(R,"failed") := recip(two)
  if (half case "failed") then
    if n < 7 then return permanent3(x)
    else return permanent2(x)
  sgn : R := 1
  a := 0$R
  w : V R := new(n,0$R)
  -- w.i will be at first x.i and later lambda.i in
  -- [Nijenhuis and Wilf, p.158, (24a) resp.(26)].
  rowi : V R := new(n,0$R)
  for i in 1..n repeat
    rowi := row(x,i) :: V R
    b := 0$R
    for j in 1..n repeat
      b := b + rowi.j
    w.i := rowi(n) - (half*b)$R
  vv : V V I := firstSubsetGray((n-1): PI)$GRAY
   -- For the meaning of the elements of vv, see GRAY.
  n :: I
  b := 1
  for i in 1..n repeat
    b := b * w.i
```

```
    a := a+b
    j := 1   -- Will be the number of the element changed in subset
    while j ^= n repeat  -- we sum over all subsets of (1,...,n-1)
      sgn := -sgn
      b := sgn
      if vv.1.j = 1 then k := -1
      else k := 1  -- was that element deleted(k=-1) or added(k=1)?
      for i in 1..n repeat
        w.i :=  w.i +$R k *$R  x(i,j)
        b := b *$R w.i
      a := a +$R b
      vv := nextSubsetGray(vv,(n-1) : PI)$GRAY
      j := vv.2.1
    if not odd?(n) then a := -a
    two * a

  permanent2(x) ==
    c : R := 0
    sgn : R := 1
    if (not (R has IntegralDomain))
      -- or (characteristic()$R = (2:NNI))
      -- compiler refuses to compile the line above !!
      or  (sgn + sgn = c)
    then return permanent3(x)
    -- This is a slight modification of permanent which is
    -- necessary if 2 is not zero or a zero-divisor in R, but has
    -- no inverse in R.
    n = 1 => x(1,1)
    two : R := (2:I) :: R
    a := 0$R
    w : V R := new(n,0$R)
    -- w.i will be at first x.i and later lambda.i in
    -- [Nijenhuis and Wilf, p.158, (24a) resp.(26)].
    rowi : V R := new(n,0$R)
    for i in 1..n repeat
      rowi := row(x,i) :: V R
      b := 0$R
      for j in 1..n repeat
        b := b + rowi.j
      w.i := (two*(rowi(n)))$R - b
    vv : V V I := firstSubsetGray((n-1): PI)$GRAY
    n :: I
    b := 1
    for i in 1..n repeat
      b := b *$R w.i
    a := a +$R b
```

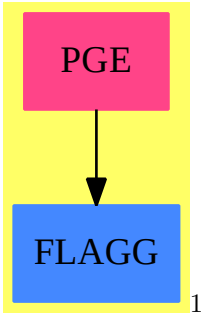
```
    j := 1   -- Will be the number of the element changed in subset
    while j ^= n repeat  -- we sum over all subsets of (1,...,n-1)
      sgn := -sgn
      b := sgn
      if vv.1.j = 1 then k := -1
      else k := 1  -- was that element deleted(k=-1) or added(k=1)?
      c := k * two
      for i in 1..n repeat
        w.i :=  w.i +$R c *$R x(i,j)
        b := b *$R w.i
      a := a +$R b
      vv := nextSubsetGray(vv,(n-1) : PI)$GRAY
      j := vv.2.1
    if not odd?(n) then a := -a
    b := two ** ((n-1):NNI)
    (a exquo b) :: R
```

⟨*PERMAN.dotabb*⟩≡

```
"PERMAN" [color="#FF4488",href="bookvol10.4.pdf#nameddest=PERMAN"]
"IVECTOR" [color="#88FF44",href="bookvol10.3.pdf#nameddest=IVECTOR"]
"PERMAN" -> "IVECTOR"
```

17.53 package PGE PermutationGroupExamples

17.54 PermutationGroupExamples



1.00

Exports:

abelianGroup	alternatingGroup	cyclicGroup	dihedralGroup	janko2
mathieu11	mathieu12	mathieu22	mathieu23	mathieu24
rubiksGroup	symmetricGroup	youngGroup		

⟨*package PGE PermutationGroupExamples*⟩≡

```
)abbrev package PGE PermutationGroupExamples
++ Authors: M. Weller, G. Schneider, J. Grabmeier
++ Date Created: 20 February 1990
++ Date Last Updated: 09 June 1990
++ Basic Operations:
++ Related Constructors:
++ Also See:
++ AMS Classifications:
++ Keywords:
++ References:
++  J. Conway, R. Curtis, S. Norton, R. Parker, R. Wilson:
++   Atlas of Finite Groups, Oxford, Clarendon Press, 1987
++ Description:
++   PermutationGroupExamples provides permutation groups for
++   some classes of groups: symmetric, alternating, dihedral, cyclic,
++   direct products of cyclic, which are in fact the finite abelian groups
++   of symmetric groups called Young subgroups.
++   Furthermore, Rubik's group as permutation group of 48 integers and a list
++   of sporadic simple groups derived from the atlas of finite groups.

PermutationGroupExamples():public == private where

    L          ==> List
    I          ==> Integer
```

```
PI         ==> PositiveInteger
NNI        ==> NonNegativeInteger
PERM       ==> Permutation
PERMGRP   ==> PermutationGroup

public ==> with

  symmetricGroup:       PI        -> PERMGRP I
    ++ symmetricGroup(n) constructs the symmetric group {\em Sn}
    ++ acting on the integers 1,...,n, generators are the
    ++ {\em n}-cycle {\em (1,...,n)} and the 2-cycle {\em (1,2)}.
  symmetricGroup:       L I       -> PERMGRP I
    ++ symmetricGroup(li) constructs the symmetric group acting on
    ++ the integers in the list {\em li}, generators are the
    ++ cycle given by {\em li} and the 2-cycle {\em (li.1,li.2)}.
    ++ Note: duplicates in the list will be removed.
  alternatingGroup:     PI        -> PERMGRP I
    ++ alternatingGroup(n) constructs the alternating group {\em An}
    ++ acting on the integers 1,...,n,  generators are in general the
    ++ {\em n-2}-cycle {\em (3,...,n)} and the 3-cycle {\em (1,2,3)}
    ++ if n is odd and the product of the 2-cycle {\em (1,2)} with
    ++ {\em n-2}-cycle {\em (3,...,n)} and the 3-cycle {\em (1,2,3)}
    ++ if n is even.
  alternatingGroup:     L I       -> PERMGRP I
    ++ alternatingGroup(li) constructs the alternating group acting
    ++ on the integers in the list {\em li}, generators are in general the
    ++ {\em n-2}-cycle {\em (li.3,...,li.n)} and the 3-cycle
    ++ {\em (li.1,li.2,li.3)}, if n is odd and
    ++ product of the 2-cycle {\em (li.1,li.2)} with
    ++ {\em n-2}-cycle {\em (li.3,...,li.n)} and the 3-cycle
    ++ {\em (li.1,li.2,li.3)}, if n is even.
    ++ Note: duplicates in the list will be removed.
  abelianGroup:         L PI      -> PERMGRP I
    ++ abelianGroup([n1,...,nk]) constructs the abelian group that
    ++ is the direct product of cyclic groups with order {\em ni}.
  cyclicGroup:          PI        -> PERMGRP I
    ++ cyclicGroup(n) constructs the cyclic group of order n acting
    ++ on the integers 1,...,n.
  cyclicGroup:          L I       -> PERMGRP I
    ++ cyclicGroup([i1,...,ik]) constructs the cyclic group of
    ++ order k acting on the integers {\em i1},...,{\em ik}.
    ++ Note: duplicates in the list will be removed.
  dihedralGroup:        PI        -> PERMGRP I
    ++ dihedralGroup(n) constructs the dihedral group of order 2n
    ++ acting on integers 1,...,N.
  dihedralGroup:        L I       -> PERMGRP I
```

```
  ++ dihedralGroup([i1,...,ik]) constructs the dihedral group of
  ++ order 2k acting on the integers out of {\em i1},...,{\em ik}.
  ++ Note: duplicates in the list will be removed.
mathieu11:            L I       -> PERMGRP I
  ++ mathieu11(li) constructs the mathieu group acting on the 11
  ++ integers given in the list {\em li}.
  ++ Note: duplicates in the list will be removed.
  ++ error, if {\em li} has less or more than 11 different entries.
mathieu11:            ()        -> PERMGRP I
  ++ mathieu11 constructs the mathieu group acting on the
  ++ integers 1,...,11.
mathieu12:            L I       -> PERMGRP I
  ++ mathieu12(li) constructs the mathieu group acting on the 12
  ++ integers given in the list {\em li}.
  ++ Note: duplicates in the list will be removed
  ++ Error: if {\em li} has less or more than 12 different entries.
mathieu12:            ()        -> PERMGRP I
  ++ mathieu12 constructs the mathieu group acting on the
  ++ integers 1,...,12.
mathieu22:            L I       -> PERMGRP I
  ++ mathieu22(li) constructs the mathieu group acting on the 22
  ++ integers given in the list {\em li}.
  ++ Note: duplicates in the list will be removed.
  ++ Error: if {\em li} has less or more than 22 different entries.
mathieu22:            ()        -> PERMGRP I
  ++ mathieu22 constructs the mathieu group acting on the
  ++ integers 1,...,22.
mathieu23:            L I       -> PERMGRP I
  ++ mathieu23(li) constructs the mathieu group acting on the 23
  ++ integers given in the list {\em li}.
  ++ Note: duplicates in the list will be removed.
  ++ Error: if {\em li} has less or more than 23 different entries.
mathieu23:            ()        -> PERMGRP I
  ++ mathieu23 constructs the mathieu group acting on the
  ++ integers 1,...,23.
mathieu24:            L I       -> PERMGRP I
  ++ mathieu24(li) constructs the mathieu group acting on the 24
  ++ integers given in the list {\em li}.
  ++ Note: duplicates in the list will be removed.
  ++ Error: if {\em li} has less or more than 24 different entries.
mathieu24:            ()        -> PERMGRP I
  ++ mathieu24 constructs the mathieu group acting on the
  ++ integers 1,...,24.
janko2:               L I       -> PERMGRP I
  ++ janko2(li) constructs the janko group acting on the 100
  ++ integers given in the list {\em li}.
```

```
  ++ Note: duplicates in the list will be removed.
  ++ Error: if {\em li} has less or more than 100 different entries
janko2:               ()        -> PERMGRP I
  ++ janko2 constructs the janko group acting on the
  ++ integers 1,...,100.
rubiksGroup:          ()        -> PERMGRP I
  ++ rubiksGroup constructs the permutation group representing
  ++ Rubic's Cube acting on integers {\em 10*i+j} for
  ++ {\em 1 <= i <= 6}, {\em 1 <= j <= 8}.
  ++ The faces of Rubik's Cube are labelled in the obvious way
  ++ Front, Right, Up, Down, Left, Back and numbered from 1 to 6
  ++ in this given ordering, the pieces on each face
  ++ (except the unmoveable center piece) are clockwise numbered
  ++ from 1 to 8 starting with the piece in the upper left
  ++ corner. The moves of the cube are represented as permutations
  ++ on these pieces, represented as a two digit
  ++ integer {\em ij} where i is the numer of theface (1 to 6)
  ++ and j is the number of the piece on this face.
  ++ The remaining ambiguities are resolved by looking
  ++ at the 6 generators, which represent a 90 degree turns of the
  ++ faces, or from the following pictorial description.
  ++ Permutation group representing Rubic's Cube acting on integers
  ++ 10*i+j for 1 <= i <= 6, 1 <= j <=8.
  ++
  ++ \begin{verbatim}
  ++ Rubik's Cube:   +-----+ +-- B   where: marks Side # :
  ++                / U   /|/
  ++               /     / |         F(ront)    <->    1
  ++       L -->  +-----+ R|         R(ight)    <->    2
  ++              |     |  +         U(p)       <->    3
  ++              |  F  | /          D(own)     <->    4
  ++              |     |/           L(eft)     <->    5
  ++              +-----+            B(ack)     <->    6
  ++                 ^
  ++                 |
  ++                 D
  ++
  ++ The Cube's surface:
  ++                                 The pieces on each side
  ++             +---+               (except the unmoveable center
  ++             |567|               piece) are clockwise numbered
  ++             |4U8|               from 1 to 8 starting with the
  ++             |321|               piece in the upper left
  ++         +---+---+---+           corner (see figure on the
  ++         |781|123|345|           left).  The moves of the cube
  ++         |6L2|8F4|2R6|           are represented as
```

```
    ++         |543|765|187|          permutations on these pieces.
    ++         +---+---+---+          Each of the pieces is
    ++             |123|              represented as a two digit
    ++             |8D4|              integer ij where i is the
    ++             |765|              # of the side ( 1 to 6 for
    ++             +---+              F to B (see table above ))
    ++             |567|              and j is the # of the piece.
    ++             |4B8|
    ++             |321|
    ++             +---+
    ++ \end{verbatim}
  youngGroup:           L I      -> PERMGRP I
    ++ youngGroup([n1,...,nk]) constructs the direct product of the
    ++ symmetric groups {\em Sn1},...,{\em Snk}.
  youngGroup:    Partition        -> PERMGRP I
    ++ youngGroup(lambda) constructs the direct product of the symmetric
    ++ groups given by the parts of the partition {\em lambda}.

private ==> add

  -- import the permutation and permutation group domains:

  import PERM I
  import PERMGRP I

  -- import the needed map function:

  import ListFunctions2(L L I,PERM I)
  -- the internal functions:

  llli2gp(l:L L L I):PERMGRP I ==
    --++ Converts an list of permutations each represented by a list
    --++ of cycles ( each of them represented as a list of Integers )
    --++ to the permutation group generated by these permutations.
    (map(cycles,l))::PERMGRP I

  li1n(n:I):L I ==
    --++ constructs the list of integers from 1 to n
    [i for i in 1..n]

  -- definition of the exported functions:
  youngGroup(l:L I):PERMGRP I ==
    gens:= nil()$(L L L I)
    element:I:= 1
    for n in l | n > 1 repeat
      gens:=cons(list [i for i in element..(element+n-1)], gens)
```

```
    if n >= 3 then gens := cons([[element,element+1]],gens)
    element:=element+n
  llli2gp
    #gens = 0 => [[[1]]]
    gens

youngGroup(lambda : Partition):PERMGRP I ==
  youngGroup(convert(lambda)$Partition)

rubiksGroup():PERMGRP I ==
  -- each generator represents a 90 degree turn of the appropriate
  -- side.
  f:L L I:=
   [[11,13,15,17],[12,14,16,18],[51,31,21,41],[53,33,23,43],[52,32,22,42]]
  r:L L I:=
   [[21,23,25,27],[22,24,26,28],[13,37,67,43],[15,31,61,45],[14,38,68,44]]
  u:L L I:=
   [[31,33,35,37],[32,34,36,38],[13,51,63,25],[11,57,61,23],[12,58,62,24]]
  d:L L I:=
   [[41,43,45,47],[42,44,46,48],[17,21,67,55],[15,27,65,53],[16,28,66,54]]
  l:L L I:=
   [[51,53,55,57],[52,54,56,58],[11,41,65,35],[17,47,63,33],[18,48,64,34]]
  b:L L I:=
   [[61,63,65,67],[62,64,66,68],[45,25,35,55],[47,27,37,57],[46,26,36,56]]
  llli2gp [f,r,u,d,l,b]

mathieu11(l:L I):PERMGRP I ==
-- permutations derived from the ATLAS
  l:=removeDuplicates l
  #l ^= 11 => error "Exactly 11 integers for mathieu11 needed !"
  a:L L I:=[[l.1,l.10],[l.2,l.8],[l.3,l.11],[l.5,l.7]]
  llli2gp [a,[[l.1,l.4,l.7,l.6],[l.2,l.11,l.10,l.9]]]

mathieu11():PERMGRP I == mathieu11 li1n 11

mathieu12(l:L I):PERMGRP I ==
-- permutations derived from the ATLAS
  l:=removeDuplicates l
  #l ^= 12 => error "Exactly 12 integers for mathieu12 needed !"
  a:L L I:=
    [[l.1,l.2,l.3,l.4,l.5,l.6,l.7,l.8,l.9,l.10,l.11]]
  llli2gp [a,[[l.1,l.6,l.5,l.8,l.3,l.7,l.4,l.2,l.9,l.10],[l.11,l.12]]]

mathieu12():PERMGRP I == mathieu12 li1n 12

mathieu22(l:L I):PERMGRP I ==
```

```
-- permutations derived from the ATLAS
  l:=removeDuplicates l
  #l ^= 22 => error "Exactly 22 integers for mathieu22 needed !"
  a:L L I:=[[1.1,1.2,1.4,1.8,1.16,1.9,1.18,1.13,1.3,1.6,1.12],   _
    [1.5,1.10,1.20,1.17,1.11,1.22,1.21,1.19,1.15,1.7,1.14]]
  b:L L I:= [[1.1,1.2,1.6,1.18],[1.3,1.15],[1.5,1.8,1.21,1.13],   _
    [1.7,1.9,1.20,1.12],[1.10,1.16],[1.11,1.19,1.14,1.22]]
  llli2gp [a,b]

mathieu22():PERMGRP I == mathieu22 li1n 22

mathieu23(l:L I):PERMGRP I ==
-- permutations derived from the ATLAS
  l:=removeDuplicates l
  #l ^= 23 => error "Exactly 23 integers for mathieu23 needed !"
  a:L L I:= [[1.1,1.2,1.3,1.4,1.5,1.6,1.7,1.8,1.9,1.10,1.11,1.12,1.13,1.14,_
             1.15,1.16,1.17,1.18,1.19,1.20,1.21,1.22,1.23]]
  b:L L I:= [[1.2,1.16,1.9,1.6,1.8],[1.3,1.12,1.13,1.18,1.4],              _
             [1.7,1.17,1.10,1.11,1.22],[1.14,1.19,1.21,1.20,1.15]]
  llli2gp [a,b]

mathieu23():PERMGRP I == mathieu23 li1n 23

mathieu24(l:L I):PERMGRP I ==
-- permutations derived from the ATLAS
  l:=removeDuplicates l
  #l ^= 24 => error "Exactly 24 integers for mathieu24 needed !"
  a:L L I:= [[1.1,1.16,1.10,1.22,1.24],[1.2,1.12,1.18,1.21,1.7],          _
             [1.4,1.5,1.8,1.6,1.17],[1.9,1.11,1.13,1.19,1.15]]
  b:L L I:= [[1.1,1.22,1.13,1.14,1.6,1.20,1.3,1.21,1.8,1.11],[1.2,1.10],  _
             [1.4,1.15,1.18,1.17,1.16,1.5,1.9,1.19,1.12,1.7],[1.23,1.24]]
  llli2gp [a,b]

mathieu24():PERMGRP I == mathieu24 li1n 24

janko2(l:L I):PERMGRP I ==
-- permutations derived from the ATLAS
  l:=removeDuplicates l
  #l ^= 100 => error "Exactly 100 integers for janko2 needed !"
  a:L L I:=[                                                          _
           [1.2,1.3,1.4,1.5,1.6,1.7,1.8],                             _
           [1.9,1.10,1.11,1.12,1.13,1.14,1.15],                       _
           [1.16,1.17,1.18,1.19,1.20,1.21,1.22],                      _
           [1.23,1.24,1.25,1.26,1.27,1.28,1.29],                      _
           [1.30,1.31,1.32,1.33,1.34,1.35,1.36],                      _
           [1.37,1.38,1.39,1.40,1.41,1.42,1.43],                      _
```

```
            [1.44,1.45,1.46,1.47,1.48,1.49,1.50],                    _
            [1.51,1.52,1.53,1.54,1.55,1.56,1.57],                    _
            [1.58,1.59,1.60,1.61,1.62,1.63,1.64],                    _
            [1.65,1.66,1.67,1.68,1.69,1.70,1.71],                    _
            [1.72,1.73,1.74,1.75,1.76,1.77,1.78],                    _
            [1.79,1.80,1.81,1.82,1.83,1.84,1.85],                    _
            [1.86,1.87,1.88,1.89,1.90,1.91,1.92],                    _
            [1.93,1.94,1.95,1.96,1.97,1.98,1.99] ]
  b:L L I:=[
          [1.1,1.74,1.83,1.21,1.36,1.77,1.44,1.80,1.64,1.2,1.34,1.75,1.48,1
          [1.3,1.15,1.31,1.52,1.19,1.11,1.73,1.79,1.26,1.56,1.41,1.99,1.39,
          [1.4,1.57,1.86,1.63,1.85,1.95,1.82,1.97,1.98,1.81,1.8,1.69,1.38,1
          [1.5,1.66,1.49,1.59,1.61],_
          [1.6,1.68,1.89,1.94,1.92,1.20,1.13,1.54,1.24,1.51,1.87,1.27,1.76,
          [1.7,1.72,1.22,1.35,1.30,1.70,1.47,1.62,1.45,1.46,1.40,1.28,1.65,
          [1.9,1.71,1.37,1.91,1.18,1.55,1.96,1.60,1.16,1.53,1.50,1.25,1.32,
          [1.10,1.78,1.88,1.29,1.12] ]
  llli2gp [a,b]

janko2():PERMGRP I == janko2 li1n 100

abelianGroup(l:L PI):PERMGRP I ==
  gens:= nil()$(L L L I)
  element:I:= 1
  for n in l | n > 1 repeat
    gens:=cons( list [i for i in element..(element+n-1) ], gens )
    element:=element+n
  llli2gp
    #gens = 0 => [[[1]]]
    gens

alternatingGroup(l:L I):PERMGRP I ==
  l:=removeDuplicates l
  #l = 0 =>
    error "Cannot construct alternating group on empty set"
  #l < 3 => llli2gp [[[l.1]]]
  #l = 3 => llli2gp [[[l.1,l.2,l.3]]]
  tmp:= [l.i for i in 3..(#l)]
  gens:L L L I:=[[tmp],[[l.1,l.2,l.3]]]
  odd?(#l) => llli2gp gens
  gens.1 := cons([l.1,l.2],gens.1)
  llli2gp gens

alternatingGroup(n:PI):PERMGRP I == alternatingGroup li1n n

symmetricGroup(l:L I):PERMGRP I ==
```

```
    l:=removeDuplicates l
    #l = 0 => error "Cannot construct symmetric group on empty set !"
    #l < 3 => llli2gp [[l]]
    llli2gp [[l],[[l.1,l.2]]]

  symmetricGroup(n:PI):PERMGRP I == symmetricGroup li1n n

  cyclicGroup(l:L I):PERMGRP I ==
    l:=removeDuplicates l
    #l = 0 => error "Cannot construct cyclic group on empty set"
    llli2gp [[l]]

  cyclicGroup(n:PI):PERMGRP I == cyclicGroup li1n n

  dihedralGroup(l:L I):PERMGRP I ==
    l:=removeDuplicates l
    #l < 3 => error "in dihedralGroup: Minimum of 3 elements needed !"
    tmp := [[l.i, l.(#l-i+1) ] for i in 1..(#l quo 2)]
    llli2gp [ [ l ], tmp ]

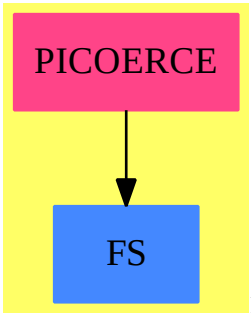
  dihedralGroup(n:PI):PERMGRP I ==
    n = 1 => symmetricGroup (2::PI)
    n = 2 => llli2gp [[[1,2]],[[3,4]]]
    dihedralGroup li1n n
```

⟨*PGE.dotabb*⟩≡

```
"PGE" [color="#FF4488",href="bookvol10.4.pdf#nameddest=PGE"]
"FLAGG" [color="#4488FF",href="bookvol10.2.pdf#nameddest=FLAGG"]
"PGE" -> "FLAGG"
```

17.55 package PICOERCE PiCoercions

17.56 PiCoercions



1.00

Exports:
coerce

⟨*package PICOERCE PiCoercions*⟩≡
```
)abbrev package PICOERCE PiCoercions
++ Coercions from %pi to symbolic or numeric domains
++ Author: Manuel Bronstein
++ Description:
++  Provides a coercion from the symbolic fractions in %pi with
++ integer coefficients to any Expression type.
++ Date Created: 21 Feb 1990
++ Date Last Updated: 21 Feb 1990
PiCoercions(R:Join(OrderedSet, IntegralDomain)): with
  coerce: Pi -> Expression R
    ++ coerce(f) returns f as an Expression(R).
 == add
  p2e: SparseUnivariatePolynomial Integer -> Expression R

  coerce(x:Pi):Expression(R) ==
    f := convert(x)@Fraction(SparseUnivariatePolynomial Integer)
    p2e(numer f) / p2e(denom f)

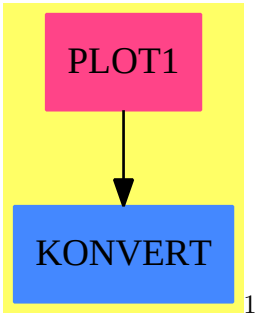
  p2e p ==
    map(#1::Expression(R), p)$SparseUnivariatePolynomialFunctions2(
        Integer, Expression R) (pi()$Expression(R))
```

⟨*PICOERCE.dotabb*⟩≡

```
"PICOERCE" [color="#FF4488",href="bookvol10.4.pdf#nameddest=PICOERCE"]
"FS" [color="#4488FF",href="bookvol10.2.pdf#nameddest=FS"]
"PICOERCE" -> "FS"
```

17.57 package PLOT1 PlotFunctions1

17.58 PlotFunctions1



1.00

Exports:
plotPolar plot

⟨*package PLOT1 PlotFunctions1*⟩≡

```
)abbrev package PLOT1 PlotFunctions1
++ Authors: R.T.M. Bronstein, C.J. Williamson
++ Date Created: Jan 1989
++ Date Last Updated: 4 Mar 1990
++ Basic Operations: plot, plotPolar
++ Related Constructors:
++ Also See:
++ AMS Classifications:
++ Keywords:
++ References:
++ Description: PlotFunctions1 provides facilities for plotting curves
++ where functions SF -> SF are specified by giving an expression
PlotFunctions1(S:ConvertibleTo InputForm): with
    plot : (S, Symbol, Segment DoubleFloat) -> Plot
      ++ plot(fcn,x,seg) plots the graph of \spad{y = f(x)} on a interval
    plot : (S, S, Symbol, Segment DoubleFloat) -> Plot
      ++ plot(f,g,t,seg) plots the graph of \spad{x = f(t)}, \spad{y = g(t)} as t
      ++ ranges over an interval.
    plotPolar : (S, Symbol, Segment DoubleFloat) -> Plot
      ++ plotPolar(f,theta,seg) plots the graph of \spad{r = f(theta)} as
      ++ theta ranges over an interval
    plotPolar : (S, Symbol) -> Plot
      ++ plotPolar(f,theta) plots the graph of \spad{r = f(theta)} as
      ++ theta ranges from 0 to 2 pi
  == add
    import MakeFloatCompiledFunction(S)
```

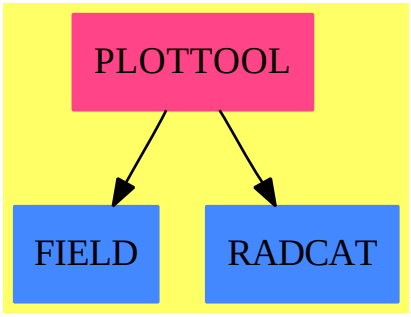
```
plot(f, x, xRange) == plot(makeFloatFunction(f, x), xRange)
plotPolar(f,theta) == plotPolar(makeFloatFunction(f,theta))
plot(f1, f2, t, tRange) ==
  plot(makeFloatFunction(f1, t), makeFloatFunction(f2, t), tRange)
plotPolar(f,theta,thetaRange) ==
  plotPolar(makeFloatFunction(f,theta),thetaRange)
```

⟨*PLOT1.dotabb*⟩≡

```
"PLOT1" [color="#FF4488",href="bookvol10.4.pdf#nameddest=PLOT1"]
"KONVERT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=KONVERT"]
"PLOT1" -> "KONVERT"
```

17.59 package PLOTTOOL PlotTools

17.60 PlotTools



1.00

Exports:
calcRanges

⟨*package PLOTTOOL PlotTools*⟩≡

```
)abbrev package PLOTTOOL PlotTools
++ Author:
++ Date Created:
++ Date Last Updated:
++ Keywords:
++ Examples:
++ References:
++ Description:
++ This package exports plotting tools
PlotTools(): Exports == Implementation where
  L   ==> List
--  Pt  ==> TwoDimensionalPoint
  SEG ==> Segment
  SF  ==> DoubleFloat
  Pt ==> Point(SF)
  PLOT ==> Plot
  DROP ==> DrawOption
  S    ==> String
  VIEW2D ==> TwoDimensionalViewport

  Exports ==> with
    calcRanges: L L Pt              -> L SEG SF
        ++ calcRanges(l) \undocumented

  Implementation ==> add
    import GraphicsDefaults
    import PLOT
```

```
    import TwoDimensionalPlotClipping
    import DrawOptionFunctions0
    import ViewportPackage
    import POINT
    import PointPackage(SF)

    --%Local functions
    xRange0: L Pt -> SEG SF
    xRange: L L Pt -> SEG SF
    yRange0: L Pt -> SEG SF
    yRange: L L Pt -> SEG SF
    drawToScaleRanges: (SEG SF,SEG SF) -> L SEG SF

    drawToScaleRanges(xVals,yVals) ==
      xDiff := (xHi := hi xVals) - (xLo := lo xVals)
      yDiff := (yHi := hi yVals) - (yLo := lo yVals)
      pad := abs(yDiff - xDiff)/2
      yDiff > xDiff => [segment(xLo - pad,xHi + pad),yVals]
      [xVals,segment(yLo - pad,yHi + pad)]

    select : (L Pt,Pt -> SF,(SF,SF) -> SF) -> SF
    select(l,f,g) ==
      m := f first l
      for p in rest l repeat m := g(m,f p)
      m

    xRange0(list:L Pt) == select(list,xCoord,min) .. select(list,xCoord,max)
    yRange0(list:L Pt) == select(list,yCoord,min) .. select(list,yCoord,max)

    select2: (L L Pt,L Pt -> SF,(SF,SF) -> SF) -> SF
    select2(l,f,g) ==
      m := f first l
      for p in rest l repeat m := g(m,f p)
      m

    xRange(list:L L Pt) ==
      select2(list,lo(xRange0(#1)),min) .. select2(list,hi(xRange0(#1)),max)

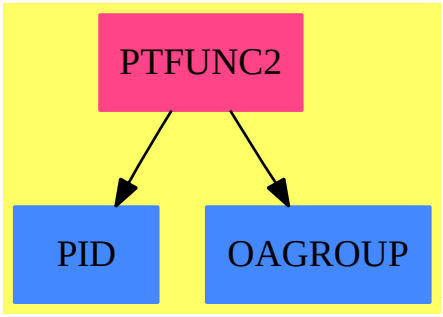
    yRange(list:L L Pt) ==
      select2(list,lo(yRange0(#1)),min) .. select2(list,hi(yRange0(#1)),max)

  --%Exported Functions
    calcRanges(llp) ==
      drawToScale() => drawToScaleRanges(xRange llp, yRange llp)
      [xRange llp, yRange llp]
```

⟨*PLOTTOOL.dotabb*⟩≡
```
"PLOTTOOL" [color="#FF4488",href="bookvol10.4.pdf#nameddest=PLOTTOOL"]
"FIELD"  [color="#4488FF",href="bookvol10.2.pdf#nameddest=FIELD"]
"RADCAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=RADCAT"]
"PLOTTOOL" -> "FIELD"
"PLOTTOOL" -> "RADCAT"
```

17.61 package PTFUNC2 PointFunctions2

17.62 PointFunctions2



1.00

Exports:
map

⟨*package PTFUNC2 PointFunctions2*⟩≡
```
)abbrev package PTFUNC2 PointFunctions2
++ Description:
++ This package \undocumented
PointFunctions2(R1:Ring,R2:Ring):Exports == Implementation where

  Exports == with
    map : ((R1->R2),Point(R1)) -> Point(R2)
        ++ map(f,p) \undocumented

  Implementation ==> add
    import Point(R1)
    import Point(R2)

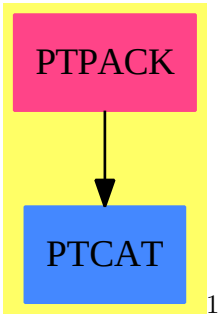
    map(mapping,p) ==
      point([mapping p.(i::PositiveInteger) for i in minIndex(p)..maxIndex(p)])$P
```

⟨*PTFUNC2.dotabb*⟩≡

```
"PTFUNC2" [color="#FF4488",href="bookvol10.4.pdf#nameddest=PTFUNC2"]
"PID" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PID"]
"OAGROUP" [color="#4488FF",href="bookvol10.2.pdf#nameddest=OAGROUP"]
"PTFUNC2" -> "PID"
"PTFUNC2" -> "OAGROUP"
```

17.63 package PTPACK PointPackage

17.64 PointPackage



1.00

Exports:

color	hue	phiCoord	rCoord	shade
thetaCoord	xCoord	yCoord	zCoord	

⟨*package PTPACK PointPackage*⟩≡

```
)abbrev package PTPACK PointPackage
++ Description:
++ This package \undocumented
PointPackage(R:Ring):Exports == Implementation where

  POINT ==> Point(R)
  I    ==> Integer
  PI   ==> PositiveInteger
  NNI  ==> NonNegativeInteger
  L    ==> List
  B    ==> Boolean

  Exports == with
    xCoord       : POINT -> R
      ++ xCoord(pt) returns the first element of the point, pt,
      ++ although no assumptions are made as to the coordinate
      ++ system being used.  This function is defined for the
      ++ convenience of the user dealing with a Cartesian
      ++ coordinate system.
    yCoord       : POINT -> R
      ++ yCoord(pt) returns the second element of the point, pt,
      ++ although no assumptions are made as to the coordinate
      ++ system being used.  This function is defined for the
      ++ convenience of the user dealing with a Cartesian
      ++ coordinate system.
    zCoord       : POINT -> R
```

```
      ++ zCoord(pt) returns the third element of the point, pt,
      ++ although no assumptions are made as to the coordinate
      ++ system being used.  This function is defined for the
      ++ convenience of the user dealing with a Cartesian
      ++ or a cylindrical coordinate system.
    rCoord       : POINT -> R
      ++ rCoord(pt) returns the first element of the point, pt,
      ++ although no assumptions are made as to the coordinate
      ++ system being used.  This function is defined for the
      ++ convenience of the user dealing with a spherical
      ++ or a cylindrical coordinate system.
    thetaCoord   : POINT -> R
      ++ thetaCoord(pt) returns the second element of the point, pt,
      ++ although no assumptions are made as to the coordinate
      ++ system being used.  This function is defined for the
      ++ convenience of the user dealing with a spherical
      ++ or a cylindrical coordinate system.
    phiCoord     : POINT -> R
      ++ phiCoord(pt) returns the third element of the point, pt,
      ++ although no assumptions are made as to the coordinate
      ++ system being used.  This function is defined for the
      ++ convenience of the user dealing with a spherical
      ++ coordinate system.
    color        : POINT -> R
      ++ color(pt) returns the fourth element of the point, pt,
      ++ although no assumptions are made with regards as to
      ++ how the components of higher dimensional points are
      ++ interpreted.  This function is defined for the
      ++ convenience of the user using specifically, color
      ++ to express a fourth dimension.
    hue : POINT -> R
      ++ hue(pt) returns the third element of the two dimensional point, pt,
      ++ although no assumptions are made with regards as to how the
      ++ components of higher dimensional points are interpreted. This
      ++ function is defined for the convenience of the user using
      ++ specifically, hue to express a third dimension.
    shade : POINT -> R
      ++ shade(pt) returns the fourth element of the two dimensional
      ++ point, pt, although no assumptions are made with regards as to
      ++ how the components of higher dimensional points are interpreted.
      ++ This function is defined for the convenience of the user using
      ++ specifically, shade to express a fourth dimension.

      -- 2D and 3D extraction of data
  Implementation ==> add
```

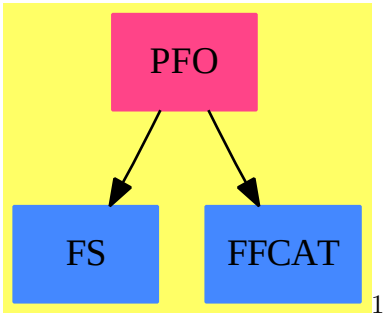
```
xCoord p == elt(p,1)
yCoord p == elt(p,2)
zCoord p == elt(p,3)
rCoord p == elt(p,1)
thetaCoord p == elt(p,2)
phiCoord p == elt(p,3)
color p ==
  #p > 3 => p.4
  p.3
hue p == elt(p,3)
  -- 4D points in 2D using extra dimensions for palette information
shade p == elt(p,4)
  -- 4D points in 2D using extra dimensions for palette information
```

⟨*PTPACK.dotabb*⟩≡

```
"PTPACK" [color="#FF4488",href="bookvol10.4.pdf#nameddest=PTPACK"]
"PTCAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PTCAT"]
"PTPACK" -> "PTCAT"
```

17.65 package PFO PointsOfFiniteOrder

17.66 PointsOfFiniteOrder



1.00

Exports:
order torsion? torsionIfCan

⟨*package PFO PointsOfFiniteOrder*⟩≡
```
)abbrev package PFO PointsOfFiniteOrder
++ Finds the order of a divisor on a curve
++ Author: Manuel Bronstein
++ Date Created: 1988
++ Date Last Updated: 22 July 1998
++ Description:
++ This package provides function for testing whether a divisor on a
++ curve is a torsion divisor.
++ Keywords: divisor, algebraic, curve.
++ Examples: )r PFO INPUT
PointsOfFiniteOrder(R0, F, UP, UPUP, R): Exports == Implementation where
  R0   : Join(OrderedSet, IntegralDomain, RetractableTo Integer)
  F    : FunctionSpace R0
  UP   : UnivariatePolynomialCategory F
  UPUP : UnivariatePolynomialCategory Fraction UP
  R    : FunctionFieldCategory(F, UP, UPUP)

  PI  ==> PositiveInteger
  N   ==> NonNegativeInteger
  Z   ==> Integer
  Q   ==> Fraction Integer
  UPF ==> SparseUnivariatePolynomial F
  UPQ ==> SparseUnivariatePolynomial Q
  QF  ==> Fraction UP
  UPUPQ ==> SparseUnivariatePolynomial Fraction UPQ
  UP2 ==> SparseUnivariatePolynomial UPQ
  UP3 ==> SparseUnivariatePolynomial UP2
```

```
FD  ==> FiniteDivisor(F, UP, UPUP, R)
K   ==> Kernel F
REC ==> Record(ncurve:UP3, disc:Z, dfpoly:UPQ)
RC0 ==> Record(ncurve:UPUPQ, disc:Z)
ID  ==> FractionalIdeal(UP, QF, UPUP, R)
SMP ==> SparseMultivariatePolynomial(R0,K)
ALGOP  ==> "%alg"

Exports ==> with
  order        : FD -> Union(N, "failed")
      ++ order(f) \undocumented
  torsion?     : FD -> Boolean
      ++ torsion?(f) \undocumented
  torsionIfCan : FD -> Union(Record(order:N, function:R), "failed")
      ++ torsionIfCan(f)\ undocumented

Implementation ==> add
  import IntegerPrimesPackage(Z)
  import PointsOfFiniteOrderTools(UPQ, UPUPQ)
  import UnivariatePolynomialCommonDenominator(Z, Q, UPQ)

  cmult: List SMP -> SMP
  raise         : (UPQ, K) -> F
  raise2        : (UP2, K) -> UP
  qmod          : F     -> Q
  fmod          : UPF   -> UPQ
  rmod          : UP    -> UPQ
  pmod          : UPUP  -> UPUPQ
  kqmod         : (F,    K) -> UPQ
  krmod         : (UP,   K) -> UP2
  kpmod         : (UPUP, K) -> UP3
  selectIntegers: K   -> REC
  selIntegers:    ()  -> RC0
  possibleOrder : FD -> N
  ratcurve      : (FD, RC0)    -> N
  algcurve      : (FD, REC, K) -> N
  kbad3Num      : (UP3, UPQ) -> Z
  kbadBadNum    : (UP2, UPQ) -> Z
  kgetGoodPrime : (REC, UPQ, UP3, UP2,UP2) -> Record(prime:PI,poly:UPQ)
  goodRed       : (REC, UPQ, UP3, UP2, UP2, PI) -> Union(UPQ, "failed")
  good?         : (UPQ, UP3, UP2, UP2, PI, UPQ) -> Boolean
  klist         : UP -> List K
  aklist        : R  -> List K
  alglist       : FD -> List K
  notIrr?       : UPQ -> Boolean
  rat           : (UPUP, FD, PI) -> N
```

```
toQ1          : (UP2, UPQ) -> UP
toQ2          : (UP3, UPQ) -> R
Q2F           : Q -> F
Q2UPUP        : UPUPQ -> UPUP

q := FunctionSpaceReduce(R0, F)

torsion? d == order(d) case N
Q2F x      == numer(x)::F / denom(x)::F
qmod x     == bringDown(x)$q
kqmod(x,k) == bringDown(x, k)$q
fmod p     == map(qmod, p)$SparseUnivariatePolynomialFunctions2(F, Q)
pmod p     == map(qmod, p)$MultipleMap(F, UP, UPUP, Q, UPQ, UPUPQ)
Q2UPUP p   == map(Q2F, p)$MultipleMap(Q, UPQ, UPUPQ, F, UP, UPUP)
klist d    == "setUnion"/[kernels c for c in coefficients d]
notIrr? d  == #(factors factor(d)$RationalFactorize(UPQ)) > 1
kbadBadNum(d, m) == mix [badNum(c rem m) for c in coefficients d]
kbad3Num(h, m)   == lcm [kbadBadNum(c, m) for c in coefficients h]

torsionIfCan d ==
  zero?(n := possibleOrder(d := reduce d)) => "failed"
  (g := generator reduce(n::Z * d)) case "failed" => "failed"
  [n, g::R]

UPQ2F(p:UPQ, k:K):F ==
  map(Q2F, p)$UnivariatePolynomialCategoryFunctions2(Q, UPQ, F, UP) (k::F)

UP22UP(p:UP2, k:K):UP ==
  map(UPQ2F(#1, k), p)$UnivariatePolynomialCategoryFunctions2(UPQ,UP2,F,UP)

UP32UPUP(p:UP3, k:K):UPUP ==
  map(UP22UP(#1,k)::QF,
      p)$UnivariatePolynomialCategoryFunctions2(UP2, UP3, QF, UPUP)

if R0 has GcdDomain then
   cmult(l:List SMP):SMP == lcm l
else
   cmult(l:List SMP):SMP == */l

doubleDisc(f:UP3):Z ==
  d := discriminant f
  g := gcd(d, differentiate d)
  d := (d exquo g)::UP2
  zero?(e := discriminant d) => 0
  gcd [retract(c)@Z for c in coefficients e]
```

```
commonDen(p:UP):SMP ==
  l1:List F := coefficients p
  l2:List SMP := [denom c for c in l1]
  cmult l2

polyred(f:UPUP):UPUP ==
  cmult([commonDen(retract(c)@UP) for c in coefficients f])::F::UP::QF * f

aklist f ==
  (r := retractIfCan(f)@Union(QF, "failed")) case "failed" =>
    "setUnion"/[klist(retract(c)@UP) for c in coefficients lift f]
  klist(retract(r::QF)@UP)

alglist d ==
  n := numer(i := ideal d)
  select_!(has?(operator #1, ALGOP),
           setUnion(klist denom i,
             "setUnion"/[aklist qelt(n,i) for i in minIndex n..maxIndex n]))

krmod(p,k) ==
   map(kqmod(#1, k),
       p)$UnivariatePolynomialCategoryFunctions2(F, UP, UPQ, UP2)

rmod p ==
   map(qmod, p)$UnivariatePolynomialCategoryFunctions2(F, UP, Q, UPQ)

raise(p, k) ==
  (map(Q2F, p)$SparseUnivariatePolynomialFunctions2(Q, F)) (k::F)

raise2(p, k) ==
  map(raise(#1, k),
      p)$UnivariatePolynomialCategoryFunctions2(UPQ, UP2, F, UP)

algcurve(d, rc, k) ==
  mn := minIndex(n := numer(i := minimize ideal d))
  h  := kpmod(lift(hh := n(mn + 1)), k)
  b2 := primitivePart
            raise2(b := krmod(retract(retract(n.mn)@QF)@UP, k), k)
  s  := kqmod(resultant(primitivePart separate(raise2(krmod(
              retract(norm hh)@UP, k), k), b2).primePart, b2), k)
  pr := kgetGoodPrime(rc, s, h, b, dd := krmod(denom i, k))
  p  := pr.prime
  pp := UP32UPUP(rc.ncurve, k)
  mm := pr.poly
  gf := InnerPrimeField p
  m  := map(retract(#1)@Z :: gf,
```

```
                          mm)$SparseUnivariatePolynomialFunctions2(Q, gf)
--      one? degree m =>
      (degree m = 1) =>
        alpha := - coefficient(m, 0) / leadingCoefficient m
        order(d, pp,
           (map(numer(#1)::gf / denom(#1)::gf,
            kqmod(#1,k))$SparseUnivariatePolynomialFunctions2(Q,gf))(alpha)
                                  )$ReducedDivisor(F, UP, UPUP, R, gf)
        -- d1 := toQ1(dd, mm)
        -- rat(pp, divisor ideal([(toQ1(b, mm) / d1)::QF::R,
                                     -- inv(d1::QF) * toQ2(h,mm)])$ID, p)
      sae:= SimpleAlgebraicExtension(gf,SparseUnivariatePolynomial gf,m)
      order(d, pp,
           reduce(map(numer(#1)::gf / denom(#1)::gf,
            kqmod(#1,k))$SparseUnivariatePolynomialFunctions2(Q,gf))$sae
                                  )$ReducedDivisor(F, UP, UPUP, R, sae)

-- returns the potential order of d, 0 if d is of infinite order
    ratcurve(d, rc) ==
      mn  := minIndex(nm := numer(i := minimize ideal d))
      h   := pmod lift(hh := nm(mn + 1))
      b   := rmod(retract(retract(nm.mn)@QF)@UP)
      s   := separate(rmod(retract(norm hh)@UP), b).primePart
      bd  := badNum rmod denom i
      r   := resultant(s, b)
      bad := lcm [rc.disc, numer r, denom r, bd.den*bd.gcdnum, badNum h]$List(Z)
      pp  := Q2UPUP(rc.ncurve)
      n   := rat(pp, d, p := getGoodPrime bad)
-- if n > 1 then it is cheaper to compute the order modulo a second prime,
-- since computing n * d could be very expensive
--      one? n => n
      (n = 1) => n
      m   := rat(pp, d, getGoodPrime(p * bad))
      n = m => n
      0

-- returns the order of d mod p
    rat(pp, d, p) ==
      gf := InnerPrimeField p
      order(d, pp, (qq := qmod #1;numer(qq)::gf / denom(qq)::gf)
                                   )$ReducedDivisor(F, UP, UPUP, R, gf)

-- returns the potential order of d, 0 if d is of infinite order
    possibleOrder d ==
--      zero?(genus()) or one?(#(numer ideal d)) => 1
      zero?(genus()) or (#(numer ideal d) = 1) => 1
```

```
  empty?(la := alglist d) => ratcurve(d, selIntegers())
  not(empty? rest la) =>
       error "PFO::possibleOrder: more than 1 algebraic constant"
  algcurve(d, selectIntegers first la, first la)

selIntegers():RC0 ==
  f := definingPolynomial()$R
  while zero?(d := doubleDisc(r := polyred pmod f)) repeat newReduc()$q
  [r, d]

selectIntegers(k:K):REC ==
  g := polyred(f := definingPolynomial()$R)
  p := minPoly k
  while zero?(d := doubleDisc(r := kpmod(g, k))) or (notIrr? fmod p)
      repeat newReduc()$q
  [r, d, splitDenominator(fmod p).num]

toQ1(p, d) ==
  map(Q2F(retract(#1 rem d)@Q),
      p)$UnivariatePolynomialCategoryFunctions2(UPQ, UP2, F, UP)

toQ2(p, d) ==
  reduce map(toQ1(#1, d)::QF,
    p)$UnivariatePolynomialCategoryFunctions2(UP2, UP3, QF, UPUP)

kpmod(p, k) ==
  map(krmod(retract(#1)@UP, k),
    p)$UnivariatePolynomialCategoryFunctions2(QF, UPUP, UP2, UP3)

order d ==
  zero?(n := possibleOrder(d := reduce d)) => "failed"
  principal? reduce(n::Z * d) => n
  "failed"

kgetGoodPrime(rec, res, h, b, d) ==
  p:PI := 3
  while (u := goodRed(rec, res, h, b, d, p)) case "failed" repeat
    p := nextPrime(p::Z)::PI
  [p, u::UPQ]

goodRed(rec, res, h, b, d, p) ==
  zero?(rec.disc rem p) => "failed"
  gf := InnerPrimeField p
  l  := [f.factor for f in factors factor(map(retract(#1)@Z :: gf,
           rec.dfpoly)$SparseUnivariatePolynomialFunctions2(Q,
             gf))$DistinctDegreeFactorize(gf,
```

```
--                  SparseUnivariatePolynomial gf) | one?(f.exponent)]
                   SparseUnivariatePolynomial gf) | (f.exponent = 1)]
      empty? l => "failed"
      mdg := first l
      for ff in rest l repeat
        if degree(ff) < degree(mdg) then mdg := ff
      md := map(convert(#1)@Z :: Q,
                mdg)$SparseUnivariatePolynomialFunctions2(gf, Q)
      good?(res, h, b, d, p, md) => md
      "failed"

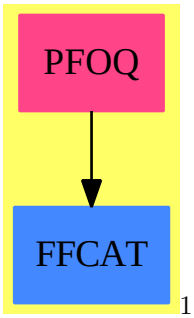
    good?(res, h, b, d, p, m) ==
      bd := badNum(res rem m)
      not (zero?(bd.den rem p) or zero?(bd.gcdnum rem p) or
        zero?(kbadBadNum(b,m) rem p) or zero?(kbadBadNum(d,m) rem p)
              or zero?(kbad3Num(h, m) rem p))
```

⟨*PFO.dotabb*⟩≡

```
"PFO" [color="#FF4488",href="bookvol10.4.pdf#nameddest=PFO"]
"FS" [color="#4488FF",href="bookvol10.2.pdf#nameddest=FS"]
"FFCAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=FFCAT"]
"PFO" -> "FS"
"PFO" -> "FFCAT"
```

17.67 package PFOQ PointsOfFiniteOrderRational

17.68 PointsOfFiniteOrderRational



1.00

Exports:
order torsion? torsionIfCan

⟨*package PFOQ PointsOfFiniteOrderRational*⟩≡

```
)abbrev package PFOQ PointsOfFiniteOrderRational
++ Finds the order of a divisor on a rational curve
++ Author: Manuel Bronstein
++ Date Created: 25 Aug 1988
++ Date Last Updated: 3 August 1993
++ Description:
++ This package provides function for testing whether a divisor on a
++ curve is a torsion divisor.
++ Keywords: divisor, algebraic, curve.
++ Examples: )r PFOQ INPUT
PointsOfFiniteOrderRational(UP, UPUP, R): Exports == Implementation where
  UP   : UnivariatePolynomialCategory Fraction Integer
  UPUP : UnivariatePolynomialCategory Fraction UP
  R    : FunctionFieldCategory(Fraction Integer, UP, UPUP)

  PI  ==> PositiveInteger
  N   ==> NonNegativeInteger
  Z   ==> Integer
  Q   ==> Fraction Integer
  FD  ==> FiniteDivisor(Q, UP, UPUP, R)

  Exports ==> with
    order       : FD -> Union(N, "failed")
        ++ order(f) \undocumented
    torsion?    : FD -> Boolean
        ++ torsion?(f) \undocumented
```

```
      torsionIfCan: FD -> Union(Record(order:N, function:R), "failed")
          ++ torsionIfCan(f) \undocumented

    Implementation ==> add
      import PointsOfFiniteOrderTools(UP, UPUP)

      possibleOrder: FD -> N
      ratcurve     : (FD, UPUP, Z) -> N
      rat          : (UPUP, FD, PI) -> N

      torsion? d  == order(d) case N

  -- returns the potential order of d, 0 if d is of infinite order
      ratcurve(d, modulus, disc) ==
        mn  := minIndex(nm := numer(i := ideal d))
        h   := lift(hh := nm(mn + 1))
        s   := separate(retract(norm hh)@UP,
                   b := retract(retract(nm.mn)@Fraction(UP))@UP).primePart
        bd  := badNum denom i
        r   := resultant(s, b)
        bad := lcm [disc, numer r, denom r, bd.den * bd.gcdnum, badNum h]$List(Z)
        n   := rat(modulus, d, p := getGoodPrime bad)
  -- if n > 1 then it is cheaper to compute the order modulo a second prime,
  -- since computing n * d could be very expensive
  --      one? n => n
        (n = 1) => n
        m   := rat(modulus, d, getGoodPrime(p * bad))
        n = m => n
        0

      rat(pp, d, p) ==
        gf := InnerPrimeField p
        order(d, pp,
          numer(#1)::gf / denom(#1)::gf)$ReducedDivisor(Q, UP, UPUP, R, gf)

  -- returns the potential order of d, 0 if d is of infinite order
      possibleOrder d ==
  --      zero?(genus()) or one?(#(numer ideal d)) => 1
        zero?(genus()) or (#(numer ideal d) = 1) => 1
        r := polyred definingPolynomial()$R
        ratcurve(d, r, doubleDisc r)

      order d ==
        zero?(n := possibleOrder(d := reduce d)) => "failed"
        principal? reduce(n::Z * d) => n
        "failed"
```

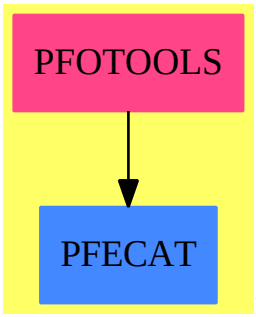
```
      torsionIfCan d ==
        zero?(n := possibleOrder(d := reduce d)) => "failed"
        (g := generator reduce(n::Z * d)) case "failed" => "failed"
        [n, g::R]
```

⟨*PFOQ.dotabb*⟩≡

```
"PFOQ" [color="#FF4488",href="bookvol10.4.pdf#nameddest=PFOQ"]
"FFCAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=FFCAT"]
"PFOQ" -> "FFCAT"
```

17.69 package PFOTOOLS PointsOfFiniteOrderTools

17.70 PointsOfFiniteOrderTools



1.00

Exports:
badNum doubleDisc getGoodPrime mix polyred

⟨*package PFOTOOLS PointsOfFiniteOrderTools*⟩≡
```
)abbrev package PFOTOOLS PointsOfFiniteOrderTools
++ Utilities for PFOQ and PFO
++ Author: Manuel Bronstein
++ Date Created: 25 Aug 1988
++ Date Last Updated: 11 Jul 1990
PointsOfFiniteOrderTools(UP, UPUP): Exports == Implementation where
  UP   : UnivariatePolynomialCategory Fraction Integer
  UPUP : UnivariatePolynomialCategory Fraction UP

  PI  ==> PositiveInteger
  N   ==> NonNegativeInteger
  Z   ==> Integer
  Q   ==> Fraction Integer

  Exports ==> with
    getGoodPrime : Z -> PI
      ++ getGoodPrime n returns the smallest prime not dividing n
    badNum       : UP   -> Record(den:Z, gcdnum:Z)
        ++ badNum(p) \undocumented
    badNum       : UPUP -> Z
        ++ badNum(u) \undocumented
    mix          : List Record(den:Z, gcdnum:Z) -> Z
        ++ mix(l) \undocumented
    doubleDisc   : UPUP -> Z
        ++ doubleDisc(u) \undocumented
    polyred      : UPUP -> UPUP
```

```
        ++ polyred(u) \undocumented

    Implementation ==> add
      import IntegerPrimesPackage(Z)
      import UnivariatePolynomialCommonDenominator(Z, Q, UP)

      mix l          == lcm(lcm [p.den for p in l], gcd [p.gcdnum for p in l])
      badNum(p:UPUP) == mix [badNum(retract(c)@UP) for c in coefficients p]

      polyred r ==
        lcm [commonDenominator(retract(c)@UP) for c in coefficients r] * r

      badNum(p:UP) ==
        cd := splitDenominator p
        [cd.den, gcd [retract(c)@Z for c in coefficients(cd.num)]]

      getGoodPrime n ==
        p:PI := 3
        while zero?(n rem p) repeat
          p := nextPrime(p::Z)::PI
        p

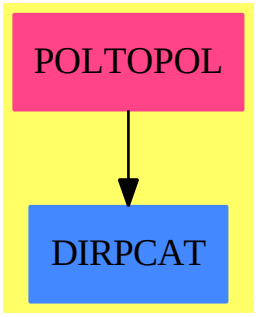
      doubleDisc r ==
        d := retract(discriminant r)@UP
        retract(discriminant((d exquo gcd(d, differentiate d))::UP))@Z
```

⟨*PFOTOOLS.dotabb*⟩≡

```
"PFOTOOLS" [color="#FF4488",href="bookvol10.4.pdf#nameddest=PFOTOOLS"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"PFOTOOLS" -> "PFECAT"
```

17.71 package POLTOPOL PolToPol

17.72 PolToPol



1.00

Exports:
dmpToHdmp dmpToP hdmpToDmp hdmpToP pToDmp pToHdmp

⟨*package POLTOPOL PolToPol*⟩≡
```
)abbrev package POLTOPOL PolToPol
++ Author : P.Gianni, Summer '88
++ Description:
++ Package with the conversion functions among different kind of polynomials
PolToPol(lv,R) : C == T

 where
  R      :    Ring
  lv     :    List Symbol
  NNI    ==>  NonNegativeInteger
  Ov     ==>  OrderedVariableList(lv)
  IES    ==>  IndexedExponents Symbol

  DP     ==>  DirectProduct(#lv,NonNegativeInteger)
  DPoly  ==>  DistributedMultivariatePolynomial(lv,R)

  HDP    ==>  HomogeneousDirectProduct(#lv,NonNegativeInteger)
  HDPoly ==>  HomogeneousDistributedMultivariatePolynomial(lv,R)
  P      ==>  Polynomial R
  VV     ==>  Vector NNI
  MPC3   ==>  MPolyCatFunctions3

  C == with
     dmpToHdmp    :    DPoly    -> HDPoly
       ++ dmpToHdmp(p) converts p from a \spadtype{DMP} to a \spadtype{HDMP}.
     hdmpToDmp    :   HDPoly    -> DPoly
       ++ hdmpToDmp(p) converts p from a \spadtype{HDMP} to a \spadtype{DMP}.
```

```
    pToHdmp      :     P       -> HDPoly
      ++ pToHdmp(p) converts p from a \spadtype{POLY} to a \spadtype{HDMP}.
    hdmpToP      :   HDPoly    -> P
      ++ hdmpToP(p) converts p from a \spadtype{HDMP} to a \spadtype{POLY}.
    dmpToP       :    DPoly    -> P
      ++ dmpToP(p) converts p from a \spadtype{DMP} to a \spadtype{POLY}.
    pToDmp       :     P       -> DPoly
      ++ pToDmp(p) converts p from a \spadtype{POLY} to a \spadtype{DMP}.
  T == add

    variable1(xx:Symbol):Ov == variable(xx)::Ov

   -- transform a P in a HDPoly --
   pToHdmp(pol:P) : HDPoly ==
     map(variable1,pol)$MPC3(Symbol,Ov,IES,HDP,R,P,HDPoly)

   -- transform an HDPoly in a P --
   hdmpToP(hdpol:HDPoly) : P ==
     map(convert,hdpol)$MPC3(Ov,Symbol,HDP,IES,R,HDPoly,P)

   -- transform an DPoly in a P --
   dmpToP(dpol:DPoly) : P ==
     map(convert,dpol)$MPC3(Ov,Symbol,DP,IES,R,DPoly,P)

   -- transform a P in a DPoly --
   pToDmp(pol:P) : DPoly ==
     map(variable1,pol)$MPC3(Symbol,Ov,IES,DP,R,P,DPoly)

   -- transform a DPoly in a HDPoly --
   dmpToHdmp(dpol:DPoly) : HDPoly ==
     dpol=0 => 0$HDPoly
     monomial(leadingCoefficient dpol,
              directProduct(degree(dpol)::VV)$HDP)$HDPoly+
                                                  dmpToHdmp(reductum dpol)

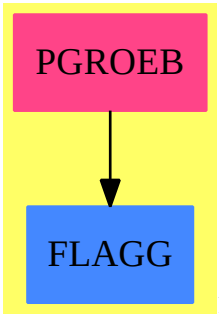
   -- transform a HDPoly in a DPoly --
   hdmpToDmp(hdpol:HDPoly) : DPoly ==
     hdpol=0 => 0$DPoly
     dd:DP:= directProduct((degree hdpol)::VV)$DP
     monomial(leadingCoefficient hdpol,dd)$DPoly+
              hdmpToDmp(reductum hdpol)
```

⟨*POLTOPOL.dotabb*⟩≡

```
"POLTOPOL" [color="#FF4488",href="bookvol10.4.pdf#nameddest=POLTOPOL"]
"DIRPCAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=DIRPCAT"]
"POLTOPOL" -> "DIRPCAT"
```

17.73 package PGROEB PolyGroebner

17.74 PolyGroebner



1.00

Exports:
lexGroebner totalGroebner

⟨*package PGROEB PolyGroebner*⟩≡

```
)abbrev package PGROEB PolyGroebner
++ Author: P. Gianni
++ Date Created: Summer 1988
++ Date Last Updated:
++ Basic Functions:
++ Related Constructors: GroebnerPackage
++ Also See:
++ AMS Classifications:
++ Keywords: groebner basis, polynomial ideals
++ References:
++ Description:
++ Groebner functions for P F
++   This package is an interface package to the groebner basis
++ package which allows you to compute groebner bases for polynomials
++ in either lexicographic ordering or total degree ordering refined
++ by reverse lex. The input is the ordinary polynomial type which
++ is internally converted to a type with the required ordering.
++ The resulting grobner basis is converted back to ordinary polynomials.
++ The ordering among the variables is controlled by an explicit list
++ of variables which is passed as a second argument. The coefficient
++ domain is allowed to be any gcd domain, but the groebner basis is
++ computed as if the polynomials were over a field.

PolyGroebner(F) : C == T

 where
  F      :   GcdDomain
```

```
NNI    ==>  NonNegativeInteger
P      ==>  Polynomial F
L      ==>  List
E      ==>  Symbol

C == with
   lexGroebner   :    (L P,L E)  ->  L P
     ++ lexGroebner(lp,lv) computes Groebner basis
     ++ for the list of polynomials lp in lexicographic order.
     ++ The variables are ordered by their position in the list lv.

   totalGroebner :    (L P, L E) ->  L P
     ++ totalGroebner(lp,lv) computes Groebner basis
     ++ for the list of polynomials lp with the terms
     ++ ordered first by total degree and then
     ++ refined by reverse lexicographic ordering.
     ++ The variables are ordered by their position in the list lv.

T == add
   lexGroebner(lp: L P,lv:L E) : L P ==
     PP:=  PolToPol(lv,F)
     DPoly := DistributedMultivariatePolynomial(lv,F)
     DP:=DirectProduct(#lv,NNI)
     OV:=OrderedVariableList lv
     b:L DPoly:=[pToDmp(pol)$PP for pol in lp]
     gb:L DPoly :=groebner(b)$GroebnerPackage(F,DP,OV,DPoly)
     [dmpToP(pp)$PP for pp in gb]

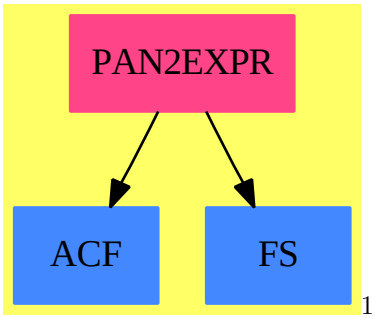
   totalGroebner(lp: L P,lv:L E) : L P ==
     PP:=  PolToPol(lv,F)
     HDPoly := HomogeneousDistributedMultivariatePolynomial(lv,F)
     HDP:=HomogeneousDirectProduct(#lv,NNI)
     OV:=OrderedVariableList lv
     b:L HDPoly:=[pToHdmp(pol)$PP for pol in lp]
     gb:=groebner(b)$GroebnerPackage(F,HDP,OV,HDPoly)
     [hdmpToP(pp)$PP for pp in gb]
```

⟨*PGROEB.dotabb*⟩≡

```
"PGROEB" [color="#FF4488",href="bookvol10.4.pdf#nameddest=PGROEB"]
"FLAGG" [color="#4488FF",href="bookvol10.2.pdf#nameddest=FLAGG"]
"PGROEB" -> "FLAGG"
```

17.75 package PAN2EXPR PolynomialAN2Expression

17.76 PolynomialAN2Expression



1.00

Exports:
coerce

⟨*package PAN2EXPR PolynomialAN2Expression*⟩≡

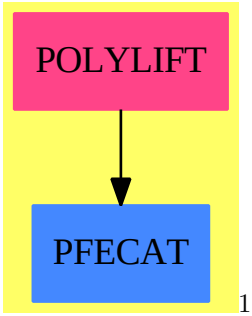
```
)abbrev package PAN2EXPR PolynomialAN2Expression
++ Author: Barry Trager
++ Date Created: 8 Oct 1991
++ Description: This package provides a coerce from polynomials over
++ algebraic numbers to \spadtype{Expression AlgebraicNumber}.
PolynomialAN2Expression():Target == Implementation where
  EXPR ==> Expression(Integer)
  AN ==> AlgebraicNumber
  PAN ==> Polynomial AN
  SY ==> Symbol
  Target ==> with
      coerce: Polynomial AlgebraicNumber -> Expression(Integer)
        ++ coerce(p) converts the polynomial \spad{p} with algebraic number
        ++ coefficients to \spadtype{Expression Integer}.
      coerce: Fraction Polynomial AlgebraicNumber -> Expression(Integer)
        ++ coerce(rf) converts \spad{rf}, a fraction of polynomial
        ++ \spad{p} with
        ++ algebraic number coefficients to \spadtype{Expression Integer}.
  Implementation ==> add
    coerce(p:PAN):EXPR ==
        map(#1::EXPR, #1::EXPR, p)$PolynomialCategoryLifting(
                                  IndexedExponents SY, SY, AN, PAN, EXPR)
    coerce(rf:Fraction PAN):EXPR ==
        numer(rf)::EXPR / denom(rf)::EXPR
```

⟨*PAN2EXPR.dotabb*⟩≡

```
"PAN2EXPR" [color="#FF4488",href="bookvol10.4.pdf#nameddest=PAN2EXPR"]
"ACF" [color="#4488FF",href="bookvol10.2.pdf#nameddest=ACF"]
"FS" [color="#4488FF",href="bookvol10.2.pdf#nameddest=FS"]
"PAN2EXPR" -> "ACF"
"PAN2EXPR" -> "FS"
```

17.77 package POLYLIFT PolynomialCategoryLifting

17.78 PolynomialCategoryLifting



1.00

Exports:
map

⟨*package POLYLIFT PolynomialCategoryLifting*⟩≡

```
)abbrev package POLYLIFT PolynomialCategoryLifting
++ Author: Manuel Bronstein
++ Date Created:
++ Date Last Updated:
++ Basic Functions:
++ Related Constructors:
++ Also See:
++ AMS Classifications:
++ Keywords:
++ References:
++ Description:
++ This package provides a very general map function, which
++ given a set S and polynomials over R with maps from the
++ variables into S and the coefficients into S, maps polynomials
++ into S. S is assumed to support \spad{+}, \spad{*} and \spad{**}.

PolynomialCategoryLifting(E,Vars,R,P,S): Exports == Implementation where
  E   : OrderedAbelianMonoidSup
  Vars: OrderedSet
  R   : Ring
  P   : PolynomialCategory(R, E, Vars)
  S   : SetCategory with
          "+" : (%, %) -> %
          "*" : (%, %) -> %
          "**": (%, NonNegativeInteger) -> %
```

```
Exports ==> with
  map: (Vars -> S, R -> S, P) -> S
    ++ map(varmap, coefmap, p) takes a
    ++ varmap, a mapping from the variables of polynomial p into S,
    ++ coefmap, a mapping from coefficients of p into S, and p, and
    ++ produces a member of S using the corresponding arithmetic.
    ++ in S

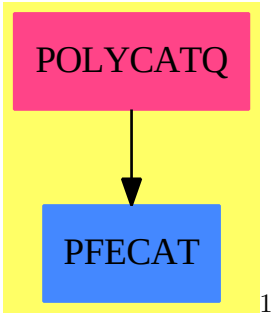
Implementation ==> add
  map(fv, fc, p) ==
    (x1 := mainVariable p) case "failed" => fc leadingCoefficient p
    up := univariate(p, x1::Vars)
    t  := fv(x1::Vars)
    ans:= fc 0
    while not ground? up repeat
      ans := ans + map(fv,fc, leadingCoefficient up) * t ** (degree up)
      up  := reductum up
    ans + map(fv, fc, leadingCoefficient up)
```

⟨*POLYLIFT.dotabb*⟩≡

```
"POLYLIFT" [color="#FF4488",href="bookvol10.4.pdf#nameddest=POLYLIFT"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"POLYLIFT" -> "PFECAT"
```

17.79 package POLYCATQ PolynomialCategoryQuotientFunctions

17.80 PolynomialCategoryQuotientFunctions



1.00

Exports:

variables	isExpt	isPlus	isPower	isTimes
mainVariable	multivariate	univariate		

⟨*package POLYCATQ PolynomialCategoryQuotientFunctions*⟩≡

```
)abbrev package POLYCATQ PolynomialCategoryQuotientFunctions
++ Manipulations on polynomial quotients
++ Author: Manuel Bronstein
++ Date Created: March 1988
++ Date Last Updated: 9 July 1990
++ Description:
++   This package transforms multivariate polynomials or fractions into
++   univariate polynomials or fractions, and back.
++ Keywords: polynomial, fraction, transformation
PolynomialCategoryQuotientFunctions(E, V, R, P, F):
 Exports == Implementation where
  E: OrderedAbelianMonoidSup
  V: OrderedSet
  R: Ring
  P: PolynomialCategory(R, E, V)
  F: Field with
    coerce: P -> %
    numer : % -> P
    denom : % -> P

  UP  ==> SparseUnivariatePolynomial F
  RF  ==> Fraction UP

  Exports ==> with
    variables   : F -> List V
```

```
      ++ variables(f) returns the list of variables appearing
      ++ in the numerator or the denominator of f.
    mainVariable: F -> Union(V, "failed")
      ++ mainVariable(f) returns the highest variable appearing
      ++ in the numerator or the denominator of f, "failed" if
      ++ f has no variables.
    univariate  : (F, V) -> RF
      ++ univariate(f, v) returns f viewed as a univariate
      ++ rational function in v.
    multivariate: (RF, V) -> F
      ++ multivariate(f, v) applies both the numerator and
      ++ denominator of f to v.
    univariate  : (F, V, UP) -> UP
      ++ univariate(f, x, p) returns f viewed as a univariate
      ++ polynomial in x, using the side-condition \spad{p(x) = 0}.
    isPlus      : F -> Union(List F, "failed")
      ++ isPlus(p) returns [m1,...,mn] if \spad{p = m1 + ... + mn} and
      ++ \spad{n > 1}, "failed" otherwise.
    isTimes     : F -> Union(List F, "failed")
      ++ isTimes(p) returns \spad{[a1,...,an]} if
      ++ \spad{p = a1 ... an} and \spad{n > 1},
      ++ "failed" otherwise.
    isExpt      : F -> Union(Record(var:V, exponent:Integer), "failed")
      ++ isExpt(p) returns \spad{[x, n]} if \spad{p = x**n} and \spad{n <> 0},
      ++ "failed" otherwise.
    isPower     : F -> Union(Record(val:F, exponent:Integer), "failed")
      ++ isPower(p) returns \spad{[x, n]} if \spad{p = x**n} and \spad{n <> 0},
      ++ "failed" otherwise.

  Implementation ==> add
    P2UP: (P, V) -> UP

    univariate(f, x) == P2UP(numer f, x) / P2UP(denom f, x)

    univariate(f, x, modulus) ==
      (bc := extendedEuclidean(P2UP(denom f, x), modulus, 1))
             case "failed" => error "univariate: denominator is 0 mod p"
      (P2UP(numer f, x) * bc.coef1) rem modulus

    multivariate(f, x) ==
      v := x::P::F
      ((numer f) v) / ((denom f) v)

    mymerge:(List V,List V) ->List V
    mymerge(l:List V,m:List V):List V==
         empty? l => m
```

```
          empty? m => l
          first l = first m => cons(first l,mymerge(rest l,rest m))
          first l > first m => cons(first l,mymerge(rest l,m))
          cons(first m,mymerge(l,rest m))

      variables f ==
        mymerge(variables numer f, variables denom f)

      isPower f ==
        (den := denom f) ^= 1 =>
          numer f ^= 1 => "failed"
          (ur := isExpt den) case "failed" => [den::F, -1]
          r := ur::Record(var:V, exponent:NonNegativeInteger)
          [r.var::P::F, - (r.exponent::Integer)]
        (ur := isExpt numer f) case "failed" => "failed"
        r := ur::Record(var:V, exponent:NonNegativeInteger)
        [r.var::P::F, r.exponent::Integer]

      isExpt f ==
        (ur := isExpt numer f) case "failed" =>
--          one? numer f =>
          (numer f) = 1 =>
            (ur := isExpt denom f) case "failed" => "failed"
            r := ur::Record(var:V, exponent:NonNegativeInteger)
            [r.var, - (r.exponent::Integer)]
          "failed"
        r := ur::Record(var:V, exponent:NonNegativeInteger)
--        one? denom f => [r.var, r.exponent::Integer]
        (denom f) = 1 => [r.var, r.exponent::Integer]
        "failed"

      isTimes f ==
        t := isTimes(num := numer f)
        l:Union(List F, "failed") :=
          t case "failed" => "failed"
          [x::F for x in t]
--        one?(den := denom f) => l
        ((den := denom f) = 1) => l
--        one? num => "failed"
        num = 1 => "failed"
        d := inv(den::F)
        l case "failed" => [num::F, d]
        concat_!(l::List(F), d)

      isPlus f ==
        denom f ^= 1 => "failed"
```

```
      (s := isPlus numer f) case "failed" => "failed"
      [x::F for x in s]

    mainVariable f ==
      a := mainVariable numer f
      (b := mainVariable denom f) case "failed" => a
      a case "failed" => b
      max(a::V, b::V)

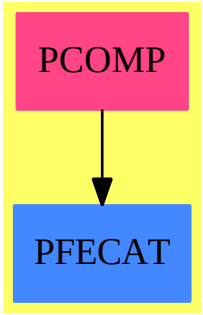
    P2UP(p, x) ==
      map(#1::F,
          univariate(p, x))$SparseUnivariatePolynomialFunctions2(P, F)
```

⟨*POLYCATQ.dotabb*⟩≡

```
"POLYCATQ" [color="#FF4488",href="bookvol10.4.pdf#nameddest=POLYCATQ"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"POLYCATQ" -> "PFECAT"
```

17.81 package PCOMP PolynomialComposition

17.82 PolynomialComposition



1.00

Exports:
compose

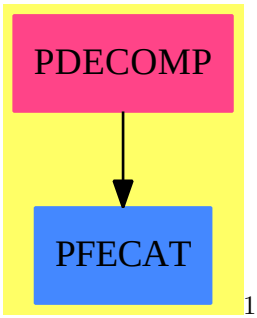
Polynomial composition and decomposition functions
If f = g o h then g = leftFactor(f, h) and h = rightFactor(f, g)

⟨*package PCOMP PolynomialComposition*⟩≡
```
)abbrev package PCOMP PolynomialComposition
++ Description:
++ This package \undocumented
PolynomialComposition(UP: UnivariatePolynomialCategory(R), R: Ring): with
        compose: (UP, UP) -> UP
                ++ compose(p,q) \undocumented
    == add
        compose(g, h) ==
            r: UP := 0
            while g ^= 0 repeat
                r := leadingCoefficient(g)*h**degree(g) + r
                g := reductum g
            r
```

⟨*PCOMP.dotabb*⟩≡
```
"PCOMP" [color="#FF4488",href="bookvol10.4.pdf#nameddest=PCOMP"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"PCOMP" -> "PFECAT"
```

17.83 package PDECOMP PolynomialDecomposition

17.84 PolynomialDecomposition



1.00

Exports:
decompose leftFactor rightFactorCandidate

Polynomial composition and decomposition functions
If f = g o h then g = leftFactor(f, h) and h = rightFactor(f, g)

⟨*package PDECOMP PolynomialDecomposition*⟩≡

```
)abbrev package PDECOMP PolynomialDecomposition
++ Description:
++ This package \undocumented
--  Ref: Kozen and Landau, Cornell University  TR 86-773
PolynomialDecomposition(UP, F): PDcat == PDdef where
    F:Field
    UP:UnivariatePolynomialCategory F
    NNI ==> NonNegativeInteger
    LR  ==> Record(left: UP, right: UP)

    PDcat == with
        decompose: UP -> List UP
                ++ decompose(up) \undocumented
        decompose: (UP, NNI, NNI) -> Union(LR, "failed")
                ++ decompose(up,m,n) \undocumented
        leftFactor: (UP, UP) -> Union(UP, "failed")
                ++ leftFactor(p,q) \undocumented
        rightFactorCandidate:  (UP, NNI) -> UP
                ++ rightFactorCandidate(p,n) \undocumented
    PDdef == add
        leftFactor(f, h) ==
             g: UP := 0
             for i in 0.. while f ^= 0 repeat
```

```
            fr := divide(f, h)
            f := fr.quotient; r := fr.remainder
            degree r > 0 => return "failed"
            g := g + r * monomial(1, i)
        g

    decompose(f, dg, dh) ==
        df := degree f
        dg*dh ^= df => "failed"
        h := rightFactorCandidate(f, dh)
        g := leftFactor(f, h)
        g case "failed" => "failed"
        [g::UP, h]

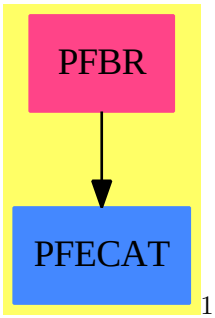
    decompose f ==
        df := degree f
        for dh in 2..df-1 | df rem dh = 0 repeat
            h := rightFactorCandidate(f, dh)
            g := leftFactor(f, h)
            g case UP => return
                append(decompose(g::UP), decompose h)
        [f]
    rightFactorCandidate(f, dh) ==
        f  := f/leadingCoefficient f
        df := degree f
        dg := df quo dh
        h  := monomial(1, dh)
        for k in 1..dh repeat
            hdg:= h**dg
            c  := (coefficient(f,(df-k)::NNI)-coefficient(hdg,(df-k)::NNI))/(
            h  := h + monomial(c, (dh-k)::NNI)
        h - monomial(coefficient(h, 0), 0) -- drop constant term
```

⟨*PDECOMP.dotabb*⟩≡

```
"PDECOMP" [color="#FF4488",href="bookvol10.4.pdf#nameddest=PDECOMP"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"PDECOMP" -> "PFECAT"
```

17.85 package PFBR PolynomialFactorization-ByRecursion

17.86 PolynomialFactorizationByRecursion



1.00

Exports:

randomR	bivariateSLPEBR	factorByRecursion
factorSFBRlcUnit	factorSquareFreeByRecursion	solveLinearPolynomialEquationByRecursion

⟨*package PFBR PolynomialFactorizationByRecursion*⟩≡

```
)abbrev package PFBR PolynomialFactorizationByRecursion
++ Description: PolynomialFactorizationByRecursion(R,E,VarSet,S)
++ is used for factorization of sparse univariate polynomials over
++ a domain S of multivariate polynomials over R.
PolynomialFactorizationByRecursion(R,E, VarSet:OrderedSet, S): public ==
 private where
  R:PolynomialFactorizationExplicit
  E:OrderedAbelianMonoidSup
  S:PolynomialCategory(R,E,VarSet)
  PI ==> PositiveInteger
  SupR ==> SparseUnivariatePolynomial R
  SupSupR ==> SparseUnivariatePolynomial SupR
  SupS ==> SparseUnivariatePolynomial S
  SupSupS ==> SparseUnivariatePolynomial SupS
  LPEBFS ==> LinearPolynomialEquationByFractions(S)
  public ==  with
     solveLinearPolynomialEquationByRecursion: (List SupS, SupS)  ->
                                                Union(List SupS,"failed")
        ++ \spad{solveLinearPolynomialEquationByRecursion([p1,...,pn],p)}
        ++ returns the list of polynomials \spad{[q1,...,qn]}
        ++ such that \spad{sum qi/pi = p / prod pi}, a
        ++ recursion step for solveLinearPolynomialEquation
        ++ as defined in \spadfun{PolynomialFactorizationExplicit} category
        ++ (see \spadfun{solveLinearPolynomialEquation}).
        ++ If no such list of qi exists, then "failed" is returned.
```

```
      factorByRecursion:  SupS -> Factored SupS
         ++ factorByRecursion(p) factors polynomial p. This function
         ++ performs the recursion step for factorPolynomial,
         ++ as defined in \spadfun{PolynomialFactorizationExplicit} category
         ++ (see \spadfun{factorPolynomial})
      factorSquareFreeByRecursion:  SupS -> Factored SupS
         ++ factorSquareFreeByRecursion(p) returns the square free
         ++ factorization of p. This functions performs
         ++ the recursion step for factorSquareFreePolynomial,
         ++ as defined in \spadfun{PolynomialFactorizationExplicit} category
         ++ (see \spadfun{factorSquareFreePolynomial}).
      randomR: -> R  -- has to be global, since has alternative definitions
         ++ randomR produces a random element of R
      bivariateSLPEBR: (List SupS, SupS,  VarSet)  -> Union(List SupS,"failed")
         ++ bivariateSLPEBR(lp,p,v) implements
         ++ the bivariate case of
         ++ \spadfunFrom{solveLinearPolynomialEquationByRecursion}{PolynomialFacto
         ++ its implementation depends on R
      factorSFBRlcUnit: (List VarSet, SupS) -> Factored SupS
         ++ factorSFBRlcUnit(p) returns the square free factorization of
         ++ polynomial p
         ++ (see \spadfun{factorSquareFreeByRecursion}{PolynomialFactorizationByRe
         ++ in the case where the leading coefficient of p
         ++ is a unit.
    private  == add
     supR: SparseUnivariatePolynomial R
     pp: SupS
     lpolys,factors: List SupS
     vv:VarSet
     lvpolys,lvpp: List VarSet
     r:R
     lr:List R
     import FactoredFunctionUtilities(SupS)
     import FactoredFunctions2(S,SupS)
     import FactoredFunctions2(SupR,SupS)
     import CommuteUnivariatePolynomialCategory(S,SupS, SupSupS)
     import UnivariatePolynomialCategoryFunctions2(S,SupS,SupS,SupSupS)
     import UnivariatePolynomialCategoryFunctions2(SupS,SupSupS,S,SupS)
     import UnivariatePolynomialCategoryFunctions2(S,SupS,R,SupR)
     import UnivariatePolynomialCategoryFunctions2(R,SupR,S,SupS)
     import UnivariatePolynomialCategoryFunctions2(S,SupS,SupR,SupSupR)
     import UnivariatePolynomialCategoryFunctions2(SupR,SupSupR,S,SupS)
     hensel: (SupS,VarSet,R,List SupS) ->
             Union(Record(fctrs:List SupS),"failed")
     chooseSLPEViableSubstitutions: (List VarSet,List SupS,SupS) ->
      Record(substnsField:List R,lpolysRField:List SupR,ppRField:SupR)
```

```
  --++ chooseSLPEViableSubstitutions(lv,lp,p) chooses substitutions
  --++ for the variables in first arg (which are all
  --++ the variables that exist) so that the polys in second argument don't
  --++ drop in degree and remain square-free, and third arg doesn't drop
  --++ drop in degree
chooseFSQViableSubstitutions: (List VarSet,SupS) ->
 Record(substnsField:List R,ppRField:SupR)
  --++ chooseFSQViableSubstitutions(lv,p) chooses substitutions for the variables in firs
  --++ the variables that exist) so that the second argument poly doesn't
  --++ drop in degree and remains square-free
raise: SupR -> SupS
lower: SupS -> SupR
SLPEBR: (List SupS, List VarSet, SupS, List VarSet)  ->
                                        Union(List SupS,"failed")
factorSFBRlcUnitInner: (List VarSet, SupS,R) ->
                                        Union(Factored SupS,"failed")
hensel(pp,vv,r,factors) ==
   origFactors:=factors
   totdegree:Integer:=0
   proddegree:Integer:=
               "max"/[degree(u,vv) for u in coefficients pp]
   n:PI:=1
   prime:=vv::S - r::S
   foundFactors:List SupS:=empty()
   while (totdegree <= proddegree) repeat
       pn:=prime**n
       Ecart:=(pp-*/factors) exquo  pn
       Ecart case "failed" =>
             error "failed lifting in hensel in PFBR"
       zero? Ecart =>
          -- then we have all the factors
          return [append(foundFactors, factors)]
       step:=solveLinearPolynomialEquation(origFactors,
                                        map(eval(#1,vv,r),
                                            Ecart))
       step case "failed" => return "failed" -- must be a false split
       factors:=[a+b*pn for a in factors for b in step]
       for a in factors for c in origFactors repeat
           pp1:= pp exquo a
           pp1 case "failed" => "next"
           pp:=pp1
           proddegree := proddegree - "max"/[degree(u,vv)
                                          for u in coefficients a]
           factors:=remove(a,factors)
           origFactors:=remove(c,origFactors)
           foundFactors:=[a,:foundFactors]
```

```
        #factors < 2 =>
           return [(empty? factors => foundFactors;
                                    [pp,:foundFactors])]
        totdegree:= +/["max"/[degree(u,vv)
                              for u in coefficients u1]
                         for u1 in factors]
        n:=n+1
     "failed" -- must have been a false split

  factorSFBRlcUnitInner(lvpp,pp,r) ==
     -- pp is square-free as a Sup, and its coefficients have precisely
     -- the variables of lvpp. Furthermore, its LC is a unit
     -- returns "failed" if the substitution is bad, else a factorization
     ppR:=map(eval(#1,first lvpp,r),pp)
     degree ppR < degree pp => "failed"
     degree gcd(ppR,differentiate ppR) >0 => "failed"
     factors:=
        empty? rest lvpp =>
           fDown:=factorSquareFreePolynomial map(retract(#1)::R,ppR)
           [raise (unit fDown * factorList(fDown).first.fctr),
            :[raise u.fctr for u in factorList(fDown).rest]]
        fSame:=factorSFBRlcUnit(rest lvpp,ppR)
        [unit fSame * factorList(fSame).first.fctr,
         :[uu.fctr for uu in factorList(fSame).rest]]
     #factors = 1 => makeFR(1,[["irred",pp,1]])
     hen:=hensel(pp,first lvpp,r,factors)
     hen case "failed" => "failed"
     makeFR(1,[["irred",u,1] for u in hen.fctrs])
  if R has StepThrough then
    factorSFBRlcUnit(lvpp,pp) ==
      val:R := init()
      while true repeat
         tempAns:=factorSFBRlcUnitInner(lvpp,pp,val)
         not (tempAns case "failed") => return tempAns
         val1:=nextItem val
         val1 case "failed" =>
           error "at this point, we know we have a finite field"
         val:=val1
  else
    factorSFBRlcUnit(lvpp,pp) ==
      val:R := randomR()
      while true repeat
         tempAns:=factorSFBRlcUnitInner(lvpp,pp,val)
         not (tempAns case "failed") => return tempAns
         val := randomR()
  if R has random: -> R then
```

```
   randomR() == random()
else randomR() == (random()$Integer)::R
if R has FiniteFieldCategory then
  bivariateSLPEBR(lpolys,pp,v) ==
    lpolysR:List SupSupR:=[map(univariate,u) for u in lpolys]
    ppR: SupSupR:=map(univariate,pp)
    ans:=solveLinearPolynomialEquation(lpolysR,ppR)$SupR
    ans case "failed" => "failed"
    [map(multivariate(#1,v),w) for w in ans]
else
  bivariateSLPEBR(lpolys,pp,v) ==
    solveLinearPolynomialEquationByFractions(lpolys,pp)$LPEBFS
chooseFSQViableSubstitutions(lvpp,pp) ==
  substns:List R
  ppR: SupR
  while true repeat
     substns:= [randomR() for v in lvpp]
     zero? eval(leadingCoefficient pp,lvpp,substns ) => "next"
     ppR:=map((retract eval(#1,lvpp,substns))::R,pp)
     degree gcd(ppR,differentiate ppR)>0 => "next"
     leave
  [substns,ppR]
chooseSLPEViableSubstitutions(lvpolys,lpolys,pp) ==
  substns:List R
  lpolysR:List SupR
  ppR: SupR
  while true repeat
     substns:= [randomR() for v in lvpolys]
     zero? eval(leadingCoefficient pp,lvpolys,substns ) => "next"
     "or"/[zero? eval(leadingCoefficient u,lvpolys,substns)
                 for u in lpolys] => "next"
     lpolysR:=[map((retract eval(#1,lvpolys,substns))::R,u)
               for u in lpolys]
     uu:=lpolysR
     while not empty? uu repeat
       "or"/[ degree(gcd(uu.first,v))>0 for v in uu.rest] => leave
       uu:=rest uu
     not empty? uu => "next"
     leave
  ppR:=map((retract eval(#1,lvpolys,substns))::R,pp)
  [substns,lpolysR,ppR]
raise(supR) == map(#1:R::S,supR)
lower(pp) == map(retract(#1)::R,pp)
SLPEBR(lpolys,lvpolys,pp,lvpp) ==
  not empty? (m:=setDifference(lvpp,lvpolys)) =>
      v:=first m
```

```
        lvpp:=remove(v,lvpp)
        pp1:SupSupS :=swap map(univariate(#1,v),pp)
        -- pp1 is mathematically equal to pp, but is in S[z][v]
        -- so we wish to operate on all of its coefficients
        ans:List SupSupS:= [0 for u in lpolys]
        for m in reverse_! monomials pp1 repeat
            ans1:=SLPEBR(lpolys,lvpolys,leadingCoefficient m,lvpp)
            ans1 case "failed" => return "failed"
            d:=degree m
            ans:=[monomial(a1,d)+a for a in ans for a1 in ans1]
        [map(multivariate(#1,v),swap pp1) for pp1 in ans]
    empty? lvpolys =>
        lpolysR:List SupR
        ppR:SupR
        lpolysR:=[map(retract,u) for u in lpolys]
        ppR:=map(retract,pp)
        ansR:=solveLinearPolynomialEquation(lpolysR,ppR)
        ansR case "failed" => return "failed"
        [map(#1::S,uu) for uu in ansR]
    cVS:=chooseSLPEViableSubstitutions(lvpolys,lpolys,pp)
    ansR:=solveLinearPolynomialEquation(cVS.lpolysRField,cVS.ppRField)
    ansR case "failed" => "failed"
    #lvpolys = 1 => bivariateSLPEBR(lpolys,pp, first lvpolys)
    solveLinearPolynomialEquationByFractions(lpolys,pp)$LPEBFS

  solveLinearPolynomialEquationByRecursion(lpolys,pp) ==
    lvpolys := removeDuplicates_!
                concat [ concat [variables z for z in coefficients u]
                                                for u in lpolys]
    lvpp := removeDuplicates_!
              concat [variables z for z in coefficients pp]
    SLPEBR(lpolys,lvpolys,pp,lvpp)

  factorByRecursion pp ==
    lv:List(VarSet) := removeDuplicates_!
                         concat [variables z for z in coefficients pp]
    empty? lv =>
        map(raise,factorPolynomial lower pp)
    c:=content pp
    unit? c => refine(squareFree pp,factorSquareFreeByRecursion)
    pp:=(pp exquo c)::SupS
    mergeFactors(refine(squareFree pp,factorSquareFreeByRecursion),
                map(#1:S::SupS,factor(c)$S))
  factorSquareFreeByRecursion pp ==
    lv:List(VarSet) := removeDuplicates_!
                         concat [variables z for z in coefficients pp]
```

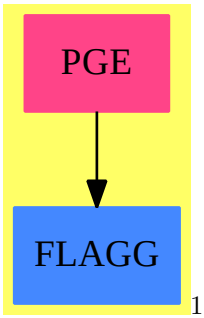
```
        empty? lv =>
            map(raise,factorPolynomial lower pp)
        unit? (lcpp := leadingCoefficient pp) => factorSFBRlcUnit(lv,pp)
        oldnfact:NonNegativeInteger:= 999999
                        -- I hope we never have to factor a polynomial
                        -- with more than this number of factors
        lcppPow:S
        while true repeat
          cVS:=chooseFSQViableSubstitutions(lv,pp)
          factorsR:=factorSquareFreePolynomial(cVS.ppRField)
          (nfact:=numberOfFactors factorsR) = 1 =>
                    return makeFR(1,[["irred",pp,1]])
          -- OK, force all leading coefficients to be equal to the leading
          -- coefficient of the input
          nfact > oldnfact => "next"   -- can't be a good reduction
          oldnfact:=nfact
          factors:=[(lcpp exquo leadingCoefficient u.fctr)::S * raise u.fctr
                    for u in factorList factorsR]
          ppAdjust:=(lcppPow:=lcpp**#(rest factors)) * pp
          lvppList:=lv
          OK:=true
          for u in lvppList for v in cVS.substnsField repeat
              hen:=hensel(ppAdjust,u,v,factors)
              hen case "failed" =>
                  OK:=false
                  "leave"
              factors:=hen.fctrs
          OK => leave
        factors:=[ (lc:=content w;
                    lcppPow:=(lcppPow exquo lc)::S;
                     (w exquo lc)::SupS)
                   for w in factors]
        not unit? lcppPow =>
            error "internal error in factorSquareFreeByRecursion"
        makeFR((recip lcppPow)::S::SupS,
                [["irred",w,1] for w in factors])
```

⟨*PFBR.dotabb*⟩≡

```
"PFBR" [color="#FF4488",href="bookvol10.4.pdf#nameddest=PFBR"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"PFBR" -> "PFECAT"
```

17.87 package PFBRU PolynomialFactorizationByRecursionUnivariate

17.88 PolynomialFactorizationByRecursionUnivariate



1.00

Exports:

randomR	factorByRecursion	factorSFBRlcUnit
factorSquareFreeByRecursion	solveLinearPolynomialEquationByRecursion	

⟨*package PFBRU PolynomialFactorizationByRecursionUnivariate*⟩≡

```
)abbrev package PFBRU PolynomialFactorizationByRecursionUnivariate
++ PolynomialFactorizationByRecursionUnivariate
++ R is a \spadfun{PolynomialFactorizationExplicit} domain,
++ S is univariate polynomials over R
++ We are interested in handling SparseUnivariatePolynomials over
++ S, is a variable we shall call z
PolynomialFactorizationByRecursionUnivariate(R, S): public == private where
  R:PolynomialFactorizationExplicit
  S:UnivariatePolynomialCategory(R)
  PI ==> PositiveInteger
  SupR ==> SparseUnivariatePolynomial R
  SupSupR ==> SparseUnivariatePolynomial SupR
  SupS ==> SparseUnivariatePolynomial S
  SupSupS ==> SparseUnivariatePolynomial SupS
  LPEBFS ==> LinearPolynomialEquationByFractions(S)
  public ==  with
     solveLinearPolynomialEquationByRecursion: (List SupS, SupS)  ->
                                               Union(List SupS,"failed")
        ++ \spad{solveLinearPolynomialEquationByRecursion([p1,...,pn],p)}
        ++ returns the list of polynomials \spad{[q1,...,qn]}
        ++ such that \spad{sum qi/pi = p / prod pi}, a
        ++ recursion step for solveLinearPolynomialEquation
        ++ as defined in \spadfun{PolynomialFactorizationExplicit} category
        ++ (see \spadfun{solveLinearPolynomialEquation}).
        ++ If no such list of qi exists, then "failed" is returned.
```

```
    factorByRecursion:  SupS -> Factored SupS
       ++ factorByRecursion(p) factors polynomial p. This function
       ++ performs the recursion step for factorPolynomial,
       ++ as defined in \spadfun{PolynomialFactorizationExplicit} category
       ++ (see \spadfun{factorPolynomial})
    factorSquareFreeByRecursion:  SupS -> Factored SupS
       ++ factorSquareFreeByRecursion(p) returns the square free
       ++ factorization of p. This functions performs
       ++ the recursion step for factorSquareFreePolynomial,
       ++ as defined in \spadfun{PolynomialFactorizationExplicit} category
       ++ (see \spadfun{factorSquareFreePolynomial}).
    randomR: -> R  -- has to be global, since has alternative definitions
       ++ randomR() produces a random element of R
    factorSFBRlcUnit: (SupS) -> Factored SupS
       ++ factorSFBRlcUnit(p) returns the square free factorization of
       ++ polynomial p
       ++ (see \spadfun{factorSquareFreeByRecursion}{PolynomialFactorizationByRecursionUniv
       ++ in the case where the leading coefficient of p
       ++ is a unit.
  private  == add
   supR: SparseUnivariatePolynomial R
   pp: SupS
   lpolys,factors: List SupS
   r:R
   lr:List R
   import FactoredFunctionUtilities(SupS)
   import FactoredFunctions2(SupR,SupS)
   import FactoredFunctions2(S,SupS)
   import UnivariatePolynomialCategoryFunctions2(S,SupS,R,SupR)
   import UnivariatePolynomialCategoryFunctions2(R,SupR,S,SupS)
   -- local function declarations
   raise: SupR -> SupS
   lower: SupS -> SupR
   factorSFBRlcUnitInner: (SupS,R) -> Union(Factored SupS,"failed")
   hensel: (SupS,R,List SupS) ->
           Union(Record(fctrs:List SupS),"failed")
   chooseFSQViableSubstitutions: (SupS) ->
    Record(substnsField:R,ppRField:SupR)
     --++ chooseFSQViableSubstitutions(p), p is a sup
     --++ ("sparse univariate polynomial")
     --++ over a sup over R, returns a record
     --++ \spad{[substnsField: r, ppRField: q]} where r is a substitution point
     --++ q is a sup over R so that the (implicit) variable in q
     --++ does not drop in degree and remains square-free.
   -- here for the moment, until it compiles
   -- N.B., we know that R is NOT a FiniteField, since
```

```
-- that is meant to have a special implementation, to break the
-- recursion
solveLinearPolynomialEquationByRecursion(lpolys,pp) ==
  lhsdeg:="max"/["max"/[degree v for v in coefficients u] for u in lpolys]
  rhsdeg:="max"/[degree v for v in coefficients pp]
  lhsdeg = 0 =>
    lpolysLower:=[lower u for u in lpolys]
    answer:List SupS := [0 for u in lpolys]
    for i in 0..rhsdeg repeat
      ppx:=map(coefficient(#1,i),pp)
      zero? ppx => "next"
      recAns:= solveLinearPolynomialEquation(lpolysLower,ppx)
      recAns case "failed" => return "failed"
      answer:=[monomial(1,i)$S * raise c + d
                 for c in recAns for d in answer]
    answer
  solveLinearPolynomialEquationByFractions(lpolys,pp)$LPEBFS
-- local function definitions
hensel(pp,r,factors) ==
   -- factors is a relatively prime factorization of pp modulo the ideal
   -- (x-r), with suitably imposed leading coefficients.
   -- This is lifted, without re-combinations, to a factorization
   -- return "failed" if this can't be done
   origFactors:=factors
   totdegree:Integer:=0
   proddegree:Integer:=
               "max"/[degree(u) for u in coefficients pp]
   n:PI:=1
   pn:=prime:=monomial(1,1) - r::S
   foundFactors:List SupS:=empty()
   while (totdegree <= proddegree) repeat
       Ecart:=(pp-*/factors) exquo  pn
       Ecart case "failed" =>
             error "failed lifting in hensel in PFBRU"
       zero? Ecart =>
          -- then we have all the factors
          return [append(foundFactors, factors)]
       step:=solveLinearPolynomialEquation(origFactors,
                                           map(elt(#1,r::S),
                                               Ecart))
       step case "failed" => return "failed" -- must be a false split
       factors:=[a+b*pn for a in factors for b in step]
       for a in factors for c in origFactors repeat
           pp1:= pp exquo a
           pp1 case "failed" => "next"
           pp:=pp1
```

```
            proddegree := proddegree - "max"/[degree(u)
                                                for u in coefficients a]
            factors:=remove(a,factors)
            origFactors:=remove(c,origFactors)
            foundFactors:=[a,:foundFactors]
          #factors < 2 =>
             return [(empty? factors => foundFactors;
                                  [pp,:foundFactors])]
          totdegree:= +/["max"/[degree(u)
                                for u in coefficients u1]
                        for u1 in factors]
          n:=n+1
          pn:=pn*prime
     "failed" -- must have been a false split
  chooseFSQViableSubstitutions(pp) ==
    substns:R
    ppR: SupR
    while true repeat
       substns:= randomR()
       zero? elt(leadingCoefficient pp,substns ) => "next"
       ppR:=map( elt(#1,substns),pp)
       degree gcd(ppR,differentiate ppR)>0 => "next"
       leave
    [substns,ppR]
  raise(supR) == map(#1:R::S,supR)
  lower(pp) == map(retract(#1)::R,pp)
  factorSFBRlcUnitInner(pp,r) ==
     -- pp is square-free as a Sup, but the Up  variable occurs.
     -- Furthermore, its LC is a unit
     -- returns "failed" if the substitution is bad, else a factorization
     ppR:=map(elt(#1,r),pp)
     degree ppR < degree pp => "failed"
     degree gcd(ppR,differentiate ppR) >0 => "failed"
     factors:=
       fDown:=factorSquareFreePolynomial ppR
       [raise (unit fDown * factorList(fDown).first.fctr),
        :[raise u.fctr for u in factorList(fDown).rest]]
     #factors = 1 => makeFR(1,[["irred",pp,1]])
     hen:=hensel(pp,r,factors)
     hen case "failed" => "failed"
     makeFR(1,[["irred",u,1] for u in hen.fctrs])
  -- exported function definitions
  if R has StepThrough then
    factorSFBRlcUnit(pp) ==
      val:R := init()
      while true repeat
```

```
        tempAns:=factorSFBRlcUnitInner(pp,val)
        not (tempAns case "failed") => return tempAns
        val1:=nextItem val
        val1 case "failed" =>
          error "at this point, we know we have a finite field"
        val:=val1
    else
      factorSFBRlcUnit(pp) ==
        val:R := randomR()
        while true repeat
           tempAns:=factorSFBRlcUnitInner(pp,val)
           not (tempAns case "failed") => return tempAns
           val := randomR()
    if R has StepThrough then
       randomCount:R:= init()
       randomR() ==
         v:=nextItem(randomCount)
         v case "failed" =>
           SAY$Lisp "Taking another set of random values"
           randomCount:=init()
           randomCount
         randomCount:=v
         randomCount
    else if R has random: -> R then
       randomR() == random()
    else randomR() == (random()$Integer rem 100)::R
    factorByRecursion pp ==
      and/[zero? degree u for u in coefficients pp] =>
          map(raise,factorPolynomial lower pp)
      c:=content pp
      unit? c => refine(squareFree pp,factorSquareFreeByRecursion)
      pp:=(pp exquo c)::SupS
      mergeFactors(refine(squareFree pp,factorSquareFreeByRecursion),
                   map(#1:S::SupS,factor(c)$S))
    factorSquareFreeByRecursion pp ==
      and/[zero? degree u for u in coefficients pp] =>
         map(raise,factorSquareFreePolynomial lower pp)
      unit? (lcpp := leadingCoefficient pp) => factorSFBRlcUnit(pp)
      oldnfact:NonNegativeInteger:= 999999
                       -- I hope we never have to factor a polynomial
                       -- with more than this number of factors
      lcppPow:S
      while true repeat  -- a loop over possible false splits
        cVS:=chooseFSQViableSubstitutions(pp)
        newppR:=primitivePart cVS.ppRField
        factorsR:=factorSquareFreePolynomial(newppR)
```

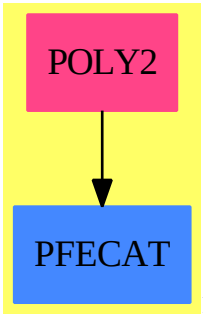
```
        (nfact:=numberOfFactors factorsR) = 1 =>
                  return makeFR(1,[["irred",pp,1]])
        -- OK, force all leading coefficients to be equal to the leading
        -- coefficient of the input
        nfact > oldnfact => "next"   -- can't be a good reduction
        oldnfact:=nfact
        lcppR:=leadingCoefficient cVS.ppRField
        factors:=[raise((lcppR exquo leadingCoefficient u.fctr) ::R * u.fctr)
                  for u in factorList factorsR]
        -- factors now multiplies to give cVS.ppRField * lcppR^(#factors-1)
        -- Now change the leading coefficient to be lcpp
        factors:=[monomial(lcpp,degree u) + reductum u for u in factors]
--      factors:=[(lcpp exquo leadingCoefficient u.fctr)::S * raise u.fctr
--                for u in factorList factorsR]
        ppAdjust:=(lcppPow:=lcpp**#(rest factors)) * pp
        OK:=true
        hen:=hensel(ppAdjust,cVS.substnsField,factors)
        hen case "failed" => "next"
        factors:=hen.fctrs
        leave
      factors:=[ (lc:=content w;
                  lcppPow:=(lcppPow exquo lc)::S;
                   (w exquo lc)::SupS)
                 for w in factors]
      not unit? lcppPow =>
          error "internal error in factorSquareFreeByRecursion"
      makeFR((recip lcppPow)::S::SupS,
              [["irred",w,1] for w in factors])
```

⟨*PFBRU.dotabb*⟩≡

```
"PFBRU" [color="#FF4488",href="bookvol10.4.pdf#nameddest=PFBRU"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"PFBRU" -> "PFECAT"
```

17.89 package POLY2 PolynomialFunctions2

17.90 PolynomialFunctions2



1.00

Exports:
map

⟨*package POLY2 PolynomialFunctions2*⟩≡
```
)abbrev package POLY2 PolynomialFunctions2
++ Author:
++ Date Created:
++ Date Last Updated:
++ Basic Functions:
++ Related Constructors:
++ Also See:
++ AMS Classifications:
++ Keywords:
++ References:
++ Description:
++   This package takes a mapping between coefficient rings, and lifts
++ it to a mapping between polynomials over those rings.

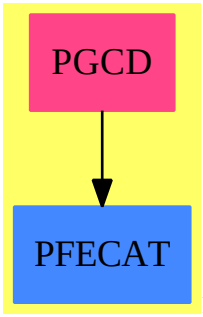
PolynomialFunctions2(R:Ring, S:Ring): with
  map: (R -> S, Polynomial R) -> Polynomial S
    ++ map(f, p) produces a new polynomial as a result of applying
    ++ the function f to every coefficient of the polynomial p.
 == add
  map(f, p) == map(#1::Polynomial(S), f(#1)::Polynomial(S),
                   p)$PolynomialCategoryLifting(IndexedExponents Symbol,
                                  Symbol, R, Polynomial R, Polynomial S)
```

⟨*POLY2.dotabb*⟩≡

```
"POLY2" [color="#FF4488",href="bookvol10.4.pdf#nameddest=POLY2"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"POLY2" -> "PFECAT"
```

17.91 package PGCD PolynomialGcdPackage

17.92 PolynomialGcdPackage



1.00

Exports:
gcd gcdPrimitive

⟨*package PGCD PolynomialGcdPackage*⟩≡

```
)abbrev package PGCD PolynomialGcdPackage
++ Author: Michael Lucks, P. Gianni
++ Date Created:
++ Date Last Updated: 17 June 1996
++ Fix History: Moved unitCanonicals for performance (BMT);
++              Fixed a problem with gcd(x,0) (Frederic Lehobey)
++ Basic Functions: gcd, content
++ Related Constructors: Polynomial
++ Also See:
++ AMS Classifications:
++ Keywords:
++ References:
++ Description:
++   This package computes multivariate polynomial gcd's using
++ a hensel lifting strategy. The contraint on the coefficient
++ domain is imposed by the lifting strategy. It is assumed that
++ the coefficient domain has the property that almost all specializations
++ preserve the degree of the gcd.

I        ==> Integer
NNI      ==> NonNegativeInteger
PI       ==> PositiveInteger

PolynomialGcdPackage(E,OV,R,P):C == T where
    R     :  EuclideanDomain
    P     :  PolynomialCategory(R,E,OV)
    OV    :  OrderedSet
```

```
E     :  OrderedAbelianMonoidSup

SUPP      ==> SparseUnivariatePolynomial P

C == with
  gcd               :   (P,P)    -> P
    ++ gcd(p,q) computes the gcd of the two polynomials p and q.
  gcd               :   List P   -> P
    ++ gcd(lp) computes the gcd of the list of polynomials lp.
  gcd               :   (SUPP,SUPP)    -> SUPP
    ++ gcd(p,q) computes the gcd of the two polynomials p and q.
  gcd               :   List SUPP   -> SUPP
    ++ gcd(lp) computes the gcd of the list of polynomials lp.
  gcdPrimitive      :   (P,P)    -> P
    ++ gcdPrimitive(p,q) computes the gcd of the primitive polynomials
    ++ p and q.
  gcdPrimitive      :   (SUPP,SUPP)    -> SUPP
    ++ gcdPrimitive(p,q) computes the gcd of the primitive polynomials
    ++ p and q.
  gcdPrimitive      :   List P   -> P
    ++ gcdPrimitive lp computes the gcd of the list of primitive
    ++ polynomials lp.

T == add

  SUP      ==> SparseUnivariatePolynomial R

  LGcd     ==> Record(locgcd:SUPP,goodint:List List R)
  UTerm    ==> Record(lpol:List SUP,lint:List List R,mpol:SUPP)
  pmod:R   :=  (prevPrime(2**26)$IntegerPrimesPackage(Integer))::R

  import MultivariateLifting(E,OV,R,P)
  import FactoringUtilities(E,OV,R,P)

            --------  Local  Functions  --------

  myran           :    Integer   -> Union(R,"failed")
  better          :    (P,P)     -> Boolean
  failtest        :   (SUPP,SUPP,SUPP)    -> Boolean
  monomContent    :   (SUPP)   -> SUPP
  gcdMonom        :  (SUPP,SUPP)    -> SUPP
  gcdTermList     :    (P,P)     -> P
  good            :  (SUPP,List OV,List List R) -> Record(upol:SUP,inval:List List R)

  chooseVal       :  (SUPP,SUPP,List OV,List List R) -> Union(UTerm,"failed")
  localgcd        :  (SUPP,SUPP,List OV,List List R) -> LGcd
```

```
    notCoprime      :  (SUPP,SUPP, List NNI,List OV,List List R)   -> SUPP
    imposelc        : (List SUP,List OV,List R,List P) -> List SUP

    lift? :(SUPP,SUPP,UTerm,List NNI,List OV) -> Union(s:SUPP,failed:"failed",n
    lift  :(SUPP,SUP,SUP,P,List OV,List NNI,List R) -> Union(SUPP,"failed")

                    ---- Local  functions ----
  -- test if something wrong happened in the gcd
    failtest(f:SUPP,p1:SUPP,p2:SUPP) : Boolean ==
      (p1 exquo f) case "failed" or (p2 exquo f) case "failed"

  -- Choose the integers
    chooseVal(p1:SUPP,p2:SUPP,lvr:List OV,ltry:List List R):Union(UTerm,"failed
      d1:=degree(p1)
      d2:=degree(p2)
      dd:NNI:=0$NNI
      nvr:NNI:=#lvr
      lval:List R :=[]
      range:I:=8
      repeat
        range:=2*range
        lval:=[ran(range) for i in 1..nvr]
        member?(lval,ltry) => "new point"
        ltry:=cons(lval,ltry)
        uf1:SUP:=completeEval(p1,lvr,lval)
        degree uf1 ^= d1 => "new point"
        uf2:SUP:= completeEval(p2,lvr,lval)
        degree uf2 ^= d2 => "new point"
        u:=gcd(uf1,uf2)
        du:=degree u
       --the univariate gcd is 1
        if du=0 then return [[1$SUP],ltry,0$SUPP]$UTerm

        ugcd:List SUP:=[u,(uf1 exquo u)::SUP,(uf2 exquo u)::SUP]
        uterm:=[ugcd,ltry,0$SUPP]$UTerm
        dd=0 => dd:=du

      --the degree is not changed
        du=dd =>

         --test if one of the polynomials is the gcd
          dd=d1 =>
            if ^((f:=p2 exquo p1) case "failed") then
              return [[u],ltry,p1]$UTerm
            if dd^=d2 then dd:=(dd-1)::NNI
```

```
            dd=d2 =>
              if ^((f:=p1 exquo p2) case "failed") then
                return [[u],ltry,p2]$UTerm
              dd:=(dd-1)::NNI
            return uterm

           --the new gcd has degree less
            du<dd => dd:=du

        good(f:SUPP,lvr:List OV,ltry:List List R):Record(upol:SUP,inval:List List R) ==
          nvr:NNI:=#lvr
          range:I:=1
          while true repeat
            range:=2*range
            lval:=[ran(range) for i in 1..nvr]
            member?(lval,ltry) => "new point"
            ltry:=cons(lval,ltry)
            uf:=completeEval(f,lvr,lval)
            if degree gcd(uf,differentiate uf)=0 then return [uf,ltry]

        -- impose the right lc
        imposelc(lipol:List SUP,
                 lvar:List OV,lval:List R,leadc:List P):List SUP ==
          result:List SUP :=[]
          for pol in lipol for leadpol in leadc repeat
             p1:= univariate eval(leadpol,lvar,lval) * pol
             result:= cons((p1 exquo leadingCoefficient pol)::SUP,result)
          reverse result

      --Compute the gcd between not coprime polynomials
        notCoprime(g:SUPP,p2:SUPP,ldeg:List NNI,lvar1:List OV,ltry:List List R) : SUPP ==
          g1:=gcd(g,differentiate g)
          l1 := (g exquo g1)::SUPP
          lg:LGcd:=localgcd(l1,p2,lvar1,ltry)
          (l,ltry):=(lg.locgcd,lg.goodint)
          lval:=ltry.first
          p2l:=(p2 exquo l)::SUPP
          (gd1,gd2):=(l,l)
          ul:=completeEval(l,lvar1,lval)
          dl:=degree ul
          if degree gcd(ul,differentiate ul) ^=0 then
            newchoice:=good(l,lvar1,ltry)
            ul:=newchoice.upol
            ltry:=newchoice.inval
            lval:=ltry.first
          ug1:=completeEval(g1,lvar1,lval)
```

```
          ulist:=[ug1,completeEval(p2l,lvar1,lval)]
          lcpol:List P:=[leadingCoefficient g1, leadingCoefficient p2]
          while true repeat
            d:SUP:=gcd(cons(ul,ulist))
            if degree d =0 then return gd1
            lquo:=(ul exquo d)::SUP
            if degree lquo ^=0 then
              lgcd:=gcd(cons(leadingCoefficient l,lcpol))
              (gdl:=lift(l,d,lquo,lgcd,lvar1,ldeg,lval)) case "failed" =>
                 return notCoprime(g,p2,ldeg,lvar1,ltry)
              l:=gd2:=gdl::SUPP
              ul:=completeEval(l,lvar1,lval)
              dl:=degree ul
            gd1:=gd1*gd2
            ulist:=[(uf exquo d)::SUP for uf in ulist]

        gcdPrimitive(p1:SUPP,p2:SUPP) : SUPP ==
          if (d1:=degree(p1)) > (d2:=degree(p2)) then
              (p1,p2):= (p2,p1)
              (d1,d2):= (d2,d1)
          degree p1 = 0 =>
              p1 = 0 => unitCanonical p2
              unitCanonical p1
          lvar:List OV:=sort(#1>#2,setUnion(variables p1,variables p2))
          empty? lvar =>
             raisePolynomial(gcd(lowerPolynomial p1,lowerPolynomial p2))
          (p2 exquo p1) case SUPP => unitCanonical p1
          ltry:List List R:=empty()
          totResult:=localgcd(p1,p2,lvar,ltry)
          result: SUPP:=totResult.locgcd
          -- special cases
          result=1 => 1$SUPP
          while failtest(result,p1,p2) repeat
--          SAY$Lisp  "retrying gcd"
            ltry:=totResult.goodint
            totResult:=localgcd(p1,p2,lvar,ltry)
            result:=totResult.locgcd
          result

      --local function for the gcd : it returns the evaluation point too
          localgcd(p1:SUPP,p2:SUPP,lvar:List(OV),ltry:List List R) : LGcd ==
            uterm:=chooseVal(p1,p2,lvar,ltry)::UTerm
            ltry:=uterm.lint
            listpol:= uterm.lpol
            ud:=listpol.first
            dd:= degree ud
```

```
      --the univariate gcd is 1
      dd=0 => [1$SUPP,ltry]$LGcd

      --one of the polynomials is the gcd
      dd=degree(p1) or dd=degree(p2) =>
                      [uterm.mpol,ltry]$LGcd
      ldeg:List NNI:=map(min,degree(p1,lvar),degree(p2,lvar))

     -- if there is a polynomial g s.t. g/gcd and gcd are coprime ...
      -- I can lift
      (h:=lift?(p1,p2,uterm,ldeg,lvar)) case notCoprime =>
        [notCoprime(p1,p2,ldeg,lvar,ltry),ltry]$LGcd
      h case failed => localgcd(p1,p2,lvar,ltry) -- skip bad values?
      [h.s,ltry]$LGcd


  -- content, internal functions return the poly if it is a monomial
    monomContent(p:SUPP):SUPP ==
      degree(p)=0 => 1
      md:= minimumDegree(p)
      monomial(gcd sort(better,coefficients p),md)

  -- Ordering for gcd purposes
    better(p1:P,p2:P):Boolean ==
      ground? p1 => true
      ground? p2 => false
      degree(p1,mainVariable(p1)::OV) < degree(p2,mainVariable(p2)::OV)

  -- Gcd between polynomial p1 and p2 with
  -- mainVariable p1 < x=mainVariable p2
    gcdTermList(p1:P,p2:P) : P ==
      termList:=sort(better,
         cons(p1,coefficients univariate(p2,(mainVariable p2)::OV)))
      q:P:=termList.first
      for term in termList.rest until q = 1$P repeat q:= gcd(q,term)
      q

  -- Gcd between polynomials with the same mainVariable
    gcd(p1:SUPP,p2:SUPP): SUPP ==
      if degree(p1) > degree(p2) then (p1,p2):= (p2,p1)
      degree p1 = 0 =>
         p1 = 0 => unitCanonical p2
         p1 = 1 => unitCanonical p1
         gcd(leadingCoefficient p1, content p2)::SUPP
      reductum(p1)=0 => gcdMonom(p1,monomContent p2)
```

```
      c1:= monomContent(p1)
      reductum(p2)=0 => gcdMonom(c1,p2)
      c2:= monomContent(p2)
      p1:= (p1 exquo c1)::SUPP
      p2:= (p2 exquo c2)::SUPP
      gcdPrimitive(p1,p2) * gcdMonom(c1,c2)

   -- gcd between 2 monomials
      gcdMonom(m1:SUPP,m2:SUPP):SUPP ==
        monomial(gcd(leadingCoefficient(m1),leadingCoefficient(m2)),
                 min(degree(m1),degree(m2)))


    --If there is a pol s.t. pol/gcd and gcd are coprime I can lift
      lift?(p1:SUPP,p2:SUPP,uterm:UTerm,ldeg:List NNI,
                    lvar:List OV) : Union(s:SUPP,failed:"failed",notCoprime:"not
        leadpol:Boolean:=false
        (listpol,lval):=(uterm.lpol,uterm.lint.first)
        d:=listpol.first
        listpol:=listpol.rest
        nolift:Boolean:=true
        for uf in listpol repeat
              --note uf and d not necessarily primitive
          degree gcd(uf,d) =0 => nolift:=false
        nolift => ["notCoprime"]
        f:SUPP:=([p1,p2]$List(SUPP)).(position(uf,listpol))
        lgcd:=gcd(leadingCoefficient p1, leadingCoefficient p2)
        (l:=lift(f,d,uf,lgcd,lvar,ldeg,lval)) case "failed" => ["failed"]
        [l :: SUPP]

   -- interface with the general "lifting" function
      lift(f:SUPP,d:SUP,uf:SUP,lgcd:P,lvar:List OV,
                       ldeg:List NNI,lval:List R):Union(SUPP,"failed") ==
        leadpol:Boolean:=false
        lcf:P
        lcf:=leadingCoefficient f
        df:=degree f
        leadlist:List(P):=[]

        if lgcd^=1 then
          leadpol:=true
          f:=lgcd*f
          ldeg:=[n0+n1 for n0 in ldeg for n1 in degree(lgcd,lvar)]
          lcd:R:=leadingCoefficient d
          if degree(lgcd)=0 then d:=((retract lgcd) *d exquo lcd)::SUP
          else d:=(retract(eval(lgcd,lvar,lval)) * d exquo lcd)::SUP
```

```
          uf:=lcd*uf
        leadlist:=[lgcd,lcf]
        lg:=imposelc([d,uf],lvar,lval,leadlist)
        (pl:=lifting(f,lvar,lg,lval,leadlist,ldeg,pmod)) case "failed" =>
               "failed"
        plist := pl :: List SUPP
        (p0:SUPP,p1:SUPP):=(plist.first,plist.2)
        if completeEval(p0,lvar,lval) ^= lg.first then
           (p0,p1):=(p1,p0)
        ^leadpol => p0
        p0 exquo content(p0)

    -- Gcd for two multivariate polynomials
        gcd(p1:P,p2:P) : P ==
          ground? p1 =>
            p1 := unitCanonical p1
            p1 = 1$P => p1
            p1 = 0$P => unitCanonical p2
            ground? p2 => gcd((retract p1)@R,(retract p2)@R)::P
            gcdTermList(p1,p2)
          ground? p2 =>
            p2 := unitCanonical p2
            p2 = 1$P => p2
            p2 = 0$P => unitCanonical p1
            gcdTermList(p2,p1)
          (p1:= unitCanonical(p1)) = (p2:= unitCanonical(p2)) => p1
          mv1:= mainVariable(p1)::OV
          mv2:= mainVariable(p2)::OV
          mv1 = mv2 => multivariate(gcd(univariate(p1,mv1),
                                        univariate(p2,mv1)),mv1)
          mv1 < mv2 => gcdTermList(p1,p2)
          gcdTermList(p2,p1)

    -- Gcd for a list of multivariate polynomials
        gcd(listp:List P) : P ==
          lf:=sort(better,listp)
          f:=lf.first
          for g in lf.rest repeat
            f:=gcd(f,g)
            if f=1$P then return f
          f

        gcd(listp:List SUPP) : SUPP ==
          lf:=sort(degree(#1)<degree(#2),listp)
          f:=lf.first
          for g in lf.rest repeat
```

```
          f:=gcd(f,g)
          if f=1 then return f
        f


    -- Gcd for primitive polynomials
       gcdPrimitive(p1:P,p2:P):P ==
         (p1:= unitCanonical(p1)) = (p2:= unitCanonical(p2)) => p1
         ground? p1 =>
           ground? p2 => gcd((retract p1)@R,(retract p2)@R)::P
           p1 = 0$P => p2
           1$P
         ground? p2 =>
           p2 = 0$P => p1
           1$P
         mv1:= mainVariable(p1)::OV
         mv2:= mainVariable(p2)::OV
         mv1 = mv2 =>
           md:=min(minimumDegree(p1,mv1),minimumDegree(p2,mv2))
           mp:=1$P
           if md>1 then
             mp:=(mv1::P)**md
             p1:=(p1 exquo mp)::P
             p2:=(p2 exquo mp)::P
           up1 := univariate(p1,mv1)
           up2 := univariate(p2,mv2)
           mp*multivariate(gcdPrimitive(up1,up2),mv1)
         1$P

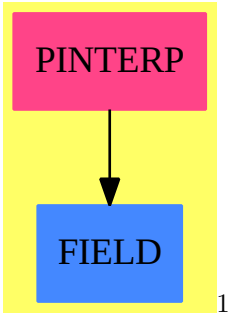
    -- Gcd for a list of primitive multivariate polynomials
        gcdPrimitive(listp:List P) : P ==
          lf:=sort(better,listp)
          f:=lf.first
          for g in lf.rest repeat
            f:=gcdPrimitive(f,g)
            if f=1$P then return f
          f
```

⟨*PGCD.dotabb*⟩≡

```
"PGCD" [color="#FF4488",href="bookvol10.4.pdf#nameddest=PGCD"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"PGCD" -> "PFECAT"
```

17.93 package PINTERP PolynomialInterpolation

17.94 PolynomialInterpolation



1.00

Exports:
interpolate

⟨*package PINTERP PolynomialInterpolation*⟩≡
```
)abbrev package PINTERP PolynomialInterpolation
++ Description:
++ This package exports interpolation algorithms
PolynomialInterpolation(xx, F): Cat == Body   where
    xx: Symbol
    F:  Field
    UP  ==> UnivariatePolynomial
    SUP ==> SparseUnivariatePolynomial

    Cat ==> with
        interpolate: (UP(xx,F), List F, List F) -> UP(xx,F)
                ++ interpolate(u,lf,lg) \undocumented
        interpolate: (List F, List F)           -> SUP F
                ++ interpolate(lf,lg) \undocumented

    Body ==> add
        PIA ==> PolynomialInterpolationAlgorithms

        interpolate(qx, lx, ly) ==
            px := LagrangeInterpolation(lx, ly)$PIA(F, UP(xx, F))
            elt(px, qx)

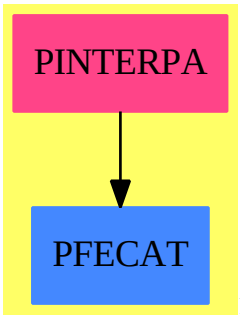
        interpolate(lx, ly) ==
            LagrangeInterpolation(lx, ly)$PIA(F, SUP F)
```

⟨*PINTERP.dotabb*⟩≡

```
"PINTERP" [color="#FF4488",href="bookvol10.4.pdf#nameddest=PINTERP"]
"FIELD"  [color="#4488FF",href="bookvol10.2.pdf#nameddest=FIELD"]
"PINTERP" -> "FIELD"
```

17.95 package PINTERPA PolynomialInterpolationAlgorithms

17.96 PolynomialInterpolationAlgorithms



1.00

Exports:
LagrangeInterpolation

⟨*package PINTERPA PolynomialInterpolationAlgorithms*⟩≡
```
)abbrev package PINTERPA PolynomialInterpolationAlgorithms
++ Description:
++ This package exports interpolation algorithms
PolynomialInterpolationAlgorithms(F, P): Cat == Body   where
    F: Field
    P: UnivariatePolynomialCategory(F)

    Cat ==> with
        LagrangeInterpolation: (List F, List F) -> P
                ++ LagrangeInterpolation(l1,l2) \undocumented

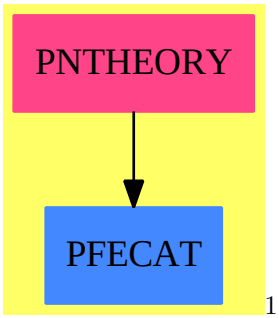
    Body ==> add
        LagrangeInterpolation(lx, ly) ==
            #lx ^= #ly =>
                error "Different number of points and values."
            ip: P := 0
            for xi in lx for yi in ly for i in 0.. repeat
                pp: P := 1
                xp: F := 1
                for xj in lx for j in 0.. | i ^= j repeat
                    pp := pp * (monomial(1,1) - monomial(xj,0))
                    xp := xp * (xi - xj)
                ip := ip + (yi/xp) * pp
            ip
```

⟨*PINTERPA.dotabb*⟩≡

```
"PINTERPA" [color="#FF4488",href="bookvol10.4.pdf#nameddest=PINTERPA"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"PINTERPA" -> "PFECAT"
```

17.97 package PNTHEORY PolynomialNumberTheoryFunctions

17.98 PolynomialNumberTheoryFunctions



1.00

Exports:

bernoulli	chebyshevT	chebyshevU	cyclotomic	euler
fixedDivisor	hermite	laguerre	legendre	

⟨*package PNTHEORY PolynomialNumberTheoryFunctions*⟩≡

```
)abbrev package PNTHEORY PolynomialNumberTheoryFunctions
++ Author: Michael Monagan, Clifton J. Williamson
++ Date Created: June 1987
++ Date Last Updated: 10 November 1996 (Claude Quitte)
++ Basic Operations:
++ Related Domains:
++ Also See:
++ AMS Classifications:
++ Keywords: polynomial, number theory
++ Examples:
++ References: Knuth, The Art of Computer Programming Vol.2
++ Description:
++ This package provides various polynomial number theoretic functions
++ over the integers.
PolynomialNumberTheoryFunctions(): Exports == Implementation where
 I ==> Integer
 RN ==> Fraction I
 SUP ==> SparseUnivariatePolynomial
 NNI ==> NonNegativeInteger

 Exports ==> with
  bernoulli : I -> SUP RN
    ++ bernoulli(n) returns the nth Bernoulli polynomial \spad{B[n](x)}.
    ++ Bernoulli polynomials denoted \spad{B(n,x)} computed by solving the
    ++ differential equation  \spad{differentiate(B(n,x),x) = n B(n-1,x)} where
```

```
    ++ \spad{B(0,x) = 1} and initial condition comes from \spad{B(n) = B(n,0)}.
  chebyshevT: I -> SUP I
    ++ chebyshevT(n) returns the nth Chebyshev polynomial \spad{T[n](x)}.
    ++ Note: Chebyshev polynomials of the first kind, denoted \spad{T[n](x)},
    ++ computed from the two term recurrence.  The generating function
    ++ \spad{(1-t*x)/(1-2*t*x+t**2) = sum(T[n](x)*t**n, n=0..infinity)}.
  chebyshevU: I -> SUP I
    ++ chebyshevU(n) returns the nth Chebyshev polynomial \spad{U[n](x)}.
    ++ Note: Chebyshev polynomials of the second kind, denoted \spad{U[n](x)},
    ++ computed from the two term recurrence.  The generating function
    ++ \spad{1/(1-2*t*x+t**2) = sum(T[n](x)*t**n, n=0..infinity)}.
  cyclotomic: I -> SUP I
    ++ cyclotomic(n) returns the nth cyclotomic polynomial \spad{phi[n](x)}.
    ++ Note: \spad{phi[n](x)} is the factor of \spad{x**n - 1} whose roots
    ++ are the primitive nth roots of unity.
  euler     : I -> SUP RN
    ++ euler(n) returns the nth Euler polynomial \spad{E[n](x)}.
    ++ Note: Euler polynomials denoted \spad{E(n,x)} computed by solving the
    ++ differential equation  \spad{differentiate(E(n,x),x) = n E(n-1,x)} where
    ++ \spad{E(0,x) = 1} and initial condition comes from \spad{E(n) = 2**n E(n,1
  fixedDivisor: SUP I -> I
    ++ fixedDivisor(a) for \spad{a(x)} in \spad{Z[x]} is the largest integer
    ++ f such that f divides \spad{a(x=k)} for all integers k.
    ++ Note: fixed divisor of \spad{a} is
    ++ \spad{reduce(gcd,[a(x=k) for k in 0..degree(a)])}.
  hermite   : I -> SUP I
    ++ hermite(n) returns the nth Hermite polynomial \spad{H[n](x)}.
    ++ Note: Hermite polynomials, denoted \spad{H[n](x)}, are computed from
    ++ the two term recurrence.  The generating function is:
    ++ \spad{exp(2*t*x-t**2) = sum(H[n](x)*t**n/n!, n=0..infinity)}.
  laguerre  : I -> SUP I
    ++ laguerre(n) returns the nth Laguerre polynomial \spad{L[n](x)}.
    ++ Note: Laguerre polynomials, denoted \spad{L[n](x)}, are computed from
    ++ the two term recurrence.  The generating function is:
    ++ \spad{exp(x*t/(t-1))/(1-t) = sum(L[n](x)*t**n/n!, n=0..infinity)}.
  legendre  : I -> SUP RN
    ++ legendre(n) returns the nth Legendre polynomial \spad{P[n](x)}.
    ++ Note: Legendre polynomials, denoted \spad{P[n](x)}, are computed from
    ++ the two term recurrence.  The generating function is:
    ++ \spad{1/sqrt(1-2*t*x+t**2) = sum(P[n](x)*t**n, n=0..infinity)}.
 Implementation ==> add
  import IntegerPrimesPackage(I)

  x := monomial(1,1)$SUP(I)
  y := monomial(1,1)$SUP(RN)
```

```
-- For functions computed via a fixed term recurrence we record
-- previous values so that the next value can be computed directly

E : Record(En:I, Ev:SUP(RN)) := [0,1]
B : Record( Bn:I, Bv:SUP(RN) ) := [0,1]
H : Record( Hn:I, H1:SUP(I), H2:SUP(I) ) := [0,1,x]
L : Record( Ln:I, L1:SUP(I), L2:SUP(I) ) := [0,1,x]
P : Record( Pn:I, P1:SUP(RN), P2:SUP(RN) ) := [0,1,y]
CT : Record( Tn:I, T1:SUP(I), T2:SUP(I) ) := [0,1,x]
U : Record( Un:I, U1:SUP(I), U2:SUP(I) ) := [0,1,0]

MonicQuotient: (SUP(I),SUP(I)) -> SUP(I)
MonicQuotient (a,b) ==
  leadingCoefficient(b) ^= 1 => error "divisor must be monic"
  b = 1 => a
  da := degree a
  db := degree b            -- assertion: degree b > 0
  q:SUP(I) := 0
  while da >= db repeat
    t := monomial(leadingCoefficient a, (da-db)::NNI)
    a := a - b * t
    q := q + t
    da := degree a
  q

cyclotomic n ==
  --++ cyclotomic polynomial denoted phi[n](x)
  p:I; q:I; r:I; s:I; m:NNI; c:SUP(I); t:SUP(I)
  n < 0 => error "cyclotomic not defined for negative integers"
  n = 0 => x
  k := n; s := p := 1
  c := x - 1
  while k > 1 repeat
    p := nextPrime p
    (q,r) := divide(k, p)
    if r = 0 then
      while r = 0 repeat (k := q; (q,r) := divide(k,p))
      t := multiplyExponents(c,p::NNI)
      c := MonicQuotient(t,c)
      s := s * p
  m := (n quo s) :: NNI
  multiplyExponents(c,m)

euler n ==
  p : SUP(RN); t : SUP(RN); c : RN; s : I
  n < 0 => error "euler not defined for negative integers"
```

```
    if n < E.En then (s,p) := (0$I,1$SUP(RN)) else (s,p) := E
    -- (s,p) := if n < E.En then (0,1) else E
    for i in s+1 .. n repeat
      t := (i::RN) * integrate p
      c := euler(i)$IntegerNumberTheoryFunctions / 2**(i::NNI) - t(1/2)
      p := t + c::SUP(RN)
    E.En := n
    E.Ev := p
    p

  bernoulli n ==
    p : SUP RN; t : SUP RN; c : RN; s : I
    n < 0 => error "bernoulli not defined for negative integers"
    if n < B.Bn then (s,p) := (0$I,1$SUP(RN)) else (s,p) := B
    -- (s,p) := if n < B.Bn then (0,1) else B
    for i in s+1 .. n repeat
      t := (i::RN) * integrate p
      c := bernoulli(i)$IntegerNumberTheoryFunctions
      p := t + c::SUP(RN)
    B.Bn := n
    B.Bv := p
    p

  fixedDivisor a ==
    g:I; d:NNI; SUP(I)
    d := degree a
    g := coefficient(a, minimumDegree a)
    for k in 1..d while g > 1 repeat g := gcd(g,a k)
    g

  hermite n ==
    s : I; p : SUP(I); q : SUP(I)
    n < 0 => error "hermite not defined for negative integers"
    -- (s,p,q) := if n < H.Hn then (0,1,x) else H
    if n < H.Hn then (s := 0; p := 1; q := x) else (s,p,q) := H
    for k in s+1 .. n repeat (p,q) := (2*x*p-2*(k-1)*q,p)
    H.Hn := n
    H.H1 := p
    H.H2 := q
    p

  legendre n ==
    s:I; t:I; p:SUP(RN); q:SUP(RN)
    n < 0 => error "legendre not defined for negative integers"
    -- (s,p,q) := if n < P.Pn then (0,1,y) else P
    if n < P.Pn then (s := 0; p := 1; q := y) else (s,p,q) := P
```

```
    for k in s+1 .. n repeat
      t := k-1
      (p,q) := ((k+t)$I/k*y*p - t/k*q,p)
    P.Pn := n
    P.P1 := p
    P.P2 := q
    p

  laguerre n ==
    k:I; s:I; t:I; p:SUP(I); q:SUP(I)
    n < 0 => error "laguerre not defined for negative integers"
    -- (s,p,q) := if n < L.Ln then (0,1,x) else L
    if n < L.Ln then (s := 0; p := 1; q := x) else (s,p,q) := L
    for k in s+1 .. n repeat
      t := k-1
      (p,q) := ((((k+t)$I)::SUP(I)-x)*p-t**2*q,p)
    L.Ln := n
    L.L1 := p
    L.L2 := q
    p

  chebyshevT n ==
    s : I; p : SUP(I); q : SUP(I)
    n < 0 => error "chebyshevT not defined for negative integers"
    -- (s,p,q) := if n < CT.Tn then (0,1,x) else CT
    if n < CT.Tn then (s := 0; p := 1; q := x) else (s,p,q) := CT
    for k in s+1 .. n repeat (p,q) := ((2*x*p - q),p)
    CT.Tn := n
    CT.T1 := p
    CT.T2 := q
    p

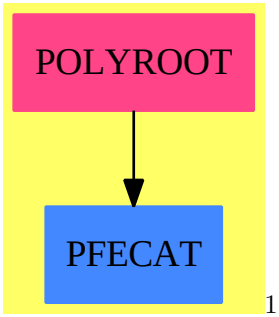
  chebyshevU n ==
    s : I; p : SUP(I); q : SUP(I)
    n < 0 => error "chebyshevU not defined for negative integers"
    if n < U.Un then (s := 0; p := 1; q := 0) else (s,p,q) := U
    for k in s+1 .. n repeat (p,q) := ((2*x*p - q),p)
    U.Un := n
    U.U1 := p
    U.U2 := q
    p
```

⟨*PNTHEORY.dotabb*⟩≡

```
"PNTHEORY" [color="#FF4488",href="bookvol10.4.pdf#nameddest=PNTHEORY"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"PNTHEORY" -> "PFECAT"
```

17.99 package POLYROOT PolynomialRoots

17.100 PolynomialRoots



1.00

Exports:
froot nthr qroot rroot

⟨*package POLYROOT PolynomialRoots*⟩≡
```
)abbrev package POLYROOT PolynomialRoots
++ Author: Manuel Bronstein
++ Date Created: 15 July 1988
++ Date Last Updated: 10 November 1993
++ Description: computes n-th roots of quotients of
++ multivariate polynomials
-- not visible to the user
PolynomialRoots(E, V, R, P, F):Exports == Implementation where
  E: OrderedAbelianMonoidSup
  V: OrderedSet
  R: IntegralDomain
  P: PolynomialCategory(R, E, V)
  F: Field with
    numer : $ -> P
        ++ numer(x) \undocumented
    denom : $ -> P
        ++ denom(x) \undocumented
    coerce: P -> $
        ++ coerce(p) \undocumented

  N   ==> NonNegativeInteger
  Z   ==> Integer
  Q   ==> Fraction Z
  REC ==> Record(exponent:N, coef:F, radicand:F)

  Exports ==> with
    rroot: (R, N) -> REC
```

```
      ++ rroot(f, n) returns \spad{[m,c,r]} such
      ++ that \spad{f**(1/n) = c * r**(1/m)}.
    qroot : (Q, N) -> REC
      ++ qroot(f, n) returns \spad{[m,c,r]} such
      ++ that \spad{f**(1/n) = c * r**(1/m)}.
    if R has GcdDomain then froot: (F, N) -> REC
      ++ froot(f, n) returns \spad{[m,c,r]} such
      ++ that \spad{f**(1/n) = c * r**(1/m)}.
    nthr: (P, N) -> Record(exponent:N,coef:P,radicand:List P)
        ++ nthr(p,n) should be local but conditional

  Implementation ==> add
    import FactoredFunctions Z
    import FactoredFunctions P

    rsplit: List P -> Record(coef:R, poly:P)
    zroot : (Z, N) -> Record(exponent:N, coef:Z, radicand:Z)

    zroot(x, n) ==
--      zero? x or one? x => [1, x, 1]
      zero? x or (x = 1) => [1, x, 1]
      s := nthRoot(squareFree x, n)
      [s.exponent, s.coef, */s.radicand]

    if R has imaginary: () -> R then
      czroot: (Z, N) -> REC

      czroot(x, n) ==
        rec := zroot(x, n)
        rec.exponent = 2 and rec.radicand < 0 =>
          [rec.exponent, rec.coef * imaginary()::P::F, (-rec.radicand)::F]
        [rec.exponent, rec.coef::F, rec.radicand::F]

      qroot(x, n) ==
        sn := czroot(numer x, n)
        sd := czroot(denom x, n)
        m  := lcm(sn.exponent, sd.exponent)::N
        [m, sn.coef / sd.coef,
                    (sn.radicand ** (m quo sn.exponent)) /
                                (sd.radicand ** (m quo sd.exponent))]
    else
      qroot(x, n) ==
        sn := zroot(numer x, n)
        sd := zroot(denom x, n)
        m  := lcm(sn.exponent, sd.exponent)::N
        [m, sn.coef::F / sd.coef::F,
```

```
                    (sn.radicand ** (m quo sn.exponent))::F /
                                (sd.radicand ** (m quo sd.exponent))::F]

    if R has RetractableTo Fraction Z then
      rroot(x, n) ==
        (r := retractIfCan(x)@Union(Fraction Z,"failed")) case "failed"
          => [n, 1, x::P::F]
        qroot(r::Q, n)

    else
      if R has RetractableTo Z then
        rroot(x, n) ==
          (r := retractIfCan(x)@Union(Z,"failed")) case "failed"
            => [n, 1, x::P::F]
          qroot(r::Z::Q, n)
      else
        rroot(x, n) == [n, 1, x::P::F]

    rsplit l ==
      r := 1$R
      p := 1$P
      for q in l repeat
        if (u := retractIfCan(q)@Union(R, "failed")) case "failed"
          then p := p * q
          else r := r * u::R
      [r, p]

    if R has GcdDomain then
      if R has RetractableTo Z then
        nthr(x, n) ==
          (r := retractIfCan(x)@Union(Z,"failed")) case "failed"
             => nthRoot(squareFree x, n)
          rec := zroot(r::Z, n)
          [rec.exponent, rec.coef::P, [rec.radicand::P]]
      else nthr(x, n) == nthRoot(squareFree x, n)

      froot(x, n) ==
--        zero? x or one? x => [1, x, 1]
        zero? x or (x = 1) => [1, x, 1]
        sn := nthr(numer x, n)
        sd := nthr(denom x, n)
        pn := rsplit(sn.radicand)
        pd := rsplit(sd.radicand)
        rn := rroot(pn.coef, sn.exponent)
        rd := rroot(pd.coef, sd.exponent)
        m := lcm([rn.exponent, rd.exponent, sn.exponent, sd.exponent])::N
```

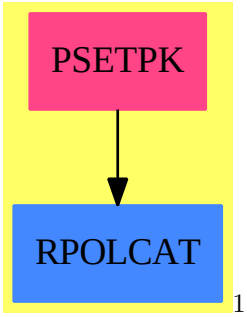
```
          [m, (sn.coef::F / sd.coef::F) * (rn.coef / rd.coef),
               ((rn.radicand ** (m quo rn.exponent)) /
                      (rd.radicand ** (m quo rd.exponent))) *
                             (pn.poly ** (m quo sn.exponent))::F /
                                      (pd.poly ** (m quo sd.exponent))::F]
```

⟨*POLYROOT.dotabb*⟩≡

```
"POLYROOT" [color="#FF4488",href="bookvol10.4.pdf#nameddest=POLYROOT"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"POLYROOT" -> "PFECAT"
```

17.101 package PSETPK PolynomialSetUtilitiesPackage

17.102 PolynomialSetUtilitiesPackage



1.00

Exports:

bivariate?	crushedSet
interReduce	linear?
univariate?	bivariatePolynomials
certainlySubVariety?	irreducibleFactors
lazyIrreducibleFactors	linearPolynomials
possiblyNewVariety?	probablyZeroDim?
quasiMonicPolynomials	removeIrreducibleRedundantFactors
removeRedundantFactors	removeRedundantFactorsInContents
removeRedundantFactorsInPols	removeRoughlyRedundantFactorsInContents
removeRoughlyRedundantFactorsInPol	removeRoughlyRedundantFactorsInPols
removeSquaresIfCan	rewriteIdealWithQuasiMonicGenerators
rewriteSetByReducingWithParticularGenerators	roughBasicSet
selectAndPolynomials	selectOrPolynomials
selectPolynomials	squareFreeFactors
univariatePolynomials	univariatePolynomialsGcds
unprotectedRemoveRedundantFactors	

⟨*package PSETPK PolynomialSetUtilitiesPackage*⟩≡

```
)abbrev package PSETPK PolynomialSetUtilitiesPackage
++ Author: Marc Moreno Maza (marc@nag.co.uk)
++ Date Created: 12/01/1995
++ Date Last Updated: 12/15/1998
++ SPARC Version
++ Basic Operations:
++ Related Domains:
++ Also See:
++ AMS Classifications:
++ Keywords:
++ Examples:
```

```
++ References:
++ Description:
++ This package provides modest routines for polynomial system solving.
++ The aim of many of the operations of this package is to remove certain
++ factors in some polynomials in order to avoid unnecessary computations
++ in algorithms involving splitting techniques by partial factorization.
++ Version: 3

PolynomialSetUtilitiesPackage (R,E,V,P) : Exports == Implementation where

  R : IntegralDomain
  E : OrderedAbelianMonoidSup
  V : OrderedSet
  P : RecursivePolynomialCategory(R,E,V)
  N ==> NonNegativeInteger
  Z ==> Integer
  B ==> Boolean
  LP ==> List P
  FP ==> Factored P
  T ==> GeneralTriangularSet(R,E,V,P)
  RRZ ==> Record(factor: P,exponent: Integer)
  RBT ==> Record(bas:T,top:LP)
  RUL ==> Record(chs:Union(T,"failed"),rfs:LP)
  GPS ==> GeneralPolynomialSet(R,E,V,P)
  pf ==> MultivariateFactorize(V, E, R, P)

  Exports ==  with

     removeRedundantFactors: LP -> LP
        ++ \axiom{removeRedundantFactors(lp)} returns \axiom{lq} such that if
        ++ \axiom{lp = [p1,...,pn]} and \axiom{lq = [q1,...,qm]}
        ++ then the product \axiom{p1*p2*...*pn} vanishes iff the product \axiom{
        ++ and the product of degrees of the \axiom{qi} is not greater than
        ++ the one of the \axiom{pj}, and no polynomial in \axiom{lq}
        ++ divides another polynomial in \axiom{lq}. In particular,
        ++ polynomials lying in the base ring \axiom{R} are removed.
        ++ Moreover, \axiom{lq} is sorted w.r.t \axiom{infRittWu?}.
        ++ Furthermore, if R is gcd-domain, the polynomials in \axiom{lq} are
        ++ pairwise without common non trivial factor.
     removeRedundantFactors: (P,P) -> LP
        ++ \axiom{removeRedundantFactors(p,q)} returns the same as
        ++ \axiom{removeRedundantFactors([p,q])}
     removeSquaresIfCan : LP -> LP
        ++ \axiom{removeSquaresIfCan(lp)} returns
        ++ \axiom{removeDuplicates [squareFreePart(p)$P for p in lp]}
        ++ if \axiom{R} is gcd-domain else returns \axiom{lp}.
```

```
unprotectedRemoveRedundantFactors: (P,P) -> LP
   ++ \axiom{unprotectedRemoveRedundantFactors(p,q)} returns the same as
   ++ \axiom{removeRedundantFactors(p,q)} but does assume that neither
   ++ \axiom{p} nor \axiom{q} lie in the base ring \axiom{R} and assumes that
   ++ \axiom{infRittWu?(p,q)} holds. Moreover, if \axiom{R} is gcd-domain,
   ++ then \axiom{p} and \axiom{q} are assumed to be square free.
removeRedundantFactors: (LP,P) -> LP
   ++ \axiom{removeRedundantFactors(lp,q)} returns the same as
   ++ \axiom{removeRedundantFactors(cons(q,lp))} assuming
   ++ that \axiom{removeRedundantFactors(lp)} returns \axiom{lp}
   ++ up to replacing some polynomial \axiom{pj} in \axiom{lp}
   ++ by some some polynomial \axiom{qj} associated to \axiom{pj}.
removeRedundantFactors : (LP,LP) -> LP
   ++ \axiom{removeRedundantFactors(lp,lq)} returns the same as
   ++ \axiom{removeRedundantFactors(concat(lp,lq))} assuming
   ++ that \axiom{removeRedundantFactors(lp)} returns \axiom{lp}
   ++ up to replacing some polynomial \axiom{pj} in \axiom{lp}
   ++ by some polynomial \axiom{qj} associated to \axiom{pj}.
removeRedundantFactors : (LP,LP,(LP -> LP)) -> LP
   ++ \axiom{removeRedundantFactors(lp,lq,remOp)} returns the same as
   ++ \axiom{concat(remOp(removeRoughlyRedundantFactorsInPols(lp,lq)),lq)}
   ++ assuming that \axiom{remOp(lq)} returns \axiom{lq} up to similarity.
certainlySubVariety? : (LP,LP) -> B
   ++ \axiom{certainlySubVariety?(newlp,lp)} returns true iff for every \axiom{p}
   ++ in \axiom{lp} the remainder of \axiom{p} by \axiom{newlp} using the division algo
   ++ of Groebner techniques is zero.
possiblyNewVariety? : (LP, List LP) -> B
   ++ \axiom{possiblyNewVariety?(newlp,llp)} returns true iff for every \axiom{lp}
   ++ in \axiom{llp} certainlySubVariety?(newlp,lp) does not hold.
probablyZeroDim?: LP -> B
   ++ \axiom{probablyZeroDim?(lp)} returns true iff the number of polynomials
   ++ in \axiom{lp} is not smaller than the number of variables occurring
   ++ in these polynomials.
selectPolynomials : ((P -> B),LP) -> Record(goodPols:LP,badPols:LP)
   ++ \axiom{selectPolynomials(pred?,ps)} returns \axiom{gps,bps} where
   ++ \axiom{gps} is a list of the polynomial \axiom{p} in \axiom{ps}
   ++ such that \axiom{pred?(p)} holds and \axiom{bps} are the other ones.
selectOrPolynomials : (List (P -> B),LP) -> Record(goodPols:LP,badPols:LP)
   ++ \axiom{selectOrPolynomials(lpred?,ps)} returns \axiom{gps,bps} where
   ++ \axiom{gps} is a list of the polynomial \axiom{p} in \axiom{ps}
   ++ such that \axiom{pred?(p)} holds for some \axiom{pred?} in \axiom{lpred?}
   ++ and \axiom{bps} are the other ones.
selectAndPolynomials : (List (P -> B),LP) -> Record(goodPols:LP,badPols:LP)
   ++ \axiom{selectAndPolynomials(lpred?,ps)} returns \axiom{gps,bps} where
   ++ \axiom{gps} is a list of the polynomial \axiom{p} in \axiom{ps}
   ++ such that \axiom{pred?(p)} holds for every \axiom{pred?} in \axiom{lpred?}
```

```
   ++ and \axiom{bps} are the other ones.
quasiMonicPolynomials : LP -> Record(goodPols:LP,badPols:LP)
   ++ \axiom{quasiMonicPolynomials(lp)} returns \axiom{qmps,nqmps} where
   ++ \axiom{qmps} is a list of the quasi-monic polynomials in \axiom{lp}
   ++ and \axiom{nqmps} are the other ones.
univariate? : P -> B
   ++ \axiom{univariate?(p)} returns true iff \axiom{p} involves one and
   ++ only one variable.
univariatePolynomials : LP -> Record(goodPols:LP,badPols:LP)
   ++ \axiom{univariatePolynomials(lp)} returns \axiom{ups,nups} where
   ++ \axiom{ups} is a list of the univariate polynomials,
   ++ and \axiom{nups} are the other ones.
linear? : P -> B
   ++ \axiom{linear?(p)} returns true iff \axiom{p} does not lie
   ++ in the base ring \axiom{R} and has main degree \axiom{1}.
linearPolynomials : LP -> Record(goodPols:LP,badPols:LP)
   ++ \axiom{linearPolynomials(lp)} returns \axiom{lps,nlps} where
   ++ \axiom{lps} is a list of the linear polynomials in lp,
   ++ and  \axiom{nlps} are the other ones.
bivariate? : P -> B
   ++ \axiom{bivariate?(p)} returns true iff \axiom{p} involves two and
   ++ only two variables.
bivariatePolynomials : LP -> Record(goodPols:LP,badPols:LP)
   ++ \axiom{bivariatePolynomials(lp)} returns \axiom{bps,nbps} where
   ++ \axiom{bps} is a list of the bivariate polynomials,
   ++ and \axiom{nbps} are the other ones.
removeRoughlyRedundantFactorsInPols : (LP, LP) -> LP
   ++ \axiom{removeRoughlyRedundantFactorsInPols(lp,lf)} returns
   ++ \axiom{newlp}where \axiom{newlp} is obtained from \axiom{lp}
   ++ by removing in every polynomial \axiom{p} of \axiom{lp}
   ++ any occurence of a polynomial \axiom{f} in \axiom{lf}.
   ++ This may involve a lot of exact-quotients computations.
removeRoughlyRedundantFactorsInPols : (LP, LP,B) -> LP
   ++ \axiom{removeRoughlyRedundantFactorsInPols(lp,lf,opt)} returns
   ++ the same as \axiom{removeRoughlyRedundantFactorsInPols(lp,lf)}
   ++ if \axiom{opt} is \axiom{false} and if the previous operation
   ++ does not return any non null and constant polynomial,
   ++ else return \axiom{[1]}.
removeRoughlyRedundantFactorsInPol : (P,LP) -> P
   ++ \axiom{removeRoughlyRedundantFactorsInPol(p,lf)} returns the same as
   ++ removeRoughlyRedundantFactorsInPols([p],lf,true)
interReduce: LP -> LP
   ++ \axiom{interReduce(lp)} returns \axiom{lq} such that \axiom{lp}
   ++ and \axiom{lq} generate the same ideal and no polynomial
   ++ in \axiom{lq} is reducuble by the others in the sense
   ++ of Groebner bases. Since no assumptions are required
```

```
      ++ the result may depend on the ordering the reductions are
      ++ performed.
    roughBasicSet: LP -> Union(Record(bas:T,top:LP),"failed")
      ++ \axiom{roughBasicSet(lp)} returns the smallest (with Ritt-Wu
      ++ ordering) triangular set contained in \axiom{lp}.
    crushedSet: LP -> LP
      ++ \axiom{crushedSet(lp)} returns \axiom{lq} such that \axiom{lp} and
      ++ and \axiom{lq} generate the same ideal and no rough basic
      ++ sets reduce (in the sense of Groebner bases) the other
      ++ polynomials in \axiom{lq}.
    rewriteSetByReducingWithParticularGenerators : (LP,(P->B),((P,P)->B),((P,P)->P)) -> LP
      ++ \axiom{rewriteSetByReducingWithParticularGenerators(lp,pred?,redOp?,redOp)}
      ++ returns \axiom{lq} where \axiom{lq} is computed by the following
      ++ algorithm. Chose a basic set w.r.t. the reduction-test \axiom{redOp?}
      ++ among the polynomials satisfying property \axiom{pred?},
      ++ if it is empty then leave, else reduce the other polynomials by
      ++ this basic set w.r.t. the reduction-operation \axiom{redOp}.
      ++ Repeat while another basic set with smaller rank can be computed.
      ++ See code. If \axiom{pred?} is \axiom{quasiMonic?} the ideal is unchanged.
    rewriteIdealWithQuasiMonicGenerators : (LP,((P,P)->B),((P,P)->P)) -> LP
      ++ \axiom{rewriteIdealWithQuasiMonicGenerators(lp,redOp?,redOp)} returns
      ++ \axiom{lq} where \axiom{lq} and \axiom{lp} generate
      ++ the same ideal in \axiom{R^(-1) P} and \axiom{lq}
      ++ has rank not higher than the one of \axiom{lp}.
      ++ Moreover, \axiom{lq} is computed by reducing \axiom{lp}
      ++ w.r.t. some basic set of the ideal generated by
      ++ the quasi-monic polynomials in \axiom{lp}.
    if R has GcdDomain
    then
      squareFreeFactors : P -> LP
        ++ \axiom{squareFreeFactors(p)} returns the square-free factors of \axiom{p}
        ++ over \axiom{R}
      univariatePolynomialsGcds : LP -> LP
        ++ \axiom{univariatePolynomialsGcds(lp)} returns \axiom{lg} where
        ++ \axiom{lg} is a list of the gcds of every pair in \axiom{lp}
        ++ of univariate polynomials in the same main variable.
      univariatePolynomialsGcds : (LP,B) -> LP
        ++ \axiom{univariatePolynomialsGcds(lp,opt)} returns the same as
        ++ \axiom{univariatePolynomialsGcds(lp)} if \axiom{opt} is
        ++ \axiom{false} and if the previous operation does not return
        ++ any non null and constant polynomial, else return \axiom{[1]}.
      removeRoughlyRedundantFactorsInContents : (LP, LP) -> LP
        ++ \axiom{removeRoughlyRedundantFactorsInContents(lp,lf)} returns
        ++ \axiom{newlp}where \axiom{newlp} is obtained from \axiom{lp}
        ++ by removing in the content of every polynomial of \axiom{lp}
        ++ any occurence of a polynomial \axiom{f} in \axiom{lf}. Moreover,
```

```
        ++ squares over \axiom{R} are first removed in the content
        ++ of every polynomial of \axiom{lp}.
      removeRedundantFactorsInContents : (LP, LP) -> LP
        ++ \axiom{removeRedundantFactorsInContents(lp,lf)} returns \axiom{newlp}
        ++ where \axiom{newlp} is obtained from \axiom{lp} by removing
        ++ in the content of every polynomial of \axiom{lp} any non trivial
        ++ factor of any polynomial \axiom{f} in \axiom{lf}. Moreover,
        ++ squares over \axiom{R} are first removed in the content
        ++ of every polynomial of \axiom{lp}.
      removeRedundantFactorsInPols : (LP, LP) -> LP
        ++ \axiom{removeRedundantFactorsInPols(lp,lf)} returns \axiom{newlp}
        ++ where \axiom{newlp} is obtained from \axiom{lp} by removing
        ++ in every polynomial \axiom{p} of \axiom{lp} any non trivial
        ++ factor of any polynomial \axiom{f} in \axiom{lf}. Moreover,
        ++ squares over \axiom{R} are first removed in every
        ++ polynomial \axiom{lp}.
    if (R has EuclideanDomain) and (R has CharacteristicZero)
    then
      irreducibleFactors : LP -> LP
        ++ \axiom{irreducibleFactors(lp)} returns \axiom{lf} such that if
        ++ \axiom{lp = [p1,...,pn]} and \axiom{lf = [f1,...,fm]} then
        ++ \axiom{p1*p2*...*pn=0} means \axiom{f1*f2*...*fm=0}, and the \axiom{
        ++ are irreducible over \axiom{R} and are pairwise distinct.
      lazyIrreducibleFactors : LP -> LP
        ++ \axiom{lazyIrreducibleFactors(lp)} returns \axiom{lf} such that if
        ++ \axiom{lp = [p1,...,pn]} and \axiom{lf = [f1,...,fm]} then
        ++ \axiom{p1*p2*...*pn=0} means \axiom{f1*f2*...*fm=0}, and the \axiom{
        ++ are irreducible over \axiom{R} and are pairwise distinct.
        ++ The algorithm tries to avoid factorization into irreducible
        ++ factors as far as possible and makes previously use of gcd
        ++ techniques over \axiom{R}.
      removeIrreducibleRedundantFactors : (LP, LP) -> LP
        ++ \axiom{removeIrreducibleRedundantFactors(lp,lq)} returns the same
        ++ as \axiom{irreducibleFactors(concat(lp,lq))} assuming
        ++ that \axiom{irreducibleFactors(lp)} returns \axiom{lp}
        ++ up to replacing some polynomial \axiom{pj} in \axiom{lp}
        ++ by some polynomial \axiom{qj} associated to \axiom{pj}.

  Implementation ==  add

    autoRemainder: T -> List(P)

    removeAssociates (lp:LP):LP ==
      removeDuplicates [primPartElseUnitCanonical(p) for p in lp]

    selectPolynomials  (pred?,ps) ==
```

```
    gps : LP := []
    bps : LP := []
    while not empty? ps repeat
      p := first ps
      ps := rest ps
      if pred?(p)
        then
          gps := cons(p,gps)
        else
          bps := cons(p,bps)
    gps := sort(infRittWu?,gps)
    bps := sort(infRittWu?,bps)
    [gps,bps]

  selectOrPolynomials (lpred?,ps) ==
    gps : LP := []
    bps : LP := []
    while not empty? ps repeat
      p := first ps
      ps := rest ps
      clpred? :=  lpred?
      while (not empty? clpred?) and (not (first clpred?)(p)) repeat
        clpred? :=  rest clpred?
      if not empty?(clpred?)
        then
          gps := cons(p,gps)
        else
          bps := cons(p,bps)
    gps := sort(infRittWu?,gps)
    bps := sort(infRittWu?,bps)
    [gps,bps]

  selectAndPolynomials (lpred?,ps) ==
    gps : LP := []
    bps : LP := []
    while not empty? ps repeat
      p := first ps
      ps := rest ps
      clpred? :=  lpred?
      while (not empty? clpred?) and ((first clpred?)(p)) repeat
        clpred? :=  rest clpred?
      if empty?(clpred?)
        then
          gps := cons(p,gps)
        else
          bps := cons(p,bps)
```

```
      gps := sort(infRittWu?,gps)
      bps := sort(infRittWu?,bps)
      [gps,bps]

    linear? p ==
      ground? p => false
--      one?(mdeg(p))
      (mdeg(p) = 1)

    linearPolynomials  ps ==
      selectPolynomials(linear?,ps)

    univariate? p ==
      ground? p => false
      not(ground?(init(p))) => false
      tp := tail(p)
      ground?(tp) => true
      not (mvar(p) = mvar(tp)) => false
      univariate?(tp)

    univariatePolynomials ps ==
      selectPolynomials(univariate?,ps)

    bivariate? p ==
      ground? p => false
      ground? tail(p) => univariate?(init(p))
      vp := mvar(p)
      vtp := mvar(tail(p))
      ((ground? init(p)) and (vp = vtp)) => bivariate? tail(p)
      ((ground? init(p)) and (vp > vtp)) => univariate? tail(p)
      not univariate?(init(p)) => false
      vip := mvar(init(p))
      vip > vtp => false
      vip = vtp => univariate? tail(p)
      vtp < vp => false
      zero? degree(tail(p),vip) => univariate? tail(p)
      bivariate? tail(p)

    bivariatePolynomials ps ==
      selectPolynomials(bivariate?,ps)

    quasiMonicPolynomials ps ==
      selectPolynomials(quasiMonic?,ps)

    removeRoughlyRedundantFactorsInPols (lp,lf,opt) ==
      empty? lp => lp
```

```
    newlp : LP := []
    stop : B := false
    lp := remove(zero?,lp)
    lf := sort(infRittWu?,lf)
    test : Union(P,"failed")
    while (not empty? lp) and (not stop) repeat
      p := first lp
      lp := rest lp
      copylf := lf
      while (not empty? copylf) and (not ground? p) and (not (mvar(p) < mvar(first copylf
        f := first copylf
        copylf := rest copylf
        while (((test := p exquo$P f)) case P) repeat
          p := test::P
      stop := opt and ground?(p)
      newlp := cons(unitCanonical(p),newlp)
    stop => [1$P]
    newlp

  removeRoughlyRedundantFactorsInPol(p,lf) ==
    zero? p => p
    lp : LP := [p]
    first removeRoughlyRedundantFactorsInPols (lp,lf,true()$B)

  removeRoughlyRedundantFactorsInPols (lp,lf) ==
    removeRoughlyRedundantFactorsInPols (lp,lf,false()$B)

  possiblyNewVariety?(newlp,llp) ==
    while (not empty? llp) and _
     (not certainlySubVariety?(newlp,first(llp))) repeat
      llp := rest llp
    empty? llp

  certainlySubVariety?(lp,lq) ==
    gs := construct(lp)$GPS
    while (not empty? lq) and _
     (zero? (remainder(first(lq),gs)$GPS).polnum) repeat
      lq := rest lq
    empty? lq

  probablyZeroDim?(lp: List P) : Boolean ==
    m := #lp
    lv : List V := variables(first lp)
    while not empty? (lp := rest lp) repeat
      lv := concat(variables(first lp),lv)
    n := #(removeDuplicates lv)
```

```
  not (n > m)

interReduce(lp: LP): LP ==
  ps := lp
  rs: List(P) := []
  repeat
    empty? ps => return rs
    ps := sort(supRittWu?, ps)
    p := first ps
    ps := rest ps
    r := remainder(p,[ps]$GPS).polnum
    zero? r => "leave"
    ground? r => return []
    associates?(r,p) => rs := cons(r,rs)
    ps := concat(ps,cons(r,rs))
    rs := []

roughRed?(p:P,q:P):B ==
  ground? p => false
  ground? q => true
  mvar(p) > mvar(q)

roughBasicSet(lp) == basicSet(lp,roughRed?)$T

autoRemainder(ts:T): List(P) ==
  empty? ts => members(ts)
  lp := sort(infRittWu?, reverse members(ts))
  newlp : List(P) := [primPartElseUnitCanonical first(lp)]
  lp := rest(lp)
  while not empty? lp repeat
    p := (remainder(first(lp),construct(newlp)$GPS)$GPS).polnum
    if not zero? p
      then
        if ground? p
          then
            newlp := [1$P]
            lp := []
          else
            newlp := cons(p,newlp)
            lp := rest(lp)
      else
        lp := rest(lp)
  newlp

crushedSet(lp) ==
  rec := roughBasicSet(lp)
```

```
        contradiction := (rec case "failed")@B
        finished : B := false
        while (not finished) and (not contradiction) repeat
          bs := (rec::RBT).bas
          rs := (rec::RBT).top
          rs :=  rewriteIdealWithRemainder(rs,bs)$T
--          contradiction := ((not empty? rs) and (one? first(rs)))
          contradiction := ((not empty? rs) and (first(rs) = 1))
          if not contradiction
            then
              rs := concat(rs,autoRemainder(bs))
              rec := roughBasicSet(rs)
              contradiction := (rec case "failed")@B
              not contradiction => finished := not infRittWu?((rec::RBT).bas,bs)
        contradiction => [1$P]
        rs

      rewriteSetByReducingWithParticularGenerators (ps,pred?,redOp?,redOp) ==
        rs : LP := remove(zero?,ps)
        any?(ground?,rs) => [1$P]
        contradiction : B := false
        bs1 : T := empty()$T
        rec : Union(RBT,"failed")
        ar : Union(T,List(P))
        stop : B := false
        while (not contradiction) and (not stop) repeat
          rec := basicSet(rs,pred?,redOp?)$T
          bs2 : T := (rec::RBT).bas
          rs := (rec::RBT).top
          -- ar := autoReduce(bs2,lazyPrem,reduced?)@Union(T,List(P))
          ar := bs2::Union(T,List(P))
          if (ar case T)@B
            then
              bs2 := ar::T
              if infRittWu?(bs2,bs1)
                then
                  rs := rewriteSetWithReduction(rs,bs2,redOp,redOp?)$T
                  bs1 := bs2
                else
                  stop := true
              rs := concat(members(bs2),rs)
            else
              rs := concat(ar::LP,rs)
          if any?(ground?,rs)
            then
              contradiction := true
```

```
        rs := [1$P]
  rs

removeRedundantFactors (lp:LP,lq :LP, remOp : (LP -> LP)) ==
  -- ASSUME remOp(lp) returns lp up to similarity
  lq := removeRoughlyRedundantFactorsInPols(lq,lp,false)
  lq := remOp lq
  sort(infRittWu?,concat(lp,lq))

removeRedundantFactors (lp:LP,lq :LP) ==
  lq := removeRoughlyRedundantFactorsInPols(lq,lp,false)
  lq := removeRedundantFactors lq
  sort(infRittWu?,concat(lp,lq))

if (R has EuclideanDomain) and (R has CharacteristicZero)
then
  irreducibleFactors lp ==
    newlp : LP := []
    lrrz : List RRZ
    rrz : RRZ
    fp : FP
    while not empty? lp repeat
      p := first lp
      lp := rest lp
      fp := factor(p)$pf
      lrrz := factors(fp)$FP
      lf := remove(ground?,[rrz.factor for rrz in lrrz])
      newlp := concat(lf,newlp)
    removeDuplicates newlp

  lazyIrreducibleFactors lp ==
    lp := removeRedundantFactors(lp)
    newlp : LP := []
    lrrz : List RRZ
    rrz : RRZ
    fp : FP
    while not empty? lp repeat
      p := first lp
      lp := rest lp
      fp := factor(p)$pf
      lrrz := factors(fp)$FP
      lf := remove(ground?,[rrz.factor for rrz in lrrz])
      newlp := concat(lf,newlp)
    newlp

  removeIrreducibleRedundantFactors (lp:LP,lq :LP) ==
```

```
    -- ASSUME lp only contains irreducible factors over R
    lq := removeRoughlyRedundantFactorsInPols(lq,lp,false)
    lq := irreducibleFactors lq
    sort(infRittWu?,concat(lp,lq))

  if R has GcdDomain
  then

    squareFreeFactors(p:P) ==
      sfp: Factored P := squareFree(p)$P
      lsf: List P := [foo.factor for foo in factors(sfp)]
      lsf

    univariatePolynomialsGcds (ps,opt) ==
      lg : LP := []
      pInV : LP
      stop : B := false
      ps := sort(infRittWu?,ps)
      p,g : P
      v : V
      while (not empty? ps) and (not stop) repeat
        while (not empty? ps) and (not univariate?((p := first(ps)))) repeat
          ps := rest ps
        if not empty? ps
          then
            v := mvar(p)$P
            pInV := [p]
            while (not empty? ps) and (mvar((p := first(ps))) = v) repeat
              if (univariate?(p))
                then
                  pInV := cons(p,pInV)
              ps := rest ps
            g := gcd(pInV)$P
            stop := opt and (ground? g)
            lg := cons(g,lg)
      stop => [1$P]
      lg

    univariatePolynomialsGcds ps ==
      univariatePolynomialsGcds (ps,false)

    removeSquaresIfCan lp ==
      empty? lp => lp
      removeDuplicates [squareFreePart(p)$P for p in lp]

    rewriteIdealWithQuasiMonicGenerators (ps,redOp?,redOp) ==
```

```
    ups := removeSquaresIfCan(univariatePolynomialsGcds(ps,true))
    ps := removeDuplicates concat(ups,ps)
    rewriteSetByReducingWithParticularGenerators(ps,quasiMonic?,redOp?,redOp

  removeRoughlyRedundantFactorsInContents (ps,lf) ==
    empty? ps => ps
    newps : LP := []
    p,newp,cp,newcp,f,g : P
    test : Union(P,"failed")
    copylf : LP
    while not empty? ps repeat
      p := first ps
      ps := rest ps
      cp := mainContent(p)$P
      newcp := squareFreePart(cp)$P
      newp := (p exquo$P cp)::P
      if not ground? newcp
        then
          copylf := [f for f in lf | mvar(f) <= mvar(newcp)]
          while (not empty? copylf) and (not ground? newcp) repeat
            f := first copylf
            copylf := rest copylf
            test := (newcp exquo$P f)
            if (test case P)@B
              then
                newcp := test::P
      if ground? newcp
        then
          newp := unitCanonical(newp)
        else
          newp := unitCanonical(newp * newcp)
      newps := cons(newp,newps)
    newps

  removeRedundantFactorsInContents (ps,lf) ==
    empty? ps => ps
    newps : LP := []
    p,newp,cp,newcp,f,g : P
    while not empty? ps repeat
      p := first ps
      ps := rest ps
      cp := mainContent(p)$P
      newcp := squareFreePart(cp)$P
      newp := (p exquo$P cp)::P
      if not ground? newcp
        then
```

```
          copylf := lf
          while (not empty? copylf) and (not ground? newcp) repeat
            f := first copylf
            copylf := rest copylf
            g := gcd(newcp,f)$P
            if not ground? g
              then
                newcp := (newcp exquo$P g)::P
        if ground? newcp
          then
            newp := unitCanonical(newp)
          else
            newp := unitCanonical(newp * newcp)
        newps := cons(newp,newps)
      newps

    removeRedundantFactorsInPols (ps,lf) ==
      empty? ps => ps
      newps : LP := []
      p,newp,cp,newcp,f,g : P
      while not empty? ps repeat
        p := first ps
        ps := rest ps
        cp := mainContent(p)$P
        newcp := squareFreePart(cp)$P
        newp := (p exquo$P cp)::P
        newp := squareFreePart(newp)$P
        copylf := lf
        while not empty? copylf repeat
          f := first copylf
          copylf := rest copylf
          if not ground? newcp
            then
              g := gcd(newcp,f)$P
              if not ground? g
                then
                  newcp := (newcp exquo$P g)::P
          if not ground? newp
            then
              g := gcd(newp,f)$P
              if not ground? g
                then
                  newp := (newp exquo$P g)::P
        if ground? newcp
          then
            newp := unitCanonical(newp)
```

```
            else
              newp := unitCanonical(newp * newcp)
          newps := cons(newp,newps)
        newps

      removeRedundantFactors (a:P,b:P) : LP ==
        a := primPartElseUnitCanonical(squareFreePart(a))
        b := primPartElseUnitCanonical(squareFreePart(b))
        if not infRittWu?(a,b)
          then
           (a,b) := (b,a)
        if ground? a
          then
            if ground? b
              then
                return([])
              else
                return([b])
          else
            if ground? b
              then
                return([a])
              else
                return(unprotectedRemoveRedundantFactors(a,b))

      unprotectedRemoveRedundantFactors (a,b) ==
        c := b exquo$P a
        if (c case P)@B
          then
            d : P := c::P
            if ground? d
              then
                return([a])
              else
                return([a,d])
          else
            g : P := gcd(a,b)$P
            if ground? g
              then
                return([a,b])
              else
                return([g,(a exquo$P g)::P,(b exquo$P g)::P])

    else

      removeSquaresIfCan lp ==
```

```
    lp

  rewriteIdealWithQuasiMonicGenerators (ps,redOp?,redOp) ==
    rewriteSetByReducingWithParticularGenerators(ps,quasiMonic?,redOp?,redOp)

  removeRedundantFactors (a:P,b:P) ==
    a := primPartElseUnitCanonical(a)
    b := primPartElseUnitCanonical(b)
    if not infRittWu?(a,b)
      then
       (a,b) := (b,a)
    if ground? a
      then
        if ground? b
          then
            return([])
          else
            return([b])
      else
        if ground? b
          then
            return([a])
          else
            return(unprotectedRemoveRedundantFactors(a,b))

  unprotectedRemoveRedundantFactors (a,b) ==
    c := b exquo$P a
    if (c case P)@B
      then
        d : P := c::P
        if ground? d
          then
            return([a])
          else
            if infRittWu?(d,a) then (a,d) := (d,a)
            return(unprotectedRemoveRedundantFactors(a,d))
       else
         return([a,b])

removeRedundantFactors (lp:LP) ==
  lp := remove(ground?, lp)
  lp := removeDuplicates [primPartElseUnitCanonical(p) for p in lp]
  lp := removeSquaresIfCan lp
  lp := removeDuplicates [unitCanonical(p) for p in lp]
  empty? lp => lp
  size?(lp,1$N)$(List P) => lp
```

```
  lp := sort(infRittWu?,lp)
  p : P := first lp
  lp := rest lp
  base : LP := unprotectedRemoveRedundantFactors(p,first lp)
  top : LP := rest lp
  while not empty? top repeat
    p := first top
    base := removeRedundantFactors(base,p)
    top := rest top
  base

removeRedundantFactors (lp:LP,a:P) ==
  lp := remove(ground?, lp)
  lp := sort(infRittWu?, lp)
  ground? a => lp
  empty? lp => [a]
  toSee : LP := lp
  toSave : LP := []
  while not empty? toSee repeat
    b := first toSee
    toSee := rest toSee
    if not infRittWu?(b,a)
      then
        (c,d) := (a,b)
      else
        (c,d) := (b,a)
    rrf := unprotectedRemoveRedundantFactors(c,d)
    empty? rrf => error"in removeRedundantFactors : (LP,P) -> LP from PSETPK
    c := first rrf
    rrf := rest rrf
    if empty? rrf
      then
        if associates?(c,b)
          then
            toSave := concat(toSave,toSee)
            a := b
            toSee := []
          else
            a := c
            toSee := concat(toSave,toSee)
            toSave := []
      else
        d := first rrf
        rrf := rest rrf
        if empty? rrf
          then
```

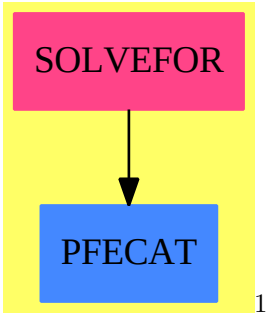
```
                if associates?(c,b)
                  then
                    toSave := concat(toSave,[b])
                    a := d
                  else
                    if associates?(d,b)
                      then
                        toSave := concat(toSave,[b])
                        a := c
                      else
                        toSave := removeRedundantFactors(toSave,c)
                        a := d
              else
                e := first rrf
                not empty? rest(rrf) => error"in removeRedundantFactors:(LP,P)->LP from PSE
                -- ASSUME that neither c, nor d, nor e may be associated to b
                toSave := removeRedundantFactors(toSave,c)
                toSave := removeRedundantFactors(toSave,d)
                a := e
          if empty? toSee
            then
              toSave := sort(infRittWu?,cons(a,toSave))
        toSave
```

⟨*PSETPK.dotabb*⟩≡

```
"PSETPK" [color="#FF4488",href="bookvol10.4.pdf#nameddest=PSETPK"]
"RPOLCAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=RPOLCAT"]
"PSETPK" -> "RPOLCAT"
```

17.103 package SOLVEFOR PolynomialSolveByFormulas

17.104 PolynomialSolveByFormulas



1.00

Exports:

aCubic	aLinear	aQuadratic	aQuartic	cubic
linear	particularSolution	quadratic	quartic	solve
mapSolve				

⟨*package SOLVEFOR PolynomialSolveByFormulas*⟩≡

```
)abbrev package SOLVEFOR PolynomialSolveByFormulas
--  Current fields with "**": (%, RationalNumber) -> % are
--     ComplexFloat, RadicalExtension(K) and RationalRadical
--  SMW June 86, BMT Sept 93
++ Description:
++ This package factors the formulas out of the general solve code,
++ allowing their recursive use over different domains.
++ Care is taken to introduce few radicals so that radical extension
++ domains can more easily simplify the results.

PolynomialSolveByFormulas(UP, F): PSFcat == PSFdef where

    UP: UnivariatePolynomialCategory F
    F:  Field with "**": (%, Fraction Integer) -> %

    L  ==> List

    PSFcat == with
        solve:      UP -> L F
                ++ solve(u) \undocumented
        particularSolution:  UP -> F
                ++ particularSolution(u) \undocumented
        mapSolve:   (UP, F -> F) -> Record(solns: L F,
                                           maps: L Record(arg:F,res:F))
```

```
            ++ mapSolve(u,f) \undocumented

        linear:     UP -> L F
            ++ linear(u) \undocumented
        quadratic:  UP -> L F
            ++ quadratic(u) \undocumented
        cubic:      UP -> L F
            ++ cubic(u) \undocumented
        quartic:    UP -> L F
            ++ quartic(u) \undocumented

        -- Arguments give coefs from high to low degree.
        linear:     (F, F)          -> L F
            ++ linear(f,g) \undocumented
        quadratic:  (F, F, F)       -> L F
            ++ quadratic(f,g,h) \undocumented
        cubic:      (F, F, F, F)    -> L F
            ++ cubic(f,g,h,i) \undocumented
        quartic:    (F, F, F, F, F) -> L F
            ++ quartic(f,g,h,i,j) \undocumented

        aLinear:    (F, F)          -> F
            ++ aLinear(f,g) \undocumented
        aQuadratic: (F, F, F)       -> F
            ++ aQuadratic(f,g,h) \undocumented
        aCubic:     (F, F, F, F)    -> F
            ++ aCubic(f,g,h,j) \undocumented
        aQuartic:   (F, F, F, F, F) -> F
            ++ aQuartic(f,g,h,i,k) \undocumented

    PSFdef == add

        ----------------------------------------------------------------
        -- Stuff for mapSolve
        ----------------------------------------------------------------
        id ==> (IDENTITY$Lisp)

        maplist: List Record(arg: F, res: F) := []
        mapSolving?: Boolean := false
        -- map: F -> F := id #1    replaced with line below
        map: Boolean := false

        mapSolve(p, fn) ==
            -- map := fn #1   replaced with line below
            locmap: F -> F := fn #1; map := id locmap
            mapSolving? := true;  maplist := []
```

```
    slist := solve p
    mapSolving? := false;
    -- map := id #1   replaced with line below
    locmap := id #1; map := id locmap
    [slist, maplist]

part(s: F): F ==
    not mapSolving? => s
    -- t := map s     replaced with line below
    t: F := SPADCALL(s, map)$Lisp
    t = s => s
    maplist := cons([t, s], maplist)
    t

------------------------------------------------------------------
-- Entry points and error handling
------------------------------------------------------------------
cc ==> coefficient

-- local intsolve
intsolve(u:UP):L(F) ==
    u := (factors squareFree u).1.factor
    n := degree u
    n=1 => linear    (cc(u,1), cc(u,0))
    n=2 => quadratic (cc(u,2), cc(u,1), cc(u,0))
    n=3 => cubic     (cc(u,3), cc(u,2), cc(u,1), cc(u,0))
    n=4 => quartic   (cc(u,4), cc(u,3), cc(u,2), cc(u,1), cc(u,0))
    error "All sqfr factors of polynomial must be of degree < 5"

solve u ==
    ls := nil$L(F)
    for f in factors squareFree u repeat
       lsf := intsolve f.factor
       for i in 1..(f.exponent) repeat ls := [:lsf,:ls]
    ls

particularSolution u ==
    u := (factors squareFree u).1.factor
    n := degree u
    n=1 => aLinear    (cc(u,1), cc(u,0))
    n=2 => aQuadratic (cc(u,2), cc(u,1), cc(u,0))
    n=3 => aCubic     (cc(u,3), cc(u,2), cc(u,1), cc(u,0))
    n=4 => aQuartic   (cc(u,4), cc(u,3), cc(u,2), cc(u,1), cc(u,0))
    error "All sqfr factors of polynomial must be of degree < 5"

needDegree(n: Integer, u: UP): Boolean ==
```

```
        degree u = n => true
        error concat("Polynomial must be of degree ", n::String)

    needLcoef(cn: F): Boolean ==
        cn ^= 0 => true
        error "Leading coefficient must not be 0."

    needChar0(): Boolean ==
        characteristic()$F = 0 => true
        error "Formula defined only for fields of characteristic 0."

    linear u ==
        needDegree(1, u)
        linear (coefficient(u,1), coefficient(u,0))

    quadratic u ==
        needDegree(2, u)
        quadratic (coefficient(u,2), coefficient(u,1),
                   coefficient(u,0))

    cubic u ==
        needDegree(3, u)
        cubic (coefficient(u,3), coefficient(u,2),
               coefficient(u,1), coefficient(u,0))

    quartic u ==
        needDegree(4, u)
        quartic (coefficient(u,4),coefficient(u,3),
                 coefficient(u,2),coefficient(u,1),coefficient(u,0))

    ------------------------------------------------------------------
    -- The formulas
    ------------------------------------------------------------------

    -- local function for testing equality of radicals.
    --  This function is necessary to detect at least some of the
    --  situations like sqrt(9)-3 = 0 --> false.
    equ(x:F,y:F):Boolean ==
        ( (recip(x-y)) case "failed" ) => true
        false

    linear(c1, c0) ==
        needLcoef c1
        [- c0/c1 ]

    aLinear(c1, c0) ==
```

```
    first linear(c1,c0)

quadratic(c2, c1, c0) ==
    needLcoef c2; needChar0()
    (c0 = 0) => [0$F,:linear(c2, c1)]
    (c1 = 0) => [(-c0/c2)**(1/2),-(-c0/c2)**(1/2)]
    D := part(c1**2 - 4*c2*c0)**(1/2)
    [(-c1+D)/(2*c2), (-c1-D)/(2*c2)]

aQuadratic(c2, c1, c0) ==
    needLcoef c2; needChar0()
    (c0 = 0) => 0$F
    (c1 = 0) => (-c0/c2)**(1/2)
    D := part(c1**2 - 4*c2*c0)**(1/2)
    (-c1+D)/(2*c2)

w3: F := (-1 + (-3::F)**(1/2)) / 2::F

cubic(c3, c2, c1, c0) ==
    needLcoef c3; needChar0()

    -- case one root = 0, not necessary but keeps result small
    (c0 = 0) => [0$F,:quadratic(c3, c2, c1)]
    a1 := c2/c3;  a2 := c1/c3;  a3 := c0/c3

    -- case x**3-a3 = 0, not necessary but keeps result small
    (a1 = 0 and a2 = 0) =>
        [ u*(-a3)**(1/3) for u in [1, w3, w3**2 ] ]

    -- case x**3 + a1*x**2 + a1**2*x/3 + a3 = 0, the general for-
    --   mula is not valid in this case, but solution is easy.
    P := part(-a1/3::F)
    equ(a1**2,3*a2) =>
      S := part((- a3 + (a1**3)/27::F)**(1/3))
      [ P + S*u for u in [1,w3,w3**2] ]

    -- general case
    Q := part((3*a2 - a1**2)/9::F)
    R := part((9*a1*a2 - 27*a3 - 2*a1**3)/54::F)
    D := part(Q**3 + R**2)**(1/2)
    S := part(R + D)**(1/3)
    -- S = 0 is done in the previous case
    [ P + S*u - Q/(S*u) for u in [1, w3, w3**2] ]

aCubic(c3, c2, c1, c0) ==
    needLcoef c3; needChar0()
```

```
        (c0 = 0) => 0$F
        a1 := c2/c3;  a2 := c1/c3;  a3 := c0/c3
        (a1 = 0 and a2 = 0) => (-a3)**(1/3)
        P := part(-a1/3::F)
        equ(a1**2,3*a2) =>
          S := part((- a3 + (a1**3)/27::F)**(1/3))
          P + S
        Q := part((3*a2 - a1**2)/9::F)
        R := part((9*a1*a2 - 27*a3 - 2*a1**3)/54::F)
        D := part(Q**3 + R**2)**(1/2)
        S := part(R + D)**(1/3)
        P + S - Q/S

    quartic(c4, c3, c2, c1, c0) ==
        needLcoef c4; needChar0()

        -- case one root = 0, not necessary but keeps result small
        (c0 = 0) => [0$F,:cubic(c4, c3, c2, c1)]
        -- Make monic:
        a1 := c3/c4; a2 := c2/c4; a3 := c1/c4; a4 := c0/c4

        -- case x**4 + a4 = 0 <=> (x**2-sqrt(-a4))*(x**2+sqrt(-a4))
        -- not necessary but keeps result small.
        (a1 = 0 and a2 = 0 and a3 = 0) =>
            append( quadratic(1, 0, (-a4)**(1/2)),_
                    quadratic(1 ,0, -((-a4)**(1/2))) )

        -- Translate w = x+a1/4 to eliminate a1:  w**4+p*w**2+q*w+r
        p := part(a2-3*a1*a1/8::F)
        q := part(a3-a1*a2/2::F + a1**3/8::F)
        r := part(a4-a1*a3/4::F + a1**2*a2/16::F - 3*a1**4/256::F)
        -- t0 := the cubic resolvent of x**3-p*x**2-4*r*x+4*p*r-q**2
        -- The roots of the translated polynomial are those of
        -- two quadratics. (What about rt=0 ?)
        -- rt=0 can be avoided by picking a root ^= p of the cubic
        -- polynomial above. This is always possible provided that
        -- the input is squarefree. In this case the two other roots
        -- are +(-) 2*r**(1/2).
        if equ(q,0)            -- this means p is a root
          then t0 := part(2*(r**(1/2)))
          else t0 := aCubic(1, -p, -4*r, 4*p*r - q**2)
        rt    := part(t0 - p)**(1/2)
        slist := append( quadratic( 1,  rt, (-q/rt + t0)/2::F ),
                         quadratic( 1, -rt, ( q/rt + t0)/2::F ))
        -- Translate back:
        [s - a1/4::F for s in slist]
```

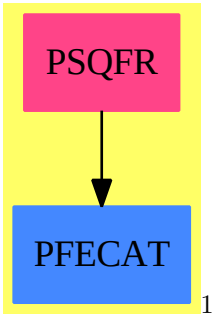
```
aQuartic(c4, c3, c2, c1, c0) ==
    needLcoef c4; needChar0()
    (c0 = 0) => 0$F
    a1 := c3/c4; a2 := c2/c4; a3 := c1/c4; a4 := c0/c4
    (a1 = 0 and a2 = 0 and a3 = 0) => (-a4)**(1/4)
    p  := part(a2-3*a1*a1/8::F)
    q  := part(a3-a1*a2/2::F + a1**2*a1/8::F)
    r  := part(a4-a1*a3/4::F + a1**2*a2/16::F - 3*a1**4/256::F)
    if equ(q,0)
      then t0 := part(2*(r**(1/2)))
      else t0 := aCubic(1, -p, -4*r, 4*p*r - q**2)
    rt := part(t0 - p)**(1/2)
    s  := aQuadratic( 1,  rt, (-q/rt + t0)/2::F )
    s - a1/4::F
```

⟨*SOLVEFOR.dotabb*⟩≡

```
"SOLVEFOR" [color="#FF4488",href="bookvol10.4.pdf#nameddest=SOLVEFOR"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"SOLVEFOR" -> "PFECAT"
```

17.105 package PSQFR PolynomialSquareFree

17.106 PolynomialSquareFree



1.00

Exports:
squareFree

⟨*package PSQFR PolynomialSquareFree*⟩≡
```
)abbrev package PSQFR PolynomialSquareFree
++ Author:
++ Date Created:
++ Date Last Updated: November 1993, (P.Gianni)
++ Basic Functions:
++ Related Constructors:
++ Also See:
++ AMS Classifications:
++ Keywords:
++ References:
++ Description:
++ This package computes square-free decomposition of multivariate
++ polynomials over a coefficient ring which is an arbitrary gcd domain.
++ The requirement on the coefficient domain guarantees that the \spadfun{content} can be
++ removed so that factors will be primitive as well as square-free.
++ Over an infinite ring of finite characteristic,it may not be possible to
++ guarantee that the factors are square-free.

PolynomialSquareFree(VarSet:OrderedSet,E,RC:GcdDomain,P):C == T where
  E:OrderedAbelianMonoidSup
  P:PolynomialCategory(RC,E,VarSet)

  C == with
    squareFree : P -> Factored P
      ++ squareFree(p) returns the square-free factorization of the
      ++ polynomial p.  Each factor has no repeated roots, and the
      ++ factors are pairwise relatively prime.
```

```
T == add
  SUP    ==> SparseUnivariatePolynomial(P)
  NNI    ==> NonNegativeInteger
  fUnion ==> Union("nil", "sqfr", "irred", "prime")
  FF     ==> Record(flg:fUnion, fctr:P, xpnt:Integer)

  finSqFr : (P,List VarSet) -> Factored P
  pthPower : P -> Factored P
  pPolRoot : P -> P
  putPth   : P -> P

  chrc:=characteristic$RC

  if RC has CharacteristicNonZero then
  -- find the p-th root of a polynomial
    pPolRoot(f:P) : P ==
      lvar:=variables f
      empty? lvar => f
      mv:=first lvar
      uf:=univariate(f,mv)
      uf:=divideExponents(uf,chrc)::SUP
      uf:=map(pPolRoot,uf)
      multivariate(uf,mv)

  -- substitute variables with their p-th power
    putPth(f:P) : P ==
      lvar:=variables f
      empty? lvar => f
      mv:=first lvar
      uf:=univariate(f,mv)
      uf:=multiplyExponents(uf,chrc)::SUP
      uf:=map(putPth,uf)
      multivariate(uf,mv)

  -- the polynomial is a perfect power
    pthPower(f:P) : Factored P ==
      proot : P := 0
      isSq  : Boolean := false
      if (g:=charthRoot f) case "failed" then proot:=pPolRoot(f)
      else
        proot := g :: P
        isSq  := true
      psqfr:=finSqFr(proot,variables f)
      isSq  =>
        makeFR((unit psqfr)**chrc,[[u.flg,u.fctr,
```

```
        (u.xpnt)*chrc] for u in factorList psqfr])
     makeFR((unit psqfr),[["nil",putPth u.fctr,u.xpnt]
                          for u in factorList psqfr])

-- compute the square free decomposition, finite characteristic case
  finSqFr(f:P,lvar:List VarSet) : Factored P ==
     empty? lvar => pthPower(f)
     mv:=first lvar
     lvar:=lvar.rest
     differentiate(f,mv)=0 => finSqFr(f,lvar)
     uf:=univariate(f,mv)
     cont := content uf
     cont1:P:=1
     uf := (uf exquo cont)::SUP
     squf := squareFree(uf)$UnivariatePolynomialSquareFree(P,SUP)
     pfaclist:List FF :=[]
     for u in factorList squf repeat
       uexp:NNI:=(u.xpnt):NNI
       u.flg = "sqfr" =>  -- the square free factor is OK
         pfaclist:= cons([u.flg,multivariate(u.fctr,mv),uexp],
                         pfaclist)
       --listfin1:= finSqFr(multivariate(u.fctr,mv),lvar)
       listfin1:= squareFree multivariate(u.fctr,mv)
       flistfin1:=[[uu.flg,uu.fctr,uu.xpnt*uexp]
                   for uu in factorList listfin1]
       cont1:=cont1*((unit listfin1)**uexp)
       pfaclist:=append(flistfin1,pfaclist)
     cont:=cont*cont1
     cont ^= 1 =>
       sqp := squareFree cont
       pfaclist:= append (factorList sqp,pfaclist)
       makeFR(unit(sqp)*coefficient(unit squf,0),pfaclist)
     makeFR(coefficient(unit squf,0),pfaclist)

squareFree(p:P) ==
   mv:=mainVariable p
   mv case "failed" => makeFR(p,[])$Factored(P)
   characteristic$RC ^=0 => finSqFr(p,variables p)
   up:=univariate(p,mv)
   cont := content up
   up := (up exquo cont)::SUP
   squp := squareFree(up)$UnivariatePolynomialSquareFree(P,SUP)
   pfaclist:List FF :=
     [[u.flg,multivariate(u.fctr,mv),u.xpnt]
                                       for u in factorList squp]
   cont ^= 1 =>
```

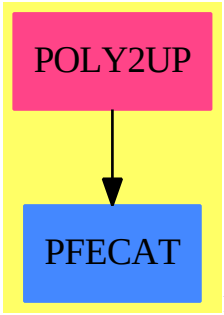
```
          sqp := squareFree cont
          makeFR(unit(sqp)*coefficient(unit squp,0),
               append(factorList sqp, pfaclist))
        makeFR(coefficient(unit squp,0),pfaclist)
```

⟨*PSQFR.dotabb*⟩≡

```
"PSQFR" [color="#FF4488",href="bookvol10.4.pdf#nameddest=PSQFR"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"PSQFR" -> "PFECAT"
```

17.107 package POLY2UP PolynomialToUnivariatePolynomial

17.108 PolynomialToUnivariatePolynomial



1.00

Exports:
univariate

⟨*package POLY2UP PolynomialToUnivariatePolynomial*⟩≡
```
)abbrev package POLY2UP PolynomialToUnivariatePolynomial
++ Author:
++ Date Created:
++ Date Last Updated:
++ Basic Functions:
++ Related Constructors:
++ Also See:
++ AMS Classifications:
++ Keywords:
++ References:
++ Description:
++ This package is primarily to help the interpreter do coercions.
++ It allows you to view a polynomial as a
++ univariate polynomial in one of its variables with
++ coefficients which are again a polynomial in all the
++ other variables.

PolynomialToUnivariatePolynomial(x:Symbol, R:Ring): with
  univariate: (Polynomial R, Variable x) ->
                                    UnivariatePolynomial(x, Polynomial R)
     ++ univariate(p, x) converts the polynomial p to a one of type
     ++ \spad{UnivariatePolynomial(x,Polynomial(R))}, ie. as a member of \spad{R[...][x]}.
 == add
  univariate(p, y) ==
    q:SparseUnivariatePolynomial(Polynomial R) := univariate(p, x)
    map(#1, q)$UnivariatePolynomialCategoryFunctions2(Polynomial R,
```

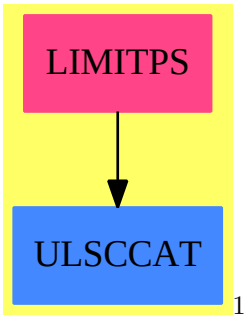
```
                SparseUnivariatePolynomial Polynomial R, Polynomial R,
                    UnivariatePolynomial(x, Polynomial R))
```

⟨*POLY2UP.dotabb*⟩≡

```
"POLY2UP" [color="#FF4488",href="bookvol10.4.pdf#nameddest=POLY2UP"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"POLY2UP" -> "PFECAT"
```

17.109 package LIMITPS PowerSeriesLimitPackage

17.110 PowerSeriesLimitPackage



1.00

Exports:
complexLimit limit

⟨*package LIMITPS PowerSeriesLimitPackage*⟩≡
```
)abbrev package LIMITPS PowerSeriesLimitPackage
++ Author: Clifton J. Williamson
++ Date Created: 21 March 1989
++ Date Last Updated: 30 March 1994
++ Basic Operations:
++ Related Domains: UnivariateLaurentSeries(FE,x,a),
++   UnivariatePuiseuxSeries(FE,x,a),ExponentialExpansion(R,FE,x,a)
++ Also See:
++ AMS Classifications:
++ Keywords: limit, functional expression, power series
++ Examples:
++ References:
++ Description:
++   PowerSeriesLimitPackage implements limits of expressions
++   in one or more variables as one of the variables approaches a
++   limiting value.  Included are two-sided limits, left- and right-
++   hand limits, and limits at plus or minus infinity.
PowerSeriesLimitPackage(R,FE): Exports == Implementation where
  R  : Join(GcdDomain,OrderedSet,RetractableTo Integer,_
            LinearlyExplicitRingOver Integer)
  FE : Join(AlgebraicallyClosedField,TranscendentalFunctionCategory,_
            FunctionSpace R)
  Z       ==> Integer
  RN      ==> Fraction Integer
  RF      ==> Fraction Polynomial R
  OFE     ==> OrderedCompletion FE
```

```
    OPF     ==> OnePointCompletion FE
    SY      ==> Symbol
    EQ      ==> Equation
    LF      ==> LiouvillianFunction
    UTS     ==> UnivariateTaylorSeries
    ULS     ==> UnivariateLaurentSeries
    UPXS    ==> UnivariatePuiseuxSeries
    EFULS   ==> ElementaryFunctionsUnivariateLaurentSeries
    EFUPXS  ==> ElementaryFunctionsUnivariatePuiseuxSeries
    FS2UPS  ==> FunctionSpaceToUnivariatePowerSeries
    FS2EXPXP ==> FunctionSpaceToExponentialExpansion
    Problem ==> Record(func:String,prob:String)
    RESULT  ==> Union(OFE,"failed")
    TwoSide ==> Record(leftHandLimit:RESULT,rightHandLimit:RESULT)
    U       ==> Union(OFE,TwoSide,"failed")
    SIGNEF  ==> ElementaryFunctionSign(R,FE)

    Exports ==> with

      limit: (FE,EQ OFE) -> U
        ++ limit(f(x),x = a) computes the real limit \spad{lim(x -> a,f(x))}.

      complexLimit: (FE,EQ OPF) -> Union(OPF, "failed")
        ++ complexLimit(f(x),x = a) computes the complex limit
        ++ \spad{lim(x -> a,f(x))}.

      limit: (FE,EQ FE,String) -> RESULT
        ++ limit(f(x),x=a,"left") computes the left hand real limit
        ++ \spad{lim(x -> a-,f(x))};
        ++ \spad{limit(f(x),x=a,"right")} computes the right hand real limit
        ++ \spad{lim(x -> a+,f(x))}.

    Implementation ==> add
      import ToolsForSign(R)
      import ElementaryFunctionStructurePackage(R,FE)

      zeroFE:FE := 0
      anyRootsOrAtrigs?   : FE -> Boolean
      complLimit  : (FE,SY) -> Union(OPF,"failed")
      okProblem?  : (String,String) -> Boolean
      realLimit   : (FE,SY) -> U
      xxpLimit    : (FE,SY) -> RESULT
      limitPlus   : (FE,SY) -> RESULT
      localsubst  : (FE,Kernel FE,Z,FE) -> FE
      locallimit  : (FE,SY,OFE) -> U
      locallimitcomplex : (FE,SY,OPF) -> Union(OPF,"failed")
```

```
poleLimit:(RN,FE,SY) -> U
poleLimitPlus:(RN,FE,SY) -> RESULT

noX?: (FE,SY) -> Boolean
noX?(fcn,x) == not member?(x,variables fcn)

constant?: FE -> Boolean
constant? fcn == empty? variables fcn

firstNonLogPtr: (FE,SY) -> List Kernel FE
firstNonLogPtr(fcn,x) ==
  -- returns a pointer to the first element of kernels(fcn) which
  -- has 'x' as a variable, which is not a logarithm, and which is
  -- not simply 'x'
  list := kernels fcn
  while not empty? list repeat
    ker := first list
    not is?(ker,"log" :: Symbol) and member?(x,variables(ker::FE)) _
           and not(x = name(ker)) =>
      return list
    list := rest list
  empty()

finiteValueAtInfinity?: Kernel FE -> Boolean
finiteValueAtInfinity? ker ==
  is?(ker,"erf" :: Symbol) => true
  is?(ker,"sech" :: Symbol) => true
  is?(ker,"csch" :: Symbol) => true
  is?(ker,"tanh" :: Symbol) => true
  is?(ker,"coth" :: Symbol) => true
  is?(ker,"atan" :: Symbol) => true
  is?(ker,"acot" :: Symbol) => true
  is?(ker,"asec" :: Symbol) => true
  is?(ker,"acsc" :: Symbol) => true
  is?(ker,"acsch" :: Symbol) => true
  is?(ker,"acoth" :: Symbol) => true
  false

knownValueAtInfinity?: Kernel FE -> Boolean
knownValueAtInfinity? ker ==
  is?(ker,"exp" :: Symbol) => true
  is?(ker,"sinh" :: Symbol) => true
  is?(ker,"cosh" :: Symbol) => true
  false

leftOrRight: (FE,SY,FE) -> SingleInteger
```

```
leftOrRight(fcn,x,limVal) ==
  -- function is called when limitPlus(fcn,x) = limVal
  -- determines whether the limiting value is approached
  -- from the left or from the right
  (value := limitPlus(inv(fcn - limVal),x)) case "failed" => 0
  (inf := whatInfinity(val := value :: OFE)) = 0 =>
     error "limit package: internal error"
  inf

specialLimit1: (FE,SY) -> RESULT
specialLimitKernel: (Kernel FE,SY) -> RESULT
specialLimitNormalize: (FE,SY) -> RESULT
specialLimit: (FE, SY) -> RESULT

specialLimit(fcn, x) ==
  xkers := [k for k in kernels fcn | member?(x,variables(k::FE))]
  #xkers = 1 => specialLimit1(fcn,x)
  num := numerator fcn
  den := denominator fcn
  for k in xkers repeat
    (fval := limitPlus(k::FE,x)) case "failed" =>
        return specialLimitNormalize(fcn,x)
    whatInfinity(val := fval::OFE) ^= 0 =>
        return specialLimitNormalize(fcn,x)
    (valu := retractIfCan(val)@Union(FE,"failed")) case "failed" =>
        return specialLimitNormalize(fcn,x)
    finVal := valu :: FE
    num := eval(num, k, finVal)
    den := eval(den, k, finVal)
    den = 0 => return specialLimitNormalize(fcn,x)
  (num/den) :: OFE :: RESULT

specialLimitNormalize(fcn,x) == -- tries to normalize result first
  nfcn := normalize(fcn)
  fcn ^= nfcn => limitPlus(nfcn,x)
  xkers := [k for k in tower fcn | member?(x,variables(k::FE))]
  # xkers ^= 2 => "failed"
  expKers := [k for k in xkers | is?(k, "exp" :: Symbol)]
  # expKers ^= 1 => "failed"
-- fcn is a rational function of x and exp(g(x)) for some rational function g
  expKer := first expKers
  (fval := limitPlus(expKer::FE,x)) case "failed" => "failed"
  vv := new()$SY; eq : EQ FE := equation(expKer :: FE,vv :: FE)
  cc := eval(fcn,eq)
  expKerLim := fval :: OFE
    -- following test for "failed" is needed due to compiler bug
```

```
      -- limVal case OFE generates EQCAR(limVal, 1) which fails on atom "failed"
    (limVal := locallimit(cc,vv,expKerLim)) case "failed" => "failed"
    limVal case OFE =>
       limm := limVal :: OFE
       (lim := retractIfCan(limm)@Union(FE,"failed")) case "failed" =>
             "failed" -- need special handling for directions at infinity
       limitPlus(lim, x)
    "failed"

  -- limit of expression having only 1 kernel involving x
  specialLimit1(fcn,x) ==
    -- find the first interesting kernel in tower(fcn)
    xkers := [k for k in kernels fcn | member?(x,variables(k::FE))]
    #xkers ^= 1 => "failed"
    ker := first xkers
    vv := new()$SY; eq : EQ FE := equation(ker :: FE,vv :: FE)
    cc := eval(fcn,eq)
    member?(x,variables cc) => "failed"
    (lim := specialLimitKernel(ker, x)) case "failed" => lim
    argLim : OFE := lim :: OFE
    (limVal := locallimit(cc,vv,argLim)) case "failed" => "failed"
    limVal case OFE => limVal :: OFE
    "failed"

  -- limit of single kernel involving x
  specialLimitKernel(ker,x) ==
    is?(ker,"log" :: Symbol) =>
        args := argument ker
        empty? args => "failed" -- error "No argument"
        not empty? rest args => "failed" -- error "Too many arugments"
        arg := first args
        -- compute limit(x -> 0+,arg)
        (limm := limitPlus(arg,x)) case "failed" => "failed"
        lim := limm :: OFE
        (inf := whatInfinity lim) = -1 => "failed"
        argLim : OFE :=
          -- log(+infinity) = +infinity
          inf = 1 => lim
          -- now 'lim' must be finite
          (li := retractIfCan(lim)@Union(FE,"failed") :: FE) = 0 =>
            -- log(0) = -infinity
            leftOrRight(arg,x,0) = 1 => minusInfinity()
            return "failed"
          log(li) :: OFE
    -- kernel should be a function of one argument f(arg)
    args := argument(ker)
```

```
empty? args => "failed"  -- error "No argument"
not empty? rest args => "failed" -- error "Too many arugments"
arg := first args
-- compute limit(x -> 0+,arg)
(limm := limitPlus(arg,x)) case "failed" => "failed"
lim := limm :: OFE
f := elt(operator ker,(var := new()$SY) :: FE)
-- compute limit(x -> 0+,f(arg))
-- case where 'lim' is finite
(inf := whatInfinity lim) = 0 =>
   is?(ker,"erf" :: Symbol) => erf(retract(lim)@FE)$LF(R,FE) :: OFE
   (kerValue := locallimit(f,var,lim)) case "failed" => "failed"
   kerValue case OFE => kerValue :: OFE
   "failed"
-- case where 'lim' is plus infinity
inf = 1 =>
  finiteValueAtInfinity? ker =>
    val : FE :=
      is?(ker,"erf" :: Symbol) => 1
      is?(ker,"sech" :: Symbol) => 0
      is?(ker,"csch" :: Symbol) => 0
      is?(ker,"tanh" :: Symbol) => 0
      is?(ker,"coth" :: Symbol) => 0
      is?(ker,"atan" :: Symbol) => pi()/(2 :: FE)
      is?(ker,"acot" :: Symbol) => 0
      is?(ker,"asec" :: Symbol) => pi()/(2 :: FE)
      is?(ker,"acsc" :: Symbol) => 0
      is?(ker,"acsch" :: Symbol) => 0
      -- ker must be acoth
      0
    val :: OFE
  knownValueAtInfinity? ker =>
    lim -- limit(exp, cosh, sinh ,x=inf) = inf
  "failed"
-- case where 'lim' is minus infinity
finiteValueAtInfinity? ker =>
  val : FE :=
    is?(ker,"erf" :: Symbol) => -1
    is?(ker,"sech" :: Symbol) => 0
    is?(ker,"csch" :: Symbol) => 0
    is?(ker,"tanh" :: Symbol) => 0
    is?(ker,"coth" :: Symbol) => 0
    is?(ker,"atan" :: Symbol) => -pi()/(2 :: FE)
    is?(ker,"acot" :: Symbol) => pi()
    is?(ker,"asec" :: Symbol) => -pi()/(2 :: FE)
    is?(ker,"acsc" :: Symbol) => -pi()
```

```
      is?(ker,"acsch" :: Symbol) => 0
      -- ker must be acoth
      0
    val :: OFE
  knownValueAtInfinity? ker =>
    is?(ker,"exp" :: Symbol) => (0@FE) :: OFE
    is?(ker,"sinh" :: Symbol) => lim
    is?(ker,"cosh" :: Symbol) => plusInfinity()
    "failed"
  "failed"

logOnlyLimit: (FE,SY) -> RESULT
logOnlyLimit(coef,x) ==
  -- this function is called when the 'constant' coefficient involves
  -- the variable 'x'. Its purpose is to compute a right hand limit
  -- of an expression involving log x. Here log x is replaced by -1/v,
  -- where v is a new variable. If the new expression no longer involves
  -- x, then take the right hand limit as v -> 0+
  vv := new()$SY
  eq : EQ FE := equation(log(x :: FE),-inv(vv :: FE))
  member?(x,variables(cc := eval(coef,eq))) => "failed"
  limitPlus(cc,vv)

locallimit(fcn,x,a) ==
  -- Here 'fcn' is a function f(x) = f(x,...) in 'x' and possibly
  -- other variables, and 'a' is a limiting value.  The function
  -- computes lim(x -> a,f(x)).
  xK := retract(x::FE)@Kernel(FE)
  (n := whatInfinity a) = 0 =>
    realLimit(localsubst(fcn,xK,1,retract(a)@FE),x)
  (u := limitPlus(eval(fcn,xK,n * inv(xK::FE)),x))
                                          case "failed" => "failed"
  u::OFE

localsubst(fcn, k, n, a) ==
  a = 0 and n = 1 => fcn
  eval(fcn,k,n * (k::FE) + a)

locallimitcomplex(fcn,x,a) ==
  xK := retract(x::FE)@Kernel(FE)
  (g := retractIfCan(a)@Union(FE,"failed")) case FE =>
    complLimit(localsubst(fcn,xK,1,g::FE),x)
  complLimit(eval(fcn,xK,inv(xK::FE)),x)

limit(fcn,eq,str) ==
  (xx := retractIfCan(lhs eq)@Union(SY,"failed")) case "failed" =>
```

```
    error "limit:left hand side must be a variable"
  x := xx :: SY; a := rhs eq
  xK := retract(x::FE)@Kernel(FE)
  limitPlus(localsubst(fcn,xK,direction str,a),x)

anyRootsOrAtrigs? fcn ==
  -- determines if 'fcn' has any kernels which are roots
  -- or if 'fcn' has any kernels which are inverse trig functions
  -- which could produce series expansions with fractional exponents
  for kernel in tower fcn repeat
    is?(kernel,"nthRoot" :: Symbol) => return true
    is?(kernel,"asin" :: Symbol) => return true
    is?(kernel,"acos" :: Symbol) => return true
    is?(kernel,"asec" :: Symbol) => return true
    is?(kernel,"acsc" :: Symbol) => return true
  false

complLimit(fcn,x) ==
  -- computes lim(x -> 0,fcn) using a Puiseux expansion of fcn,
  -- if fcn is an expression involving roots, and using a Laurent
  -- expansion of fcn otherwise
  lim : FE :=
    anyRootsOrAtrigs? fcn =>
      ppack := FS2UPS(R,FE,RN,_
          UPXS(FE,x,zeroFE),EFUPXS(FE,ULS(FE,x,zeroFE),UPXS(FE,x,zeroFE),_
          EFULS(FE,UTS(FE,x,zeroFE),ULS(FE,x,zeroFE))),x)
      pseries := exprToUPS(fcn,false,"complex")$ppack
      pseries case %problem => return "failed"
      if pole?(upxs := pseries.%series) then upxs := map(normalize,upxs)
      pole? upxs => return infinity()
      coefficient(upxs,0)
    lpack := FS2UPS(R,FE,Z,ULS(FE,x,zeroFE),_
             EFULS(FE,UTS(FE,x,zeroFE),ULS(FE,x,zeroFE)),x)
    lseries := exprToUPS(fcn,false,"complex")$lpack
    lseries case %problem => return "failed"
    if pole?(uls := lseries.%series) then uls := map(normalize,uls)
    pole? uls => return infinity()
    coefficient(uls,0)
  -- can the following happen?
  member?(x,variables lim) =>
    member?(x,variables(answer := normalize lim)) =>
      error "limit: can't evaluate limit"
    answer :: OPF
  lim :: FE :: OPF

okProblem?(function,problem) ==
```

```
    (function = "log") or (function = "nth root") =>
      (problem = "series of non-zero order") or _
             (problem = "negative leading coefficient")
    (function = "atan") => problem = "branch problem"
    (function = "erf") => problem = "unknown kernel"
    problem = "essential singularity"

  poleLimit(order,coef,x) ==
    -- compute limit for function with pole
    not member?(x,variables coef) =>
      (s := sign(coef)$SIGNEF) case Integer =>
        rtLim := (s :: Integer) * plusInfinity()
        even? numer order => rtLim
        even? denom order => ["failed",rtLim]$TwoSide
        [-rtLim,rtLim]$TwoSide
      -- infinite limit, but cannot determine sign
      "failed"
    error "limit: can't evaluate limit"

  poleLimitPlus(order,coef,x) ==
    -- compute right hand limit for function with pole
    not member?(x,variables coef) =>
      (s := sign(coef)$SIGNEF) case Integer =>
        (s :: Integer) * plusInfinity()
      -- infinite limit, but cannot determine sign
      "failed"
    (clim := specialLimit(coef,x)) case "failed" => "failed"
    zero? (lim := clim :: OFE) =>
      -- in this event, we need to determine if the limit of
      -- the coef is 0+ or 0-
      (cclim := specialLimit(inv coef,x)) case "failed" => "failed"
      ss := whatInfinity(cclim :: OFE) :: Z
      zero? ss =>
        error "limit: internal error"
      ss * plusInfinity()
    t := whatInfinity(lim :: OFE) :: Z
    zero? t =>
      (tt := sign(coef)$SIGNEF) case Integer =>
        (tt :: Integer) * plusInfinity()
      -- infinite limit, but cannot determine sign
      "failed"
    t * plusInfinity()

  realLimit(fcn,x) ==
    -- computes lim(x -> 0,fcn) using a Puiseux expansion of fcn,
    -- if fcn is an expression involving roots, and using a Laurent
```

```
-- expansion of fcn otherwise
lim : Union(FE,"failed") :=
  anyRootsOrAtrigs? fcn =>
    ppack := FS2UPS(R,FE,RN,_
         UPXS(FE,x,zeroFE),EFUPXS(FE,ULS(FE,x,zeroFE),UPXS(FE,x,zeroFE),_
         EFULS(FE,UTS(FE,x,zeroFE),ULS(FE,x,zeroFE))),x)
    pseries := exprToUPS(fcn,true,"real: two sides")$ppack
    pseries case %problem =>
      trouble := pseries.%problem
      function := trouble.func; problem := trouble.prob
      okProblem?(function,problem) =>
        left :=
          xK : Kernel FE := kernel x
          fcn0 := eval(fcn,xK,-(xK :: FE))
          limitPlus(fcn0,x)
        right := limitPlus(fcn,x)
        (left case "failed") and (right case "failed") =>
          return "failed"
        if (left case OFE) and (right case OFE) then
          (left :: OFE) = (right :: OFE) => return (left :: OFE)
        return([left,right]$TwoSide)
      return "failed"
    if pole?(upxs := pseries.%series) then upxs := map(normalize,upxs)
    pole? upxs =>
      cp := coefficient(upxs,ordp := order upxs)
      return poleLimit(ordp,cp,x)
    coefficient(upxs,0)
  lpack := FS2UPS(R,FE,Z,ULS(FE,x,zeroFE),_
           EFULS(FE,UTS(FE,x,zeroFE),ULS(FE,x,zeroFE)),x)
  lseries := exprToUPS(fcn,true,"real: two sides")$lpack
  lseries case %problem =>
    trouble := lseries.%problem
    function := trouble.func; problem := trouble.prob
    okProblem?(function,problem) =>
      left :=
        xK : Kernel FE := kernel x
        fcn0 := eval(fcn,xK,-(xK :: FE))
        limitPlus(fcn0,x)
      right := limitPlus(fcn,x)
      (left case "failed") and (right case "failed") =>
        return "failed"
      if (left case OFE) and (right case OFE) then
        (left :: OFE) = (right :: OFE) => return (left :: OFE)
      return([left,right]$TwoSide)
    return "failed"
  if pole?(uls := lseries.%series) then uls := map(normalize,uls)
```

```
      pole? uls =>
        cl := coefficient(uls,ordl := order uls)
        return poleLimit(ordl :: RN,cl,x)
      coefficient(uls,0)
    lim case "failed" => "failed"
    member?(x,variables(lim :: FE)) =>
      member?(x,variables(answer := normalize(lim :: FE))) =>
        error "limit: can't evaluate limit"
      answer :: OFE
    lim :: FE :: OFE

  xxpLimit(fcn,x) ==
    -- computes lim(x -> 0+,fcn) using an exponential expansion of fcn
    xpack := FS2EXPXP(R,FE,x,zeroFE)
    xxp := exprToXXP(fcn,true)$xpack
    xxp case %problem => "failed"
    limitPlus(xxp.%expansion)

  limitPlus(fcn,x) ==
    -- computes lim(x -> 0+,fcn) using a generalized Puiseux expansion
    -- of fcn, if fcn is an expression involving roots, and using a
    -- generalized Laurent expansion of fcn otherwise
    lim : Union(FE,"failed") :=
      anyRootsOrAtrigs? fcn =>
        ppack := FS2UPS(R,FE,RN,_
             UPXS(FE,x,zeroFE),EFUPXS(FE,ULS(FE,x,zeroFE),UPXS(FE,x,zeroFE),_
             EFULS(FE,UTS(FE,x,zeroFE),ULS(FE,x,zeroFE))),x)
        pseries := exprToGenUPS(fcn,true,"real: right side")$ppack
        pseries case %problem =>
          trouble := pseries.%problem
          ff := trouble.func; pp := trouble.prob
          (pp = "negative leading coefficient") => return "failed"
          "failed"
        -- pseries case %problem => return "failed"
        if pole?(upxs := pseries.%series) then upxs := map(normalize,upxs)
        pole? upxs =>
          cp := coefficient(upxs,ordp := order upxs)
          return poleLimitPlus(ordp,cp,x)
        coefficient(upxs,0)
      lpack := FS2UPS(R,FE,Z,ULS(FE,x,zeroFE),_
               EFULS(FE,UTS(FE,x,zeroFE),ULS(FE,x,zeroFE)),x)
      lseries := exprToGenUPS(fcn,true,"real: right side")$lpack
      lseries case %problem =>
        trouble := lseries.%problem
        ff := trouble.func; pp := trouble.prob
        (pp = "negative leading coefficient") => return "failed"
```

```
          "failed"
        -- lseries case %problem => return "failed"
        if pole?(uls := lseries.%series) then uls := map(normalize,uls)
        pole? uls =>
          cl := coefficient(uls,ordl := order uls)
          return poleLimitPlus(ordl :: RN,cl,x)
        coefficient(uls,0)
      lim case "failed" =>
        (xLim := xxpLimit(fcn,x)) case "failed" => specialLimit(fcn,x)
        xLim
      member?(x,variables(lim :: FE)) =>
        member?(x,variables(answer := normalize(lim :: FE))) =>
          (xLim := xxpLimit(answer,x)) case "failed" => specialLimit(answer,x)
          xLim
        answer :: OFE
      lim :: FE :: OFE

    limit(fcn:FE,eq:EQ OFE) ==
      (f := retractIfCan(lhs eq)@Union(FE,"failed")) case "failed" =>
        error "limit:left hand side must be a variable"
      (xx := retractIfCan(f)@Union(SY,"failed")) case "failed" =>
        error "limit:left hand side must be a variable"
      x := xx :: SY; a := rhs eq
      locallimit(fcn,x,a)

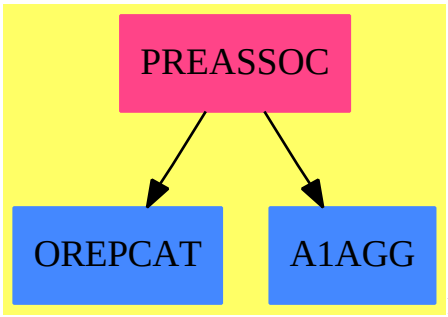
    complexLimit(fcn:FE,eq:EQ OPF) ==
      (f := retractIfCan(lhs eq)@Union(FE,"failed")) case "failed" =>
        error "limit:left hand side must be a variable"
      (xx := retractIfCan(f)@Union(SY,"failed")) case "failed" =>
        error "limit:left hand side must be a variable"
      x := xx :: SY; a := rhs eq
      locallimitcomplex(fcn,x,a)
```

⟨*LIMITPS.dotabb*⟩≡

```
"LIMITPS" [color="#FF4488",href="bookvol10.4.pdf#nameddest=LIMITPS"]
"ULSCCAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=ULSCCAT"]
"LIMITPS" -> "ULSCCAT"
```

17.111 package PREASSOC PrecomputedAssociatedEquations

17.112 PrecomputedAssociatedEquations



1.00

Exports:
firstUncouplingMatrix

⟨*package PREASSOC PrecomputedAssociatedEquations*⟩≡

```
)abbrev package PREASSOC PrecomputedAssociatedEquations
++ Author: Manuel Bronstein
++ Date Created: 13 January 1994
++ Date Last Updated: 3 February 1994
++ Description:
++ \spadtype{PrecomputedAssociatedEquations} stores some generic
++ precomputations which speed up the computations of the
++ associated equations needed for factoring operators.
PrecomputedAssociatedEquations(R, L): Exports == Implementation where
  R: IntegralDomain
  L: LinearOrdinaryDifferentialOperatorCategory R

  PI  ==> PositiveInteger
  N   ==> NonNegativeInteger
  A   ==> PrimitiveArray R
  U   ==> Union(Matrix R, "failed")

  Exports ==> with
    firstUncouplingMatrix: (L, PI) -> U
      ++ firstUncouplingMatrix(op, m) returns the matrix A such that
      ++ \spad{A w = (W',W'',...,W^N)} in the corresponding associated
      ++ equations for right-factors of order m of op.
      ++ Returns "failed" if the matrix A has not been precomputed for
      ++ the particular combination \spad{degree(L), m}.

  Implementation ==> add
```

```
A32:  L -> U
A42:  L -> U
A425: (A, A, A) -> List R
A426: (A, A, A) -> List R
makeMonic: L -> Union(A, "failed")

diff:L := D()

firstUncouplingMatrix(op, m) ==
  n := degree op
  n = 3 and m = 2 => A32 op
  n = 4 and m = 2 => A42 op
  "failed"

makeMonic op ==
  lc := leadingCoefficient op
  a:A := new(n := degree op, 0)
  for i in 0..(n-1)::N repeat
    (u := coefficient(op, i) exquo lc) case "failed" => return "failed"
    a.i := - (u::R)
  a

A32 op ==
  (u := makeMonic op) case "failed" => "failed"
  a := u::A
  matrix [[0, 1, 0], [a.1, a.2, 1],
          [diff(a.1) + a.1 * a.2 - a.0, diff(a.2) + a.2**2 + a.1, 2 * a.2]]

A42 op ==
  (u := makeMonic op) case "failed" => "failed"
  a := u::A
  a':A := new(4, 0)
  a'':A := new(4, 0)
  for i in 0..3 repeat
    a'.i := diff(a.i)
    a''.i := diff(a'.i)
  matrix [[0, 1, 0, 0, 0, 0], [0, 0, 1, 1, 0, 0], [a.1,a.2,0,a.3,2::R,0],
          [a'.1 + a.1 * a.3 - 2 * a.0, a'.2 + a.2 * a.3 + a.1, 3 * a.2,
           a'.3 + a.3 ** 2 + a.2, 3 * a.3, 2::R],
            A425(a, a', a''), A426(a, a', a'')]

A425(a, a', a'') ==
  [a''.1 + 2 * a.1 * a'.3 + a.3 * a'.1 - 2 * a'.0 + a.1 * a.3 ** 2
   - 3 * a.0 * a.3 + a.1 * a.2,
    a''.2 + 2 * a.2 * a'.3 + a.3 * a'.2 + 2 * a'.1 + a.2 * a.3 ** 2
     + a.1 * a.3 + a.2 ** 2 - 4 * a.0,
```

```
          4 * a'.2 + 4 * a.2 * a.3 - a.1,
           a''.3 + 3 * a.3 * a'.3 + 2 * a'.2 + a.3 ** 3 + 2 * a.2 * a.3 + a.1,
            4 * a'.3 + 4 * a.3 ** 2 + 4 * a.2, 5 * a.3]

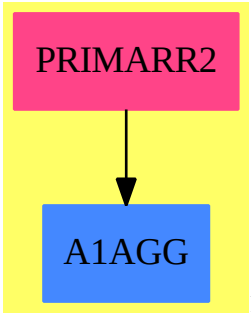
    A426(a, a', a'') ==
      [diff(a''.1) + 3 * a.1 * a''.3 + a.3 * a''.1 - 2 * a''.0
       + (3 * a'.1 + 5 * a.1 * a.3 - 7 * a.0) * a'.3 + 3 * a.1 * a'.2
        + (a.3 ** 2 + a.2) * a'.1 - 3 * a.3 * a'.0 + a.1 * a.3 ** 3
         - 4 * a.0 * a.3 ** 2 + 2 * a.1 * a.2 * a.3 - 4 * a.0 * a.2 + a.1 ** 2,
          diff(a''.2) + 3 * a.2 * a''.3 + a.3 * a''.2 + 3 * a''.1
           + (3*a'.2 + 5*a.2 * a.3 + 3 * a.1) * a'.3 + (a.3**2 + 4*a.2)*a'.2
            + 2 * a.3 * a'.1 - 6 * a'.0 + a.2 * a.3 ** 3 + a.1 * a.3 ** 2
             + (2 * a.2**2 - 8 * a.0) * a.3 + 2 * a.1 * a.2,
              5 * a''.2 + 10 * a.2 * a'.3 + 5 * a.3 * a'.2 + a'.1
               + 5 * a.2 * a.3 ** 2 - 4 * a.1 * a.3 + 5 * a.2**2 - 4 * a.0,
                diff(a''.3) + 4 * a.3 * a''.3 + 3*a''.2 + 3 * a'.3**2
                 + (6 * a.3**2 + 4 * a.2) * a'.3 + 5 * a.3 * a'.2 + 3 * a'.1
                  + a.3**4 + 3 * a.2 * a.3**2 + 2 * a.1 * a.3 + a.2**2 - 4*a.0,
                   5 * a''.3 + 15 * a.3 * a'.3 + 10 * a'.2 + 5 * a.3**3
                    + 10 * a.2 * a.3, 9 * a'.3 + 9 * a.3**2 + 4 * a.2]
```

⟨*PREASSOC.dotabb*⟩≡

```
"PREASSOC" [color="#FF4488",href="bookvol10.4.pdf#nameddest=PREASSOC"]
"OREPCAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=OREPCAT"]
"A1AGG" [color="#4488FF",href="bookvol10.2.pdf#nameddest=A1AGG"]
"PREASSOC" -> "OREPCAT"
"PREASSOC" -> "A1AGG"
```

17.113 package PRIMARR2 PrimitiveArrayFunctions2

17.114 PrimitiveArrayFunctions2



1.00

Exports:
map reduce scan

⟨*package PRIMARR2 PrimitiveArrayFunctions2*⟩≡

```
)abbrev package PRIMARR2 PrimitiveArrayFunctions2
++ This package provides tools for operating on primitive arrays
++ with unary and binary functions involving different underlying types
PrimitiveArrayFunctions2(A, B): Exports == Implementation where
  A, B: Type

  VA ==> PrimitiveArray A
  VB ==> PrimitiveArray B
  O2 ==> FiniteLinearAggregateFunctions2(A, VA, B, VB)
  Exports ==> with
   scan : ((A, B) -> B, VA, B) -> VB
    ++ scan(f,a,r) successively applies
    ++ \spad{reduce(f,x,r)} to more and more leading sub-arrays
    ++ x of primitive array \spad{a}.
    ++ More precisely, if \spad{a} is \spad{[a1,a2,...]}, then
    ++ \spad{scan(f,a,r)} returns
    ++ \spad{[reduce(f,[a1],r),reduce(f,[a1,a2],r),...]}.
    ++
    ++X T1:=PrimitiveArrayFunctions2(Integer,Integer)
    ++X adder(a:Integer,b:Integer):Integer == a+b
    ++X scan(adder,[i for i in 1..10],0)$T1

   reduce : ((A, B) -> B, VA, B) -> B
    ++ reduce(f,a,r) applies function f to each
    ++ successive element of the
    ++ primitive array \spad{a} and an accumulant initialized to r.
```

```
      ++ For example, \spad{reduce(_+$Integer,[1,2,3],0)}
      ++ does \spad{3+(2+(1+0))}. Note: third argument r
      ++ may be regarded as the identity element for the function f.
      ++
      ++X T1:=PrimitiveArrayFunctions2(Integer,Integer)
      ++X adder(a:Integer,b:Integer):Integer == a+b
      ++X reduce(adder,[i for i in 1..10],0)$T1

     map : (A -> B, VA) -> VB
      ++ map(f,a) applies function f to each member of primitive array
      ++ \spad{a} resulting in a new primitive array over a
      ++ possibly different underlying domain.
      ++
      ++X T1:=PrimitiveArrayFunctions2(Integer,Integer)
      ++X map(x+->x+2,[i for i in 1..10])$T1

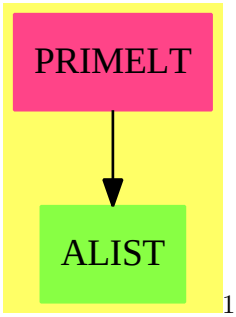
    Implementation ==> add
      map(f, v)       == map(f, v)$O2
      scan(f, v, b)   == scan(f, v, b)$O2
      reduce(f, v, b) == reduce(f, v, b)$O2
```

⟨*PRIMARR2.dotabb*⟩≡

```
"PRIMARR2" [color="#FF4488",href="bookvol10.4.pdf#nameddest=PRIMARR2"]
"A1AGG" [color="#4488FF",href="bookvol10.2.pdf#nameddest=A1AGG"]
"PRIMARR2" -> "A1AGG"
```

17.115 package PRIMELT PrimitiveElement

17.116 PrimitiveElement



1.00

Exports:
primitiveElement

⟨*package PRIMELT PrimitiveElement*⟩≡

```
)abbrev package PRIMELT PrimitiveElement
++ Computation of primitive elements.
++ Author: Manuel Bronstein
++ Date Created: 6 Jun 1990
++ Date Last Updated: 25 April 1991
++ Description:
++   PrimitiveElement provides functions to compute primitive elements
++   in algebraic extensions;
++ Keywords: algebraic, extension, primitive.
PrimitiveElement(F): Exports == Implementation where
  F : Join(Field, CharacteristicZero)

  SY  ==> Symbol
  P   ==> Polynomial F
  UP  ==> SparseUnivariatePolynomial F
  RC  ==> Record(coef1: Integer, coef2: Integer, prim:UP)
  REC ==> Record(coef: List Integer, poly:List UP, prim: UP)

  Exports ==> with
    primitiveElement: (P, SY, P, SY) -> RC
      ++ primitiveElement(p1, a1, p2, a2) returns \spad{[c1, c2, q]}
      ++ such that \spad{k(a1, a2) = k(a)}
      ++ where \spad{a = c1 a1 + c2 a2, and q(a) = 0}.
      ++ The pi's are the defining polynomials for the ai's.
      ++ The p2 may involve a1, but p1 must not involve a2.
      ++ This operation uses \spadfun{resultant}.
    primitiveElement: (List P, List SY) -> REC
```

```
      ++ primitiveElement([p1,...,pn], [a1,...,an]) returns
      ++ \spad{[[c1,...,cn], [q1,...,qn], q]}
      ++ such that then \spad{k(a1,...,an) = k(a)},
      ++ where \spad{a = a1 c1 + ... + an cn},
      ++ \spad{ai = qi(a)}, and \spad{q(a) = 0}.
      ++ The pi's are the defining polynomials for the ai's.
      ++ This operation uses the technique of
      ++ \spadglossSee{groebner bases}{Groebner basis}.
    primitiveElement: (List P, List SY, SY) -> REC
      ++ primitiveElement([p1,...,pn], [a1,...,an], a) returns
      ++ \spad{[[c1,...,cn], [q1,...,qn], q]}
      ++ such that then \spad{k(a1,...,an) = k(a)},
      ++ where \spad{a = a1 c1 + ... + an cn},
      ++ \spad{ai = qi(a)}, and \spad{q(a) = 0}.
      ++ The pi's are the defining polynomials for the ai's.
      ++ This operation uses the technique of
      ++ \spadglossSee{groebner bases}{Groebner basis}.

  Implementation ==> add
    import PolyGroebner(F)

    multi     : (UP, SY) -> P
    randomInts: (NonNegativeInteger, NonNegativeInteger) -> List Integer
    findUniv  : (List P, SY, SY) -> Union(P, "failed")
    incl?     : (List SY, List SY) -> Boolean
    triangularLinearIfCan:(List P,List SY,SY) -> Union(List UP,"failed")
    innerPrimitiveElement: (List P, List SY, SY) -> REC

    multi(p, v)            == multivariate(map(#1, p), v)
    randomInts(n, m)       == [symmetricRemainder(random()$Integer, m) for i in 1..n]
    incl?(a, b)            == every?(member?(#1, b), a)
    primitiveElement(l, v) == primitiveElement(l, v, new()$SY)

    primitiveElement(p1, a1, p2, a2) ==
--      one? degree(p2, a1) => [0, 1, univariate resultant(p1, p2, a1)]
      (degree(p2, a1) = 1) => [0, 1, univariate resultant(p1, p2, a1)]
      u := (new()$SY)::P
      b := a2::P
      for i in 10.. repeat
        c := symmetricRemainder(random()$Integer, i)
        w := u - c * b
        r := univariate resultant(eval(p1, a1, w), eval(p2, a1, w), a2)
        not zero? r and r = squareFreePart r => return [1, c, r]

    findUniv(l, v, opt) ==
      for p in l repeat
```

```
      degree(p, v) > 0 and incl?(variables p, [v, opt]) => return p
    "failed"

  triangularLinearIfCan(l, lv, w) ==
    (u := findUniv(l, w, w)) case "failed" => "failed"
    pw := univariate(u::P)
    ll := nil()$List(UP)
    for v in lv repeat
      ((u := findUniv(l, v, w)) case "failed") or
        (degree(p := univariate(u::P, v)) ^= 1) => return "failed"
      (bc := extendedEuclidean(univariate leadingCoefficient p, pw,1))
         case "failed" => error "Should not happen"
      ll := concat(map(#1,
              (- univariate(coefficient(p,0)) * bc.coef1) rem pw), ll)
    concat(map(#1, pw), reverse_! ll)

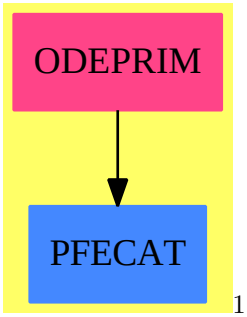
  primitiveElement(l, vars, uu) ==
    u    := uu::P
    vv   := [v::P for v in vars]
    elim := concat(vars, uu)
    w    := uu::P
    n    := #l
    for i in 10.. repeat
      cf := randomInts(n, i)
      (tt := triangularLinearIfCan(lexGroebner(
           concat(w - +/[c * t for c in cf for t in vv], l), elim),
              vars, uu)) case List(UP) =>
                 ltt := tt::List(UP)
                 return([cf, rest ltt, first ltt])
```

⟨*PRIMELT.dotabb*⟩≡

```
"PRIMELT" [color="#FF4488",href="bookvol10.4.pdf#nameddest=PRIMELT"]
"ALIST" [color="#88FF44",href="bookvol10.3.pdf#nameddest=ALIST"]
"PRIMELT" -> "ALIST"
```

17.117 package ODEPRIM PrimitiveRatDE

17.118 PrimitiveRatDE



1.00

Exports:
indicialEquation denomLODE indicialEquations splitDenominator

⟨*package ODEPRIM PrimitiveRatDE*⟩≡
```
)abbrev package ODEPRIM PrimitiveRatDE
++ Author: Manuel Bronstein
++ Date Created: 1 March 1991
++ Date Last Updated: 1 February 1994
++ Description:
++  \spad{PrimitiveRatDE} provides functions for in-field solutions of linear
++   ordinary differential equations, in the transcendental case.
++   The derivation to use is given by the parameter \spad{L}.
PrimitiveRatDE(F, UP, L, LQ): Exports == Implementation where
  F  : Join(Field, CharacteristicZero, RetractableTo Fraction Integer)
  UP : UnivariatePolynomialCategory F
  L  : LinearOrdinaryDifferentialOperatorCategory UP
  LQ : LinearOrdinaryDifferentialOperatorCategory Fraction UP

  N   ==> NonNegativeInteger
  Z   ==> Integer
  RF  ==> Fraction UP
  UP2 ==> SparseUnivariatePolynomial UP
  REC ==> Record(center:UP, equation:UP)

  Exports ==> with
    denomLODE: (L, RF) -> Union(UP, "failed")
      ++ denomLODE(op, g) returns a polynomial d such that
      ++ any rational solution of \spad{op y = g}
      ++ is of the form \spad{p/d} for some polynomial p, and
      ++ "failed", if the equation has no rational solution.
    denomLODE: (L, List RF) -> UP
```

```
    ++ denomLODE(op, [g1,...,gm]) returns a polynomial
    ++ d such that any rational solution of \spad{op y = c1 g1 + ... + cm gm}
    ++ is of the form \spad{p/d} for some polynomial p.
  indicialEquations: L -> List REC
    ++ indicialEquations op returns \spad{[[d1,e1],...,[dq,eq]]} where
    ++ the \spad{d_i}'s are the affine singularities of \spad{op},
    ++ and the \spad{e_i}'s are the indicial equations at each \spad{d_i}.
  indicialEquations: (L, UP) -> List REC
    ++ indicialEquations(op, p) returns \spad{[[d1,e1],...,[dq,eq]]} where
    ++ the \spad{d_i}'s are the affine singularities of \spad{op}
    ++ above the roots of \spad{p},
    ++ and the \spad{e_i}'s are the indicial equations at each \spad{d_i}.
  indicialEquation: (L, F) -> UP
    ++ indicialEquation(op, a) returns the indicial equation of \spad{op}
    ++ at \spad{a}.
  indicialEquations: LQ -> List REC
    ++ indicialEquations op returns \spad{[[d1,e1],...,[dq,eq]]} where
    ++ the \spad{d_i}'s are the affine singularities of \spad{op},
    ++ and the \spad{e_i}'s are the indicial equations at each \spad{d_i}.
  indicialEquations: (LQ, UP) -> List REC
    ++ indicialEquations(op, p) returns \spad{[[d1,e1],...,[dq,eq]]} where
    ++ the \spad{d_i}'s are the affine singularities of \spad{op}
    ++ above the roots of \spad{p},
    ++ and the \spad{e_i}'s are the indicial equations at each \spad{d_i}.
  indicialEquation: (LQ, F) -> UP
    ++ indicialEquation(op, a) returns the indicial equation of \spad{op}
    ++ at \spad{a}.
  splitDenominator: (LQ, List RF) -> Record(eq:L, rh:List RF)
    ++ splitDenominator(op, [g1,...,gm]) returns \spad{op0, [h1,...,hm]}
    ++ such that the equations \spad{op y = c1 g1 + ... + cm gm} and
    ++ \spad{op0 y = c1 h1 + ... + cm hm} have the same solutions.

Implementation ==> add
  import BoundIntegerRoots(F, UP)
  import BalancedFactorisation(F, UP)
  import InnerCommonDenominator(UP, RF, List UP, List RF)
  import UnivariatePolynomialCategoryFunctions2(F, UP, UP, UP2)

  tau          : (UP, UP, UP, N) -> UP
  NPbound      : (UP, L, UP) -> N
  hdenom       : (L, UP, UP) -> UP
  denom0       : (Z, L, UP, UP, UP) -> UP
  indicialEq   : (UP, List N, List UP) -> UP
  separateZeros: (UP, UP) -> UP
  UPfact       : N -> UP
  UP2UP2       : UP -> UP2
```

```
    indeq        : (UP, L) -> UP
    NPmulambda   : (UP, L) -> Record(mu:Z, lambda:List N, func:List UP)

    diff := D()$L

    UP2UP2 p                    == map(#1::UP, p)
    indicialEquations(op:L)     == indicialEquations(op, leadingCoefficient op)
    indicialEquation(op:L, a:F) == indeq(monomial(1, 1) - a::UP, op)

    splitDenominator(op, lg) ==
      cd := splitDenominator coefficients op
      f  := cd.den / gcd(cd.num)
      l:L := 0
      while op ^= 0 repeat
          l  := l + monomial(retract(f * leadingCoefficient op), degree op)
          op := reductum op
      [l, [f * g for g in lg]]

    tau(p, pp, q, n) ==
      ((pp ** n) * ((q exquo (p ** order(q, p)))::UP)) rem p

    indicialEquations(op:LQ) ==
      indicialEquations(splitDenominator(op, empty()).eq)

    indicialEquations(op:LQ, p:UP) ==
      indicialEquations(splitDenominator(op, empty()).eq, p)

    indicialEquation(op:LQ, a:F) ==
      indeq(monomial(1, 1) - a::UP, splitDenominator(op, empty()).eq)

-- returns z(z-1)...(z-(n-1))
    UPfact n ==
      zero? n => 1
      z := monomial(1, 1)$UP
      */[z - i::F::UP for i in 0..(n-1)::N]

    indicialEq(c, lamb, lf) ==
      cp := diff c
      cc := UP2UP2 c
      s:UP2 := 0
      for i in lamb for f in lf repeat
        s := s + (UPfact i) * UP2UP2 tau(c, cp, f, i)
      primitivePart resultant(cc, s)

    NPmulambda(c, l) ==
      lamb:List(N) := [d := degree l]
```

```
      lf:List(UP) := [a := leadingCoefficient l]
      mup := d::Z - order(a, c)
      while (l := reductum l) ^= 0 repeat
          a := leadingCoefficient l
          if (m := (d := degree l)::Z - order(a, c)) > mup then
              mup := m
              lamb := [d]
              lf := [a]
          else if (m = mup) then
              lamb := concat(d, lamb)
              lf := concat(a, lf)
      [mup, lamb, lf]

  -- e = 0 means homogeneous equation
      NPbound(c, l, e) ==
        rec := NPmulambda(c, l)
        n := max(0, - integerBound indicialEq(c, rec.lambda, rec.func))
        zero? e => n::N
        max(n, order(e, c)::Z - rec.mu)::N

      hdenom(l, d, e) ==
        */[dd.factor ** NPbound(dd.factor, l, e)
                      for dd in factors balancedFactorisation(d, coefficients l)]

      denom0(n, l, d, e, h) ==
        hdenom(l, d, e) * */[hh.factor ** max(0, order(e, hh.factor) - n)::N
                                    for hh in factors balancedFactorisation(h, e)]

  -- returns a polynomials whose zeros are the zeros of e which are not
  -- zeros of d
      separateZeros(d, e) ==
        ((g := squareFreePart e) exquo gcd(g, squareFreePart d))::UP

      indeq(c, l) ==
        rec := NPmulambda(c, l)
        indicialEq(c, rec.lambda, rec.func)

      indicialEquations(op:L, p:UP) ==
        [[dd.factor, indeq(dd.factor, op)]
                     for dd in factors balancedFactorisation(p, coefficients op)]

  -- cannot return "failed" in the homogeneous case
      denomLODE(l:L, g:RF) ==
        d := leadingCoefficient l
        zero? g => hdenom(l, d, 0)
        h := separateZeros(d, e := denom g)
```

```
        n := degree l
        (e exquo (h**(n + 1))) case "failed" => "failed"
        denom0(n, l, d, e, h)

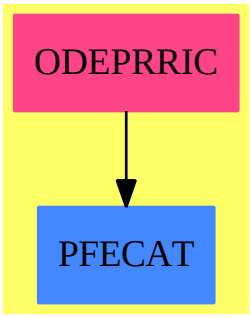
      denomLODE(l:L, lg:List RF) ==
        empty? lg => denomLODE(l, 0)::UP
        d := leadingCoefficient l
        h := separateZeros(d, e := "lcm"/[denom g for g in lg])
        denom0(degree l, l, d, e, h)
```

⟨*ODEPRIM.dotabb*⟩≡

```
"ODEPRIM" [color="#FF4488",href="bookvol10.4.pdf#nameddest=ODEPRIM"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"ODEPRIM" -> "PFECAT"
```

17.119 package ODEPRRIC PrimitiveRatRicDE

17.120 PrimitiveRatRicDE



1.00

Exports:
changeVar denomRicDE constantCoefficientRicDE leadingCoefficientRicDE polyRicDE singRicDE

⟨*package ODEPRRIC PrimitiveRatRicDE*⟩≡

```
)abbrev package ODEPRRIC PrimitiveRatRicDE
++ Author: Manuel Bronstein
++ Date Created: 22 October 1991
++ Date Last Updated: 2 February 1993
++ Description: In-field solution of Riccati equations, primitive case.
PrimitiveRatRicDE(F, UP, L, LQ): Exports == Implementation where
  F  : Join(Field, CharacteristicZero, RetractableTo Fraction Integer)
  UP : UnivariatePolynomialCategory F
  L  : LinearOrdinaryDifferentialOperatorCategory UP
  LQ : LinearOrdinaryDifferentialOperatorCategory Fraction UP

  N    ==> NonNegativeInteger
  Z    ==> Integer
  RF   ==> Fraction UP
  UP2  ==> SparseUnivariatePolynomial UP
  REC  ==> Record(deg:N, eq:UP)
  REC2 ==> Record(deg:N, eq:UP2)
  POL  ==> Record(poly:UP, eq:L)
  FRC  ==> Record(frac:RF, eq:L)
  CNT  ==> Record(constant:F, eq:L)
  IJ   ==> Record(ij: List Z, deg:N)

  Exports ==> with
    denomRicDE: L -> UP
      ++ denomRicDE(op) returns a polynomial \spad{d} such that any rational
```

```
    ++ solution of the associated Riccati equation of \spad{op y = 0} is
    ++ of the form \spad{p/d + q'/q + r} for some polynomials p and q
    ++ and a reduced r. Also, \spad{deg(p) < deg(d)} and {gcd(d,q) = 1}.
  leadingCoefficientRicDE:  L -> List REC
    ++ leadingCoefficientRicDE(op) returns
    ++ \spad{[[m1, p1], [m2, p2], ... , [mk, pk]]} such that the polynomial
    ++ part of any rational solution of the associated Riccati equation of
    ++ \spad{op y = 0} must have degree mj for some j, and its leading
    ++ coefficient is then a zero of pj. In addition,\spad{m1>m2> ... >mk}.
  constantCoefficientRicDE: (L, UP -> List F) -> List CNT
    ++ constantCoefficientRicDE(op, ric) returns
    ++ \spad{[[a1, L1], [a2, L2], ... , [ak, Lk]]} such that any rational
    ++ solution with no polynomial part of the associated Riccati equation of
    ++ \spad{op y = 0} must be one of the ai's in which case the equation for
    ++ \spad{z = y e^{-int ai}} is \spad{Li z = 0}.
    ++ \spad{ric} is a Riccati equation solver over \spad{F}, whose input
    ++ is the associated linear equation.
  polyRicDE: (L, UP -> List F) -> List POL
    ++ polyRicDE(op, zeros) returns
    ++ \spad{[[p1, L1], [p2, L2], ... , [pk, Lk]]} such that the polynomial
    ++ part of any rational solution of the associated Riccati equation of
    ++ \spad{op y=0} must be one of the pi's (up to the constant coefficient),
    ++ in which case the equation for \spad{z=y e^{-int p}} is \spad{Li z =0}.
    ++ \spad{zeros} is a zero finder in \spad{UP}.
  singRicDE: (L, (UP, UP2) -> List UP, UP -> Factored UP) -> List FRC
    ++ singRicDE(op, zeros, ezfactor) returns
    ++ \spad{[[f1, L1], [f2, L2], ... , [fk, Lk]]} such that the singular
    ++ part of any rational solution of the associated Riccati equation of
    ++ \spad{op y=0} must be one of the fi's (up to the constant coefficient),
    ++ in which case the equation for \spad{z=y e^{-int p}} is \spad{Li z=0}.
    ++ \spad{zeros(C(x),H(x,y))} returns all the \spad{P_i(x)}'s such that
    ++ \spad{H(x,P_i(x)) = 0 modulo C(x)}.
    ++ Argument \spad{ezfactor} is a factorisation in \spad{UP},
    ++ not necessarily into irreducibles.
  changeVar: (L, UP) -> L
    ++ changeVar(+/[ai D^i], a) returns the operator \spad{+/[ai (D+a)^i]}.
  changeVar: (L, RF) -> L
    ++ changeVar(+/[ai D^i], a) returns the operator \spad{+/[ai (D+a)^i]}.

Implementation ==> add
  import PrimitiveRatDE(F, UP, L, LQ)
  import BalancedFactorisation(F, UP)

  bound             : (UP, L) -> N
  lambda            : (UP, L) -> List IJ
  infmax            : (IJ, L) -> List Z
```

```
    dmax              : (IJ, UP, L) -> List Z
    getPoly           : (IJ, L, List Z) -> UP
    getPol            : (IJ, UP, L, List Z) -> UP2
    innerlb           : (L, UP -> Z) -> List IJ
    innermax          : (IJ, L, UP -> Z) -> List Z
    tau0              : (UP, UP) -> UP
    poly1             : (UP, UP, Z) -> UP2
    getPol1           : (List Z, UP, L) -> UP2
    getIndices        : (N, List IJ) -> List Z
    refine            : (List UP, UP -> Factored UP) -> List UP
    polysol           : (L, N, Boolean, UP -> List F) -> List POL
    fracsol           : (L, (UP, UP2) -> List UP, List UP) -> List FRC
    padicsol      l   : (UP, L, N, Boolean, (UP, UP2) -> List UP) -> List FRC
    leadingDenomRicDE : (UP, L) -> List REC2
    factoredDenomRicDE: L -> List UP
    constantCoefficientOperator: (L, N) -> UP
    infLambda: L -> List IJ
      -- infLambda(op) returns
      -- \spad{[[[i,j], (\deg(a_i)-\deg(a_j))/(i-j) ]]} for all the pairs
      -- of indices \spad{i,j} such that \spad{(\deg(a_i)-\deg(a_j))/(i-j)} is
      -- an integer.

    diff  := D()$L
    diffq := D()$LQ

    lambda(c, l)        == innerlb(l, order(#1, c)::Z)
    infLambda l         == innerlb(l, -(degree(#1)::Z))
    infmax(rec, l)      == innermax(rec, l, degree(#1)::Z)
    dmax(rec, c, l)     == innermax(rec, l, - order(#1, c)::Z)
    tau0(p, q)          == ((q exquo (p ** order(q, p)))::UP) rem p
    poly1(c, cp, i)     == */[monomial(1,1)$UP2 - (j * cp)::UP2 for j in 0..i-1]
    getIndices(n, l)    == removeDuplicates_! concat [r.ij for r in l | r.deg=n]
    denomRicDE l        == */[c ** bound(c, l) for c in factoredDenomRicDE l]
    polyRicDE(l, zeros) == concat([0, l], polysol(l, 0, false, zeros))

  -- refine([p1,...,pn], foo) refines the list of factors using foo
    refine(l, ezfactor) ==
      concat [[r.factor for r in factors ezfactor p] for p in l]

  -- returns [] if the solutions of l have no p-adic component at c
    padicsol(c, op, b, finite?, zeros) ==
      ans:List(FRC) := empty()
      finite? and zero? b => ans
      lc := leadingDenomRicDE(c, op)
      if finite? then lc := select_!(#1.deg <= b, lc)
      for rec in lc repeat
```

```
          for r in zeros(c, rec.eq) | r ^= 0 repeat
            rcn := r /$RF (c ** rec.deg)
            neweq := changeVar(op, rcn)
            sols := padicsol(c, neweq, (rec.deg-1)::N, true, zeros)
            ans :=
              empty? sols => concat([rcn, neweq], ans)
              concat_!([[rcn + sol.frac, sol.eq] for sol in sols], ans)
        ans

      leadingDenomRicDE(c, l) ==
        ind:List(Z)          -- to cure the compiler... (won't compile without)
        lb := lambda(c, l)
        done:List(N) := empty()
        ans:List(REC2) := empty()
        for rec in lb | (not member?(rec.deg, done)) and
          not(empty?(ind := dmax(rec, c, l))) repeat
            ans := concat([rec.deg, getPol(rec, c, l, ind)], ans)
            done := concat(rec.deg, done)
        sort_!(#1.deg > #2.deg, ans)

      getPol(rec, c, l, ind) ==
--      one?(rec.deg) => getPol1(ind, c, l)
        (rec.deg = 1) => getPol1(ind, c, l)
        +/[monomial(tau0(c, coefficient(l, i::N)), i::N)$UP2 for i in ind]

      getPol1(ind, c, l) ==
        cp := diff c
        +/[tau0(c, coefficient(l, i::N)) * poly1(c, cp, i) for i in ind]

      constantCoefficientRicDE(op, ric) ==
        m := "max"/[degree p for p in coefficients op]
        [[a, changeVar(op,a::UP)] for a in ric constantCoefficientOperator(op,m)]

      constantCoefficientOperator(op, m) ==
        ans:UP := 0
        while op ^= 0 repeat
          if degree(p := leadingCoefficient op) = m then
            ans := ans + monomial(leadingCoefficient p, degree op)
          op := reductum op
        ans

      getPoly(rec, l, ind) ==
        +/[monomial(leadingCoefficient coefficient(l,i::N),i::N)$UP for i in ind]

  -- returns empty() if rec is does not reach the max,
  -- the list of indices (including rec) that reach the max otherwise
```

```
innermax(rec, l, nu) ==
  n := degree l
  i := first(rec.ij)
  m := i * (d := rec.deg) + nu coefficient(l, i::N)
  ans:List(Z) := empty()
  for j in 0..n | (f := coefficient(l, j)) ^= 0 repeat
    if ((k := (j * d + nu f)) > m) then return empty()
    else if (k = m) then ans := concat(j, ans)
  ans

leadingCoefficientRicDE l ==
  ind:List(Z)          -- to cure the compiler... (won't compile without)
  lb := infLambda l
  done:List(N) := empty()
  ans:List(REC) := empty()
  for rec in lb | (not member?(rec.deg, done)) and
    not(empty?(ind := infmax(rec, l))) repeat
      ans := concat([rec.deg, getPoly(rec, l, ind)], ans)
      done := concat(rec.deg, done)
  sort_!(#1.deg > #2.deg, ans)

factoredDenomRicDE l ==
  bd := factors balancedFactorisation(leadingCoefficient l, coefficients l)
  [dd.factor for dd in bd]

changeVar(l:L, a:UP) ==
  dpa := diff + a::L            -- the operator (D + a)
  dpan:L := 1                   -- will accumulate the powers of (D + a)
  op:L := 0
  for i in 0..degree l repeat
    op   := op + coefficient(l, i) * dpan
    dpan := dpa * dpan
  primitivePart op

changeVar(l:L, a:RF) ==
  dpa := diffq + a::LQ          -- the operator (D + a)
  dpan:LQ := 1                  -- will accumulate the powers of (D + a)
  op:LQ := 0
  for i in 0..degree l repeat
    op   := op + coefficient(l, i)::RF * dpan
    dpan := dpa * dpan
  splitDenominator(op, empty()).eq

bound(c, l) ==
  empty?(lb := lambda(c, l)) => 1
  "max"/[rec.deg for rec in lb]
```

```
-- returns all the pairs [[i, j], n] such that
-- n = (nu(i) - nu(j)) / (i - j) is an integer
    innerlb(l, nu) ==
      lb:List(IJ) := empty()
      n := degree l
      for i in 0..n | (li := coefficient(l, i)) ^= 0repeat
        for j in i+1..n | (lj := coefficient(l, j)) ^= 0 repeat
          u := (nu li - nu lj) exquo (i-j)
          if (u case Z) and ((b := u::Z) > 0) then
            lb := concat([[i, j], b::N], lb)
      lb

    singRicDE(l, zeros, ezfactor) ==
      concat([0, l], fracsol(l, zeros, refine(factoredDenomRicDE l, ezfactor)))

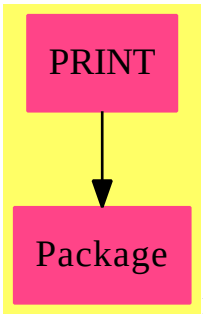
-- returns [] if the solutions of l have no singular component
    fracsol(l, zeros, lc) ==
      ans:List(FRC) := empty()
      empty? lc => ans
      empty?(sols := padicsol(first lc, l, 0, false, zeros)) =>
        fracsol(l, zeros, rest lc)
      for rec in sols repeat
        neweq := changeVar(l, rec.frac)
        sols := fracsol(neweq, zeros, rest lc)
        ans :=
          empty? sols => concat(rec, ans)
          concat_!([[rec.frac + sol.frac, sol.eq] for sol in sols], ans)
      ans

-- returns [] if the solutions of l have no polynomial component
    polysol(l, b, finite?, zeros) ==
      ans:List(POL) := empty()
      finite? and zero? b => ans
      lc := leadingCoefficientRicDE l
      if finite? then lc := select_!(#1.deg <= b, lc)
      for rec in lc repeat
        for a in zeros(rec.eq) | a ^= 0 repeat
          atn:UP := monomial(a, rec.deg)
          neweq := changeVar(l, atn)
          sols := polysol(neweq, (rec.deg - 1)::N, true, zeros)
          ans :=
            empty? sols => concat([atn, neweq], ans)
            concat_!([[atn + sol.poly, sol.eq] for sol in sols], ans)
      ans
```

⟨*ODEPRRIC.dotabb*⟩≡
```
"ODEPRRIC" [color="#FF4488",href="bookvol10.4.pdf#nameddest=ODEPRRIC"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"ODEPRRIC" -> "PFECAT"
```

17.121 package PRINT PrintPackage

17.122 PrintPackage



1.00

Exports:
print

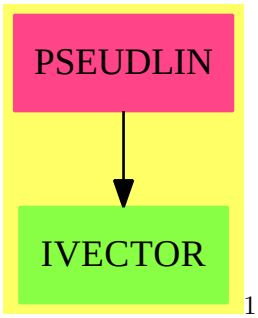
⟨*package PRINT PrintPackage*⟩≡
```
)abbrev package PRINT PrintPackage
++ Author: Scott Morrison
++ Date Created: Aug. 1, 1990
++ Date Last Updated:
++ Basic Operations: print
++ Related Constructors:
++ Also See:
++ AMS Classifications:
++ Keywords: print
++ References:
++ Description: PrintPackage provides a print function for output forms.
PrintPackage(): with
    print : OutputForm ->  Void
      ++ print(o) writes the output form o on standard output using the
      ++ two-dimensional formatter.
 == add
    print(x) == print(x)$OutputForm
```

⟨*PRINT.dotabb*⟩≡

```
"PRINT" [color="#FF4488",href="bookvol10.4.pdf#nameddest=PRINT"]
"Package" [color="#FF4488"]
"PRINT" -> "Package"
```

17.123 package PSEUDLIN PseudoLinearNormalForm

17.124 PseudoLinearNormalForm



1.00

Exports:
changeBase companionBlocks normalForm

⟨*package PSEUDLIN PseudoLinearNormalForm*⟩≡

```
)abbrev package PSEUDLIN PseudoLinearNormalForm
++ Normal forms of pseudo-linear operators
++ Author: Bruno Zuercher
++ Date Created: November 1993
++ Date Last Updated: 12 April 1994
++ Description:
++   PseudoLinearNormalForm provides a function for computing a block-companion
++   form for pseudo-linear operators.
PseudoLinearNormalForm(K:Field): Exports == Implementation where
  ER  ==> Record(C: Matrix K, g: Vector K)
  REC ==> Record(R: Matrix K, A: Matrix K, Ainv: Matrix K)

  Exports ==> with
    normalForm: (Matrix K, Automorphism K, K -> K) -> REC
      ++ normalForm(M, sig, der) returns \spad{[R, A, A^{-1}]} such that
      ++ the pseudo-linear operator whose matrix in the basis \spad{y} is
      ++ \spad{M} had matrix \spad{R} in the basis \spad{z = A y}.
      ++ \spad{der} is a \spad{sig}-derivation.
    changeBase: (Matrix K, Matrix K, Automorphism K, K -> K) -> Matrix K
      ++ changeBase(M, A, sig, der): computes the new matrix of a pseudo-linear
      ++ transform given by the matrix M under the change of base A
    companionBlocks: (Matrix K, Vector K) -> List ER
      ++ companionBlocks(m, v) returns \spad{[[C_1, g_1],...,[C_k, g_k]]}
      ++ such that each \spad{C_i} is a companion block and
      ++ \spad{m = diagonal(C_1,...,C_k)}.
```

```
Implementation ==> add
  normalForm0: (Matrix K, Automorphism K, Automorphism K, K -> K) -> REC
  mulMatrix: (Integer, Integer, K) -> Matrix K
    -- mulMatrix(N, i, a): under a change of base with the resulting matrix of
    -- size N*N the following operations are performed:
    -- D1: column i will be multiplied by sig(a)
    -- D2: row i will be multiplied by 1/a
    -- D3: addition of der(a)/a to the element at position (i,i)
  addMatrix: (Integer, Integer, Integer, K) -> Matrix K
    -- addMatrix(N, i, k, a): under a change of base with the resulting matrix
    -- of size N*N the following operations are performed:
    -- C1: addition of column i multiplied by sig(a) to column k
    -- C2: addition of row k multiplied by -a to row i
    -- C3: addition of -a*der(a) to the element at position (i,k)
  permutationMatrix: (Integer, Integer, Integer) -> Matrix K
    -- permutationMatrix(N, i, k): under a change of base with the resulting
    -- permutation matrix of size N*N the following operations are performed:
    -- P1: columns i and k will be exchanged
    -- P2: rows i and k will be exchanged
  inv: Matrix K -> Matrix K
    -- inv(M): computes the inverse of a invertable matrix M.
    -- avoids possible type conflicts

  inv m                       == inverse(m) :: Matrix K
  changeBase(M, A, sig, der) == inv(A) * (M * map(sig #1, A) + map(der, A))
  normalForm(M, sig, der)    == normalForm0(M, sig, inv sig, der)

  companionBlocks(R, w) ==
    -- decomposes the rational matrix R into single companion blocks
    -- and the inhomogenity w as well
    i:Integer := 1
    n := nrows R
    l:List(ER) := empty()
    while i <= n repeat
      j := i
      while j+1 <= n and R(j,j+1) = 1 repeat j := j+1
      --split block now
      v:Vector K := new((j-i+1)::NonNegativeInteger, 0)
      for k in i..j repeat v(k-i+1) := w k
      l := concat([subMatrix(R,i,j,i,j), v], l)
      i := j+1
    l

  normalForm0(M, sig, siginv, der) ==
    -- the changes of base will be incremented in B and Binv,
    -- where B**(-1)=Binv; E defines an elementary matrix
```

```
B, Binv, E    : Matrix K
recOfMatrices : REC
N := nrows M
B := diagonalMatrix [1 for k in 1..N]
Binv := copy B
-- avoid unnecessary recursion
if diagonal?(M) then return [M, B, Binv]
i : Integer := 1
while i < N repeat
  j := i + 1
  while j <= N and M(i, j) = 0 repeat  j := j + 1
  if j <= N then
    -- expand companionblock by lemma 5
    if j ^= i+1 then
      -- perform first a permutation
      E := permutationMatrix(N, i+1, j)
      M := changeBase(M, E, sig, der)
      B := B*E
      Binv := E*Binv
    -- now is M(i, i+1) ^= 0
    E := mulMatrix(N, i+1, siginv inv M(i,i+1))
    M := changeBase(M, E, sig, der)
    B := B*E
    Binv := inv(E)*Binv
    for j in 1..N repeat
      if j ^= i+1 then
        E := addMatrix(N, i+1, j, siginv(-M(i,j)))
        M := changeBase(M, E, sig, der)
        B := B*E
        Binv := inv(E)*Binv
    i := i + 1
  else
    -- apply lemma 6
    for j in i..2 by -1 repeat
      for k in (i+1)..N repeat
        E := addMatrix(N, k, j-1, M(k,j))
        M := changeBase(M, E, sig, der)
        B := B*E
        Binv := inv(E)*Binv
    j := i + 1
    while j <= N and M(j,1) = 0 repeat  j := j + 1
    if j <= N then
      -- expand companionblock by lemma 8
      E := permutationMatrix(N, 1, j)
      M := changeBase(M, E, sig, der)
      B := B*E
```

```
            Binv := E*Binv
            -- start again to establish rational form
            i := 1
          else
            -- split a direct factor
            recOfMatrices :=
              normalForm(subMatrix(M, i+1, N, i+1, N), sig, der)
            setsubMatrix!(M, i+1, i+1, recOfMatrices.R)
            E := diagonalMatrix [1 for k in 1..N]
            setsubMatrix!(E, i+1, i+1, recOfMatrices.A)
            B := B*E
            setsubMatrix!(E, i+1, i+1, recOfMatrices.Ainv)
            Binv := E*Binv
            -- M in blockdiagonalform, stop program
            i := N
        [M, B, Binv]

      mulMatrix(N, i, a) ==
        M : Matrix K := diagonalMatrix [1 for j in 1..N]
        M(i, i) := a
        M

      addMatrix(N, i, k, a) ==
        A : Matrix K := diagonalMatrix [1 for j in 1..N]
        A(i, k) := a
        A

      permutationMatrix(N, i, k) ==
        P : Matrix K := diagonalMatrix [1 for j in 1..N]
        P(i, i) := P(k, k) := 0
        P(i, k) := P(k, i) := 1
        P
```

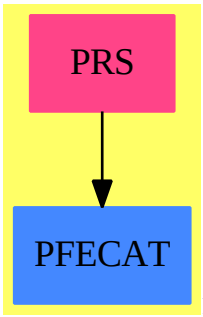
⟨*PSEUDLIN.dotabb*⟩≡

```
"PSEUDLIN" [color="#FF4488",href="bookvol10.4.pdf#nameddest=PSEUDLIN"]
"IVECTOR" [color="#88FF44",href="bookvol10.3.pdf#nameddest=IVECTOR"]
"PSEUDLIN" -> "IVECTOR"
```

17.125 package PRS PseudoRemainderSequence

The package constructor **PseudoRemainderSequence** provides efficient algorithms by Lionel Ducos (University of Poitiers, France) for computing subresultants. This leads to a speed up in many places in Axiom where subresultants are computed (polynomial system solving, algebraic factorization, integration).

17.126 PseudoRemainderSequence



1.00

Exports:

discriminant	Lazard
Lazard2	chainSubResultants
degreeSubResultant	degreeSubResultantEuclidean
discriminantEuclidean	divide
exquo	gcd
indiceSubResultant	indiceSubResultantEuclidean
lastSubResultant	lastSubResultantEuclidean
nextsousResultant2	pseudoDivide
resultant	resultantEuclidean
resultantEuclideannaif	resultantnaif
resultantReduit	resultantReduitEuclidean
schema	semiDegreeSubResultantEuclidean
semiDiscriminantEuclidean	semiIndiceSubResultantEuclidean
semiLastSubResultantEuclidean	semiResultantEuclidean1
semiResultantEuclidean2	semiResultantEuclideannaif
semiResultantReduitEuclidean	semiSubResultantGcdEuclidean1
semiSubResultantGcdEuclidean2	subResultantGcd
subResultantGcdEuclidean	?*?

⟨*package PRS PseudoRemainderSequence*⟩≡

```
)abbrev package PRS PseudoRemainderSequence
++ Author: Ducos Lionel
```

```
++ Date Created: january 1995
++ Date Last Updated: 5 february 1999
++ Basic Functions:
++ Related Constructors:
++ Also See:
++ AMS Classifications:
++ Keywords:
++ Description: This package contains some functions:
++ \axiomOpFrom{discriminant}{PseudoRemainderSequence},
++ \axiomOpFrom{resultant}{PseudoRemainderSequence},
++ \axiomOpFrom{subResultantGcd}{PseudoRemainderSequence},
++ \axiomOpFrom{chainSubResultants}{PseudoRemainderSequence},
++ \axiomOpFrom{degreeSubResultant}{PseudoRemainderSequence},
++ \axiomOpFrom{lastSubResultant}{PseudoRemainderSequence},
++ \axiomOpFrom{resultantEuclidean}{PseudoRemainderSequence},
++ \axiomOpFrom{subResultantGcdEuclidean}{PseudoRemainderSequence},
++ \axiomOpFrom{semiSubResultantGcdEuclidean1}{PseudoRemainderSequence},
++ \axiomOpFrom{semiSubResultantGcdEuclidean2}{PseudoRemainderSequence}, etc.
++ This procedures are coming from improvements
++ of the subresultants algorithm.
++   Version : 7
++   References : Lionel Ducos "Optimizations of the subresultant algorithm"
++   to appear in the Journal of Pure and Applied Algebra.
++   Author : Ducos Lionel \axiom{Lionel.Ducos@mathlabo.univ-poitiers.fr}

PseudoRemainderSequence(R, polR) : Specification == Implementation where
  R : IntegralDomain
  polR : UnivariatePolynomialCategory(R)
  NNI ==> NonNegativeInteger
  LC  ==> leadingCoefficient

  Specification == with
       resultant : (polR, polR) -> R
         ++ \axiom{resultant(P, Q)} returns the resultant
         ++ of \axiom{P} and \axiom{Q}

       resultantEuclidean : (polR, polR) ->
                            Record(coef1 : polR, coef2 : polR, resultant : R)
         ++ \axiom{resultantEuclidean(P,Q)} carries out the equality
         ++ \axiom{coef1*P + coef2*Q = resultant(P,Q)}

       semiResultantEuclidean2 : (polR, polR) ->
                            Record(coef2 : polR, resultant : R)
         ++ \axiom{semiResultantEuclidean2(P,Q)} carries out the equality
         ++ \axiom{...P + coef2*Q = resultant(P,Q)}.
         ++ Warning: \axiom{degree(P) >= degree(Q)}.
```

```
semiResultantEuclidean1 : (polR, polR) ->
                    Record(coef1 : polR, resultant : R)
  ++ \axiom{semiResultantEuclidean1(P,Q)} carries out the equality
  ++ \axiom{coef1.P + ? Q = resultant(P,Q)}.

indiceSubResultant : (polR, polR, NNI) -> polR
  ++ \axiom{indiceSubResultant(P, Q, i)} returns
  ++ the subresultant of indice \axiom{i}

indiceSubResultantEuclidean : (polR, polR, NNI) ->
                 Record(coef1 : polR, coef2 : polR, subResultant : polR)
  ++ \axiom{indiceSubResultant(P, Q, i)} returns
  ++ the subresultant \axiom{S_i(P,Q)} and carries out the equality
  ++ \axiom{coef1*P + coef2*Q = S_i(P,Q)}

semiIndiceSubResultantEuclidean : (polR, polR, NNI) ->
                    Record(coef2 : polR, subResultant : polR)
  ++ \axiom{semiIndiceSubResultantEuclidean(P, Q, i)} returns
  ++ the subresultant \axiom{S_i(P,Q)} and carries out the equality
  ++ \axiom{...P + coef2*Q = S_i(P,Q)}
  ++ Warning: \axiom{degree(P) >= degree(Q)}.

degreeSubResultant : (polR, polR, NNI) -> polR
  ++ \axiom{degreeSubResultant(P, Q, d)} computes
  ++ a subresultant of degree \axiom{d}.

degreeSubResultantEuclidean : (polR, polR, NNI) ->
                 Record(coef1 : polR, coef2 : polR, subResultant : polR)
  ++ \axiom{indiceSubResultant(P, Q, i)} returns
  ++ a subresultant \axiom{S} of degree \axiom{d}
  ++ and carries out the equality \axiom{coef1*P + coef2*Q = S_i}.

semiDegreeSubResultantEuclidean : (polR, polR, NNI) ->
                    Record(coef2 : polR, subResultant : polR)
  ++ \axiom{indiceSubResultant(P, Q, i)} returns
  ++ a subresultant \axiom{S} of degree \axiom{d}
  ++ and carries out the equality \axiom{...P + coef2*Q = S_i}.
  ++ Warning: \axiom{degree(P) >= degree(Q)}.

lastSubResultant : (polR, polR) -> polR
  ++ \axiom{lastSubResultant(P, Q)} computes
  ++ the last non zero subresultant of \axiom{P} and \axiom{Q}

lastSubResultantEuclidean : (polR, polR) ->
                 Record(coef1 : polR, coef2 : polR, subResultant : polR)
```

```
  ++ \axiom{lastSubResultantEuclidean(P, Q)} computes
  ++ the last non zero subresultant \axiom{S}
  ++ and carries out the equality \axiom{coef1*P + coef2*Q = S}.

semiLastSubResultantEuclidean : (polR, polR) ->
                Record(coef2 : polR, subResultant : polR)
  ++ \axiom{semiLastSubResultantEuclidean(P, Q)} computes
  ++ the last non zero subresultant \axiom{S}
  ++ and carries out the equality \axiom{...P + coef2*Q = S}.
  ++ Warning: \axiom{degree(P) >= degree(Q)}.

subResultantGcd : (polR, polR) -> polR
  ++ \axiom{subResultantGcd(P, Q)} returns the gcd
  ++ of two primitive polynomials \axiom{P} and \axiom{Q}.

subResultantGcdEuclidean : (polR, polR)
              -> Record(coef1 : polR, coef2 : polR, gcd : polR)
  ++ \axiom{subResultantGcdEuclidean(P,Q)} carries out the equality
  ++ \axiom{coef1*P + coef2*Q = +/- S_i(P,Q)}
  ++ where the degree (not the indice)
  ++ of the subresultant \axiom{S_i(P,Q)} is the smaller as possible.

semiSubResultantGcdEuclidean2 : (polR, polR)
                            -> Record(coef2 : polR, gcd : polR)
  ++ \axiom{semiSubResultantGcdEuclidean2(P,Q)} carries out the equality
  ++ \axiom{...P + coef2*Q = +/- S_i(P,Q)}
  ++ where the degree (not the indice)
  ++ of the subresultant \axiom{S_i(P,Q)} is the smaller as possible.
  ++ Warning: \axiom{degree(P) >= degree(Q)}.

semiSubResultantGcdEuclidean1: (polR, polR)->Record(coef1: polR, gcd: polR)
  ++ \axiom{semiSubResultantGcdEuclidean1(P,Q)} carries out the equality
  ++ \axiom{coef1*P + ? Q = +/- S_i(P,Q)}
  ++ where the degree (not the indice)
  ++ of the subresultant \axiom{S_i(P,Q)} is the smaller as possible.

discriminant : polR -> R
  ++ \axiom{discriminant(P, Q)} returns the discriminant
  ++ of \axiom{P} and \axiom{Q}.

discriminantEuclidean : polR ->
                    Record(coef1 : polR, coef2 : polR, discriminant : R)
  ++ \axiom{discriminantEuclidean(P)} carries out the equality
  ++ \axiom{coef1 * P + coef2 * D(P) = discriminant(P)}.

semiDiscriminantEuclidean : polR ->
```

```
                    Record(coef2 : polR, discriminant : R)
  ++ \axiom{discriminantEuclidean(P)} carries out the equality
  ++ \axiom{...P + coef2 * D(P) = discriminant(P)}.
  ++ Warning: \axiom{degree(P) >= degree(Q)}.

chainSubResultants : (polR, polR) -> List(polR)
  ++ \axiom{chainSubResultants(P, Q)} computes the list
  ++ of non zero subresultants of \axiom{P} and \axiom{Q}.

schema : (polR, polR) -> List(NNI)
  ++ \axiom{schema(P,Q)} returns the list of degrees of
  ++ non zero subresultants of \axiom{P} and \axiom{Q}.

if R has GcdDomain then
  resultantReduit : (polR, polR) -> R
  ++ \axiom{resultantReduit(P,Q)} returns the "reduce resultant"
  ++ of \axiom{P} and \axiom{Q}.

  resultantReduitEuclidean : (polR, polR) ->
                Record(coef1 : polR, coef2 : polR, resultantReduit : R)
  ++ \axiom{resultantReduitEuclidean(P,Q)} returns
  ++ the "reduce resultant" and carries out the equality
  ++ \axiom{coef1*P + coef2*Q = resultantReduit(P,Q)}.

  semiResultantReduitEuclidean : (polR, polR) ->
                Record(coef2 : polR, resultantReduit : R)
  ++ \axiom{semiResultantReduitEuclidean(P,Q)} returns
  ++ the "reduce resultant" and carries out the equality
  ++ \axiom{...P + coef2*Q = resultantReduit(P,Q)}.

  gcd : (polR, polR) -> polR
  ++ \axiom{gcd(P, Q)} returns the gcd of \axiom{P} and \axiom{Q}.

-- sub-routines exported for convenience ----------------------------

"*" : (R, Vector(polR)) -> Vector(polR)
  ++ \axiom{r * v} computes the product of \axiom{r} and \axiom{v}

"exquo" : (Vector(polR), R) -> Vector(polR)
  ++ \axiom{v exquo r} computes
  ++ the exact quotient of \axiom{v} by \axiom{r}

pseudoDivide : (polR, polR) ->
                          Record(coef:R, quotient:polR, remainder:polR)
  ++ \axiom{pseudoDivide(P,Q)} computes the pseudoDivide
  ++ of \axiom{P} by \axiom{Q}.
```

```
        divide : (polR, polR) -> Record(quotient : polR, remainder : polR)
          ++ \axiom{divide(F,G)} computes quotient and rest
          ++ of the exact euclidean division of \axiom{F} by \axiom{G}.

        Lazard : (R, R, NNI) -> R
          ++ \axiom{Lazard(x, y, n)} computes \axiom{x**n/y**(n-1)}

        Lazard2 : (polR, R, R, NNI) -> polR
          ++ \axiom{Lazard2(F, x, y, n)} computes  \axiom{(x/y)**(n-1) * F}

        next_sousResultant2 : (polR, polR, polR, R) -> polR
          ++ \axiom{nextsousResultant2(P, Q, Z, s)} returns
          ++ the subresultant \axiom{S_{e-1}} where
          ++ \axiom{P ~ S_d,  Q = S_{d-1},  Z = S_e,  s = lc(S_d)}

        resultant_naif : (polR, polR) -> R
          ++ \axiom{resultantEuclidean_naif(P,Q)} returns
          ++ the resultant of \axiom{P} and \axiom{Q} computed
          ++ by means of the naive algorithm.

        resultantEuclidean_naif : (polR, polR) ->
                            Record(coef1 : polR, coef2 : polR, resultant : R)
          ++ \axiom{resultantEuclidean_naif(P,Q)} returns
          ++ the extended resultant of \axiom{P} and \axiom{Q} computed
          ++ by means of the naive algorithm.

        semiResultantEuclidean_naif : (polR, polR) ->
                            Record(coef2 : polR, resultant : R)
          ++ \axiom{resultantEuclidean_naif(P,Q)} returns
          ++ the semi-extended resultant of \axiom{P} and \axiom{Q} computed
          ++ by means of the naive algorithm.

      Implementation == add
        X : polR := monomial(1$R,1)

        r : R * v : Vector(polR) == r::polR * v
                  -- the instruction  map(r * #1, v) is slower !?

        v : Vector(polR) exquo r : R == map((#1 exquo r)::polR, v)

        pseudoDivide(P : polR, Q : polR) :
                                      Record(coef:R,quotient:polR,remainder:polR) ==
        -- computes the pseudoDivide of P by Q
           zero?(Q) => error("PseudoDivide$PRS : division by 0")
           zero?(P) => construct(1, 0, P)
```

```
    lcQ : R := LC(Q)
    (degP, degQ) := (degree(P), degree(Q))
    degP < degQ => construct(1, 0, P)
    Q := reductum(Q)
    i : NNI := (degP - degQ + 1)::NNI
    co : R := lcQ**i
    quot : polR := 0$polR
    while (delta : Integer := degree(P) - degQ) >= 0 repeat
      i := (i - 1)::NNI
      mon := monomial(LC(P), delta::NNI)$polR
      quot := quot + lcQ**i * mon
      P := lcQ * reductum(P) - mon * Q
    P := lcQ**i * P
    return construct(co, quot, P)

  divide(F : polR, G : polR) : Record(quotient : polR, remainder : polR)==
  -- computes quotient and rest of the exact euclidean division of F by G
      lcG : R := LC(G)
      degG : NNI := degree(G)
      zero?(degG) => ( F := (F exquo lcG)::polR; return construct(F, 0))
      G : polR := reductum(G)
      quot : polR := 0
      while (delta := degree(F) - degG) >= 0 repeat
         mon : polR := monomial((LC(F) exquo lcG)::R, delta::NNI)
         quot := quot + mon
         F := reductum(F) - mon * G
      return construct(quot, F)

  resultant_naif(P : polR, Q : polR) : R ==
  -- valid over a field
    a : R := 1
    repeat
       zero?(Q) => return 0
       (degP, degQ) := (degree(P), degree(Q))
       if odd?(degP) and odd?(degQ) then a := - a
       zero?(degQ) => return (a * LC(Q)**degP)
       U : polR := divide(P, Q).remainder
       a := a * LC(Q)**(degP - degree(U))::NNI
       (P, Q) := (Q, U)

  resultantEuclidean_naif(P : polR, Q : polR) :
                     Record(coef1 : polR, coef2 : polR, resultant : R) ==
  -- valid over a field.
    a : R := 1
    old_cf1 : polR := 1 ; cf1 : polR := 0
    old_cf2 : polR := 0 ; cf2 : polR := 1
```

```
        repeat
           zero?(Q) => construct(0::polR, 0::polR, 0::R)
           (degP, degQ) := (degree(P), degree(Q))
           if odd?(degP) and odd?(degQ) then a := -a
           if zero?(degQ) then
              a := a * LC(Q)**(degP-1)::NNI
              return construct(a*cf1, a*cf2, a*LC(Q))
           divid := divide(P,Q)
           a := a * LC(Q)**(degP - degree(divid.remainder))::NNI
           (P, Q) := (Q, divid.remainder)
           (old_cf1, old_cf2, cf1, cf2) := (cf1, cf2,
                 old_cf1 - divid.quotient * cf1, old_cf2 - divid.quotient * cf2)

     semiResultantEuclidean_naif(P : polR, Q : polR) :
                       Record(coef2 : polR, resultant : R) ==
     -- valid over a field
        a : R := 1
        old_cf2 : polR := 0 ; cf2 : polR := 1
        repeat
           zero?(Q) => construct(0::polR, 0::R)
           (degP, degQ) := (degree(P), degree(Q))
           if odd?(degP) and odd?(degQ) then a := -a
           if zero?(degQ) then
              a := a * LC(Q)**(degP-1)::NNI
              return construct(a*cf2, a*LC(Q))
           divid := divide(P,Q)
           a := a * LC(Q)**(degP - degree(divid.remainder))::NNI
           (P, Q) := (Q, divid.remainder)
           (old_cf2, cf2) := (cf2, old_cf2 - divid.quotient * cf2)

     Lazard(x : R, y : R, n : NNI) : R ==
        zero?(n) => error("Lazard$PRS : n = 0")
--        one?(n) => x
        (n = 1) => x
        a : NNI := 1
        while n >= (b := 2*a) repeat a := b
        c : R := x
        n := (n - a)::NNI
        repeat                    --  c = x**i / y**(i-1),  i=n_0 quo a,  a=2**?
--           one?(a) => return c
           (a = 1) => return c
           a := a quo 2
           c := ((c * c) exquo y)::R
           if n >= a then ( c := ((c * x) exquo y)::R ; n := (n - a)::NNI )

     Lazard2(F : polR, x : R, y : R, n : NNI) : polR ==
```

```
        zero?(n) => error("Lazard2$PRS : n = 0")
--        one?(n) => F
        (n = 1) => F
        x := Lazard(x, y, (n-1)::NNI)
        return ((x * F) exquo y)::polR

    Lazard3(V : Vector(polR), x : R, y : R, n : NNI) : Vector(polR) ==
    -- computes x**(n-1) * V / y**(n-1)
        zero?(n) => error("Lazard2$prs : n = 0")
--        one?(n) => V
        (n = 1) => V
        x := Lazard(x, y, (n-1)::NNI)
        return ((x * V) exquo y)

    next_sousResultant2(P : polR, Q : polR, Z : polR, s : R) : polR ==
        (lcP, c, se) := (LC(P), LC(Q), LC(Z))
        (d, e) := (degree(P), degree(Q))
        (P, Q, H) := (reductum(P), reductum(Q), - reductum(Z))
        A : polR := coefficient(P, e) * H
        for i in e+1..d-1 repeat
           H := if degree(H) = e-1 then
                   X * reductum(H) - ((LC(H) * Q) exquo c)::polR
                else
                   X * H
           -- H = s_e * X^i mod S_d-1
           A := coefficient(P, i) * H + A
        while degree(P) >= e repeat P := reductum(P)
        A := A + se * P            --  A = s_e * reductum(P_0)       mod S_d-1
        A := (A exquo lcP)::polR   --  A = s_e * reductum(S_d) / s_d mod S_d-1
        A := if degree(H) = e-1 then
                c * (X * reductum(H) + A) - LC(H) * Q
             else
                c * (X * H + A)
        A := (A exquo s)::polR                    -- A = +/- S_e-1
        return (if odd?(d-e) then A else - A)

    next_sousResultant3(VP : Vector(polR), VQ : Vector(polR), s : R, ss : R) :
                                                        Vector(polR) ==
    -- P ~ S_d,  Q = S_d-1,  s = lc(S_d),  ss = lc(S_e)
        (P, Q) := (VP.1, VQ.1)
        (lcP, c) := (LC(P), LC(Q))
        e : NNI := degree(Q)
--        if one?(delta := degree(P) - e) then                  -- algo_new
        if ((delta := degree(P) - e) = 1) then                  -- algo_new
          VP := c * VP - coefficient(P, e) * VQ
          VP := VP exquo lcP
```

```
      VP := c * (VP - X * VQ) + coefficient(Q, (e-1)::NNI) * VQ
      VP := VP exquo s
    else                                    -- algorithm of Lickteig - Roy
      (r, rr) := (s * lcP, ss * c)
      divid := divide(rr * P, Q)
      VP.1 := (divid.remainder exquo r)::polR
      for i in 2..#VP repeat
        VP.i := rr * VP.i - VQ.i * divid.quotient
        VP.i := (VP.i exquo r)::polR
    return (if odd?(delta) then VP else - VP)

  algo_new(P : polR, Q : polR) : R ==
    delta : NNI := (degree(P) - degree(Q))::NNI
    s : R := LC(Q)**delta
    (P, Q) := (Q, pseudoRemainder(P, -Q))
    repeat
      -- P = S_c-1 (except the first turn : P ~ S_c-1),
      -- Q = S_d-1,  s = lc(S_d)
      zero?(Q) => return 0
      delta := (degree(P) - degree(Q))::NNI
      Z : polR := Lazard2(Q, LC(Q), s, delta)
      -- Z = S_e ~ S_d-1
      zero?(degree(Z)) => return LC(Z)
      (P, Q) := (Q, next_sousResultant2(P, Q, Z, s))
      s := LC(Z)

  resultant(P : polR, Q : polR) : R ==
    zero?(Q) or zero?(P) => 0
    if degree(P) < degree(Q) then
      (P, Q) := (Q, P)
      if odd?(degree(P)) and odd?(degree(Q)) then Q := - Q
    zero?(degree(Q)) => LC(Q)**degree(P)
    -- degree(P) >= degree(Q) > 0
    R has Finite => resultant_naif(P, Q)
    return algo_new(P, Q)

  subResultantEuclidean(P : polR, Q : polR) :
                        Record(coef1 : polR, coef2 : polR, resultant : R) ==
    s : R := LC(Q)**(degree(P) - degree(Q))::NNI
    VP : Vector(polR) := [Q, 0::polR, 1::polR]
    pdiv := pseudoDivide(P, -Q)
    VQ : Vector(polR) := [pdiv.remainder, pdiv.coef::polR, pdiv.quotient]
    repeat
      --  VP.1 = S_{c-1},  VQ.1 = S_{d-1},  s=lc(S_d)
      --  S_{c-1} = VP.2 P_0 + VP.3 Q_0,  S_{d-1} = VQ.2 P_0 + VQ.3 Q_0
      (P, Q) := (VP.1, VQ.1)
```

```
      zero?(Q) => return construct(0::polR, 0::polR, 0::R)
      e : NNI := degree(Q)
      delta : NNI := (degree(P) - e)::NNI
      if zero?(e) then
         l : Vector(polR) := Lazard3(VQ, LC(Q), s, delta)
         return construct(l.2, l.3, LC(l.1))
      ss : R := Lazard(LC(Q), s, delta)
      (VP, VQ) := (VQ, next_sousResultant3(VP, VQ, s, ss))
      s := ss

resultantEuclidean(P : polR, Q : polR) :
                  Record(coef1 : polR, coef2 : polR, resultant : R) ==
   zero?(P) or zero?(Q) => construct(0::polR, 0::polR, 0::R)
   if degree(P) < degree(Q) then
      e : Integer := if odd?(degree(P)) and odd?(degree(Q)) then -1 else 1
      l := resultantEuclidean(Q, e * P)
      return construct(e * l.coef2, l.coef1, l.resultant)
   if zero?(degree(Q)) then
      degP : NNI := degree(P)
      zero?(degP) => error("resultantEuclidean$PRS : constant polynomials")
      s : R := LC(Q)**(degP-1)::NNI
      return construct(0::polR, s::polR, s * LC(Q))
   R has Finite => resultantEuclidean_naif(P, Q)
   return subResultantEuclidean(P,Q)

semiSubResultantEuclidean(P : polR, Q : polR) :
                  Record(coef2 : polR, resultant : R) ==
   s : R := LC(Q)**(degree(P) - degree(Q))::NNI
   VP : Vector(polR) := [Q, 1::polR]
   pdiv := pseudoDivide(P, -Q)
   VQ : Vector(polR) := [pdiv.remainder, pdiv.quotient]
   repeat
      --  VP.1 = S_{c-1},  VQ.1 = S_{d-1},  s=lc(S_d)
      --  S_{c-1} = ...P_0 + VP.3 Q_0,  S_{d-1} = ...P_0 + VQ.3 Q_0
      (P, Q) := (VP.1, VQ.1)
      zero?(Q) => return construct(0::polR, 0::R)
      e : NNI := degree(Q)
      delta : NNI := (degree(P) - e)::NNI
      if zero?(e) then
         l : Vector(polR) := Lazard3(VQ, LC(Q), s, delta)
         return construct(l.2, LC(l.1))
      ss : R := Lazard(LC(Q), s, delta)
      (VP, VQ) := (VQ, next_sousResultant3(VP, VQ, s, ss))
      s := ss

semiResultantEuclidean2(P : polR, Q : polR) :
```

```
                    Record(coef2 : polR, resultant : R) ==
    zero?(P) or zero?(Q) => construct(0::polR, 0::R)
    degree(P) < degree(Q) => error("semiResultantEuclidean2 : bad degrees")
    if zero?(degree(Q)) then
       degP : NNI := degree(P)
       zero?(degP) => error("semiResultantEuclidean2 : constant polynomials")
       s : R := LC(Q)**(degP-1)::NNI
       return construct(s::polR, s * LC(Q))
    R has Finite => semiResultantEuclidean_naif(P, Q)
    return semiSubResultantEuclidean(P,Q)

semiResultantEuclidean1(P : polR, Q : polR) :
                    Record(coef1 : polR, resultant : R) ==
    result := resultantEuclidean(P,Q)
    [result.coef1, result.resultant]

indiceSubResultant(P : polR, Q : polR, i : NNI) : polR ==
    zero?(Q) or zero?(P) => 0
    if degree(P) < degree(Q) then
       (P, Q) := (Q, P)
       if odd?(degree(P)-i) and odd?(degree(Q)-i) then Q := - Q
    if i = degree(Q) then
       delta : NNI := (degree(P)-degree(Q))::NNI
       zero?(delta) => error("indiceSubResultant$PRS : bad degrees")
       s : R := LC(Q)**(delta-1)::NNI
       return s*Q
    i > degree(Q) => 0
    s : R := LC(Q)**(degree(P) - degree(Q))::NNI
    (P, Q) := (Q, pseudoRemainder(P, -Q))
    repeat
       -- P = S_{c-1} ~ S_d , Q = S_{d-1},  s = lc(S_d),  i < d
       (degP, degQ) := (degree(P), degree(Q))
       i = degP-1 => return Q
       zero?(Q) or (i > degQ) => return 0
       Z : polR := Lazard2(Q, LC(Q), s, (degP - degQ)::NNI)
       --  Z = S_e ~ S_d-1
       i = degQ => return Z
       (P, Q) := (Q, next_sousResultant2(P, Q, Z, s))
       s := LC(Z)

indiceSubResultantEuclidean(P : polR, Q : polR, i : NNI) :
                Record(coef1 : polR, coef2 : polR, subResultant : polR) ==
    zero?(Q) or zero?(P) => construct(0::polR, 0::polR, 0::polR)
    if degree(P) < degree(Q) then
       e := if odd?(degree(P)-i) and odd?(degree(Q)-i) then -1 else 1
       l := indiceSubResultantEuclidean(Q, e * P, i)
```

```
            return construct(e * l.coef2, l.coef1, l.subResultant)
         if i = degree(Q) then
            delta : NNI := (degree(P)-degree(Q))::NNI
            zero?(delta) =>
                        error("indiceSubResultantEuclidean$PRS : bad degrees")
            s : R := LC(Q)**(delta-1)::NNI
            return construct(0::polR, s::polR, s * Q)
         i > degree(Q) => construct(0::polR, 0::polR, 0::polR)
         s : R := LC(Q)**(degree(P) - degree(Q))::NNI
         VP : Vector(polR) := [Q, 0::polR, 1::polR]
         pdiv := pseudoDivide(P, -Q)
         VQ : Vector(polR) := [pdiv.remainder, pdiv.coef::polR, pdiv.quotient]
         repeat
            --  VP.1 = S_{c-1},  VQ.1 = S_{d-1},  s=lc(S_d),  i < d
            --  S_{c-1} = VP.2 P_0 + VP.3 Q_0,  S_{d-1} = VQ.2 P_0 + VQ.3 Q_0
            (P, Q) := (VP.1, VQ.1)
            zero?(Q) => return construct(0::polR, 0::polR, 0::polR)
            (degP, degQ) := (degree(P), degree(Q))
            i = degP-1 => return construct(VQ.2, VQ.3, VQ.1)
            (i > degQ) => return construct(0::polR, 0::polR, 0::polR)
            VZ := Lazard3(VQ, LC(Q), s, (degP - degQ)::NNI)
            i = degQ => return construct(VZ.2, VZ.3, VZ.1)
            ss : R := LC(VZ.1)
            (VP, VQ) := (VQ, next_sousResultant3(VP, VQ, s, ss))
            s := ss

      semiIndiceSubResultantEuclidean(P : polR, Q : polR, i : NNI) :
                      Record(coef2 : polR, subResultant : polR) ==
         zero?(Q) or zero?(P) => construct(0::polR, 0::polR)
         degree(P) < degree(Q) =>
                     error("semiIndiceSubResultantEuclidean$PRS : bad degrees")
         if i = degree(Q) then
            delta : NNI := (degree(P)-degree(Q))::NNI
            zero?(delta) =>
                     error("semiIndiceSubResultantEuclidean$PRS : bad degrees")
            s : R := LC(Q)**(delta-1)::NNI
            return construct(s::polR, s * Q)
         i > degree(Q) => construct(0::polR, 0::polR)
         s : R := LC(Q)**(degree(P) - degree(Q))::NNI
         VP : Vector(polR) := [Q, 1::polR]
         pdiv := pseudoDivide(P, -Q)
         VQ : Vector(polR) := [pdiv.remainder, pdiv.quotient]
         repeat
            --  VP.1 = S_{c-1},  VQ.1 = S_{d-1},  s = lc(S_d),  i < d
            --  S_{c-1} = ...P_0 + VP.2 Q_0,  S_{d-1} = ...P_0 + ...Q_0
            (P, Q) := (VP.1, VQ.1)
```

```
      zero?(Q) => return construct(0::polR, 0::polR)
      (degP, degQ) := (degree(P), degree(Q))
      i = degP-1 => return construct(VQ.2, VQ.1)
      (i > degQ) => return construct(0::polR, 0::polR)
      VZ := Lazard3(VQ, LC(Q), s, (degP - degQ)::NNI)
      i = degQ => return construct(VZ.2, VZ.1)
      ss : R := LC(VZ.1)
      (VP, VQ) := (VQ, next_sousResultant3(VP, VQ, s, ss))
      s := ss

  degreeSubResultant(P : polR, Q : polR, i : NNI) : polR ==
    zero?(Q) or zero?(P) => 0
    if degree(P) < degree(Q) then (P, Q) := (Q, P)
    if i = degree(Q) then
      delta : NNI := (degree(P)-degree(Q))::NNI
      zero?(delta) => error("degreeSubResultant$PRS : bad degrees")
      s : R := LC(Q)**(delta-1)::NNI
      return s*Q
    i > degree(Q) => 0
    s : R := LC(Q)**(degree(P) - degree(Q))::NNI
    (P, Q) := (Q, pseudoRemainder(P, -Q))
    repeat
      -- P = S_{c-1},  Q = S_{d-1},  s = lc(S_d)
      zero?(Q) or (i > degree(Q)) => return 0
      i = degree(Q) => return Q
      Z : polR := Lazard2(Q, LC(Q), s, (degree(P) - degree(Q))::NNI)
      --  Z = S_e ~ S_d-1
      (P, Q) := (Q, next_sousResultant2(P, Q, Z, s))
      s := LC(Z)

  degreeSubResultantEuclidean(P : polR, Q : polR, i : NNI) :
                  Record(coef1 : polR, coef2 : polR, subResultant : polR) ==
    zero?(Q) or zero?(P) => construct(0::polR, 0::polR, 0::polR)
    if degree(P) < degree(Q) then
      l := degreeSubResultantEuclidean(Q, P, i)
      return construct(l.coef2, l.coef1, l.subResultant)
    if i = degree(Q) then
      delta : NNI := (degree(P)-degree(Q))::NNI
      zero?(delta) =>
                  error("degreeSubResultantEuclidean$PRS : bad degrees")
      s : R := LC(Q)**(delta-1)::NNI
      return construct(0::polR, s::polR, s * Q)
    i > degree(Q) => construct(0::polR, 0::polR, 0::polR)
    s : R := LC(Q)**(degree(P) - degree(Q))::NNI
    VP : Vector(polR) := [Q, 0::polR, 1::polR]
    pdiv := pseudoDivide(P, -Q)
```

```
      VQ : Vector(polR) := [pdiv.remainder, pdiv.coef::polR, pdiv.quotient]
      repeat
         --  VP.1 = S_{c-1},  VQ.1 = S_{d-1},  s=lc(S_d)
         --  S_{c-1} = ...P_0 + VP.3 Q_0,  S_{d-1} = ...P_0 + VQ.3 Q_0
         (P, Q) := (VP.1, VQ.1)
         zero?(Q) or (i > degree(Q)) =>
              return construct(0::polR, 0::polR, 0::polR)
         i = degree(Q) => return construct(VQ.2, VQ.3, VQ.1)
         ss : R := Lazard(LC(Q), s, (degree(P)-degree(Q))::NNI)
         (VP, VQ) := (VQ, next_sousResultant3(VP, VQ, s, ss))
         s := ss

   semiDegreeSubResultantEuclidean(P : polR, Q : polR, i : NNI) :
                    Record(coef2 : polR, subResultant : polR) ==
      zero?(Q) or zero?(P) => construct(0::polR, 0::polR)
      degree(P) < degree(Q) =>
                 error("semiDegreeSubResultantEuclidean$PRS : bad degrees")
      if i = degree(Q) then
         delta : NNI := (degree(P)-degree(Q))::NNI
         zero?(delta) =>
                 error("semiDegreeSubResultantEuclidean$PRS : bad degrees")
         s : R := LC(Q)**(delta-1)::NNI
         return construct(s::polR, s * Q)
      i > degree(Q) => construct(0::polR, 0::polR)
      s : R := LC(Q)**(degree(P) - degree(Q))::NNI
      VP : Vector(polR) := [Q, 1::polR]
      pdiv := pseudoDivide(P, -Q)
      VQ : Vector(polR) := [pdiv.remainder, pdiv.quotient]
      repeat
         --  VP.1 = S_{c-1},  VQ.1 = S_{d-1},  s=lc(S_d)
         --  S_{c-1} = ...P_0 + VP.3 Q_0,  S_{d-1} = ...P_0 + VQ.3 Q_0
         (P, Q) := (VP.1, VQ.1)
         zero?(Q) or (i > degree(Q)) =>
              return construct(0::polR, 0::polR)
         i = degree(Q) => return construct(VQ.2, VQ.1)
         ss : R := Lazard(LC(Q), s, (degree(P)-degree(Q))::NNI)
         (VP, VQ) := (VQ, next_sousResultant3(VP, VQ, s, ss))
         s := ss

   lastSubResultant(P : polR, Q : polR) : polR ==
      zero?(Q) or zero?(P) => 0
      if degree(P) < degree(Q) then (P, Q) := (Q, P)
      zero?(degree(Q)) => (LC(Q)**degree(P))::polR
      s : R := LC(Q)**(degree(P) - degree(Q))::NNI
      (P, Q) := (Q, pseudoRemainder(P, -Q))
      Z : polR := P
```

```
      repeat
         -- Z = S_d  (except the first turn : Z = P)
         -- P = S_{c-1} ~ S_d,  Q = S_{d-1},  s = lc(S_d)
         zero?(Q) => return Z
         Z := Lazard2(Q, LC(Q), s, (degree(P) - degree(Q))::NNI)
         -- Z = S_e ~ S_{d-1}
         zero?(degree(Z)) => return Z
         (P, Q) := (Q, next_sousResultant2(P, Q, Z, s))
         s := LC(Z)

   lastSubResultantEuclidean(P : polR, Q : polR) :
                  Record(coef1 : polR, coef2 : polR, subResultant : polR) ==
      zero?(Q) or zero?(P) => construct(0::polR, 0::polR, 0::polR)
      if degree(P) < degree(Q) then
         l := lastSubResultantEuclidean(Q, P)
         return construct(l.coef2, l.coef1, l.subResultant)
      if zero?(degree(Q)) then
         degP : NNI := degree(P)
         zero?(degP) =>
             error("lastSubResultantEuclidean$PRS : constant polynomials")
         s : R := LC(Q)**(degP-1)::NNI
         return construct(0::polR, s::polR, s * Q)
      s : R := LC(Q)**(degree(P) - degree(Q))::NNI
      VP : Vector(polR) := [Q, 0::polR, 1::polR]
      pdiv := pseudoDivide(P, -Q)
      VQ : Vector(polR) := [pdiv.remainder, pdiv.coef::polR, pdiv.quotient]
      VZ : Vector(polR) := copy(VP)
      repeat
         --  VZ.1 = S_d,  VP.1 = S_{c-1},  VQ.1 = S_{d-1},  s = lc(S_d)
         --  S_{c-1} = VP.2 P_0 + VP.3 Q_0
         --  S_{d-1} = VQ.2 P_0 + VQ.3 Q_0
         --  S_d     = VZ.2 P_0 + VZ.3 Q_0
         (Q, Z) := (VQ.1, VZ.1)
         zero?(Q) => return construct(VZ.2, VZ.3, VZ.1)
         VZ := Lazard3(VQ, LC(Q), s, (degree(Z) - degree(Q))::NNI)
         zero?(degree(Q)) => return construct(VZ.2, VZ.3, VZ.1)
         ss : R := LC(VZ.1)
         (VP, VQ) := (VQ, next_sousResultant3(VP, VQ, s, ss))
         s := ss

   semiLastSubResultantEuclidean(P : polR, Q : polR) :
                  Record(coef2 : polR, subResultant : polR) ==
      zero?(Q) or zero?(P) => construct(0::polR, 0::polR)
      degree(P) < degree(Q) =>
             error("semiLastSubResultantEuclidean$PRS : bad degrees")
      if zero?(degree(Q)) then
```

```
            degP : NNI := degree(P)
            zero?(degP) =>
                error("semiLastSubResultantEuclidean$PRS : constant polynomials")
            s : R := LC(Q)**(degP-1)::NNI
            return construct(s::polR, s * Q)
         s : R := LC(Q)**(degree(P) - degree(Q))::NNI
         VP : Vector(polR) := [Q, 1::polR]
         pdiv := pseudoDivide(P, -Q)
         VQ : Vector(polR) := [pdiv.remainder, pdiv.quotient]
         VZ : Vector(polR) := copy(VP)
         repeat
            --  VZ.1 = S_d,  VP.1 = S_{c-1},  VQ.1 = S_{d-1},  s = lc(S_d)
            --  S_{c-1} = ... P_0 + VP.2 Q_0
            --  S_{d-1} = ... P_0 + VQ.2 Q_0
            --  S_d     = ... P_0 + VZ.2 Q_0
            (Q, Z) := (VQ.1, VZ.1)
            zero?(Q) => return construct(VZ.2, VZ.1)
            VZ := Lazard3(VQ, LC(Q), s, (degree(Z) - degree(Q))::NNI)
            zero?(degree(Q)) => return construct(VZ.2, VZ.1)
            ss : R := LC(VZ.1)
            (VP, VQ) := (VQ, next_sousResultant3(VP, VQ, s, ss))
            s := ss

      chainSubResultants(P : polR, Q : polR) : List(polR) ==
         zero?(Q) or zero?(P) => []
         if degree(P) < degree(Q) then
            (P, Q) := (Q, P)
            if odd?(degree(P)) and odd?(degree(Q)) then Q := - Q
         L : List(polR) := []
         zero?(degree(Q)) => L
         L := [Q]
         s : R := LC(Q)**(degree(P) - degree(Q))::NNI
         (P, Q) := (Q, pseudoRemainder(P, -Q))
         repeat
            -- P = S_{c-1},  Q = S_{d-1},  s = lc(S_d)
            -- L = [S_d,....,S_{q-1}]
            zero?(Q) => return L
            L := concat(Q, L)
            -- L = [S_{d-1},....,S_{q-1}]
            delta : NNI := (degree(P) - degree(Q))::NNI
            Z : polR := Lazard2(Q, LC(Q), s, delta)            -- Z = S_e ~ S_d-1
            if delta > 1 then L := concat(Z, L)
            -- L = [S_e,....,S_{q-1}]
            zero?(degree(Z)) => return L
            (P, Q) := (Q, next_sousResultant2(P, Q, Z, s))
            s := LC(Z)
```

```
schema(P : polR, Q : polR) : List(NNI) ==
   zero?(Q) or zero?(P) => []
   if degree(P) < degree(Q) then (P, Q) := (Q, P)
   zero?(degree(Q)) => [0]
   L : List(NNI) := []
   s : R := LC(Q)**(degree(P) - degree(Q))::NNI
   (P, Q) := (Q, pseudoRemainder(P, Q))
   repeat
      -- P = S_{c-1} ~ S_d,  Q = S_{d-1},  s = lc(S_d)
      zero?(Q) => return L
      e : NNI := degree(Q)
      L := concat(e, L)
      delta : NNI := (degree(P) - e)::NNI
      Z : polR := Lazard2(Q, LC(Q), s, delta)            -- Z = S_e ~ S_d-1
      if delta > 1 then L := concat(e, L)
      zero?(e) => return L
      (P, Q) := (Q, next_sousResultant2(P, Q, Z, s))
      s := LC(Z)

subResultantGcd(P : polR, Q : polR) : polR ==
   zero?(P) and zero?(Q) => 0
   zero?(P) => Q
   zero?(Q) => P
   if degree(P) < degree(Q) then (P, Q) := (Q, P)
   zero?(degree(Q)) => 1$polR
   s : R := LC(Q)**(degree(P) - degree(Q))::NNI
   (P, Q) := (Q, pseudoRemainder(P, -Q))
   repeat
      -- P = S_{c-1},  Q = S_{d-1},  s = lc(S_d)
      zero?(Q) => return P
      zero?(degree(Q)) => return 1$polR
      Z : polR := Lazard2(Q, LC(Q), s, (degree(P) - degree(Q))::NNI)
      -- Z = S_e ~ S_d-1
      (P, Q) := (Q, next_sousResultant2(P, Q, Z, s))
      s := LC(Z)

subResultantGcdEuclidean(P : polR, Q : polR) :
               Record(coef1 : polR, coef2 : polR, gcd : polR) ==
   zero?(P) and zero?(Q) => construct(0::polR, 0::polR, 0::polR)
   zero?(P) => construct(0::polR, 1::polR, Q)
   zero?(Q) => construct(1::polR, 0::polR, P)
   if degree(P) < degree(Q) then
      l := subResultantGcdEuclidean(Q, P)
      return construct(l.coef2, l.coef1, l.gcd)
   zero?(degree(Q)) => construct(0::polR, 1::polR, Q)
```

```
      s : R := LC(Q)**(degree(P) - degree(Q))::NNI
      VP : Vector(polR) := [Q, 0::polR, 1::polR]
      pdiv := pseudoDivide(P, -Q)
      VQ : Vector(polR) := [pdiv.remainder, pdiv.coef::polR, pdiv.quotient]
      repeat
         --  VP.1 = S_{c-1},  VQ.1 = S_{d-1},  s=lc(S_d)
         --  S_{c-1} = VP.2 P_0 + VP.3 Q_0,  S_{d-1} = VQ.2 P_0 + VQ.3 Q_0
         (P, Q) := (VP.1, VQ.1)
         zero?(Q) => return construct(VP.2, VP.3, P)
         e : NNI := degree(Q)
         zero?(e) => return construct(VQ.2, VQ.3, Q)
         ss := Lazard(LC(Q), s, (degree(P) - e)::NNI)
         (VP,VQ) := (VQ, next_sousResultant3(VP, VQ, s, ss))
         s := ss

   semiSubResultantGcdEuclidean2(P : polR, Q : polR) :
                                 Record(coef2 : polR, gcd : polR) ==
      zero?(P) and zero?(Q) => construct(0::polR, 0::polR)
      zero?(P) => construct(1::polR, Q)
      zero?(Q) => construct(0::polR, P)
      degree(P) < degree(Q) =>
                     error("semiSubResultantGcdEuclidean2$PRS : bad degrees")
      zero?(degree(Q)) => construct(1::polR, Q)
      s : R := LC(Q)**(degree(P) - degree(Q))::NNI
      VP : Vector(polR) := [Q, 1::polR]
      pdiv := pseudoDivide(P, -Q)
      VQ : Vector(polR) := [pdiv.remainder, pdiv.quotient]
      repeat
         --  P=S_{c-1},  Q=S_{d-1},  s=lc(S_d)
         --  S_{c-1} = ? P_0 + old_cf2 Q_0,  S_{d-1} = ? P_0 + cf2 Q_0
         (P, Q) := (VP.1, VQ.1)
         zero?(Q) => return construct(VP.2, P)
         e : NNI := degree(Q)
         zero?(e) => return construct(VQ.2, Q)
         ss := Lazard(LC(Q), s, (degree(P) - e)::NNI)
         (VP,VQ) := (VQ, next_sousResultant3(VP, VQ, s, ss))
         s := ss

   semiSubResultantGcdEuclidean1(P : polR, Q : polR) :
                     Record(coef1 : polR, gcd : polR) ==
      result := subResultantGcdEuclidean(P,Q)
      [result.coef1, result.gcd]

   discriminant(P : polR) : R ==
      d : Integer := degree(P)
      zero?(d) => error "cannot take discriminant of constants"
```

```
      a : Integer := (d * (d-1)) quo 2
      a := (-1)**a::NonNegativeInteger
      dP : polR := differentiate P
      r : R := resultant(P, dP)
      d := d - degree(dP) - 1
      return (if zero?(d) then a * (r exquo LC(P))::R
              else a * r * LC(P)**(d-1)::NNI)

   discriminantEuclidean(P : polR) :
                       Record(coef1 : polR, coef2 : polR, discriminant : R) ==
      d : Integer := degree(P)
      zero?(d) => error "cannot take discriminant of constants"
      a : Integer := (d * (d-1)) quo 2
      a := (-1)**a::NonNegativeInteger
      dP : polR := differentiate P
      rE := resultantEuclidean(P, dP)
      d := d - degree(dP) - 1
      if zero?(d) then
         c1 : polR := a * (rE.coef1 exquo LC(P))::polR
         c2 : polR := a * (rE.coef2 exquo LC(P))::polR
         cr : R := a * (rE.resultant exquo LC(P))::R
      else
         c1 : polR := a * rE.coef1 * LC(P)**(d-1)::NNI
         c2 : polR := a * rE.coef2 * LC(P)**(d-1)::NNI
         cr : R := a * rE.resultant * LC(P)**(d-1)::NNI
      return construct(c1, c2, cr)

   semiDiscriminantEuclidean(P : polR) :
                            Record(coef2 : polR, discriminant : R) ==
      d : Integer := degree(P)
      zero?(d) => error "cannot take discriminant of constants"
      a : Integer := (d * (d-1)) quo 2
      a := (-1)**a::NonNegativeInteger
      dP : polR := differentiate P
      rE := semiResultantEuclidean2(P, dP)
      d := d - degree(dP) - 1
      if zero?(d) then
         c2 : polR := a * (rE.coef2 exquo LC(P))::polR
         cr : R := a * (rE.resultant exquo LC(P))::R
      else
         c2 : polR := a * rE.coef2 * LC(P)**(d-1)::NNI
         cr : R := a * rE.resultant * LC(P)**(d-1)::NNI
      return construct(c2, cr)

   if R has GcdDomain then
      resultantReduit(P : polR, Q : polR) : R ==
```

```
      UV := subResultantGcdEuclidean(P, Q)
      UVs : polR := UV.gcd
      degree(UVs) > 0 => 0
      l : List(R) := concat(coefficients(UV.coef1), coefficients(UV.coef2))
      return (LC(UVs) exquo gcd(l))::R

   resultantReduitEuclidean(P : polR, Q : polR) :
                 Record(coef1 : polR, coef2 : polR, resultantReduit : R) ==
      UV := subResultantGcdEuclidean(P, Q)
      UVs : polR := UV.gcd
      degree(UVs) > 0 => construct(0::polR, 0::polR, 0::R)
      l : List(R) := concat(coefficients(UV.coef1), coefficients(UV.coef2))
      gl : R := gcd(l)
      c1 : polR := (UV.coef1 exquo gl)::polR
      c2 : polR := (UV.coef2 exquo gl)::polR
      rr : R := (LC(UVs) exquo gl)::R
      return construct(c1, c2, rr)

   semiResultantReduitEuclidean(P : polR, Q : polR) :
                               Record(coef2 : polR, resultantReduit : R) ==
      UV := subResultantGcdEuclidean(P, Q)
      UVs : polR := UV.gcd
      degree(UVs) > 0 => construct(0::polR, 0::R)
      l : List(R) := concat(coefficients(UV.coef1), coefficients(UV.coef2))
      gl : R := gcd(l)
      c2 : polR := (UV.coef2 exquo gl)::polR
      rr : R := (LC(UVs) exquo gl)::R
      return construct(c2, rr)

   gcd_naif(P : polR, Q : polR) : polR ==
   -- valid over a field
      zero?(P) => (Q exquo LC(Q))::polR
      repeat
         zero?(Q) => return (P exquo LC(P))::polR
         zero?(degree(Q)) => return 1$polR
         (P, Q) := (Q, divide(P, Q).remainder)

   gcd(P : polR, Q : polR) : polR ==
      R has Finite => gcd_naif(P,Q)
      zero?(P) => Q
      zero?(Q) => P
      cP : R := content(P)
      cQ : R := content(Q)
      P := (P exquo cP)::polR
      Q := (Q exquo cQ)::polR
      G : polR := subResultantGcd(P, Q)
```

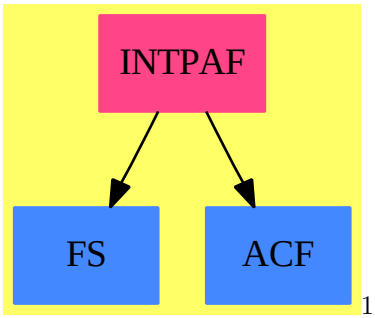
```
            return gcd(cP,cQ) * primitivePart(G)
```

⟨*PRS.dotabb*⟩≡

```
"PRS" [color="#FF4488",href="bookvol10.4.pdf#nameddest=PRS"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"PRS" -> "PFECAT"
```

17.127 package INTPAF PureAlgebraicIntegration

17.128 PureAlgebraicIntegration



1.00

Exports:
palgLODE palgRDE palgextint palgint palglimint

⟨*package INTPAF PureAlgebraicIntegration*⟩≡

```
)abbrev package INTPAF PureAlgebraicIntegration
++ Integration of pure algebraic functions;
++ Author: Manuel Bronstein
++ Date Created: 27 May 1988
++ Date Last Updated: 24 June 1994
++ Description:
++ This package provides functions for integration, limited integration,
++ extended integration and the risch differential equation for
++ pure algebraic integrands;
PureAlgebraicIntegration(R, F, L): Exports == Implementation where
  R: Join(GcdDomain,RetractableTo Integer,OrderedSet, CharacteristicZero,
          LinearlyExplicitRingOver Integer)
  F: Join(FunctionSpace R, AlgebraicallyClosedField,
          TranscendentalFunctionCategory)
  L: SetCategory

  SY  ==> Symbol
  N   ==> NonNegativeInteger
  K   ==> Kernel F
  P   ==> SparseMultivariatePolynomial(R, K)
  UP  ==> SparseUnivariatePolynomial F
  RF  ==> Fraction UP
  UPUP==> SparseUnivariatePolynomial RF
  IR  ==> IntegrationResult F
  IR2 ==> IntegrationResultFunctions2(curve, F)
  ALG ==> AlgebraicIntegrate(R, F, UP, UPUP, curve)
```

```
LDALG ==> LinearOrdinaryDifferentialOperator1 curve
RDALG ==> PureAlgebraicLODE(F, UP, UPUP, curve)
LOG ==> Record(coeff:F, logand:F)
REC ==> Record(particular:U1, basis:List F)
CND ==> Record(left:UP, right:UP)
CHV ==> Record(int:UPUP, left:UP, right:UP, den:RF, deg:N)
U1  ==> Union(F, "failed")
U2  ==> Union(Record(ratpart:F, coeff:F),"failed")
U3  ==> Union(Record(mainpart:F, limitedlogs:List LOG), "failed")
FAIL==> error "failed - cannot handle that integrand"

Exports ==> with
  palgint   : (F, K, K)    -> IR
    ++ palgint(f, x, y) returns the integral of \spad{f(x,y)dx}
    ++ where y is an algebraic function of x.
  palgextint: (F, K, K, F) -> U2
    ++ palgextint(f, x, y, g) returns functions \spad{[h, c]} such that
    ++ \spad{dh/dx = f(x,y) - c g}, where y is an algebraic function of x;
    ++ returns "failed" if no such functions exist.
  palglimint: (F, K, K, List F) -> U3
    ++ palglimint(f, x, y, [u1,...,un]) returns functions
    ++ \spad{[h,[[ci, ui]]]} such that the ui's are among \spad{[u1,...,un]}
    ++ and \spad{d(h + sum(ci log(ui)))/dx = f(x,y)} if such functions exist,
    ++ "failed" otherwise;
    ++ y is an algebraic function of x.
  palgRDE   : (F, F, F, K, K, (F, F, SY) -> U1) -> U1
    ++ palgRDE(nfp, f, g, x, y, foo) returns a function \spad{z(x,y)}
    ++ such that \spad{dz/dx + n * df/dx z(x,y) = g(x,y)} if such a z exists,
    ++ "failed" otherwise;
    ++ y is an algebraic function of x;
    ++ \spad{foo(a, b, x)} is a function that solves
    ++ \spad{du/dx + n * da/dx u(x) = u(x)}
    ++ for an unknown \spad{u(x)} not involving y.
    ++ \spad{nfp} is \spad{n * df/dx}.
  if L has LinearOrdinaryDifferentialOperatorCategory F then
    palgLODE: (L, F, K, K, SY) -> REC
      ++ palgLODE(op, g, kx, y, x) returns the solution of \spad{op f = g}.
      ++ y is an algebraic function of x.

Implementation ==> add
  import IntegrationTools(R, F)
  import RationalIntegration(F, UP)
  import GenusZeroIntegration(R, F, L)
  import ChangeOfVariable(F, UP, UPUP)
  import IntegrationResultFunctions2(F, F)
  import IntegrationResultFunctions2(RF, F)
```

```
    import SparseUnivariatePolynomialFunctions2(F, RF)
    import UnivariatePolynomialCommonDenominator(UP, RF, UPUP)
    import PolynomialCategoryQuotientFunctions(IndexedExponents K,
                                                  K, R, P, F)

    quadIfCan      : (K, K) -> Union(Record(coef:F, poly:UP), "failed")
    linearInXIfCan : (K, K) -> Union(Record(xsub:F, dxsub:RF), "failed")
    prootintegrate : (F, K, K) -> IR
    prootintegrate1: (UPUP, K, K, UPUP) -> IR
    prootextint    : (F, K, K, F) -> U2
    prootlimint    : (F, K, K, List F) -> U3
    prootRDE       : (F, F, F, K, K, (F, F, SY) -> U1) -> U1
    palgRDE1       : (F, F, K, K) -> U1
    palgLODE1      : (List F, F, K, K, SY) -> REC
    palgintegrate  : (F, K, K) -> IR
    palgext        : (F, K, K, F) -> U2
    palglim        : (F, K, K, List F) -> U3
    UPUP2F1        : (UPUP, RF, RF, K, K) -> F
    UPUP2F0        : (UPUP, K, K) -> F
    RF2UPUP        : (RF, UPUP) -> UPUP
    algaddx        : (IR, F) -> IR
    chvarIfCan     : (UPUP, RF, UP, RF) -> Union(UPUP, "failed")
    changeVarIfCan : (UPUP, RF, N) -> Union(CHV, "failed")
    rationalInt    : (UPUP, N, UP) -> IntegrationResult RF
    chv            : (UPUP, N, F, F) -> RF
    chv0           : (UPUP, N, F, F) -> F
    candidates     : UP -> List CND

    dummy := new()$SY
    dumk  := kernel(dummy)@K

    UPUP2F1(p, t, cf, kx, k) == UPUP2F0(eval(p, t, cf), kx, k)
    UPUP2F0(p, kx, k)        == multivariate(p, kx, k::F)
    chv(f, n, a, b)          == univariate(chv0(f, n, a, b), dumk)

    RF2UPUP(f, modulus) ==
      bc := extendedEuclidean(map(#1::UP::RF, denom f), modulus,
                                      1)::Record(coef1:UPUP, coef2:UPUP)
      (map(#1::UP::RF, numer f) * bc.coef1) rem modulus

-- returns "failed", or (xx, c) such that f(x, y)dx = f(xx, y) c dy
-- if p(x, y) = 0 is linear in x
    linearInXIfCan(x, y) ==
      a := b := 0$UP
      p := clearDenominator lift(minPoly y, x)
      while p ^= 0 repeat
```

```
        degree(q := numer leadingCoefficient p) > 1 => return "failed"
        a := a + monomial(coefficient(q, 1), d := degree p)
        b := b - monomial(coefficient(q, 0), d)
        p := reductum p
      xx:RF := b / a
      [xx(dumk::F), differentiate(xx, differentiate)]

-- return Int(f(x,y)dx) where y is an n^th root of a rational function in x
    prootintegrate(f, x, y) ==
      modulus := lift(p := minPoly y, x)
      rf      := reductum(ff := univariate(f, x, y, p))
      ((r := retractIfCan(rf)@Union(RF,"failed")) case RF) and rf ^= 0 =>
            -- in this case, ff := lc(ff) y^i + r so we integrate both terms
            -- separately to gain time
            map(#1(x::F), integrate(r::RF)) +
                 prootintegrate1(leadingMonomial ff, x, y, modulus)
      prootintegrate1(ff, x, y, modulus)

    prootintegrate1(ff, x, y, modulus) ==
      chv:CHV
      r := radPoly(modulus)::Record(radicand:RF, deg:N)
      (uu := changeVarIfCan(ff, r.radicand, r.deg)) case CHV =>
        chv := uu::CHV
        newalg := nthRoot((chv.left)(dumk::F), chv.deg)
        kz := retract(numer newalg)@K
        newf := multivariate(chv.int, ku := dumk, newalg)
        vu := (chv.right)(x::F)
        vz := (chv.den)(x::F) * (y::F) * denom(newalg)::F
        map(eval(#1, [ku, kz], [vu, vz]), palgint(newf, ku, kz))
      cv     := chvar(ff, modulus)
      r      := radPoly(cv.poly)::Record(radicand:RF, deg:N)
      qprime := differentiate(q := retract(r.radicand)@UP)::RF
      not zero? qprime and
       ((u := chvarIfCan(cv.func, 1, q, inv qprime)) case UPUP) =>
         m := monomial(1, r.deg)$UPUP - q::RF::UPUP
         map(UPUP2F1(RF2UPUP(#1, m), cv.c1, cv.c2, x, y),
            rationalInt(u::UPUP, r.deg, monomial(1, 1)))
      curve  := RadicalFunctionField(F, UP, UPUP, q::RF, r.deg)
      algaddx(map(UPUP2F1(lift #1, cv.c1, cv.c2, x, y),
        palgintegrate(reduce(cv.func), differentiate$UP)$ALG)$IR2, x::F)

-- Do the rationalizing change of variable
-- Int(f(x, y) dx) --> Int(n u^(n-1) f((u^n - b)/a, u) / a du) where
-- u^n = y^n = g(x) = a x + b
-- returns the integral as an integral of a rational function in u
    rationalInt(f, n, g) ==
```

```
--      not one? degree g => error "rationalInt: radicand must be linear"
      not ((degree g) = 1) => error "rationalInt: radicand must be linear"
      a := leadingCoefficient g
      integrate(n * monomial(inv a, (n-1)::N)$UP
                  * chv(f, n, a, leadingCoefficient reductum g))

-- Do the rationalizing change of variable f(x,y) --> f((u^n - b)/a, u) where
-- u = y = (a x + b)^(1/n).
-- Returns f((u^n - b)/a,u) as an element of F
    chv0(f, n, a, b) ==
      d := dumk::F
      (f (d::UP::RF)) ((d ** n - b) / a)

-- candidates(p) returns a list of pairs [g, u] such that p(x) = g(u(x)),
-- those u's are candidates for change of variables
-- currently uses a dumb heuristic where the candidates u's are p itself
-- and all the powers x^2, x^3, ..., x^{deg(p)},
-- will use polynomial decomposition in smarter days   MB 8/93
    candidates p ==
      l:List(CND) := empty()
      ground? p => l
      for i in 2..degree p repeat
        if (u := composite(p, xi := monomial(1, i))) case UP then
          l := concat([u::UP, xi], l)
      concat([monomial(1, 1), p], l)

-- checks whether Int(p(x, y) dx) can be rewritten as
-- Int(r(u, z) du) where u is some polynomial of x,
-- z = d y for some polynomial d, and z^m = g(u)
-- returns either [r(u, z), g, u, d, m] or "failed"
-- we have y^n = radi
    changeVarIfCan(p, radi, n) ==
      rec := rootPoly(radi, n)
      for cnd in candidates(rec.radicand) repeat
        (u := chvarIfCan(p, rec.coef, cnd.right,
              inv(differentiate(cnd.right)::RF))) case UPUP =>
                 return [u::UPUP, cnd.left, cnd.right, rec.coef, rec.exponent]
      "failed"

-- checks whether Int(p(x, y) dx) can be rewritten as
-- Int(r(u, z) du) where u is some polynomial of x and z = d y
-- we have y^n = a(x)/d(x)
-- returns either "failed" or r(u, z)
    chvarIfCan(p, d, u, u1) ==
      ans:UPUP := 0
      while p ^= 0 repeat
```

```
        (v := composite(u1 * leadingCoefficient(p) / d ** degree(p), u))
          case "failed" => return "failed"
        ans := ans + monomial(v::RF, degree p)
        p   := reductum p
      ans

    algaddx(i, xx) ==
      elem? i => i
      mkAnswer(ratpart i, logpart i,
                [[- ne.integrand / (xx**2), xx] for ne in notelem i])

    prootRDE(nfp, f, g, x, k, rde) ==
      modulus := lift(p := minPoly k, x)
      r       := radPoly(modulus)::Record(radicand:RF, deg:N)
      rec     := rootPoly(r.radicand, r.deg)
      dqdx    := inv(differentiate(q := rec.radicand)::RF)
      ((uf := chvarIfCan(ff := univariate(f,x,k,p),rec.coef,q,1)) case UPUP) and
        ((ug:=chvarIfCan(gg:=univariate(g,x,k,p),rec.coef,q,dqdx)) case UPUP) =>
          (u := rde(chv0(uf::UPUP, rec.exponent, 1, 0), rec.exponent *
                    (dumk::F) ** (rec.exponent * (rec.exponent - 1))
                      * chv0(ug::UPUP, rec.exponent, 1, 0),
                       symbolIfCan(dumk)::SY)) case "failed" => "failed"
          eval(u::F, dumk, k::F)
--      one?(rec.coef) =>
      ((rec.coef) = 1) =>
        curve  := RadicalFunctionField(F, UP, UPUP, q::RF, rec.exponent)
        rc := algDsolve(D()$LDALG + reduce(univariate(nfp, x, k, p))::LDALG,
                        reduce univariate(g, x, k, p))$RDALG
        rc.particular case "failed" => "failed"
        UPUP2F0(lift((rc.particular)::curve), x, k)
      palgRDE1(nfp, g, x, k)

    prootlimint(f, x, k, lu) ==
      modulus := lift(p := minPoly k, x)
      r       := radPoly(modulus)::Record(radicand:RF, deg:N)
      rec     := rootPoly(r.radicand, r.deg)
      dqdx    := inv(differentiate(q := rec.radicand)::RF)
      (uf := chvarIfCan(ff := univariate(f,x,k,p),rec.coef,q,dqdx)) case UPUP =>
        l := empty()$List(RF)
        n := rec.exponent * monomial(1, (rec.exponent - 1)::N)$UP
        for u in lu repeat
          if ((v:=chvarIfCan(uu:=univariate(u,x,k,p),rec.coef,q,dqdx))case UPUP)
            then l := concat(n * chv(v::UPUP,rec.exponent, 1, 0), l) else FAIL
        m := monomial(1, rec.exponent)$UPUP - q::RF::UPUP
        map(UPUP2F0(RF2UPUP(#1,m), x, k),
            limitedint(n * chv(uf::UPUP, rec.exponent, 1, 0), reverse_! l))
```

```
    cv     := chvar(ff, modulus)
    r      := radPoly(cv.poly)::Record(radicand:RF, deg:N)
    dqdx   := inv(differentiate(q := retract(r.radicand)@UP)::RF)
    curve  := RadicalFunctionField(F, UP, UPUP, q::RF, r.deg)
    (ui := palginfieldint(reduce(cv.func), differentiate$UP)$ALG)
      case "failed" => FAIL
    [UPUP2F1(lift(ui::curve), cv.c1, cv.c2, x, k), empty()]

  prootextint(f, x, k, g) ==
    modulus := lift(p := minPoly k, x)
    r       := radPoly(modulus)::Record(radicand:RF, deg:N)
    rec     := rootPoly(r.radicand, r.deg)
    dqdx    := inv(differentiate(q := rec.radicand)::RF)
    ((uf:=chvarIfCan(ff:=univariate(f,x,k,p),rec.coef,q,dqdx)) case UPUP) and
      ((ug:=chvarIfCan(gg:=univariate(g,x,k,p),rec.coef,q,dqdx)) case UPUP) =>
        m := monomial(1, rec.exponent)$UPUP - q::RF::UPUP
        n := rec.exponent * monomial(1, (rec.exponent - 1)::N)$UP
        map(UPUP2F0(RF2UPUP(#1,m), x, k),
            extendedint(n * chv(uf::UPUP, rec.exponent, 1, 0),
                        n * chv(ug::UPUP, rec.exponent, 1, 0)))
    cv     := chvar(ff, modulus)
    r      := radPoly(cv.poly)::Record(radicand:RF, deg:N)
    dqdx   := inv(differentiate(q := retract(r.radicand)@UP)::RF)
    curve  := RadicalFunctionField(F, UP, UPUP, q::RF, r.deg)
    (u := palginfieldint(reduce(cv.func), differentiate$UP)$ALG)
      case "failed" => FAIL
    [UPUP2F1(lift(u::curve), cv.c1, cv.c2, x, k), 0]

  palgRDE1(nfp, g, x, y) ==
    palgLODE1([nfp, 1], g, x, y, symbolIfCan(x)::SY).particular

  palgLODE1(eq, g, kx, y, x) ==
    modulus:= lift(p := minPoly y, kx)
    curve  := AlgebraicFunctionField(F, UP, UPUP, modulus)
    neq:LDALG := 0
    for f in eq for i in 0.. repeat
        neq := neq + monomial(reduce univariate(f, kx, y, p), i)
    empty? remove_!(y, remove_!(kx, varselect(kernels g, x))) =>
      rec := algDsolve(neq, reduce univariate(g, kx, y, p))$RDALG
      bas:List(F) := [UPUP2F0(lift h, kx, y) for h in rec.basis]
      rec.particular case "failed" => ["failed", bas]
      [UPUP2F0(lift((rec.particular)::curve), kx, y), bas]
    rec := algDsolve(neq, 0)
    ["failed", [UPUP2F0(lift h, kx, y) for h in rec.basis]]

  palgintegrate(f, x, k) ==
```

```
    modulus:= lift(p := minPoly k, x)
    cv     := chvar(univariate(f, x, k, p), modulus)
    curve  := AlgebraicFunctionField(F, UP, UPUP, cv.poly)
    knownInfBasis(cv.deg)
    algaddx(map(UPUP2F1(lift #1, cv.c1, cv.c2, x, k),
      palgintegrate(reduce(cv.func), differentiate$UP)$ALG)$IR2, x::F)

  palglim(f, x, k, lu) ==
    modulus:= lift(p := minPoly k, x)
    cv     := chvar(univariate(f, x, k, p), modulus)
    curve  := AlgebraicFunctionField(F, UP, UPUP, cv.poly)
    knownInfBasis(cv.deg)
    (u := palginfieldint(reduce(cv.func), differentiate$UP)$ALG)
      case "failed" => FAIL
    [UPUP2F1(lift(u::curve), cv.c1, cv.c2, x, k), empty()]

  palgext(f, x, k, g) ==
    modulus:= lift(p := minPoly k, x)
    cv     := chvar(univariate(f, x, k, p), modulus)
    curve  := AlgebraicFunctionField(F, UP, UPUP, cv.poly)
    knownInfBasis(cv.deg)
    (u := palginfieldint(reduce(cv.func), differentiate$UP)$ALG)
      case "failed" => FAIL
    [UPUP2F1(lift(u::curve), cv.c1, cv.c2, x, k), 0]

  palgint(f, x, y) ==
    (v := linearInXIfCan(x, y)) case "failed" =>
      (u := quadIfCan(x, y)) case "failed" =>
        is?(y, "nthRoot"::SY) => prootintegrate(f, x, y)
        is?(y,  "rootOf"::SY) => palgintegrate(f, x, y)
        FAIL
      palgint0(f, x, y, u.coef, u.poly)
    palgint0(f, x, y, dumk, v.xsub, v.dxsub)

  palgextint(f, x, y, g) ==
    (v := linearInXIfCan(x, y)) case "failed" =>
      (u := quadIfCan(x, y)) case "failed" =>
        is?(y, "nthRoot"::SY) => prootextint(f, x, y, g)
        is?(y,  "rootOf"::SY) => palgext(f, x, y, g)
        FAIL
      palgextint0(f, x, y, g, u.coef, u.poly)
    palgextint0(f, x, y, g, dumk, v.xsub, v.dxsub)

  palglimint(f, x, y, lu) ==
    (v := linearInXIfCan(x, y)) case "failed" =>
      (u := quadIfCan(x, y)) case "failed" =>
```

```
            is?(y, "nthRoot"::SY) => prootlimint(f, x, y, lu)
            is?(y,  "rootOf"::SY) => palglim(f, x, y, lu)
            FAIL
          palglimint0(f, x, y, lu, u.coef, u.poly)
        palglimint0(f, x, y, lu, dumk, v.xsub, v.dxsub)

      palgRDE(nfp, f, g, x, y, rde) ==
        (v := linearInXIfCan(x, y)) case "failed" =>
          (u := quadIfCan(x, y)) case "failed" =>
            is?(y, "nthRoot"::SY) => prootRDE(nfp, f, g, x, y, rde)
            palgRDE1(nfp, g, x, y)
          palgRDE0(f, g, x, y, rde, u.coef, u.poly)
        palgRDE0(f, g, x, y, rde, dumk, v.xsub, v.dxsub)

      -- returns "failed", or (d, P) such that (dy)**2 = P(x)
      -- and degree(P) = 2
      quadIfCan(x, y) ==
        (degree(p := minPoly y) = 2) and zero?(coefficient(p, 1)) =>
          d := denom(ff :=
                   univariate(- coefficient(p, 0) / coefficient(p, 2), x))
          degree(radi := d * numer ff) = 2 => [d(x::F), radi]
          "failed"
        "failed"

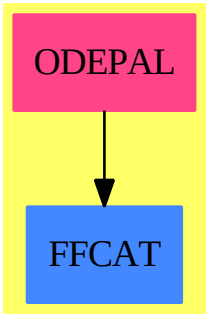
      if L has LinearOrdinaryDifferentialOperatorCategory F then
        palgLODE(eq, g, kx, y, x) ==
          (v := linearInXIfCan(kx, y)) case "failed" =>
            (u := quadIfCan(kx, y)) case "failed" =>
              palgLODE1([coefficient(eq, i) for i in 0..degree eq], g, kx, y, x)
            palgLODE0(eq, g, kx, y, u.coef, u.poly)
          palgLODE0(eq, g, kx, y, dumk, v.xsub, v.dxsub)
```

⟨*INTPAF.dotabb*⟩≡

```
"INTPAF" [color="#FF4488",href="bookvol10.4.pdf#nameddest=INTPAF"]
"FS" [color="#4488FF",href="bookvol10.2.pdf#nameddest=FS"]
"ACF" [color="#4488FF",href="bookvol10.2.pdf#nameddest=ACF"]
"INTPAF" -> "FS"
"INTPAF" -> "ACF"
```

17.129 package ODEPAL PureAlgebraicLODE

17.130 PureAlgebraicLODE



1.00

Exports:
algDsolve

⟨*package ODEPAL PureAlgebraicLODE*⟩≡
```
)abbrev package ODEPAL PureAlgebraicLODE
++ Author: Manuel Bronstein
++ Date Created: 21 August 1991
++ Date Last Updated: 3 February 1994
++ Description: In-field solution of an linear ordinary differential equation,
++ pure algebraic case.
PureAlgebraicLODE(F, UP, UPUP, R): Exports == Implementation where
  F   : Join(Field, CharacteristicZero,
             RetractableTo Integer, RetractableTo Fraction Integer)
  UP  : UnivariatePolynomialCategory F
  UPUP: UnivariatePolynomialCategory Fraction UP
  R   : FunctionFieldCategory(F, UP, UPUP)

  RF  ==> Fraction UP
  V   ==> Vector RF
  U   ==> Union(R, "failed")
  REC ==> Record(particular: Union(RF, "failed"), basis: List RF)
  L   ==> LinearOrdinaryDifferentialOperator1 R
  LQ  ==> LinearOrdinaryDifferentialOperator1 RF

  Exports ==> with
    algDsolve: (L, R) -> Record(particular: U, basis: List R)
      ++ algDsolve(op, g) returns \spad{["failed", []]} if the equation
      ++ \spad{op y = g} has no solution in \spad{R}. Otherwise, it returns
      ++ \spad{[f, [y1,...,ym]]} where \spad{f} is a particular rational
      ++ solution and the \spad{y_i's} form a basis for the solutions in
      ++ \spad{R} of the homogeneous equation.
```

```
Implementation ==> add
  import RationalLODE(F, UP)
  import SystemODESolver(RF, LQ)
  import ReduceLODE(RF, LQ, UPUP, R, L)

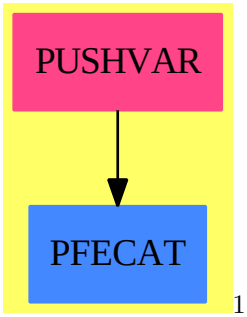
  algDsolve(l, g) ==
    rec := reduceLODE(l, g)
    sol := solveInField(rec.mat, rec.vec, ratDsolve)
    bas:List(R) := [represents v for v in sol.basis]
    (u := sol.particular) case V => [represents(u::V), bas]
    ["failed", bas]
```

⟨*ODEPAL.dotabb*⟩≡

```
"ODEPAL" [color="#FF4488",href="bookvol10.4.pdf#nameddest=ODEPAL"]
"FFCAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=FFCAT"]
"ODEPAL" -> "FFCAT"
```

17.131 package PUSHVAR PushVariables

17.132 PushVariables



1.00

Exports:

⟨*package PUSHVAR PushVariables*⟩≡

```
)abbrev package PUSHVAR PushVariables
++ This package \undocumented{}
PushVariables(R,E,OV,PPR):C == T where
   E : OrderedAbelianMonoidSup
   OV: OrderedSet with
        convert: % -> Symbol
          ++ convert(x) converts x to a symbol
        variable: Symbol -> Union(%, "failed")
          ++ variable(s) makes an element from symbol s or fails
   R  : Ring
   PR ==> Polynomial R
   PPR: PolynomialCategory(PR,E,OV)
   SUP ==> SparseUnivariatePolynomial
   C == with
     pushdown : (PPR, OV) -> PPR
        ++ pushdown(p,v) \undocumented{}
     pushdown : (PPR, List OV) -> PPR
        ++ pushdown(p,lv) \undocumented{}
     pushup   : (PPR, OV) -> PPR
        ++ pushup(p,v) \undocumented{}
     pushup   : (PPR, List OV) -> PPR
        ++ pushup(p,lv) \undocumented{}
     map      : ((PR -> PPR), PPR) -> PPR
        ++ map(f,p) \undocumented{}

   T == add
     pushdown(g:PPR,x:OV) : PPR ==
```

```
    eval(g,x,monomial(1,convert x,1)$PR)

  pushdown(g:PPR, lv:List OV) : PPR ==
    vals:=[monomial(1,convert x,1)$PR for x in lv]
    eval(g,lv,vals)

  map(f:(PR -> PPR), p: PPR) : PPR ==
    ground? p => f(retract p)
    v:=mainVariable(p)::OV
    multivariate(map(map(f,#1),univariate(p,v)),v)

            ----  push back the variable  ----
  pushupCoef(c:PR, lv:List OV): PPR ==
    ground? c => c::PPR
    v:=mainVariable(c)::Symbol
    v2 := variable(v)$OV
    uc := univariate(c,v)
    ppr : PPR := 0
    v2 case OV =>
       while not zero? uc repeat
          ppr := ppr + monomial(1,v2,degree(uc))$PPR *
                          pushupCoef(leadingCoefficient uc, lv)
          uc := reductum uc
       ppr
    while not zero? uc repeat
       ppr := ppr + monomial(1,v,degree(uc))$PR *
                          pushupCoef(leadingCoefficient uc, lv)
       uc := reductum uc
    ppr

  pushup(f:PPR,x:OV) :PPR ==
    map(pushupCoef(#1,[x]), f)

  pushup(g:PPR, lv:List OV) : PPR ==
    map(pushupCoef(#1, lv), g)
```

⟨*PUSHVAR.dotabb*⟩≡

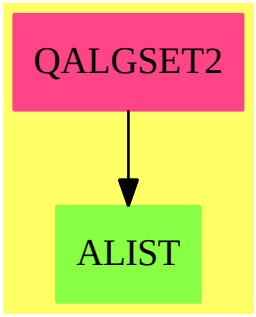
```
"PUSHVAR" [color="#FF4488",href="bookvol10.4.pdf#nameddest=PUSHVAR"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"PUSHVAR" -> "PFECAT"
```

Chapter 18

Chapter Q

18.1 package QALGSET2 QuasiAlgebraicSet2

18.2 QuasiAlgebraicSet2



1.00

Exports:
radicalSimplify

⟨*package QALGSET2 QuasiAlgebraicSet2*⟩≡
```
)abbrev package QALGSET2 QuasiAlgebraicSet2
++ Author:  William Sit
++ Date Created: March 13, 1992
++ Date Last Updated: June 12, 1992
++ Basic Operations:
++ Related Constructors:GroebnerPackage, IdealDecompositionPackage,
++                       PolynomialIdeals
++ See Also: QuasiAlgebraicSet
++ AMS Classifications:
++ Keywords: Zariski closed sets, quasi-algebraic sets
++ References:William Sit, "An Algorithm for Parametric Linear Systems"
```

```
++            J. Sym. Comp., April, 1992
++ Description:
++ \spadtype{QuasiAlgebraicSet2} adds a function \spadfun{radicalSimplify}
++ which uses \spadtype{IdealDecompositionPackage} to simplify
++ the representation of a quasi-algebraic set.  A quasi-algebraic set
++ is the intersection of a Zariski
++ closed set, defined as the common zeros of a given list of
++ polynomials (the defining polynomials for equations), and a principal
++ Zariski open set, defined as the complement of the common
++ zeros of a polynomial f (the defining polynomial for the inequation).
++ Quasi-algebraic sets are implemented in the domain
++ \spadtype{QuasiAlgebraicSet}, where two simplification routines are
++ provided:
++ \spadfun{idealSimplify} and \spadfun{simplify}.
++ The function
++ \spadfun{radicalSimplify} is added
++ for comparison study only.  Because the domain
++ \spadtype{IdealDecompositionPackage} provides facilities for
++ computing with radical ideals, it is necessary to restrict
++ the ground ring to the domain \spadtype{Fraction Integer},
++ and the polynomial ring to be of type
++ \spadtype{DistributedMultivariatePolynomial}.
++ The routine \spadfun{radicalSimplify} uses these to compute groebner
++ basis of radical ideals and
++ is inefficient and restricted when compared to the
++ two in \spadtype{QuasiAlgebraicSet}.
QuasiAlgebraicSet2(vl,nv) : C == T where
   vl       :   List Symbol
   nv       :   NonNegativeInteger
   R        ==> Integer
   F        ==> Fraction R
   Var      ==> OrderedVariableList vl
   NNI      ==> NonNegativeInteger
   Expon    ==> DirectProduct(nv,NNI)
   Dpoly    ==> DistributedMultivariatePolynomial(vl,F)
   QALG     ==> QuasiAlgebraicSet(F, Var, Expon, Dpoly)
   newExpon ==> DirectProduct(#newvl, NNI)
   newPoly  ==> DistributedMultivariatePolynomial(newvl,F)
   newVar   ==> OrderedVariableList newvl
   Status   ==> Union(Boolean,"failed") -- empty or not, or don't know

   C == with
     radicalSimplify:QALG -> QALG
       ++ radicalSimplify(s) returns a different and presumably simpler
       ++ representation of s with the defining polynomials for the
       ++ equations
```

```
    ++ forming a groebner basis, and the defining polynomial for the
    ++ inequation reduced with respect to the basis, using
    ++ using groebner basis of radical ideals
T  == add
            ----  Local Functions  ----
  ts:=new()$Symbol
  newvl:=concat(ts, vl)
  tv:newVar:=(variable ts)::newVar
  npoly         :     Dpoly            ->  newPoly
  oldpoly       :     newPoly          ->  Union(Dpoly,"failed")
  f             :     Var              ->  newPoly
  g             :     newVar           ->  Dpoly

  import PolynomialIdeals(F,newExpon,newVar,newPoly)
  import GroebnerPackage(F,Expon,Var,Dpoly)
  import GroebnerPackage(F,newExpon,newVar,newPoly)
  import IdealDecompositionPackage(newvl,#newvl)
  import QuasiAlgebraicSet(F, Var, Expon, Dpoly)
  import PolynomialCategoryLifting(Expon,Var,F,Dpoly,newPoly)
  import PolynomialCategoryLifting(newExpon,newVar,F,newPoly,Dpoly)
  f(v:Var):newPoly ==
    variable((convert v)@Symbol)@Union(newVar,"failed")::newVar
      ::newPoly
  g(v:newVar):Dpoly ==
    v = tv => 0
    variable((convert v)@Symbol)@Union(Var,"failed")::Var::Dpoly

  npoly(p:Dpoly) : newPoly ==  map(f #1, #1::newPoly, p)

  oldpoly(q:newPoly) : Union(Dpoly,"failed") ==
    (x:=mainVariable q) case "failed" => (leadingCoefficient q)::Dpoly
    (x::newVar = tv) => "failed"
    map(g #1,#1::Dpoly, q)

  radicalSimplify x ==
    status(x)$QALG = true => x     -- x is empty
    z0:=definingEquations x
    n0:=definingInequation x
    t:newPoly:= coerce(tv)$newPoly
    tp:newPoly:= t * (npoly n0) - 1$newPoly
    gen:List newPoly:= concat(tp, [npoly g for g in z0])
    id:=ideal gen
    ngb:=generators radical(id)
    member? (1$newPoly, ngb) => empty()$QALG
    gb:List Dpoly:=nil
    while not empty? ngb repeat
```

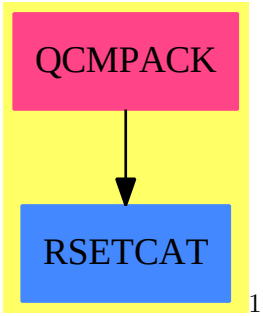
```
      if ((k:=oldpoly ngb.first) case Dpoly) then gb:=concat(k, gb)
      ngb:=ngb.rest
    y:=quasiAlgebraicSet(gb, primitivePart normalForm(n0, gb))
    setStatus(y,false::Status)
```

⟨*QALGSET2.dotabb*⟩≡

```
"QALGSET2" [color="#FF4488",href="bookvol10.4.pdf#nameddest=QALGSET2"]
"ALIST" [color="#88FF44",href="bookvol10.3.pdf#nameddest=ALIST"]
"QALGSET2" -> "ALIST"
```

18.3 package QCMPACK QuasiComponentPackage

18.4 QuasiComponentPackage



1.00

Exports:

algebraicSort	branchIfCan
infRittWu?	internalInfRittWu?
internalSubPolSet?	internalSubQuasiComponent?
moreAlgebraic?	prepareDecompose
removeSuperfluousCases	removeSuperfluousQuasiComponents
startTable!	subCase?
subPolSet?	subQuasiComponent?
subQuasiComponent?	subTriSet?
supDimElseRittWu?	

⟨*package QCMPACK QuasiComponentPackage*⟩≡

```
)abbrev package QCMPACK QuasiComponentPackage
++ Author: Marc Moreno Maza
++   marc@nag.co.uk
++ Date Created: 08/30/1998
++ Date Last Updated: 12/16/1998
++ Basic Functions:
++ Related Constructors:
++ Also See: 'tosedom.spad'
++ AMS Classifications:
++ Keywords:
++ Description:
++ A package for removing redundant quasi-components and redundant
++ branches when decomposing a variety by means of quasi-components
++ of regular triangular sets. \newline
++ References :
++  [1] D. LAZARD "A new method for solving algebraic systems of
++      positive dimension" Discr. App. Math. 33:147-160,1991
++  [2] M. MORENO MAZA "Calculs de pgcd au-dessus des tours
```

```
++      d'extensions simples et resolution des systemes d'equations
++      algebriques" These, Universite P.etM. Curie, Paris, 1997.
++  [3] M. MORENO MAZA "A new algorithm for computing triangular
++      decomposition of algebraic varieties" NAG Tech. Rep. 4/98.
++ Version: 3.

QuasiComponentPackage(R,E,V,P,TS): Exports == Implementation where

  R : GcdDomain
  E : OrderedAbelianMonoidSup
  V : OrderedSet
  P : RecursivePolynomialCategory(R,E,V)
  TS : RegularTriangularSetCategory(R,E,V,P)
  N ==> NonNegativeInteger
  Z ==> Integer
  B ==> Boolean
  S ==> String
  LP ==> List P
  PtoP ==> P -> P
  PS ==> GeneralPolynomialSet(R,E,V,P)
  PWT ==> Record(val : P, tower : TS)
  BWT ==> Record(val : Boolean, tower : TS)
  LpWT ==> Record(val : (List P), tower : TS)
  Branch ==> Record(eq: List P, tower: TS, ineq: List P)
  UBF ==> Union(Branch,"failed")
  Split ==> List TS
  Key ==>  Record(left:TS, right:TS)
  Entry ==> Boolean
  H ==> TabulatedComputationPackage(Key, Entry)
  polsetpack ==> PolynomialSetUtilitiesPackage(R,E,V,P)

  Exports ==  with
     startTable!: (S,S,S) -> Void
         ++ \axiom{startTableGcd!(s1,s2,s3)}
         ++ is an internal subroutine, exported only for developement.
     stopTable!: () -> Void
         ++ \axiom{stopTableGcd!()}
         ++ is an internal subroutine, exported only for developement.
     supDimElseRittWu?: (TS,TS) -> Boolean
         ++ \axiom{supDimElseRittWu(ts,us)} returns true iff \axiom{ts}
         ++ has less elements than \axiom{us} otherwise if \axiom{ts}
         ++ has higher rank than \axiom{us} w.r.t. Riit and Wu ordering.
     algebraicSort: Split -> Split
         ++ \axiom{algebraicSort(lts)} sorts \axiom{lts} w.r.t
         ++ \axiomOpFrom{supDimElseRittWu?}{QuasiComponentPackage}.
     moreAlgebraic?: (TS,TS) -> Boolean
```

```
    ++ \axiom{moreAlgebraic?(ts,us)} returns false iff \axiom{ts}
    ++ and \axiom{us} are both empty, or \axiom{ts}
    ++ has less elements than \axiom{us}, or some variable is
    ++ algebraic w.r.t. \axiom{us} and is not w.r.t. \axiom{ts}.
subTriSet?: (TS,TS) -> Boolean
    ++ \axiom{subTriSet?(ts,us)} returns true iff \axiom{ts} is
    ++ a sub-set of \axiom{us}.
subPolSet?: (LP, LP) -> Boolean
    ++ \axiom{subPolSet?(lp1,lp2)} returns true iff \axiom{lp1} is
    ++ a sub-set of \axiom{lp2}.
internalSubPolSet?: (LP, LP) -> Boolean
    ++ \axiom{internalSubPolSet?(lp1,lp2)} returns true iff \axiom{lp1} is
    ++ a sub-set of \axiom{lp2} assuming that these lists are sorted
    ++ increasingly w.r.t. \axiomOpFrom{infRittWu?}{RecursivePolynomialCategory}.
internalInfRittWu?: (LP, LP) -> Boolean
    ++ \axiom{internalInfRittWu?(lp1,lp2)}
    ++ is an internal subroutine, exported only for developement.
infRittWu?: (LP, LP) -> Boolean
    ++ \axiom{infRittWu?(lp1,lp2)}
    ++ is an internal subroutine, exported only for developement.
internalSubQuasiComponent?: (TS,TS) -> Union(Boolean,"failed")
    ++ \axiom{internalSubQuasiComponent?(ts,us)} returns a boolean \spad{b} value
    ++ if the fact that the regular zero set of \axiom{us} contains that of
    ++ \axiom{ts} can be decided (and in that case \axiom{b} gives this
    ++ inclusion) otherwise returns \axiom{"failed"}.
subQuasiComponent?: (TS,TS) -> Boolean
    ++ \axiom{subQuasiComponent?(ts,us)} returns true iff
    ++ \axiomOpFrom{internalSubQuasiComponent?}{QuasiComponentPackage}
    ++ returns true.
subQuasiComponent?: (TS,Split) -> Boolean
    ++ \axiom{subQuasiComponent?(ts,lus)} returns true iff
    ++ \axiom{subQuasiComponent?(ts,us)} holds for one \spad{us} in \spad{lus}.
removeSuperfluousQuasiComponents: Split -> Split
    ++ \axiom{removeSuperfluousQuasiComponents(lts)} removes from \axiom{lts}
    ++ any \spad{ts} such that \axiom{subQuasiComponent?(ts,us)} holds for
    ++ another \spad{us} in \axiom{lts}.
subCase?: (LpWT,LpWT) -> Boolean
    ++ \axiom{subCase?(lpwt1,lpwt2)}
    ++ is an internal subroutine, exported only for developement.
removeSuperfluousCases: List LpWT -> List LpWT
    ++ \axiom{removeSuperfluousCases(llpwt)}
    ++ is an internal subroutine, exported only for developement.
prepareDecompose: (LP, List(TS),B,B) -> List Branch
    ++ \axiom{prepareDecompose(lp,lts,b1,b2)}
    ++ is an internal subroutine, exported only for developement.
branchIfCan: (LP,TS,LP,B,B,B,B,B) -> Union(Branch,"failed")
```

```
        ++ \axiom{branchIfCan(leq,ts,lineq,b1,b2,b3,b4,b5)}
        ++ is an internal subroutine, exported only for developement.

    Implementation == add

      squareFreeFactors(lp: LP): LP ==
        lsflp: LP := []
        for p in lp repeat
          lsfp := squareFreeFactors(p)$polsetpack
          lsflp := concat(lsfp,lsflp)
        sort(infRittWu?,removeDuplicates lsflp)

      startTable!(ok: S, ko: S, domainName: S): Void ==
        initTable!()$H
        if (not empty? ok) and (not empty? ko) then printInfo!(ok,ko)$H
        if (not empty? domainName) then startStats!(domainName)$H
        void()

      stopTable!(): Void ==
        if makingStats?()$H then printStats!()$H
        clearTable!()$H

      supDimElseRittWu? (ts:TS,us:TS): Boolean ==
        #ts < #us => true
        #ts > #us => false
        lp1 :LP := members(ts)
        lp2 :LP := members(us)
        while (not empty? lp1) and (not infRittWu?(first(lp2),first(lp1))) repeat
          lp1 := rest lp1
          lp2 := rest lp2
        not empty? lp1

      algebraicSort (lts:Split): Split ==
        lts := removeDuplicates lts
        sort(supDimElseRittWu?,lts)

      moreAlgebraic?(ts:TS,us:TS): Boolean  ==
        empty? ts => empty? us
        empty? us => true
        #ts < #us => false
        for p in (members us) repeat
           not algebraic?(mvar(p),ts) => return false
        true

      subTriSet?(ts:TS,us:TS): Boolean  ==
        empty? ts => true
```

```
    empty? us => false
    mvar(ts) > mvar(us) => false
    mvar(ts) < mvar(us) => subTriSet?(ts,rest(us)::TS)
    first(ts)::P = first(us)::P => subTriSet?(rest(ts)::TS,rest(us)::TS)
    false

  internalSubPolSet?(lp1: LP, lp2: LP): Boolean  ==
    empty? lp1 => true
    empty? lp2 => false
    associates?(first lp1, first lp2) =>
      internalSubPolSet?(rest lp1, rest lp2)
    infRittWu?(first lp1, first lp2) => false
    internalSubPolSet?(lp1, rest lp2)

  subPolSet?(lp1: LP, lp2: LP): Boolean  ==
    lp1 := sort(infRittWu?, lp1)
    lp2 := sort(infRittWu?, lp2)
    internalSubPolSet?(lp1,lp2)

  infRittWu?(lp1: LP, lp2: LP): Boolean ==
    lp1 := sort(infRittWu?, lp1)
    lp2 := sort(infRittWu?, lp2)
    internalInfRittWu?(lp1,lp2)

  internalInfRittWu?(lp1: LP, lp2: LP): Boolean ==
    empty? lp1 => not empty? lp2
    empty? lp2 => false
    infRittWu?(first lp1, first lp2)$P => true
    infRittWu?(first lp2, first lp1)$P => false
    infRittWu?(rest lp1, rest lp2)$$

  subCase? (lpwt1:LpWT,lpwt2:LpWT): Boolean ==
    -- ASSUME lpwt.{1,2}.val is sorted w.r.t. infRittWu?
    not internalSubPolSet?(lpwt2.val, lpwt1.val) => false
    subQuasiComponent?(lpwt1.tower,lpwt2.tower)

  internalSubQuasiComponent?(ts:TS,us:TS): Union(Boolean,"failed") ==
    -- "failed" is false iff saturate(us) is radical
    subTriSet?(us,ts) => true
    not moreAlgebraic?(ts,us) => false::Union(Boolean,"failed")
    for p in (members us) repeat
      mdeg(p) < mdeg(select(ts,mvar(p))::P) =>
        return("failed"::Union(Boolean,"failed"))
    for p in (members us) repeat
      not zero? initiallyReduce(p,ts) =>
        return("failed"::Union(Boolean,"failed"))
```

```
  lsfp := squareFreeFactors(initials us)
  for p in lsfp repeat
    not invertible?(p,ts)@B =>
      return(false::Union(Boolean,"failed"))
  true::Union(Boolean,"failed")

subQuasiComponent?(ts:TS,us:TS): Boolean ==
  k: Key := [ts, us]
  e := extractIfCan(k)$H
  e case Entry => e::Entry
  ubf: Union(Boolean,"failed") := internalSubQuasiComponent?(ts,us)
  b: Boolean := (ubf case Boolean) and (ubf::Boolean)
  insert!(k,b)$H
  b

subQuasiComponent?(ts:TS,lus:Split): Boolean ==
  for us in lus repeat
     subQuasiComponent?(ts,us)@B => return true
  false

removeSuperfluousCases (cases:List LpWT) ==
  #cases < 2 => cases
  toSee := sort(supDimElseRittWu?(#1.tower,#2.tower),cases)
  lpwt1,lpwt2 : LpWT
  toSave,headmaxcases,maxcases,copymaxcases : List LpWT
  while not empty? toSee repeat
    lpwt1 := first toSee
    toSee := rest toSee
    toSave := []
    for lpwt2 in toSee repeat
       if subCase?(lpwt1,lpwt2)
         then
           lpwt1 := lpwt2
         else
           if not subCase?(lpwt2,lpwt1)
             then
               toSave := cons(lpwt2,toSave)
    if empty? maxcases
      then
        headmaxcases := [lpwt1]
        maxcases := headmaxcases
      else
        copymaxcases := maxcases
        while (not empty? copymaxcases) and _
          (not subCase?(lpwt1,first(copymaxcases))) repeat
            copymaxcases := rest copymaxcases
```

```
        if empty? copymaxcases
          then
            setrest!(headmaxcases,[lpwt1])
            headmaxcases := rest headmaxcases
    toSee := reverse toSave
  maxcases

removeSuperfluousQuasiComponents(lts: Split): Split ==
  lts := removeDuplicates lts
  #lts < 2 => lts
  toSee := algebraicSort lts
  toSave,headmaxlts,maxlts,copymaxlts : Split
  while not empty? toSee repeat
    ts := first toSee
    toSee := rest toSee
    toSave := []
    for us in toSee repeat
       if subQuasiComponent?(ts,us)@B
         then
           ts := us
         else
           if not subQuasiComponent?(us,ts)@B
             then
               toSave := cons(us,toSave)
    if empty? maxlts
      then
        headmaxlts := [ts]
        maxlts := headmaxlts
      else
        copymaxlts := maxlts
        while (not empty? copymaxlts) and _
          (not subQuasiComponent?(ts,first(copymaxlts))@B) repeat
            copymaxlts := rest copymaxlts
        if empty? copymaxlts
          then
            setrest!(headmaxlts,[ts])
            headmaxlts := rest headmaxlts
    toSee := reverse toSave
  algebraicSort maxlts

removeAssociates (lp:LP):LP ==
  removeDuplicates [primitivePart(p) for p in lp]

branchIfCan(leq: LP,ts: TS,lineq: LP, b1:B,b2:B,b3:B,b4:B,b5:B):UBF ==
   -- ASSUME pols in leq are squarefree and mainly primitive
   -- if b1 then CLEAN UP leq
```

```
        -- if b2 then CLEAN UP lineq
        -- if b3 then SEARCH for ZERO in lineq with leq
        -- if b4 then SEARCH for ZERO in lineq with ts
        -- if b5 then SEARCH for ONE in leq with lineq
        if b1
          then
            leq := removeAssociates(leq)
            leq := remove(zero?,leq)
            any?(ground?,leq) =>
              return("failed"::Union(Branch,"failed"))
        if b2
          then
            any?(zero?,lineq) =>
              return("failed"::Union(Branch,"failed"))
            lineq := removeRedundantFactors(lineq)$polsetpack
        if b3
          then
            ps: PS := construct(leq)$PS
            for q in lineq repeat
              zero? remainder(q,ps).polnum =>
                return("failed"::Union(Branch,"failed"))
        (empty? leq) or (empty? lineq) => ([leq, ts, lineq]$Branch)::UBF
        if b4
          then
            for q in lineq repeat
              zero? initiallyReduce(q,ts) =>
                return("failed"::Union(Branch,"failed"))
        if b5
          then
            newleq: LP := []
            for p in leq repeat
              for q in lineq repeat
                if mvar(p) = mvar(q)
                  then
                    g := gcd(p,q)
                    newp := (p exquo g)::P
                    ground? newp =>
                      return("failed"::Union(Branch,"failed"))
                    newleq := cons(newp,newleq)
                  else
                    newleq := cons(p,newleq)
            leq := newleq
        leq := sort(infRittWu?, removeDuplicates leq)
        ([leq, ts, lineq]$Branch)::UBF

      prepareDecompose(lp: LP, lts: List(TS), b1: B, b2: B): List Branch ==
```

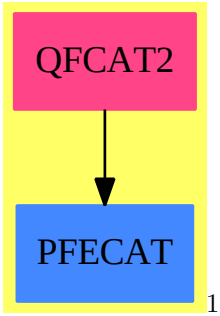
```
        -- if b1 then REMOVE REDUNDANT COMPONENTS in lts
        -- if b2 then SPLIT the input system with squareFree
        lp := sort(infRittWu?, remove(zero?,removeAssociates(lp)))
        any?(ground?,lp) => []
        empty? lts => []
        if b1 then lts := removeSuperfluousQuasiComponents lts
        not b2 =>
          [[lp,ts,squareFreeFactors(initials ts)]$Branch for ts in lts]
        toSee: List Branch
        lq: LP := []
        toSee := [[lq,ts,squareFreeFactors(initials ts)]$Branch for ts in lts]
        empty? lp => toSee
        for p in lp repeat
          lsfp := squareFreeFactors(p)$polsetpack
          branches: List Branch := []
          lq := []
          for f in lsfp repeat
            for branch in toSee repeat
              leq : LP := branch.eq
              ts := branch.tower
              lineq : LP := branch.ineq
              ubf1: UBF := branchIfCan(leq,ts,lq,false,false,true,true,true)@UBF
              ubf1 case "failed" => "leave"
              ubf2: UBF := branchIfCan([f],ts,lineq,false,false,true,true,true)@UBF
              ubf2 case "failed" => "leave"
              leq := sort(infRittWu?,removeDuplicates concat(ubf1.eq,ubf2.eq))
              lineq := sort(infRittWu?,removeDuplicates concat(ubf1.ineq,ubf2.ineq))
              newBranch := branchIfCan(leq,ts,lineq,false,false,false,false,false)
              branches:= cons(newBranch::Branch,branches)
            lq := cons(f,lq)
          toSee := branches
        sort(supDimElseRittWu?(#1.tower,#2.tower),toSee)
```

⟨*QCMPACK.dotabb*⟩≡

```
"QCMPACK" [color="#FF4488",href="bookvol10.4.pdf#nameddest=QCMPACK"]
"RSETCAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=RSETCAT"]
"QCMPACK" -> "RSETCAT"
```

18.5 package QFCAT2 QuotientFieldCategoryFunctions2

18.6 QuotientFieldCategoryFunctions2



1.00

Exports:
map

⟨*package QFCAT2 QuotientFieldCategoryFunctions2*⟩≡

```
)abbrev package QFCAT2 QuotientFieldCategoryFunctions2
++ Author:
++ Date Created:
++ Date Last Updated:
++ Basic Functions:
++ Related Constructors:
++ Also See:
++ AMS Classifications:
++ Keywords:
++ References:
++ Description:
++ This package extends a function between integral domains
++ to a mapping between their quotient fields.
QuotientFieldCategoryFunctions2(A, B, R, S): Exports == Impl where
  A, B: IntegralDomain
  R   : QuotientFieldCategory(A)
  S   : QuotientFieldCategory(B)

  Exports ==> with
    map: (A -> B, R) -> S
      ++ map(func,frac) applies the function func to the numerator
      ++ and denominator of frac.

  Impl ==> add
    map(f, r) == f(numer r) / f(denom r)
```

⟨*QFCAT2.dotabb*⟩≡

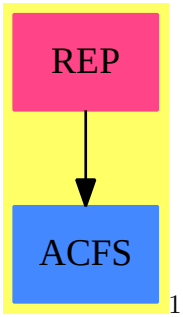
```
"QFCAT2" [color="#FF4488",href="bookvol10.4.pdf#nameddest=QFCAT2"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"QFCAT2" -> "PFECAT"
```

Chapter 19

Chapter R

19.1 package REP RadicalEigenPackage

19.2 RadicalEigenPackage



1.00

Exports:

eigenMatrix	gramschmidt	normalise	orthonormalBasis
radicalEigenvalues	radicalEigenvectors		

⟨*package REP RadicalEigenPackage*⟩≡

```
)abbrev package REP RadicalEigenPackage
++ Author: P.Gianni
++ Date Created: Summer 1987
++ Date Last Updated: October 1992
++ Basic Functions:
++ Related Constructors: EigenPackage, RadicalSolve
++ Also See:
++ AMS Classifications:
++ Keywords:
++ References:
```

```
++ Description:
++ Package for the computation of eigenvalues and eigenvectors.
++ This package works for matrices with coefficients which are
++ rational functions over the integers.
++ (see \spadtype{Fraction Polynomial Integer}).
++ The eigenvalues and eigenvectors are expressed in terms of radicals.
RadicalEigenPackage() : C == T
 where
   R     ==> Integer
   P     ==> Polynomial R
   F     ==> Fraction P
   RE    ==> Expression R
   SE    ==> Symbol()
   M     ==> Matrix(F)
   MRE   ==> Matrix(RE)
   ST    ==> SuchThat(SE,P)
   NNI   ==> NonNegativeInteger

   EigenForm    ==> Record(eigval:Union(F,ST),eigmult:NNI,eigvec:List(M))
   RadicalForm  ==> Record(radval:RE,radmult:Integer,radvect:List(MRE))



   C == with
     radicalEigenvectors    :  M     ->  List(RadicalForm)
       ++ radicalEigenvectors(m) computes
       ++ the eigenvalues and the corresponding eigenvectors of the
       ++ matrix m;
       ++ when possible, values are expressed in terms of radicals.

     radicalEigenvector       :  (RE,M)     ->  List(MRE)
       ++ radicalEigenvector(c,m) computes the eigenvector(s) of the
       ++ matrix m corresponding to the eigenvalue c;
       ++ when possible, values are
       ++ expressed in terms of radicals.

     radicalEigenvalues     :  M     ->  List RE
       ++ radicalEigenvalues(m) computes the eigenvalues of the matrix m;
       ++ when possible, the eigenvalues are expressed in terms of radicals.

     eigenMatrix       :    M        ->  Union(MRE,"failed")
       ++ eigenMatrix(m) returns the matrix b
       ++ such that \spad{b*m*(inverse b)} is diagonal,
       ++ or "failed" if no such b exists.

     normalise         :    MRE      ->  MRE
```

```
    ++ normalise(v) returns the column
    ++ vector v
    ++ divided by its euclidean norm;
    ++ when possible, the vector v is expressed in terms of radicals.

  gramschmidt      : List(MRE)   ->  List(MRE)
    ++ gramschmidt(lv) converts the list of column vectors lv into
    ++ a set of orthogonal column vectors
    ++ of euclidean length 1 using the Gram-Schmidt algorithm.

  orthonormalBasis  :    M        ->  List(MRE)
    ++ orthonormalBasis(m)  returns the orthogonal matrix b such that
    ++ \spad{b*m*(inverse b)} is diagonal.
    ++ Error: if m is not a symmetric matrix.

T == add
  PI       ==> PositiveInteger
  RSP := RadicalSolvePackage R
  import EigenPackage R

             ----  Local  Functions  ----
  evalvect         :  (M,RE,SE)  ->  MRE
  innerprod        :   (MRE,MRE)   ->  RE

      ----  eval a vector of F in a radical expression  ----
  evalvect(vect:M,alg:RE,x:SE) : MRE ==
    n:=nrows vect
    xx:=kernel(x)$Kernel(RE)
    w:MRE:=zero(n,1)$MRE
    for i in 1..n repeat
      v:=eval(vect(i,1) :: RE,xx,alg)
      setelt(w,i,1,v)
    w
                  ---- inner product ----
  innerprod(v1:MRE,v2:MRE): RE == (((transpose v1)* v2)::MRE)(1,1)

             ----  normalization of a vector  ----
  normalise(v:MRE) : MRE ==
    normv:RE := sqrt(innerprod(v,v))
    normv = 0$RE => v
    (1/normv)*v

           ----  Eigenvalues of the matrix A  ----
  radicalEigenvalues(A:M): List(RE) ==
    x:SE :=new()$SE
    pol:= characteristicPolynomial(A,x) :: F
```

```
  radicalRoots(pol,x)$RSP

 ----  Eigenvectors belonging to a given eigenvalue  ----
       ----  expressed in terms of radicals ----
radicalEigenvector(alpha:RE,A:M) : List(MRE) ==
  n:=nrows A
  B:MRE := zero(n,n)$MRE
  for i in 1..n repeat
    for j in 1..n repeat B(i,j):=(A(i,j))::RE
    B(i,i):= B(i,i) - alpha
  [v::MRE  for v in nullSpace B]

       ----  eigenvectors and eigenvalues  ----
radicalEigenvectors(A:M) : List(RadicalForm) ==
  leig:List EigenForm := eigenvectors A
  n:=nrows A
  sln:List RadicalForm := empty()
  veclist: List MRE
  for eig in leig repeat
    eig.eigval case F =>
      veclist := empty()
      for ll in eig.eigvec repeat
        m:MRE:=zero(n,1)
        for i in 1..n repeat m(i,1):=(ll(i,1))::RE
        veclist:=cons(m,veclist)
      sln:=cons([(eig.eigval)::F::RE,eig.eigmult,veclist]$RadicalForm,sln)
    sym := eig.eigval :: ST
    xx:= lhs sym
    lval : List RE := radicalRoots((rhs sym) :: F ,xx)$RSP
    for alg in lval repeat
      nsl:=[alg,eig.eigmult,
            [evalvect(ep,alg,xx) for ep in eig.eigvec]]$RadicalForm
      sln:=cons(nsl,sln)
  sln

      ----  orthonormalization of a list of vectors  ----
            ----  Grahm - Schmidt process  ----

gramschmidt(lvect:List(MRE)) : List(MRE) ==
  lvect=[]  => []
  v:=lvect.first
  n := nrows v
  RMR:=RectangularMatrix(n:PI,1,RE)
  orth:List(MRE):=[(normalise v)]
  for v in lvect.rest repeat
    pol:=((v:RMR)-(+/[(innerprod(w,v)*w):RMR for w in orth])):MRE
```

```
      orth:=cons(normalise pol,orth)
    orth


          ----  The matrix of eigenvectors  ----

  eigenMatrix(A:M) : Union(MRE,"failed") ==
    lef:List(MRE):=[:eiv.radvect  for eiv in radicalEigenvectors(A)]
    n:=nrows A
    #lef <n => "failed"
    d:MRE:=copy(lef.first)
    for v in lef.rest repeat d:=(horizConcat(d,v))::MRE
    d

      ----  orthogonal basis for a symmetric matrix  ----

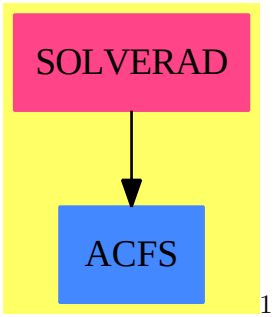
  orthonormalBasis(A:M):List(MRE) ==
    ^symmetric?(A) => error "the matrix is not symmetric"
    basis:List(MRE):=[]
    lvec:List(MRE) := []
    alglist:List(RadicalForm):=radicalEigenvectors(A)
    n:=nrows A
    for alterm in alglist repeat
      if (lvec:=alterm.radvect)=[] then error "sorry "
      if #(lvec)>1  then
        lvec:= gramschmidt(lvec)
        basis:=[:lvec,:basis]
      else basis:=[normalise(lvec.first),:basis]
    basis
```

⟨*REP.dotabb*⟩≡

```
"REP" [color="#FF4488",href="bookvol10.4.pdf#nameddest=REP"]
"ACFS" [color="#4488FF",href="bookvol10.2.pdf#nameddest=ACFS"]
"REP" -> "ACFS"
```

19.3 package SOLVERAD RadicalSolvePackage

19.4 RadicalSolvePackage



1.00

Exports:
contractSolve radicalRoots radicalSolve

⟨*package SOLVERAD RadicalSolvePackage*⟩≡
```
)abbrev package SOLVERAD RadicalSolvePackage
++ Author: P.Gianni
++ Date Created: Summer 1990
++ Date Last Updated: October 1991
++ Basic Functions:
++ Related Constructors: SystemSolvePackage, FloatingRealPackage,
++ FloatingComplexPackage
++ Also See:
++ AMS Classifications:
++ Keywords:
++ References:
++ Description:
++ This package tries to find solutions
++ expressed in terms of radicals for systems of equations
++ of rational functions with coefficients in an integral domain R.
RadicalSolvePackage(R): Cat == Capsule where
    R   :  Join(EuclideanDomain, OrderedSet, CharacteristicZero)
    PI ==> PositiveInteger
    NNI==> NonNegativeInteger
    Z  ==> Integer
    B  ==> Boolean
    ST ==> String
    PR ==> Polynomial R
    UP ==> SparseUnivariatePolynomial PR
    LA ==> LocalAlgebra(PR, Z, Z)
    RF ==> Fraction PR
    RE ==> Expression R
```

```
EQ ==> Equation
SY ==> Symbol
SU ==> SuchThat(List RE, List Equation RE)
SUP==> SparseUnivariatePolynomial
L  ==> List
P  ==> Polynomial

SOLVEFOR ==> PolynomialSolveByFormulas(SUP RE, RE)
UPF2     ==> SparseUnivariatePolynomialFunctions2(PR,RE)

Cat ==> with

    radicalSolve :     (RF,SY)      -> L EQ RE
      ++ radicalSolve(rf,x) finds the solutions expressed in terms of
      ++ radicals of the equation rf = 0 with respect to the symbol x,
      ++ where rf is a rational function.
    radicalSolve :       RF         -> L EQ RE
      ++ radicalSolve(rf) finds the solutions expressed in terms of
      ++ radicals of the equation rf = 0, where rf is a
      ++ univariate rational function.
    radicalSolve :    (EQ RF,SY)    -> L EQ RE
      ++ radicalSolve(eq,x) finds the solutions expressed in terms of
      ++ radicals of the equation of rational functions eq
      ++ with respect to the symbol x.
    radicalSolve :      EQ RF       -> L EQ RE
      ++ radicalSolve(eq) finds the solutions expressed in terms of
      ++ radicals of the equation of rational functions eq
      ++ with respect to the unique symbol x appearing in eq.
    radicalSolve :    (L RF,L SY)   -> L L EQ RE
      ++ radicalSolve(lrf,lvar) finds the solutions expressed in terms of
      ++ radicals of the system of equations lrf = 0 with
      ++ respect to the list of symbols lvar,
      ++ where lrf is a list of rational functions.
    radicalSolve :       L RF       -> L L EQ RE
      ++ radicalSolve(lrf) finds the solutions expressed in terms of
      ++ radicals of the system of equations lrf = 0, where lrf is a
      ++ system of univariate rational functions.
    radicalSolve :   (L EQ RF,L SY) -> L L EQ RE
      ++ radicalSolve(leq,lvar) finds the solutions expressed in terms of
      ++ radicals of the system of equations of rational functions leq
      ++ with respect to the list of symbols lvar.
    radicalSolve :     L EQ RF      -> L L EQ RE
      ++ radicalSolve(leq) finds the solutions expressed in terms of
      ++ radicals of the system of equations of rational functions leq
      ++ with respect to the unique symbol x appearing in leq.
    radicalRoots :      (RF,SY)     -> L RE
```

```
      ++ radicalRoots(rf,x) finds the roots expressed in terms of radicals
      ++ of the rational function rf with respect to the symbol x.
    radicalRoots :    (L RF,L SY)   -> L L RE
      ++ radicalRoots(lrf,lvar) finds the roots expressed in terms of
      ++ radicals of the list of rational functions lrf
      ++ with respect to the list of symbols lvar.
    contractSolve:    (EQ RF,SY)  -> SU
      ++ contractSolve(eq,x) finds the solutions expressed in terms of
      ++ radicals of the equation of rational functions eq
      ++ with respect to the symbol x.  The result contains new
      ++ symbols for common subexpressions in order to reduce the
      ++ size of the output.
    contractSolve:    (RF,SY)     -> SU
      ++ contractSolve(rf,x) finds the solutions expressed in terms of
      ++ radicals of the equation rf = 0 with respect to the symbol x,
      ++ where rf is a rational function. The result contains  new
      ++ symbols for common subexpressions in order to reduce the
      ++ size of the output.
  Capsule ==> add
    import DegreeReductionPackage(PR, R)
    import SOLVEFOR

    SideEquations: List EQ RE := []
    ContractSoln:  B := false

    ---- Local Function Declarations ----
    solveInner:(PR, SY, B) -> SU
    linear:    UP -> List RE
    quadratic: UP -> List RE
    cubic:     UP -> List RE
    quartic:   UP -> List RE
    rad:       PI -> RE
    wrap:      RE -> RE
    New:       RE -> RE
    makeEq : (List RE,L SY) -> L EQ RE
    select :    L L RE      -> L L RE
    isGeneric? :  (L PR,L SY)  ->  Boolean
    findGenZeros :  (L PR,L SY) -> L L RE
    findZeros   :   (L PR,L SY) -> L L RE


    New s ==
        s = 0 => 0
        S := new()$Symbol ::PR::RF::RE
        SideEquations := append([S = s], SideEquations)
        S
```

```
 linear u    == [(-coefficient(u,0))::RE /(coefficient(u,1))::RE]
 quadratic u == quadratic(map(coerce,u)$UPF2)$SOLVEFOR
 cubic u     == cubic(map(coerce,u)$UPF2)$SOLVEFOR
 quartic u   == quartic(map(coerce,u)$UPF2)$SOLVEFOR
 rad n       == n::Z::RE
 wrap s      == (ContractSoln => New s; s)


 ---- Exported Functions ----


-- find the zeros of components in "generic" position --
 findGenZeros(rlp:L PR,rlv:L SY) : L L RE ==
  pp:=rlp.first
  v:=first rlv
  rlv:=rest rlv
  res:L L RE:=[]
  res:=append([reverse cons(r,[eval(
    (-coefficient(univariate(p,vv),0)::RE)/(leadingCoefficient univariate(p,vv))::RE,
       kernel(v)@Kernel(RE),r) for vv in rlv for p in rlp.rest])
         for r in radicalRoots(pp::RF,v)],res)
  res


 findZeros(rlp:L PR,rlv:L SY) : L L RE ==
  parRes:=[radicalRoots(p::RF,v) for p in rlp for v in rlv]
  parRes:=select parRes
  res:L L RE :=[]
  res1:L RE
  for par in parRes repeat
    res1:=[par.first]
    lv1:L Kernel(RE):=[kernel rlv.first]
    rlv1:=rlv.rest
    p1:=par.rest
    while p1^=[] repeat
      res1:=cons(eval(p1.first,lv1,res1),res1)
      p1:=p1.rest
      lv1:=cons(kernel rlv1.first,lv1)
      rlv1:=rlv1.rest
    res:=cons(res1,res)
  res

 radicalSolve(pol:RF,v:SY) ==
   [equation(v::RE,r) for r in radicalRoots(pol,v)]
```

```
radicalSolve(p:RF) ==
  zero? p =>
     error "equation is always satisfied"
  lv:=removeDuplicates
     concat(variables numer p, variables denom p)
  empty? lv => error "inconsistent equation"
  #lv>1 => error "too many variables"
  radicalSolve(p,lv.first)

radicalSolve(eq: EQ RF) ==
  radicalSolve(lhs eq -rhs eq)

radicalSolve(eq: EQ RF,v:SY) ==
   radicalSolve(lhs eq - rhs eq,v)

radicalRoots(lp: L RF,lv: L SY) ==
  parRes:=triangularSystems(lp,lv)$SystemSolvePackage(R)
  parRes= list [] => []
   -- select the components in "generic" form
  rlv:=reverse lv
  rpRes:=[reverse res for res in parRes]
  listGen:= [res for res in rpRes|isGeneric?(res,rlv)]
  result:L L RE:=[]
  if listGen^=[] then
    result:="append"/[findGenZeros(res,rlv) for res in listGen]
    for res in listGen repeat
        rpRes:=delete(rpRes,position(res,rpRes))
   --  non-generic components
  rpRes = [] => result
  append("append"/[findZeros(res,rlv) for res in rpRes],
                result)

radicalSolve(lp:L RF,lv:L SY) ==
  [makeEq(lres,lv) for lres in radicalRoots(lp,lv)]

radicalSolve(lp: L RF) ==
  lv:="setUnion"/[setUnion(variables numer p,variables denom p)
                  for p in lp]
  [makeEq(lres,lv) for lres in radicalRoots(lp,lv)]

radicalSolve(le:L EQ RF,lv:L SY) ==
  lp:=[rhs p -lhs p for p in le]
  [makeEq(lres,lv) for lres in radicalRoots(lp,lv)]

radicalSolve(le: L EQ RF) ==
  lp:=[rhs p -lhs p for p in le]
```

```
      lv:="setUnion"/[setUnion(variables numer p,variables denom p)
                      for p in lp]
      [makeEq(lres,lv) for lres in radicalRoots(lp,lv)]

    contractSolve(eq:EQ RF, v:SY)==
       solveInner(numer(lhs eq - rhs eq), v, true)

    contractSolve(pq:RF, v:SY) == solveInner(numer pq, v, true)

    radicalRoots(pq:RF, v:SY) == lhs solveInner(numer pq, v, false)


   -- test if the ideal is radical in generic position --
    isGeneric?(rlp:L PR,rlv:L SY) : Boolean ==
      "and"/[degree(f,x)=1 for f in rest rlp  for x in rest rlv]

    ---- select  the univariate factors
    select(lp:L L RE) : L L RE ==
      lp=[] => list []
      [:[cons(f,lsel) for lsel in select lp.rest] for f in lp.first]

    ---- Local Functions ----
   -- construct the equation
    makeEq(nres:L RE,lv:L SY) : L EQ RE ==
      [equation(x :: RE,r) for x in lv for r in nres]

    solveInner(pq:PR,v:SY,contractFlag:B) ==
        SideEquations := []
        ContractSoln  := contractFlag

        factors:= factors
           (factor pq)$MultivariateFactorize(SY,IndexedExponents SY,R,PR)

        constants:  List PR     := []
        unsolved:   List PR     := []
        solutions:  List RE     := []

        for f in factors repeat
            ff:=f.factor
            ^ member?(v, variables (ff)) =>
                constants := cons(ff, constants)
            u := univariate(ff, v)
            t := reduce u
            u := t.pol
            n := degree u
            l: List RE :=
```

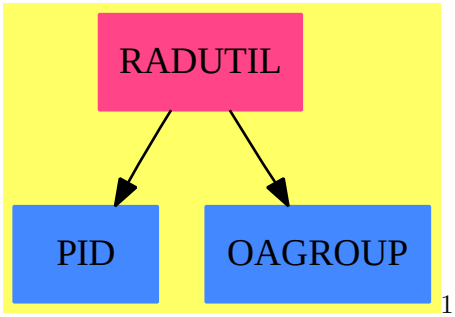
```
                    n = 1 => linear u
                    n = 2 => quadratic u
                    n = 3 => cubic u
                    n = 4 => quartic u
                    unsolved := cons(ff, unsolved)
                    []
                for s in l repeat
                    if t.deg > 1 then s := wrap s
                    T0 := expand(s, t.deg)
                    for i in 1..f.exponent repeat
                        solutions := append(T0, solutions)
                    re := SideEquations
            [solutions, SideEquations]$SU
```

⟨*SOLVERAD.dotabb*⟩≡

```
"SOLVERAD" [color="#FF4488",href="bookvol10.4.pdf#nameddest=SOLVERAD"]
"ACFS" [color="#4488FF",href="bookvol10.2.pdf#nameddest=ACFS"]
"SOLVERAD" -> "ACFS"
```

19.5 package RADUTIL RadixUtilities

19.6 RadixUtilities



1.00

Exports:
radix

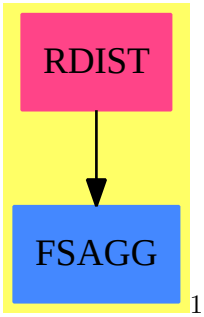
⟨*package RADUTIL RadixUtilities*⟩≡
```
)abbrev package RADUTIL RadixUtilities
++ Author: Stephen M. Watt
++ Date Created: October 1986
++ Date Last Updated: May 15, 1991
++ Basic Operations:
++ Related Domains: RadixExpansion
++ Also See:
++ AMS Classifications:
++ Keywords: radix, base, repeading decimal
++ Examples:
++ References:
++ Description:
++   This package provides tools for creating radix expansions.
RadixUtilities: Exports == Implementation where
  Exports ==> with
    radix: (Fraction Integer,Integer) -> Any
      ++ radix(x,b) converts x to a radix expansion in base b.
  Implementation ==> add
    radix(q, b) ==
      coerce(q :: RadixExpansion(b))$AnyFunctions1(RadixExpansion b)
```

⟨*RADUTIL.dotabb*⟩≡

```
"RADUTIL" [color="#FF4488",href="bookvol10.4.pdf#nameddest=RADUTIL"]
"PID" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PID"]
"OAGROUP" [color="#4488FF",href="bookvol10.2.pdf#nameddest=OAGROUP"]
"RADUTIL" -> "PID"
"RADUTIL" -> "OAGROUP"
```

19.7 package RDIST RandomDistributions

19.8 RandomDistributions



1.00

Exports:
uniform rdHack1 weighted

⟨*package RDIST RandomDistributions*⟩≡

```
)abbrev package RDIST RandomDistributions
++ Description:
++ This package exports random distributions
RandomDistributions(S: SetCategory): with
        uniform:  Set S -> (() -> S)
                ++ uniform(s) \undocumented
        weighted: List Record(value: S, weight: Integer) -> (()->S)
                ++ weighted(l) \undocumented
        rdHack1:  (Vector S,Vector Integer,Integer)->(()->S)
                ++ rdHack1(v,u,n) \undocumented
    == add
        import RandomNumberSource()

        weighted lvw ==
            -- Collapse duplicates, adding weights.
            t: Table(S, Integer) := table()
            for r in lvw repeat
                u := search(r.value,t)
                w := (u case "failed" => 0; u::Integer)
                t r.value := w + r.weight

            -- Construct vectors of values and cumulative weights.
            kl := keys t
            n  := (#kl)::NonNegativeInteger
            n = 0 => error "Cannot select from empty set"
            kv: Vector(S)       := new(n, kl.0)
            wv: Vector(Integer) := new(n, 0)
```

```
            totwt: Integer := 0
            for k in kl for i in 1..n repeat
                kv.i := k
                totwt:= totwt + t k
                wv.i := totwt

            -- Function to generate an integer and lookup.
            rdHack1(kv, wv, totwt)

        rdHack1(kv, wv, totwt) ==
            w := randnum totwt
            -- do binary search in wv
            kv.1

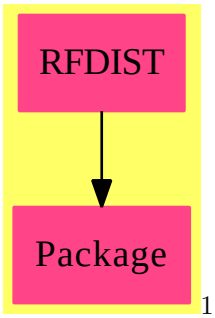
        uniform fset ==
            l := members fset
            n := #l
            l.(randnum(n)+1)
```

⟨*RDIST.dotabb*⟩≡

```
"RDIST" [color="#FF4488",href="bookvol10.4.pdf#nameddest=RDIST"]
"FSAGG" [color="#4488FF",href="bookvol10.2.pdf#nameddest=FSAGG"]
"RDIST" -> "FSAGG"
```

19.9 package RFDIST RandomFloatDistributions

19.10 RandomFloatDistributions



1.00

Exports:

Beta	chiSquare	chiSquare1	exponential	exponential1
F	normal	t	normal01	uniform01
uniform				

⟨*package RFDIST RandomFloatDistributions*⟩≡

```
)abbrev package RFDIST RandomFloatDistributions
++ Description:
++ This package exports random floating-point distributions
RationalNumber==> Fraction Integer
RandomFloatDistributions(): Cat == Body where
    NNI ==> NonNegativeInteger

    Cat ==> with
        uniform01:   ()  -> Float
                ++ uniform01() \undocumented
        normal01:    ()  -> Float
                ++ normal01() \undocumented
        exponential1:()  -> Float
                ++ exponential1() \undocumented
        chiSquare1:  NNI -> Float
                ++ chiSquare1(n) \undocumented

        uniform:     (Float, Float) -> (() -> Float)
                ++ uniform(f,g) \undocumented
        normal:      (Float, Float) -> (() -> Float)
                ++ normal(f,g) \undocumented
        exponential: (Float)        -> (() -> Float)
                ++ exponential(f) \undocumented
        chiSquare:   (NNI)          -> (() -> Float)
```

```
            ++ chiSquare(n) \undocumented
        Beta:       (NNI, NNI)     -> (() -> Float)
            ++ Beta(n,m) \undocumented
        F:          (NNI, NNI)     -> (() -> Float)
            ++ F(n,m) \undocumented
        t:          (NNI)          -> (() -> Float)
            ++ t(n) \undocumented


    Body ==> add
        import RandomNumberSource()
--      FloatPackage0()

        -- random()  generates numbers in 0..rnmax
        rnmax := (size()$RandomNumberSource() - 1)::Float

        uniform01() ==
            randnum()::Float/rnmax
        uniform(a,b) ==
            a + uniform01()*(b-a)

        exponential1() ==
            u: Float := 0
            -- This test should really be  u < m where m is
            -- the minumum acceptible argument to log.
            while u = 0 repeat u := uniform01()
            - log u
        exponential(mean) ==
            mean*exponential1()

        -- This method is correct but slow.
        normal01() ==
            s := 2::Float
            while s >= 1 repeat
                v1 := 2 * uniform01() - 1
                v2 := 2 * uniform01() - 1
                s  := v1**2 + v2**2
            v1 * sqrt(-2 * log s/s)
        normal(mean, stdev) ==
            mean + stdev*normal01()

        chiSquare1 dgfree ==
            x: Float := 0
            for i in 1..dgfree quo 2 repeat
                x := x + 2*exponential1()
            if odd? dgfree then
```

```
            x := x + normal01()**2
        x
    chiSquare dgfree ==
        chiSquare1 dgfree

    Beta(dgfree1, dgfree2) ==
        y1 := chiSquare1 dgfree1
        y2 := chiSquare1 dgfree2
        y1/(y1 + y2)

    F(dgfree1, dgfree2) ==
        y1 := chiSquare1 dgfree1
        y2 := chiSquare1 dgfree2
        (dgfree2 * y1)/(dgfree1 * y2)

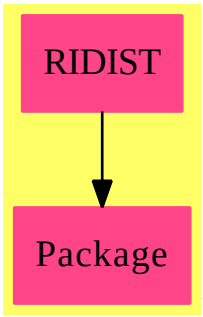
    t dgfree ==
        n := normal01()
        d := chiSquare1(dgfree) / (dgfree::Float)
        n / sqrt d
```

⟨*RFDIST.dotabb*⟩≡

```
"RFDIST" [color="#FF4488",href="bookvol10.4.pdf#nameddest=RFDIST"]
"Package" [color="#FF4488"]
"RFDIST" -> "Package"
```

19.11 package RIDIST RandomIntegerDistributions

19.12 RandomIntegerDistributions



1.00

Exports:
binomial geometric poisson ridHack1 uniform

⟨*package RIDIST RandomIntegerDistributions*⟩≡
```
)abbrev package RIDIST RandomIntegerDistributions
++ Description:
++ This package exports integer distributions
RandomIntegerDistributions(): with
   uniform:   Segment Integer           -> (() -> Integer)
     ++ uniform(s) as
     ++     l + u0 + w*u1 + w**2*u2 +...+ w**(n-1)*u-1 + w**n*m
     ++ where
     ++     s = a..b
     ++     l = min(a,b)
     ++     m = abs(b-a) + 1
     ++     w**n < m < w**(n+1)
     ++     u0,...,un-1  are uniform on  0..w-1
     ++     m            is  uniform on  0..(m quo w**n)-1
   binomial:  (Integer, RationalNumber) -> (() -> Integer)
     ++ binomial(n,f) \undocumented
   poisson:   RationalNumber          -> (() -> Integer)
     ++ poisson(f) \undocumented
   geometric: RationalNumber          -> (() -> Integer)
     ++ geometric(f) \undocumented
   ridHack1:  (Integer,Integer,Integer,Integer) -> Integer
     ++ ridHack1(i,j,k,l) \undocumented
  == add
    import RandomNumberSource()
    import IntegerBits()
```

```
    uniform aTob ==
        a := lo aTob;  b := hi aTob
        l := min(a,b); m := abs(a-b) + 1

        w := 2**(bitLength size() quo 2)::NonNegativeInteger

        n  := 0
        mq := m  -- m quo w**n
        while (mqnext := mq quo w) > 0 repeat
            n  := n + 1
            mq := mqnext
        ridHack1(mq, n, w, l)

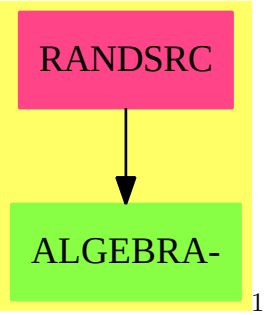
    ridHack1(mq, n, w, l) ==
        r := randnum mq
        for i in 1..n repeat r := r*w + randnum w
        r + l
```

⟨*RIDIST.dotabb*⟩≡

```
"RIDIST" [color="#FF4488",href="bookvol10.4.pdf#nameddest=RIDIST"]
"Package" [color="#FF4488"]
"RIDIST" -> "Package"
```

19.13 package RANDSRC RandomNumberSource

19.14 RandomNumberSource



1.00

Exports:
randnum reseed seed size

⟨*package RANDSRC RandomNumberSource*⟩≡
```
)abbrev package RANDSRC RandomNumberSource
++ Author:S.M.Watt
++ Date Created: April 87
++ Date Last Updated:Jan 92, May 1995 (MCD)
++ Basic Operations:
++ Related Domains:
++ Also See:
++ AMS Classifications:
++ Keywords:
++ Examples:
++ References:
++ Description:Random number generators
--% RandomNumberSource
++  All random numbers used in the system should originate from
++  the same generator.  This package is intended to be the source.
--
--  Possible improvements:
--  1) Start where the user left off
--  2) Be able to switch between methods in the random number source.
RandomNumberSource(): with
    -- If r := randnum() then  0 <= r < size().
        randnum:  () -> Integer
           ++ randnum() is a random number between 0 and size().
    --   If r := randnum() then  0 <= r < size().
        size:     () -> Integer
          ++ size() is the base of the random number generator
```

```
        -- If r := randnum n and n <= size()  then  0 <= r < n.
        randnum:  Integer -> Integer
           ++ randnum(n) is a random number between 0 and n.
        reseed:   Integer -> Void
           ++ reseed(n) restarts the random number generator at n.
        seed : () -> Integer
           ++ seed() returns the current seed value.

    == add
        -- This random number generator passes the spectral test
        -- with flying colours. [Knuth vol2, 2nd ed, p105]
        ranbase: Integer := 2**31-1
        x0:   Integer := 1231231231
        x1:   Integer := 3243232987

        randnum() ==
            t := (271828183 * x1 - 314159269 * x0) rem ranbase
            if t < 0 then t := t + ranbase
            x0:= x1
            x1:= t

        size() == ranbase
        reseed n ==
            x0 := n rem ranbase
            -- x1 := (n quo ranbase) rem ranbase
            x1 := n quo ranbase

        seed() == x1*ranbase + x0

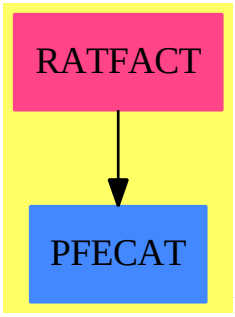
        -- Compute an integer in 0..n-1.
        randnum n ==
            (n * randnum()) quo ranbase
```

⟨*RANDSRC.dotabb*⟩≡

```
"RANDSRC" [color="#FF4488",href="bookvol10.4.pdf#nameddest=RANDSRC"]
"ALGEBRA-" [color="#88FF44",href="bookvol10.3.pdf#nameddest=ALGEBRA"]
"RANDSRC" -> "ALGEBRA-"
```

19.15 package RATFACT RationalFactorize

19.16 RationalFactorize



1.00

Exports:
factor factorSquareFree

⟨*package RATFACT RationalFactorize*⟩≡
```
)abbrev package RATFACT RationalFactorize
++ Author: P. Gianni
++ Date created: ??
++ Date last updated: December 1993
++ Factorization of extended polynomials with rational coefficients.
++ This package implements factorization of extended polynomials
++ whose coefficients are rational numbers. It does this by taking the
++ lcm of the coefficients of the polynomial and creating a polynomial
++ with integer coefficients. The algorithm in \spadtype{GaloisGroupFactorizer} i
++ used to factor the integer polynomial. The result is normalized
++ with respect to the original lcm of the denominators.
++ Keywords: factorization, hensel, rational number
I  ==> Integer
RN ==> Fraction Integer

RationalFactorize(RP) : public == private where
  BP ==> SparseUnivariatePolynomial(I)
  RP : UnivariatePolynomialCategory RN

  public   ==> with

     factor           : RP ->  Factored RP
      ++ factor(p) factors an extended polynomial p over the rational numbers.
     factorSquareFree : RP -> Factored RP
      ++ factorSquareFree(p) factors an extended squareFree
      ++ polynomial p over the rational numbers.
```

```
private  ==> add
   import GaloisGroupFactorizer (BP)
   ParFact   ==> Record(irr:BP,pow:I)
   FinalFact ==> Record(contp:I,factors:List(ParFact))
   URNI      ==> UnivariatePolynomialCategoryFunctions2(RN,RP,I,BP)
   UIRN      ==> UnivariatePolynomialCategoryFunctions2(I,BP,RN,RP)
   fUnion    ==> Union("nil", "sqfr", "irred", "prime")
   FFE       ==> Record(flg:fUnion, fctr:RP, xpnt:I)

   factor(p:RP) : Factored(RP) ==
     p = 0 => 0
     pden: I := lcm([denom c for c in coefficients p])
     pol : RP := pden*p
     ipol: BP := map(numer,pol)$URNI
     ffact: FinalFact := henselFact(ipol,false)
     makeFR(((ffact.contp)/pden)::RP,
       [["prime",map(coerce,u.irr)$UIRN,u.pow]$FFE
                          for u in ffact.factors])

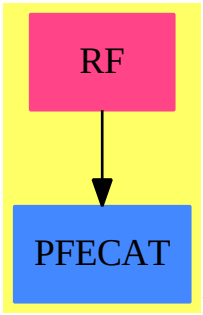
   factorSquareFree(p:RP) : Factored(RP) ==
     p = 0 => 0
     pden: I := lcm([denom c for c in coefficients p])
     pol : RP := pden*p
     ipol: BP := map(numer,pol)$URNI
     ffact: FinalFact := henselFact(ipol,true)
     makeFR(((ffact.contp)/pden)::RP,
       [["prime",map(coerce,u.irr)$UIRN,u.pow]$FFE
                          for u in ffact.factors])
```

⟨*RATFACT.dotabb*⟩≡

```
"RATFACT" [color="#FF4488",href="bookvol10.4.pdf#nameddest=RATFACT"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"RATFACT" -> "PFECAT"
```

19.17 package RF RationalFunction

19.18 RationalFunction



1.00

Exports:
coerce eval mainVariable multivariate univariate variables

⟨*package RF RationalFunction*⟩≡
```
)abbrev package RF RationalFunction
++ Top-level manipulations of rational functions
++ Author: Manuel Bronstein
++ Date Created: 1987
++ Date Last Updated: 18 April 1991
++ Description:
++   Utilities that provide the same top-level manipulations on
++   fractions than on polynomials.
++ Keywords: polynomial, fraction
-- Do not make into a domain!
RationalFunction(R:IntegralDomain): Exports == Implementation where
  V  ==> Symbol
  P  ==> Polynomial R
  Q  ==> Fraction P
  QF ==> PolynomialCategoryQuotientFunctions(IndexedExponents Symbol,
                                                    Symbol, R, P, Q)

  Exports ==> with
    variables   : Q -> List V
      ++ variables(f) returns the list of variables appearing
      ++ in the numerator or the denominator of f.
    mainVariable: Q -> Union(V, "failed")
      ++ mainVariable(f) returns the highest variable appearing
      ++ in the numerator or the denominator of f, "failed" if
      ++ f has no variables.
    univariate  : (Q, V) -> Fraction SparseUnivariatePolynomial Q
      ++ univariate(f, v) returns f viewed as a univariate
```

```
      ++ rational function in v.
    multivariate: (Fraction SparseUnivariatePolynomial Q, V) -> Q
      ++ multivariate(f, v) applies both the numerator and
      ++ denominator of f to v.
    eval        : (Q, V, Q) -> Q
      ++ eval(f, v, g) returns f with v replaced by g.
    eval        : (Q, List V, List Q) -> Q
      ++ eval(f, [v1,...,vn], [g1,...,gn]) returns f with
      ++ each vi replaced by gi in parallel, i.e. vi's appearing
      ++ inside the gi's are not replaced.
    eval        : (Q, Equation Q) -> Q
      ++ eval(f, v = g) returns f with v replaced by g.
      ++ Error: if v is not a symbol.
    eval        : (Q, List Equation Q) -> Q
      ++ eval(f, [v1 = g1,...,vn = gn]) returns f with
      ++ each vi replaced by gi in parallel, i.e. vi's appearing
      ++ inside the gi's are not replaced.
      ++ Error: if any vi is not a symbol.
    coerce      : R -> Q
      ++ coerce(r) returns r viewed as a rational function over R.

  Implementation ==> add
    foo  : (List V, List Q, V) -> Q
    peval: (P, List V, List Q) -> Q

    coerce(r:R):Q            == r::P::Q
    variables f              == variables(f)$QF
    mainVariable f           == mainVariable(f)$QF
    univariate(f, x)         == univariate(f, x)$QF
    multivariate(f, x)       == multivariate(f, x)$QF
    eval(x:Q, s:V, y:Q)      == eval(x, [s], [y])
    eval(x:Q, eq:Equation Q) == eval(x, [eq])
    foo(ls, lv, x)           == match(ls, lv, x, x::Q)$ListToMap(V, Q)

    eval(x:Q, l:List Equation Q) ==
      eval(x, [retract(lhs eq)@V for eq in l]$List(V),
               [rhs eq for eq in l]$List(Q))

    eval(x:Q, ls:List V, lv:List Q) ==
      peval(numer x, ls, lv) / peval(denom x, ls, lv)

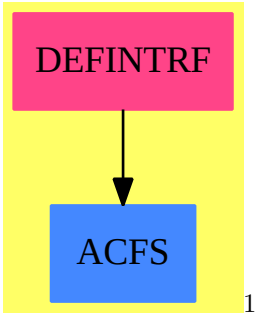
    peval(p, ls, lv) ==
      map(foo(ls, lv, #1), #1::Q,
          p)$PolynomialCategoryLifting(IndexedExponents V,V,R,P,Q)
```

⟨*RF.dotabb*⟩≡

```
"RF" [color="#FF4488",href="bookvol10.4.pdf#nameddest=RF"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"RF" -> "PFECAT"
```

19.19 package DEFINTRF RationalFunctionDefiniteIntegration

19.20 RationalFunctionDefiniteIntegration



1.00

Exports:
integrate

⟨*package DEFINTRF RationalFunctionDefiniteIntegration*⟩≡

```
)abbrev package DEFINTRF RationalFunctionDefiniteIntegration
++ Definite integration of rational functions.
++ Author: Manuel Bronstein
++ Date Created: 2 October 1989
++ Date Last Updated: 2 February 1993
++ Description:
++   \spadtype{RationalFunctionDefiniteIntegration} provides functions to
++   compute definite integrals of rational functions.


RationalFunctionDefiniteIntegration(R): Exports == Implementation where
  R : Join(EuclideanDomain, OrderedSet, CharacteristicZero,
           RetractableTo Integer, LinearlyExplicitRingOver Integer)

  SE  ==> Symbol
  RF  ==> Fraction Polynomial R
  FE  ==> Expression R
  ORF ==> OrderedCompletion RF
  OFE ==> OrderedCompletion FE
  U   ==> Union(f1:OFE, f2:List OFE, fail:"failed", pole:"potentialPole")

  Exports ==> with
    integrate: (RF, SegmentBinding OFE) -> U
      ++ integrate(f, x = a..b) returns the integral of
      ++ \spad{f(x)dx} from a to b.
      ++ Error: if f has a pole for x between a and b.
```

```
    integrate: (RF, SegmentBinding OFE, String) -> U
      ++ integrate(f, x = a..b, "noPole") returns the
      ++ integral of \spad{f(x)dx} from a to b.
      ++ If it is not possible to check whether f has a pole for x
      ++ between a and b (because of parameters), then this function
      ++ will assume that f has no such pole.
      ++ Error: if f has a pole for x between a and b or
      ++ if the last argument is not "noPole".
  -- the following two are contained in the above, but they are for the
  -- interpreter... DO NOT COMMENT OUT UNTIL THE INTERPRETER IS BETTER!
    integrate: (RF, SegmentBinding ORF) -> U
      ++ integrate(f, x = a..b) returns the integral of
      ++ \spad{f(x)dx} from a to b.
      ++ Error: if f has a pole for x between a and b.
    integrate: (RF, SegmentBinding ORF, String) -> U
      ++ integrate(f, x = a..b, "noPole") returns the
      ++ integral of \spad{f(x)dx} from a to b.
      ++ If it is not possible to check whether f has a pole for x
      ++ between a and b (because of parameters), then this function
      ++ will assume that f has no such pole.
      ++ Error: if f has a pole for x between a and b or
      ++ if the last argument is not "noPole".

  Implementation ==> add
    import DefiniteIntegrationTools(R, FE)
    import IntegrationResultRFToFunction(R)
    import OrderedCompletionFunctions2(RF, FE)

    int   : (RF, SE, OFE, OFE, Boolean) -> U
    nopole: (RF, SE, OFE, OFE) -> U

    integrate(f:RF, s:SegmentBinding OFE) ==
      int(f, variable s, lo segment s, hi segment s, false)

    nopole(f, x, a, b) ==
      k := kernel(x)@Kernel(FE)
      (u := integrate(f, x)) case FE =>
        (v := computeInt(k, u::FE, a, b, true)) case "failed" => ["failed"]
        [v::OFE]
      ans := empty()$List(OFE)
      for g in u::List(FE) repeat
        (v := computeInt(k, g, a, b, true)) case "failed" => return ["failed"]
        ans := concat_!(ans, [v::OFE])
      [ans]

    integrate(f:RF, s:SegmentBinding ORF) ==
```

```
    int(f, variable s, map(#1::FE, lo segment s),
                       map(#1::FE, hi segment s), false)

  integrate(f:RF, s:SegmentBinding ORF, str:String) ==
    int(f, variable s, map(#1::FE, lo segment s),
                       map(#1::FE, hi segment s), ignore? str)

  integrate(f:RF, s:SegmentBinding OFE, str:String) ==
    int(f, variable s, lo segment s, hi segment s, ignore? str)

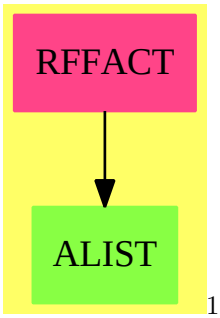
  int(f, x, a, b, ignor?) ==
    a = b => [0::OFE]
    (z := checkForZero(denom f, x, a, b, true)) case "failed" =>
      ignor? => nopole(f, x, a, b)
      ["potentialPole"]
    z::Boolean => error "integrate: pole in path of integration"
    nopole(f, x, a, b)
```

⟨*DEFINTRF.dotabb*⟩≡

```
"DEFINTRF" [color="#FF4488",href="bookvol10.4.pdf#nameddest=DEFINTRF"]
"ACFS" [color="#4488FF",href="bookvol10.2.pdf#nameddest=ACFS"]
"DEFINTRF" -> "ACFS"
```

19.21 package RFFACT RationalFunctionFactor

19.22 RationalFunctionFactor



1.00

Exports:
factor

⟨*package RFFACT RationalFunctionFactor*⟩≡

```
)abbrev package RFFACT RationalFunctionFactor
++ Factorisation in UP FRAC POLY INT
++ Author: Patrizia Gianni
++ Date Created: ???
++ Date Last Updated: ???
++ Description:
++ Factorization of univariate polynomials with coefficients which
++ are rational functions with integer coefficients.

RationalFunctionFactor(UP): Exports == Implementation where
  UP: UnivariatePolynomialCategory Fraction Polynomial Integer

  SE ==> Symbol
  P  ==> Polynomial Integer
  RF ==> Fraction P
  UPCF2 ==> UnivariatePolynomialCategoryFunctions2

  Exports ==> with
    factor: UP -> Factored UP
      ++ factor(p) returns a prime factorisation of p.

  Implementation ==> add
    likuniv: (P, SE, P) -> UP

    dummy := new()$SE
```

```
likuniv(p, x, d) ==
  map(#1 / d, univariate(p, x))$UPCF2(P,SparseUnivariatePolynomial P,
                                      RF, UP)

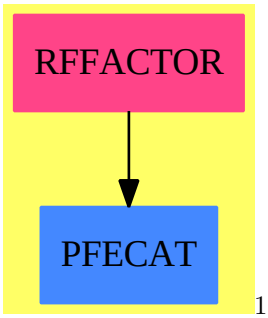
factor p ==
  d  := denom(q := elt(p,dummy::P :: RF))
  map(likuniv(#1,dummy,d),
      factor(numer q)$MultivariateFactorize(SE,
           IndexedExponents SE,Integer,P))$FactoredFunctions2(P, UP)
```

⟨*RFFACT.dotabb*⟩≡

```
"RFFACT" [color="#FF4488",href="bookvol10.4.pdf#nameddest=RFFACT"]
"ALIST" [color="#88FF44",href="bookvol10.3.pdf#nameddest=ALIST"]
"RFFACT" -> "ALIST"
```

19.23 package RFFACTOR RationalFunctionFactorizer

19.24 RationalFunctionFactorizer



1.00

Exports:
factor

⟨*package RFFACTOR RationalFunctionFactorizer*⟩≡

```
)abbrev package RFFACTOR RationalFunctionFactorizer
++ Author: P. Gianni
++ Date Created:
++ Date Last Updated: March 1995
++ Basic Functions:
++ Related Constructors: Fraction, Polynomial
++ Also See:
++ AMS Classifications:
++ Keywords:
++ References:
++ Description:
++ \spadtype{RationalFunctionFactorizer} contains the factor function
++ (called factorFraction) which factors fractions of polynomials by factoring
++ the numerator and denominator. Since any non zero fraction is a unit
++ the usual factor operation will just return the original fraction.

RationalFunctionFactorizer(R) : C == T
 where
  R  :    EuclideanDomain  -- R with factor for R[X]
  P  ==>  Polynomial R
  FP ==>  Fraction P
  SE ==>  Symbol

  C  == with
     factorFraction    : FP  ->   Fraction Factored(P)
       ++ factorFraction(r) factors the numerator and the denominator of
```

```
      ++ the polynomial fraction r.
  T  == add

    factorFraction(p:FP) : Fraction Factored(P) ==
      R is Fraction Integer =>
        MR:=MRationalFactorize(IndexedExponents SE,SE,
                               Integer,P)
        (factor(numer p)$MR)/ (factor(denom p)$MR)

      R has FiniteFieldCategory =>
        FF:=MultFiniteFactorize(SE,IndexedExponents SE,R,P)
        (factor(numer p))$FF/(factor(denom p))$FF

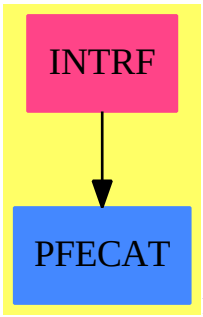
      R has CharacteristicZero =>
         MFF:=MultivariateFactorize(SE,IndexedExponents SE,R,P)
         (factor(numer p))$MFF/(factor(denom p))$MFF
      error "case not handled"
```

⟨*RFFACTOR.dotabb*⟩≡

```
"RFFACTOR" [color="#FF4488",href="bookvol10.4.pdf#nameddest=RFFACTOR"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"RFFACTOR" -> "PFECAT"
```

19.25 package INTRF RationalFunctionIntegration

19.26 RationalFunctionIntegration



1.00

Exports:
extendedIntegrate infieldIntegrate internalIntegrate limitedIntegrate

⟨*package INTRF RationalFunctionIntegration*⟩≡

```
)abbrev package INTRF RationalFunctionIntegration
++ Integration of rational functions
++ Author: Manuel Bronstein
++ Date Created: 1987
++ Date Last Updated: 29 Mar 1990
++ Keywords: polynomial, fraction, integration.
++ Description:
++   This package provides functions for the integration
++   of rational functions.
++ Examples: )r INTRF INPUT
RationalFunctionIntegration(F): Exports == Implementation where
  F: Join(IntegralDomain, RetractableTo Integer, CharacteristicZero)

  SE  ==> Symbol
  P   ==> Polynomial F
  Q   ==> Fraction P
  UP  ==> SparseUnivariatePolynomial Q
  QF  ==> Fraction UP
  LGQ ==> List Record(coeff:Q, logand:Q)
  UQ  ==> Union(Record(ratpart:Q, coeff:Q), "failed")
  ULQ ==> Union(Record(mainpart:Q, limitedlogs:LGQ), "failed")

  Exports ==> with
    internalIntegrate: (Q, SE) -> IntegrationResult Q
       ++ internalIntegrate(f, x) returns g such that \spad{dg/dx = f}.
    infieldIntegrate : (Q, SE) -> Union(Q, "failed")
```

```
        ++ infieldIntegrate(f, x) returns a fraction
        ++ g such that \spad{dg/dx = f}
        ++ if g exists, "failed" otherwise.
      limitedIntegrate : (Q, SE, List Q) -> ULQ
        ++ \spad{limitedIntegrate(f, x, [g1,...,gn])} returns fractions
        ++ \spad{[h, [[ci,gi]]]} such that the gi's are among
        ++ \spad{[g1,...,gn]},
        ++ \spad{dci/dx = 0}, and  \spad{d(h + sum(ci log(gi)))/dx = f}
        ++ if possible, "failed" otherwise.
      extendedIntegrate: (Q, SE, Q) -> UQ
        ++ extendedIntegrate(f, x, g) returns fractions \spad{[h, c]} such that
        ++ \spad{dc/dx = 0} and \spad{dh/dx = f - cg}, if \spad{(h, c)} exist,
        ++ "failed" otherwise.

    Implementation ==> add
      import RationalIntegration(Q, UP)
      import IntegrationResultFunctions2(QF, Q)
      import PolynomialCategoryQuotientFunctions(IndexedExponents SE,
                                                 SE, F, P, Q)

      infieldIntegrate(f, x) ==
        map(multivariate(#1, x), infieldint univariate(f, x))

      internalIntegrate(f, x) ==
        map(multivariate(#1, x), integrate univariate(f, x))

      extendedIntegrate(f, x, g) ==
        map(multivariate(#1, x),
            extendedint(univariate(f, x), univariate(g, x)))

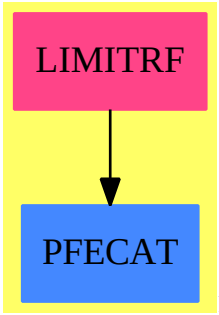
      limitedIntegrate(f, x, lu) ==
        map(multivariate(#1, x),
            limitedint(univariate(f, x), [univariate(u, x) for u in lu]))
```

⟨*INTRF.dotabb*⟩≡

```
"INTRF" [color="#FF4488",href="bookvol10.4.pdf#nameddest=INTRF"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"INTRF" -> "PFECAT"
```

19.27 package LIMITRF RationalFunctionLimitPackage

19.28 RationalFunctionLimitPackage



1.00

Exports:
complexLimit limit

⟨*package LIMITRF RationalFunctionLimitPackage*⟩≡

```
)abbrev package LIMITRF RationalFunctionLimitPackage
++ Computation of limits for rational functions
++ Author: Manuel Bronstein
++ Date Created: 4 October 1989
++ Date Last Updated: 26 November 1991
++ Description: Computation of limits for rational functions.
++ Keywords: limit, rational function.
RationalFunctionLimitPackage(R:GcdDomain):Exports==Implementation where
  Z       ==> Integer
  P       ==> Polynomial R
  RF      ==> Fraction P
  EQ      ==> Equation
  ORF     ==> OrderedCompletion RF
  OPF     ==> OnePointCompletion RF
  UP      ==> SparseUnivariatePolynomial RF
  SE      ==> Symbol
  QF      ==> Fraction SparseUnivariatePolynomial RF
  Result  ==> Union(ORF, "failed")
  TwoSide ==> Record(leftHandLimit:Result, rightHandLimit:Result)
  U       ==> Union(ORF, TwoSide, "failed")
  RFSGN   ==> RationalFunctionSign(R)

  Exports ==> with
-- The following are the one we really want, but the interpreter cannot
-- handle them...
--  limit: (RF,EQ ORF) -> U
```

```
--  ++ limit(f(x),x,a) computes the real two-sided limit lim(x -> a,f(x))

--  complexLimit: (RF,EQ OPF) -> OPF
--  ++ complexLimit(f(x),x,a) computes the complex limit lim(x -> a,f(x))

-- ... so we replace them by the following 4:
    limit: (RF,EQ OrderedCompletion P) -> U
      ++ limit(f(x),x = a) computes the real two-sided limit
      ++ of f as its argument x approaches \spad{a}.
    limit: (RF,EQ RF) -> U
      ++ limit(f(x),x = a) computes the real two-sided limit
      ++ of f as its argument x approaches \spad{a}.
    complexLimit: (RF,EQ OnePointCompletion P) -> OPF
      ++ \spad{complexLimit(f(x),x = a)} computes the complex limit
      ++ of \spad{f} as its argument x approaches \spad{a}.
    complexLimit: (RF,EQ RF) -> OPF
      ++ complexLimit(f(x),x = a) computes the complex limit
      ++ of f as its argument x approaches \spad{a}.
    limit: (RF,EQ RF,String) -> Result
      ++ limit(f(x),x,a,"left") computes the real limit
      ++ of f as its argument x approaches \spad{a} from the left;
      ++ limit(f(x),x,a,"right") computes the corresponding limit as x
      ++ approaches \spad{a} from the right.

  Implementation ==> add
    import ToolsForSign R
    import InnerPolySign(RF, UP)
    import RFSGN
    import PolynomialCategoryQuotientFunctions(IndexedExponents SE,
                                                    SE, R, P, RF)

    finiteComplexLimit: (QF, RF) -> OPF
    finiteLimit       : (QF, RF) -> U
    fLimit            : (Z, UP, RF, Z) -> Result

-- These 2 should be exported, see comment above
    locallimit       : (RF, SE, ORF) -> U
    locallimitcomplex: (RF, SE, OPF) -> OPF

    limit(f:RF,eq:EQ RF) ==
      (xx := retractIfCan(lhs eq)@Union(SE,"failed")) case "failed" =>
        error "limit: left hand side must be a variable"
      x := xx :: SE; a := rhs eq
      locallimit(f,x,a::ORF)

    complexLimit(f:RF,eq:EQ RF) ==
```

```
    (xx := retractIfCan(lhs eq)@Union(SE,"failed")) case "failed" =>
      error "limit: left hand side must be a variable"
    x := xx :: SE; a := rhs eq
    locallimitcomplex(f,x,a::OPF)

  limit(f:RF,eq:EQ OrderedCompletion P) ==
    (p := retractIfCan(lhs eq)@Union(P,"failed")) case "failed" =>
      error "limit: left hand side must be a variable"
    (xx := retractIfCan(p)@Union(SE,"failed")) case "failed" =>
      error "limit: left hand side must be a variable"
    x := xx :: SE
    a := map(#1::RF,rhs eq)$OrderedCompletionFunctions2(P,RF)
    locallimit(f,x,a)

  complexLimit(f:RF,eq:EQ OnePointCompletion P) ==
    (p := retractIfCan(lhs eq)@Union(P,"failed")) case "failed" =>
      error "limit: left hand side must be a variable"
    (xx := retractIfCan(p)@Union(SE,"failed")) case "failed" =>
      error "limit: left hand side must be a variable"
    x := xx :: SE
    a := map(#1::RF,rhs eq)$OnePointCompletionFunctions2(P,RF)
    locallimitcomplex(f,x,a)

  fLimit(n, d, a, dir) ==
    (s := signAround(d, a, dir, sign$RFSGN)) case "failed" => "failed"
    n * (s::Z) * plusInfinity()

  finiteComplexLimit(f, a) ==
    zero?(n := (numer f) a) => 0
    zero?(d := (denom f) a) => infinity()
    (n / d)::OPF

  finiteLimit(f, a) ==
    zero?(n := (numer f) a) => 0
    zero?(d := (denom f) a) =>
      (s := sign(n)$RFSGN) case "failed" => "failed"
      rhsl := fLimit(s::Z, denom f, a, 1)
      lhsl := fLimit(s::Z, denom f, a, -1)
      rhsl case "failed" =>
        lhsl case "failed" => "failed"
        [lhsl, rhsl]
      lhsl case "failed" => [lhsl, rhsl]
      rhsl::ORF = lhsl::ORF => lhsl::ORF
      [lhsl, rhsl]
    (n / d)::ORF
```

```
locallimit(f,x,a) ==
  g := univariate(f, x)
  zero?(n := whatInfinity a) => finiteLimit(g, retract a)
  (dn := degree numer g) > (dd := degree denom g) =>
    (sn := signAround(numer g, n, sign$RFSGN)) case "failed" => "failed"
    (sd := signAround(denom g, n, sign$RFSGN)) case "failed" => "failed"
    (sn::Z) * (sd::Z) * plusInfinity()
  dn < dd => 0
  ((leadingCoefficient numer g) / (leadingCoefficient denom g))::ORF

limit(f,eq,st) ==
  (xx := retractIfCan(lhs eq)@Union(SE,"failed")) case "failed" =>
    error "limit: left hand side must be a variable"
  x := xx :: SE; a := rhs eq
  zero?(n := (numer(g := univariate(f, x))) a) => 0
  zero?(d := (denom g) a) =>
    (s := sign(n)$RFSGN) case "failed" => "failed"
    fLimit(s::Z, denom g, a, direction st)
  (n / d)::ORF

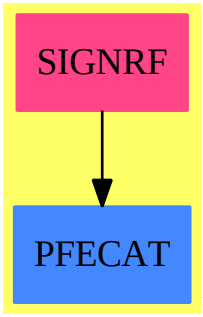
locallimitcomplex(f,x,a) ==
  g := univariate(f, x)
  (r := retractIfCan(a)@Union(RF, "failed")) case RF =>
    finiteComplexLimit(g, r::RF)
  (dn := degree numer g) > (dd := degree denom g) => infinity()
  dn < dd => 0
  ((leadingCoefficient numer g) / (leadingCoefficient denom g))::OPF
```

⟨*LIMITRF.dotabb*⟩≡

```
"LIMITRF" [color="#FF4488",href="bookvol10.4.pdf#nameddest=LIMITRF"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"LIMITRF" -> "PFECAT"
```

19.29 package SIGNRF RationalFunctionSign

19.30 RationalFunctionSign



1.00

Exports:
sign

⟨*package SIGNRF RationalFunctionSign*⟩≡

```
)abbrev package SIGNRF RationalFunctionSign
--%% RationalFunctionSign
++ Author: Manuel Bronstein
++ Date Created: 23 August 1989
++ Date Last Updated: 26 November 1991
++ Description:
++ Find the sign of a rational function around a point or infinity.
RationalFunctionSign(R:GcdDomain): Exports == Implementation where
  SE  ==> Symbol
  P   ==> Polynomial R
  RF  ==> Fraction P
  ORF ==> OrderedCompletion RF
  UP  ==> SparseUnivariatePolynomial RF
  U   ==> Union(Integer, "failed")
  SGN ==> ToolsForSign(R)

  Exports ==> with
    sign: RF -> U
      ++ sign f returns the sign of f if it is constant everywhere.
    sign: (RF, SE, ORF) -> U
      ++ sign(f, x, a) returns the sign of f as x approaches \spad{a},
      ++ from both sides if \spad{a} is finite.
    sign: (RF, SE, RF, String) -> U
      ++ sign(f, x, a, s) returns the sign of f as x nears \spad{a} from
      ++ the left (below) if s is the string \spad{"left"},
      ++ or from the right (above) if s is the string \spad{"right"}.
```

```
Implementation ==> add
  import SGN
  import InnerPolySign(RF, UP)
  import PolynomialCategoryQuotientFunctions(IndexedExponents SE,
                                                SE, R, P, RF)

  psign     : P -> U
  sqfrSign  : P -> U
  termSign  : P -> U
  listSign  : (List P, Integer) -> U
  finiteSign: (Fraction UP, RF) -> U

  sign f ==
    (un := psign numer f) case "failed" => "failed"
    (ud := psign denom f) case "failed" => "failed"
    (un::Integer) * (ud::Integer)

  finiteSign(g, a) ==
    (ud := signAround(denom g, a, sign$%)) case "failed" => "failed"
    (un := signAround(numer g, a, sign$%)) case "failed" => "failed"
    (un::Integer) * (ud::Integer)

  sign(f, x, a) ==
    g := univariate(f, x)
    zero?(n := whatInfinity a) => finiteSign(g, retract a)
    (ud := signAround(denom g, n, sign$%)) case "failed" => "failed"
    (un := signAround(numer g, n, sign$%)) case "failed" => "failed"
    (un::Integer) * (ud::Integer)

  sign(f, x, a, st) ==
    (ud := signAround(denom(g := univariate(f, x)), a,
                  d := direction st, sign$%)) case "failed" => "failed"
    (un := signAround(numer g, a, d, sign$%)) case "failed" => "failed"
    (un::Integer) * (ud::Integer)

  psign p ==
    (r := retractIfCan(p)@Union(R, "failed")) case R => sign(r::R)$SGN
    (u := sign(retract(unit(s := squareFree p))@R)$SGN) case "failed" =>
      "failed"
    ans := u::Integer
    for term in factors s | odd?(term.exponent) repeat
      (u := sqfrSign(term.factor)) case "failed" => return "failed"
      ans := ans * (u::Integer)
    ans

  sqfrSign p ==
```

```
        (u := termSign first(l := monomials p)) case "failed" => "failed"
        listSign(rest l, u::Integer)

      listSign(l, s) ==
        for term in l repeat
          (u := termSign term) case "failed" => return "failed"
          u::Integer ^= s => return "failed"
        s

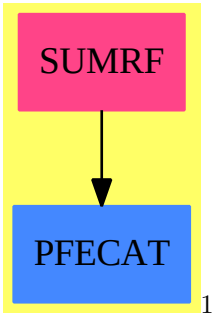
      termSign term ==
        for var in variables term repeat
          odd? degree(term, var) => return "failed"
        sign(leadingCoefficient term)$SGN
```

⟨*SIGNRF.dotabb*⟩≡

```
"SIGNRF" [color="#FF4488",href="bookvol10.4.pdf#nameddest=SIGNRF"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"SIGNRF" -> "PFECAT"
```

19.31 package SUMRF RationalFunctionSum

19.32 RationalFunctionSum



1.00

Exports:
sum

⟨*package SUMRF RationalFunctionSum*⟩≡

```
)abbrev package SUMRF RationalFunctionSum
++ Summation of rational functions
++ Author: Manuel Bronstein
++ Date Created: ???
++ Date Last Updated: 19 April 1991
++ Description: Computes sums of rational functions;
RationalFunctionSum(R): Exports == Impl where
    R: Join(IntegralDomain, OrderedSet, RetractableTo Integer)

    P   ==> Polynomial R
    RF  ==> Fraction P
    FE  ==> Expression R
    SE  ==> Symbol

    Exports ==> with
        sum: (P, SE) -> RF
            ++ sum(a(n), n) returns \spad{A} which
            ++ is the indefinite sum of \spad{a} with respect to
            ++ upward difference on \spad{n}, i.e. \spad{A(n+1) - A(n) = a(n)}.
        sum: (RF, SE) -> Union(RF, FE)
            ++ sum(a(n), n) returns \spad{A} which
            ++ is the indefinite sum of \spad{a} with respect to
            ++ upward difference on \spad{n}, i.e. \spad{A(n+1) - A(n) = a(n)}.
        sum: (P, SegmentBinding P) -> RF
            ++ sum(f(n), n = a..b) returns \spad{f(a) + f(a+1) + ... f(b)}.
        sum: (RF, SegmentBinding RF) -> Union(RF, FE)
            ++ sum(f(n), n = a..b) returns \spad{f(a) + f(a+1) + ... f(b)}.
```

```
Impl ==> add
  import RationalFunction R
  import GosperSummationMethod(IndexedExponents SE, SE, R, P, RF)

  innersum    : (RF, SE) -> Union(RF, "failed")
  innerpolysum: (P, SE) -> RF

  sum(f:RF, s:SegmentBinding RF) ==
    (indef := innersum(f, v := variable s)) case "failed" =>
      summation(f::FE,map(#1::FE,s)$SegmentBindingFunctions2(RF,FE))
    eval(indef::RF, v, 1 + hi segment s)
      - eval(indef::RF, v,lo segment s)

  sum(an:RF, n:SE) ==
    (u := innersum(an, n)) case "failed" => summation(an::FE, n)
    u::RF

  sum(p:P, s:SegmentBinding P) ==
    f := sum(p, v := variable s)
    eval(f, v, (1 + hi segment s)::RF) - eval(f,v,lo(segment s)::RF)

  innersum(an, n) ==
    (r := retractIfCan(an)@Union(P, "failed")) case "failed" =>
       an1 := eval(an, n, -1 + n::RF)
       (u := GospersMethod(an/an1, n, new$SE)) case "failed" =>
         "failed"
       an1 * eval(u::RF, n, -1 + n::RF)
    sum(r::P, n)

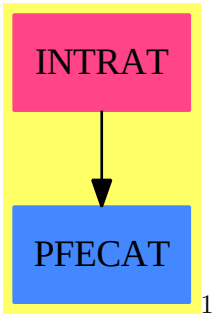
  sum(p:P, n:SE) ==
    rec := sum(p, n)$InnerPolySum(IndexedExponents SE, SE, R, P)
    rec.num / (rec.den :: P)
```

⟨*SUMRF.dotabb*⟩≡

```
"SUMRF" [color="#FF4488",href="bookvol10.4.pdf#nameddest=SUMRF"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"SUMRF" -> "PFECAT"
```

19.33 package INTRAT RationalIntegration

19.34 RationalIntegration



1.00

Exports:
extendedint infieldint integrate limitedint

⟨*package INTRAT RationalIntegration*⟩≡
```
)abbrev package INTRAT RationalIntegration
++ Rational function integration
++ Author: Manuel Bronstein
++ Date Created: 1987
++ Date Last Updated: 24 October 1995
++ Description:
++   This package provides functions for the base
++   case of the Risch algorithm.
-- Used internally bt the integration packages
RationalIntegration(F, UP): Exports == Implementation where
  F : Join(Field, CharacteristicZero, RetractableTo Integer)
  UP: UnivariatePolynomialCategory F

  RF  ==> Fraction UP
  IR  ==> IntegrationResult RF
  LLG ==> List Record(coeff:RF, logand:RF)
  URF ==> Union(Record(ratpart:RF, coeff:RF), "failed")
  U   ==> Union(Record(mainpart:RF, limitedlogs:LLG), "failed")

  Exports ==> with
    integrate  : RF -> IR
      ++ integrate(f) returns g such that \spad{g' = f}.
    infieldint : RF -> Union(RF, "failed")
      ++ infieldint(f) returns g such that \spad{g' = f} or "failed"
      ++ if the integral of f is not a rational function.
    extendedint: (RF, RF) -> URF
       ++ extendedint(f, g) returns fractions \spad{[h, c]} such that
```

```
        ++ \spad{c' = 0} and \spad{h' = f - cg},
        ++ if \spad{(h, c)} exist, "failed" otherwise.
      limitedint : (RF, List RF) -> U
        ++ \spad{limitedint(f, [g1,...,gn])} returns
        ++ fractions \spad{[h,[[ci, gi]]]}
        ++ such that the gi's are among \spad{[g1,...,gn]}, \spad{ci' = 0}, and
        ++ \spad{(h+sum(ci log(gi)))' = f}, if possible, "failed" otherwise.

    Implementation ==> add
      import TranscendentalIntegration(F, UP)

      infieldint f ==
        rec := baseRDE(0, f)$TranscendentalRischDE(F, UP)
        rec.nosol => "failed"
        rec.ans

      integrate f ==
        rec := monomialIntegrate(f, differentiate)
        integrate(rec.polypart)::RF::IR + rec.ir

      limitedint(f, lu) ==
        quorem := divide(numer f, denom f)
        (u := primlimintfrac(quorem.remainder / (denom f), differentiate,
          lu)) case "failed" => "failed"
        [u.mainpart + integrate(quorem.quotient)::RF, u.limitedlogs]

      extendedint(f, g) ==
        fqr := divide(numer f, denom f)
        gqr := divide(numer g, denom g)
        (i1 := primextintfrac(fqr.remainder / (denom f), differentiate,
                    gqr.remainder / (denom g))) case "failed" => "failed"
        i2:=integrate(fqr.quotient-retract(i1.coeff)@UP *gqr.quotient)::RF
        [i2 + i1.ratpart, i1.coeff]
```

⟨*INTRAT.dotabb*⟩≡

```
"INTRAT" [color="#FF4488",href="bookvol10.4.pdf#nameddest=INTRAT"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"INTRAT" -> "PFECAT"
```

19.35 package RINTERP RationalInterpolation

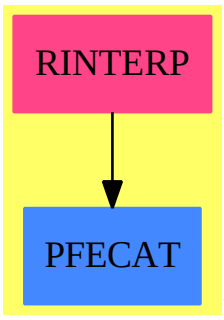
19.35.1 Introduction

This file contains a crude naïve implementation of rational interpolation, where the coefficients of the rational function are in any given field.

19.35.2 Questions and Outlook

- Maybe this file should be joined with pinterp.spad, where polynomial Lagrange interpolation is implemented. I have a second version that parallels the structure of pinterp.spad closely.
- There are probably better ways to implement rational interpolation. Maybe http://www.cs.ucsb.edu/ omer/personal/abstracts/rational.html contains something useful, but I don't know.
- Comments welcome!

19.36 RationalInterpolation



1.00

Exports:
interpolate

⟨*package RINTERP RationalInterpolation*⟩≡
```
)abbrev package RINTERP RationalInterpolation
++ Description:
++ This package exports rational interpolation algorithms
RationalInterpolation(xx,F): Exports == Implementation  where
    xx: Symbol
    F: Field
    Exports == with
        interpolate: (List F, List F, NonNegativeInteger,
                                  NonNegativeInteger) -> Fraction Polynomial F
```

The implementation sets up a system of linear equations and solves it.

⟨*package RINTERP RationalInterpolation*⟩+≡

```
Implementation == add
    interpolate(xlist, ylist, m, k) ==
```

First we check whether we have the right number of points and values. Clearly the number of points and the number of values must be identical. Note that we want to determine the numerator and denominator polynomials only up to a factor. Thus, we want to determine $m + k + 1$ coefficients, where m is the degree of the polynomial in the numerator and k is the degree of the polynomial in the denominator.

In fact, we could also leave – for example – k unspecified and determine it as $k = \#\texttt{xlist} - m - 1$: I don't know whether this would be better.

⟨*package RINTERP RationalInterpolation*⟩+≡

```
#xlist ^= #ylist =>
    error "Different number of points and values."
#xlist ^= m+k+1 =>
    error "wrong number of points"
```

The next step is to set up the matrix. Suppose that our numerator polynomial is $p(x) = a_0 + a_1 x + \ldots + a_m x^m$ and that our denominator polynomial is $q(x) = b_0 + b_1 x + \ldots + b_m x^m$. Then we have the following equations, writing n for $m + k + 1$:

$$\begin{aligned}
p(x_1) - y_1 q(x_1) &= a_0 + a_1 x_1 + \ldots + a_m x_1^m - y_1(b_0 + b_1 x_1 + \ldots + b_k x_1^k) = 0 \\
p(x_2) - y_2 q(x_2) &= a_0 + a_1 x_2 + \ldots + a_m x_2^m - y_2(b_0 + b_1 x_2 + \ldots + b_k x_2^k) = 0 \\
&\vdots \\
p(x_n) - y_n q(x_n) &= a_0 + a_1 x_n + \ldots + a_m x_n^m - y_n(b_0 + b_1 x_n + \ldots + b_k x_n^k) = 0
\end{aligned}$$

This can be written as

$$\begin{bmatrix}
1 & x_1 & \ldots & x_1^m & -y_1 & -y_1 x_1 & \ldots & -y_1 x_1^k \\
1 & x_2 & \ldots & x_2^m & -y_2 & -y_2 x_2 & \ldots & -y_2 x_2^k \\
 & & & \vdots & & & & \\
1 & x_n & \ldots & x_n^m & -y_n & -y_n x_n & \ldots & -y_n x_2^k
\end{bmatrix}
\begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_m \\ b_0 \\ b_1 \\ \vdots \\ b_k \end{bmatrix} = \mathbf{0}$$

We generate this matrix columnwise:

⟨*package RINTERP RationalInterpolation*⟩+≡

```
tempvec: List F := [1 for i in 1..(m+k+1)]

collist: List List F := cons(tempvec,
                             [(tempvec := [tempvec.i * xlist.i _
                                           for i in 1..(m+k+1)]) _
                              for j in 1..max(m,k)])

collist := append([collist.j for j in 1..(m+1)], _
                  [[- collist.j.i * ylist.i for i in 1..(m+k+1)] _
                   for j in 1..(k+1)])
```

Now we can solve the system:

⟨*package RINTERP RationalInterpolation*⟩+≡

```
res: List Vector F := nullSpace((transpose matrix collist) _
                                ::Matrix F)
```

Note that it may happen that the system has several solutions. In this case, some of the data points may not be interpolated correctly. However, the solution is often still useful, thus we do not signal an error.

⟨*package RINTERP RationalInterpolation*⟩+≡

```
if #res~=1 then output("Warning: unattainable points!" _
                       ::OutputForm)$OutputPackage
```

In this situation, all the solutions will be equivalent, thus we can always simply take the first one:

⟨*package RINTERP RationalInterpolation*⟩+≡

```
reslist: List List Polynomial F := _
          [[(res.1).(i+1)*(xx::Polynomial F)**i for i in 0..m], _
          [(res.1).(i+m+2)*(xx::Polynomial F)**i for i in 0..k]]
```

Finally, we generate the rational function:

⟨*package RINTERP RationalInterpolation*⟩+≡

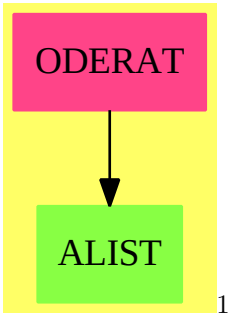
```
reduce((_+),reslist.1)/reduce((_+),reslist.2)
```

⟨*RINTERP.dotabb*⟩≡

```
"RINTERP" [color="#FF4488",href="bookvol10.4.pdf#nameddest=RINTERP"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"RINTERP" -> "PFECAT"
```

19.37 package ODERAT RationalLODE

19.38 RationalLODE



1.00

Exports:
indicialEquationAtInfinity ratDsolve

⟨*package ODERAT RationalLODE*⟩≡

```
)abbrev package ODERAT RationalLODE
++ Author: Manuel Bronstein
++ Date Created: 13 March 1991
++ Date Last Updated: 13 April 1994
++ Description:
++  \spad{RationalLODE} provides functions for in-field solutions of linear
++   ordinary differential equations, in the rational case.
RationalLODE(F, UP): Exports == Implementation where
  F  : Join(Field, CharacteristicZero, RetractableTo Integer,
                                       RetractableTo Fraction Integer)
  UP : UnivariatePolynomialCategory F

  N   ==> NonNegativeInteger
  Z   ==> Integer
  RF  ==> Fraction UP
  U   ==> Union(RF, "failed")
  V   ==> Vector F
  M   ==> Matrix F
  LODO ==> LinearOrdinaryDifferentialOperator1 RF
  LODO2==> LinearOrdinaryDifferentialOperator2(UP, RF)

  Exports ==> with
    ratDsolve: (LODO, RF) -> Record(particular: U, basis: List RF)
      ++ ratDsolve(op, g) returns \spad{["failed", []]} if the equation
      ++ \spad{op y = g} has no rational solution. Otherwise, it returns
      ++ \spad{[f, [y1,...,ym]]} where f is a particular rational solution
      ++ and the yi's form a basis for the rational solutions of the
```

```
    ++ homogeneous equation.
  ratDsolve: (LODO, List RF) -> Record(basis:List RF, mat:Matrix F)
    ++ ratDsolve(op, [g1,...,gm]) returns \spad{[[h1,...,hq], M]} such
    ++ that any rational solution of \spad{op y = c1 g1 + ... + cm gm}
    ++ is of the form \spad{d1 h1 + ... + dq hq} where
    ++ \spad{M [d1,...,dq,c1,...,cm] = 0}.
  ratDsolve: (LODO2, RF) -> Record(particular: U, basis: List RF)
    ++ ratDsolve(op, g) returns \spad{["failed", []]} if the equation
    ++ \spad{op y = g} has no rational solution. Otherwise, it returns
    ++ \spad{[f, [y1,...,ym]]} where f is a particular rational solution
    ++ and the yi's form a basis for the rational solutions of the
    ++ homogeneous equation.
  ratDsolve: (LODO2, List RF) -> Record(basis:List RF, mat:Matrix F)
    ++ ratDsolve(op, [g1,...,gm]) returns \spad{[[h1,...,hq], M]} such
    ++ that any rational solution of \spad{op y = c1 g1 + ... + cm gm}
    ++ is of the form \spad{d1 h1 + ... + dq hq} where
    ++ \spad{M [d1,...,dq,c1,...,cm] = 0}.
  indicialEquationAtInfinity: LODO -> UP
    ++ indicialEquationAtInfinity op returns the indicial equation of
    ++ \spad{op} at infinity.
  indicialEquationAtInfinity: LODO2 -> UP
    ++ indicialEquationAtInfinity op returns the indicial equation of
    ++ \spad{op} at infinity.

Implementation ==> add
  import BoundIntegerRoots(F, UP)
  import RationalIntegration(F, UP)
  import PrimitiveRatDE(F, UP, LODO2, LODO)
  import LinearSystemMatrixPackage(F, V, V, M)
  import InnerCommonDenominator(UP, RF, List UP, List RF)

  nzero?             : V -> Boolean
  evenodd            : N -> F
  UPfact             : N -> UP
  infOrder           : RF -> Z
  infTau             : (UP, N) -> F
  infBound           : (LODO2, List RF) -> N
  regularPoint       : (LODO2, List RF) -> Z
  infIndicialEquation: (List N, List UP) -> UP
  makeDot            : (Vector F, List RF) -> RF
  unitlist           : (N, N) -> List F
  infMuLambda: LODO2 -> Record(mu:Z, lambda:List N, func:List UP)
  ratDsolve0: (LODO2, RF) -> Record(particular: U, basis: List RF)
  ratDsolve1: (LODO2, List RF) -> Record(basis:List RF, mat:Matrix F)
  candidates: (LODO2,List RF,UP) -> Record(basis:List RF,particular:List RF)
```

```
dummy := new()$Symbol

infOrder f == (degree denom f) - (degree numer f)
evenodd n  == (even? n => 1; -1)

ratDsolve1(op, lg) ==
  d := denomLODE(op, lg)
  rec := candidates(op, lg, d)
  l := concat([op q for q in rec.basis],
              [op(rec.particular.i) - lg.i for i in 1..#(rec.particular)])
  sys1 := reducedSystem(matrix [l])@Matrix(UP)
  [rec.basis, reducedSystem sys1]

ratDsolve0(op, g) ==
  zero? degree op => [inv(leadingCoefficient(op)::RF) * g, empty()]
  minimumDegree op > 0 =>
    sol := ratDsolve0(monicRightDivide(op, monomial(1, 1)).quotient, g)
    b:List(RF) := [1]
    for f in sol.basis repeat
      if (uu := infieldint f) case RF then b := concat(uu::RF, b)
    sol.particular case "failed" => ["failed", b]
    [infieldint(sol.particular::RF), b]
  (u := denomLODE(op, g)) case "failed" => ["failed", empty()]
  rec := candidates(op, [g], u::UP)
  l := lb := lsol := empty()$List(RF)
  for q in rec.basis repeat
      if zero?(opq := op q) then lsol := concat(q, lsol)
      else (l := concat(opq, l); lb := concat(q, lb))
  h:RF := (zero? g => 0; first(rec.particular))
  empty? l =>
      zero? g => [0, lsol]
      [(g = op h => h; "failed"), lsol]
  m:M
  v:V
  if zero? g then
      m := reducedSystem(reducedSystem(matrix [l])@Matrix(UP))@M
      v := new(ncols m, 0)$V
  else
      sys1 := reducedSystem(matrix [l], vector [g - op h]
                           )@Record(mat: Matrix UP, vec: Vector UP)
      sys2 := reducedSystem(sys1.mat, sys1.vec)@Record(mat:M, vec:V)
      m := sys2.mat
      v := sys2.vec
  sol := solve(m, v)
  part:U :=
    zero? g => 0
```

```
    sol.particular case "failed" => "failed"
    makeDot(sol.particular::V, lb) + first(rec.particular)
  [part,
   concat_!(lsol, [makeDot(v, lb) for v in sol.basis | nzero? v])]

indicialEquationAtInfinity(op:LODO2) ==
  rec := infMuLambda op
  infIndicialEquation(rec.lambda, rec.func)

indicialEquationAtInfinity(op:LODO) ==
  rec := splitDenominator(op, empty())
  indicialEquationAtInfinity(rec.eq)

regularPoint(l, lg) ==
  a := leadingCoefficient(l) * commonDenominator lg
  coefficient(a, 0) ^= 0 => 0
  for i in 1.. repeat
    a(j := i::F) ^= 0 => return i
    a(-j) ^= 0 => return(-i)

unitlist(i, q) ==
  v := new(q, 0)$Vector(F)
  v.i := 1
  parts v

candidates(op, lg, d) ==
  n := degree d + infBound(op, lg)
  m := regularPoint(op, lg)
  uts := UnivariateTaylorSeries(F, dummy, m::F)
  tools := UTSodetools(F, UP, LODO2, uts)
  solver := UnivariateTaylorSeriesODESolver(F, uts)
  dd := UP2UTS(d)$tools
  f := LODO2FUN(op)$tools
  q := degree op
  e := unitlist(1, q)
  hom := [UTS2UP(dd * ode(f, unitlist(i, q))$solver, n)$tools /$RF d
              for i in 1..q]$List(RF)
  a1 := inv(leadingCoefficient(op)::RF)
  part := [UTS2UP(dd * ode(RF2UTS(a1 * g)$tools + f #1, e)$solver, n)$tools
            /$RF d for g in lg | g ^= 0]$List(RF)
  [hom, part]

nzero? v ==
  for i in minIndex v .. maxIndex v repeat
    not zero? qelt(v, i) => return true
  false
```

```
-- returns z(z+1)...(z+(n-1))
    UPfact n ==
      zero? n => 1
      z := monomial(1, 1)$UP
      */[z + i::F::UP for i in 0..(n-1)::N]

    infMuLambda l ==
      lamb:List(N) := [d := degree l]
      lf:List(UP) := [a := leadingCoefficient l]
      mup := degree(a)::Z - d
      while (l := reductum l) ^= 0 repeat
          a := leadingCoefficient l
          if (m := degree(a)::Z - (d := degree l)) > mup then
            mup := m
            lamb := [d]
            lf := [a]
          else if (m = mup) then
            lamb := concat(d, lamb)
            lf := concat(a, lf)
      [mup, lamb, lf]

    infIndicialEquation(lambda, lf) ==
      ans:UP := 0
      for i in lambda for f in lf repeat
        ans := ans + evenodd i * leadingCoefficient f * UPfact i
      ans

    infBound(l, lg) ==
      rec := infMuLambda l
      n := min(- degree(l)::Z - 1,
               integerBound infIndicialEquation(rec.lambda, rec.func))
      while not(empty? lg) and zero? first lg repeat lg := rest lg
      empty? lg => (-n)::N
      m := infOrder first lg
      for g in rest lg repeat
        if not(zero? g) and (mm := infOrder g) < m then m := mm
      (-min(n, rec.mu - degree(leadingCoefficient l)::Z + m))::N

    makeDot(v, bas) ==
      ans:RF := 0
      for i in 1.. for b in bas repeat ans := ans + v.i::UP * b
      ans

    ratDsolve(op:LODO, g:RF) ==
      rec := splitDenominator(op, [g])
```

```
        ratDsolve0(rec.eq, first(rec.rh))

      ratDsolve(op:LODO, lg:List RF) ==
        rec := splitDenominator(op, lg)
        ratDsolve1(rec.eq, rec.rh)

      ratDsolve(op:LODO2, g:RF) ==
        unit?(c := content op) => ratDsolve0(op, g)
        ratDsolve0((op exquo c)::LODO2, inv(c::RF) * g)

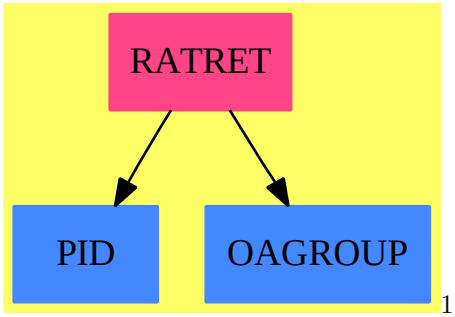
      ratDsolve(op:LODO2, lg:List RF) ==
        unit?(c := content op) => ratDsolve1(op, lg)
        ratDsolve1((op exquo c)::LODO2, [inv(c::RF) * g for g in lg])
```

⟨*ODERAT.dotabb*⟩≡

```
"ODERAT" [color="#FF4488",href="bookvol10.4.pdf#nameddest=ODERAT"]
"ALIST" [color="#88FF44",href="bookvol10.3.pdf#nameddest=ALIST"]
"ODERAT" -> "ALIST"
```

19.39 package RATRET RationalRetractions

19.40 RationalRetractions



1.00

Exports:
rational rational? rationalIfCan

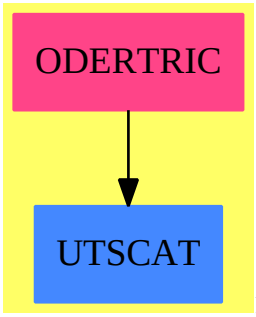
⟨*package RATRET RationalRetractions*⟩≡
```
)abbrev package RATRET RationalRetractions
++ Author: Manuel Bronstein
++ Description: rational number testing and retraction functions.
++ Date Created: March 1990
++ Date Last Updated: 9 April 1991
RationalRetractions(S:RetractableTo(Fraction Integer)): with
    rational     : S -> Fraction Integer
      ++ rational(x) returns x as a rational number;
      ++ error if x is not a rational number;
    rational?    : S -> Boolean
      ++ rational?(x) returns true if x is a rational number,
      ++ false otherwise;
    rationalIfCan: S -> Union(Fraction Integer, "failed")
      ++ rationalIfCan(x) returns x as a rational number,
      ++ "failed" if x is not a rational number;
  == add
    rational s      == retract s
    rational? s     == retractIfCan(s) case Fraction(Integer)
    rationalIfCan s == retractIfCan s
```

⟨*RATRET.dotabb*⟩≡

```
"RATRET" [color="#FF4488",href="bookvol10.4.pdf#nameddest=RATRET"]
"PID" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PID"]
"OAGROUP" [color="#4488FF",href="bookvol10.2.pdf#nameddest=OAGROUP"]
"RATRET" -> "PID"
"RATRET" -> "OAGROUP"
```

19.41 package ODERTRIC RationalRicDE

19.42 RationalRicDE



1.00

Exports:
polyRicDE ricDsolve singRicDE

⟨*package ODERTRIC RationalRicDE*⟩≡

```
)abbrev package ODERTRIC RationalRicDE
++ Author: Manuel Bronstein
++ Date Created: 22 October 1991
++ Date Last Updated: 11 April 1994
++ Description: In-field solution of Riccati equations, rational case.
RationalRicDE(F, UP): Exports == Implementation where
  F  : Join(Field, CharacteristicZero, RetractableTo Integer,
                                       RetractableTo Fraction Integer)
  UP : UnivariatePolynomialCategory F

  N   ==> NonNegativeInteger
  Z   ==> Integer
  SY  ==> Symbol
  P   ==> Polynomial F
  RF  ==> Fraction P
  EQ  ==> Equation RF
  QF  ==> Fraction UP
  UP2 ==> SparseUnivariatePolynomial UP
  SUP ==> SparseUnivariatePolynomial P
  REC ==> Record(poly:SUP, vars:List SY)
  SOL ==> Record(var:List SY, val:List F)
  POL ==> Record(poly:UP, eq:L)
  FRC ==> Record(frac:QF, eq:L)
  CNT ==> Record(constant:F, eq:L)
  UTS ==> UnivariateTaylorSeries(F, dummy, 0)
  UPS ==> SparseUnivariatePolynomial UTS
  L   ==> LinearOrdinaryDifferentialOperator2(UP, QF)
```

```
LQ  ==> LinearOrdinaryDifferentialOperator1 QF

Exports ==> with
  ricDsolve: (LQ, UP -> List F) -> List QF
    ++ ricDsolve(op, zeros) returns the rational solutions of the associated
    ++ Riccati equation of \spad{op y = 0}.
    ++ \spad{zeros} is a zero finder in \spad{UP}.
  ricDsolve: (LQ, UP -> List F, UP -> Factored UP) -> List QF
    ++ ricDsolve(op, zeros, ezfactor) returns the rational
    ++ solutions of the associated Riccati equation of \spad{op y = 0}.
    ++ \spad{zeros} is a zero finder in \spad{UP}.
    ++ Argument \spad{ezfactor} is a factorisation in \spad{UP},
    ++ not necessarily into irreducibles.
  ricDsolve: (L, UP -> List F) -> List QF
    ++ ricDsolve(op, zeros) returns the rational solutions of the associated
    ++ Riccati equation of \spad{op y = 0}.
    ++ \spad{zeros} is a zero finder in \spad{UP}.
  ricDsolve: (L, UP -> List F, UP -> Factored UP) -> List QF
    ++ ricDsolve(op, zeros, ezfactor) returns the rational
    ++ solutions of the associated Riccati equation of \spad{op y = 0}.
    ++ \spad{zeros} is a zero finder in \spad{UP}.
    ++ Argument \spad{ezfactor} is a factorisation in \spad{UP},
    ++ not necessarily into irreducibles.
  singRicDE: (L, UP -> Factored UP) -> List FRC
    ++ singRicDE(op, ezfactor) returns \spad{[[f1,L1], [f2,L2],..., [fk,Lk]]}
    ++ such that the singular ++ part of any rational solution of the
    ++ associated Riccati equation of \spad{op y = 0} must be one of the fi's
    ++ (up to the constant coefficient), in which case the equation for
    ++ \spad{z = y e^{-int ai}} is \spad{Li z = 0}.
    ++ Argument \spad{ezfactor} is a factorisation in \spad{UP},
    ++ not necessarily into irreducibles.
  polyRicDE: (L, UP -> List F) -> List POL
    ++ polyRicDE(op, zeros) returns \spad{[[p1, L1], [p2, L2], ... , [pk,Lk]]}
    ++ such that the polynomial part of any rational solution of the
    ++ associated Riccati equation of \spad{op y = 0} must be one of the pi's
    ++ (up to the constant coefficient), in which case the equation for
    ++ \spad{z = y e^{-int p}} is \spad{Li z = 0}.
    ++ \spad{zeros} is a zero finder in \spad{UP}.
  if F has AlgebraicallyClosedField then
    ricDsolve: LQ -> List QF
      ++ ricDsolve(op) returns the rational solutions of the associated
      ++ Riccati equation of \spad{op y = 0}.
    ricDsolve: (LQ, UP -> Factored UP) -> List QF
      ++ ricDsolve(op, ezfactor) returns the rational solutions of the
      ++ associated Riccati equation of \spad{op y = 0}.
      ++ Argument \spad{ezfactor} is a factorisation in \spad{UP},
```

```
      ++ not necessarily into irreducibles.
    ricDsolve: L -> List QF
      ++ ricDsolve(op) returns the rational solutions of the associated
      ++ Riccati equation of \spad{op y = 0}.
    ricDsolve: (L, UP -> Factored UP) -> List QF
      ++ ricDsolve(op, ezfactor) returns the rational solutions of the
      ++ associated Riccati equation of \spad{op y = 0}.
      ++ Argument \spad{ezfactor} is a factorisation in \spad{UP},
      ++ not necessarily into irreducibles.

  Implementation ==> add
    import RatODETools(P, SUP)
    import RationalLODE(F, UP)
    import NonLinearSolvePackage F
    import PrimitiveRatDE(F, UP, L, LQ)
    import PrimitiveRatRicDE(F, UP, L, LQ)

    FifCan           : RF -> Union(F, "failed")
    UP2SUP           : UP -> SUP
    innersol         : (List UP, Boolean) -> List QF
    mapeval          : (SUP, List SY, List F) -> UP
    ratsol           : List List EQ -> List SOL
    ratsln           : List EQ -> Union(SOL, "failed")
    solveModulo      : (UP, UP2) -> List UP
    logDerOnly       : L -> List QF
    nonSingSolve     : (N, L, UP -> List F) -> List QF
    constantRic      : (UP, UP -> List F) -> List F
    nopoly           : (N, UP, L, UP -> List F) -> List QF
    reverseUP        : UP -> UTS
    reverseUTS       : (UTS, N) -> UP
    newtonSolution   : (L, F, N, UP -> List F) -> UP
    newtonSolve      : (UPS, F, N) -> Union(UTS, "failed")
    genericPolynomial: (SY, Z) -> Record(poly:SUP, vars:List SY)
      -- genericPolynomial(s, n) returns
      -- \spad{[[s0 + s1 X +...+ sn X^n],[s0,...,sn]]}.

    dummy := new()$SY

    UP2SUP p == map(#1::P,p)$UnivariatePolynomialCategoryFunctions2(F,UP,P,SUP)
    logDerOnly l == [differentiate(s) / s for s in ratDsolve(l, 0).basis]
    ricDsolve(l:LQ, zeros:UP -> List F) == ricDsolve(l, zeros, squareFree)
    ricDsolve(l:L,  zeros:UP -> List F) == ricDsolve(l, zeros, squareFree)
    singRicDE(l, ezfactor)              == singRicDE(l, solveModulo, ezfactor)

    ricDsolve(l:LQ, zeros:UP -> List F, ezfactor:UP -> Factored UP) ==
      ricDsolve(splitDenominator(l, empty()).eq, zeros, ezfactor)
```

```
    mapeval(p, ls, lv) ==
      map(ground eval(#1, ls, lv),
          p)$UnivariatePolynomialCategoryFunctions2(P, SUP, F, UP)

    FifCan f ==
      ((n := retractIfCan(numer f))@Union(F, "failed") case F) and
        ((d := retractIfCan(denom f))@Union(F, "failed") case F) =>
           (n::F) / (d::F)
      "failed"

  -- returns [0, []] if n < 0
    genericPolynomial(s, n) ==
      ans:SUP := 0
      l:List(SY) := empty()
      for i in 0..n repeat
        ans := ans + monomial((sy := new s)::P, i::N)
        l := concat(sy, l)
      [ans, reverse_! l]

    ratsln l ==
      ls:List(SY) := empty()
      lv:List(F) := empty()
      for eq in l repeat
        ((u := FifCan rhs eq) case "failed") or
          ((v := retractIfCan(lhs eq)@Union(SY, "failed")) case "failed")
            => return "failed"
        lv := concat(u::F, lv)
        ls := concat(v::SY, ls)
      [ls, lv]

    ratsol l ==
      ans:List(SOL) := empty()
      for sol in l repeat
        if ((u := ratsln sol) case SOL) then ans := concat(u::SOL, ans)
      ans

  -- returns [] if the solutions of l have no polynomial component
    polyRicDE(l, zeros) ==
      ans:List(POL) := [[0, l]]
      empty?(lc := leadingCoefficientRicDE l) => ans
      rec := first lc                          -- one with highest degree
      for a in zeros(rec.eq) | a ^= 0 repeat
        if (p := newtonSolution(l, a, rec.deg, zeros)) ^= 0 then
            ans := concat([p, changeVar(l, p)], ans)
      ans
```

```
-- reverseUP(a_0 + a_1 x + ... + an x^n) = a_n + ... + a_0 x^n
    reverseUP p ==
      ans:UTS := 0
      n := degree(p)::Z
      while p ^= 0 repeat
        ans := ans + monomial(leadingCoefficient p, (n - degree p)::N)
        p   := reductum p
      ans

-- reverseUTS(a_0 + a_1 x + ..., n) = a_n + ... + a_0 x^n
    reverseUTS(s, n) ==
      +/[monomial(coefficient(s, i), (n - i)::N)$UP for i in 0..n]

-- returns a potential polynomial solution p with leading coefficient a*?**n
    newtonSolution(l, a, n, zeros) ==
      i:N
      m:Z := 0
      aeq:UPS := 0
      op := l
      while op ^= 0 repeat
        mu := degree(op) * n + degree leadingCoefficient op
        op := reductum op
        if mu > m then m := mu
      while l ^= 0 repeat
        c := leadingCoefficient l
        d := degree l
        s:UTS := monomial(1, (m - d * n - degree c)::N)$UTS * reverseUP c
        aeq := aeq + monomial(s, d)
        l := reductum l
      (u := newtonSolve(aeq, a, n)) case UTS => reverseUTS(u::UTS, n)
      -- newton lifting failed, so revert to traditional method
      atn := monomial(a, n)$UP
      neq := changeVar(l, atn)
      sols := [sol.poly for sol in polyRicDE(neq, zeros) | degree(sol.poly) < n]
      empty? sols => atn
      atn + first sols

-- solves the algebraic equation eq for y, returns a solution of degree n with
-- initial term a
-- uses naive newton approximation for now
-- an example where this fails is   y^2 + 2 x y + 1 + x^2 = 0
-- which arises from the differential operator D^2 + 2 x D + 1 + x^2
    newtonSolve(eq, a, n) ==
      deq := differentiate eq
      sol := a::UTS
```

```
      for i in 1..n repeat
        (xquo := eq(sol) exquo deq(sol)) case "failed" => return "failed"
        sol := truncate(sol - xquo::UTS, i)
      sol

  -- there could be the same solutions coming in different ways, so we
  -- stop when the number of solutions reaches the order of the equation
    ricDsolve(l:L, zeros:UP -> List F, ezfactor:UP -> Factored UP) ==
      n := degree l
      ans:List(QF) := empty()
      for rec in singRicDE(l, ezfactor) repeat
        ans := removeDuplicates_! concat_!(ans,
                        [rec.frac + f for f in nonSingSolve(n, rec.eq, zeros)])
        #ans = n => return ans
      ans

  -- there could be the same solutions coming in different ways, so we
  -- stop when the number of solutions reaches the order of the equation
    nonSingSolve(n, l, zeros) ==
      ans:List(QF) := empty()
      for rec in polyRicDE(l, zeros) repeat
        ans := removeDuplicates_! concat_!(ans, nopoly(n,rec.poly,rec.eq,zeros))
        #ans = n => return ans
      ans

    constantRic(p, zeros) ==
      zero? degree p => empty()
      zeros squareFreePart p

  -- there could be the same solutions coming in different ways, so we
  -- stop when the number of solutions reaches the order of the equation
    nopoly(n, p, l, zeros) ==
      ans:List(QF) := empty()
      for rec in constantCoefficientRicDE(l, constantRic(#1, zeros)) repeat
        ans := removeDuplicates_! concat_!(ans,
                  [(rec.constant::UP + p)::QF + f for f in logDerOnly(rec.eq)])
        #ans = n => return ans
      ans

  -- returns [p1,...,pn] s.t. h(x,pi(x)) = 0 mod c(x)
    solveModulo(c, h) ==
      rec := genericPolynomial(dummy, degree(c)::Z - 1)
      unk:SUP := 0
      while not zero? h repeat
        unk := unk + UP2SUP(leadingCoefficient h) * (rec.poly ** degree h)
        h   := reductum h
```

```
        sol := ratsol solve(coefficients(monicDivide(unk,UP2SUP c).remainder),
                            rec.vars)
        [mapeval(rec.poly, s.var, s.val) for s in sol]

      if F has AlgebraicallyClosedField then
        zro1: UP -> List F
        zro : (UP, UP -> Factored UP) -> List F

        ricDsolve(l:L)  == ricDsolve(l, squareFree)
        ricDsolve(l:LQ) == ricDsolve(l, squareFree)

        ricDsolve(l:L, ezfactor:UP -> Factored UP) ==
          ricDsolve(l, zro(#1, ezfactor), ezfactor)

        ricDsolve(l:LQ, ezfactor:UP -> Factored UP) ==
          ricDsolve(l, zro(#1, ezfactor), ezfactor)

        zro(p, ezfactor) ==
          concat [zro1(r.factor) for r in factors ezfactor p]

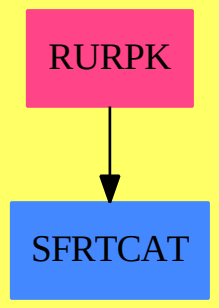
        zro1 p ==
          [zeroOf(map(#1, p)$UnivariatePolynomialCategoryFunctions2(F, UP,
                                                F, SparseUnivariatePolynomial F))]
```

⟨*ODERTRIC.dotabb*⟩≡

```
"ODERTRIC" [color="#FF4488",href="bookvol10.4.pdf#nameddest=ODERTRIC"]
"UTSCAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=UTSCAT"]
"ODERTRIC" -> "UTSCAT"
```

19.43 package RURPK RationalUnivariateRepresentationPackage

19.44 RationalUnivariateRepresentationPackage



1.00

Exports:
rur

⟨*package RURPK RationalUnivariateRepresentationPackage*⟩≡

```
)abbrev package RURPK RationalUnivariateRepresentationPackage
++ Author: Marc Moreno Maza
++ Date Created: 01/1999
++ Date Last Updated: 23/01/1999
++ Basic Functions:
++ Related Constructors:
++ Also See:
++ AMS Classifications:
++ Description:
++   A package for computing the rational univariate representation
++   of a zero-dimensional algebraic variety given by a regular
++   triangular set. This package is essentially an interface for the
++  \spadtype{InternalRationalUnivariateRepresentationPackage} constructor.
++  It is used in the \spadtype{ZeroDimensionalSolvePackage}
++  for solving polynomial systems with finitely many solutions.
++ Version: 1.

RationalUnivariateRepresentationPackage(R,ls): Exports == Implementation where
  R : Join(EuclideanDomain,CharacteristicZero)
  ls: List Symbol
  N ==> NonNegativeInteger
  Z ==> Integer
  P ==> Polynomial R
  LP ==> List P
  U ==> SparseUnivariatePolynomial(R)
  RUR ==> Record(complexRoots: U, coordinates: LP)
```

```
Exports ==  with

   rur: (LP,Boolean) -> List RUR
     ++ \spad{rur(lp,univ?)} returns a rational univariate representation
     ++ of \spad{lp}. This assumes that \spad{lp} defines a regular
     ++ triangular \spad{ts} whose associated variety is zero-dimensional
     ++ over \spad{R}. \spad{rur(lp,univ?)} returns a list of items
     ++ \spad{[u,lc]} where \spad{u} is an irreducible univariate polynomial
     ++ and each \spad{c} in \spad{lc} involves two variables: one from \spad{ls},
     ++ called the coordinate of \spad{c}, and an extra variable which
     ++ represents any root of \spad{u}. Every root of \spad{u} leads to
     ++ a tuple of values for the coordinates of \spad{lc}. Moreover,
     ++ a point \spad{x} belongs to the variety associated with \spad{lp} iff
     ++ there exists an item \spad{[u,lc]} in \spad{rur(lp,univ?)} and
     ++ a root \spad{r} of \spad{u} such that \spad{x} is given by the
     ++ tuple of values for the coordinates of \spad{lc} evaluated at \spad{r}.
     ++ If \spad{univ?} is \spad{true} then each polynomial \spad{c}
     ++ will have a constant leading coefficient w.r.t. its coordinate.
     ++ See the example which illustrates the \spadtype{ZeroDimensionalSolvePackage}
     ++ package constructor.
   rur: (LP) -> List RUR
     ++ \spad{rur(lp)} returns the same as \spad{rur(lp,true)}
   rur: (LP,Boolean,Boolean) -> List RUR
     ++ \spad{rur(lp,univ?,check?)} returns the same as \spad{rur(lp,true)}.
     ++ Moreover, if \spad{check?} is \spad{true} then the result is checked.

Implementation == add
   news: Symbol := new()$Symbol
   lv: List Symbol := concat(ls,news)
   V ==> OrderedVariableList(lv)
   Q ==> NewSparseMultivariatePolynomial(R,V)
   E ==> IndexedExponents V
   TS ==> SquareFreeRegularTriangularSet(R,E,V,Q)
   QWT ==> Record(val: Q, tower: TS)
   LQWT ==> Record(val: List Q, tower: TS)
   polsetpack ==> PolynomialSetUtilitiesPackage(R,E,V,Q)
   normpack ==> NormalizationPackage(R,E,V,Q,TS)
   rurpack ==> InternalRationalUnivariateRepresentationPackage(R,E,V,Q,TS)
   newv: V := variable(news)::V
   newq : Q := newv :: Q

   rur(lp: List P, univ?: Boolean, check?: Boolean): List RUR ==
     lp := remove(zero?,lp)
     empty? lp =>
       error "rur$RURPACK: #1 is empty"
```

```
any?(ground?,lp) =>
  error "rur$RURPACK: #1 is not a triangular set"
ts: TS := [[newq]$(List Q)]
lq: List Q := []
for p in lp repeat
  rif: Union(Q,"failed") := retractIfCan(p)$Q
  rif case "failed" =>
    error "rur$RURPACK: #1 is not a subset of R[ls]"
  q: Q := rif::Q
  lq := cons(q,lq)
lq := sort(infRittWu?,lq)
toSee: List LQWT := [[lq,ts]$LQWT]
toSave: List TS := []
while not empty? toSee repeat
  lqwt := first toSee; toSee := rest toSee
  lq := lqwt.val; ts := lqwt.tower
  empty? lq =>
    -- output(ts::OutputForm)$OutputPackage
    toSave := cons(ts,toSave)
  q := first lq; lq := rest lq
  not (mvar(q) > mvar(ts)) =>
    error "rur$RURPACK: #1 is not a triangular set"
  empty? (rest(ts)::TS) =>
    lfq := irreducibleFactors([q])$polsetpack
    for fq in lfq repeat
      newts := internalAugment(fq,ts)
      newlq := [remainder(q,newts).polnum for q in lq]
      toSee := cons([newlq,newts]$LQWT,toSee)
  lsfqwt: List QWT := squareFreePart(q,ts)
  for qwt in lsfqwt repeat
    q := qwt.val; ts := qwt.tower
    if not ground? init(q)
      then
        q := normalizedAssociate(q,ts)$normpack
    newts := internalAugment(q,ts)
    newlq := [remainder(q,newts).polnum for q in lq]
    toSee := cons([newlq,newts]$LQWT,toSee)
toReturn: List RUR := []
for ts in toSave repeat
  lus := rur(ts,univ?)$rurpack
  check? and (not checkRur(ts,lus)$rurpack) =>
    output("RUR for: ")$OutputPackage
    output(ts::OutputForm)$OutputPackage
    output("Is: ")$OutputPackage
    for us in lus repeat output(us::OutputForm)$OutputPackage
    error "rur$RURPACK: bad result with function rur$IRURPK"
```

```
            for us in lus repeat
               g: U  := univariate(select(us,newv)::Q)$Q
               lc: LP := [convert(q)@P for q in parts(collectUpper(us,newv))]
               toReturn := cons([g,lc]$RUR, toReturn)
          toReturn

        rur(lp: List P, univ?: Boolean): List RUR ==
          rur(lp,univ?,false)

        rur(lp: List P): List RUR == rur(lp,true)
```

⟨*RURPK.dotabb*⟩≡

```
"RURPK" [color="#FF4488",href="bookvol10.4.pdf#nameddest=RURPK"]
"SFRTCAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=SFRTCAT"]
"RURPK" -> "SFRTCAT"
```

19.45 package POLUTIL RealPolynomialUtilitiesPackage

This file describes the Real Closure 1.0 package which consists of different packages, categoris and domains :

The package RealPolynomialUtilitiesPackage whichs receives a field and a univariate polynomial domain with coefficients in the field. It computes some simple functions such as Strum and Sylvester sequences.

The category RealRootCharacterizationCategory provides abstarct functionalities to work with "real roots" of univariate polynomials. These resemble variables with some functionalities needed to compute important operations.

RealClosedField is a category with provides common operations available over real closed fiels. These include finding all the roots of univariate polynomial, taking square roots, ...

CAVEATS

Since real algebraic expressions are stored as depending on "real roots" which are managed like variables, there is an ordering on these. This ordering is dynamical in the sense that any new algebraic takes precedence over older ones. In particular every cretaion function raises a new "real root". This has the effect that when you type something like sqrt(2) + sqrt(2) you have two new variables which happen to be equal. To avoid this name the expression such as in s2 := sqrt(2) ; s2 + s2

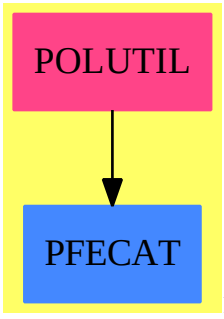
Also note that computing times depend strongly on the ordering you implicitly provide. Please provide algebraics in the order which most natural to you.

LIMITATIONS

The file reclos.input show some basic use of the package. This packages uses algorithms which are published in [1] and [2] which are based on field arithmetics, in particular for polynomial gcd related algorithms. This can be quite slow for high degree polynomials and subresultants methods usually work best. Betas versions of the package try to use these techniques in a better way and work significantly faster. These are mostly based on unpublished algorithms and cannot be distributed. Please contact the author if you have a particular problem to solve or want to use these versions.

Be aware that approximations behave as post-processing and that all computations are done excatly. They can thus be quite time consuming when depending on several “real roots”.

19.46 RealPolynomialUtilitiesPackage



1.00

Exports:
boundOfCauchy lazyVariations sturmSequence sturmVariationsOf sylvesterSequence

⟨*package POLUTIL RealPolynomialUtilitiesPackage*⟩≡
```
)abbrev package POLUTIL RealPolynomialUtilitiesPackage
++ Author: Renaud Rioboo
++ Date Created: summer 1992
++ Basic Functions: provides polynomial utilities
++ Related Constructors: RealClosure,
++ Date Last Updated: July 2004
++ Also See:
++ AMS Classifications:
++ Keywords: Sturm sequences
++ References:
++ Description:
++ \axiomType{RealPolynomialUtilitiesPackage} provides common functions used
++ by interval coding.
RealPolynomialUtilitiesPackage(TheField,ThePols) : PUB == PRIV where

    TheField : Field
    ThePols : UnivariatePolynomialCategory(TheField)

    Z ==> Integer
    N ==> NonNegativeInteger
    P ==> ThePols

    PUB == with

       sylvesterSequence : (ThePols,ThePols) -> List ThePols
         ++ \axiom{sylvesterSequence(p,q)} is the negated remainder sequence
         ++ of p and q divided by the last computed term
       sturmSequence : ThePols -> List ThePols
         ++ \axiom{sturmSequence(p) = sylvesterSequence(p,p')}
```

```
      if TheField has OrderedRing then
        boundOfCauchy : ThePols -> TheField
          ++ \axiom{boundOfCauchy(p)} bounds the roots of p
        sturmVariationsOf : List TheField -> N
          ++ \axiom{sturmVariationsOf(l)} is the number of sign variations
          ++ in the list of numbers l,
          ++ note that the first term counts as a sign
        lazyVariations : (List(TheField), Z, Z) -> N
          ++ \axiom{lazyVariations(l,s1,sn)} is the number of sign variations
          ++ in the list of non null numbers [s1::l]@sn,


    PRIV == add

     sturmSequence(p) ==
       sylvesterSequence(p,differentiate(p))

     sylvesterSequence(p1,p2) ==
       res : List(ThePols) := [p1]
       while (p2 ^= 0) repeat
         res := cons(p2 , res)
         (p1 , p2) := (p2 , -(p1 rem p2))
       if degree(p1) > 0
       then
         p1 := unitCanonical(p1)
         res := [ term quo p1 for term in res ]
       reverse! res

     if TheField has OrderedRing
     then

       boundOfCauchy(p) ==
         c :TheField := inv(leadingCoefficient(p))
         l := [ c*term for term in rest(coefficients(p))]
         null(l) => 1
         1 + ("max" / [ abs(t) for t in l ])

--       sturmVariationsOf(l) ==
--         res : N := 0
--         lsg := sign(first(l))
--         for term in l repeat
--           if ^( (sg := sign(term) ) = 0 ) then
--             if (sg ^= lsg) then res := res + 1
--             lsg := sg
--         res
```

```
sturmVariationsOf(l) ==
  null(l) => error "POLUTIL: sturmVariationsOf: empty list !"
  l1 := first(l)
  -- first 0 counts as a sign
  ll : List(TheField) := []
  for term in rest(l) repeat
    -- zeros don't count
    if not(zero?(term)) then ll := cons(term,ll)
  -- if l1 is not zero then ll = reverse(l)
  null(ll) => error "POLUTIL: sturmVariationsOf: Bad sequence"
  ln := first(ll)
  ll := reverse(rest(ll))
  -- if l1 is not zero then first(l) = first(ll)
  -- if l1 is zero then first zero should count as a sign
  zero?(l1) => 1 + lazyVariations(rest(ll),sign(first(ll)),sign(ln))
  lazyVariations(ll, sign(l1), sign(ln))

lazyVariations(l,sl,sh) ==
  zero?(sl) or zero?(sh) => error "POLUTIL: lazyVariations: zero sign!"
  null(l) =>
    if sl = sh then 0 else 1
  null(rest(l)) =>
    if zero?(first(l))
    then error "POLUTIL: lazyVariations: zero sign!"
    else
      if sl = sh
      then
        if (sl = sign(first(l)))
        then 0
        else 2
      -- in this case we save one test
      else 1
  s := sign(l.2)
  lazyVariations([first(l)],sl,s) +
    lazyVariations(rest(rest(l)),s,sh)
```

⟨*POLUTIL.dotabb*⟩≡

```
"POLUTIL" [color="#FF4488",href="bookvol10.4.pdf#nameddest=POLUTIL"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"POLUTIL" -> "PFECAT"
```

19.47 package REALSOLV RealSolvePackage

⟨*RealSolvePackage.input*⟩≡

```
)set break resume
)sys rm -f RealSolvePackage.output
)spool RealSolvePackage.output
)set message test on
)set message auto off
)clear all
--S 1 of 13
p := 4*x^3 - 3*x^2 + 2*x - 4
--R
--R
--R          3     2
--R   (1)  4x  - 3x  + 2x - 4
--R                                                    Type: Polynomial Integer
--E 1

--S 2 of 13
ans1 := solve(p,0.01)$REALSOLV
--R
--R
--R   (2)  [1.11328125]
--R                                                          Type: List Float
--E 2

--S 3 of 13
ans2 := solve(p::POLY(FRAC(INT)),0.01)$REALSOLV
--R
--R
--R   (3)  [1.11328125]
--R                                                          Type: List Float
--E 3

--S 4 of 13
R := Integer
--R
--R
--R   (4)  Integer
--R                                                             Type: Domain
--E 4

--S 5 of 13
ls : List Symbol := [x,y,z,t]
--R
--R
```

```
--R   (5)  [x,y,z,t]
--R                                                    Type: List Symbol
--E 5

--S 6 of 13
ls2 : List Symbol := [x,y,z,t,new()$Symbol]
--R
--R
--R   (6)  [x,y,z,t,%A]
--R                                                    Type: List Symbol
--E 6

--S 7 of 13
pack := ZDSOLVE(R,ls,ls2)
--R
--R
--R   (7)  ZeroDimensionalSolvePackage(Integer,[x,y,z,t],[x,y,z,t,%A])
--R                                                          Type: Domain
--E 7

--S 8 of 13
p1 := x**2*y*z + y*z
--R
--R
--R          2
--R   (8)  (x  + 1)y z
--R                                              Type: Polynomial Integer
--E 8

--S 9 of 13
p2 := x**2*y**2*z + x + z
--R
--R
--R          2 2
--R   (9)  (x y  + 1)z + x
--R                                              Type: Polynomial Integer
--E 9

--S 10 of 13
p3 := x**2*y**2*z**2 +  z + 1
--R
--R
--R          2 2 2
--R   (10)  x y z  + z + 1
--R                                              Type: Polynomial Integer
--E 10
```

```
--S 11 of 13
lp := [p1, p2, p3]
--R
--R
--R            2           2 2            2 2 2
--R   (11)  [(x  + 1)y z,(x y  + 1)z + x,x y z  + z + 1]
--R                                                 Type: List Polynomial Integer
--E 11

--S 12 of 13
lsv:List(Symbol):=[x,y,z]
--R
--R
--R   (12)  [x,y,z]
--R                                                      Type: List Symbol
--E 12

--S 13 of 13
ans3 := realSolve(lp,lsv,0.01)$REALSOLV
--R
--R
--R   (13)  [[1.0,0.0,- 1.0]]
--R                                                    Type: List List Float
--E 13
)spool
```

⟨*RealSolvePackage.help*⟩≡

```
==========================================================================
RealSolvePackage examples
==========================================================================

  p := 4*x^3 - 3*x^2 + 2*x - 4

  ans1 := solve(p,0.01)$REALSOLV

  ans2 := solve(p::POLY(FRAC(INT)),0.01)$REALSOLV

  R := Integer

  ls : List Symbol := [x,y,z,t]

  ls2 : List Symbol := [x,y,z,t,new()$Symbol]

  pack := ZDSOLVE(R,ls,ls2)

  p1 := x**2*y*z + y*z

  p2 := x**2*y**2*z + x + z

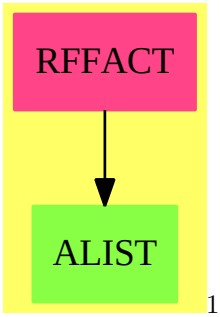
  p3 := x**2*y**2*z**2 +  z + 1

  lp := [p1, p2, p3]

  ans3 := realSolve(lp,[x,y,z],0.01)

See Also:
o )show RealSolvePackage
```

19.48 RealSolvePackage



1.00

Exports:
realSolve solve

⟨*package REALSOLV RealSolvePackage*⟩≡
```
)abbrev package REALSOLV RealSolvePackage
++ Description: This package provides numerical solutions of systems of
++   polynomial equations for use in ACPLOT
RealSolvePackage(): Exports == Implementation where
  I    ==> Integer
  IE   ==> IndexedExponents Symbol
  L    ==> List
  NF   ==> Float
  P    ==> Polynomial
  RN   ==> Fraction Integer
  SE   ==> Symbol
  RFI  ==> Fraction Polynomial Integer
  LIFT ==> PolynomialCategoryLifting(IE,SE,RN,P RN, RFI)
  SOLV ==> FloatingRealPackage Float

  Exports ==> with
    solve: (P RN,NF) -> L NF
      ++ solve(p,eps) finds the real zeroes of a
      ++ univariate rational polynomial p with precision eps.
      ++
      ++X p := 4*x^3 - 3*x^2 + 2*x - 4
      ++X solve(p::POLY(FRAC(INT)),0.01)$REALSOLV

    solve: (P I,NF) -> L NF
      ++ solve(p,eps) finds the real zeroes of a univariate
      ++ integer polynomial p with precision eps.
      ++
      ++X p := 4*x^3 - 3*x^2 + 2*x - 4
      ++X solve(p,0.01)$REALSOLV
```

```
    realSolve: (L P I,L SE,NF) -> L L NF
      ++ realSolve(lp,lv,eps) = compute the list of the real
      ++ solutions of the list lp of polynomials with integer
      ++ coefficients with respect to the variables in lv,
      ++ with precision eps.
      ++
      ++X p1 := x**2*y*z + y*z
      ++X p2 := x**2*y**2*z + x + z
      ++X p3 := x**2*y**2*z**2 +  z + 1
      ++X lp := [p1, p2, p3]
      ++X realSolve(lp,[x,y,z],0.01)

  Implementation ==>  add

    prn2rfi: P RN -> RFI
    prn2rfi p ==
      map(x+->x::RFI, x+->(numer(x)::RFI)/(denom(x)::RFI), p)$LIFT

    pi2rfi: P I -> RFI
    pi2rfi p == p :: RFI

    solve(p:P RN,eps:NF) == realRoots(prn2rfi p, eps)$SOLV

    solve(p:P I,eps:NF)  == realRoots(p::RFI, eps)$SOLV

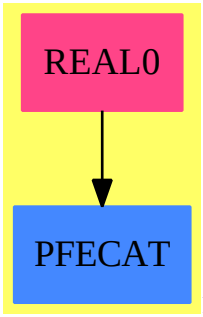
    realSolve(lp,lv,eps) ==
      realRoots(map(pi2rfi, lp)$ListFunctions2(P I,RFI),lv,eps)$SOLV
```

⟨*REALSOLV.dotabb*⟩≡

```
"REALSOLV" [color="#FF4488",href="bookvol10.4.pdf#nameddest=REALSOLV"]
"Package" [color="#FF4488"]
"REALSOLV" -> "Package"
```

19.49 package REAL0 RealZeroPackage

19.50 RealZeroPackage



1.00

Exports:
midpoint midpoints realZeros refine

⟨*package REAL0 RealZeroPackage*⟩≡

```
)abbrev package REAL0 RealZeroPackage
++ Author: Andy Neff
++ Date Created:
++ Date Last Updated:
++ Basic Functions:
++ Related Constructors: UnivariatePolynomial, RealZeroPackageQ
++ Also See:
++ AMS Classifications:
++ Keywords:
++ References:
++ Description:
++   This package provides functions for finding the real zeros
++ of univariate polynomials over the integers to arbitrary user-specified
++ precision. The results are returned as a list of
++ isolating intervals which are expressed as records with "left" and "right" rat
++ components.

RealZeroPackage(Pol): T == C where
   Pol: UnivariatePolynomialCategory Integer
   RN ==> Fraction Integer
   Interval ==> Record(left : RN, right : RN)
   isoList ==> List(Interval)
   T == with
    -- next two functions find isolating intervals
      realZeros: (Pol) -> isoList
        ++ realZeros(pol) returns a list of isolating intervals for
        ++ all the real zeros of the univariate polynomial pol.
```

```
    realZeros: (Pol, Interval) -> isoList
      ++ realZeros(pol, range) returns a list of isolating intervals
      ++ for all the real zeros of the univariate polynomial pol which
      ++ lie in the interval expressed by the record range.
  -- next two functions return intervals smaller then tolerence
    realZeros: (Pol, RN) -> isoList
      ++ realZeros(pol, eps) returns a list of intervals of length less
      ++ than the rational number eps for all the real roots of the
      ++ polynomial pol.
    realZeros: (Pol, Interval, RN) -> isoList
      ++ realZeros(pol, int, eps) returns a list of intervals of length
      ++ less than the rational number eps for all the real roots of the
      ++ polynomial pol which lie in the interval expressed by the
      ++ record int.
    refine: (Pol, Interval, RN) -> Interval
      ++ refine(pol, int, eps) refines the interval int containing
      ++ exactly one root of the univariate polynomial pol to size less
      ++ than the rational number eps.
    refine: (Pol, Interval, Interval) -> Union(Interval,"failed")
      ++ refine(pol, int, range) takes a univariate polynomial pol and
      ++ and isolating interval int containing exactly one real
      ++ root of pol; the operation returns an isolating interval which
      ++ is contained within range, or "failed" if no such isolating interval exists.
    midpoint: Interval -> RN
      ++ midpoint(int) returns the midpoint of the interval int.
    midpoints: isoList -> List RN
      ++ midpoints(isolist) returns the list of midpoints for the list
      ++ of intervals isolist.
  C == add
    --Local Functions
    makeSqfr: Pol -> Pol
    ReZeroSqfr: (Pol) -> isoList
    PosZero: (Pol) -> isoList
    Zero1: (Pol) -> isoList
    transMult: (Integer, Pol) -> Pol
    transMultInv: (Integer, Pol) -> Pol
    transAdd1: (Pol) -> Pol
    invert: (Pol) -> Pol
    minus: (Pol) -> Pol
    negate: Interval -> Interval
    rootBound: (Pol) -> Integer
    var: (Pol) -> Integer

    negate(int : Interval):Interval == [-int.right,-int.left]

    midpoint(i : Interval):RN ==  (1/2)*(i.left + i.right)
```

```
midpoints(li : isoList) : List RN ==
  [midpoint x for x in li]

makeSqfr(F : Pol):Pol ==
   sqfr := squareFree F
   F := */[s.factor for s in factors(sqfr)]

realZeros(F : Pol) ==
   ReZeroSqfr makeSqfr F

realZeros(F : Pol, rn : RN) ==
   F := makeSqfr F
   [refine(F,int,rn) for int in ReZeroSqfr(F)]

realZeros(F : Pol, bounds : Interval) ==
   F := makeSqfr F
   [rint::Interval for int in ReZeroSqfr(F) |
       (rint:=refine(F,int,bounds)) case Interval]

realZeros(F : Pol, bounds : Interval, rn : RN) ==
   F := makeSqfr F
   [refine(F,int,rn) for int in realZeros(F,bounds)]

ReZeroSqfr(F : Pol) ==
   F = 0 => error "ReZeroSqfr: zero polynomial"
   L : isoList := []
   degree(F) = 0 => L
   if (r := minimumDegree(F)) > 0 then
        L := [[0,0]$Interval]
        tempF := F exquo monomial(1, r)
        if not (tempF case "failed") then
             F := tempF
   J:isoList := [negate int for int in reverse(PosZero(minus(F)))]
   K : isoList := PosZero(F)
   append(append(J, L), K)

PosZero(F : Pol) ==   --F is square free, primitive
                    --and F(0) ^= 0; returns isoList for positive
                    --roots of F

   b : Integer := rootBound(F)
   F := transMult(b,F)
   L : isoList := Zero1(F)
   int : Interval
   L := [[b*int.left, b*int.right]$Interval for int in L]
```

```
Zero1(F : Pol) ==   --returns isoList for roots of F in (0,1)
   J : isoList
   K : isoList
   L : isoList
   L := []
   (v := var(transAdd1(invert(F)))) = 0 => []
   v = 1 => L := [[0,1]$Interval]
   G : Pol := transMultInv(2, F)
   H : Pol := transAdd1(G)
   if minimumDegree H > 0 then
           -- H has a root at 0 => F has one at 1/2, and G at 1
        L := [[1/2,1/2]$Interval]
        Q : Pol := monomial(1, 1)
        tempH : Union(Pol, "failed") := H exquo Q
        if not (tempH case "failed") then H := tempH
        Q := Q + monomial(-1, 0)
        tempG : Union(Pol, "failed") := G exquo Q
        if not (tempG case "failed") then G := tempG
   int : Interval
   J := [[(int.left+1)* (1/2),(int.right+1) * (1/2)]$Interval
                                              for int in Zero1(H)]
   K := [[int.left * (1/2), int.right * (1/2)]$Interval
                                              for int in Zero1(G)]
   append(append(J, L), K)

rootBound(F : Pol) ==  --returns power of 2 that is a bound
                       --for the positive roots of F
   if leadingCoefficient(F) < 0 then F := -F
   lcoef := leadingCoefficient(F)
   F := reductum(F)
   i : Integer := 0
   while not (F = 0) repeat
        if (an := leadingCoefficient(F)) < 0 then i := i - an
        F := reductum(F)
   b : Integer := 1
   while (b * lcoef) <= i repeat
        b := 2 * b
   b

transMult(c : Integer, F : Pol) ==
           --computes Pol G such that G(x) = F(c*x)
   G : Pol := 0
   while not (F = 0) repeat
        n := degree(F)
        G := G + monomial((c**n) * leadingCoefficient(F), n)
```

```
            F := reductum(F)
         G

      transMultInv(c : Integer, F : Pol) ==
                  --computes Pol G such that G(x) = (c**n) * F(x/c)
         d := degree(F)
         cc : Integer := 1
         G : Pol := monomial(leadingCoefficient F,d)
         while (F:=reductum(F)) ^= 0 repeat
              n := degree(F)
              cc := cc*(c**(d-n):NonNegativeInteger)
              G := G + monomial(cc * leadingCoefficient(F), n)
              d := n
         G

--    otransAdd1(F : Pol) ==
--                --computes Pol G such that G(x) = F(x+1)
--       G : Pol := F
--       n : Integer := 1
--       while (F := differentiate(F)) ^= 0 repeat
--            if not ((tempF := F exquo n) case "failed") then F := tempF
--            G := G + F
--            n := n + 1
--       G

      transAdd1(F : Pol) ==
                  --computes Pol G such that G(x) = F(x+1)
         n := degree F
         v := vectorise(F, n+1)
         for i in 0..(n-1) repeat
            for j in (n-i)..n repeat
               qsetelt_!(v,j, qelt(v,j) + qelt(v,(j+1)))
         ans : Pol := 0
         for i in 0..n repeat
            ans := ans + monomial(qelt(v,(i+1)),i)
         ans


      minus(F : Pol) ==
                  --computes Pol G such that G(x) = F(-x)
         G : Pol := 0
         while not (F = 0) repeat
              n := degree(F)
              coef := leadingCoefficient(F)
              odd? n =>
                   G := G + monomial(-coef, n)
```

```
            F := reductum(F)
        G := G + monomial(coef, n)
        F := reductum(F)
   G

invert(F : Pol) ==
           --computes Pol G such that G(x) = (x**n) * F(1/x)
   G : Pol := 0
   n := degree(F)
   while not (F = 0) repeat
        G := G + monomial(leadingCoefficient(F),
                          (n-degree(F))::NonNegativeInteger)
        F := reductum(F)
   G

var(F : Pol) ==    --number of sign variations in coefs of F
   i : Integer := 0
   LastCoef : Boolean
   next : Boolean
   LastCoef := leadingCoefficient(F) < 0
   while not ((F := reductum(F)) = 0) repeat
        next := leadingCoefficient(F) < 0
        if ((not LastCoef) and next) or
            ((not next) and LastCoef) then i := i+1
        LastCoef := next
   i

refine(F : Pol, int : Interval, bounds : Interval) ==
   lseg := min(int.right,bounds.right) - max(int.left,bounds.left)
   lseg < 0 => "failed"
   lseg = 0 =>
      pt :=
         int.left = bounds.right => int.left
         int.right
      elt(transMultInv(denom(pt),F),numer pt) = 0 => [pt,pt]
      "failed"
   lseg = int.right - int.left => int
   refine(F, refine(F, int, lseg), bounds)

refine(F : Pol, int : Interval, eps : RN) ==
   a := int.left
   b := int.right
   a=b => [a,b]$Interval
   an : Integer := numer(a)
   ad : Integer := denom(a)
   bn : Integer := numer(b)
```

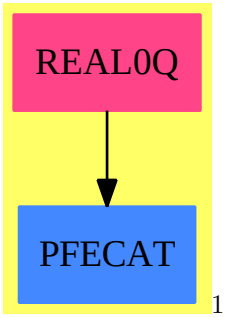
```
bd : Integer := denom(b)
xfl : Boolean := false
if (u:=elt(transMultInv(ad, F), an)) = 0 then
    F := (F exquo (monomial(ad,1)-monomial(an,0)))::Pol
    u:=elt(transMultInv(ad, F), an)
if (v:=elt(transMultInv(bd, F), bn)) = 0 then
    F := (F exquo (monomial(bd,1)-monomial(bn,0)))::Pol
    v:=elt(transMultInv(bd, F), bn)
    u:=elt(transMultInv(ad, F), an)
if u > 0 then (F:=-F;v:=-v)
if v < 0 then
   error [int, "is not a valid isolation interval for", F]
if eps <= 0 then error "precision must be positive"
while (b - a) >= eps repeat
     mid : RN := (b + a) * (1/2)
     midn : Integer := numer(mid)
     midd : Integer := denom(mid)
     (v := elt(transMultInv(midd, F), midn)) < 0 =>
          a := mid
          an := midn
          ad := midd
     v > 0 =>
          b := mid
          bn := midn
          bd := midd
     v = 0 =>
          a := mid
          b := mid
          an := midn
          ad := midd
          xfl := true
[a, b]$Interval
```

⟨*REAL0.dotabb*⟩≡

```
"REAL0" [color="#FF4488",href="bookvol10.4.pdf#nameddest=REAL0"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"REAL0" -> "PFECAT"
```

19.51 package REAL0Q RealZeroPackageQ

19.52 RealZeroPackageQ



1.00

Exports:
realZeros refine

⟨*package REAL0Q RealZeroPackageQ*⟩≡

```
)abbrev package REAL0Q RealZeroPackageQ
++ Author: Andy Neff, Barry Trager
++ Date Created:
++ Date Last Updated: 7 April 1991
++ Basic Functions:
++ Related Constructors: UnivariatePolynomial, RealZeroPackageQ
++ Also See:
++ AMS Classifications:
++ Keywords:
++ References:
++ Description:
++   This package provides functions for finding the real zeros
++ of univariate polynomials over the rational numbers to arbitrary user-specified
++ precision. The results are returned as a list of
++ isolating intervals, expressed as records with "left" and "right" rational number compone

RealZeroPackageQ(Pol): T == C where
   RN  ==> Fraction Integer
   I   ==> Integer
   SUP ==> SparseUnivariatePolynomial
   Pol: UnivariatePolynomialCategory RN
   Interval ==> Record(left : RN, right : RN)
   isoList ==> List(Interval)
   ApproxInfo ==> Record(approx : RN, exFlag : Boolean)
   T == with
    -- next two functions find isolating intervals
      realZeros: (Pol) -> isoList
```

```
      ++ realZeros(pol) returns a list of isolating intervals for
      ++ all the real zeros of the univariate polynomial pol.
    realZeros: (Pol, Interval) -> isoList
      ++ realZeros(pol, range) returns a list of isolating intervals
      ++ for all the real zeros of the univariate polynomial pol which
      ++ lie in the interval expressed by the record range.
  -- next two functions return intervals smaller then tolerence
    realZeros: (Pol, RN) -> isoList
      ++ realZeros(pol, eps) returns a list of intervals of length less
      ++ than the rational number eps for all the real roots of the
      ++ polynomial pol.
    realZeros: (Pol, Interval, RN) -> isoList
      ++ realZeros(pol, int, eps) returns a list of intervals of length
      ++ less than the rational number eps for all the real roots of the
      ++ polynomial pol which lie in the interval expressed by the
      ++ record int.
    refine: (Pol, Interval, RN) -> Interval
      ++ refine(pol, int, eps) refines the interval int containing
      ++ exactly one root of the univariate polynomial pol to size less
      ++ than the rational number eps.
    refine: (Pol, Interval, Interval) -> Union(Interval,"failed")
      ++ refine(pol, int, range) takes a univariate polynomial pol and
      ++ and isolating interval int which must contain exactly one real
      ++ root of pol, and returns an isolating interval which
      ++ is contained within range, or "failed" if no such isolating interval e
  C == add
    import RealZeroPackage SparseUnivariatePolynomial Integer

    convert2PolInt: Pol -> SparseUnivariatePolynomial Integer

    convert2PolInt(f : Pol) ==
       pden:I :=lcm([denom c for c in coefficients f])
       map(numer,pden * f)$UnivariatePolynomialCategoryFunctions2(RN,Pol,I,SUP

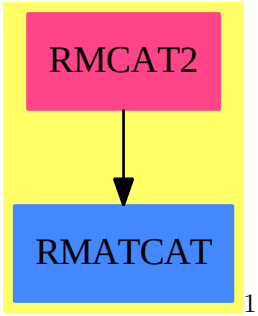
    realZeros(f : Pol) == realZeros(convert2PolInt f)
    realZeros(f : Pol, rn : RN) == realZeros(convert2PolInt f, rn)
    realZeros(f : Pol, bounds : Interval) ==
             realZeros(convert2PolInt f, bounds)
    realZeros(f : Pol, bounds : Interval, rn : RN) ==
             realZeros(convert2PolInt f, bounds, rn)
    refine(f : Pol, int : Interval, eps : RN) ==
             refine(convert2PolInt f, int, eps)
    refine(f : Pol, int : Interval, bounds : Interval) ==
             refine(convert2PolInt f, int, bounds)
```

⟨*REAL0Q.dotabb*⟩≡

```
"REAL0Q" [color="#FF4488",href="bookvol10.4.pdf#nameddest=REAL0Q"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"REAL0Q" -> "PFECAT"
```

19.53 package RMCAT2 RectangularMatrixCategoryFunctions2

19.54 RectangularMatrixCategoryFunctions2



1.00

Exports:
map reduce

⟨*package RMCAT2 RectangularMatrixCategoryFunctions2*⟩≡

```
)abbrev package RMCAT2 RectangularMatrixCategoryFunctions2
++ Author: Clifton J. Williamson
++ Date Created: 21 November 1989
++ Date Last Updated: 12 June 1991
++ Basic Operations:
++ Related Domains: IndexedMatrix(R,minRow,minCol), Matrix(R),
++    RectangularMatrix(n,m,R), SquareMatrix(n,R)
++ Also See:
++ AMS Classifications:
++ Keywords:
++ Keywords: matrix, map, reduce
++ Examples:
++ References:
++ Description:
++ \spadtype{RectangularMatrixCategoryFunctions2} provides functions between
++ two matrix domains.  The functions provided are \spadfun{map} and \spadfun{red

RectangularMatrixCategoryFunctions2(m,n,R1,Row1,Col1,M1,R2,Row2,Col2,M2):_
         Exports == Implementation where
  m,n  : NonNegativeInteger
  R1   : Ring
  Row1 : DirectProductCategory(n, R1)
  Col1 : DirectProductCategory(m, R1)
  M1   : RectangularMatrixCategory(m,n,R1,Row1,Col1)
  R2   : Ring
  Row2 : DirectProductCategory(n, R2)
```

```
    Col2 : DirectProductCategory(m, R2)
    M2   : RectangularMatrixCategory(m,n,R2,Row2,Col2)

    Exports ==> with
      map: (R1 -> R2,M1) -> M2
        ++ \spad{map(f,m)} applies the function \spad{f} to the elements of the
        ++ matrix \spad{m}.
      reduce: ((R1,R2) -> R2,M1,R2) -> R2
        ++ \spad{reduce(f,m,r)} returns a matrix \spad{n} where
        ++ \spad{n[i,j] = f(m[i,j],r)} for all indices spad{i} and \spad{j}.

    Implementation ==> add
      minr ==> minRowIndex
      maxr ==> maxRowIndex
      minc ==> minColIndex
      maxc ==> maxColIndex

      map(f,mat) ==
        ans : M2 := new(m,n,0)$Matrix(R2) pretend M2
        for i in minr(mat)..maxr(mat) for k in minr(ans)..maxr(ans) repeat
          for j in minc(mat)..maxc(mat) for l in minc(ans)..maxc(ans) repeat
            qsetelt_!(ans pretend Matrix R2,k,l,f qelt(mat,i,j))
        ans

      reduce(f,mat,ident) ==
        s := ident
        for i in minr(mat)..maxr(mat) repeat
         for j in minc(mat)..maxc(mat) repeat
           s := f(qelt(mat,i,j),s)
        s
```

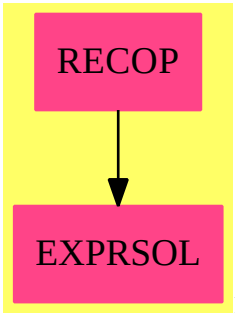
⟨*RMCAT2.dotabb*⟩≡

```
"RMCAT2" [color="#FF4488",href="bookvol10.4.pdf#nameddest=RMCAT2"]
"RMATCAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=RMATCAT"]
"RMCAT2" -> "RMATCAT"
```

19.55 package RECOP RecurrenceOperator

The package defined in this file provide an operator for the n^{th} term of a recurrence and an operator for the coefficient of x^n in a function specified by a functional equation.

19.56 RecurrenceOperator



1.00

Exports:

getEq	getOp	evalADE	evalRec
getShiftRec	numberOfValuesNeeded	shiftInfoRec	

⟨*package RECOP RecurrenceOperator*⟩≡

```
)abbrev package RECOP RecurrenceOperator
++ Author: Martin Rubey
++ Description:
++ This package provides an operator for the n-th term of a recurrence and an
++ operator for the coefficient of x^n in a function specified by a functional
++ equation.
RecurrenceOperator(R, F): Exports == Implementation where
  R: Join(OrderedSet, IntegralDomain, ConvertibleTo InputForm)
  F: Join(FunctionSpace R, AbelianMonoid, RetractableTo Integer, _
          RetractableTo Symbol, PartialDifferentialRing Symbol, _
          CombinatorialOpsCategory)
--RecurrenceOperator(F): Exports == Implementation where
--  F: Join(ExpressionSpace, AbelianMonoid, RetractableTo Integer,
--          RetractableTo Symbol, PartialDifferentialRing Symbol)

  Exports == with

    evalRec: (BasicOperator, Symbol, F, F, F, List F) -> F
      ++ \spad{evalRec(u, dummy, n, n0, eq, values)} creates an expression that
      ++ stands for u(n0), where u(n) is given by the equation eq. However, for
      ++ technical reasons the variable n has to be replaced by a dummy
      ++ variable dummy in eq. The argument values specifies the initial values
      ++ of the recurrence u(0), u(1),...
      ++ For the moment we don't allow recursions that contain u inside of
      ++ another operator.

    evalADE: (BasicOperator, Symbol, F, F, F, List F) -> F
      ++ \spad{evalADE(f, dummy, x, n, eq, values)} creates an expression that
      ++ stands for the coefficient of x^n in the Taylor expansion of f(x),
```

```
      ++ where f(x) is given by the functional equation eq. However, for
      ++ technical reasons the variable x has to be replaced by a dummy
      ++ variable dummy in eq. The argument values specifies the first few
      ++ Taylor coefficients.

    getEq: F -> F
      ++ \spad{getEq f} returns the defining equation, if f represents the
      ++ coefficient of an ADE or a recurrence.

    getOp: F -> BasicOperator
      ++ \spad{getOp f}, if f represents the coefficient of a recurrence or
      ++ ADE, returns the operator representing the solution


  -- should be local
    numberOfValuesNeeded: (Integer, BasicOperator, Symbol, F) -> Integer

  -- should be local
    if R has Ring then
      getShiftRec: (BasicOperator, Kernel F, Symbol) -> Union(Integer, "failed")

      shiftInfoRec: (BasicOperator, Symbol, F) ->
                     Record(max: Union(Integer, "failed"),
                            ord: Union(Integer, "failed"),
                            ker: Kernel F)

  Implementation == add
```

⟨*implementation: RecurrenceOperator*⟩

19.56.1 Defining new operators

We define two new operators, one for recurrences, the other for functional equations. The operators for recurrences represents the n^{th} term of the corresponding sequence, the other the coefficient of x^n in the Taylor series expansion.

⟨*implementation: RecurrenceOperator*⟩≡

```
    oprecur := operator("rootOfRec"::Symbol)$BasicOperator

    opADE := operator("rootOfADE"::Symbol)$BasicOperator

    setProperty(oprecur, "%dummyVar", 2 pretend None)
    setProperty(opADE, "%dummyVar", 2 pretend None)
```

Setting these properties implies that the second and third arguments of oprecur are dummy variables and affects `tower$ES`: the second argument will not appear in `tower$ES`, if it does not appear in any argument but the first and second. The third argument will not appear in `tower$ES`, if it does not appear in any other argument. (`%defsum` is a good example)

The arguments of the two operators are as follows:

1. `eq`, i.e. the vanishing expression

   ⟨*implementation: RecurrenceOperator*⟩+≡
   ```
   eqAsF: List F -> F
   eqAsF l == l.1
   ```

2. `dummy`, a dummy variable to make substitutions possible

   ⟨*implementation: RecurrenceOperator*⟩+≡
   ```
   dummy: List F -> Symbol
   dummy l == retract(l.2)@Symbol

   dummyAsF: List F -> F
   dummyAsF l == l.2
   ```

3. the variable for display

   ⟨*implementation: RecurrenceOperator*⟩+≡
   ```
   displayVariable: List F -> F
   displayVariable l == l.3
   ```

4. `operatorName(argument)`

   ⟨*implementation: RecurrenceOperator*⟩+≡
   ```
   operatorName: List F -> BasicOperator
   operatorName l == operator(kernels(l.4).1)

   operatorNameAsF: List F -> F
   operatorNameAsF l == l.4

   operatorArgument: List F -> F
   operatorArgument l == argument(kernels(l.4).1).1
   ```

Concerning `rootOfADE`, note that although we have `arg` as argument of the operator, it is intended to indicate the coefficient, not the argument of the power series.

5. `values` in reversed order.

   - `rootOfRec`: maybe `values` should be preceded by the index of the first given value. Currently, the last value is interpreted as $f(0)$.
   - `rootOfADE`: values are the first few coefficients of the power series expansion in order.

⟨*implementation: RecurrenceOperator*⟩+≡

```
initialValues: List F -> List F
initialValues l == rest(l, 4)
```

19.56.2 Recurrences

Extracting some information from the recurrence

We need to find out wether we can determine the next term of the sequence, and how many initial values are necessary.

⟨*implementation: RecurrenceOperator*⟩+≡

```
      if R has Ring then
        getShiftRec(op: BasicOperator, f: Kernel F, n: Symbol)
                    : Union(Integer, "failed") ==
            a := argument f
            if every?(freeOf?(#1, n::F), a) then return 0

            if #a ~= 1 then error "RECOP: operator should have only one argument"

            p := univariate(a.1, retract(n::F)@Kernel(F))
            if denominator p ~= 1 then return "failed"

            num := numer p

            if degree num = 1 and coefficient(num, 1) = 1
               and every?(freeOf?(#1, n::F), coefficients num)
            then return retractIfCan(coefficient(num, 0))
            else return "failed"

  -- if the recurrence is of the form
  -- $p(n, f(n+m-o), f(n+m-o+1), \dots, f(n+m)) = 0$
  -- in which case shiftInfoRec returns [m, o, f(n+m)].

        shiftInfoRec(op: BasicOperator, argsym: Symbol, eq: F):
                     Record(max: Union(Integer, "failed"),
                            ord: Union(Integer, "failed"),
                            ker: Kernel F) ==

  -- ord and ker are valid only if all shifts are Integers
  -- ker is the kernel of the maximal shift.
          maxShift: Integer
          minShift: Integer
          nextKernel: Kernel F

  -- We consider only those kernels that have op as operator. If there is none,
  -- we raise an error. For the moment we don't allow recursions that contain op
  -- inside of another operator.

          error? := true
```

```
for f in kernels eq repeat
  if is?(f, op) then
    shift := getShiftRec(op, f, argsym)
    if error? then
      error? := false
      nextKernel := f
      if shift case Integer then
        maxShift := shift
        minShift := shift
      else return ["failed", "failed", nextKernel]
    else
      if shift case Integer then
        if maxShift < shift then
          maxShift := shift
          nextKernel := f
        if minShift > shift then
          minShift := shift
      else return ["failed", "failed", nextKernel]

if error? then error "evalRec: equation does not contain operator"

[maxShift, maxShift - minShift, nextKernel]
```

Evaluating a recurrence

⟨*implementation: RecurrenceOperator*⟩+≡

```
        evalRec(op, argsym, argdisp, arg, eq, values) ==
          if ((n := retractIfCan(arg)@Union(Integer, "failed")) case "failed")
             or (n < 0)
          then
            shiftInfo := shiftInfoRec(op, argsym, eq)

            if (shiftInfo.ord case "failed") or ((shiftInfo.ord)::Integer > 0)
            then
              kernel(oprecur,
                     append([eq, argsym::F, argdisp, op(arg)], values))
            else
              p := univariate(eq, shiftInfo.ker)
              num := numer p

  -- If the degree is 1, we can return the function explicitly.

              if degree num = 1 then
                eval(-coefficient(num, 0)/coefficient(num, 1), argsym::F,
                     arg::F-(shiftInfo.max)::Integer::F)
              else
                kernel(oprecur,
                       append([eq, argsym::F, argdisp, op(arg)], values))
          else
            len: Integer := #values
            if n < len
            then values.(len-n)
            else
              shiftInfo := shiftInfoRec(op, argsym, eq)

              if shiftInfo.max case Integer then
                p := univariate(eq, shiftInfo.ker)

                num := numer p

                if degree num = 1 then

                  next := -coefficient(num, 0)/coefficient(num, 1)
                  nextval := eval(next, argsym::F,
                                  (len-(shiftInfo.max)::Integer)::F)
                  newval := eval(nextval, op,
                                 evalRec(op, argsym, argdisp, #1, eq, values))
                  evalRec(op, argsym, argdisp, arg, eq, cons(newval, values))
                else
```

```
            kernel(oprecur,
                   append([eq, argsym::F, argdisp, op(arg)], values))

          else
            kernel(oprecur,
                   append([eq, argsym::F, argdisp, op(arg)], values))

      numberOfValuesNeeded(numberOfValues: Integer,
                           op: BasicOperator, argsym: Symbol, eq: F): Integer ==
        order := shiftInfoRec(op, argsym, eq).ord
        if order case Integer
        then min(numberOfValues, retract(order)@Integer)
        else numberOfValues


    else
      evalRec(op, argsym, argdisp, arg, eq, values) ==
        kernel(oprecur,
               append([eq, argsym::F, argdisp, op(arg)], values))

      numberOfValuesNeeded(numberOfValues: Integer,
                           op: BasicOperator, argsym: Symbol, eq: F): Integer ==
        numberOfValues
```

Setting the evaluation property of `oprecur`

`irecur` is just a wrapper that allows us to write a recurrence relation as an operator.

⟨*implementation: RecurrenceOperator*⟩+≡

```
    irecur: List F -> F
    irecur l ==
      evalRec(operatorName l,
              dummy l, displayVariable l,
              operatorArgument l, eqAsF l, initialValues l)

    evaluate(oprecur, irecur)$BasicOperatorFunctions1(F)
```

Displaying a recurrence relation

⟨*implementation: RecurrenceOperator*⟩+≡

```
ddrec: List F -> OutputForm
ddrec l ==
  op := operatorName l
  values := reverse l
  eq := eqAsF l

  numberOfValues := numberOfValuesNeeded(#values-4, op, dummy l, eq)

  vals: List OutputForm
       := cons(eval(eq, dummyAsF l, displayVariable l)::OutputForm = _
               0::OutputForm,
               [elt(op::OutputForm, [(i-1)::OutputForm]) = _
                (values.i)::OutputForm
                for i in 1..numberOfValues])

  bracket(hconcat([(operatorNameAsF l)::OutputForm,
                   ": ",
                   commaSeparate vals]))

setProperty(oprecur, "%specialDisp",
            ddrec@(List F -> OutputForm) pretend None)
```

19.56.3 Functional Equations

Determining the number of initial values for ADE's

We use Joris van der Hoeven's instructions for ADE's. Given $Q = p(f, f', \ldots, f^{(r)})$ we first need to differentiate Q with respect to $f^{(i)}$ for $i \in \{0, 1, \ldots, r\}$, plug in the given truncated power series solution and determine the valuation.

⟨*NOTYET implementation: RecurrenceOperator*⟩≡

```
getValuation(op, argsym, eq, maxorder, values): Integer ==
    max: Integer := -1;
    ker: Kernel F
    for i in 0..maxorder repeat
        ker := D(op(argsym), argsym, i)::Kernel F
        pol := univariate(eq, ker)
        dif := D pol
        ground numer(dif.D(op(argsym), argsym, i))
```

Extracting some information from the functional equation

`getOrder` returns the maximum derivative of `op` occurring in `f`.

⟨*implementation: RecurrenceOperator*⟩+≡

```
getOrder(op: BasicOperator, f: Kernel F): NonNegativeInteger ==
  res: NonNegativeInteger := 0
  g := f
  while is?(g, %diff) repeat
    g := kernels(argument(g).1).1
    res := res+1

  if is?(g, op) then res else 0
```

Extracting a coefficient given a functional equation

⟨*implementation: RecurrenceOperator*⟩+≡

```
      evalADE(op, argsym, argdisp, arg, eq, values) ==
        if not freeOf?(eq, retract(argdisp)@Symbol)
        then error "RECOP: The argument should not be used in the equation of the_
   ADE"

        if ((n := retractIfCan(arg)@Union(Integer, "failed")) case "failed")
        then
  -- try to determine the necessary number of initial values
          keq := kernels eq
          order := getOrder(op, keq.1)
          for k in rest keq repeat order := max(order, getOrder(op, k))

          p: Fraction SparseUnivariatePolynomial F
            := univariate(eq, kernels(D(op(argsym::F), argsym, order)).1)$F

          if one? degree numer p
  -- the equation is holonomic
          then kernel(opADE,
                      append([eq, argsym::F, argdisp, op(arg)],
                             reverse first(reverse values, order)))
          else kernel(opADE,
                      append([eq, argsym::F, argdisp, op(arg)], values))
        else
          if n < 0
          then 0
          else
            keq := kernels eq
            order := getOrder(op, keq.1)
  --          output(hconcat("The order is ", order::OutputForm))$OutputPackage
            for k in rest keq repeat order := max(order, getOrder(op, k))

            p: Fraction SparseUnivariatePolynomial F
              := univariate(eq, kernels(D(op(argsym::F), argsym, order)).1)$F

  --          output(hconcat("p: ", p::OutputForm))$OutputPackage

            if degree numer p > 1
            then
  --            kernel(opADE,
  --                   append([eq, argsym::F, argdisp, op(arg)], values))

              s := seriesSolve(eq, op, argsym, reverse values)
                             $ExpressionSolve(R, F,
```

```
                                          UnivariateFormalPowerSeries F,
                                          UnivariateFormalPowerSeries
                                            SparseUnivariatePolynomialExpressions F)

            elt(s, n::Integer::NonNegativeInteger)

          else
            s := seriesSolve(eq, op, argsym, first(reverse values, order))
                          $ExpressionSolve(R, F,
                                          UnivariateFormalPowerSeries F,
                                          UnivariateFormalPowerSeries
                                            SparseUnivariatePolynomialExpressions F)

            elt(s, n::Integer::NonNegativeInteger)


    iADE: List F -> F
-- This is just a wrapper that allows us to write a recurrence relation as an
-- operator.
    iADE l ==
      evalADE(operatorName l,
              dummy l, displayVariable l,
              operatorArgument l, eqAsF l, initialValues l)

    evaluate(opADE, iADE)$BasicOperatorFunctions1(F)

    getEq(f: F): F ==
      ker := kernels f
      if one?(#ker) and _
         (is?(operator(ker.1), "rootOfADE"::Symbol) or _
          is?(operator(ker.1), "rootOfRec"::Symbol)) then
        l := argument(ker.1)
        eval(eqAsF l, dummyAsF l, displayVariable l)
      else
        error "getEq: argument should be a single rootOfADE or rootOfRec object"

    getOp(f: F): BasicOperator ==
      ker := kernels f
      if one?(#ker) and _
         (is?(operator(ker.1), "rootOfADE"::Symbol) or _
          is?(operator(ker.1), "rootOfRec"::Symbol)) then
        operatorName argument(ker.1)
      else
        error "getOp: argument should be a single rootOfADE or rootOfRec object"
```

Displaying a functional equation

⟨*implementation: RecurrenceOperator*⟩+≡

```
ddADE: List F -> OutputForm
ddADE l ==
  op := operatorName l
  values := reverse l

  vals: List OutputForm
       := cons(eval(eqAsF l, dummyAsF l, displayVariable l)::OutputForm = _
               0::OutputForm,
               [eval(D(op(dummyAsF l), dummy l, i), _
                     dummyAsF l=0)::OutputForm = _
                (values.(i+1))::OutputForm * _
                factorial(box(i::R::F)$F)::OutputForm _
                for i in 0..min(4,#values-5)])

  bracket(hconcat([bracket((displayVariable l)::OutputForm ** _
                           (operatorArgument l)::OutputForm),
                   (op(displayVariable l))::OutputForm, ": ",
                   commaSeparate vals]))

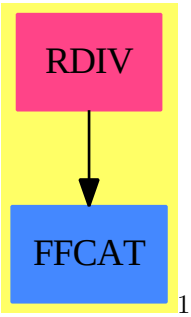
setProperty(opADE, "%specialDisp",
            ddADE@(List F -> OutputForm) pretend None)
```

⟨*RECOP.dotabb*⟩≡

```
"RECOP" [color="#FF4488",href="bookvol10.4.pdf#nameddest=RECOP"]
"EXPRSOL" [color="#FF4488",href="bookvol10.4.pdf#nameddest=EXPRSOL"]
"RECOP" -> "EXPRSOL"
```

19.57 package RDIV ReducedDivisor

19.58 ReducedDivisor



1.00

Exports:
order

⟨*package RDIV ReducedDivisor*⟩≡

```
)abbrev package RDIV ReducedDivisor
++ Finds the order of a divisor over a finite field
++ Author: Manuel Bronstein
++ Date Created: 1988
++ Date Last Updated: 8 November 1994
ReducedDivisor(F1, UP, UPUP, R, F2): Exports == Implementation where
  F1    : Field
  UP    : UnivariatePolynomialCategory F1
  UPUP  : UnivariatePolynomialCategory Fraction UP
  R     : FunctionFieldCategory(F1, UP, UPUP)
  F2    : Join(Finite, Field)

  N     ==> NonNegativeInteger
  FD    ==> FiniteDivisor(F1, UP, UPUP, R)
  UP2   ==> SparseUnivariatePolynomial F2
  UPUP2 ==> SparseUnivariatePolynomial Fraction UP2

  Exports ==> with
    order: (FD, UPUP, F1 -> F2) -> N
        ++ order(f,u,g) \undocumented

  Implementation ==> add
    algOrder : (FD, UPUP, F1 -> F2)  -> N
    rootOrder: (FD, UP, N, F1 -> F2) -> N

-- pp is not necessarily monic
    order(d, pp, f) ==
```

```
        (r := retractIfCan(reductum pp)@Union(Fraction UP, "failed"))
          case "failed" => algOrder(d, pp, f)
        rootOrder(d, - retract(r::Fraction(UP) / leadingCoefficient pp)@UP,
                  degree pp, f)

      algOrder(d, modulus, reduce) ==
        redmod := map(reduce, modulus)$MultipleMap(F1,UP,UPUP,F2,UP2,UPUP2)
        curve  := AlgebraicFunctionField(F2, UP2, UPUP2, redmod)
        order(map(reduce,
                d)$FiniteDivisorFunctions2(F1,UP,UPUP,R,F2,UP2,UPUP2,curve)
                                    )$FindOrderFinite(F2, UP2, UPUP2, curve)

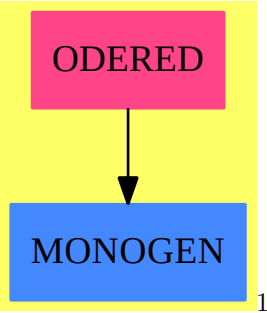
      rootOrder(d, radicand, n, reduce) ==
        redrad := map(reduce,
             radicand)$UnivariatePolynomialCategoryFunctions2(F1,UP,F2,UP2)
        curve  := RadicalFunctionField(F2, UP2, UPUP2, redrad::Fraction UP2, n)
        order(map(reduce,
                d)$FiniteDivisorFunctions2(F1,UP,UPUP,R,F2,UP2,UPUP2,curve)
                                    )$FindOrderFinite(F2, UP2, UPUP2, curve)
```

⟨*RDIV.dotabb*⟩≡

```
"RDIV" [color="#FF4488",href="bookvol10.4.pdf#nameddest=RDIV"]
"FFCAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=FFCAT"]
"RDIV" -> "FFCAT"
```

19.59 package ODERED ReduceLODE

19.60 ReduceLODE



1.00

Exports:
reduceLODE

⟨*package ODERED ReduceLODE*⟩≡
```
)abbrev package ODERED ReduceLODE
++ Author: Manuel Bronstein
++ Date Created: 19 August 1991
++ Date Last Updated: 11 April 1994
++ Description: Elimination of an algebraic from the coefficentss
++ of a linear ordinary differential equation.
ReduceLODE(F, L, UP, A, LO): Exports == Implementation where
  F : Field
  L : LinearOrdinaryDifferentialOperatorCategory F
  UP: UnivariatePolynomialCategory F
  A : MonogenicAlgebra(F, UP)
  LO: LinearOrdinaryDifferentialOperatorCategory A

  V ==> Vector F
  M ==> Matrix L

  Exports ==> with
    reduceLODE: (LO, A) -> Record(mat:M, vec:V)
      ++ reduceLODE(op, g) returns \spad{[m, v]} such that
      ++ any solution in \spad{A} of \spad{op z = g}
      ++ is of the form \spad{z = (z_1,...,z_m) . (b_1,...,b_m)} where
      ++ the \spad{b_i's} are the basis of \spad{A} over \spad{F} returned
      ++ by \spadfun{basis}() from \spad{A}, and the \spad{z_i's} satisfy the
      ++ differential system \spad{M.z = v}.

  Implementation ==> add
    matF2L: Matrix F -> M
```

```
      diff := D()$L

  -- coerces a matrix of elements of F into a matrix of (order 0) L.O.D.O's
      matF2L m ==
        map(#1::L, m)$MatrixCategoryFunctions2(F, V, V, Matrix F,
                                               L, Vector L, Vector L, M)

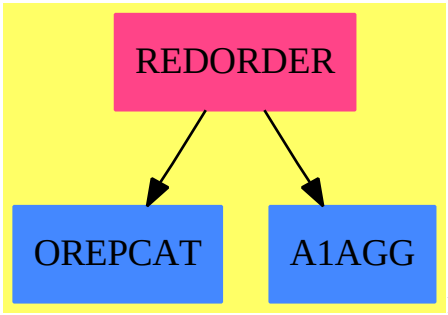
  -- This follows the algorithm and notation of
  --  "The Risch Differential Equation on an Algebraic Curve", M. Bronstein,
  -- in 'Proceedings of ISSAC '91', Bonn, BRD, ACM Press, pp.241-246, July 1991.
      reduceLODE(l, g) ==
        n := rank()$A
  -- md is the basic differential matrix (D x I + Dy)
        md := matF2L transpose derivationCoordinates(basis(), diff #1)
        for i in minRowIndex md .. maxRowIndex md
          for j in minColIndex md .. maxColIndex md repeat
            md(i, j) := diff + md(i, j)
  -- mdi will go through the successive powers of md
        mdi := copy md
        sys := matF2L(transpose regularRepresentation coefficient(l, 0))
        for i in 1..degree l repeat
          sys := sys +
                  matF2L(transpose regularRepresentation coefficient(l, i)) * mdi
          mdi := md * mdi
        [sys, coordinates g]
```

⟨*ODERED.dotabb*⟩≡

```
  "ODERED" [color="#FF4488",href="bookvol10.4.pdf#nameddest=ODERED"]
  "MONOGEN" [color="#4488FF",href="bookvol10.2.pdf#nameddest=MONOGEN"]
  "ODERED" -> "MONOGEN"
```

19.61 package REDORDER ReductionOfOrder

19.62 ReductionOfOrder



1.00

Exports:
ReduceOrder

⟨*package REDORDER ReductionOfOrder*⟩≡

```
)abbrev package REDORDER ReductionOfOrder
++ Author: Manuel Bronstein
++ Date Created: 4 November 1991
++ Date Last Updated: 3 February 1994
++ Description:
++ \spadtype{ReductionOfOrder} provides
++ functions for reducing the order of linear ordinary differential equations
++ once some solutions are known.
++ Keywords: differential equation, ODE
ReductionOfOrder(F, L): Exports == Impl where
  F: Field
  L: LinearOrdinaryDifferentialOperatorCategory F

  Z ==> Integer
  A ==> PrimitiveArray F

  Exports ==> with
    ReduceOrder: (L, F) -> L
      ++ ReduceOrder(op, s) returns \spad{op1} such that for any solution
      ++ \spad{z} of \spad{op1 z = 0}, \spad{y = s \int z} is a solution of
      ++ \spad{op y = 0}. \spad{s} must satisfy \spad{op s = 0}.
    ReduceOrder: (L, List F) -> Record(eq:L, op:List F)
      ++ ReduceOrder(op, [f1,...,fk]) returns \spad{[op1,[g1,...,gk]]} such that
      ++ for any solution \spad{z} of \spad{op1 z = 0},
      ++ \spad{y = gk \int(g_{k-1} \int(... \int(g1 \int z)...)} is a solution
      ++ of \spad{op y = 0}. Each \spad{fi} must satisfy \spad{op fi = 0}.
```

```
Impl ==> add
  ithcoef   : (L, Z, A) -> F
  locals    : (A, Z, Z) -> F
  localbinom: (Z, Z) -> Z

  diff := D()$L

  localbinom(j, i) == (j > i => binomial(j, i+1); 0)
  locals(s, j, i)  == (j > i => qelt(s, j - i - 1); 0)

  ReduceOrder(l:L, sols:List F) ==
    empty? sols => [l, empty()]
    neweq := ReduceOrder(l, sol := first sols)
    rec := ReduceOrder(neweq, [diff(s / sol) for s in rest sols])
    [rec.eq, concat_!(rec.op, sol)]

  ithcoef(eq, i, s) ==
    ans:F := 0
    while eq ^= 0 repeat
        j   := degree eq
        ans := ans + localbinom(j, i) * locals(s,j,i) * leadingCoefficient eq
        eq  := reductum eq
    ans

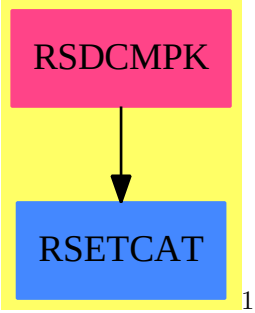
  ReduceOrder(eq:L, sol:F) ==
    s:A := new(n := degree eq, 0)         -- will contain derivatives of sol
    si := sol                             -- will run through the derivatives
    qsetelt_!(s, 0, si)
    for i in 1..(n-1)::NonNegativeInteger repeat
        qsetelt_!(s, i, si := diff si)
    ans:L := 0
    for i in 0..(n-1)::NonNegativeInteger repeat
        ans := ans + monomial(ithcoef(eq, i, s), i)
    ans
```

⟨*REDORDER.dotabb*⟩≡

```
"REDORDER" [color="#FF4488",href="bookvol10.4.pdf#nameddest=REDORDER"]
"OREPCAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=OREPCAT"]
"A1AGG" [color="#4488FF",href="bookvol10.2.pdf#nameddest=A1AGG"]
"REDORDER" -> "OREPCAT"
"REDORDER" -> "A1AGG"
```

19.63 package RSDCMPK RegularSetDecompositionPackage

19.64 RegularSetDecompositionPackage



1.00

Exports:

KrullNumber	algebraicDecompose	convert	decompose
internalDecompose	numberOfVariables	printInfo	transcendentalDecompose
upDateBranches			

⟨*package RSDCMPK RegularSetDecompositionPackage*⟩≡

```
)abbrev package RSDCMPK RegularSetDecompositionPackage
++ Author: Marc Moreno Maza
++ Date Created: 09/16/1998
++ Date Last Updated: 12/16/1998
++ Basic Functions:
++ Related Constructors:
++ Also See:
++ AMS Classifications:
++ Keywords:
++ Description:
++ A package providing a new algorithm for solving polynomial systems
++ by means of regular chains. Two ways of solving are proposed:
++ in the sense of Zariski closure (like in Kalkbrener's algorithm)
++ or in the sense of the regular zeros (like in Wu, Wang or Lazard
++ methods). This algorithm is valid for nay type
++ of regular set. It does not care about the way a polynomial is
++ added in an regular set, or how two quasi-components are compared
++ (by an inclusion-test), or how the invertibility test is made in
++ the tower of simple extensions associated with a regular set.
++ These operations are realized respectively by the domain \spad{TS}
++ and the packages
++ \axiomType{QCMPACK}(R,E,V,P,TS) and \axiomType{RSETGCD}(R,E,V,P,TS).
++ The same way it does not care about the way univariate polynomial
++ gcd (with coefficients in the tower of simple extensions associated
```

```
++ with a regular set) are computed. The only requirement is that these
++ gcd need to have invertible initials (normalized or not).
++ WARNING. There is no need for a user to call diectly any operation
++ of this package since they can be accessed by the domain \axiom{TS}.
++ Thus, the operations of this package are not documented.\newline
++ References :
++  [1] M. MORENO MAZA "A new algorithm for computing triangular
++      decomposition of algebraic varieties" NAG Tech. Rep. 4/98.
++ Version: 5. Same as 4 but Does NOT use any unproved criteria.

RegularSetDecompositionPackage(R,E,V,P,TS): Exports == Implementation where

  R : GcdDomain
  E : OrderedAbelianMonoidSup
  V : OrderedSet
  P : RecursivePolynomialCategory(R,E,V)
  TS : RegularTriangularSetCategory(R,E,V,P)
  N ==> NonNegativeInteger
  Z ==> Integer
  B ==> Boolean
  LP ==> List P
  PS ==> GeneralPolynomialSet(R,E,V,P)
  PWT ==> Record(val : P, tower : TS)
  BWT ==> Record(val : Boolean, tower : TS)
  LpWT ==> Record(val : (List P), tower : TS)
  Wip ==> Record(done: Split, todo: List LpWT)
  Branch ==> Record(eq: List P, tower: TS, ineq: List P)
  UBF ==> Union(Branch,"failed")
  Split ==> List TS
  iprintpack ==> InternalPrintPackage()
  polsetpack ==> PolynomialSetUtilitiesPackage(R,E,V,P)
  quasicomppack ==> QuasiComponentPackage(R,E,V,P,TS)
  regsetgcdpack ==> RegularTriangularSetGcdPackage(R,E,V,P,TS)

  Exports ==  with

     KrullNumber: (LP, Split) -> N
     numberOfVariables: (LP, Split) -> N
     algebraicDecompose: (P,TS,B) -> Record(done: Split, todo: List LpWT)
     transcendentalDecompose: (P,TS,N) -> Record(done: Split, todo: List LpWT)
     transcendentalDecompose: (P,TS) -> Record(done: Split, todo: List LpWT)
     internalDecompose: (P,TS,N,B) -> Record(done: Split, todo: List LpWT)
     internalDecompose: (P,TS,N) -> Record(done: Split, todo: List LpWT)
     internalDecompose: (P,TS) -> Record(done: Split, todo: List LpWT)
     decompose: (LP, Split, B, B) -> Split
     decompose: (LP, Split, B, B, B, B, B) -> Split
```

```
    upDateBranches: (LP,Split,List LpWT,Wip,N) -> List LpWT
    convert: Record(val: List P,tower: TS) -> String
    printInfo: (List Record(val: List P,tower: TS), N) -> Void

  Implementation == add

    KrullNumber(lp: LP, lts: Split): N ==
      ln: List N := [#(ts) for ts in lts]
      n := #lp + reduce(max,ln)

    numberOfVariables(lp: LP, lts: Split): N ==
      lv: List V := variables([lp]$PS)
      for ts in lts repeat lv := concat(variables(ts), lv)
      # removeDuplicates(lv)

    algebraicDecompose(p: P, ts: TS, clos?: B): Record(done: Split, todo: List LpWT) ==
      ground? p =>
        error " in algebraicDecompose$REGSET: should never happen !"
      v := mvar(p); n := #ts
      ts_v_- := collectUnder(ts,v)
      ts_v_+ := collectUpper(ts,v)
      ts_v := select(ts,v)::P
      if mdeg(p) < mdeg(ts_v)
        then
          lgwt := internalLastSubResultant(ts_v,p,ts_v_-,true,false)$regsetgcdpack
        else
          lgwt := internalLastSubResultant(p,ts_v,ts_v_-,true,false)$regsetgcdpack
      lts: Split := []
      llpwt: List LpWT := []
      for gwt in lgwt repeat
        g := gwt.val; us := gwt.tower
        zero? g =>
          error " in algebraicDecompose$REGSET: should never happen !!"
        ground? g => "leave"
        if mvar(g) = v then lts := concat(augment(members(ts_v_+),augment(g,us)),lts)
        h := leadingCoefficient(g,v)
        b: Boolean := purelyAlgebraic?(us)
        lsfp := squareFreeFactors(h)$polsetpack
        lus := augment(members(ts_v_+),augment(ts_v,us)@Split)
        for f in lsfp repeat
          ground? f => "leave"
          b and purelyAlgebraic?(f,us) => "leave"
          for vs in lus repeat
             llpwt := cons([[f,p],vs]$LpWT, llpwt)
      [lts,llpwt]
```

```
transcendentalDecompose(p: P, ts: TS,bound: N): Record(done: Split, todo: Li
  lts: Split
  if #ts < bound
    then
      lts := augment(p,ts)
    else
      lts := []
  llpwt: List LpWT := []
  [lts,llpwt]

transcendentalDecompose(p: P, ts: TS): Record(done: Split, todo: List LpWT)
  lts: Split:= augment(p,ts)
  llpwt: List LpWT := []
  [lts,llpwt]

internalDecompose(p: P, ts: TS,bound: N,clos?:B): Record(done: Split, todo:
  clos? => internalDecompose(p,ts,bound)
  internalDecompose(p,ts)

internalDecompose(p: P, ts: TS,bound: N): Record(done: Split, todo: List LpW
  -- ASSUME p not constant
  llpwt: List LpWT := []
  lts: Split := []
  -- EITHER mvar(p) is null
  if (not zero? tail(p)) and (not ground? (lmp := leastMonomial(p)))
    then
      llpwt := cons([[mvar(p)::P],ts]$LpWT,llpwt)
      p := (p exquo lmp)::P
  ip := squareFreePart init(p); tp := tail p
  p := mainPrimitivePart p
  -- OR init(p) is null or not
  lbwt := invertible?(ip,ts)@(List BWT)
  for bwt in lbwt repeat
    bwt.val =>
      if algebraic?(mvar(p),bwt.tower)
        then
          rsl := algebraicDecompose(p,bwt.tower,true)
        else
          rsl := transcendentalDecompose(p,bwt.tower,bound)
      lts := concat(rsl.done,lts)
      llpwt := concat(rsl.todo,llpwt)
      -- purelyAlgebraicLeadingMonomial?(ip,bwt.tower) => "leave"  -- UNPROV
      purelyAlgebraic?(ip,bwt.tower) and purelyAlgebraic?(bwt.tower) => "lea
      (not ground? ip) =>
        zero? tp => llpwt := cons([[ip],bwt.tower]$LpWT, llpwt)
        (not ground? tp) => llpwt := cons([[ip,tp],bwt.tower]$LpWT, llpwt)
```

```
    riv := removeZero(ip,bwt.tower)
    (zero? riv) =>
      zero? tp => lts := cons(bwt.tower,lts)
      (not ground? tp) => llpwt := cons([[tp],bwt.tower]$LpWT, llpwt)
    llpwt := cons([[riv * mainMonomial(p) + tp],bwt.tower]$LpWT, llpwt)
  [lts,llpwt]

internalDecompose(p: P, ts: TS): Record(done: Split, todo: List LpWT) ==
  -- ASSUME p not constant
  llpwt: List LpWT := []
  lts: Split := []
  -- EITHER mvar(p) is null
  if (not zero? tail(p)) and (not ground? (lmp := leastMonomial(p)))
    then
      llpwt := cons([[mvar(p)::P],ts]$LpWT,llpwt)
      p := (p exquo lmp)::P
  ip := squareFreePart init(p); tp := tail p
  p := mainPrimitivePart p
  -- OR init(p) is null or not
  lbwt := invertible?(ip,ts)@(List BWT)
  for bwt in lbwt repeat
    bwt.val =>
      if algebraic?(mvar(p),bwt.tower)
        then
          rsl := algebraicDecompose(p,bwt.tower,false)
        else
          rsl := transcendentalDecompose(p,bwt.tower)
      lts := concat(rsl.done,lts)
      llpwt :=  concat(rsl.todo,llpwt)
      purelyAlgebraic?(ip,bwt.tower) and purelyAlgebraic?(bwt.tower) => "leave"
      (not ground? ip) =>
        zero? tp => llpwt := cons([[ip],bwt.tower]$LpWT, llpwt)
        (not ground? tp) => llpwt := cons([[ip,tp],bwt.tower]$LpWT, llpwt)
    riv := removeZero(ip,bwt.tower)
    (zero? riv) =>
      zero? tp => lts := cons(bwt.tower,lts)
      (not ground? tp) => llpwt := cons([[tp],bwt.tower]$LpWT, llpwt)
    llpwt := cons([[riv * mainMonomial(p) + tp],bwt.tower]$LpWT, llpwt)
  [lts,llpwt]

decompose(lp: LP, lts: Split, clos?: B, info?: B): Split ==
  decompose(lp,lts,false,false,clos?,true,info?)

convert(lpwt: LpWT): String ==
  ls: List String := ["<", string((#(lpwt.val))::Z), ",", string((#(lpwt.tower))::Z), '
  concat ls
```

```
printInfo(toSee: List LpWT, n: N): Void ==
  lpwt := first toSee
  s: String := concat ["[", string((#toSee)::Z), " ", convert(lpwt)@String]
  m: N := #(lpwt.val)
  toSee := rest toSee
  for lpwt in toSee repeat
    m := m + #(lpwt.val)
    s := concat [s, ",", convert(lpwt)@String]
  s := concat [s, " -> |", string(m::Z), "|; {", string(n::Z),"}]"]
  iprint(s)$iprintpack
  void()

decompose(lp: LP, lts: Split, cleanW?: B, sqfr?: B, clos?: B, rem?: B, info?
  -- if cleanW? then REMOVE REDUNDANT COMPONENTS in lts
  -- if sqfr? then SPLIT the system with SQUARE-FREE FACTORIZATION
  -- if clos? then SOLVE in the closure sense
  -- if rem? then REDUCE the current p by using remainder
  -- if info? then PRINT info
  empty? lp => lts
  branches: List Branch := prepareDecompose(lp,lts,cleanW?,sqfr?)$quasicompp
  empty? branches => []
  toSee: List LpWT := [[br.eq,br.tower]$LpWT for br in branches]
  toSave: Split := []
  if clos? then bound := KrullNumber(lp,lts) else bound := numberOfVariables
  while (not empty? toSee) repeat
    if info? then printInfo(toSee,#toSave)
    lpwt := first toSee; toSee := rest toSee
    lp := lpwt.val; ts := lpwt.tower
    empty? lp =>
      toSave := cons(ts, toSave)
    p := first lp;  lp := rest lp
    if rem? and (not ground? p) and (not empty? ts)
       then
         p := remainder(p,ts).polnum
    p := removeZero(p,ts)
    zero? p => toSee := cons([lp,ts]$LpWT, toSee)
    ground? p => "leave"
    rsl := internalDecompose(p,ts,bound,clos?)
    toSee := upDateBranches(lp,toSave,toSee,rsl,bound)
  removeSuperfluousQuasiComponents(toSave)$quasicomppack

upDateBranches(leq:LP,lts:Split,current:List LpWT,wip: Wip,n:N): List LpWT =
  newBranches: List LpWT := wip.todo
  newComponents: Split := wip.done
  branches1, branches2:  List LpWT
```

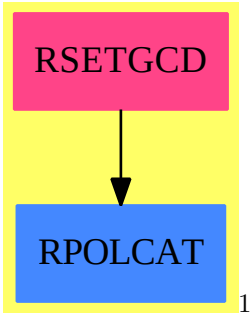
```
        branches1 := []; branches2  := []
        for branch in newBranches repeat
          us := branch.tower
          #us > n => "leave"
          newleq := sort(infRittWu?,concat(leq,branch.val))
          --foo := rewriteSetWithReduction(newleq,us,initiallyReduce,initiallyReduced?)
          --any?(ground?,foo)  => "leave"
          branches1 := cons([newleq,us]$LpWT, branches1)
        for us in newComponents repeat
          #us > n => "leave"
          subQuasiComponent?(us,lts)$quasicomppack => "leave"
          --newleq := leq
          --foo := rewriteSetWithReduction(newleq,us,initiallyReduce,initiallyReduced?)
          --any?(ground?,foo)  => "leave"
          branches2 := cons([leq,us]$LpWT, branches2)
        empty? branches1 =>
          empty? branches2 => current
          concat(branches2, current)
        branches := concat [branches2, branches1, current]
        -- branches := concat(branches,current)
        removeSuperfluousCases(branches)$quasicomppack
```

⟨*RSDCMPK.dotabb*⟩≡

```
"RSDCMPK" [color="#FF4488",href="bookvol10.4.pdf#nameddest=RSDCMPK"]
"RSETCAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=RSETCAT"]
"RSDCMPK" -> "RSETCAT"
```

19.65 package RSETGCD RegularTriangularSetGcdPackage

19.66 RegularTriangularSetGcdPackage



1.00

Exports:

stopTableGcd!	stopTableInvSet!	integralLastSubResultant
prepareSubResAlgo	startTableGcd!	startTableInvSet!
toseInvertible?	toseInvertibleSet	toseLastSubResultant
toseSquareFreePart		

⟨*package RSETGCD RegularTriangularSetGcdPackage*⟩≡

```
)abbrev package RSETGCD RegularTriangularSetGcdPackage
++ Author: Marc Moreno Maza (marc@nag.co.uk)
++ Date Created: 08/30/1998
++ Date Last Updated: 12/15/1998
++ Basic Functions:
++ Related Constructors:
++ Also See:
++ AMS Classifications:
++ Keywords:
++ Description:
++ An internal package for computing gcds and resultants of univariate
++ polynomials with coefficients in a tower of simple extensions of a field.\newl
++ References :
++  [1] M. MORENO MAZA and R. RIOBOO "Computations of gcd over
++      algebraic towers of simple extensions" In proceedings of AAECC11
++      Paris, 1995.
++  [2] M. MORENO MAZA "Calculs de pgcd au-dessus des tours
++      d'extensions simples et resolution des systemes d'equations
++      algebriques" These, Universite P.etM. Curie, Paris, 1997.
++  [3] M. MORENO MAZA "A new algorithm for computing triangular
++      decomposition of algebraic varieties" NAG Tech. Rep. 4/98.
++ Version: 4.
```

```
RegularTriangularSetGcdPackage(R,E,V,P,TS): Exports == Implementation where

  R : GcdDomain
  E : OrderedAbelianMonoidSup
  V : OrderedSet
  P : RecursivePolynomialCategory(R,E,V)
  TS : RegularTriangularSetCategory(R,E,V,P)
  N ==> NonNegativeInteger
  Z ==> Integer
  B ==> Boolean
  S ==> String
  LP ==> List P
  PtoP ==> P -> P
  PS ==> GeneralPolynomialSet(R,E,V,P)
  PWT ==> Record(val : P, tower : TS)
  BWT ==> Record(val : Boolean, tower : TS)
  LpWT ==> Record(val : (List P), tower : TS)
  Branch ==> Record(eq: List P, tower: TS, ineq: List P)
  UBF ==> Union(Branch,"failed")
  Split ==> List TS
  KeyGcd ==> Record(arg1: P, arg2: P, arg3: TS, arg4: B)
  EntryGcd ==> List PWT
  HGcd ==> TabulatedComputationPackage(KeyGcd, EntryGcd)
  KeyInvSet ==> Record(arg1: P, arg3: TS)
  EntryInvSet ==> List TS
  HInvSet ==> TabulatedComputationPackage(KeyInvSet, EntryInvSet)
  polsetpack ==> PolynomialSetUtilitiesPackage(R,E,V,P)
  quasicomppack ==> QuasiComponentPackage(R,E,V,P,TS)

  Exports ==  with
     startTableGcd!: (S,S,S) -> Void
         ++ \axiom{startTableGcd!(s1,s2,s3)}
         ++ is an internal subroutine, exported only for developement.
     stopTableGcd!: () -> Void
         ++ \axiom{stopTableGcd!()}
         ++ is an internal subroutine, exported only for developement.
     startTableInvSet!: (S,S,S) -> Void
         ++ \axiom{startTableInvSet!(s1,s2,s3)}
         ++ is an internal subroutine, exported only for developement.
     stopTableInvSet!: () -> Void
         ++ \axiom{stopTableInvSet!()} is an internal subroutine,
         ++ exported only for developement.
     prepareSubResAlgo: (P,P,TS) -> List LpWT
         ++ \axiom{prepareSubResAlgo(p1,p2,ts)}
         ++ is an internal subroutine, exported only for developement.
     internalLastSubResultant: (P,P,TS,B,B) -> List PWT
```

```
        ++ \axiom{internalLastSubResultant(p1,p2,ts,inv?,break?)}
        ++ is an internal subroutine, exported only for developement.
    internalLastSubResultant: (List LpWT,V,B) -> List PWT
        ++ \axiom{internalLastSubResultant(lpwt,v,flag)} is an internal
        ++ subroutine, exported only for developement.
    integralLastSubResultant: (P,P,TS) -> List PWT
        ++ \axiom{integralLastSubResultant(p1,p2,ts)}
        ++ is an internal subroutine, exported only for developement.
    toseLastSubResultant: (P,P,TS) -> List PWT
        ++ \axiom{toseLastSubResultant(p1,p2,ts)} has the same specifications as
        ++ \axiomOpFrom{lastSubResultant}{RegularTriangularSetCategory}.
    toseInvertible?: (P,TS) -> B
        ++ \axiom{toseInvertible?(p1,p2,ts)} has the same specifications as
        ++ \axiomOpFrom{invertible?}{RegularTriangularSetCategory}.
    toseInvertible?: (P,TS) -> List BWT
        ++ \axiom{toseInvertible?(p1,p2,ts)} has the same specifications as
        ++ \axiomOpFrom{invertible?}{RegularTriangularSetCategory}.
    toseInvertibleSet: (P,TS) -> Split
        ++ \axiom{toseInvertibleSet(p1,p2,ts)} has the same specifications as
        ++ \axiomOpFrom{invertibleSet}{RegularTriangularSetCategory}.
    toseSquareFreePart: (P,TS) -> List PWT
        ++ \axiom{toseSquareFreePart(p,ts)} has the same specifications as
        ++ \axiomOpFrom{squareFreePart}{RegularTriangularSetCategory}.

  Implementation == add

    startTableGcd!(ok: S, ko: S, domainName: S): Void ==
      initTable!()$HGcd
      printInfo!(ok,ko)$HGcd
      startStats!(domainName)$HGcd
      void()

    stopTableGcd!(): Void ==
      if makingStats?()$HGcd then printStats!()$HGcd
      clearTable!()$HGcd

    startTableInvSet!(ok: S, ko: S, domainName: S): Void ==
      initTable!()$HInvSet
      printInfo!(ok,ko)$HInvSet
      startStats!(domainName)$HInvSet
      void()

    stopTableInvSet!(): Void ==
      if makingStats?()$HInvSet then printStats!()$HInvSet
      clearTable!()$HInvSet
```

```
toseInvertible?(p:P,ts:TS): Boolean ==
  q := primitivePart initiallyReduce(p,ts)
  zero? q => false
  normalized?(q,ts) => true
  v := mvar(q)
  not algebraic?(v,ts) =>
    toCheck: List BWT := toseInvertible?(p,ts)@(List BWT)
    for bwt in toCheck repeat
      bwt.val = false => return false
    return true
  ts_v := select(ts,v)::P
  ts_v_- := collectUnder(ts,v)
  lgwt := internalLastSubResultant(ts_v,q,ts_v_-,false,true)
  for gwt in lgwt repeat
    g := gwt.val;
    (not ground? g) and (mvar(g) = v) =>
      return false
  true

toseInvertible?(p:P,ts:TS): List BWT ==
  q := primitivePart initiallyReduce(p,ts)
  zero? q => [[false,ts]$BWT]
  normalized?(q,ts) => [[true,ts]$BWT]
  v := mvar(q)
  not algebraic?(v,ts) =>
    lbwt: List BWT := []
    toCheck: List BWT := toseInvertible?(init(q),ts)@(List BWT)
    for bwt in toCheck repeat
      bwt.val => lbwt := cons(bwt,lbwt)
      newq := removeZero(q,bwt.tower)
      zero? newq => lbwt := cons(bwt,lbwt)
      lbwt := concat(toseInvertible?(newq,bwt.tower)@(List BWT), lbwt)
    return lbwt
  ts_v := select(ts,v)::P
  ts_v_- := collectUnder(ts,v)
  ts_v_+ := collectUpper(ts,v)
  lgwt := internalLastSubResultant(ts_v,q,ts_v_-,false,false)
  lbwt: List BWT := []
  for gwt in lgwt repeat
    g := gwt.val; ts := gwt.tower
    (ground? g) or (mvar(g) < v) =>
      ts := internalAugment(ts_v,ts)
      ts := internalAugment(members(ts_v_+),ts)
      lbwt := cons([true, ts]$BWT,lbwt)
    g := mainPrimitivePart g
    ts_g := internalAugment(g,ts)
```

```
      ts_g := internalAugment(members(ts_v_+),ts_g)
      -- USE internalAugment with parameters ??
      lbwt := cons([false, ts_g]$BWT,lbwt)
      h := lazyPquo(ts_v,g)
      (ground? h) or (mvar(h) < v) => "leave"
      h := mainPrimitivePart h
      ts_h := internalAugment(h,ts)
      ts_h := internalAugment(members(ts_v_+),ts_h)
      -- USE internalAugment with parameters ??
      -- CAN BE OPTIMIZED if the input tower is separable
      inv := toseInvertible?(q,ts_h)@(List BWT)
      lbwt := concat([bwt for bwt in inv | bwt.val],lbwt)
    sort(#1.val < #2.val,lbwt)

  toseInvertibleSet(p:P,ts:TS): Split ==
    k: KeyInvSet := [p,ts]
    e := extractIfCan(k)$HInvSet
    e case EntryInvSet => e::EntryInvSet
    q := primitivePart initiallyReduce(p,ts)
    zero? q => []
    normalized?(q,ts) => [ts]
    v := mvar(q)
    toSave: Split := []
    not algebraic?(v,ts) =>
      toCheck: List BWT := toseInvertible?(init(q),ts)@(List BWT)
      for bwt in toCheck repeat
        bwt.val => toSave := cons(bwt.tower,toSave)
        newq := removeZero(q,bwt.tower)
        zero? newq => "leave"
        toSave := concat(toseInvertibleSet(newq,bwt.tower), toSave)
      toSave := removeDuplicates toSave
      return algebraicSort(toSave)$quasicomppack
    ts_v := select(ts,v)::P
    ts_v_- := collectUnder(ts,v)
    ts_v_+ := collectUpper(ts,v)
    lgwt := internalLastSubResultant(ts_v,q,ts_v_-,false,false)
    for gwt in lgwt repeat
      g := gwt.val; ts := gwt.tower
      (ground? g) or (mvar(g) < v) =>
        ts := internalAugment(ts_v,ts)
        ts := internalAugment(members(ts_v_+),ts)
        toSave := cons(ts,toSave)
      g := mainPrimitivePart g
      h := lazyPquo(ts_v,g)
      h := mainPrimitivePart h
      (ground? h) or (mvar(h) < v) => "leave"
```

```
          ts_h := internalAugment(h,ts)
          ts_h := internalAugment(members(ts_v_+),ts_h)
          inv := toseInvertibleSet(q,ts_h)
          toSave := removeDuplicates concat(inv,toSave)
        toSave := algebraicSort(toSave)$quasicomppack
        insert!(k,toSave)$HInvSet
        toSave

      toseSquareFreePart_wip(p:P, ts: TS): List PWT ==
      -- ASSUME p is not constant and mvar(p) > mvar(ts)
      -- ASSUME init(p) is invertible w.r.t. ts
      -- ASSUME p is mainly primitive
--        one? mdeg(p) => [[p,ts]$PWT]
        mdeg(p) = 1 => [[p,ts]$PWT]
        v := mvar(p)$P
        q: P := mainPrimitivePart D(p,v)
        lgwt: List PWT := internalLastSubResultant(p,q,ts,true,false)
        lpwt : List PWT := []
        sfp : P
        for gwt in lgwt repeat
          g := gwt.val; us := gwt.tower
          (ground? g) or (mvar(g) < v) =>
            lpwt := cons([p,us],lpwt)
          g := mainPrimitivePart g
          sfp := lazyPquo(p,g)
          sfp := mainPrimitivePart stronglyReduce(sfp,us)
          lpwt := cons([sfp,us],lpwt)
        lpwt

      toseSquareFreePart_base(p:P, ts: TS): List PWT == [[p,ts]$PWT]

      toseSquareFreePart(p:P, ts: TS): List PWT == toseSquareFreePart_wip(p,ts)

      prepareSubResAlgo(p1:P,p2:P,ts:TS): List LpWT ==
        -- ASSUME mvar(p1) = mvar(p2) > mvar(ts) and mdeg(p1) >= mdeg(p2)
        -- ASSUME init(p1) invertible modulo ts !!!
        toSee: List LpWT := [[[p1,p2],ts]$LpWT]
        toSave: List LpWT := []
        v := mvar(p1)
        while (not empty? toSee) repeat
          lpwt := first toSee; toSee := rest toSee
          p1 := lpwt.val.1; p2 := lpwt.val.2
          ts := lpwt.tower
          lbwt := toseInvertible?(leadingCoefficient(p2,v),ts)@(List BWT)
          for bwt in lbwt repeat
            (bwt.val = true) and (degree(p2,v) > 0) =>
```

```
            p3 := prem(p1, -p2)
            s: P := init(p2)**(mdeg(p1) - mdeg(p2))::N
            toSave := cons([[p2,p3,s],bwt.tower]$LpWT,toSave)
          -- p2 := initiallyReduce(p2,bwt.tower)
          newp2 := primitivePart initiallyReduce(p2,bwt.tower)
          (bwt.val = true) =>
            -- toSave := cons([[p2,0,1],bwt.tower]$LpWT,toSave)
            toSave := cons([[p2,0,1],bwt.tower]$LpWT,toSave)
          -- zero? p2 =>
          zero? newp2 =>
            toSave := cons([[p1,0,1],bwt.tower]$LpWT,toSave)
          -- toSee := cons([[p1,p2],ts]$LpWT,toSee)
          toSee := cons([[p1,newp2],bwt.tower]$LpWT,toSee)
        toSave

      integralLastSubResultant(p1:P,p2:P,ts:TS): List PWT ==
        -- ASSUME mvar(p1) = mvar(p2) > mvar(ts) and mdeg(p1) >= mdeg(p2)
        -- ASSUME p1 and p2 have no algebraic coefficients
        lsr := lastSubResultant(p1, p2)
        ground?(lsr) => [[lsr,ts]$PWT]
        mvar(lsr) < mvar(p1) => [[lsr,ts]$PWT]
        gi1i2 := gcd(init(p1),init(p2))
        ex: Union(P,"failed") := (gi1i2 * lsr) exquo$P init(lsr)
        ex case "failed" => [[lsr,ts]$PWT]
        [[ex::P,ts]$PWT]

      internalLastSubResultant(p1:P,p2:P,ts:TS,b1:B,b2:B): List PWT ==
        -- ASSUME mvar(p1) = mvar(p2) > mvar(ts) and mdeg(p1) >= mdeg(p2)
        -- if b1 ASSUME init(p2) invertible w.r.t. ts
        -- if b2 BREAK with the first non-trivial gcd
        k: KeyGcd := [p1,p2,ts,b2]
        e := extractIfCan(k)$HGcd
        e case EntryGcd => e::EntryGcd
        toSave: List PWT
        empty? ts =>
          toSave := integralLastSubResultant(p1,p2,ts)
          insert!(k,toSave)$HGcd
          return toSave
        toSee: List LpWT
        if b1
          then
            p3 := prem(p1, -p2)
            s: P := init(p2)**(mdeg(p1) - mdeg(p2))::N
            toSee := [[[p2,p3,s],ts]$LpWT]
          else
            toSee := prepareSubResAlgo(p1,p2,ts)
```

```
    toSave := internalLastSubResultant(toSee,mvar(p1),b2)
    insert!(k,toSave)$HGcd
    toSave

  internalLastSubResultant(llpwt: List LpWT,v:V,b2:B): List PWT ==
    toReturn: List PWT := []; toSee: List LpWT;
    while (not empty? llpwt) repeat
      toSee := llpwt; llpwt := []
      -- CONSIDER FIRST the vanishing current last subresultant
      for lpwt in toSee repeat
        p1 := lpwt.val.1; p2 := lpwt.val.2; s := lpwt.val.3; ts := lpwt.tower
        lbwt := toseInvertible?(leadingCoefficient(p2,v),ts)@(List BWT)
        for bwt in lbwt repeat
          bwt.val = false =>
            toReturn := cons([p1,bwt.tower]$PWT, toReturn)
            b2 and positive?(degree(p1,v)) => return toReturn
          llpwt := cons([[p1,p2,s],bwt.tower]$LpWT, llpwt)
      empty? llpwt => "leave"
      -- CONSIDER NOW the branches where the computations continue
      toSee := llpwt; llpwt := []
      lpwt := first toSee; toSee := rest toSee
      p1 := lpwt.val.1; p2 := lpwt.val.2; s := lpwt.val.3
      delta: N := (mdeg(p1) - degree(p2,v))::N
      p3: P := LazardQuotient2(p2, leadingCoefficient(p2,v), s, delta)
      zero?(degree(p3,v)) =>
        toReturn := cons([p3,lpwt.tower]$PWT, toReturn)
        for lpwt in toSee repeat
          toReturn := cons([p3,lpwt.tower]$PWT, toReturn)
      (p1, p2) := (p3, next_subResultant2(p1, p2, p3, s))
      s := leadingCoefficient(p1,v)
      llpwt := cons([[p1,p2,s],lpwt.tower]$LpWT, llpwt)
      for lpwt in toSee repeat
        llpwt := cons([[p1,p2,s],lpwt.tower]$LpWT, llpwt)
    toReturn

  toseLastSubResultant(p1:P,p2:P,ts:TS): List PWT ==
    ground? p1 =>
      error"in toseLastSubResultantElseSplit$TOSEGCD  : bad #1"
    ground? p2 =>
      error"in toseLastSubResultantElseSplit$TOSEGCD : bad #2"
    not (mvar(p2) = mvar(p1)) =>
      error"in toseLastSubResultantElseSplit$TOSEGCD : bad #2"
    algebraic?(mvar(p1),ts) =>
      error"in toseLastSubResultantElseSplit$TOSEGCD : bad #1"
    not initiallyReduced?(p1,ts) =>
      error"in toseLastSubResultantElseSplit$TOSEGCD : bad #1"
```

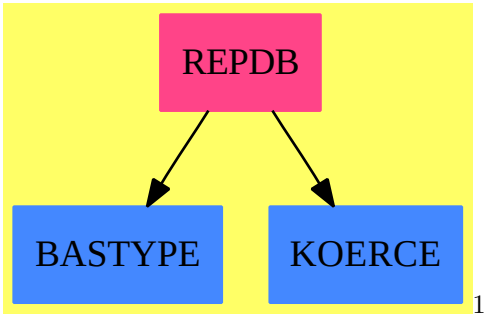
```
        not initiallyReduced?(p2,ts) =>
          error"in toseLastSubResultantElseSplit$TOSEGCD : bad #2"
        purelyTranscendental?(p1,ts) and purelyTranscendental?(p2,ts) =>
          integralLastSubResultant(p1,p2,ts)
        if mdeg(p1) < mdeg(p2) then
           (p1, p2) := (p2, p1)
           if odd?(mdeg(p1)) and odd?(mdeg(p2)) then p2 := - p2
        internalLastSubResultant(p1,p2,ts,false,false)
```

⟨*RSETGCD.dotabb*⟩≡

```
"RSETGCD" [color="#FF4488",href="bookvol10.4.pdf#nameddest=RSETGCD"]
"RPOLCAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=RPOLCAT"]
"RSETGCD" -> "RPOLCAT"
```

19.67 package REPDB RepeatedDoubling

19.68 RepeatedDoubling



1.00

Exports:
double

⟨*package REPDB RepeatedDoubling*⟩≡

```
)abbrev package REPDB RepeatedDoubling
++ Repeated Doubling
++ Integer multiplication by repeated doubling.
++ Description:
++ Implements multiplication by repeated addition
++ RelatedOperations: *

-- the following package is only instantiated over %
-- thus shouldn't be cached. We prevent it
-- from being cached by declaring it to be mutableDomains

)bo PUSH('RepeatedDoubling, $mutableDomains)

RepeatedDoubling(S):Exports ==Implementation where
   S: SetCategory with
        "+":(%,%)->%
                ++ x+y returns the sum of x and y
   Exports == with
     double: (PositiveInteger,S) -> S
       ++ double(i, r) multiplies r by i using repeated doubling.
   Implementation == add
     x: S
     n: PositiveInteger
     double(n,x) ==
--         one? n => x
        (n = 1) => x
        odd?(n)$Integer =>
```

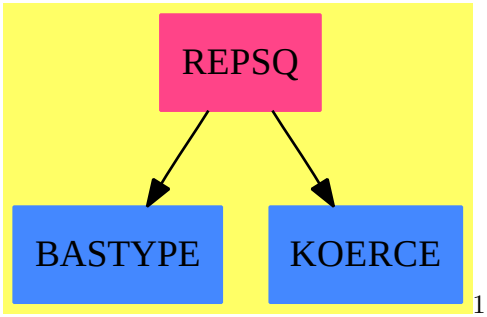
```
            x + double(shift(n,-1) pretend PositiveInteger,(x+x))
          double(shift(n,-1) pretend PositiveInteger,(x+x))
```

⟨*REPDB.dotabb*⟩≡

```
"REPDB" [color="#FF4488",href="bookvol10.4.pdf#nameddest=REPDB"]
"BASTYPE" [color="#4488FF",href="bookvol10.2.pdf#nameddest=BASTYPE"]
"KOERCE" [color="#4488FF",href="bookvol10.2.pdf#nameddest=KOERCE"]
"REPDB" -> "BASTYPE"
"REPDB" -> "KOERCE"
```

19.69 package REPSQ RepeatedSquaring

19.70 RepeatedSquaring



1.00

Exports:
expt

⟨*package REPSQ RepeatedSquaring*⟩≡
```
)abbrev package REPSQ RepeatedSquaring
++ Repeated Squaring
++ Description:
++ Implements exponentiation by repeated squaring
++ RelatedOperations: expt
-- the following package is only instantiated over %
-- thus shouldn't be cached. We prevent it
-- from being cached by declaring it to be mutableDomains

)bo PUSH('RepeatedSquaring, $mutableDomains)

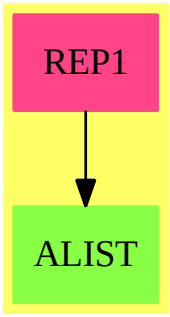
RepeatedSquaring(S): Exports == Implementation where
   S: SetCategory with
        "*":(%,%)->%
             ++ x*y returns the product of x and y
   Exports == with
     expt: (S,PositiveInteger) -> S
       ++ expt(r, i) computes r**i  by repeated squaring
   Implementation == add
     x: S
     n: PositiveInteger
     expt(x, n) ==
--         one? n => x
        (n = 1) => x
        odd?(n)$Integer=> x * expt(x*x,shift(n,-1) pretend PositiveInteger)
        expt(x*x,shift(n,-1) pretend PositiveInteger)
```

⟨*REPSQ.dotabb*⟩≡

```
"REPSQ" [color="#FF4488",href="bookvol10.4.pdf#nameddest=REPSQ"]
"BASTYPE" [color="#4488FF",href="bookvol10.2.pdf#nameddest=BASTYPE"]
"KOERCE" [color="#4488FF",href="bookvol10.2.pdf#nameddest=KOERCE"]
"REPSQ" -> "BASTYPE"
"REPSQ" -> "KOERCE"
```

19.71 package REP1 RepresentationPackage1

19.72 RepresentationPackage1



1.00

Exports:

antisymmetricTensors	createGenericMatrix	permutationRepresentation
symmetricTensors	tensorProduct	

⟨*package REP1 RepresentationPackage1*⟩≡

```
)abbrev package REP1 RepresentationPackage1
++ Authors: Holger Gollan, Johannes Grabmeier, Thorsten Werther
++ Date Created: 12 September 1987
++ Date Last Updated: 24 May 1991
++ Basic Operations: antisymmetricTensors,symmetricTensors,
++   tensorProduct, permutationRepresentation
++ Related Constructors: RepresentationPackage1, Permutation
++ Also See: IrrRepSymNatPackage
++ AMS Classifications:
++ Keywords: representation, symmetrization, tensor product
++ References:
++   G. James, A. Kerber: The Representation Theory of the Symmetric
++    Group. Encycl. of Math. and its Appl. Vol 16., Cambr. Univ Press 1981;
++   J. Grabmeier, A. Kerber: The Evaluation of Irreducible
++    Polynomial Representations of the General Linear Groups
++    and of the Unitary Groups over Fields of Characteristic 0,
++    Acta Appl. Math. 8 (1987), 271-291;
++   H. Gollan, J. Grabmeier: Algorithms in Representation Theory and
++    their Realization in the Computer Algebra System Scratchpad,
++    Bayreuther Mathematische Schriften, Heft 33, 1990, 1-23
++ Description:
++   RepresentationPackage1 provides functions for representation theory
++   for finite groups and algebras.
++   The package creates permutation representations and uses tensor products
++   and its symmetric and antisymmetric components to create new
++   representations of larger degree from given ones.
```

```
++   Note: instead of having parameters from \spadtype{Permutation}
++   this package allows list notation of permutations as well:
++   e.g. \spad{[1,4,3,2]} denotes permutes 2 and 4 and fixes 1 and 3.

RepresentationPackage1(R): public == private where

   R   : Ring
   OF    ==> OutputForm
   NNI   ==> NonNegativeInteger
   PI    ==> PositiveInteger
   I     ==> Integer
   L     ==> List
   M     ==> Matrix
   P     ==> Polynomial
   SM    ==> SquareMatrix
   V     ==> Vector
   ICF   ==> IntegerCombinatoricFunctions Integer
   SGCF  ==> SymmetricGroupCombinatoricFunctions
   PERM  ==> Permutation

   public ==> with

     if R has commutative("*") then
       antisymmetricTensors : (M R,PI) ->  M R
         ++ antisymmetricTensors(a,n) applies to the square matrix
         ++ {\em a} the irreducible, polynomial representation of the
         ++ general linear group {\em GLm}, where m is the number of
         ++ rows of {\em a}, which corresponds to the partition
         ++ {\em (1,1,...,1,0,0,...,0)} of n.
         ++ Error: if n is greater than m.
         ++ Note: this corresponds to the symmetrization of the representation
         ++ with the sign representation of the symmetric group {\em Sn}.
         ++ The carrier spaces of the representation are the antisymmetric
         ++ tensors of the n-fold tensor product.
     if R has commutative("*") then
       antisymmetricTensors : (L M R, PI) -> L M R
         ++ antisymmetricTensors(la,n) applies to each
         ++ m-by-m square matrix in
         ++ the list {\em la} the irreducible, polynomial representation
         ++ of the general linear group {\em GLm}
         ++ which corresponds
         ++ to the partition {\em (1,1,...,1,0,0,...,0)} of n.
         ++ Error: if n is greater than m.
         ++ Note: this corresponds to the symmetrization of the representation
         ++ with the sign representation of the symmetric group {\em Sn}.
         ++ The carrier spaces of the representation are the antisymmetric
```

```
    ++ tensors of the n-fold tensor product.
createGenericMatrix : NNI -> M P R
  ++ createGenericMatrix(m) creates a square matrix of dimension k
  ++ whose entry at the i-th row and j-th column is the
  ++ indeterminate {\em x[i,j]} (double subscripted).
symmetricTensors : (M R, PI) -> M R
  ++ symmetricTensors(a,n) applies to the m-by-m
  ++ square matrix {\em a} the
  ++ irreducible, polynomial representation of the general linear
  ++ group {\em GLm}
  ++ which corresponds to the partition {\em (n,0,...,0)} of n.
  ++ Error: if {\em a} is not a square matrix.
  ++ Note: this corresponds to the symmetrization of the representation
  ++ with the trivial representation of the symmetric group {\em Sn}.
  ++ The carrier spaces of the representation are the symmetric
  ++ tensors of the n-fold tensor product.
symmetricTensors : (L M R, PI)  ->  L M R
  ++ symmetricTensors(la,n) applies to each m-by-m square matrix in the
  ++ list {\em la} the irreducible, polynomial representation
  ++ of the general linear group {\em GLm}
  ++ which corresponds
  ++ to the partition {\em (n,0,...,0)} of n.
  ++ Error: if the matrices in {\em la} are not square matrices.
  ++ Note: this corresponds to the symmetrization of the representation
  ++ with the trivial representation of the symmetric group {\em Sn}.
  ++ The carrier spaces of the representation are the symmetric
  ++ tensors of the n-fold tensor product.
tensorProduct : (M R, M R) -> M R
  ++ tensorProduct(a,b) calculates the Kronecker product
  ++ of the matrices {\em a} and b.
  ++ Note: if each matrix corresponds to a group representation
  ++ (repr. of  generators) of one group, then these matrices
  ++ correspond to the tensor product of the two representations.
tensorProduct : (L M R, L M R) -> L M R
  ++ tensorProduct([a1,...,ak],[b1,...,bk]) calculates the list of
  ++ Kronecker products of the matrices {\em ai} and {\em bi}
  ++ for {1 <= i <= k}.
  ++ Note: If each list of matrices corresponds to a group representation
  ++ (repr. of generators) of one group, then these matrices
  ++ correspond to the tensor product of the two representations.
tensorProduct : M R -> M R
  ++ tensorProduct(a) calculates the Kronecker product
  ++ of the matrix {\em a} with itself.
tensorProduct : L M R -> L M R
  ++ tensorProduct([a1,...ak]) calculates the list of
  ++ Kronecker products of each matrix {\em ai} with itself
```

```
      ++ for {1 <= i <= k}.
      ++ Note: If the list of matrices corresponds to a group representation
      ++ (repr. of generators) of one group, then these matrices correspond
      ++ to the tensor product of the representation with itself.
    permutationRepresentation : (PERM I, I) -> M I
      ++ permutationRepresentation(pi,n) returns the matrix
      ++ {\em (deltai,pi(i))} (Kronecker delta) for a permutation
      ++ {\em pi} of {\em {1,2,...,n}}.
    permutationRepresentation : L I -> M I
      ++ permutationRepresentation(pi,n) returns the matrix
      ++ {\em (deltai,pi(i))} (Kronecker delta) if the permutation
      ++ {\em pi} is in list notation and permutes {\em {1,2,...,n}}.
    permutationRepresentation : (L PERM I, I) -> L M I
      ++ permutationRepresentation([pi1,...,pik],n) returns the list
      ++ of matrices {\em [(deltai,pi1(i)),...,(deltai,pik(i))]}
      ++ (Kronecker delta) for the permutations {\em pi1,...,pik}
      ++ of {\em {1,2,...,n}}.
    permutationRepresentation : L L I -> L M I
      ++ permutationRepresentation([pi1,...,pik],n) returns the list
      ++ of matrices {\em [(deltai,pi1(i)),...,(deltai,pik(i))]}
      ++ if the permutations {\em pi1},...,{\em pik} are in
      ++ list notation and are permuting {\em {1,2,...,n}}.

  private ==> add

    -- import of domains and packages

    import OutputForm

    -- declaration of local functions:


    calcCoef : (L I, M I) -> I
      -- calcCoef(beta,C) calculates the term
      -- |S(beta) gamma S(alpha)| / |S(beta)|


    invContent : L I -> V I
      -- invContent(alpha) calculates the weak monoton function f with
      -- f : m -> n with invContent alpha. f is stored in the returned
      -- vector


    -- definition of local functions
```

```
calcCoef(beta,C) ==
  prod : I := 1
  for i in 1..maxIndex beta repeat
    prod := prod * multinomial(beta(i), entries row(C,i))$ICF
  prod


invContent(alpha) ==
  n : NNI := (+/alpha)::NNI
  f : V I  := new(n,0)
  i : NNI := 1
  j : I   := - 1
  for og in alpha repeat
    j := j + 1
    for k in 1..og repeat
      f(i) := j
      i := i + 1
  f


-- exported functions:



if R has commutative("*") then
  antisymmetricTensors ( a : M R , k : PI ) ==

    n      : NNI   := nrows a
    k = 1 => a
    k > n =>
      error("second parameter for antisymmetricTensors is too large")
    m      :   I   := binomial(n,k)$ICF
    il     : L L I   := [subSet(n,k,i)$SGCF for i in 0..m-1]
    b      :  M R   := zero(m::NNI, m::NNI)
    for i in 1..m repeat
      for j in 1..m repeat
        c : M R := zero(k,k)
        lr: L I := il.i
        lt: L I := il.j
        for  r in 1..k repeat
          for t in 1..k repeat
            rr : I := lr.r
            tt : I := lt.t
            --c.r.t := a.(1+rr).(1+tt)
            setelt(c,r,t,elt(a, 1+rr, 1+tt))
        setelt(b, i, j, determinant c)
```

```
    b


if R has commutative("*") then
  antisymmetricTensors(la: L M R, k: PI) ==
    [antisymmetricTensors(ma,k) for ma in la]



symmetricTensors (a : M R, n : PI) ==

  m : NNI := nrows a
  m ^= ncols a =>
    error("Input to symmetricTensors is no square matrix")
  n = 1 => a

  dim : NNI := (binomial(m+n-1,n)$ICF)::NNI
  c : M R := new(dim,dim,0)
  f : V I := new(n,0)
  g : V I := new(n,0)
  nullMatrix : M I := new(1,1,0)
  colemanMatrix : M I

  for i in 1..dim repeat
    -- unrankImproperPartitions1 starts counting from 0
    alpha := unrankImproperPartitions1(n,m,i-1)$SGCF
    f := invContent(alpha)
    for j in 1..dim repeat
      -- unrankImproperPartitions1 starts counting from 0
      beta := unrankImproperPartitions1(n,m,j-1)$SGCF
      g := invContent(beta)
      colemanMatrix := nextColeman(alpha,beta,nullMatrix)$SGCF
      while colemanMatrix ^= nullMatrix repeat
        gamma := inverseColeman(alpha,beta,colemanMatrix)$SGCF
        help : R := calcCoef(beta,colemanMatrix)::R
        for k in 1..n repeat
          help := help * a( (1+f k)::NNI, (1+g(gamma k))::NNI )
        c(i,j) := c(i,j) + help
        colemanMatrix := nextColeman(alpha,beta,colemanMatrix)$SGCF
      -- end of while
    -- end of j-loop
  -- end of i-loop

  c
```

```
symmetricTensors(la : L M R, k : PI) ==
  [symmetricTensors (ma, k) for ma in la]


tensorProduct(a: M R, b: M R) ==
  n      : NNI := nrows a
  m      : NNI := nrows b
  nc     : NNI := ncols a
  mc     : NNI := ncols b
  c      : M R   := zero(n * m, nc * mc)
  indexr : NNI := 1                              --   row index
  for i in 1..n repeat
     for k in 1..m repeat
        indexc : NNI := 1                        --   column index
        for j in 1..nc repeat
           for l in 1..mc repeat
              c(indexr,indexc) := a(i,j) * b(k,l)
              indexc          := indexc + 1
        indexr := indexr + 1
  c


tensorProduct (la: L M R, lb: L M R) ==
  [tensorProduct(la.i, lb.i) for i in 1..maxIndex la]


tensorProduct(a : M R) == tensorProduct(a, a)

tensorProduct(la : L M R) ==
  tensorProduct(la :: L M R, la :: L M R)

permutationRepresentation (p : PERM I, n : I) ==
  -- permutations are assumed to permute {1,2,...,n}
  a : M I := zero(n :: NNI, n :: NNI)
  for i in 1..n repeat
     a(eval(p,i)$(PERM I),i) := 1
  a


permutationRepresentation (p : L I) ==
  -- permutations are assumed to permute {1,2,...,n}
  n : I := #p
  a : M I := zero(n::NNI, n::NNI)
  for i in 1..n repeat
     a(p.i,i) := 1
  a
```

```
permutationRepresentation(listperm : L PERM I, n : I) ==
  -- permutations are assumed to permute {1,2,...,n}
  [permutationRepresentation(perm, n) for perm in listperm]

permutationRepresentation (listperm : L L I) ==
  -- permutations are assumed to permute {1,2,...,n}
  [permutationRepresentation perm for perm in listperm]

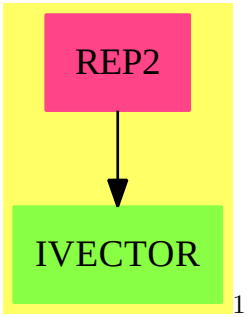
createGenericMatrix(m) ==
  res : M P R := new(m,m,0$(P R))
  for i in 1..m repeat
    for j in 1..m repeat
       iof : OF := coerce(i)$Integer
       jof : OF := coerce(j)$Integer
       le : L OF := cons(iof,list jof)
       sy : Symbol := subscript(x::Symbol, le)$Symbol
       res(i,j) := (sy :: P R)
  res
```

⟨*REP1.dotabb*⟩≡

```
"REP1" [color="#FF4488",href="bookvol10.4.pdf#nameddest=REP1"]
"ALIST" [color="#88FF44",href="bookvol10.3.pdf#nameddest=ALIST"]
"REP1" -> "ALIST"
```

19.73 package REP2 RepresentationPackage2

19.74 RepresentationPackage2



1.00

Exports:

areEquivalent?	completeEchelonBasis
createRandomElement	cyclicSubmodule
isAbsolutelyIrreducible?	meatAxe
scanOneDimSubspaces	split
standardBasisOfCyclicSubmodule	

⟨*package REP2 RepresentationPackage2*⟩≡

```
)abbrev package REP2 RepresentationPackage2
++ Authors: Holger Gollan, Johannes Grabmeier
++ Date Created: 10 September 1987
++ Date Last Updated: 20 August 1990
++ Basic Operations: areEquivalent?, isAbsolutelyIrreducible?,
++   split, meatAxe
++ Related Constructors: RepresentationTheoryPackage1
++ Also See: IrrRepSymNatPackage
++ AMS Classifications:
++ Keywords: meat-axe, modular representation
++ Reference:
++   R. A. Parker: The Computer Calculation of Modular Characters
++   (The Meat-Axe), in M. D. Atkinson (Ed.), Computational Group Theory
++   Academic Press, Inc., London 1984
++   H. Gollan, J. Grabmeier: Algorithms in Representation Theory and
++    their Realization in the Computer Algebra System Scratchpad,
++    Bayreuther Mathematische Schriften, Heft 33, 1990, 1-23.
++ Description:
++   RepresentationPackage2 provides functions for working with
++   modular representations of finite groups and algebra.
++   The routines in this package are created, using ideas of R. Parker,
++   (the meat-Axe) to get smaller representations from bigger ones,
++   i.e. finding sub- and factormodules, or to show, that such the
```

```
++   representations are irreducible.
++   Note: most functions are randomized functions of Las Vegas type
++   i.e. every answer is correct, but with small probability
++   the algorithm fails to get an answer.
RepresentationPackage2(R): public == private where

  R    : Ring
  OF    ==> OutputForm
  I     ==> Integer
  L     ==> List
  SM    ==> SquareMatrix
  M     ==> Matrix
  NNI   ==> NonNegativeInteger
  V     ==> Vector
  PI    ==> PositiveInteger
  B     ==> Boolean
  RADIX ==> RadixExpansion

  public ==> with

    completeEchelonBasis : V V R  ->  M R
      ++ completeEchelonBasis(lv) completes the basis {\em lv} assumed
      ++ to be in echelon form of a subspace of {\em R**n} (n the length
      ++ of all the vectors in {\em lv}) with unit vectors to a basis of
      ++ {\em R**n}. It is assumed that the argument is not an empty
      ++ vector and that it is not the basis of the 0-subspace.
      ++ Note: the rows of the result correspond to the vectors of the basis.
    createRandomElement : (L M R, M R) -> M R
      ++ createRandomElement(aG,x) creates a random element of the group
      ++ algebra generated by {\em aG}.
    -- randomWord : (L L I, L M)  ->  M R
      --++ You can create your own 'random' matrix with "randomWord(lli, lm)".
      --++ Each li in lli determines a product of matrices, the entries in li
      --++ determine which matrix from lm is chosen. Finally we sum over all
      --++ products. The result "sm" can be used to call split with (e.g.)
      --++ second parameter "first nullSpace sm"
    if R has EuclideanDomain then   -- using rowEchelon
      cyclicSubmodule  : (L M R, V R) ->   V V R
        ++ cyclicSubmodule(lm,v) generates a basis as follows.
        ++ It is assumed that the size n of the vector equals the number
        ++ of rows and columns of the matrices. Then the matrices generate
        ++ a subalgebra, say \spad{A}, of the algebra of all square matrices of
        ++ dimension n. {\em V R} is an \spad{A}-module in the natural way.
        ++ cyclicSubmodule(lm,v) generates the R-Basis of {\em Av} as
        ++ described in section 6 of R. A. Parker's "The Meat-Axe".
        ++ Note: in contrast to the description in "The Meat-Axe" and to
```

```
      ++ {\em standardBasisOfCyclicSubmodule} the result is in
      ++ echelon form.
    standardBasisOfCyclicSubmodule  : (L M R, V R) ->   M R
      ++ standardBasisOfCyclicSubmodule(lm,v) returns a matrix as follows.
      ++ It is assumed that the size n of the vector equals the number
      ++ of rows and columns of the matrices.  Then the matrices generate
      ++ a subalgebra, say \spad{A},
      ++ of the algebra of all square matrices of
      ++ dimension n. {\em V R} is an \spad{A}-module in the natural way.
      ++ standardBasisOfCyclicSubmodule(lm,v) calculates a matrix whose
      ++ non-zero column vectors are the R-Basis of {\em Av} achieved
      ++ in the way as described in section 6
      ++ of R. A. Parker's "The Meat-Axe".
      ++ Note: in contrast to {\em cyclicSubmodule}, the result is not
      ++ in echelon form.
  if R has Field then  -- only because of inverse in SM
    areEquivalent? : (L M R, L M R, B, I) -> M R
      ++ areEquivalent?(aG0,aG1,randomelements,numberOfTries) tests
      ++ whether the two lists of matrices, all assumed of same
      ++ square shape, can be simultaneously conjugated by a non-singular
      ++ matrix. If these matrices represent the same group generators,
      ++ the representations are equivalent.
      ++ The algorithm tries
      ++ {\em numberOfTries} times to create elements in the
      ++ generated algebras in the same fashion. If their ranks differ,
      ++ they are not equivalent. If an
      ++ isomorphism is assumed, then
      ++ the kernel of an element of the first algebra
      ++ is mapped to the kernel of the corresponding element in the
      ++ second algebra. Now consider the one-dimensional ones.
      ++ If they generate the whole space (e.g. irreducibility !)
      ++ we use {\em standardBasisOfCyclicSubmodule} to create the
      ++ only possible transition matrix. The method checks whether the
      ++ matrix conjugates all corresponding matrices from {\em aGi}.
      ++ The way to choose the singular matrices is as in {\em meatAxe}.
      ++ If the two representations are equivalent, this routine
      ++ returns the transformation matrix {\em TM} with
      ++ {\em aG0.i * TM = TM * aG1.i} for all i. If the representations
      ++ are not equivalent, a small 0-matrix is returned.
      ++ Note: the case
      ++ with different sets of group generators cannot be handled.
    areEquivalent? : (L M R, L M R) -> M R
      ++ areEquivalent?(aG0,aG1) calls {\em areEquivalent?(aG0,aG1,true,25)}.
      ++ Note: the choice of 25 was rather arbitrary.
    areEquivalent? : (L M R, L M R, I) -> M R
      ++ areEquivalent?(aG0,aG1,numberOfTries) calls
```

```
      ++ {\em areEquivalent?(aG0,aG1,true,25)}.
      ++ Note: the choice of 25 was rather arbitrary.
    isAbsolutelyIrreducible? : (L M R, I) -> B
      ++ isAbsolutelyIrreducible?(aG, numberOfTries) uses
      ++ Norton's irreducibility test to check for absolute
      ++ irreduciblity, assuming if a one-dimensional kernel is found.
      ++ As no field extension changes create "new" elements
      ++ in a one-dimensional space, the criterium stays true
      ++ for every extension. The method looks for one-dimensionals only
      ++ by creating random elements (no fingerprints) since
      ++ a run of {\em meatAxe} would have proved absolute irreducibility
      ++ anyway.
    isAbsolutelyIrreducible? : L M R -> B
      ++ isAbsolutelyIrreducible?(aG) calls
      ++ {\em isAbsolutelyIrreducible?(aG,25)}.
      ++ Note: the choice of 25 was rather arbitrary.
    split : (L M R, V R)  ->  L L M R
      ++ split(aG, vector) returns a subalgebra \spad{A} of all
      ++ square matrix of dimension n as a list of list of matrices,
      ++ generated by the list of matrices aG, where n denotes both
      ++ the size of vector as well as the dimension of each of the
      ++ square matrices.
      ++ {\em V R} is an A-module in the natural way.
      ++ split(aG, vector) then checks whether the cyclic submodule
      ++ generated by {\em vector} is a proper submodule of {\em V R}.
      ++ If successful, it returns a two-element list, which contains
      ++ first the list of the representations of the submodule,
      ++ then the list of the representations of the factor module.
      ++ If the vector generates the whole module, a one-element list
      ++ of the old representation is given.
      ++ Note: a later version this should call the other split.
    split:  (L M R, V V R) -> L L M R
      ++ split(aG,submodule) uses a proper submodule of {\em R**n}
      ++ to create the representations of the submodule and of
      ++ the factor module.
  if (R has Finite) and (R has Field) then
    meatAxe  : (L M R, B, I, I)  ->  L L M R
      ++ meatAxe(aG,randomElements,numberOfTries, maxTests) returns
      ++ a 2-list of representations as follows.
      ++ All matrices of argument aG are assumed to be square
      ++ and of equal size.
      ++ Then \spad{aG} generates a subalgebra, say \spad{A}, of the algebra
      ++ of all square matrices of dimension n. {\em V R} is an A-module
      ++ in the usual way.
      ++ meatAxe(aG,numberOfTries, maxTests) creates at most
      ++ {\em numberOfTries} random elements of the algebra, tests
```

```
    ++ them for singularity. If singular, it tries at most {\em maxTests}
    ++ elements of its kernel to generate a proper submodule.
    ++ If successful, a 2-list is returned: first, a list
    ++ containing first the list of the
    ++ representations of the submodule, then a list of the
    ++ representations of the factor module.
    ++ Otherwise, if we know that all the kernel is already
    ++ scanned, Norton's irreducibility test can be used either
    ++ to prove irreducibility or to find the splitting.
    ++ If {\em randomElements} is {\em false}, the first 6 tries
    ++ use Parker's fingerprints.
  meatAxe : L M R -> L L M R
    ++ meatAxe(aG) calls {\em meatAxe(aG,false,25,7)} returns
    ++ a 2-list of representations as follows.
    ++ All matrices of argument \spad{aG} are assumed to be square
    ++ and of
    ++ equal size. Then \spad{aG} generates a subalgebra,
    ++ say \spad{A}, of the algebra
    ++ of all square matrices of dimension n. {\em V R} is an A-module
    ++ in the usual way.
    ++ meatAxe(aG) creates at most 25 random elements
    ++ of the algebra, tests
    ++ them for singularity. If singular, it tries at most 7
    ++ elements of its kernel to generate a proper submodule.
    ++ If successful a list which contains first the list of the
    ++ representations of the submodule, then a list of the
    ++ representations of the factor module is returned.
    ++ Otherwise, if we know that all the kernel is already
    ++ scanned, Norton's irreducibility test can be used either
    ++ to prove irreducibility or to find the splitting.
    ++ Notes: the first 6 tries use Parker's fingerprints.
    ++ Also, 7 covers the case of three-dimensional kernels over
    ++ the field with 2 elements.
  meatAxe: (L M R, B) ->  L L M R
    ++ meatAxe(aG, randomElements) calls {\em meatAxe(aG,false,6,7)},
    ++ only using Parker's fingerprints, if {\em randomElemnts} is false.
    ++ If it is true, it calls {\em  meatAxe(aG,true,25,7)},
    ++ only using random elements.
    ++ Note: the choice of 25 was rather arbitrary.
    ++ Also, 7 covers the case of three-dimensional kernels over the field
    ++ with 2 elements.
  meatAxe : (L M R, PI)  ->  L L M R
    ++ meatAxe(aG, numberOfTries) calls
    ++ {\em meatAxe(aG,true,numberOfTries,7)}.
    ++ Notes: 7 covers the case of three-dimensional
    ++ kernels over the field with 2 elements.
```

```
    scanOneDimSubspaces: (L V R, I)  ->  V R
      ++ scanOneDimSubspaces(basis,n) gives a canonical representative
      ++ of the {\em n}-th one-dimensional subspace of the vector space
      ++ generated by the elements of {\em basis}, all from {\em R**n}.
      ++ The coefficients of the representative are of shape
      ++ {\em (0,...,0,1,*,...,*)}, {\em *} in R. If the size of R
      ++ is q, then there are {\em (q**n-1)/(q-1)} of them.
      ++ We first reduce n modulo this number, then find the
      ++ largest i such that {\em +/[q**i for i in 0..i-1] <= n}.
      ++ Subtracting this sum of powers from n results in an
      ++ i-digit number to basis q. This fills the positions of the
      ++ stars.
      -- would prefer to have (V V R,....   but nullSpace  results
      -- in L V R

  private ==> add

    -- import of domain and packages
    import OutputForm

    -- declarations and definitions of local variables and
    -- local function

    blockMultiply: (M R, M R, L I, I) -> M R
      -- blockMultiply(a,b,li,n) assumes that a has n columns
      -- and b has n rows, li is a sublist of the rows of a and
      -- a sublist of the columns of b. The result is the
      -- multiplication of the (li x n) part of a with the
      -- (n x li) part of b. We need this, because just matrix
      -- multiplying the parts would require extra storage.
    blockMultiply(a, b, li, n) ==
      matrix([[ +/[a(i,s) * b(s,j) for s in 1..n ] _
        for j in li  ]  for i in li])

    fingerPrint: (NNI, M R, M R, M R) -> M R
      -- is local, because one should know all the results for smaller i
    fingerPrint (i : NNI, a : M R, b : M R, x :M R) ==
      -- i > 2 only gives the correct result if the value of x from
      -- the parameter list equals the result of fingerprint(i-1,...)
      (i::PI) = 1 => x := a + b + a*b
      (i::PI) = 2 => x := (x + a*b)*b
      (i::PI) = 3 => x := a + b*x
      (i::PI) = 4 => x := x + b
      (i::PI) = 5 => x := x + a*b
      (i::PI) = 6 => x := x - a + b*a
      error "Sorry, but there are only 6 fingerprints!"
```

```
    x


  -- definition of exported functions


  --randomWord(lli,lm)  ==
  --  -- we assume that all matrices are square of same size
  --  numberOfMatrices := #lm
  --  +/[*/[lm.(1+i rem numberOfMatrices) for i in li ] for li in lli]

  completeEchelonBasis(basis) ==

    dimensionOfSubmodule : NNI := #basis
    n : NNI := # basis.1
    indexOfVectorToBeScanned : NNI := 1
    row : NNI := dimensionOfSubmodule

    completedBasis : M R := zero(n, n)
    for i in 1..dimensionOfSubmodule repeat
      completedBasis := setRow_!(completedBasis, i, basis.i)
    if #basis <= n then
      newStart : NNI := 1
      for j in 1..n
        while indexOfVectorToBeScanned <= dimensionOfSubmodule repeat
          if basis.indexOfVectorToBeScanned.j = 0 then
            completedBasis(1+row,j) := 1  --put unit vector into basis
            row := row + 1
          else
            indexOfVectorToBeScanned := indexOfVectorToBeScanned + 1
          newStart : NNI := j + 1
      for j in newStart..n repeat
        completedBasis(j,j) := 1  --put unit vector into basis
    completedBasis


  createRandomElement(aG,algElt) ==
    numberOfGenerators : NNI := #aG
    -- randomIndex := randnum numberOfGenerators
    randomIndex := 1+(random()$Integer rem numberOfGenerators)
    algElt := algElt * aG.randomIndex
    -- randomIndxElement := randnum numberOfGenerators
    randomIndex := 1+(random()$Integer rem numberOfGenerators)
    algElt + aG.randomIndex
```

```
if R has EuclideanDomain then
  cyclicSubmodule (lm : L M R, v : V R)  ==
    basis : M R := rowEchelon matrix list entries v
    -- normalizing the vector
    -- all these elements lie in the submodule generated by v
    furtherElts : L V R := [(lm.i*v)::V R for i in 1..maxIndex lm]
    --furtherElts has elements of the generated submodule. It will
    --will be checked whether they are in the span of the vectors
    --computed so far. Of course we stop if we have got the whole
    --space.
    while (^null furtherElts) and (nrows basis < #v)  repeat
      w : V R := first furtherElts
      nextVector : M R := matrix list entries w -- normalizing the vector
      -- will the rank change if we add this nextVector
      -- to the basis so far computed?
      addedToBasis : M R := vertConcat(basis, nextVector)
      if rank addedToBasis ^= nrows basis then
         basis := rowEchelon addedToBasis  -- add vector w to basis
         updateFurtherElts : L V R := _
           [(lm.i*w)::V R for i in 1..maxIndex lm]
         furtherElts := append (rest furtherElts, updateFurtherElts)
      else
         -- the vector w lies in the span of matrix, no updating
         -- of the basis
         furtherElts := rest furtherElts
    vector [row(basis, i) for i in 1..maxRowIndex basis]


  standardBasisOfCyclicSubmodule (lm : L M R, v : V R)  ==
    dim   : NNI := #v
    standardBasis : L L R := list(entries v)
    basis : M R := rowEchelon matrix list entries v
    -- normalizing the vector
    -- all these elements lie in the submodule generated by v
    furtherElts : L V R := [(lm.i*v)::V R for i in 1..maxIndex lm]
    --furtherElts has elements of the generated submodule. It will
    --will be checked whether they are in the span of the vectors
    --computed so far. Of course we stop if we  have got the whole
    --space.
    while (^null furtherElts) and (nrows basis < #v)  repeat
      w : V R := first furtherElts
      nextVector : M R := matrix list entries w  -- normalizing the vector
      -- will the rank change if we add this nextVector
      -- to the basis so far computed?
      addedToBasis : M R := vertConcat(basis, nextVector)
      if rank addedToBasis ^= nrows basis then
```

```
          standardBasis := cons(entries w, standardBasis)
          basis := rowEchelon addedToBasis  -- add vector w to basis
          updateFurtherElts : L V R := _
            [lm.i*w for i in 1..maxIndex lm]
          furtherElts := append (rest furtherElts, updateFurtherElts)
        else
          -- the vector w lies in the span of matrix, therefore
          -- no updating of matrix
          furtherElts := rest furtherElts
      transpose matrix standardBasis


    if R has Field then  -- only because of inverse in Matrix

      -- as conditional local functions, *internal have to be here

      splitInternal: (L M R, V R, B) -> L L M R
      splitInternal(algebraGenerators : L M R, vector: V R,doSplitting? : B) ==

        n : I := # vector    -- R-rank of representation module =
                             -- degree of representation
        submodule : V V R := cyclicSubmodule (algebraGenerators,vector)
        rankOfSubmodule : I := # submodule  -- R-Rank of submodule
        submoduleRepresentation    : L M R := nil()
        factormoduleRepresentation : L M R := nil()
        if n ^= rankOfSubmodule then
          messagePrint "  A proper cyclic submodule is found."
          if doSplitting? then   -- no else !!
            submoduleIndices : L I := [i for i in 1..rankOfSubmodule]
            factormoduleIndices : L I := [i for i in (1+rankOfSubmodule)..n]
            transitionMatrix : M R := _
              transpose completeEchelonBasis submodule
            messagePrint "  Transition matrix computed"
            inverseTransitionMatrix : M R :=  _
             autoCoerce(inverse transitionMatrix)$Union(M R,"failed")
            messagePrint "  The inverse of the transition matrix computed"
            messagePrint "  Now transform the matrices"
            for i in 1..maxIndex algebraGenerators repeat
              helpMatrix : M R := inverseTransitionMatrix * algebraGenerators.i
              -- in order to not create extra space and regarding the fact
              -- that we only want the two blocks in the main diagonal we
              -- multiply with the aid of the local function blockMultiply
              submoduleRepresentation := cons( blockMultiply( _
                helpMatrix,transitionMatrix,submoduleIndices,n), _
                submoduleRepresentation)
              factormoduleRepresentation := cons( blockMultiply( _
```

```
            helpMatrix,transitionMatrix,factormoduleIndices,n), _
            factormoduleRepresentation)
      [reverse submoduleRepresentation, reverse _
        factormoduleRepresentation]
    else -- represesentation is irreducible
      messagePrint "  The generated cyclic submodule was not proper"
      [algebraGenerators]



irreducibilityTestInternal: (L M R, M R, B) -> L L M R
irreducibilityTestInternal(algebraGenerators,_
    singularMatrix,split?) ==
  algebraGeneratorsTranspose : L M R := [transpose _
     algebraGenerators.j for j in 1..maxIndex algebraGenerators]
  xt : M R := transpose singularMatrix
  messagePrint "  We know that all the cyclic submodules generated by all"
  messagePrint "    non-trivial element of the singular matrix under view a
  messagePrint "    not proper, hence Norton's irreducibility test can be d
  -- actually we only would need one (!) non-trivial element from
  -- the kernel of xt, such an element must exist as the transpose
  -- of a singular matrix is of course singular. Question: Can
  -- we get it more easily from the kernel of x = singularMatrix?
  kernel : L V R := nullSpace xt
  result : L L M R :=  _
    splitInternal(algebraGeneratorsTranspose,first kernel,split?)
  if null rest result then  -- this means first kernel generates
    -- the whole module
    if 1 = #kernel then
       messagePrint "  Representation is absolutely irreducible"
    else
       messagePrint "  Representation is irreducible, but we don't know "
       messagePrint "    whether it is absolutely irreducible"
  else
    if split? then
      messagePrint "  Representation is not irreducible and it will be spli
      -- these are the dual representations, so calculate the
      -- dual to get the desired result, i.e. "transpose inverse"
      -- improvements??
      for i in 1..maxIndex result repeat
        for j in 1..maxIndex (result.i) repeat
          mat : M R := result.i.j
          result.i.j := _
            transpose autoCoerce(inverse mat)$Union(M R,"failed")
    else
      messagePrint "  Representation is not irreducible, use meatAxe to spl
```

```
    -- if "split?" then dual representation interchange factor
    -- and submodules, hence reverse
    reverse result



  -- exported functions for FiniteField-s.


  areEquivalent? (aG0, aG1) ==
    areEquivalent? (aG0, aG1, true, 25)


  areEquivalent? (aG0, aG1, numberOfTries) ==
    areEquivalent? (aG0, aG1, true, numberOfTries)


  areEquivalent? (aG0, aG1, randomelements, numberOfTries) ==
      result : B := false
      transitionM : M R := zero(1, 1)
      numberOfGenerators  : NNI := #aG0
      -- need a start value for creating random matrices:
      -- if we switch to randomelements later, we take the last
      -- fingerprint.
      if randomelements then   -- random should not be from I
         --randomIndex  : I   := randnum numberOfGenerators
         randomIndex := 1+(random()$Integer rem numberOfGenerators)
         x0 : M R := aG0.randomIndex
         x1 : M R := aG1.randomIndex
      n : NNI := #row(x0,1)   -- degree  of representation
      foundResult : B := false
      for i in 1..numberOfTries until foundResult repeat
        -- try to create a non-singular element of the algebra
        -- generated by "aG". If only two generators,
        -- i < 7 and not "randomelements" use Parker's  fingerprints
        -- i >= 7 create random elements recursively:
        -- x_i+1 :=x_i * mr1 + mr2, where mr1 and mr2 are randomly
        -- chosen elements form "aG".
        if i = 7 then randomelements := true
        if randomelements then
           --randomIndex := randnum numberOfGenerators
           randomIndex := 1+(random()$Integer rem numberOfGenerators)
           x0 := x0 * aG0.randomIndex
           x1 := x1 * aG1.randomIndex
           --randomIndex := randnum numberOfGenerators
           randomIndex := 1+(random()$Integer rem numberOfGenerators)
```

```
   x0 := x0 + aG0.randomIndex
   x1 := x1 + aG1.randomIndex
else
   x0 := fingerPrint (i, aG0.0, aG0.1 ,x0)
   x1 := fingerPrint (i, aG1.0, aG1.1 ,x1)
-- test singularity of x0 and x1
rk0 : NNI := rank x0
rk1 : NNI := rank x1
rk0 ^= rk1 =>
  messagePrint  "Dimensions of kernels differ"
  foundResult := true
  result := false
-- can assume dimensions are equal
rk0 ^= n - 1 =>
  -- not of any use here if kernel not one-dimensional
  if randomelements then
    messagePrint  "Random element in generated algebra does"
    messagePrint  "  not have a one-dimensional kernel"
  else
    messagePrint  "Fingerprint element in generated algebra does"
    messagePrint  "  not have a one-dimensional kernel"
-- can assume dimensions are equal and equal to n-1
if randomelements then
  messagePrint  "Random element in generated algebra has"
  messagePrint  "  one-dimensional kernel"
else
  messagePrint  "Fingerprint element in generated algebra has"
  messagePrint  "  one-dimensional kernel"
kernel0 : L V R := nullSpace x0
kernel1 : L V R := nullSpace x1
baseChange0 : M R := standardBasisOfCyclicSubmodule(_
  aG0,kernel0.1)
baseChange1 : M R := standardBasisOfCyclicSubmodule(_
  aG1,kernel1.1)
(ncols baseChange0) ^= (ncols baseChange1) =>
  messagePrint  "  Dimensions of generated cyclic submodules differ"
  foundResult := true
  result := false
-- can assume that dimensions of cyclic submodules are equal
(ncols baseChange0) = n =>   -- full dimension
  transitionM := baseChange0 * _
    autoCoerce(inverse baseChange1)$Union(M R,"failed")
  foundResult := true
  result := true
  for j in 1..numberOfGenerators while result repeat
    if (aG0.j*transitionM) ^= (transitionM*aG1.j) then
```

```
            result := false
            transitionM := zero(1 ,1)
            messagePrint "  There is no isomorphism, as the only possible one"
            messagePrint "    fails to do the necessary base change"
        -- can assume that dimensions of cyclic submodules are not "n"
        messagePrint  "  Generated cyclic submodules have equal, but not full"
        messagePrint  "    dimension, hence we can not draw any conclusion"
      -- here ends the for-loop
      if not foundResult then
        messagePrint  " "
        messagePrint  "Can neither prove equivalence nor inequivalence."
        messagePrint  "  Try again."
      else
        if result then
          messagePrint  " "
          messagePrint  "Representations are equivalent."
        else
          messagePrint  " "
          messagePrint  "Representations are not equivalent."
      transitionM


isAbsolutelyIrreducible?(aG) == isAbsolutelyIrreducible?(aG,25)


isAbsolutelyIrreducible?(aG, numberOfTries) ==
  result : B := false
  numberOfGenerators  : NNI := #aG
  -- need a start value for creating random matrices:
  -- randomIndex  : I   := randnum numberOfGenerators
  randomIndex := 1+(random()$Integer rem numberOfGenerators)
  x : M R := aG.randomIndex
  n : NNI := #row(x,1)   -- degree  of representation
  foundResult : B := false
  for i in 1..numberOfTries until foundResult repeat
    -- try to create a non-singular element of the algebra
    -- generated by "aG", dimension of its kernel being 1.
    -- create random elements recursively:
    -- x_i+1 :=x_i * mr1 + mr2, where mr1 and mr2 are randomly
    -- chosen elements form "aG".
    -- randomIndex := randnum numberOfGenerators
    randomIndex := 1+(random()$Integer rem numberOfGenerators)
    x := x * aG.randomIndex
    --randomIndex := randnum numberOfGenerators
    randomIndex := 1+(random()$Integer rem numberOfGenerators)
    x := x + aG.randomIndex
```

```
    -- test whether rank of x is n-1
    rk : NNI := rank x
    if rk = n - 1 then
      foundResult := true
      messagePrint "Random element in generated algebra has"
      messagePrint "  one-dimensional kernel"
      kernel : L V R := nullSpace x
      if n=#cyclicSubmodule(aG, first kernel) then
        result := (irreducibilityTestInternal(aG,x,false)).1 ^= nil()$(L M
        -- result := not null? first irreducibilityTestInternal(aG,x,false)
      else -- we found a proper submodule
        result := false
        --split(aG,kernel.1) -- to get the splitting
    else -- not of any use here if kernel not one-dimensional
      messagePrint "Random element in generated algebra does"
      messagePrint "  not have a one-dimensional kernel"
  -- here ends the for-loop
  if not foundResult then
    messagePrint "We have not found a one-dimensional kernel so far,"
    messagePrint "  as we do a random search you could try again"
  --else
  --  if not result then
  --    messagePrint "Representation is not irreducible."
  --  else
  --    messagePrint "Representation is irreducible."
  result



split(algebraGenerators: L M R, vector: V R) ==
  splitInternal(algebraGenerators, vector, true)


split(algebraGenerators : L M R, submodule: V V R) == --not zero submodule
  n : NNI := #submodule.1 -- R-rank of representation module =
                          -- degree of representation
  rankOfSubmodule : I := (#submodule) :: I --R-Rank of submodule
  submoduleRepresentation    : L M R := nil()
  factormoduleRepresentation : L M R := nil()
  submoduleIndices : L I := [i for i in 1..rankOfSubmodule]
  factormoduleIndices : L I := [i for i in (1+rankOfSubmodule)..(n::I)]
  transitionMatrix : M R := _
    transpose completeEchelonBasis submodule
  messagePrint "  Transition matrix computed"
  inverseTransitionMatrix : M R :=
    autoCoerce(inverse transitionMatrix)$Union(M R,"failed")
```

```
      messagePrint "  The inverse of the transition matrix computed"
      messagePrint "  Now transform the matrices"
      for i in 1..maxIndex algebraGenerators repeat
        helpMatrix : M R := inverseTransitionMatrix * algebraGenerators.i
        -- in order to not create extra space and regarding the fact
        -- that we only want the two blocks in the main diagonal we
        -- multiply with the aid of the local function blockMultiply
        submoduleRepresentation := cons( blockMultiply( _
          helpMatrix,transitionMatrix,submoduleIndices,n), _
          submoduleRepresentation)
        factormoduleRepresentation := cons( blockMultiply( _
          helpMatrix,transitionMatrix,factormoduleIndices,n), _
          factormoduleRepresentation)
      cons(reverse submoduleRepresentation, list( reverse _
        factormoduleRepresentation)::(L L M R))


    -- the following is "under"  "if R has Field", as there are compiler
    -- problems with conditinally defined local functions, i.e. it
    -- doesn't know, that "FiniteField" has "Field".


      -- we are scanning through the vectorspaces
      if (R has Finite) and (R has Field) then

        meatAxe(algebraGenerators, randomelements, numberOfTries, _
           maxTests) ==
          numberOfGenerators  : NNI := #algebraGenerators
          result : L L M R := nil()$(L L M R)
          q   : PI  := size()$R:PI
          -- need a start value for creating random matrices:
          -- if we switch to randomelements later, we take the last
          -- fingerprint.
          if randomelements then   -- random should not be from I
             --randomIndex  : I   := randnum numberOfGenerators
             randomIndex := 1+(random()$Integer rem numberOfGenerators)
             x : M R := algebraGenerators.randomIndex
          foundResult : B := false
          for i in 1..numberOfTries until foundResult repeat
            -- try to create a non-singular element of the algebra
            -- generated by "algebraGenerators". If only two generators,
            -- i < 7 and not "randomelements" use Parker's  fingerprints
            -- i >= 7 create random elements recursively:
            -- x_i+1 :=x_i * mr1 + mr2, where mr1 and mr2 are randomly
            -- chosen elements form "algebraGenerators".
            if i = 7 then randomelements := true
```

```
if randomelements then
   --randomIndex := randnum numberOfGenerators
   randomIndex := 1+(random()$Integer rem numberOfGenerators)
   x := x * algebraGenerators.randomIndex
   --randomIndex := randnum numberOfGenerators
   randomIndex := 1+(random()$Integer rem numberOfGenerators)
   x := x + algebraGenerators.randomIndex
else
   x := fingerPrint (i, algebraGenerators.1,_
     algebraGenerators.2 , x)
-- test singularity of x
n : NNI := #row(x, 1)  -- degree  of representation
if (rank x) ^= n then  -- x singular
  if randomelements then
     messagePrint "Random element in generated algebra is singular"
  else
     messagePrint "Fingerprint element in generated algebra is singul
  kernel : L V R := nullSpace x
  -- the first number is the maximal number of one dimensional
  -- subspaces of the kernel, the second is a user given
  -- constant
  numberOfOneDimSubspacesInKernel : I := (q**(#kernel)-1)quo(q-1)
  numberOfTests : I :=  _
    min(numberOfOneDimSubspacesInKernel, maxTests)
  for j in 1..numberOfTests repeat
     --we create an element in the kernel, there is a good
     --probability for it to generate a proper submodule, the
     --called "split" does the further work:
     result := _
       split(algebraGenerators,scanOneDimSubspaces(kernel,j))
     -- we had "not null rest result" directly in the following
     -- if .. then, but the statment there foundResult := true
     -- didn't work properly
     foundResult :=  not null rest result
     if foundResult then
       leave -- inner for-loop
       -- finish here with result
     else  -- no proper submodule
       -- we were not successfull, i.e gen. submodule was
       -- not proper, if the whole kernel is already scanned,
       -- Norton's irreducibility test is used now.
       if (j+1)>numberOfOneDimSubspacesInKernel then
         -- we know that all the cyclic submodules generated
         -- by all non-trivial elements of the kernel are proper.
         foundResult := true
         result : L L M R := irreducibilityTestInternal (_
```

```
                algebraGenerators,x,true)
              leave  -- inner for-loop
        -- here ends the inner for-loop
      else  -- x non-singular
        if randomelements then
          messagePrint "Random element in generated algebra is non-singular"
        else
          messagePrint "Fingerprint element in generated algebra is non-singular"
    -- here ends the outer for-loop
    if not foundResult then
       result : L L M R := [nil()$(L M R), nil()$(L M R)]
       messagePrint " "
       messagePrint "Sorry, no result, try meatAxe(...,true)"
       messagePrint "  or consider using an extension field."
    result


  meatAxe (algebraGenerators) ==
    meatAxe(algebraGenerators, false, 25, 7)


  meatAxe (algebraGenerators, randomElements?) ==
    randomElements? => meatAxe (algebraGenerators, true, 25, 7)
    meatAxe(algebraGenerators, false, 6, 7)


  meatAxe (algebraGenerators:L M R, numberOfTries:PI) ==
    meatAxe (algebraGenerators, true, numberOfTries, 7)



  scanOneDimSubspaces(basis,n) ==
    -- "dimension" of subspace generated by "basis"
    dim : NNI := #basis
    -- "dimension of the whole space:
    nn : NNI := #(basis.1)
    q : NNI := size()$R
    -- number of all one-dimensional subspaces:
    nred : I := n rem ((q**dim -1) quo (q-1))
    pos : I := nred
    i : I := 0
    for i in 0..dim-1 while nred >= 0 repeat
      pos := nred
      nred := nred - (q**i)
    i  := if i = 0 then 0 else i-1
    coefficients : V R := new(dim,0$R)
```

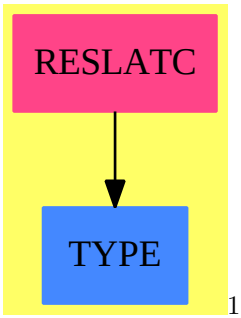
```
            coefficients.(dim-i) := 1$R
            iR : L I := wholeRagits(pos::RADIX q)
            for j in 1..(maxIndex iR) repeat
              coefficients.(dim-((#iR)::I) +j) := index((iR.j+(q::I))::PI)$R
            result : V R := new(nn,0)
            for i in 1..maxIndex coefficients repeat
              newAdd : V R := coefficients.i * basis.i
              for j in 1..nn repeat
                result.j := result.j + newAdd.j
            result
```

⟨*REP2.dotabb*⟩≡

```
"REP2" [color="#FF4488",href="bookvol10.4.pdf#nameddest=REP2"]
"IVECTOR" [color="#88FF44",href="bookvol10.3.pdf#nameddest=IVECTOR"]
"REP2" -> "IVECTOR"
```

19.75 package RESLATC ResolveLatticeCompletion

19.76 ResolveLatticeCompletion



1.00

Exports:
coerce

⟨*package RESLATC ResolveLatticeCompletion*⟩≡

```
)abbrev package RESLATC ResolveLatticeCompletion
++ Author: Stephen M. Watt
++ Date Created: 1986
++ Date Last Updated: May 30, 1991
++ Basic Operations:
++ Related Domains: ErrorFunctions, Exit, Void
++ Also See:
++ AMS Classifications:
++ Keywords: mode, resolve, type lattice
++ Examples:
++ References:
++ Description:
++   This package provides coercions for the special types \spadtype{Exit}
++   and \spadtype{Void}.
ResolveLatticeCompletion(S: Type): with
        coerce: S -> Void
             ++ coerce(s) throws all information about s away.
             ++ This coercion allows values of any type to appear
             ++ in contexts where they will not be used.
             ++ For example, it allows the resolution of different types in
             ++ the \spad{then} and \spad{else} branches when an \spad{if}
             ++ is in a context where the resulting value is not used.
        coerce: Exit -> S
             ++ coerce(e) is never really evaluated.  This coercion is
             ++ used for formal type correctness when a function will not
             ++ return directly to its caller.
```

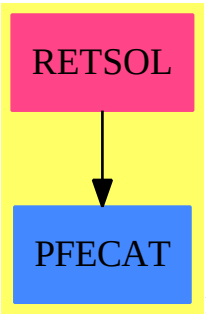
```
      == add
          coerce(s: S): Void == void()
          coerce(e: Exit): S ==
              error "Bug: Should not be able to obtain value of type Exit"
```

⟨*RESLATC.dotabb*⟩≡

```
"RESLATC" [color="#FF4488",href="bookvol10.4.pdf#nameddest=RESLATC"]
"TYPE" [color="#4488FF",href="bookvol10.2.pdf#nameddest=TYPE"]
"RESLATC" -> "TYPE"
```

19.77 package RETSOL RetractSolvePackage

19.78 RetractSolvePackage



1.00

Exports:
solveRetract

⟨*package RETSOL RetractSolvePackage*⟩≡
```
)abbrev package RETSOL RetractSolvePackage
++ Author: Manuel Bronstein
++ Date Created: 31 October 1991
++ Date Last Updated: 31 October 1991
++ Description:
++ RetractSolvePackage is an interface to \spadtype{SystemSolvePackage}
++ that attempts to retract the coefficients of the equations before
++ solving.

RetractSolvePackage(Q, R): Exports == Implementation where
  Q: IntegralDomain
  R: Join(IntegralDomain, RetractableTo Q)

  PQ  ==> Polynomial Q
  FQ  ==> Fraction PQ
  SY  ==> Symbol
  P   ==> Polynomial R
  F   ==> Fraction P
  EQ  ==> Equation
  SSP ==> SystemSolvePackage

  Exports ==> with
    solveRetract: (List P, List SY) -> List List EQ F
      ++ solveRetract(lp,lv) finds the solutions of the list lp of
      ++ rational functions with respect to the list of symbols lv.
      ++ The function tries to retract all the coefficients of the equations
      ++ to Q before solving if possible.
```

```
Implementation ==> add
  LEQQ2F : List EQ FQ -> List EQ F
  FQ2F   : FQ -> F
  PQ2P   : PQ -> P
  QIfCan : List P -> Union(List FQ, "failed")
  PQIfCan: P -> Union(FQ, "failed")

  PQ2P p   == map(#1::R, p)$PolynomialFunctions2(Q, R)
  FQ2F f   == PQ2P numer f / PQ2P denom f
  LEQQ2F l == [equation(FQ2F lhs eq, FQ2F rhs eq) for eq in l]

  solveRetract(lp, lv) ==
    (u := QIfCan lp) case "failed" =>
      solve([p::F for p in lp]$List(F), lv)$SSP(R)
    [LEQQ2F l for l in solve(u::List(FQ), lv)$SSP(Q)]

  QIfCan l ==
    ans:List(FQ) := empty()
    for p in l repeat
      (u := PQIfCan p) case "failed" => return "failed"
      ans := concat(u::FQ, ans)
    ans

  PQIfCan p ==
    (u := mainVariable p) case "failed" =>
      (r := retractIfCan(ground p)@Union(Q,"failed")) case Q => r::Q::PQ::FQ
      "failed"
    up := univariate(p, s := u::SY)
    ans:FQ := 0
    while up ^= 0 repeat
      (v := PQIfCan leadingCoefficient up) case "failed" => return "failed"
      ans := ans + monomial(1, s, degree up)$PQ * (v::FQ)
      up  := reductum up
    ans
```

⟨*RETSOL.dotabb*⟩≡

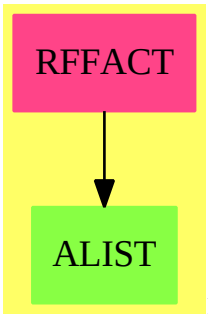
```
"RETSOL" [color="#FF4488",href="bookvol10.4.pdf#nameddest=RETSOL"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"RETSOL" -> "PFECAT"
```

Chapter 20

Chapter S

20.1 package SAERFFC SAERationalFunction-AlgFactor

20.2 SAERationalFunctionAlgFactor



1.00

Exports:
factor

⟨*package SAERFFC SAERationalFunctionAlgFactor*⟩≡
```
)abbrev package SAERFFC SAERationalFunctionAlgFactor
++ Factorisation in UP SAE FRAC POLY INT
++ Author: Patrizia Gianni
++ Date Created: ???
++ Date Last Updated: ???
++ Description:
++ Factorization of univariate polynomials with coefficients in an
++ algebraic extension of \spadtype{Fraction Polynomial Integer}.
++ Keywords: factorization, algebraic extension, univariate polynomial
```

```
SAERationalFunctionAlgFactor(UP, SAE, UPA): Exports == Implementation where
  UP : UnivariatePolynomialCategory Fraction Polynomial Integer
  SAE : Join(Field, CharacteristicZero,
                    MonogenicAlgebra(Fraction Polynomial Integer, UP))
  UPA: UnivariatePolynomialCategory SAE

  Exports ==> with
    factor: UPA -> Factored UPA
      ++ factor(p) returns a prime factorisation of p.

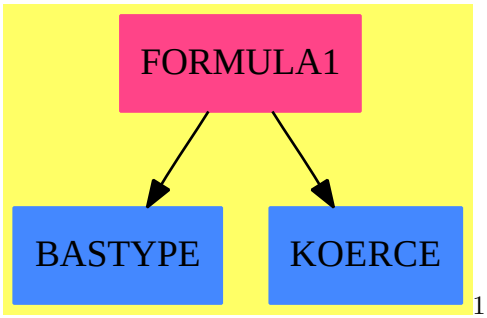
  Implementation ==> add
    factor q ==
      factor(q, factor$RationalFunctionFactor(UP)
              )$InnerAlgFactor(Fraction Polynomial Integer, UP, SAE, UPA)
```

⟨*SAERFFC.dotabb*⟩≡

```
"SAERFFC" [color="#FF4488",href="bookvol10.4.pdf#nameddest=SAERFFC"]
"MONOGEN" [color="#4488FF",href="bookvol10.2.pdf#nameddest=MONOGEN"]
"SAERFFC" -> "MONOGEN"
```

20.3 package FORMULA1 ScriptFormulaFormat1

20.4 ScriptFormulaFormat1



1.00

Exports:
coerce

⟨*package FORMULA1 ScriptFormulaFormat1*⟩≡

```
)abbrev package FORMULA1 ScriptFormulaFormat1
++ Author: Robert S. Sutor
++ Date Created: 1987 through 1990
++ Change History:
++ Basic Operations: coerce
++ Related Constructors: ScriptFormulaFormat
++ Also See: TexFormat, TexFormat1
++ AMS Classifications:
++ Keywords: output, format, SCRIPT, BookMaster, formula
++ References:
++   SCRIPT Mathematical Formula Formatter User's Guide, SH20-6453,
++   IBM Corporation, Publishing Systems Information Development,
++   Dept. G68, P.O. Box 1900, Boulder, Colorado, USA 80301-9191.
++ Description:
++   \spadtype{ScriptFormulaFormat1} provides a utility coercion for
++   changing to SCRIPT formula format anything that has a coercion to
++   the standard output format.

ScriptFormulaFormat1(S : SetCategory): public == private where
  public  ==  with
    coerce: S -> ScriptFormulaFormat()
      ++ coerce(s) provides a direct coercion from an expression s of domain S
      ++ to SCRIPT formula format. This allows the user to skip the step of
      ++ first manually coercing the object to standard output format
      ++ before it is coerced to SCRIPT formula format.
```

```
    private == add
      import ScriptFormulaFormat()

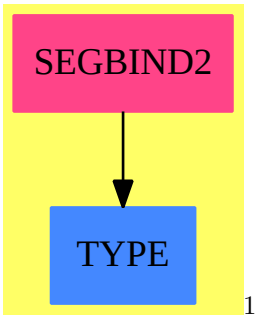
      coerce(s : S): ScriptFormulaFormat ==
        coerce(s :: OutputForm)$ScriptFormulaFormat
```

⟨*FORMULA1.dotabb*⟩≡

```
"FORMULA1" [color="#FF4488",href="bookvol10.4.pdf#nameddest=FORMULA1"]
"BASTYPE" [color="#4488FF",href="bookvol10.2.pdf#nameddest=BASTYPE"]
"KOERCE" [color="#4488FF",href="bookvol10.2.pdf#nameddest=KOERCE"]
"FORMULA1" -> "BASTYPE"
"FORMULA1" -> "KOERCE"
```

20.5 package SEGBIND2 SegmentBinding-Functions2

20.6 SegmentBindingFunctions2



1.00

Exports:
map

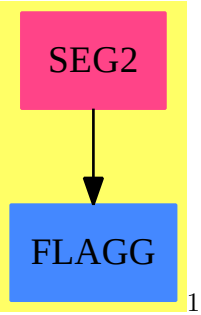
⟨*package SEGBIND2 SegmentBindingFunctions2*⟩≡
```
)abbrev package SEGBIND2 SegmentBindingFunctions2
++ Author:
++ Date Created:
++ Date Last Updated: June 4, 1991
++ Basic Operations:
++ Related Domains: SegmentBinding, Segment, Equation
++ Also See:
++ AMS Classifications:
++ Keywords: equation
++ Examples:
++ References:
++ Description:
++   This package provides operations for mapping functions onto
++   \spadtype{SegmentBinding}s.
SegmentBindingFunctions2(R:Type, S:Type): with
  map: (R -> S, SegmentBinding R) -> SegmentBinding S
      ++ map(f,v=a..b) returns the value given by \spad{v=f(a)..f(b)}.
 == add
  map(f, b) ==
    equation(variable b, map(f, segment b)$SegmentFunctions2(R, S))
```

⟨*SEGBIND2.dotabb*⟩≡

```
"SEGBIND2" [color="#FF4488",href="bookvol10.4.pdf#nameddest=SEGBIND2"]
"TYPE" [color="#4488FF",href="bookvol10.2.pdf#nameddest=TYPE"]
"SEGBIND2" -> "TYPE"
```

20.7 package SEG2 SegmentFunctions2

20.8 SegmentFunctions2



1.00

Exports:
map

⟨*package SEG2 SegmentFunctions2*⟩≡
```
)abbrev package SEG2 SegmentFunctions2
++ Author:
++ Date Created:
++ Date Last Updated: June 4, 1991
++ Basic Operations:
++ Related Domains: Segment, UniversalSegment
++ Also See:
++ AMS Classifications:
++ Keywords: equation
++ Examples:
++ References:
++ Description:
++   This package provides operations for mapping functions onto segments.

SegmentFunctions2(R:Type, S:Type): public == private where
  public ==> with
    map: (R -> S, Segment R) -> Segment S
        ++ map(f,l..h) returns a new segment \spad{f(l)..f(h)}.

    if R has OrderedRing then
      map: (R -> S, Segment R) -> List S
        ++ map(f,s) expands the segment s, applying \spad{f} to each
        ++ value.  For example, if \spad{s = l..h by k}, then the list
        ++ \spad{[f(l), f(l+k),..., f(lN)]} is computed, where
        ++ \spad{lN <= h < lN+k}.
```

```
    private ==> add
      map(f : R->S, r : Segment R): Segment S ==
        SEGMENT(f lo r,f hi r)$Segment(S)

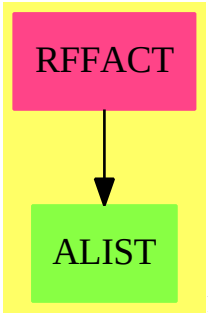
      if R has OrderedRing then
       map(f : R->S, r : Segment R): List S ==
         lr := nil()$List(S)
         l := lo r
         h := hi r
         inc := (incr r)::R
         if inc > 0 then
           while l <= h repeat
             lr := concat(f(l), lr)
             l := l + inc
         else
           while l >= h repeat
             lr := concat(f(l), lr)
             l := l + inc
         reverse_! lr
```

⟨*SEG2.dotabb*⟩≡

```
"SEG2" [color="#FF4488",href="bookvol10.4.pdf#nameddest=SEG2"]
"FLAGG" [color="#4488FF",href="bookvol10.2.pdf#nameddest=FLAGG"]
"SEG2" -> "FLAGG"
```

20.9 package SAEFACT SimpleAlgebraicExtensionAlgFactor

20.10 SimpleAlgebraicExtensionAlgFactor



1.00

Exports:
factor

⟨*package SAEFACT SimpleAlgebraicExtensionAlgFactor*⟩≡
```
)abbrev package SAEFACT SimpleAlgebraicExtensionAlgFactor
++ Factorisation in a simple algebraic extension;
++ Author: Patrizia Gianni
++ Date Created: ???
++ Date Last Updated: ???
++ Description:
++ Factorization of univariate polynomials with coefficients in an
++ algebraic extension of the rational numbers (\spadtype{Fraction Integer}).
++ Keywords: factorization, algebraic extension, univariate polynomial

SimpleAlgebraicExtensionAlgFactor(UP,SAE,UPA):Exports==Implementation where
  UP : UnivariatePolynomialCategory Fraction Integer
  SAE : Join(Field, CharacteristicZero,
                        MonogenicAlgebra(Fraction Integer, UP))
  UPA: UnivariatePolynomialCategory SAE

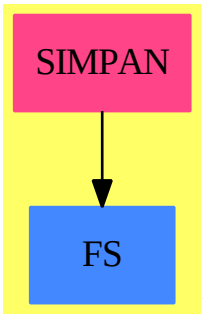
  Exports ==> with
    factor: UPA -> Factored UPA
      ++ factor(p) returns a prime factorisation of p.

  Implementation ==> add
    factor q ==
      factor(q, factor$RationalFactorize(UP)
                      )$InnerAlgFactor(Fraction Integer, UP, SAE, UPA)
```

⟨*SAEFACT.dotabb*⟩≡
```
"SAEFACT" [color="#FF4488",href="bookvol10.4.pdf#nameddest=SAEFACT"]
"MONOGEN" [color="#4488FF",href="bookvol10.2.pdf#nameddest=MONOGEN"]
"SAEFACT" -> "MONOGEN"
```

20.11 package SIMPAN SimplifyAlgebraicNumberConvertPackage

20.12 SimplifyAlgebraicNumberConvertPackage



1.00

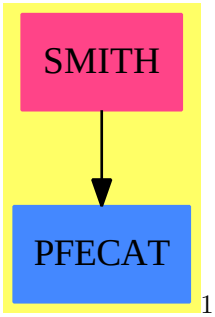
Exports:
simplify

⟨*package SIMPAN SimplifyAlgebraicNumberConvertPackage*⟩≡
```
)abbrev package SIMPAN SimplifyAlgebraicNumberConvertPackage
++ Package to allow simplify to be called on AlgebraicNumbers
++ by converting to EXPR(INT)
SimplifyAlgebraicNumberConvertPackage(): with
  simplify: AlgebraicNumber -> Expression(Integer)
        ++ simplify(an) applies simplifications to an
 == add
  simplify(a:AlgebraicNumber) ==
    simplify(a::Expression(Integer))$TranscendentalManipulations(Integer, Express
```

⟨*SIMPAN.dotabb*⟩≡
```
"SIMPAN" [color="#FF4488",href="bookvol10.4.pdf#nameddest=SIMPAN"]
"FS" [color="#4488FF",href="bookvol10.2.pdf#nameddest=FS"]
"SIMPAN" -> "FS"
```

20.13 package SMITH SmithNormalForm

20.14 SmithNormalForm



1.00

Exports:
completeHermite completeSmith diophantineSystem hermite smith

⟨*package SMITH SmithNormalForm*⟩≡

```
)abbrev package SMITH SmithNormalForm
++ Author: Patrizia Gianni
++ Date Created: October 1992
++ Date Last Updated:
++ Basic Operations:
++ Related Domains: Matrix(R)
++ Also See:
++ AMS Classifications:
++ Keywords: matrix, canonical forms, linear algebra
++ Examples:
++ References:
++ Description:
++   \spadtype{SmithNormalForm} is a package
++   which provides some standard canonical forms for matrices.

SmithNormalForm(R,Row,Col,M) : Exports == Implementation where

  R   : EuclideanDomain
  Row : FiniteLinearAggregate R
  Col : FiniteLinearAggregate R
  M   : MatrixCategory(R,Row,Col)

  I            ==> Integer
  NNI          ==> NonNegativeInteger
  HermiteForm  ==>   Record(Hermite:M,eqMat:M)
  SmithForm    ==>   Record(Smith : M, leftEqMat : M, rightEqMat : M)
  PartialV     ==> Union(Col, "failed")
```

```
Both         ==> Record(particular: PartialV, basis: List Col)

Exports ==  with
  hermite : M -> M
    ++ \spad{hermite(m)} returns the Hermite normal form of the
    ++ matrix m.
  completeHermite : M -> HermiteForm
    ++ \spad{completeHermite} returns a record  that contains
    ++ the Hermite normal form H of the matrix and the equivalence matrix
    ++ U such that U*m = H
  smith : M -> M
    ++ \spad{smith(m)} returns the Smith Normal form of the matrix m.
  completeSmith : M ->  SmithForm
    ++ \spad{completeSmith} returns a record  that contains
    ++ the Smith normal form H of the matrix and the left and right
    ++ equivalence matrices U and V such that U*m*v = H
  diophantineSystem : (M,Col) -> Both
    ++ \spad{diophantineSystem(A,B)} returns a particular integer solution and
    ++ an integer  basis  of the equation \spad{AX = B}.

Implementation == add
  MATCAT1 ==> MatrixCategoryFunctions2(R,Row,Col,M,QF,Row2,Col2,M2)
  MATCAT2 ==> MatrixCategoryFunctions2(QF,Row2,Col2,M2,R,Row,Col,M)
  QF      ==> Fraction R
  Row2    ==> Vector QF
  Col2    ==> Vector QF
  M2      ==> Matrix QF

              ------  Local Functions -----
  elRow1       :   (M,I,I)         ->  M
  elRow2       :  (M,R,I,I)        ->  M
  elColumn2    :  (M,R,I,I)        ->  M
  isDiagonal?  :      M            ->  Boolean
  ijDivide     : (SmithForm ,I,I)  ->  SmithForm
  lastStep     :   SmithForm       ->  SmithForm
  test1        :  (M,Col,NNI)      ->  Union(NNI, "failed")
  test2        : (M, Col,NNI,NNI)  ->  Union( Col, "failed")

   -- inconsistent system : case  0 = c --
  test1(sm:M,b:Col,m1 : NNI) : Union(NNI , "failed") ==
    km:=m1
    while zero? sm(km,km) repeat
      if not zero?(b(km)) then return "failed"
      km:= (km - 1) :: NNI
    km
```

```
if Col has shallowlyMutable then

  test2(sm : M ,b : Col, n1:NNI,dk:NNI) : Union( Col, "failed") ==
    -- test divisibility --
    sol:Col := new(n1,0)
    for k in 1..dk repeat
      if (c:=(b(k) exquo sm(k,k))) case "failed" then return "failed"
      sol(k):= c::R
    sol

 -- test if the matrix is diagonal or pseudo-diagonal --
isDiagonal?(m : M) : Boolean ==
  m1:= nrows m
  n1:= ncols m
  for i in 1..m1 repeat
    for j in 1..n1 | (j ^= i) repeat
      if  not zero?(m(i,j)) then return false
  true

   -- elementary operation of first kind: exchange two rows --
elRow1(m:M,i:I,j:I) : M ==
  vec:=row(m,i)
  setRow!(m,i,row(m,j))
  setRow!(m,j,vec)
  m

         -- elementary operation of second kind: add to row i--
                    -- a*row j  (i^=j) --
elRow2(m : M,a:R,i:I,j:I) : M ==
  vec:= map(a*#1,row(m,j))
  vec:=map("+",row(m,i),vec)
  setRow!(m,i,vec)
  m
         -- elementary operation of second kind: add to column i --
                      -- a*column j (i^=j) --
elColumn2(m : M,a:R,i:I,j:I) : M ==
  vec:= map(a*#1,column(m,j))
  vec:=map("+",column(m,i),vec)
  setColumn!(m,i,vec)
  m

   -- modify SmithForm in such a way that the term m(i,i) --
       -- divides the term m(j,j). m is diagonal --
ijDivide(sf : SmithForm , i : I,j : I) : SmithForm ==
  m:=sf.Smith
  mii:=m(i,i)
```

```
      mjj:=m(j,j)
      extGcd:=extendedEuclidean(mii,mjj)
      d := extGcd.generator
      mii:=(mii exquo d)::R
      mjj := (mjj exquo d) :: R
      -- add to row j extGcd.coef1*row i --
      lMat:=elRow2(sf.leftEqMat,extGcd.coef1,j,i)
      -- switch rows i and j --
      lMat:=elRow1(lMat,i,j)
      -- add to row j -mii*row i --
      lMat := elRow2(lMat,-mii,j,i)
--      lMat := ijModify(mii,mjj,extGcd.coef1,extGcd.coef2,sf.leftEqMat,i,j)
      m(j,j):= m(i,i) * mjj
      m(i,i):= d
      -- add to column i extGcd.coef2 * column j --
      rMat := elColumn2(sf.rightEqMat,extGcd.coef2,i,j)
      -- add to column j -mjj*column i --
      rMat:=elColumn2(rMat,-mjj,j,i)
      -- multiply by -1 column j --
      setColumn!(rMat,j,map(-1 * #1,column(rMat,j)))
      [m,lMat,rMat]


   -- given a diagonal matrix compute its Smith form --
  lastStep(sf : SmithForm) : SmithForm ==
    m:=sf.Smith
    m1:=min(nrows m,ncols m)
    for i in 1..m1 while (mii:=m(i,i)) ^=0 repeat
      for j in i+1..m1 repeat
        if (m(j,j) exquo mii) case "failed" then return
           lastStep(ijDivide(sf,i,j))
    sf

  -- given m and t row-equivalent matrices, with t in upper triangular --
       -- form  compute the matrix u such that u*m=t --
  findEqMat(m :  M,t : M) : Record(Hermite : M, eqMat : M) ==
    m1:=nrows m
    n1:=ncols m
    "and"/[zero? t(m1,j) for j in 1..n1] => -- there are 0 rows
       if "and"/[zero? t(1,j) for j in 1..n1]
       then return [m,scalarMatrix(m1,1)]  -- m is the zero matrix
       mm:=horizConcat(m,scalarMatrix(m1,1))
       mmh:=rowEchelon mm
       [subMatrix(mmh,1,m1,1,n1), subMatrix(mmh,1,m1,n1+1,n1+m1)]
    u:M:=zero(m1,m1)
    j:=1
```

```
  while t(1,j)=0 repeat j:=j+1  -- there are 0 columns
  t1:=copy t
  mm:=copy m
  if j>1 then
    t1:=subMatrix(t,1,m1,j,n1)
    mm:=subMatrix(m,1,m1,j,n1)
  t11:=t1(1,1)
  for i in 1..m1 repeat
    u(i,1) := (mm(i,1) exquo t11) :: R
    for j in 2..m1 repeat
      j0:=j
      while zero?(tjj:=t1(j,j0)) repeat j0:=j0+1
      u(i,j) :=((mm(i,j0) - ("+"/[u(i,k) * t1(k,j0) for k in 1..(j-1)])) exquo
                tjj) :: R
  u1:M2:= map(#1 :: QF,u)$MATCAT1
  [t,map(retract$QF,(inverse u1)::M2)$MATCAT2]

            --- Hermite normal form of m ---
hermite(m:M) : M == rowEchelon m

 -- Hermite normal form and equivalence matrix --
completeHermite(m : M) : Record(Hermite : M, eqMat : M) ==
  findEqMat(m,rowEchelon m)

smith(m : M) : M == completeSmith(m).Smith

completeSmith(m : M) : Record(Smith : M, leftEqMat : M, rightEqMat : M) ==
  cm1:=completeHermite m
  leftm:=cm1.eqMat
  m1:=cm1.Hermite
  isDiagonal? m1 => lastStep([m1,leftm,scalarMatrix(ncols m,1)])
  nr:=nrows m
  cm1:=completeHermite transpose m1
  rightm:= transpose cm1.eqMat
  m1:=cm1.Hermite
  isDiagonal? m1 =>
    cm2:=lastStep([m1,leftm,rightm])
    nrows(m:=cm2.Smith) = nr => cm2
    [transpose m,cm2.leftEqMat, cm2.rightEqMat]
  cm2:=completeSmith m1
  cm2:=lastStep([cm2.Smith,transpose(cm2.rightEqMat)*leftm,
            rightm*transpose(cm2.leftEqMat)])
  nrows(m:=cm2.Smith) = nr => cm2
  [transpose m, cm2.leftEqMat, cm2.rightEqMat]

-- Find the solution in R of the linear system mX = b --
```

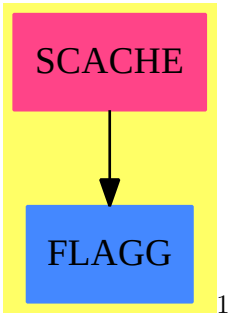
```
diophantineSystem(m : M, b : Col) : Both  ==
  sf:=completeSmith m
  sm:=sf.Smith
  m1:=nrows sm
  lm:=sf.leftEqMat
  b1:Col:= lm* b
  (t1:=test1(sm,b1,m1)) case "failed" => ["failed",empty()]
  dk:=t1 :: NNI
  n1:=ncols sm
  (t2:=test2(sm,b1,n1,dk)) case "failed" => ["failed",empty()]
  rm := sf.rightEqMat
  sol:=rm*(t2 :: Col)  -- particular solution
  dk = n1  => [sol,list new(n1,0)]
  lsol:List Col := [column(rm,i) for i in (dk+1)..n1]
  [sol,lsol]
```

⟨*SMITH.dotabb*⟩≡

```
"SMITH" [color="#FF4488",href="bookvol10.4.pdf#nameddest=SMITH"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"SMITH" -> "PFECAT"
```

20.15 package SCACHE SortedCache

20.16 SortedCache



1.00

Exports:
cache clearCache enterInCache

⟨*package SCACHE SortedCache*⟩≡
```
)abbrev package SCACHE SortedCache
++ Cache of elements in a set
++ Author: Manuel Bronstein
++ Date Created: 31 Oct 1988
++ Date Last Updated: 14 May 1991
++   A sorted cache of a cachable set S is a dynamic structure that
++   keeps the elements of S sorted and assigns an integer to each
++   element of S once it is in the cache. This way, equality and ordering
++   on S are tested directly on the integers associated with the elements
++   of S, once they have been entered in the cache.
SortedCache(S:CachableSet): Exports == Implementation where
  N    ==> NonNegativeInteger
  DIFF ==> 1024

  Exports ==> with
    clearCache  : () -> Void
      ++ clearCache() empties the cache.
    cache       : () -> List S
      ++ cache() returns the current cache as a list.
    enterInCache: (S, S -> Boolean) -> S
      ++ enterInCache(x, f) enters x in the cache, calling \spad{f(y)} to
      ++ determine whether x is equal to y. It returns x with an integer
      ++ associated with it.
    enterInCache: (S, (S, S) -> Integer) -> S
      ++ enterInCache(x, f) enters x in the cache, calling \spad{f(x, y)} to
      ++ determine whether \spad{x < y (f(x,y) < 0), x = y (f(x,y) = 0)}, or
      ++ \spad{x > y (f(x,y) > 0)}.
```

```
    ++ It returns x with an integer associated with it.

  Implementation ==> add
    shiftCache   : (List S, N) -> Void
    insertInCache: (List S, List S, S, N) -> S

    cach := [nil()]$Record(cche:List S)

    cache() == cach.cche

    shiftCache(l, n) ==
      for x in l repeat setPosition(x, n + position x)
      void

    clearCache() ==
      for x in cache repeat setPosition(x, 0)
      cach.cche := nil()
      void

    enterInCache(x:S, equal?:S -> Boolean) ==
      scan := cache()
      while not null scan repeat
        equal?(y := first scan) =>
          setPosition(x, position y)
          return y
        scan := rest scan
      setPosition(x, 1 + #cache())
      cach.cche := concat(cache(), x)
      x

    enterInCache(x:S, triage:(S, S) -> Integer) ==
      scan := cache()
      pos:N:= 0
      for i in 1..#scan repeat
        zero?(n := triage(x, y := first scan)) =>
          setPosition(x, position y)
          return y
        n<0 => return insertInCache(first(cache(),(i-1)::N),scan,x,pos)
        scan := rest scan
        pos  := position y
      setPosition(x, pos + DIFF)
      cach.cche := concat(cache(), x)
      x

    insertInCache(before, after, x, pos) ==
      if ((pos+1) = position first after) then shiftCache(after, DIFF)
```

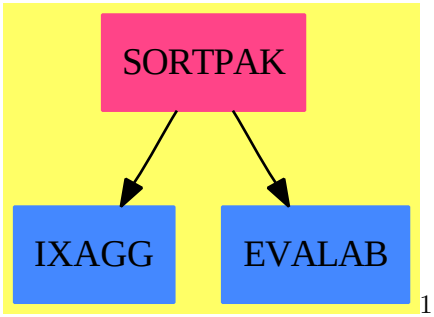
```
        setPosition(x, pos + (((position first after) - pos)::N quo 2))
        cach.cche := concat(before, concat(x, after))
        x
```

⟨*SCACHE.dotabb*⟩≡

```
  "SCACHE" [color="#FF4488",href="bookvol10.4.pdf#nameddest=SCACHE"]
  "FLAGG" [color="#4488FF",href="bookvol10.2.pdf#nameddest=FLAGG"]
  "SCACHE" -> "FLAGG"
```

20.17 package SORTPAK SortPackage

20.18 SortPackage



1.00

Exports:
bubbleSort! insertionSort!

⟨*package SORTPAK SortPackage*⟩≡
```
)abbrev package SORTPAK SortPackage
++ Description:
++ This package exports sorting algorithnms
SortPackage(S,A) : Exports == Implementation where
  S: Type
  A: IndexedAggregate(Integer,S)
    with (finiteAggregate; shallowlyMutable)

  Exports == with
    bubbleSort_!: (A,(S,S) -> Boolean) -> A
        ++ bubbleSort!(a,f) \undocumented
    insertionSort_!: (A, (S,S) -> Boolean) -> A
        ++ insertionSort!(a,f) \undocumented
    if S has OrderedSet then
      bubbleSort_!: A -> A
        ++ bubbleSort!(a) \undocumented
      insertionSort_!: A -> A
        ++ insertionSort! \undocumented

  Implementation == add
    bubbleSort_!(m,f) ==
      n := #m
      for i in 1..(n-1) repeat
        for j in n..(i+1) by -1 repeat
          if f(m.j,m.(j-1)) then swap_!(m,j,j-1)
      m
    insertionSort_!(m,f) ==
```

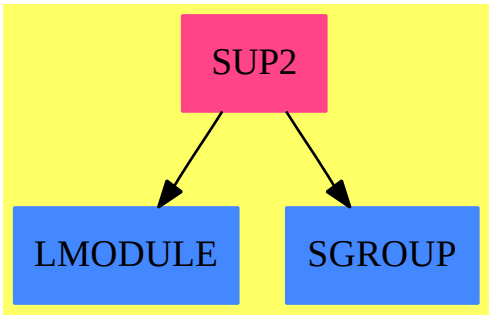
```
      for i in 2..#m repeat
        j := i
        while j > 1 and f(m.j,m.(j-1)) repeat
          swap_!(m,j,j-1)
          j := (j - 1) pretend PositiveInteger
      m
    if S has OrderedSet then
      bubbleSort_!(m) == bubbleSort_!(m,_<$S)
      insertionSort_!(m) == insertionSort_!(m,_<$S)
    if A has UnaryRecursiveAggregate(S) then
      bubbleSort_!(m,fn) ==
        empty? m => m
        l := m
        while not empty? (r := l.rest) repeat
           r := bubbleSort_!(r,fn)
           x := l.first
           if fn(r.first,x) then
             l.first := r.first
             r.first := x
           l.rest := r
           l := l.rest
        m
```

⟨*SORTPAK.dotabb*⟩≡

```
"SORTPAK" [color="#FF4488",href="bookvol10.4.pdf#nameddest=SORTPAK"]
"IXAGG" [color="#4488FF",href="bookvol10.2.pdf#nameddest=IXAGG"]
"EVALAB" [color="#4488FF",href="bookvol10.2.pdf#nameddest=EVALAB"]
"SORTPAK" -> "IXAGG"
"SORTPAK" -> "EVALAB"
```

20.19 package SUP2 SparseUnivariatePolynomialFunctions2

20.20 SparseUnivariatePolynomialFunctions2



1.00

Exports:
map

⟨*package SUP2 SparseUnivariatePolynomialFunctions2*⟩≡

```
)abbrev package SUP2 SparseUnivariatePolynomialFunctions2
++ Author:
++ Date Created:
++ Date Last Updated:
++ Basic Functions:
++ Related Constructors:
++ Also See:
++ AMS Classifications:
++ Keywords:
++ References:
++ Description:
++ This package lifts a mapping from coefficient rings R to S to
++ a mapping from sparse univariate polynomial over R to
++ a sparse univariate polynomial over S.
++ Note that the mapping is assumed
++ to send zero to zero, since it will only be applied to the non-zero
++ coefficients of the polynomial.

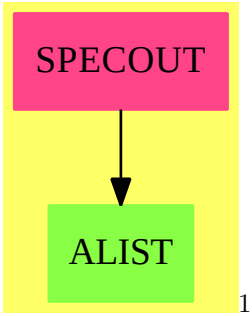
SparseUnivariatePolynomialFunctions2(R:Ring, S:Ring): with
  map:(R->S,SparseUnivariatePolynomial R) -> SparseUnivariatePolynomial S
    ++ map(func, poly) creates a new polynomial by applying func to
    ++ every non-zero coefficient of the polynomial poly.
 == add
  map(f, p) == map(f, p)$UnivariatePolynomialCategoryFunctions2(R,
           SparseUnivariatePolynomial R, S, SparseUnivariatePolynomial S)
```

⟨*SUP2.dotabb*⟩≡

```
"SUP2" [color="#FF4488",href="bookvol10.4.pdf#nameddest=SUP2"]
"LMODULE" [color="#4488FF",href="bookvol10.2.pdf#nameddest=LMODULE"]
"SGROUP" [color="#4488FF",href="bookvol10.2.pdf#nameddest=SGROUP"]
"SUP2" -> "LMODULE"
"SUP2" -> "SGROUP"
```

20.21 package SPECOUT SpecialOutputPackage

20.22 SpecialOutputPackage



1.00

Exports:
outputAsTex outputAsFortran outputAsScript

⟨*package SPECOUT SpecialOutputPackage*⟩≡

```
)abbrev package SPECOUT SpecialOutputPackage
++ Author: Stephen M. Watt
++ Date Created: September 1986
++ Date Last Updated: May 23, 1991
++ Basic Operations: outputAsFortran, outputAsScript, outputAsTex
++ Related Constructors:
++ Also See:
++ AMS Classifications:
++ Keywords:
++ References:
++ Description: SpecialOutputPackage allows FORTRAN, Tex and
++   Script Formula Formatter output from programs.

SpecialOutputPackage: public == private where
  public == with
    outputAsFortran: (String,OutputForm) -> Void
      ++ outputAsFortran(v,o) sends output v = o in FORTRAN format
      ++ to the destination defined by \spadsyscom{set output fortran}.
    outputAsFortran: OutputForm          -> Void
      ++ outputAsFortran(o) sends output o in FORTRAN format.
    outputAsScript:  OutputForm          -> Void
      ++ outputAsScript(o) sends output o in Script Formula Formatter format
      ++ to the destination defined by \spadsyscom{set output formula}.
    outputAsTex:     OutputForm          -> Void
      ++ outputAsTex(o) sends output o in Tex format to the destination
      ++ defined by \spadsyscom{set output tex}.
```

```
    outputAsFortran: List OutputForm     -> Void
      ++ outputAsFortran(l) sends (for each expression in the list l)
      ++ output in FORTRAN format to the destination defined by
      ++ \spadsyscom{set output fortran}.
    outputAsScript:  List OutputForm     -> Void
      ++ outputAsScript(l) sends (for each expression in the list l)
      ++ output in Script Formula Formatter format to the destination defined.
      ++ by \spadsyscom{set output forumula}.
    outputAsTex:     List OutputForm     -> Void
      ++ outputAsTex(l) sends (for each expression in the list l)
      ++ output in Tex format to the destination as defined by
      ++ \spadsyscom{set output tex}.

  private == add
    e : OutputForm
    l : List OutputForm
    var : String
    --ExpressionPackage()

    juxtaposeTerms: List OutputForm -> OutputForm
    juxtaposeTerms l == blankSeparate l

    outputAsFortran e ==
      dispfortexp$Lisp e
      void()$Void

    outputAsFortran(var,e) ==
      e := var::Symbol::OutputForm  = e
      dispfortexp(e)$Lisp
      void()$Void

    outputAsFortran l ==
      dispfortexp$Lisp juxtaposeTerms l
      void()$Void

    outputAsScript e ==
      formulaFormat$Lisp e
      void()$Void

    outputAsScript l ==
      formulaFormat$Lisp juxtaposeTerms l
      void()$Void

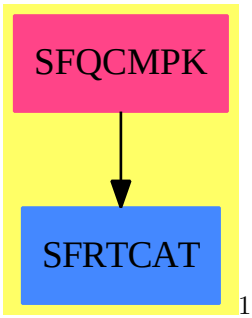
    outputAsTex e ==
      texFormat$Lisp e
      void()$Void
```

```
outputAsTex l ==
  texFormat$Lisp juxtaposeTerms l
  void()$Void
```

⟨*SPECOUT.dotabb*⟩≡
```
"SPECOUT" [color="#FF4488",href="bookvol10.4.pdf#nameddest=SPECOUT"]
"ALIST" [color="#88FF44",href="bookvol10.3.pdf#nameddest=ALIST"]
"SPECOUT" -> "ALIST"
```

20.23 package SFQCMPK SquareFreeQuasiComponentPackage

20.24 SquareFreeQuasiComponentPackage



1.00

Exports:

algebraicSort	branchIfCan
infRittWu?	internalInfRittWu?
internalSubPolSet?	internalSubQuasiComponent?
moreAlgebraic?	prepareDecompose
removeSuperfluousCases	removeSuperfluousQuasiComponents
startTable!	stopTable!
subCase?	subPolSet?
subQuasiComponent?	subTriSet?
supDimElseRittWu?	

⟨*SFQCMPK.dotabb*⟩≡
```
"SFQCMPK" [color="#FF4488",href="bookvol10.4.pdf#nameddest=SFQCMPK"]
```

⟨*package SFQCMPK SquareFreeQuasiComponentPackage*⟩≡

```
)abbrev package SFQCMPK SquareFreeQuasiComponentPackage
++ Author: Marc Moreno Maza
++ Date Created: 09/23/1998
++ Date Last Updated: 12/16/1998
++ Basic Functions:
++ Related Constructors:
++ Also See:
++ AMS Classifications:
++ Keywords:
++ Description:
++   A internal package for removing redundant quasi-components and redundant
++   branches when decomposing a variety by means of quasi-components
++   of regular triangular sets. \newline
++ References :
++  [1] D. LAZARD "A new method for solving algebraic systems of
++      positive dimension" Discr. App. Math. 33:147-160,1991
++      Tech. Report (PoSSo project)
++  [2] M. MORENO MAZA "Calculs de pgcd au-dessus des tours
++      d'extensions simples et resolution des systemes d'equations
++      algebriques" These, Universite P.etM. Curie, Paris, 1997.
++  [3] M. MORENO MAZA "A new algorithm for computing triangular
++      decomposition of algebraic varieties" NAG Tech. Rep. 4/98.
++ Version: 1.

SquareFreeQuasiComponentPackage(R,E,V,P,TS): Exports == Implementation where

  R : GcdDomain
  E : OrderedAbelianMonoidSup
  V : OrderedSet
  P : RecursivePolynomialCategory(R,E,V)
  TS : RegularTriangularSetCategory(R,E,V,P)
  N ==> NonNegativeInteger
  Z ==> Integer
  B ==> Boolean
  S ==> String
  LP ==> List P
  PtoP ==> P -> P
  PS ==> GeneralPolynomialSet(R,E,V,P)
  PWT ==> Record(val : P, tower : TS)
  BWT ==> Record(val : Boolean, tower : TS)
  LpWT ==> Record(val : (List P), tower : TS)
  Branch ==> Record(eq: List P, tower: TS, ineq: List P)
  UBF ==> Union(Branch,"failed")
  Split ==> List TS
  Key ==>  Record(left:TS, right:TS)
```

```
Entry ==> Boolean
H ==> TabulatedComputationPackage(Key, Entry)
polsetpack ==> PolynomialSetUtilitiesPackage(R,E,V,P)
SQUAREFREE ==> SquareFreeRegularTriangularSetCategory(R,E,V,P)

Exports ==  with
   startTable!: (S,S,S) -> Void
       ++ \axiom{startTableGcd!(s1,s2,s3)}
       ++ is an internal subroutine, exported only for developement.
   stopTable!: () -> Void
       ++ \axiom{stopTableGcd!()}
       ++ is an internal subroutine, exported only for developement.
   supDimElseRittWu?: (TS,TS) -> Boolean
       ++ \axiom{supDimElseRittWu(ts,us)} returns true iff \axiom{ts}
       ++ has less elements than \axiom{us} otherwise if \axiom{ts}
       ++ has higher rank than \axiom{us} w.r.t. Riit and Wu ordering.
   algebraicSort: Split -> Split
       ++ \axiom{algebraicSort(lts)} sorts \axiom{lts} w.r.t
       ++ \axiomOpFrom{supDimElseRittWu}{QuasiComponentPackage}.
   moreAlgebraic?: (TS,TS) -> Boolean
       ++ \axiom{moreAlgebraic?(ts,us)} returns false iff \axiom{ts}
       ++ and \axiom{us} are both empty, or \axiom{ts}
       ++ has less elements than \axiom{us}, or some variable is
       ++ algebraic w.r.t. \axiom{us} and is not w.r.t. \axiom{ts}.
   subTriSet?: (TS,TS) -> Boolean
       ++ \axiom{subTriSet?(ts,us)} returns true iff \axiom{ts} is
       ++ a sub-set of \axiom{us}.
   subPolSet?: (LP, LP) -> Boolean
       ++ \axiom{subPolSet?(lp1,lp2)} returns true iff \axiom{lp1} is
       ++ a sub-set of \axiom{lp2}.
   internalSubPolSet?: (LP, LP) -> Boolean
       ++ \axiom{internalSubPolSet?(lp1,lp2)} returns true iff \axiom{lp1} is
       ++ a sub-set of \axiom{lp2} assuming that these lists are sorted
       ++ increasingly w.r.t. \axiomOpFrom{infRittWu?}{RecursivePolynomialCateg
   internalInfRittWu?: (LP, LP) -> Boolean
       ++ \axiom{internalInfRittWu?(lp1,lp2)}
       ++ is an internal subroutine, exported only for developement.
   infRittWu?: (LP, LP) -> Boolean
       ++ \axiom{infRittWu?(lp1,lp2)}
       ++ is an internal subroutine, exported only for developement.
   internalSubQuasiComponent?: (TS,TS) -> Union(Boolean,"failed")
       ++ \axiom{internalSubQuasiComponent?(ts,us)} returns a boolean \spad{b}
       ++ if the fact the regular zero set of \axiom{us} contains that of
       ++ \axiom{ts} can be decided (and in that case \axiom{b} gives this
       ++ inclusion) otherwise returns \axiom{"failed"}.
   subQuasiComponent?: (TS,TS) -> Boolean
```

```
        ++ \axiom{subQuasiComponent?(ts,us)} returns true iff
        ++ \axiomOpFrom{internalSubQuasiComponent?(ts,us)}{QuasiComponentPackage}
        ++ returns true.
    subQuasiComponent?: (TS,Split) -> Boolean
        ++ \axiom{subQuasiComponent?(ts,lus)} returns true iff
        ++ \axiom{subQuasiComponent?(ts,us)} holds for one \spad{us} in \spad{lus}.
    removeSuperfluousQuasiComponents: Split -> Split
        ++ \axiom{removeSuperfluousQuasiComponents(lts)} removes from \axiom{lts}
        ++ any \spad{ts} such that \axiom{subQuasiComponent?(ts,us)} holds for
        ++ another \spad{us} in \axiom{lts}.
    subCase?: (LpWT,LpWT) -> Boolean
        ++ \axiom{subCase?(lpwt1,lpwt2)}
        ++ is an internal subroutine, exported only for developement.
    removeSuperfluousCases: List LpWT -> List LpWT
        ++ \axiom{removeSuperfluousCases(llpwt)}
        ++ is an internal subroutine, exported only for developement.
    prepareDecompose: (LP, List(TS),B,B) -> List Branch
        ++ \axiom{prepareDecompose(lp,lts,b1,b2)}
        ++ is an internal subroutine, exported only for developement.
    branchIfCan: (LP,TS,LP,B,B,B,B,B) -> Union(Branch,"failed")
        ++ \axiom{branchIfCan(leq,ts,lineq,b1,b2,b3,b4,b5)}
        ++ is an internal subroutine, exported only for developement.

Implementation == add

  squareFreeFactors(lp: LP): LP ==
    lsflp: LP := []
    for p in lp repeat
      lsfp := squareFreeFactors(p)$polsetpack
      lsflp := concat(lsfp,lsflp)
    sort(infRittWu?,removeDuplicates lsflp)

  startTable!(ok: S, ko: S, domainName: S): Void ==
    initTable!()$H
    if (not empty? ok) and (not empty? ko) then printInfo!(ok,ko)$H
    if (not empty? domainName) then startStats!(domainName)$H
    void()

  stopTable!(): Void ==
    if makingStats?()$H then printStats!()$H
    clearTable!()$H

  supDimElseRittWu? (ts:TS,us:TS): Boolean ==
    #ts < #us => true
    #ts > #us => false
    lp1 :LP := members(ts)
```

```
  lp2 :LP := members(us)
  while (not empty? lp1) and (not infRittWu?(first(lp2),first(lp1))) repeat
    lp1 := rest lp1
    lp2 := rest lp2
  not empty? lp1

algebraicSort (lts:Split): Split ==
  lts := removeDuplicates lts
  sort(supDimElseRittWu?,lts)

moreAlgebraic?(ts:TS,us:TS): Boolean  ==
  empty? ts => empty? us
  empty? us => true
  #ts < #us => false
  for p in (members us) repeat
     not algebraic?(mvar(p),ts) => return false
  true

subTriSet?(ts:TS,us:TS): Boolean  ==
  empty? ts => true
  empty? us => false
  mvar(ts) > mvar(us) => false
  mvar(ts) < mvar(us) => subTriSet?(ts,rest(us)::TS)
  first(ts)::P = first(us)::P => subTriSet?(rest(ts)::TS,rest(us)::TS)
  false

internalSubPolSet?(lp1: LP, lp2: LP): Boolean  ==
  empty? lp1 => true
  empty? lp2 => false
  associates?(first lp1, first lp2) =>
    internalSubPolSet?(rest lp1, rest lp2)
  infRittWu?(first lp1, first lp2) => false
  internalSubPolSet?(lp1, rest lp2)

subPolSet?(lp1: LP, lp2: LP): Boolean  ==
  lp1 := sort(infRittWu?, lp1)
  lp2 := sort(infRittWu?, lp2)
  internalSubPolSet?(lp1,lp2)

infRittWu?(lp1: LP, lp2: LP): Boolean ==
  lp1 := sort(infRittWu?, lp1)
  lp2 := sort(infRittWu?, lp2)
  internalInfRittWu?(lp1,lp2)

internalInfRittWu?(lp1: LP, lp2: LP): Boolean ==
  empty? lp1 => not empty? lp2
```

```
    empty? lp2 => false
    infRittWu?(first lp1, first lp2)$P => true
    infRittWu?(first lp2, first lp1)$P => false
    infRittWu?(rest lp1, rest lp2)$$

  subCase? (lpwt1:LpWT,lpwt2:LpWT): Boolean ==
    -- ASSUME lpwt.{1,2}.val is sorted w.r.t. infRittWu?
    not internalSubPolSet?(lpwt2.val, lpwt1.val) => false
    subQuasiComponent?(lpwt1.tower,lpwt2.tower)

  if TS has SquareFreeRegularTriangularSetCategory(R,E,V,P)
  then

    internalSubQuasiComponent?(ts:TS,us:TS): Union(Boolean,"failed") ==
      subTriSet?(us,ts) => true
      not moreAlgebraic?(ts,us) => false::Union(Boolean,"failed")
      for p in (members us) repeat
        mdeg(p) < mdeg(select(ts,mvar(p))::P) =>
          return("failed"::Union(Boolean,"failed"))
      for p in (members us) repeat
        not zero? initiallyReduce(p,ts) =>
          return("failed"::Union(Boolean,"failed"))
      lsfp := squareFreeFactors(initials us)
      for p in lsfp repeat
        b: B := invertible?(p,ts)$TS
        not b =>
          return(false::Union(Boolean,"failed"))
      true::Union(Boolean,"failed")

  else

    internalSubQuasiComponent?(ts:TS,us:TS): Union(Boolean,"failed") ==
      subTriSet?(us,ts) => true
      not moreAlgebraic?(ts,us) => false::Union(Boolean,"failed")
      for p in (members us) repeat
        mdeg(p) < mdeg(select(ts,mvar(p))::P) =>
          return("failed"::Union(Boolean,"failed"))
      for p in (members us) repeat
        not zero? reduceByQuasiMonic(p,ts) =>
          return("failed"::Union(Boolean,"failed"))
      true::Union(Boolean,"failed")

  subQuasiComponent?(ts:TS,us:TS): Boolean ==
    k: Key := [ts, us]
    e := extractIfCan(k)$H
    e case Entry => e::Entry
```

```
  ubf: Union(Boolean,"failed") := internalSubQuasiComponent?(ts,us)
  b: Boolean := (ubf case Boolean) and (ubf::Boolean)
  insert!(k,b)$H
  b

subQuasiComponent?(ts:TS,lus:Split): Boolean ==
  for us in lus repeat
     subQuasiComponent?(ts,us)@B => return true
  false

removeSuperfluousCases (cases:List LpWT) ==
  #cases < 2 => cases
  toSee := sort(supDimElseRittWu?(#1.tower,#2.tower),cases)
  lpwt1,lpwt2 : LpWT
  toSave,headmaxcases,maxcases,copymaxcases : List LpWT
  while not empty? toSee repeat
    lpwt1 := first toSee
    toSee := rest toSee
    toSave := []
    for lpwt2 in toSee repeat
       if subCase?(lpwt1,lpwt2)
         then
           lpwt1 := lpwt2
         else
           if not subCase?(lpwt2,lpwt1)
             then
               toSave := cons(lpwt2,toSave)
    if empty? maxcases
      then
        headmaxcases := [lpwt1]
        maxcases := headmaxcases
      else
        copymaxcases := maxcases
        while (not empty? copymaxcases) and _
          (not subCase?(lpwt1,first(copymaxcases))) repeat
            copymaxcases := rest copymaxcases
        if empty? copymaxcases
          then
            setrest!(headmaxcases,[lpwt1])
            headmaxcases := rest headmaxcases
    toSee := reverse toSave
  maxcases

removeSuperfluousQuasiComponents(lts: Split): Split ==
  lts := removeDuplicates lts
  #lts < 2 => lts
```

```
    toSee := algebraicSort lts
    toSave,headmaxlts,maxlts,copymaxlts : Split
    while not empty? toSee repeat
      ts := first toSee
      toSee := rest toSee
      toSave := []
      for us in toSee repeat
         if subQuasiComponent?(ts,us)@B
           then
             ts := us
           else
             if not subQuasiComponent?(us,ts)@B
               then
                 toSave := cons(us,toSave)
      if empty? maxlts
        then
          headmaxlts := [ts]
          maxlts := headmaxlts
        else
          copymaxlts := maxlts
          while (not empty? copymaxlts) and _
            (not subQuasiComponent?(ts,first(copymaxlts))@B) repeat
              copymaxlts := rest copymaxlts
          if empty? copymaxlts
            then
              setrest!(headmaxlts,[ts])
              headmaxlts := rest headmaxlts
      toSee := reverse toSave
    algebraicSort maxlts

  removeAssociates (lp:LP):LP ==
    removeDuplicates [primitivePart(p) for p in lp]

  branchIfCan(leq: LP,ts: TS,lineq: LP, b1:B,b2:B,b3:B,b4:B,b5:B):UBF ==
     -- ASSUME pols in leq are squarefree and mainly primitive
     -- if b1 then CLEAN UP leq
     -- if b2 then CLEAN UP lineq
     -- if b3 then SEARCH for ZERO in lineq with leq
     -- if b4 then SEARCH for ZERO in lineq with ts
     -- if b5 then SEARCH for ONE in leq with lineq
     if b1
       then
         leq := removeAssociates(leq)
         leq := remove(zero?,leq)
         any?(ground?,leq) =>
           return("failed"::Union(Branch,"failed"))
```

```
      if b2
        then
          any?(zero?,lineq) =>
            return("failed"::Union(Branch,"failed"))
          lineq := removeRedundantFactors(lineq)$polsetpack
      if b3
        then
          ps: PS := construct(leq)$PS
          for q in lineq repeat
            zero? remainder(q,ps).polnum =>
              return("failed"::Union(Branch,"failed"))
      (empty? leq) or (empty? lineq) => ([leq, ts, lineq]$Branch)::UBF
      if b4
        then
          for q in lineq repeat
            zero? initiallyReduce(q,ts) =>
              return("failed"::Union(Branch,"failed"))
      if b5
        then
          newleq: LP := []
          for p in leq repeat
            for q in lineq repeat
              if mvar(p) = mvar(q)
                then
                  g := gcd(p,q)
                  newp := (p exquo g)::P
                  ground? newp =>
                    return("failed"::Union(Branch,"failed"))
                  newleq := cons(newp,newleq)
                else
                  newleq := cons(p,newleq)
          leq := newleq
      leq := sort(infRittWu?, removeDuplicates leq)
      ([leq, ts, lineq]$Branch)::UBF

    prepareDecompose(lp: LP, lts: List(TS), b1: B, b2: B): List Branch ==
      -- if b1 then REMOVE REDUNDANT COMPONENTS in lts
      -- if b2 then SPLIT the input system with squareFree
      lp := sort(infRittWu?, remove(zero?,removeAssociates(lp)))
      any?(ground?,lp) => []
      empty? lts => []
      if b1 then lts := removeSuperfluousQuasiComponents lts
      not b2 =>
        [[lp,ts,squareFreeFactors(initials ts)]$Branch for ts in lts]
      toSee: List Branch
      lq: LP := []
```

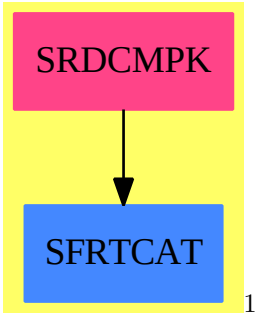
```
        toSee := [[lq,ts,squareFreeFactors(initials ts)]$Branch for ts in lts]
        empty? lp => toSee
        for p in lp repeat
          lsfp := squareFreeFactors(p)$polsetpack
          branches: List Branch := []
          lq := []
          for f in lsfp repeat
            for branch in toSee repeat
              leq : LP := branch.eq
              ts := branch.tower
              lineq : LP := branch.ineq
              ubf1: UBF := branchIfCan(leq,ts,lq,false,false,true,true,true)@UBF
              ubf1 case "failed" => "leave"
              ubf2: UBF := branchIfCan([f],ts,lineq,false,false,true,true,true)@UBF
              ubf2 case "failed" => "leave"
              leq := sort(infRittWu?,removeDuplicates concat(ubf1.eq,ubf2.eq))
              lineq := sort(infRittWu?,removeDuplicates concat(ubf1.ineq,ubf2.ineq))
              newBranch := branchIfCan(leq,ts,lineq,false,false,false,false,false)
              branches:= cons(newBranch::Branch,branches)
            lq := cons(f,lq)
          toSee := branches
        sort(supDimElseRittWu?(#1.tower,#2.tower),toSee)
```

⟨*SFQCMPK.dotabb*⟩+≡

```
"SFQCMPK" [color="#FF4488",href="bookvol10.4.pdf#nameddest=SFQCMPK"]
"SFRTCAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=SFRTCAT"]
"SFQCMPK" -> "SFRTCAT"
```

20.25 package SRDCMPK SquareFreeRegularSetDecompositionPackage

20.26 SquareFreeRegularSetDecompositionPackage



1.00

Exports:

algebraicDecompose	convert	decompose
internalDecompose	KrullNumber	numberOfVariables
printInfo	transcendentalDecompose	upDateBranches

⟨*package SRDCMPK SquareFreeRegularSetDecompositionPackage*⟩≡

```
)abbrev package SRDCMPK SquareFreeRegularSetDecompositionPackage
++ Author: Marc Moreno Maza
++ Date Created: 09/23/1998
++ Date Last Updated: 12/16/1998
++ Basic Functions:
++ Related Constructors:
++ Also See:
++ AMS Classifications:
++ Keywords:
++ Description:
++ A package providing a new algorithm for solving polynomial systems
++ by means of regular chains. Two ways of solving are provided:
++ in the sense of Zariski closure (like in Kalkbrener's algorithm)
++ or in the sense of the regular zeros (like in Wu, Wang or Lazard-
++ Moreno methods). This algorithm is valid for nay type
++ of regular set. It does not care about the way a polynomial is
++ added in an regular set, or how two quasi-components are compared
++ (by an inclusion-test), or how the invertibility test is made in
++ the tower of simple extensions associated with a regular set.
++ These operations are realized respectively by the domain \spad{TS}
++ and the packages \spad{QCMPPK(R,E,V,P,TS)} and \spad{RSETGCD(R,E,V,P,TS)}.
++ The same way it does not care about the way univariate polynomial
++ gcds (with coefficients in the tower of simple extensions associated
++ with a regular set) are computed. The only requirement is that these
```

```
++ gcds need to have invertible initials (normalized or not).
++ WARNING. There is no need for a user to call diectly any operation
++ of this package since they can be accessed by the domain \axiomType{TS}.
++ Thus, the operations of this package are not documented.\newline
++ References :
++  [1] M. MORENO MAZA "A new algorithm for computing triangular
++      decomposition of algebraic varieties" NAG Tech. Rep. 4/98.
++ Version: 2. Does not use any unproved criteria.

SquareFreeRegularSetDecompositionPackage(R,E,V,P,TS): Exports == Implementation where

  R : GcdDomain
  E : OrderedAbelianMonoidSup
  V : OrderedSet
  P : RecursivePolynomialCategory(R,E,V)
  TS : SquareFreeRegularTriangularSetCategory(R,E,V,P)
  N ==> NonNegativeInteger
  Z ==> Integer
  B ==> Boolean
  LP ==> List P
  PS ==> GeneralPolynomialSet(R,E,V,P)
  PWT ==> Record(val : P, tower : TS)
  BWT ==> Record(val : Boolean, tower : TS)
  LpWT ==> Record(val : (List P), tower : TS)
  Wip ==> Record(done: Split, todo: List LpWT)
  Branch ==> Record(eq: List P, tower: TS, ineq: List P)
  UBF ==> Union(Branch,"failed")
  Split ==> List TS
  iprintpack ==> InternalPrintPackage()
  polsetpack ==> PolynomialSetUtilitiesPackage(R,E,V,P)
  quasicomppack ==> SquareFreeQuasiComponentPackage(R,E,V,P,TS)
  regsetgcdpack ==> SquareFreeRegularTriangularSetGcdPackage(R,E,V,P,TS)

  Exports ==  with

     KrullNumber: (LP, Split) -> N
     numberOfVariables: (LP, Split) -> N
     algebraicDecompose: (P,TS) -> Record(done: Split, todo: List LpWT)
     transcendentalDecompose: (P,TS,N) -> Record(done: Split, todo: List LpWT)
     transcendentalDecompose: (P,TS) -> Record(done: Split, todo: List LpWT)
     internalDecompose: (P,TS,N,B) -> Record(done: Split, todo: List LpWT)
     internalDecompose: (P,TS,N) -> Record(done: Split, todo: List LpWT)
     internalDecompose: (P,TS) -> Record(done: Split, todo: List LpWT)
     decompose: (LP, Split, B, B) -> Split
     decompose: (LP, Split, B, B, B, B, B) -> Split
     upDateBranches: (LP,Split,List LpWT,Wip,N) -> List LpWT
```

```
      convert: Record(val: List P,tower: TS) -> String
      printInfo: (List Record(val: List P,tower: TS), N) -> Void

   Implementation == add

      KrullNumber(lp: LP, lts: Split): N ==
        ln: List N := [#(ts) for ts in lts]
        n := #lp + reduce(max,ln)

      numberOfVariables(lp: LP, lts: Split): N ==
        lv: List V := variables([lp]$PS)
        for ts in lts repeat lv := concat(variables(ts), lv)
        # removeDuplicates(lv)

      algebraicDecompose(p: P, ts: TS): Record(done: Split, todo: List LpWT) ==
        ground? p =>
          error " in algebraicDecompose$REGSET: should never happen !"
        v := mvar(p); n := #ts
        ts_v_- := collectUnder(ts,v)
        ts_v_+ := collectUpper(ts,v)
        ts_v := select(ts,v)::P
        lgwt: List PWT
        if mdeg(p) < mdeg(ts_v)
          then
            lgwt := stoseInternalLastSubResultant(ts_v,p,ts_v_-,true,false)$regset
          else
            lgwt := stoseInternalLastSubResultant(p,ts_v,ts_v_-,true,false)$regset
        lts: Split := []
        llpwt: List LpWT := []
        for gwt in lgwt repeat
          g := gwt.val; us := gwt.tower
          zero? g =>
            error " in algebraicDecompose$REGSET: should never happen !!"
          ground? g => "leave"
          h := leadingCoefficient(g,v)
          lus := augment(members(ts_v_+),augment(ts_v,us)$TS)$TS
          lsfp := squareFreeFactors(h)$polsetpack
          for f in lsfp repeat
            ground? f => "leave"
            for vs in lus repeat
              llpwt := cons([[f,p],vs]$LpWT, llpwt)
          n < #us =>
            error " in algebraicDecompose$REGSET: should never happen !!!"
          mvar(g) = v =>
            lts := concat(augment(members(ts_v_+),augment(g,us)$TS)$TS,lts)
        [lts,llpwt]
```

```
transcendentalDecompose(p: P, ts: TS,bound: N): Record(done: Split, todo: List LpWT) ==
  lts: Split
  if #ts < bound
    then
      lts := augment(p,ts)$TS
    else
      lts := []
  llpwt: List LpWT := []
  [lts,llpwt]

transcendentalDecompose(p: P, ts: TS): Record(done: Split, todo: List LpWT) ==
  lts: Split:= augment(p,ts)$TS
  llpwt: List LpWT := []
  [lts,llpwt]

internalDecompose(p: P, ts: TS,bound: N,clos?:B): Record(done: Split, todo: List LpWT)
  clos? => internalDecompose(p,ts,bound)
  internalDecompose(p,ts)

internalDecompose(p: P, ts: TS,bound: N): Record(done: Split, todo: List LpWT) ==
  -- ASSUME p not constant
  llpwt: List LpWT := []
  lts: Split := []
  -- EITHER mvar(p) is null
  if (not zero? tail(p)) and (not ground? (lmp := leastMonomial(p)))
    then
      llpwt := cons([[mvar(p)::P],ts]$LpWT,llpwt)
      p := (p exquo lmp)::P
  ip := squareFreePart init(p); tp := tail p
  p := mainPrimitivePart p
  -- OR init(p) is null or not
  lbwt: List BWT := stoseInvertible?_sqfreg(ip,ts)$regsetgcdpack
  for bwt in lbwt repeat
    bwt.val =>
      if algebraic?(mvar(p),bwt.tower)
        then
          rsl := algebraicDecompose(p,bwt.tower)
        else
          rsl := transcendentalDecompose(p,bwt.tower,bound)
      lts := concat(rsl.done,lts)
      llpwt :=  concat(rsl.todo,llpwt)
      (not ground? ip) =>
        zero? tp => llpwt := cons([[ip],bwt.tower]$LpWT, llpwt)
        (not ground? tp) => llpwt := cons([[ip,tp],bwt.tower]$LpWT, llpwt)
    riv := removeZero(ip,bwt.tower)
```

```
    (zero? riv) =>
      zero? tp => lts := cons(bwt.tower,lts)
      (not ground? tp) => llpwt := cons([[tp],bwt.tower]$LpWT, llpwt)
    llpwt := cons([[riv * mainMonomial(p) + tp],bwt.tower]$LpWT, llpwt)
  [lts,llpwt]

internalDecompose(p: P, ts: TS): Record(done: Split, todo: List LpWT) ==
  -- ASSUME p not constant
  llpwt: List LpWT := []
  lts: Split := []
  -- EITHER mvar(p) is null
  if (not zero? tail(p)) and (not ground? (lmp := leastMonomial(p)))
    then
      llpwt := cons([[mvar(p)::P],ts]$LpWT,llpwt)
      p := (p exquo lmp)::P
  ip := squareFreePart init(p); tp := tail p
  p := mainPrimitivePart p
  -- OR init(p) is null or not
  lbwt: List BWT := stoseInvertible?_sqfreg(ip,ts)$regsetgcdpack
  for bwt in lbwt repeat
    bwt.val =>
      if algebraic?(mvar(p),bwt.tower)
        then
          rsl := algebraicDecompose(p,bwt.tower)
        else
          rsl := transcendentalDecompose(p,bwt.tower)
      lts := concat(rsl.done,lts)
      llpwt :=  concat(rsl.todo,llpwt)
      (not ground? ip) =>
        zero? tp => llpwt := cons([[ip],bwt.tower]$LpWT, llpwt)
        (not ground? tp) => llpwt := cons([[ip,tp],bwt.tower]$LpWT, llpwt)
    riv := removeZero(ip,bwt.tower)
    (zero? riv) =>
      zero? tp => lts := cons(bwt.tower,lts)
      (not ground? tp) => llpwt := cons([[tp],bwt.tower]$LpWT, llpwt)
    llpwt := cons([[riv * mainMonomial(p) + tp],bwt.tower]$LpWT, llpwt)
  [lts,llpwt]

decompose(lp: LP, lts: Split, clos?: B, info?: B): Split ==
  decompose(lp,lts,false,false,clos?,true,info?)

convert(lpwt: LpWT): String ==
  ls: List String := ["<", string((#(lpwt.val))::Z), ",", string((#(lpwt.tow
  concat ls

printInfo(toSee: List LpWT, n: N): Void ==
```

```
    lpwt := first toSee
    s: String := concat ["[", string((#toSee)::Z), " ", convert(lpwt)@String]
    m: N := #(lpwt.val)
    toSee := rest toSee
    for lpwt in toSee repeat
      m := m + #(lpwt.val)
      s := concat [s, ",", convert(lpwt)@String]
    s := concat [s, " -> |", string(m::Z), "|; {", string(n::Z),"}]"]
    iprint(s)$iprintpack
    void()

  decompose(lp: LP, lts: Split, cleanW?: B, sqfr?: B, clos?: B, rem?: B, info?: B): Split
    -- if cleanW? then REMOVE REDUNDANT COMPONENTS in lts
    -- if sqfr? then SPLIT the system with SQUARE-FREE FACTORIZATION
    -- if clos? then SOLVE in the closure sense
    -- if rem? then REDUCE the current p by using remainder
    -- if info? then PRINT info
    empty? lp => lts
    branches: List Branch := prepareDecompose(lp,lts,cleanW?,sqfr?)$quasicomppack
    empty? branches => []
    toSee: List LpWT := [[br.eq,br.tower]$LpWT for br in branches]
    toSave: Split := []
    if clos? then bound := KrullNumber(lp,lts) else bound := numberOfVariables(lp,lts)
    while (not empty? toSee) repeat
      if info? then printInfo(toSee,#toSave)
      lpwt := first toSee; toSee := rest toSee
      lp := lpwt.val; ts := lpwt.tower
      empty? lp =>
        toSave := cons(ts, toSave)
      p := first lp;  lp := rest lp
      if rem? and (not ground? p) and (not empty? ts)
         then
           p := remainder(p,ts).polnum
      p := removeZero(p,ts)
      zero? p => toSee := cons([lp,ts]$LpWT, toSee)
      ground? p => "leave"
      rsl := internalDecompose(p,ts,bound,clos?)
      toSee := upDateBranches(lp,toSave,toSee,rsl,bound)
    removeSuperfluousQuasiComponents(toSave)$quasicomppack

  upDateBranches(leq:LP,lts:Split,current:List LpWT,wip: Wip,n:N): List LpWT ==
    newBranches: List LpWT := wip.todo
    newComponents: Split := wip.done
    branches1, branches2:  List LpWT
    branches1 := []; branches2  := []
    for branch in newBranches repeat
```

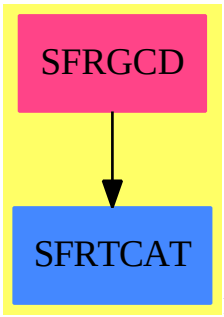
```
      us := branch.tower
      #us > n => "leave"
      newleq := sort(infRittWu?,concat(leq,branch.val))
      --foo := rewriteSetWithReduction(newleq,us,initiallyReduce,initiallyRedu
      --any?(ground?,foo)  => "leave"
      branches1 := cons([newleq,us]$LpWT, branches1)
    for us in newComponents repeat
      #us > n => "leave"
      subQuasiComponent?(us,lts)$quasicomppack => "leave"
      --newleq := leq
      --foo := rewriteSetWithReduction(newleq,us,initiallyReduce,initiallyRedu
      --any?(ground?,foo)  => "leave"
      branches2 := cons([leq,us]$LpWT, branches2)
    empty? branches1 =>
      empty? branches2 => current
      concat(branches2, current)
    branches := concat [branches2, branches1, current]
    -- branches := concat(branches,current)
    removeSuperfluousCases(branches)$quasicomppack
```

⟨*SRDCMPK.dotabb*⟩≡

```
"SRDCMPK" [color="#FF4488",href="bookvol10.4.pdf#nameddest=SRDCMPK"]
"SFRTCAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=SFRTCAT"]
"SRDCMPK" -> "SFRTCAT"
```

20.27 package SFRGCD SquareFreeRegularTriangularSetGcdPackage

20.28 SquareFreeRegularTriangularSetGcdPackage



1.00

Exports:

stopTableGcd!	stopTableInvSet!
startTableGcd!	startTableInvSet!
stoseIntegralLastSubResultant	stoseInternalLastSubResultant
stoseInvertible?	stoseInvertible?reg
stoseInvertible?sqfreg	stoseInvertibleSet
stoseInvertibleSetreg	stoseInvertibleSetsqfreg
stoseLastSubResultant	stosePrepareSubResAlgo
stoseSquareFreePart	

⟨*package SFRGCD SquareFreeRegularTriangularSetGcdPackage*⟩≡

```
)abbrev package SFRGCD SquareFreeRegularTriangularSetGcdPackage
++ Author: Marc Moreno Maza
++ Date Created: 09/23/1998
++ Date Last Updated: 10/01/1998
++ Basic Functions:
++ Related Constructors:
++ Also See:
++ AMS Classifications:
++ Keywords:
++ Description:
++ A internal package for computing gcds and resultants of univariate polynomials
++ with coefficients in a tower of simple extensions of a field.
++ There is no need to use directly this package since its main operations are
++ available from \spad{TS}. \newline
++ References :
++  [1] M. MORENO MAZA and R. RIOBOO "Computations of gcd over
++      algebraic towers of simple extensions" In proceedings of AAECC11
++      Paris, 1995.
++  [2] M. MORENO MAZA "Calculs de pgcd au-dessus des tours
```

```
++      d'extensions simples et resolution des systemes d'equations
++      algebriques" These, Universite P.etM. Curie, Paris, 1997.
++  [3] M. MORENO MAZA "A new algorithm for computing triangular
++      decomposition of algebraic varieties" NAG Tech. Rep. 4/98.
++ Version: 1.

SquareFreeRegularTriangularSetGcdPackage(R,E,V,P,TS): Exports == Implementation wi

  R : GcdDomain
  E : OrderedAbelianMonoidSup
  V : OrderedSet
  P : RecursivePolynomialCategory(R,E,V)
  TS : RegularTriangularSetCategory(R,E,V,P)
  N ==> NonNegativeInteger
  Z ==> Integer
  B ==> Boolean
  S ==> String
  LP ==> List P
  PtoP ==> P -> P
  PS ==> GeneralPolynomialSet(R,E,V,P)
  PWT ==> Record(val : P, tower : TS)
  BWT ==> Record(val : Boolean, tower : TS)
  LpWT ==> Record(val : (List P), tower : TS)
  Branch ==> Record(eq: List P, tower: TS, ineq: List P)
  UBF ==> Union(Branch,"failed")
  Split ==> List TS
  KeyGcd ==> Record(arg1: P, arg2: P, arg3: TS, arg4: B)
  EntryGcd ==> List PWT
  HGcd ==> TabulatedComputationPackage(KeyGcd, EntryGcd)
  KeyInvSet ==> Record(arg1: P, arg3: TS)
  EntryInvSet ==> List TS
  HInvSet ==> TabulatedComputationPackage(KeyInvSet, EntryInvSet)
  iprintpack ==> InternalPrintPackage()
  polsetpack ==> PolynomialSetUtilitiesPackage(R,E,V,P)
  quasicomppack ==> SquareFreeQuasiComponentPackage(R,E,V,P,TS)

  SQUAREFREE ==> SquareFreeRegularTriangularSetCategory(R,E,V,P)

  Exports ==  with
     startTableGcd!: (S,S,S) -> Void
     stopTableGcd!: () -> Void
     startTableInvSet!: (S,S,S) -> Void
     stopTableInvSet!: () -> Void
     stosePrepareSubResAlgo: (P,P,TS) -> List LpWT
     stoseInternalLastSubResultant: (P,P,TS,B,B) -> List PWT
     stoseInternalLastSubResultant: (List LpWT,V,B) -> List PWT
```

```
    stoseIntegralLastSubResultant: (P,P,TS) -> List PWT
    stoseLastSubResultant: (P,P,TS) -> List PWT
    stoseInvertible?: (P,TS) -> B
    stoseInvertible?_sqfreg: (P,TS) -> List BWT
    stoseInvertibleSet_sqfreg: (P,TS) -> Split
    stoseInvertible?_reg: (P,TS) -> List BWT
    stoseInvertibleSet_reg: (P,TS) -> Split
    stoseInvertible?: (P,TS) -> List BWT
    stoseInvertibleSet: (P,TS) -> Split
    stoseSquareFreePart: (P,TS) -> List PWT

  Implementation == add

    startTableGcd!(ok: S, ko: S, domainName: S): Void ==
      initTable!()$HGcd
      printInfo!(ok,ko)$HGcd
      startStats!(domainName)$HGcd
      void()

    stopTableGcd!(): Void ==
      if makingStats?()$HGcd then printStats!()$HGcd
      clearTable!()$HGcd

    startTableInvSet!(ok: S, ko: S, domainName: S): Void ==
      initTable!()$HInvSet
      printInfo!(ok,ko)$HInvSet
      startStats!(domainName)$HInvSet
      void()

    stopTableInvSet!(): Void ==
      if makingStats?()$HInvSet then printStats!()$HInvSet
      clearTable!()$HInvSet

    stoseInvertible?(p:P,ts:TS): Boolean ==
      q := primitivePart initiallyReduce(p,ts)
      zero? q => false
      normalized?(q,ts) => true
      v := mvar(q)
      not algebraic?(v,ts) =>
        toCheck: List BWT := stoseInvertible?(p,ts)@(List BWT)
        for bwt in toCheck repeat
          bwt.val = false => return false
        return true
      ts_v := select(ts,v)::P
      ts_v_- := collectUnder(ts,v)
      lgwt := stoseInternalLastSubResultant(ts_v,q,ts_v_-,false,true)
```

```
  for gwt in lgwt repeat
    g := gwt.val;
    (not ground? g) and (mvar(g) = v) =>
      return false
  true

stosePrepareSubResAlgo(p1:P,p2:P,ts:TS): List LpWT ==
  -- ASSUME mvar(p1) = mvar(p2) > mvar(ts) and mdeg(p1) >= mdeg(p2)
  -- ASSUME init(p1) invertible modulo ts !!!
  toSee: List LpWT := [[[p1,p2],ts]$LpWT]
  toSave: List LpWT := []
  v := mvar(p1)
  while (not empty? toSee) repeat
    lpwt := first toSee; toSee := rest toSee
    p1 := lpwt.val.1; p2 := lpwt.val.2
    ts := lpwt.tower
    lbwt := stoseInvertible?(leadingCoefficient(p2,v),ts)@(List BWT)
    for bwt in lbwt repeat
      (bwt.val = true) and (degree(p2,v) > 0) =>
        p3 := prem(p1, -p2)
        s: P := init(p2)**(mdeg(p1) - mdeg(p2))::N
        toSave := cons([[p2,p3,s],bwt.tower]$LpWT,toSave)
      -- p2 := initiallyReduce(p2,bwt.tower)
      newp2 := primitivePart initiallyReduce(p2,bwt.tower)
      (bwt.val = true) =>
        -- toSave := cons([[p2,0,1],bwt.tower]$LpWT,toSave)
        toSave := cons([[p2,0,1],bwt.tower]$LpWT,toSave)
      -- zero? p2 =>
      zero? newp2 =>
        toSave := cons([[p1,0,1],bwt.tower]$LpWT,toSave)
      -- toSee := cons([[p1,p2],bwt.tower]$LpWT,toSee)
      toSee := cons([[p1,newp2],bwt.tower]$LpWT,toSee)
  toSave

stoseIntegralLastSubResultant(p1:P,p2:P,ts:TS): List PWT ==
  -- ASSUME mvar(p1) = mvar(p2) > mvar(ts) and mdeg(p1) >= mdeg(p2)
  -- ASSUME p1 and p2 have no algebraic coefficients
  lsr := lastSubResultant(p1, p2)
  ground?(lsr) => [[lsr,ts]$PWT]
  mvar(lsr) < mvar(p1) => [[lsr,ts]$PWT]
  gi1i2 := gcd(init(p1),init(p2))
  ex: Union(P,"failed") := (gi1i2 * lsr) exquo$P init(lsr)
  ex case "failed" => [[lsr,ts]$PWT]
  [[ex::P,ts]$PWT]

stoseInternalLastSubResultant(p1:P,p2:P,ts:TS,b1:B,b2:B): List PWT ==
```

```
    -- ASSUME mvar(p1) = mvar(p2) > mvar(ts) and mdeg(p1) >= mdeg(p2)
    -- if b1 ASSUME init(p2) invertible w.r.t. ts
    -- if b2 BREAK with the first non-trivial gcd
    k: KeyGcd := [p1,p2,ts,b2]
    e := extractIfCan(k)$HGcd
    e case EntryGcd => e::EntryGcd
    toSave: List PWT
    empty? ts =>
      toSave := stoseIntegralLastSubResultant(p1,p2,ts)
      insert!(k,toSave)$HGcd
      return toSave
    toSee: List LpWT
    if b1
      then
        p3 := prem(p1, -p2)
        s: P := init(p2)**(mdeg(p1) - mdeg(p2))::N
        toSee := [[[p2,p3,s],ts]$LpWT]
      else
        toSee := stosePrepareSubResAlgo(p1,p2,ts)
    toSave := stoseInternalLastSubResultant(toSee,mvar(p1),b2)
    insert!(k,toSave)$HGcd
    toSave

  stoseInternalLastSubResultant(llpwt: List LpWT,v:V,b2:B): List PWT ==
    toReturn: List PWT := []; toSee: List LpWT;
    while (not empty? llpwt) repeat
      toSee := llpwt; llpwt := []
      -- CONSIDER FIRST the vanishing current last subresultant
      for lpwt in toSee repeat
        p1 := lpwt.val.1; p2 := lpwt.val.2; s := lpwt.val.3; ts := lpwt.tower
        lbwt := stoseInvertible?(leadingCoefficient(p2,v),ts)@(List BWT)
        for bwt in lbwt repeat
          bwt.val = false =>
            toReturn := cons([p1,bwt.tower]$PWT, toReturn)
            b2 and positive?(degree(p1,v)) => return toReturn
          llpwt := cons([[p1,p2,s],bwt.tower]$LpWT, llpwt)
      empty? llpwt => "leave"
      -- CONSIDER NOW the branches where the computations continue
      toSee := llpwt; llpwt := []
      lpwt := first toSee; toSee := rest toSee
      p1 := lpwt.val.1; p2 := lpwt.val.2; s := lpwt.val.3
      delta: N := (mdeg(p1) - degree(p2,v))::N
      p3: P := LazardQuotient2(p2, leadingCoefficient(p2,v), s, delta)
      zero?(degree(p3,v)) =>
        toReturn := cons([p3,lpwt.tower]$PWT, toReturn)
        for lpwt in toSee repeat
```

```
            toReturn := cons([p3,lpwt.tower]$PWT, toReturn)
          (p1, p2) := (p3, next_subResultant2(p1, p2, p3, s))
          s := leadingCoefficient(p1,v)
          llpwt := cons([[p1,p2,s],lpwt.tower]$LpWT, llpwt)
          for lpwt in toSee repeat
            llpwt := cons([[p1,p2,s],lpwt.tower]$LpWT, llpwt)
        toReturn

      stoseLastSubResultant(p1:P,p2:P,ts:TS): List PWT ==
        ground? p1 =>
          error"in stoseLastSubResultantElseSplit$SFRGCD  : bad #1"
        ground? p2 =>
          error"in stoseLastSubResultantElseSplit$SFRGCD : bad #2"
        not (mvar(p2) = mvar(p1)) =>
          error"in stoseLastSubResultantElseSplit$SFRGCD : bad #2"
        algebraic?(mvar(p1),ts) =>
          error"in stoseLastSubResultantElseSplit$SFRGCD : bad #1"
        not initiallyReduced?(p1,ts) =>
          error"in stoseLastSubResultantElseSplit$SFRGCD : bad #1"
        not initiallyReduced?(p2,ts) =>
          error"in stoseLastSubResultantElseSplit$SFRGCD : bad #2"
        purelyTranscendental?(p1,ts) and purelyTranscendental?(p2,ts) =>
          stoseIntegralLastSubResultant(p1,p2,ts)
        if mdeg(p1) < mdeg(p2) then
           (p1, p2) := (p2, p1)
           if odd?(mdeg(p1)) and odd?(mdeg(p2)) then p2 := - p2
        stoseInternalLastSubResultant(p1,p2,ts,false,false)

      stoseSquareFreePart_wip(p:P, ts: TS): List PWT ==
      -- ASSUME p is not constant and mvar(p) > mvar(ts)
      -- ASSUME init(p) is invertible w.r.t. ts
      -- ASSUME p is mainly primitive
--      one? mdeg(p) => [[p,ts]$PWT]
        mdeg(p) = 1 => [[p,ts]$PWT]
        v := mvar(p)$P
        q: P := mainPrimitivePart D(p,v)
        lgwt: List PWT := stoseInternalLastSubResultant(p,q,ts,true,false)
        lpwt : List PWT := []
        sfp : P
        for gwt in lgwt repeat
          g := gwt.val; us := gwt.tower
          (ground? g) or (mvar(g) < v) =>
            lpwt := cons([p,us],lpwt)
          g := mainPrimitivePart g
          sfp := lazyPquo(p,g)
          sfp := mainPrimitivePart stronglyReduce(sfp,us)
```

```
    lpwt := cons([sfp,us],lpwt)
  lpwt

stoseSquareFreePart_base(p:P, ts: TS): List PWT == [[p,ts]$PWT]

stoseSquareFreePart(p:P, ts: TS): List PWT == stoseSquareFreePart_wip(p,ts)

stoseInvertible?_sqfreg(p:P,ts:TS): List BWT ==
  --iprint("+")$iprintpack
  q := primitivePart initiallyReduce(p,ts)
  zero? q => [[false,ts]$BWT]
  normalized?(q,ts) => [[true,ts]$BWT]
  v := mvar(q)
  not algebraic?(v,ts) =>
    lbwt: List BWT := []
    toCheck: List BWT := stoseInvertible?_sqfreg(init(q),ts)@(List BWT)
    for bwt in toCheck repeat
      bwt.val => lbwt := cons(bwt,lbwt)
      newq := removeZero(q,bwt.tower)
      zero? newq => lbwt := cons(bwt,lbwt)
      lbwt := concat(stoseInvertible?_sqfreg(newq,bwt.tower)@(List BWT), lbwt)
    return lbwt
  ts_v := select(ts,v)::P
  ts_v_- := collectUnder(ts,v)
  ts_v_+ := collectUpper(ts,v)
  lgwt := stoseInternalLastSubResultant(ts_v,q,ts_v_-,false,false)
  lbwt: List BWT := []
  lts, lts_g, lts_h: Split
  for gwt in lgwt repeat
    g := gwt.val; ts := gwt.tower
    (ground? g) or (mvar(g) < v) =>
      lts := augment(ts_v,ts)$TS
      lts := augment(members(ts_v_+),lts)$TS
      for ts in lts repeat
        lbwt := cons([true, ts]$BWT,lbwt)
    g := mainPrimitivePart g
    lts_g := augment(g,ts)$TS
    lts_g := augment(members(ts_v_+),lts_g)$TS
    -- USE stoseInternalAugment with parameters ??
    for ts_g in lts_g repeat
      lbwt := cons([false, ts_g]$BWT,lbwt)
    h := lazyPquo(ts_v,g)
    (ground? h) or (mvar(h) < v) => "leave"
    h := mainPrimitivePart h
    lts_h := augment(h,ts)$TS
    lts_h := augment(members(ts_v_+),lts_h)$TS
```

```
    -- USE stoseInternalAugment with parameters ??
    for ts_h in lts_h repeat
      lbwt := cons([true, ts_h]$BWT,lbwt)
  sort(#1.val < #2.val,lbwt)

stoseInvertibleSet_sqfreg(p:P,ts:TS): Split ==
  --iprint("*")$iprintpack
  k: KeyInvSet := [p,ts]
  e := extractIfCan(k)$HInvSet
  e case EntryInvSet => e::EntryInvSet
  q := primitivePart initiallyReduce(p,ts)
  zero? q => []
  normalized?(q,ts) => [ts]
  v := mvar(q)
  toSave: Split := []
  not algebraic?(v,ts) =>
    toCheck: List BWT := stoseInvertible?_sqfreg(init(q),ts)@(List BWT)
    for bwt in toCheck repeat
      bwt.val => toSave := cons(bwt.tower,toSave)
      newq := removeZero(q,bwt.tower)
      zero? newq => "leave"
      toSave := concat(stoseInvertibleSet_sqfreg(newq,bwt.tower), toSave)
    toSave := removeDuplicates toSave
    return algebraicSort(toSave)$quasicomppack
  ts_v := select(ts,v)::P
  ts_v_- := collectUnder(ts,v)
  ts_v_+ := collectUpper(ts,v)
  lgwt := stoseInternalLastSubResultant(ts_v,q,ts_v_-,false,false)
  lts, lts_h: Split
  for gwt in lgwt repeat
    g := gwt.val; ts := gwt.tower
    (ground? g) or (mvar(g) < v) =>
      lts := augment(ts_v,ts)$TS
      lts := augment(members(ts_v_+),lts)$TS
      toSave := concat(lts,toSave)
    g := mainPrimitivePart g
    h := lazyPquo(ts_v,g)
    h := mainPrimitivePart h
    (ground? h) or (mvar(h) < v) => "leave"
    lts_h := augment(h,ts)$TS
    lts_h := augment(members(ts_v_+),lts_h)$TS
    toSave := concat(lts_h,toSave)
  toSave := algebraicSort(toSave)$quasicomppack
  insert!(k,toSave)$HInvSet
  toSave
```

```
stoseInvertible?_reg(p:P,ts:TS): List BWT ==
  --iprint("-")$iprintpack
  q := primitivePart initiallyReduce(p,ts)
  zero? q => [[false,ts]$BWT]
  normalized?(q,ts) => [[true,ts]$BWT]
  v := mvar(q)
  not algebraic?(v,ts) =>
    lbwt: List BWT := []
    toCheck: List BWT := stoseInvertible?_reg(init(q),ts)@(List BWT)
    for bwt in toCheck repeat
      bwt.val => lbwt := cons(bwt,lbwt)
      newq := removeZero(q,bwt.tower)
      zero? newq => lbwt := cons(bwt,lbwt)
      lbwt := concat(stoseInvertible?_reg(newq,bwt.tower)@(List BWT), lbwt)
    return lbwt
  ts_v := select(ts,v)::P
  ts_v_- := collectUnder(ts,v)
  ts_v_+ := collectUpper(ts,v)
  lgwt := stoseInternalLastSubResultant(ts_v,q,ts_v_-,false,false)
  lbwt: List BWT := []
  lts, lts_g, lts_h: Split
  for gwt in lgwt repeat
    g := gwt.val; ts := gwt.tower
    (ground? g) or (mvar(g) < v) =>
      lts := augment(ts_v,ts)$TS
      lts := augment(members(ts_v_+),lts)$TS
      for ts in lts repeat
        lbwt := cons([true, ts]$BWT,lbwt)
    g := mainPrimitivePart g
    lts_g := augment(g,ts)$TS
    lts_g := augment(members(ts_v_+),lts_g)$TS
    -- USE internalAugment with parameters ??
    for ts_g in lts_g repeat
      lbwt := cons([false, ts_g]$BWT,lbwt)
    h := lazyPquo(ts_v,g)
    (ground? h) or (mvar(h) < v) => "leave"
    h := mainPrimitivePart h
    lts_h := augment(h,ts)$TS
    lts_h := augment(members(ts_v_+),lts_h)$TS
    -- USE internalAugment with parameters ??
    for ts_h in lts_h repeat
      inv := stoseInvertible?_reg(q,ts_h)@(List BWT)
      lbwt := concat([bwt for bwt in inv | bwt.val],lbwt)
  sort(#1.val < #2.val,lbwt)

stoseInvertibleSet_reg(p:P,ts:TS): Split ==
```

```
    --iprint("/")$iprintpack
    k: KeyInvSet := [p,ts]
    e := extractIfCan(k)$HInvSet
    e case EntryInvSet => e::EntryInvSet
    q := primitivePart initiallyReduce(p,ts)
    zero? q => []
    normalized?(q,ts) => [ts]
    v := mvar(q)
    toSave: Split := []
    not algebraic?(v,ts) =>
      toCheck: List BWT := stoseInvertible?_reg(init(q),ts)@(List BWT)
      for bwt in toCheck repeat
        bwt.val => toSave := cons(bwt.tower,toSave)
        newq := removeZero(q,bwt.tower)
        zero? newq => "leave"
        toSave := concat(stoseInvertibleSet_reg(newq,bwt.tower), toSave)
      toSave := removeDuplicates toSave
      return algebraicSort(toSave)$quasicomppack
    ts_v := select(ts,v)::P
    ts_v_- := collectUnder(ts,v)
    ts_v_+ := collectUpper(ts,v)
    lgwt := stoseInternalLastSubResultant(ts_v,q,ts_v_-,false,false)
    lts, lts_h: Split
    for gwt in lgwt repeat
      g := gwt.val; ts := gwt.tower
      (ground? g) or (mvar(g) < v) =>
        lts := augment(ts_v,ts)$TS
        lts := augment(members(ts_v_+),lts)$TS
        toSave := concat(lts,toSave)
      g := mainPrimitivePart g
      h := lazyPquo(ts_v,g)
      h := mainPrimitivePart h
      (ground? h) or (mvar(h) < v) => "leave"
      lts_h := augment(h,ts)$TS
      lts_h := augment(members(ts_v_+),lts_h)$TS
      for ts_h in lts_h repeat
        inv := stoseInvertibleSet_reg(q,ts_h)
        toSave := removeDuplicates concat(inv,toSave)
    toSave := algebraicSort(toSave)$quasicomppack
    insert!(k,toSave)$HInvSet
    toSave

  if TS has SquareFreeRegularTriangularSetCategory(R,E,V,P)
  then

    stoseInvertible?(p:P,ts:TS): List BWT == stoseInvertible?_sqfreg(p,ts)
```

```
      stoseInvertibleSet(p:P,ts:TS): Split == stoseInvertibleSet_sqfreg(p,ts)

    else

      stoseInvertible?(p:P,ts:TS): List BWT == stoseInvertible?_reg(p,ts)

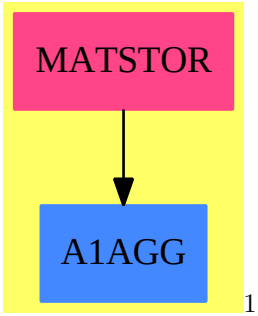
      stoseInvertibleSet(p:P,ts:TS): Split == stoseInvertibleSet_reg(p,ts)
```

⟨*SFRGCD.dotabb*⟩≡

```
"SFRGCD" [color="#FF4488",href="bookvol10.4.pdf#nameddest=SFRGCD"]
"SFRTCAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=SFRTCAT"]
"SFRGCD" -> "SFRTCAT"
```

20.29 package MATSTOR StorageEfficientMatrixOperations

20.30 StorageEfficientMatrixOperations



1.00

Exports:

copy!	leftScalarTimes!	minus!	plus!	power!
rightScalarTimes!	times!	?**?		

⟨*package MATSTOR StorageEfficientMatrixOperations*⟩≡

```
)abbrev package MATSTOR StorageEfficientMatrixOperations
++ Author: Clifton J. Williamson
++ Date Created: 18 July 1990
++ Date Last Updated: 18 July 1990
++ Basic Operations:
++ Related Domains: Matrix(R)
++ Also See:
++ AMS Classifications:
++ Keywords: matrix, linear algebra
++ Examples:
++ References:
++ Description:
++   This package provides standard arithmetic operations on matrices.
++   The functions in this package store the results of computations
++   in existing matrices, rather than creating new matrices.  This
++   package works only for matrices of type Matrix and uses the
++   internal representation of this type.
StorageEfficientMatrixOperations(R): Exports == Implementation where
  R     : Ring
  M   ==> Matrix R
  NNI ==> NonNegativeInteger
  ARR ==> PrimitiveArray R
  REP ==> PrimitiveArray PrimitiveArray R

  Exports ==> with
```

```
    copy_! : (M,M) -> M
     ++ \spad{copy!(c,a)} copies the matrix \spad{a} into the matrix c.
     ++ Error: if \spad{a} and c do not have the same
     ++ dimensions.
    plus_! : (M,M,M) -> M
     ++ \spad{plus!(c,a,b)} computes the matrix sum \spad{a + b} and stores the
     ++ result in the matrix c.
     ++ Error: if \spad{a}, b, and c do not have the same dimensions.
    minus_! : (M,M) -> M
     ++ \spad{minus!(c,a)} computes \spad{-a} and stores the result in the
     ++ matrix c.
     ++ Error: if a and c do not have the same dimensions.
    minus_! : (M,M,M) -> M
     ++ \spad{!minus!(c,a,b)} computes the matrix difference \spad{a - b}
     ++ and stores the result in the matrix c.
     ++ Error: if \spad{a}, b, and c do not have the same dimensions.
    leftScalarTimes_! : (M,R,M) -> M
     ++ \spad{leftScalarTimes!(c,r,a)} computes the scalar product
     ++ \spad{r * a} and stores the result in the matrix c.
     ++ Error: if \spad{a} and c do not have the same dimensions.
    rightScalarTimes_! : (M,M,R) -> M
     ++ \spad{rightScalarTimes!(c,a,r)} computes the scalar product
     ++ \spad{a * r} and stores the result in the matrix c.
     ++ Error: if \spad{a} and c do not have the same dimensions.
    times_! : (M,M,M) -> M
     ++ \spad{times!(c,a,b)} computes the matrix product \spad{a * b}
     ++ and stores the result in the matrix c.
     ++ Error: if \spad{a}, b, and c do not have
     ++ compatible dimensions.
    power_! : (M,M,M,M,NNI) -> M
     ++ \spad{power!(a,b,c,m,n)} computes m ** n and stores the result in
     ++ \spad{a}. The matrices b and c are used to store intermediate results.
     ++ Error: if \spad{a}, b, c, and m are not square
     ++ and of the same dimensions.
    "**" : (M,NNI) -> M
     ++ \spad{x ** n} computes the n-th power
     ++ of a square matrix. The power n is assumed greater than 1.

  Implementation ==> add

    rep : M -> REP
    rep m == m pretend REP

    copy_!(c,a) ==
      m := nrows a; n := ncols a
      not((nrows c) = m and (ncols c) = n) =>
```

```
    error "copy!: matrices of incompatible dimensions"
  aa := rep a; cc := rep c
  for i in 0..(m-1) repeat
    aRow := qelt(aa,i); cRow := qelt(cc,i)
    for j in 0..(n-1) repeat
      qsetelt_!(cRow,j,qelt(aRow,j))
  c

plus_!(c,a,b) ==
  m := nrows a; n := ncols a
  not((nrows b) = m and (ncols b) = n) =>
    error "plus!: matrices of incompatible dimensions"
  not((nrows c) = m and (ncols c) = n) =>
    error "plus!: matrices of incompatible dimensions"
  aa := rep a; bb := rep b; cc := rep c
  for i in 0..(m-1) repeat
    aRow := qelt(aa,i); bRow := qelt(bb,i); cRow := qelt(cc,i)
    for j in 0..(n-1) repeat
      qsetelt_!(cRow,j,qelt(aRow,j) + qelt(bRow,j))
  c

minus_!(c,a) ==
  m := nrows a; n := ncols a
  not((nrows c) = m and (ncols c) = n) =>
    error "minus!: matrices of incompatible dimensions"
  aa := rep a; cc := rep c
  for i in 0..(m-1) repeat
    aRow := qelt(aa,i); cRow := qelt(cc,i)
    for j in 0..(n-1) repeat
      qsetelt_!(cRow,j,-qelt(aRow,j))
  c

minus_!(c,a,b) ==
  m := nrows a; n := ncols a
  not((nrows b) = m and (ncols b) = n) =>
    error "minus!: matrices of incompatible dimensions"
  not((nrows c) = m and (ncols c) = n) =>
    error "minus!: matrices of incompatible dimensions"
  aa := rep a; bb := rep b; cc := rep c
  for i in 0..(m-1) repeat
    aRow := qelt(aa,i); bRow := qelt(bb,i); cRow := qelt(cc,i)
    for j in 0..(n-1) repeat
      qsetelt_!(cRow,j,qelt(aRow,j) - qelt(bRow,j))
  c

leftScalarTimes_!(c,r,a) ==
```

```
    m := nrows a; n := ncols a
    not((nrows c) = m and (ncols c) = n) =>
      error "leftScalarTimes!: matrices of incompatible dimensions"
    aa := rep a; cc := rep c
    for i in 0..(m-1) repeat
      aRow := qelt(aa,i); cRow := qelt(cc,i)
      for j in 0..(n-1) repeat
        qsetelt_!(cRow,j,r * qelt(aRow,j))
    c

  rightScalarTimes_!(c,a,r) ==
    m := nrows a; n := ncols a
    not((nrows c) = m and (ncols c) = n) =>
      error "rightScalarTimes!: matrices of incompatible dimensions"
    aa := rep a; cc := rep c
    for i in 0..(m-1) repeat
      aRow := qelt(aa,i); cRow := qelt(cc,i)
      for j in 0..(n-1) repeat
        qsetelt_!(cRow,j,qelt(aRow,j) * r)
    c

  copyCol_!: (ARR,REP,Integer,Integer) -> ARR
  copyCol_!(bCol,bb,j,n1) ==
    for i in 0..n1 repeat qsetelt_!(bCol,i,qelt(qelt(bb,i),j))

  times_!(c,a,b) ==
    m := nrows a; n := ncols a; p := ncols b
    not((nrows b) = n and (nrows c) = m and (ncols c) = p) =>
      error "times!: matrices of incompatible dimensions"
    aa := rep a; bb := rep b; cc := rep c
    bCol : ARR := new(n,0)
    m1 := (m :: Integer) - 1; n1 := (n :: Integer) - 1
    for j in 0..(p-1) repeat
      copyCol_!(bCol,bb,j,n1)
      for i in 0..m1 repeat
        aRow := qelt(aa,i); cRow := qelt(cc,i)
        sum : R := 0
        for k in 0..n1 repeat
          sum := sum + qelt(aRow,k) * qelt(bCol,k)
        qsetelt_!(cRow,j,sum)
    c

  power_!(a,b,c,m,p) ==
    mm := nrows a; nn := ncols a
    not(mm = nn) =>
      error "power!: matrix must be square"
```

```
        not((nrows b) = mm and (ncols b) = nn) =>
          error "power!: matrices of incompatible dimensions"
        not((nrows c) = mm and (ncols c) = nn) =>
          error "power!: matrices of incompatible dimensions"
        not((nrows m) = mm and (ncols m) = nn) =>
          error "power!: matrices of incompatible dimensions"
        flag := false
        copy_!(b,m)
        repeat
          if odd? p then
            flag =>
              times_!(c,b,a)
              copy_!(a,c)
            flag := true
            copy_!(a,b)
  --        one? p => return a
          (p = 1) => return a
          p := p quo 2
          times_!(c,b,b)
          copy_!(b,c)

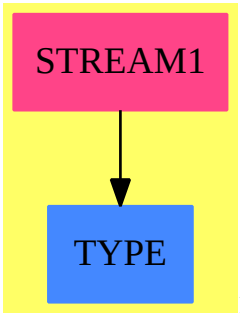
      m ** n ==
        not square? m => error "**: matrix must be square"
        a := copy m; b := copy m; c := copy m
        power_!(a,b,c,m,n)
```

⟨*MATSTOR.dotabb*⟩≡

```
"MATSTOR" [color="#FF4488",href="bookvol10.4.pdf#nameddest=MATSTOR"]
"A1AGG" [color="#4488FF",href="bookvol10.2.pdf#nameddest=A1AGG"]
"MATSTOR" -> "A1AGG"
```

20.31 package STREAM1 StreamFunctions1

20.32 StreamFunctions1



1.00

Exports:
concat

⟨*package STREAM1 StreamFunctions1*⟩≡

```
)abbrev package STREAM1 StreamFunctions1
++ Authors: Burge, Watt; updated by Clifton J. Williamson
++ Date Created: July 1986
++ Date Last Updated: 29 January 1990
++ Keywords: stream, infinite list, infinite sequence
StreamFunctions1(S:Type): Exports == Implementation where
  ++ Functions defined on streams with entries in one set.
  ST  ==> Stream

  Exports ==> with
    concat: ST ST S -> ST S
      ++ concat(u) returns the left-to-right concatentation of the
      ++ streams in u. Note: \spad{concat(u) = reduce(concat,u)}.
      ++
      ++X m:=[i for i in 10..]
      ++X n:=[j for j in 1.. | prime? j]
      ++X p:=[m,n]::Stream(Stream(PositiveInteger))
      ++X concat(p)

  Implementation ==> add

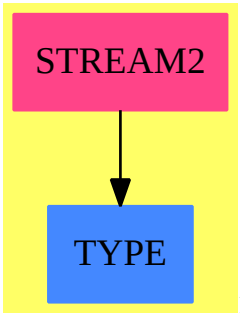
    concat z == delay
      empty? z => empty()
      empty?(x := frst z) => concat rst z
      concat(frst x,concat(rst x,concat rst z))
```

⟨*STREAM1.dotabb*⟩≡

```
"STREAM1" [color="#FF4488",href="bookvol10.4.pdf#nameddest=STREAM1"]
"TYPE" [color="#4488FF",href="bookvol10.2.pdf#nameddest=TYPE"]
"STREAM1" -> "TYPE"
```

20.33 package STREAM2 StreamFunctions2

20.34 StreamFunctions2



1.00

Exports:
map reduce scan

⟨*package STREAM2 StreamFunctions2*⟩≡

```
)abbrev package STREAM2 StreamFunctions2
++ Authors: Burge, Watt; updated by Clifton J. Williamson
++ Date Created: July 1986
++ Date Last Updated: 29 January 1990
++ Keywords: stream, infinite list, infinite sequence
StreamFunctions2(A:Type,B:Type): Exports == Implementation where
  ++ Functions defined on streams with entries in two sets.
  ST   ==> Stream

  Exports ==> with
    map: ((A -> B),ST A) -> ST B
      ++ map(f,s) returns a stream whose elements are the function f applied
      ++ to the corresponding elements of s.
      ++ Note: \spad{map(f,[x0,x1,x2,...]) = [f(x0),f(x1),f(x2),..]}.
      ++
      ++X m:=[i for i in 1..]
      ++X f(i:PositiveInteger):PositiveInteger==i**2
      ++X map(f,m)

    scan: (B,((A,B) -> B),ST A) -> ST B
      ++ scan(b,h,[x0,x1,x2,...]) returns \spad{[y0,y1,y2,...]}, where
      ++ \spad{y0 = h(x0,b)},
      ++ \spad{y1 = h(x1,y0)},\spad{...}
      ++ \spad{yn = h(xn,y(n-1))}.
      ++
      ++X m:=[i for i in 1..]::Stream(Integer)
      ++X f(i:Integer,j:Integer):Integer==i+j
```

```
      ++X scan(1,f,m)

    reduce:  (B,(A,B) -> B,ST A) -> B
      ++ reduce(b,f,u), where u is a finite stream \spad{[x0,x1,...,xn]},
      ++ returns the value \spad{r(n)} computed as follows:
      ++ \spad{r0 = f(x0,b),
      ++ r1 = f(x1,r0),...,
      ++ r(n) = f(xn,r(n-1))}.
      ++
      ++X m:=[i for i in 1..300]::Stream(Integer)
      ++X f(i:Integer,j:Integer):Integer==i+j
      ++X reduce(1,f,m)

--  rreduce: (B,(A,B) -> B,ST A) -> B
--    ++ reduce(b,h,[x0,x1,..,xn]) = h(x1,h(x2(..,h(x(n-1),h(xn,b))..)
--  reshape: (ST B,ST A) -> ST B
--    ++ reshape(y,x) = y

  Implementation ==> add

    mapp: (A -> B,ST A) -> ST B
    mapp(f,x)== delay
      empty? x => empty()
      concat(f frst x, map(f,rst x))

    map(f,x) ==
      explicitlyEmpty? x => empty()
      eq?(x,rst x) => repeating([f frst x])
      mapp(f, x)

--  reshape(y,x) == y

    scan(b,h,x) == delay
      empty? x => empty()
      c := h(frst x,b)
      concat(c,scan(c,h,rst x))

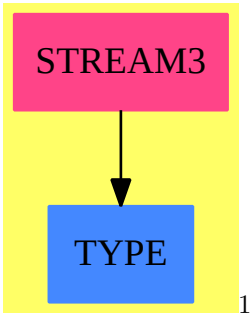
    reduce(b,h,x) ==
      empty? x => b
      reduce(h(frst x,b),h,rst x)
--  rreduce(b,h,x) ==
--    empty? x => b
--    h(frst x,rreduce(b,h,rst x))
```

⟨*STREAM2.dotabb*⟩≡

```
"STREAM2" [color="#FF4488",href="bookvol10.4.pdf#nameddest=STREAM2"]
"TYPE" [color="#4488FF",href="bookvol10.2.pdf#nameddest=TYPE"]
"STREAM2" -> "TYPE"
```

20.35 package STREAM3 StreamFunctions3

20.36 StreamFunctions3



1.00

Exports:
map

⟨*package STREAM3 StreamFunctions3*⟩≡
```
)abbrev package STREAM3 StreamFunctions3
++ Authors: Burge, Watt; updated by Clifton J. Williamson
++ Date Created: July 1986
++ Date Last Updated: 29 January 1990
++ Keywords: stream, infinite list, infinite sequence
StreamFunctions3(A,B,C): Exports == Implementation where
  ++ Functions defined on streams with entries in three sets.
  A  : Type
  B  : Type
  C  : Type
  ST ==> Stream

  Exports ==> with
    map: ((A,B) -> C,ST A,ST B) -> ST C
      ++ map(f,st1,st2) returns the stream whose elements are the
      ++ function f applied to the corresponding elements of st1 and st2.
      ++ \spad{map(f,[x0,x1,x2,..],[y0,y1,y2,..]) = [f(x0,y0),f(x1,y1),..]}.
      ++
      ++S
      ++X m:=[i for i in 1..]::Stream(Integer)
      ++X n:=[i for i in 1..]::Stream(Integer)
      ++X f(i:Integer,j:Integer):Integer == i+j
      ++X map(f,m,n)

  Implementation ==> add

    mapp:((A,B) -> C,ST A,ST B) -> ST C
```

```
mapp(g,x,y) == delay
  empty? x or empty? y => empty()
  concat(g(frst x,frst y), map(g,rst x,rst y))

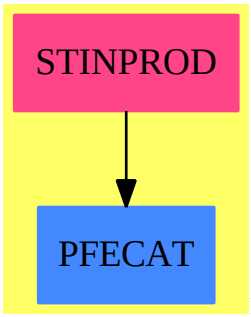
map(g,x,y) ==
  explicitlyEmpty? x => empty()
  eq?(x,rst x) => map(g(frst x,#1),y)$StreamFunctions2(B,C)
  explicitlyEmpty? y => empty()
  eq?(y,rst y) => map(g(#1,frst y),x)$StreamFunctions2(A,C)
  mapp(g,x,y)
```

⟨*STREAM3.dotabb*⟩≡

```
"STREAM3" [color="#FF4488",href="bookvol10.4.pdf#nameddest=STREAM3"]
"TYPE" [color="#4488FF",href="bookvol10.2.pdf#nameddest=TYPE"]
"STREAM3" -> "TYPE"
```

20.37 package STINPROD StreamInfiniteProduct

20.38 StreamInfiniteProduct



1.00

Exports:

evenInfiniteProduct	generalInfiniteProduct
infiniteProduct	oddInfiniteProduct

⟨*package STINPROD StreamInfiniteProduct*⟩≡

```
)abbrev package STINPROD StreamInfiniteProduct
++ Author: Clifton J. Williamson
++ Date Created: 23 February 1990
++ Date Last Updated: 23 February 1990
++ Basic Operations: infiniteProduct, evenInfiniteProduct, oddInfiniteProduct,
++   generalInfiniteProduct
++ Related Domains: UnivariateTaylorSeriesCategory
++ Also See:
++ AMS Classifications:
++ Keywords: Taylor series, infinite product
++ Examples:
++ References:
++ Description:
++   This package computes infinite products of Taylor series over an
++   integral domain of characteristic 0.  Here Taylor series are
++   represented by streams of Taylor coefficients.
StreamInfiniteProduct(Coef): Exports == Implementation where
  Coef: Join(IntegralDomain,CharacteristicZero)
  I  ==> Integer
  QF ==> Fraction
  ST ==> Stream

  Exports ==> with

    infiniteProduct: ST Coef -> ST Coef
```

```
    ++ infiniteProduct(f(x)) computes \spad{product(n=1,2,3...,f(x**n))}.
    ++ The series \spad{f(x)} should have constant coefficient 1.
  evenInfiniteProduct: ST Coef -> ST Coef
    ++ evenInfiniteProduct(f(x)) computes \spad{product(n=2,4,6...,f(x**n))}.
    ++ The series \spad{f(x)} should have constant coefficient 1.
  oddInfiniteProduct: ST Coef -> ST Coef
    ++ oddInfiniteProduct(f(x)) computes \spad{product(n=1,3,5...,f(x**n))}.
    ++ The series \spad{f(x)} should have constant coefficient 1.
  generalInfiniteProduct: (ST Coef,I,I) -> ST Coef
    ++ generalInfiniteProduct(f(x),a,d) computes
    ++ \spad{product(n=a,a+d,a+2*d,...,f(x**n))}.
    ++ The series \spad{f(x)} should have constant coefficient 1.

Implementation ==> add

  if Coef has Field then

    import StreamTaylorSeriesOperations(Coef)
    import StreamTranscendentalFunctions(Coef)

    infiniteProduct st             == exp lambert log st
    evenInfiniteProduct st         == exp evenlambert log st
    oddInfiniteProduct st          == exp oddlambert log st
    generalInfiniteProduct(st,a,d) == exp generalLambert(log st,a,d)

  else

    import StreamTaylorSeriesOperations(QF Coef)
    import StreamTranscendentalFunctions(QF Coef)

    applyOverQF:(ST QF Coef -> ST QF Coef,ST Coef) -> ST Coef
    applyOverQF(f,st) ==
      stQF := map(#1 :: QF(Coef),st)$StreamFunctions2(Coef,QF Coef)
      map(retract(#1)@Coef,f stQF)$StreamFunctions2(QF Coef,Coef)

    infiniteProduct st     == applyOverQF(exp lambert log #1,st)
    evenInfiniteProduct st == applyOverQF(exp evenlambert log #1,st)
    oddInfiniteProduct st  == applyOverQF(exp oddlambert log #1,st)
    generalInfiniteProduct(st,a,d) ==
      applyOverQF(exp generalLambert(log #1,a,d),st)
```

⟨*STINPROD.dotabb*⟩≡

```
"STINPROD" [color="#FF4488",href="bookvol10.4.pdf#nameddest=STINPROD"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"STINPROD" -> "PFECAT"
```

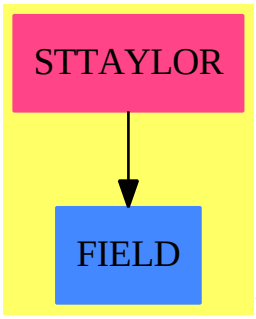
20.39 package STTAYLOR StreamTaylorSeriesOperations

Problems raising a UTS to a negative integer power.

The code in `powern(rn,x)` which raises an unnecessary error where no distinction between rational and integer powers are made.

The fix is easy. Since the problem does not exist in SUPS we can just take the definition there.

20.40 StreamTaylorSeriesOperations



1.00

Exports:

addiag	coerce	compose	deriv	eval
evenlambert	exquo	gderiv	generalLambert	int
integers	integrate	invmultisect	lagrange	lambert
lazyGintegrate	lazyIntegrate	mapdiv	mapmult	monom
multisect	nlde	oddintegers	oddlambert	power
powern	recip	revert	?*?	?+?
?-?	?/?	?*?	-?	

⟨*package STTAYLOR StreamTaylorSeriesOperations*⟩≡

```
)abbrev package STTAYLOR StreamTaylorSeriesOperations
++ Author: William Burge, Stephen Watt, Clifton J. Williamson
++ Date Created: 1986
++ Date Last Updated: 26 May 1994
++ Basic Operations:
++ Related Domains: Stream(A), ParadoxicalCombinatorsForStreams(A),
++   StreamTranscendentalFunctions(A),
++   StreamTranscendentalFunctionsNonCommutative(A)
++ Also See:
++ AMS Classifications:
++ Keywords: stream, Taylor series
++ Examples:
```

```
++ References:
++ Description:
++   StreamTaylorSeriesOperations implements Taylor series arithmetic,
++   where a Taylor series is represented by a stream of its coefficients.
StreamTaylorSeriesOperations(A): Exports == Implementation where
  A :        Ring
  RN     ==> Fraction Integer
  I      ==> Integer
  NNI    ==> NonNegativeInteger
  ST     ==> Stream
  SP2    ==> StreamFunctions2
  SP3    ==> StreamFunctions3
  L      ==> List
  LA     ==> List A
  YS     ==> Y$ParadoxicalCombinatorsForStreams(A)
  UN     ==> Union(ST A,"failed")
  Exports ==> with
    "+"          : (ST A,ST A) -> ST A
      ++ a + b returns the power series sum of \spad{a} and \spad{b}:
      ++ \spad{[a0,a1,..] + [b0,b1,..] = [a0 + b0,a1 + b1,..]}
    "-"          : (ST A,ST A) -> ST A
      ++ a - b returns the power series difference of \spad{a} and
      ++ \spad{b}: \spad{[a0,a1,..] - [b0,b1,..] = [a0 - b0,a1 - b1,..]}
    "-"          : ST A -> ST A
      ++ - a returns the power series negative of \spad{a}:
      ++ \spad{- [a0,a1,...] = [- a0,- a1,...]}
    "*"          : (ST A,ST A) -> ST A
      ++ a * b returns the power series (Cauchy) product of \spad{a} and b:
      ++ \spad{[a0,a1,...] * [b0,b1,...] = [c0,c1,...]} where
      ++ \spad{ck = sum(i + j = k,ai * bk)}.
    "*"          : (A,ST A) -> ST A
      ++ r * a returns the power series scalar multiplication of r by \spad{a}:
      ++ \spad{r * [a0,a1,...] = [r * a0,r * a1,...]}
    "*"          : (ST A,A) -> ST A
      ++ a * r returns the power series scalar multiplication of \spad{a} by r:
      ++ \spad{[a0,a1,...] * r = [a0 * r,a1 * r,...]}
    "exquo"      : (ST A,ST A) -> Union(ST A,"failed")
      ++ exquo(a,b) returns the power series quotient of \spad{a} by b,
      ++ if the quotient exists, and "failed" otherwise
    "/"          : (ST A,ST A) -> ST A
      ++ a / b returns the power series quotient of \spad{a} by b.
      ++ An error message is returned if \spad{b} is not invertible.
      ++ This function is used in fixed point computations.
    recip        : ST A -> UN
      ++ recip(a) returns the power series reciprocal of \spad{a}, or
      ++ "failed" if not possible.
```

```
monom        : (A,I) -> ST A
  ++ monom(deg,coef) is a monomial of degree deg with coefficient
  ++ coef.
integers     : I -> ST I
  ++ integers(n) returns \spad{[n,n+1,n+2,...]}.
oddintegers  : I -> ST I
  ++ oddintegers(n) returns \spad{[n,n+2,n+4,...]}.
int          : A -> ST A
  ++ int(r) returns [r,r+1,r+2,...], where r is a ring element.
mapmult      : (ST A,ST A) -> ST A
  ++ mapmult([a0,a1,..],[b0,b1,..])
  ++ returns \spad{[a0*b0,a1*b1,..]}.
deriv        : ST A -> ST A
  ++ deriv(a) returns the derivative of the power series with
  ++ respect to the power series variable. Thus
  ++ \spad{deriv([a0,a1,a2,...])} returns \spad{[a1,2 a2,3 a3,...]}.
gderiv       : (I -> A,ST A)  -> ST A
  ++ gderiv(f,[a0,a1,a2,..]) returns
  ++ \spad{[f(0)*a0,f(1)*a1,f(2)*a2,..]}.
coerce       : A -> ST A
  ++ coerce(r) converts a ring element r to a stream with one element.
eval         : (ST A,A) -> ST A
  ++ eval(a,r) returns a stream of partial sums of the power series
  ++ \spad{a} evaluated at the power series variable equal to r.
compose      : (ST A,ST A) -> ST A
  ++ compose(a,b) composes the power series \spad{a} with
  ++ the power series b.
lagrange     : ST A -> ST A
  ++ lagrange(g) produces the power series for f where f is
  ++ implicitly defined as \spad{f(z) = z*g(f(z))}.
revert       : ST A -> ST A
  ++ revert(a) computes the inverse of a power series \spad{a}
  ++ with respect to composition.
  ++ the series should have constant coefficient 0 and first
  ++ order coefficient 1.
addiag       : ST ST A -> ST A
  ++ addiag(x) performs diagonal addition of a stream of streams. if x =
  ++ \spad{[[a<0,0>,a<0,1>,..],[a<1,0>,a<1,1>,..],[a<2,0>,a<2,1>,..],..]}
  ++ and \spad{addiag(x) = [b<0,b<1>,...], then b<k> = sum(i+j=k,a<i,j>)}.
lambert      : ST A -> ST A
  ++ lambert(st) computes \spad{f(x) + f(x**2) + f(x**3) + ...}
  ++ if st is a stream representing \spad{f(x)}.
  ++ This function is used for computing infinite products.
  ++ If \spad{f(x)} is a power series with constant coefficient 1 then
  ++ \spad{prod(f(x**n),n = 1..infinity) = exp(lambert(log(f(x))))}.
oddlambert   : ST A -> ST A
```

```
    ++ oddlambert(st) computes \spad{f(x) + f(x**3) + f(x**5) + ...}
    ++ if st is a stream representing \spad{f(x)}.
    ++ This function is used for computing infinite products.
    ++ If f(x) is a power series with constant coefficient 1 then
    ++ \spad{prod(f(x**(2*n-1)),n=1..infinity) = exp(oddlambert(log(f(x))))}.
  evenlambert  : ST A -> ST A
    ++ evenlambert(st) computes \spad{f(x**2) + f(x**4) + f(x**6) + ...}
    ++ if st is a stream representing \spad{f(x)}.
    ++ This function is used for computing infinite products.
    ++ If \spad{f(x)} is a power series with constant coefficient 1, then
    ++ \spad{prod(f(x**(2*n)),n=1..infinity) = exp(evenlambert(log(f(x))))}.
  generalLambert : (ST A,I,I) -> ST A
    ++ generalLambert(f(x),a,d) returns
    ++ \spad{f(x**a) + f(x**(a + d)) + f(x**(a + 2 d)) + ...}.
    ++ \spad{f(x)} should have zero constant
    ++ coefficient and \spad{a} and d should be positive.
  multisect    : (I,I,ST A) -> ST A
    ++ multisect(a,b,st)
    ++ selects the coefficients of \spad{x**((a+b)*n+a)},
    ++ and changes them to \spad{x**n}.
  invmultisect : (I,I,ST A) -> ST A
    ++ invmultisect(a,b,st) substitutes \spad{x**((a+b)*n)} for \spad{x**n}
    ++ and multiplies by \spad{x**b}.
  if A has Algebra RN then
    integrate  : (A,ST A) -> ST A
      ++ integrate(r,a) returns the integral of the power series \spad{a}
      ++ with respect to the power series variableintegration where
      ++ r denotes the constant of integration. Thus
      ++ \spad{integrate(a,[a0,a1,a2,...]) = [a,a0,a1/2,a2/3,...]}.
    lazyIntegrate  : (A,() -> ST A) -> ST A
      ++ lazyIntegrate(r,f) is a local function
      ++ used for fixed point computations.
    nlde       : ST ST A -> ST A
      ++ nlde(u) solves a
      ++ first order non-linear differential equation described by u of the
      ++ form \spad{[[b<0,0>,b<0,1>,...],[b<1,0>,b<1,1>,.],...]}.
      ++ the differential equation has the form
      ++ \spad{y' = sum(i=0 to infinity,j=0 to infinity,b<i,j>*(x**i)*(y**j))}.
    powern : (RN,ST A) -> ST A
      ++ powern(r,f) raises power series f to the power r.
  if A has Field then
    mapdiv     : (ST A,ST A) -> ST A
      ++ mapdiv([a0,a1,..],[b0,b1,..]) returns
      ++ \spad{[a0/b0,a1/b1,..]}.
    lazyGintegrate : (I -> A,A,() -> ST A) -> ST A
      ++ lazyGintegrate(f,r,g) is used for fixed point computations.
```

```
      power      : (A,ST A) -> ST A
        ++ power(a,f) returns the power series f raised to the power \spad{a}.

  Implementation ==> add

--% definitions

    zro: () -> ST A
    -- returns a zero power series
    zro() == empty()$ST(A)

--% arithmetic

    x + y == delay
      empty? y => x
      empty? x => y
      eq?(x,rst x) => map(frst x + #1,y)
      eq?(y,rst y) => map(frst y + #1,x)
      concat(frst x + frst y,rst x + rst y)

    x - y == delay
      empty? y => x
      empty? x => -y
      eq?(x,rst x) => map(frst x - #1,y)
      eq?(y,rst y) => map(#1 - frst y,x)
      concat(frst x - frst y,rst x - rst y)

    -y == map(_-#1,y)

    (x:ST A) * (y:ST A) == delay
      empty? y => zro()
      empty? x => zro()
      concat(frst x * frst y,frst x * rst y + rst x * y)

    (s:A) * (x:ST A) ==
      zero? s => zro()
      map(s * #1,x)

    (x:ST A) * (s:A) ==
      zero? s => zro()
      map(#1 * s,x)

    iDiv: (ST A,ST A,A) -> ST A
    iDiv(x,y,ry0) == delay
      empty? x => empty()
      c0 := frst x * ry0
```

```
      concat(c0,iDiv(rst x - c0 * rst y,y,ry0))

    x exquo y ==
      for n in 1.. repeat
        n > 1000 => return "failed"
        empty? y => return "failed"
        empty? x => return empty()
        frst y = 0 =>
          frst x = 0 => (x := rst x; y := rst y)
          return "failed"
        leave "first entry in y is non-zero"
      (ry0 := recip frst y) case "failed" => "failed"
      empty? rst y => map(#1 * (ry0 :: A),x)
      iDiv(x,y,ry0 :: A)

    (x:ST A) / (y:ST A) == delay
      empty? y => error "/: division by zero"
      empty? x => empty()
      (ry0 := recip frst y) case "failed" =>
        error "/: second argument is not invertible"
      empty? rst y => map(#1 * (ry0 :: A),x)
      iDiv(x,y,ry0 :: A)

    recip x ==
      empty? x => "failed"
      rh1 := recip frst x
      rh1 case "failed" => "failed"
      rh := rh1 :: A
      delay
        concat(rh,iDiv(- rh * rst x,x,rh))

  --% coefficients

    rp: (I,A) -> L A
    -- rp(z,s) is a list of length z each of whose entries is s.
    rp(z,s) ==
      z <= 0 => empty()
      concat(s,rp(z-1,s))

    rpSt: (I,A) -> ST A
    -- rpSt(z,s) is a stream of length z each of whose entries is s.
    rpSt(z,s) == delay
      z <= 0 => empty()
      concat(s,rpSt(z-1,s))

    monom(s,z) ==
```

```
      z < 0 => error "monom: cannot create monomial of negative degree"
      concat(rpSt(z,0),concat(s,zro()))

  --% some streams of integers
      nnintegers: NNI -> ST NNI
      nnintegers zz == generate(#1 + 1,zz)
      integers z == generate(#1 + 1,z)
      oddintegers z == generate(#1 + 2,z)
      int s == generate(#1 + 1,s)

  --% derivatives

      mapmult(x,y) == delay
        empty? y => zro()
        empty? x => zro()
        concat(frst x * frst y,mapmult(rst x,rst y))

      deriv x ==
        empty? x => zro()
        mapmult(int 1,rest x)

      gderiv(f,x) ==
        empty? x => zro()
        mapmult(map(f,integers 0)$SP2(I,A),x)

  --% coercions

      coerce(s:A) ==
        zero? s => zro()
        concat(s,zro())

  --% evaluations and compositions

      eval(x,at) == scan(0,#1 + #2,mapmult(x,generate(at * #1,1)))$SP2(A,A)

      compose(x,y) == delay
        empty? y => concat(frst x,zro())
        not zero? frst y =>
          error "compose: 2nd argument should have 0 constant coefficient"
        empty? x => zro()
        concat(frst x,compose(rst x,y) * rst(y))

  --% reversion

      lagrangere:(ST A,ST A) -> ST A
      lagrangere(x,c) == delay(concat(0,compose(x,c)))
```

```
    lagrange x == YS(lagrangere(x,#1))

    revert x ==
      empty? x => error "revert should start 0,1,..."
      zero? frst x =>
        empty? rst x => error "revert: should start 0,1,..."
--        one? frst rst x => lagrange(recip(rst x) :: (ST A))
        (frst rst x) = 1 => lagrange(recip(rst x) :: (ST A))
      error "revert:should start 0,1,..."

--% lambert functions

    addiag(ststa:ST ST A) == delay
      empty? ststa => zro()
      empty? frst ststa => concat(0,addiag rst ststa)
      concat(frst(frst ststa),rst(frst ststa) + addiag(rst ststa))

-- lambert operates on a series +/[a[i]x**i for i in 1..] , and produces
-- the series +/[a[i](x**i/(1-x**i)) for i in 1..] i.e. forms the
-- coefficients A[n] which is the sum of a[i] for all divisors i of n
-- (including 1 and n)

    rptg1:(I,A) -> ST A
    --                                  ---------
    -- returns the repeating stream [s,0,...,0]; (there are z zeroes)
    rptg1(z,s) == repeating concat(s,rp(z,0))

    rptg2:(I,A) -> ST A
    --                                          ---------
    -- returns the repeating stream [0,...,0,s,0,...,0]
    -- there are z leading zeroes and z-1 in the period
    rptg2(z,s) == repeating concat(rp(z,0),concat(s,rp(z-1,0)))

    rptg3:(I,I,I,A) -> ST A
    rptg3(a,d,n,s) ==
      concat(rpSt(n*(a-1),0),repeating(concat(s,rp(d*n-1,0))))

    lambert x == delay
      empty? x => zro()
      zero? frst x =>
        concat(0,addiag(map(rptg1,integers 0,rst x)$SP3(I,A,ST A)))
      error "lambert:constant coefficient should be zero"

    oddlambert x == delay
      empty? x => zro()
      zero? frst x =>
```

```
        concat(0,addiag(map(rptg1,oddintegers 1,rst x)$SP3(I,A,ST A)))
      error "oddlambert: constant coefficient should be zero"

    evenlambert x == delay
      empty? x => zro()
      zero? frst x =>
        concat(0,addiag(map(rptg2,integers 1,rst x)$SP3(I,A,ST A)))
      error "evenlambert: constant coefficient should be zero"

    generalLambert(st,a,d) == delay
      a < 1 or d < 1 =>
        error "generalLambert: both integer arguments must be positive"
      empty? st => zro()
      zero? frst st =>
        concat(0,addiag(map(rptg3(a,d,#1,#2),_
                  integers 1,rst st)$SP3(I,A,ST A)))
      error "generalLambert: constant coefficient should be zero"

  --% misc. functions

    ms: (I,I,ST A) -> ST A
    ms(m,n,s) == delay
      empty? s => zro()
      zero? n => concat(frst s,ms(m,m-1,rst s))
      ms(m,n-1,rst s)

    multisect(b,a,x) == ms(a+b,0,rest(x,a :: NNI))

    altn: (ST A,ST A) -> ST A
    altn(zs,s) == delay
      empty? s => zro()
      concat(frst s,concat(zs,altn(zs,rst s)))

    invmultisect(a,b,x) ==
      concat(rpSt(b,0),altn(rpSt(a + b - 1,0),x))

  -- comps(ststa,y) forms the composition of +/b[i,j]*y**i*x**j
  -- where y is a power series in y.

    cssa ==> concat$(ST ST A)
    mapsa ==> map$SP2(ST A,ST A)
    comps: (ST ST A,ST A) -> ST ST A
    comps(ststa,x) == delay$(ST ST A)
       empty? ststa => empty()$(ST ST A)
       empty? x => cssa(frst ststa,empty()$(ST ST A))
       cssa(frst ststa,mapsa((rst x) * #1,comps(rst ststa,x)))
```

```
    if A has Algebra RN then
      integre: (ST A,I) -> ST A
      integre(x,n) == delay
        empty? x => zro()
        concat((1$I/n) * frst(x),integre(rst x,n + 1))

      integ: ST A -> ST A
      integ x == integre(x,1)

      integrate(a,x) == concat(a,integ x)
      lazyIntegrate(s,xf) == concat(s,integ(delay xf))

      nldere:(ST ST A,ST A) -> ST A
      nldere(lslsa,c) == lazyIntegrate(0,addiag(comps(lslsa,c)))
      nlde lslsa == YS(nldere(lslsa,#1))

      RATPOWERS : Boolean := A has "**": (A,RN) -> A

      smult: (RN,ST A) -> ST A
      smult(rn,x) == map(rn * #1,x)
      powerrn:(RN,ST A,ST A) -> ST A
      powerrn(rn,x,c) == delay
        concat(1,integ(smult(rn + 1,c * deriv x)) - rst x * c)
      powern(rn,x) ==
        order : I := 0
        for n in 0.. repeat
          empty? x => return zro()
          not zero? frst x => (order := n; leave x)
          x := rst x
          n = 1000 =>
            error "**: series with many leading zero coefficients"
        (ord := (order exquo denom(rn))) case "failed" =>
          error "**: rational power does not exist"
        co := frst x
        (invCo := recip co) case "failed" =>
           error "** rational power of coefficient undefined"
-- This error message is misleading, isn't it? see sups.spad/cRationalPower
        power :=
--          one? co => YS(powerrn(rn,x,#1))
          (co = 1) => YS(powerrn(rn,x,#1))
          (denom rn) = 1 =>
            not negative?(num := numer rn) =>
-- It seems that this cannot happen, but I don't know why
              (co**num::NNI) * YS(powerrn(rn,(invCo :: A) * x,#1))
            (invCo :: A)**((-num)::NNI) * YS(powerrn(rn,(invCo :: A) * x,#1))
```

```
          RATPOWERS => co**rn * YS(powerrn(rn,(invCo :: A) * x,#1))
          error "** rational power of coefficient undefined"

      if A has Field then
        mapdiv(x,y) == delay
          empty? y => error "stream division by zero"
          empty? x => zro()
          concat(frst x/frst y,mapdiv(rst x,rst y))

        ginteg: (I -> A,ST A) -> ST A
        ginteg(f,x) == mapdiv(x,map(f,integers 1)$SP2(I,A))

        lazyGintegrate(fntoa,s,xf) == concat(s,ginteg(fntoa,delay xf))

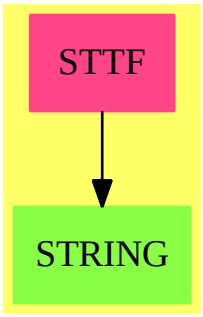
        finteg: ST A -> ST A
        finteg x == mapdiv(x,int 1)
        powerre: (A,ST A,ST A) -> ST A
        powerre(s,x,c) == delay
          empty? x => zro()
          frst x^=1 => error "**:constant coefficient should be 1"
          concat(frst x,finteg((s+1)*(c*deriv x))-rst x * c)
        power(s,x) == YS(powerre(s,x,#1))
```

⟨*STTAYLOR.dotabb*⟩≡

```
"STTAYLOR" [color="#FF4488",href="bookvol10.4.pdf#nameddest=STTAYLOR"]
"FIELD"  [color="#4488FF",href="bookvol10.2.pdf#nameddest=FIELD"]
"STTAYLOR" -> "FIELD"
```

20.41 package STTF StreamTranscendentalFunctions

20.42 StreamTranscendentalFunctions



1.00

Exports:

acos	acot	acsc	asec	asin
atan	cos	cot	csc	exp
sec	sin	tan	sincos	sinhcosh
?**?				

⟨*package STTF StreamTranscendentalFunctions*⟩≡

```
)abbrev package STTF StreamTranscendentalFunctions
++ Author: William Burge, Clifton J. Williamson
++ Date Created: 1986
++ Date Last Updated: 6 March 1995
++ Basic Operations:
++ Related Domains:
++ Also See:
++ AMS Classifications:
++ Keywords: Taylor series, elementary function, transcendental function
++ Examples:
++ References:
++ Description:
++   StreamTranscendentalFunctions implements transcendental functions on
++   Taylor series, where a Taylor series is represented by a stream of
++   its coefficients.
StreamTranscendentalFunctions(Coef): Exports == Implementation where
  Coef : Algebra Fraction Integer
  L   ==> List
  I   ==> Integer
  RN  ==> Fraction Integer
  SG  ==> String
  ST  ==> Stream Coef
  STT ==> StreamTaylorSeriesOperations Coef
```

```
   YS  ==> Y$ParadoxicalCombinatorsForStreams(Coef)

   Exports ==> with
 --% Exponentials and Logarithms
     exp     : ST -> ST
       ++ exp(st) computes the exponential of a power series st.
     log     : ST -> ST
       ++ log(st) computes the log of a power series.
     "**"    : (ST,ST) -> ST
       ++ st1 ** st2 computes the power of a power series st1 by another
       ++ power series st2.

 --% TrigonometricFunctionCategory
     sincos  : ST -> Record(sin:ST, cos:ST)
       ++ sincos(st) returns a record containing the sine and cosine
       ++ of a power series st.
     sin     : ST -> ST
       ++ sin(st) computes sine of a power series st.
     cos     : ST -> ST
       ++ cos(st) computes cosine of a power series st.
     tan     : ST -> ST
       ++ tan(st) computes tangent of a power series st.
     cot     : ST -> ST
       ++ cot(st) computes cotangent of a power series st.
     sec     : ST -> ST
       ++ sec(st) computes secant of a power series st.
     csc     : ST -> ST
       ++ csc(st) computes cosecant of a power series st.
     asin    : ST -> ST
       ++ asin(st) computes arcsine of a power series st.
     acos    : ST -> ST
       ++ acos(st) computes arccosine of a power series st.
     atan    : ST -> ST
       ++ atan(st) computes arctangent of a power series st.
     acot    : ST -> ST
       ++ acot(st) computes arccotangent of a power series st.
     asec    : ST -> ST
       ++ asec(st) computes arcsecant of a power series st.
     acsc    : ST -> ST
       ++ acsc(st) computes arccosecant of a power series st.

 --% HyperbloicTrigonometricFunctionCategory
     sinhcosh: ST -> Record(sinh:ST, cosh:ST)
       ++ sinhcosh(st) returns a record containing
       ++ the hyperbolic sine and cosine
       ++ of a power series st.
```

```
    sinh    : ST -> ST
      ++ sinh(st) computes the hyperbolic sine of a power series st.
    cosh    : ST -> ST
      ++ cosh(st) computes the hyperbolic cosine of a power series st.
    tanh    : ST -> ST
      ++ tanh(st) computes the hyperbolic tangent of a power series st.
    coth    : ST -> ST
      ++ coth(st) computes the hyperbolic cotangent of a power series st.
    sech    : ST -> ST
      ++ sech(st) computes the hyperbolic secant of a power series st.
    csch    : ST -> ST
      ++ csch(st) computes the hyperbolic cosecant of a power series st.
    asinh   : ST -> ST
      ++ asinh(st) computes the inverse hyperbolic sine of a power series st.
    acosh   : ST -> ST
      ++ acosh(st) computes the inverse hyperbolic cosine
      ++ of a power series st.
    atanh   : ST -> ST
      ++ atanh(st) computes the inverse hyperbolic tangent
      ++ of a power series st.
    acoth   : ST -> ST
      ++ acoth(st) computes the inverse hyperbolic
      ++ cotangent of a power series st.
    asech   : ST -> ST
      ++ asech(st) computes the inverse hyperbolic secant of a
      ++ power series st.
    acsch   : ST -> ST
      ++ acsch(st) computes the inverse hyperbolic
      ++ cosecant of a power series st.

  Implementation ==> add
    import StreamTaylorSeriesOperations Coef

    TRANSFCN : Boolean := Coef has TranscendentalFunctionCategory

  --% Error Reporting

    TRCONST : SG := "series expansion involves transcendental constants"
    NPOWERS : SG := "series expansion has terms of negative degree"
    FPOWERS : SG := "series expansion has terms of fractional degree"
    MAYFPOW : SG := "series expansion may have terms of fractional degree"
    LOGS : SG := "series expansion has logarithmic term"
    NPOWLOG : SG :=
       "series expansion has terms of negative degree or logarithmic term"
    FPOWLOG : SG :=
       "series expansion has terms of fractional degree or logarithmic term"
```

```
    NOTINV : SG := "leading coefficient not invertible"

  --% Exponentials and Logarithms

    expre:(Coef,ST,ST) -> ST
    expre(r,e,dx) == lazyIntegrate(r,e*dx)

    exp z ==
      empty? z => 1 :: ST
      (coef := frst z) = 0 => YS expre(1,#1,deriv z)
      TRANSFCN => YS expre(exp coef,#1,deriv z)
      error concat("exp: ",TRCONST)

    log z ==
      empty? z => error "log: constant coefficient should not be 0"
      (coef := frst z) = 0 => error "log: constant coefficient should not be 0"
      coef = 1 => lazyIntegrate(0,deriv z/z)
      TRANSFCN => lazyIntegrate(log coef,deriv z/z)
      error concat("log: ",TRCONST)

    z1:ST ** z2:ST == exp(z2 * log z1)

  --% Trigonometric Functions

    sincosre:(Coef,Coef,L ST,ST,Coef) -> L ST
    sincosre(rs,rc,sc,dx,sign) ==
      [lazyIntegrate(rs,(second sc)*dx),lazyIntegrate(rc,sign*(first sc)*dx)]

    -- When the compiler had difficulties with the above definition,
    -- I did the following to help it:

    -- sincosre:(Coef,Coef,L ST,ST,Coef) -> L ST
    -- sincosre(rs,rc,sc,dx,sign) ==
      -- st1 : ST := (second sc) * dx
      -- st2 : ST := (first sc) * dx
      -- st2 := sign * st2
      -- [lazyIntegrate(rs,st1),lazyIntegrate(rc,st2)]

    sincos z ==
      empty? z => [0 :: ST,1 :: ST]
      l :=
        (coef := frst z) = 0 => YS(sincosre(0,1,#1,deriv z,-1),2)
        TRANSFCN => YS(sincosre(sin coef,cos coef,#1,deriv z,-1),2)
        error concat("sincos: ",TRCONST)
      [first l,second l]
```

```
sin z == sincos(z).sin
cos z == sincos(z).cos

tanre:(Coef,ST,ST,Coef) -> ST
tanre(r,t,dx,sign) == lazyIntegrate(r,((1 :: ST) + sign*t*t)*dx)

-- When the compiler had difficulties with the above definition,
-- I did the following to help it:

-- tanre:(Coef,ST,ST,Coef) -> ST
-- tanre(r,t,dx,sign) ==
  -- st1 : ST := t * t
  -- st1 := sign * st1
  -- st2 : ST := 1 :: ST
  -- st1 := st2 + st1
  -- st1 := st1 * dx
  -- lazyIntegrate(r,st1)

tan z ==
  empty? z => 0 :: ST
  (coef := frst z) = 0 => YS tanre(0,#1,deriv z,1)
  TRANSFCN => YS tanre(tan coef,#1,deriv z,1)
  error concat("tan: ",TRCONST)

cotre:(Coef,ST,ST) -> ST
cotre(r,t,dx) == lazyIntegrate(r,-((1 :: ST) + t*t)*dx)

-- When the compiler had difficulties with the above definition,
-- I did the following to help it:

-- cotre:(Coef,ST,ST) -> ST
-- cotre(r,t,dx) ==
  -- st1 : ST := t * t
  -- st2 : ST := 1 :: ST
  -- st1 := st2 + st1
  -- st1 := st1 * dx
  -- st1 := -st1
  -- lazyIntegrate(r,st1)

cot z ==
  empty? z => error "cot: cot(0) is undefined"
  (coef := frst z) = 0 => error concat("cot: ",NPOWERS)
  TRANSFCN => YS cotre(cot coef,#1,deriv z)
  error concat("cot: ",TRCONST)

sec z ==
```

```
    empty? z => 1 :: ST
    frst z = 0 => recip(cos z) :: ST
    TRANSFCN =>
      cosz := cos z
      first cosz = 0 => error concat("sec: ",NPOWERS)
      recip(cosz) :: ST
    error concat("sec: ",TRCONST)

  csc z ==
    empty? z => error "csc: csc(0) is undefined"
    TRANSFCN =>
      sinz := sin z
      first sinz = 0 => error concat("csc: ",NPOWERS)
      recip(sinz) :: ST
    error concat("csc: ",TRCONST)

  orderOrFailed : ST -> Union(I,"failed")
  orderOrFailed x ==
  -- returns the order of x or "failed"
  -- if -1 is returned, the series is identically zero
    for n in 0..1000 repeat
      empty? x => return -1
      not zero? frst x => return n :: I
      x := rst x
    "failed"

  asin z ==
    empty? z => 0 :: ST
    (coef := frst z) = 0 =>
      integrate(0,powern(-1/2,(1 :: ST) - z*z) * (deriv z))
    TRANSFCN =>
      coef = 1 or coef = -1 =>
        x := (1 :: ST) - z*z
        -- compute order of 'x'
        (ord := orderOrFailed x) case "failed" =>
          error concat("asin: ",MAYFPOW)
        (order := ord :: I) = -1 => return asin(coef) :: ST
        odd? order => error concat("asin: ",FPOWERS)
        squirt := powern(1/2,x)
        (quot := (deriv z) exquo squirt) case "failed" =>
           error concat("asin: ",NOTINV)
        integrate(asin coef,quot :: ST)
      integrate(asin coef,powern(-1/2,(1 :: ST) - z*z) * (deriv z))
    error concat("asin: ",TRCONST)

  acos z ==
```

```
    empty? z =>
      TRANSFCN => acos(0)$Coef :: ST
      error concat("acos: ",TRCONST)
    TRANSFCN =>
      coef := frst z
      coef = 1 or coef = -1 =>
        x := (1 :: ST) - z*z
        -- compute order of 'x'
        (ord := orderOrFailed x) case "failed" =>
          error concat("acos: ",MAYFPOW)
        (order := ord :: I) = -1 => return acos(coef) :: ST
        odd? order => error concat("acos: ",FPOWERS)
        squirt := powern(1/2,x)
        (quot := (-deriv z) exquo squirt) case "failed" =>
           error concat("acos: ",NOTINV)
        integrate(acos coef,quot :: ST)
      integrate(acos coef,-powern(-1/2,(1 :: ST) - z*z) * (deriv z))
    error concat("acos: ",TRCONST)

  atan z ==
    empty? z => 0 :: ST
    (coef := frst z) = 0 =>
      integrate(0,(recip((1 :: ST) + z*z) :: ST) * (deriv z))
    TRANSFCN =>
      (y := recip((1 :: ST) + z*z)) case "failed" =>
        error concat("atan: ",LOGS)
      integrate(atan coef,(y :: ST) * (deriv z))
    error concat("atan: ",TRCONST)

  acot z ==
    empty? z =>
      TRANSFCN => acot(0)$Coef :: ST
      error concat("acot: ",TRCONST)
    TRANSFCN =>
      (y := recip((1 :: ST) + z*z)) case "failed" =>
        error concat("acot: ",LOGS)
      integrate(acot frst z,-(y :: ST) * (deriv z))
    error concat("acot: ",TRCONST)

  asec z ==
    empty? z => error "asec: constant coefficient should not be 0"
    TRANSFCN =>
      (coef := frst z) = 0 =>
        error "asec: constant coefficient should not be 0"
      coef = 1 or coef = -1 =>
        x := z*z - (1 :: ST)
```

```
          -- compute order of 'x'
          (ord := orderOrFailed x) case "failed" =>
            error concat("asec: ",MAYFPOW)
          (order := ord :: I) = -1 => return asec(coef) :: ST
          odd? order => error concat("asec: ",FPOWERS)
          squirt := powern(1/2,x)
          (quot := (deriv z) exquo squirt) case "failed" =>
            error concat("asec: ",NOTINV)
          (quot2 := (quot :: ST) exquo z) case "failed" =>
            error concat("asec: ",NOTINV)
          integrate(asec coef,quot2 :: ST)
        integrate(asec coef,(powern(-1/2,z*z-(1::ST))*(deriv z)) / z)
      error concat("asec: ",TRCONST)

    acsc z ==
      empty? z => error "acsc: constant coefficient should not be zero"
      TRANSFCN =>
        (coef := frst z) = 0 =>
          error "acsc: constant coefficient should not be zero"
        coef = 1 or coef = -1 =>
          x := z*z - (1 :: ST)
          -- compute order of 'x'
          (ord := orderOrFailed x) case "failed" =>
            error concat("acsc: ",MAYFPOW)
          (order := ord :: I) = -1 => return acsc(coef) :: ST
          odd? order => error concat("acsc: ",FPOWERS)
          squirt := powern(1/2,x)
          (quot := (-deriv z) exquo squirt) case "failed" =>
            error concat("acsc: ",NOTINV)
          (quot2 := (quot :: ST) exquo z) case "failed" =>
            error concat("acsc: ",NOTINV)
          integrate(acsc coef,quot2 :: ST)
        integrate(acsc coef,-(powern(-1/2,z*z-(1::ST))*(deriv z)) / z)
      error concat("acsc: ",TRCONST)

  --% Hyperbolic Trigonometric Functions

    sinhcosh z ==
      empty? z => [0 :: ST,1 :: ST]
      l :=
        (coef := frst z) = 0 => YS(sincosre(0,1,#1,deriv z,1),2)
        TRANSFCN => YS(sincosre(sinh coef,cosh coef,#1,deriv z,1),2)
        error concat("sinhcosh: ",TRCONST)
      [first l,second l]

    sinh z == sinhcosh(z).sinh
```

```
cosh z == sinhcosh(z).cosh

tanh z ==
  empty? z => 0 :: ST
  (coef := frst z) = 0 => YS tanre(0,#1,deriv z,-1)
  TRANSFCN => YS tanre(tanh coef,#1,deriv z,-1)
  error concat("tanh: ",TRCONST)

coth z ==
  tanhz := tanh z
  empty? tanhz => error "coth: coth(0) is undefined"
  (frst tanhz) = 0 => error concat("coth: ",NPOWERS)
  recip(tanhz) :: ST

sech z ==
  coshz := cosh z
  (empty? coshz) or (frst coshz = 0) => error concat("sech: ",NPOWERS)
  recip(coshz) :: ST

csch z ==
  sinhz := sinh z
  (empty? sinhz) or (frst sinhz = 0) => error concat("csch: ",NPOWERS)
  recip(sinhz) :: ST

asinh z ==
  empty? z => 0 :: ST
  (coef := frst z) = 0 => log(z + powern(1/2,(1 :: ST) + z*z))
  TRANSFCN =>
    x := (1 :: ST) + z*z
    -- compute order of 'x', in case coefficient(z,0) = +- %i
    (ord := orderOrFailed x) case "failed" =>
      error concat("asinh: ",MAYFPOW)
    (order := ord :: I) = -1 => return asinh(coef) :: ST
    odd? order => error concat("asinh: ",FPOWERS)
    -- the argument to 'log' must have a non-zero constant term
    log(z + powern(1/2,x))
  error concat("asinh: ",TRCONST)

acosh z ==
  empty? z =>
    TRANSFCN => acosh(0)$Coef :: ST
    error concat("acosh: ",TRCONST)
  TRANSFCN =>
    coef := frst z
    coef = 1 or coef = -1 =>
      x := z*z - (1 :: ST)
```

```
      -- compute order of 'x'
      (ord := orderOrFailed x) case "failed" =>
        error concat("acosh: ",MAYFPOW)
      (order := ord :: I) = -1 => return acosh(coef) :: ST
      odd? order => error concat("acosh: ",FPOWERS)
      -- the argument to 'log' must have a non-zero constant term
      log(z + powern(1/2,x))
    log(z + powern(1/2,z*z - (1 :: ST)))
  error concat("acosh: ",TRCONST)

atanh z ==
  empty? z => 0 :: ST
  (coef := frst z) = 0 =>
    (inv(2::RN)::Coef) * log(((1 :: ST) + z)/((1 :: ST) - z))
  TRANSFCN =>
    coef = 1 or coef = -1 => error concat("atanh: ",LOGS)
    (inv(2::RN)::Coef) * log(((1 :: ST) + z)/((1 :: ST) - z))
  error concat("atanh: ",TRCONST)

acoth z ==
  empty? z =>
    TRANSFCN => acoth(0)$Coef :: ST
    error concat("acoth: ",TRCONST)
  TRANSFCN =>
    frst z = 1 or frst z = -1 => error concat("acoth: ",LOGS)
    (inv(2::RN)::Coef) * log((z + (1 :: ST))/(z - (1 :: ST)))
  error concat("acoth: ",TRCONST)

asech z ==
  empty? z => error "asech: asech(0) is undefined"
  TRANSFCN =>
    (coef := frst z) = 0 => error concat("asech: ",NPOWLOG)
    coef = 1 or coef = -1 =>
      x := (1 :: ST) - z*z
      -- compute order of 'x'
      (ord := orderOrFailed x) case "failed" =>
        error concat("asech: ",MAYFPOW)
      (order := ord :: I) = -1 => return asech(coef) :: ST
      odd? order => error concat("asech: ",FPOWERS)
      log(((1 :: ST) + powern(1/2,x))/z)
    log(((1 :: ST) + powern(1/2,(1 :: ST) - z*z))/z)
  error concat("asech: ",TRCONST)

acsch z ==
  empty? z => error "acsch: acsch(0) is undefined"
  TRANSFCN =>
```

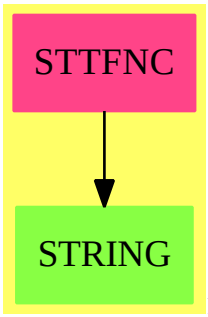
```
        frst z = 0 => error concat("acsch: ",NPOWLOG)
        x := z*z + (1 :: ST)
        -- compute order of 'x'
        (ord := orderOrFailed x) case "failed" =>
          error concat("acsc: ",MAYFPOW)
        (order := ord :: I) = -1 => return acsch(frst z) :: ST
        odd? order => error concat("acsch: ",FPOWERS)
        log(((1 :: ST) + powern(1/2,x))/z)
      error concat("acsch: ",TRCONST)
```

⟨*STTF.dotabb*⟩≡

```
"STTF" [color="#FF4488",href="bookvol10.4.pdf#nameddest=STTF"]
"STRING" [color="#88FF44",href="bookvol10.3.pdf#nameddest=STRING"]
"STTF" -> "STRING"
```

20.43 package STTFNC StreamTranscendental-FunctionsNonCommutative

20.44 StreamTranscendentalFunctionsNonCommutative



1.00

Exports:

acos	acot	acsc	asec	asin
atan	cos	cot	csc	exp
sec	sin	tan	?**?	

⟨*package STTFNC StreamTranscendentalFunctionsNonCommutative*⟩≡

```
)abbrev package STTFNC StreamTranscendentalFunctionsNonCommutative
++ Author: Clifton J. Williamson
++ Date Created: 26 May 1994
++ Date Last Updated: 26 May 1994
++ Basic Operations:
++ Related Domains:
++ Also See:
++ AMS Classifications:
++ Keywords: Taylor series, transcendental function, non-commutative
++ Examples:
++ References:
++ Description:
++   StreamTranscendentalFunctionsNonCommutative implements transcendental
++   functions on Taylor series over a non-commutative ring, where a Taylor
++   series is represented by a stream of its coefficients.
StreamTranscendentalFunctionsNonCommutative(Coef): _
         Exports == Implementation where
  Coef :   Algebra Fraction Integer
  I    ==> Integer
  SG   ==> String
  ST   ==> Stream Coef
  STTF ==> StreamTranscendentalFunctions Coef

  Exports ==> with
```

```
--% Exponentials and Logarithms
    exp     : ST -> ST
      ++ exp(st) computes the exponential of a power series st.
    log     : ST -> ST
      ++ log(st) computes the log of a power series.
    "**"    : (ST,ST) -> ST
      ++ st1 ** st2 computes the power of a power series st1 by another
      ++ power series st2.

--% TrigonometricFunctionCategory
    sin     : ST -> ST
      ++ sin(st) computes sine of a power series st.
    cos     : ST -> ST
      ++ cos(st) computes cosine of a power series st.
    tan     : ST -> ST
      ++ tan(st) computes tangent of a power series st.
    cot     : ST -> ST
      ++ cot(st) computes cotangent of a power series st.
    sec     : ST -> ST
      ++ sec(st) computes secant of a power series st.
    csc     : ST -> ST
      ++ csc(st) computes cosecant of a power series st.
    asin    : ST -> ST
      ++ asin(st) computes arcsine of a power series st.
    acos    : ST -> ST
      ++ acos(st) computes arccosine of a power series st.
    atan    : ST -> ST
      ++ atan(st) computes arctangent of a power series st.
    acot    : ST -> ST
      ++ acot(st) computes arccotangent of a power series st.
    asec    : ST -> ST
      ++ asec(st) computes arcsecant of a power series st.
    acsc    : ST -> ST
      ++ acsc(st) computes arccosecant of a power series st.

--% HyperbloicTrigonometricFunctionCategory
    sinh    : ST -> ST
      ++ sinh(st) computes the hyperbolic sine of a power series st.
    cosh    : ST -> ST
      ++ cosh(st) computes the hyperbolic cosine of a power series st.
    tanh    : ST -> ST
      ++ tanh(st) computes the hyperbolic tangent of a power series st.
    coth    : ST -> ST
      ++ coth(st) computes the hyperbolic cotangent of a power series st.
    sech    : ST -> ST
      ++ sech(st) computes the hyperbolic secant of a power series st.
```

```
    csch    : ST -> ST
      ++ csch(st) computes the hyperbolic cosecant of a power series st.
    asinh   : ST -> ST
      ++ asinh(st) computes the inverse hyperbolic sine of a power series st.
    acosh   : ST -> ST
      ++ acosh(st) computes the inverse hyperbolic cosine
      ++ of a power series st.
    atanh   : ST -> ST
      ++ atanh(st) computes the inverse hyperbolic tangent
      ++ of a power series st.
    acoth   : ST -> ST
      ++ acoth(st) computes the inverse hyperbolic
      ++ cotangent of a power series st.
    asech   : ST -> ST
      ++ asech(st) computes the inverse hyperbolic secant of a
      ++ power series st.
    acsch   : ST -> ST
      ++ acsch(st) computes the inverse hyperbolic
      ++ cosecant of a power series st.

  Implementation ==> add
    import StreamTaylorSeriesOperations(Coef)

--% Error Reporting

    ZERO    : SG := "series must have constant coefficient zero"
    ONE     : SG := "series must have constant coefficient one"
    NPOWERS : SG := "series expansion has terms of negative degree"

--% Exponentials and Logarithms

    exp z ==
      empty? z => 1 :: ST
      (frst z) = 0 =>
        expx := exp(monom(1,1))$STTF
        compose(expx,z)
      error concat("exp: ",ZERO)

    log z ==
      empty? z => error concat("log: ",ONE)
      (frst z) = 1 =>
        log1PlusX := log(monom(1,0) + monom(1,1))$STTF
        compose(log1PlusX,z - monom(1,0))
      error concat("log: ",ONE)

    (z1:ST) ** (z2:ST) == exp(log(z1) * z2)
```

```
--% Trigonometric Functions

    sin z ==
      empty? z => 0 :: ST
      (frst z) = 0 =>
        sinx := sin(monom(1,1))$STTF
        compose(sinx,z)
      error concat("sin: ",ZERO)

    cos z ==
      empty? z => 1 :: ST
      (frst z) = 0 =>
        cosx := cos(monom(1,1))$STTF
        compose(cosx,z)
      error concat("cos: ",ZERO)

    tan z ==
      empty? z => 0 :: ST
      (frst z) = 0 =>
        tanx := tan(monom(1,1))$STTF
        compose(tanx,z)
      error concat("tan: ",ZERO)

    cot z ==
      empty? z => error "cot: cot(0) is undefined"
      (frst z) = 0 => error concat("cot: ",NPOWERS)
      error concat("cot: ",ZERO)

    sec z ==
      empty? z => 1 :: ST
      (frst z) = 0 =>
        secx := sec(monom(1,1))$STTF
        compose(secx,z)
      error concat("sec: ",ZERO)

    csc z ==
      empty? z => error "csc: csc(0) is undefined"
      (frst z) = 0 => error concat("csc: ",NPOWERS)
      error concat("csc: ",ZERO)

    asin z ==
      empty? z => 0 :: ST
      (frst z) = 0 =>
        asinx := asin(monom(1,1))$STTF
        compose(asinx,z)
```

```
        error concat("asin: ",ZERO)

      atan z ==
        empty? z => 0 :: ST
        (frst z) = 0 =>
          atanx := atan(monom(1,1))$STTF
          compose(atanx,z)
        error concat("atan: ",ZERO)

      acos z == error "acos: acos undefined on this coefficient domain"
      acot z == error "acot: acot undefined on this coefficient domain"
      asec z == error "asec: asec undefined on this coefficient domain"
      acsc z == error "acsc: acsc undefined on this coefficient domain"

  --% Hyperbolic Trigonometric Functions

      sinh z ==
        empty? z => 0 :: ST
        (frst z) = 0 =>
          sinhx := sinh(monom(1,1))$STTF
          compose(sinhx,z)
        error concat("sinh: ",ZERO)

      cosh z ==
        empty? z => 1 :: ST
        (frst z) = 0 =>
          coshx := cosh(monom(1,1))$STTF
          compose(coshx,z)
        error concat("cosh: ",ZERO)

      tanh z ==
        empty? z => 0 :: ST
        (frst z) = 0 =>
          tanhx := tanh(monom(1,1))$STTF
          compose(tanhx,z)
        error concat("tanh: ",ZERO)

      coth z ==
        empty? z => error "coth: coth(0) is undefined"
        (frst z) = 0 => error concat("coth: ",NPOWERS)
        error concat("coth: ",ZERO)

      sech z ==
        empty? z => 1 :: ST
        (frst z) = 0 =>
          sechx := sech(monom(1,1))$STTF
```

```
      compose(sechx,z)
    error concat("sech: ",ZERO)

  csch z ==
    empty? z => error "csch: csch(0) is undefined"
    (frst z) = 0 => error concat("csch: ",NPOWERS)
    error concat("csch: ",ZERO)

  asinh z ==
    empty? z => 0 :: ST
    (frst z) = 0 =>
      asinhx := asinh(monom(1,1))$STTF
      compose(asinhx,z)
    error concat("asinh: ",ZERO)

  atanh z ==
    empty? z => 0 :: ST
    (frst z) = 0 =>
      atanhx := atanh(monom(1,1))$STTF
      compose(atanhx,z)
    error concat("atanh: ",ZERO)

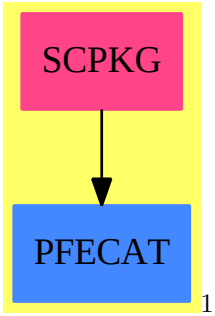
  acosh z == error "acosh: acosh undefined on this coefficient domain"
  acoth z == error "acoth: acoth undefined on this coefficient domain"
  asech z == error "asech: asech undefined on this coefficient domain"
  acsch z == error "acsch: acsch undefined on this coefficient domain"
```

⟨*STTFNC.dotabb*⟩≡

```
"STTFNC" [color="#FF4488",href="bookvol10.4.pdf#nameddest=STTFNC"]
"STRING" [color="#88FF44",href="bookvol10.3.pdf#nameddest=STRING"]
"STTFNC" -> "STRING"
```

20.45 package SCPKG StructuralConstantsPackage

20.46 StructuralConstantsPackage



1.00

Exports:
coordinates structuralConstants

⟨*package SCPKG StructuralConstantsPackage*⟩≡

```
)abbrev package SCPKG StructuralConstantsPackage
++ Authors: J. Grabmeier
++ Date Created: 02 April 1992
++ Date Last Updated: 14 April 1992
++ Basic Operations:
++ Related Constructors: AlgebraPackage, AlgebraGivenByStructuralConstants
++ Also See:
++ AMS Classifications:
++ Keywords: structural constants
++ Reference:
++ Description:
++  StructuralConstantsPackage provides functions creating
++  structural constants from a multiplication tables or a basis
++  of a matrix algebra and other useful functions in this context.
StructuralConstantsPackage(R:Field): public == private where

  L  ==> List
  S  ==> Symbol
  FRAC ==> Fraction
  POLY ==> Polynomial
  V  ==> Vector
  M  ==> Matrix
  REC  ==> Record(particular: Union(V R,"failed"),basis: List V R)
  LSMP ==> LinearSystemMatrixPackage(R,V R,V R, M R)

  public ==>  with
```

```
    -- what we really want to have here is a matrix over
    -- linear polynomials in the list of symbols, having arbitrary
    -- coefficients from a ring extension of R, e.g. FRAC POLY R.
    structuralConstants : (L S, M FRAC POLY R) -> V M FRAC POLY R
      ++ structuralConstants(ls,mt) determines the structural constants
      ++ of an algebra with generators ls and multiplication table mt, the
      ++ entries of which must be given as linear polynomials in the
      ++ indeterminates given by ls. The result is in particular useful
      ++  as fourth argument for \spadtype{AlgebraGivenByStructuralConstants}
      ++  and \spadtype{GenericNonAssociativeAlgebra}.
    structuralConstants : (L S, M POLY R) -> V M POLY R
      ++ structuralConstants(ls,mt) determines the structural constants
      ++ of an algebra with generators ls and multiplication table mt, the
      ++ entries of which must be given as linear polynomials in the
      ++ indeterminates given by ls. The result is in particular useful
      ++  as fourth argument for \spadtype{AlgebraGivenByStructuralConstants}
      ++  and \spadtype{GenericNonAssociativeAlgebra}.
    structuralConstants: L M R -> V M R
      ++ structuralConstants(basis)  takes the basis of a matrix
      ++ algebra, e.g. the result of \spadfun{basisOfCentroid} and calculates
      ++ the structural constants.
      ++ Note, that the it is not checked, whether basis really is a
      ++ basis of a matrix algebra.
    coordinates: (M R, L M R) -> V R
      ++ coordinates(a,[v1,...,vn]) returns the coordinates of \spad{a}
      ++ with respect to the \spad{R}-module basis \spad{v1},...,\spad{vn}.

  private ==> add

    matrix2Vector: M R -> V R
    matrix2Vector m ==
      lili : L L R := listOfLists m
      --li : L R  := reduce(concat, listOfLists m)
      li : L R  := reduce(concat, lili)
      construct(li)$(V R)

    coordinates(x,b) ==
      m : NonNegativeInteger := (maxIndex b) :: NonNegativeInteger
      n : NonNegativeInteger := nrows(b.1) * ncols(b.1)
      transitionMatrix   : Matrix R := new(n,m,0$R)$Matrix(R)
      for i in 1..m repeat
        setColumn_!(transitionMatrix,i,matrix2Vector(b.i))
      res : REC := solve(transitionMatrix,matrix2Vector(x))$LSMP
      if (not every?(zero?$R,first res.basis)) then
        error("coordinates: the second argument is linearly dependent")
      (res.particular  case "failed") =>
```

```
          error("coordinates: first argument is not in linear span of _
 second argument")
        (res.particular) :: (Vector R)

      structuralConstants b ==
        --n := rank()
        -- be careful with the possibility that b is not a basis
        m : NonNegativeInteger := (maxIndex b) :: NonNegativeInteger
        sC : Vector Matrix R := [new(m,m,0$R) for k in 1..m]
        for i in 1..m repeat
          for j in 1..m repeat
            covec : Vector R := coordinates(b.i * b.j, b)$%
            for k in 1..m repeat
               setelt( sC.k, i, j, covec.k )
        sC

      structuralConstants(ls:L S, mt: M POLY R)  ==
        nn := #(ls)
        nrows(mt) ^= nn or ncols(mt) ^= nn =>
          error "structuralConstants: size of second argument does not _
 agree with number of generators"
        gamma : L M POLY R := []
        lscopy : L S := copy ls
        while not null lscopy repeat
          mat : M POLY R := new(nn,nn,0)
          s : S := first lscopy
          for i in 1..nn repeat
            for j in 1..nn repeat
              p := qelt(mt,i,j)
              totalDegree(p,ls) > 1 =>
                error "structuralConstants: entries of second argument _
 must be linear polynomials in the generators"
              if (c := coefficient(p, s, 1) ) ^= 0 then qsetelt_!(mat,i,j,c)
          gamma := cons(mat, gamma)
          lscopy := rest lscopy
        vector reverse gamma

      structuralConstants(ls:L S, mt: M FRAC POLY R)  ==
        nn := #(ls)
        nrows(mt) ^= nn or ncols(mt) ^= nn =>
          error "structuralConstants: size of second argument does not _
 agree with number of generators"
        gamma : L M FRAC(POLY R) := []
        lscopy : L S := copy ls
        while not null lscopy repeat
          mat : M FRAC(POLY R) := new(nn,nn,0)
```

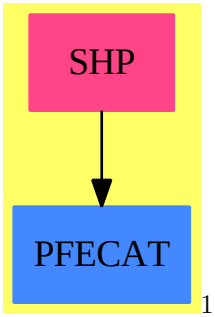
```
            s : S := first lscopy
            for i in 1..nn repeat
              for j in 1..nn repeat
                r := qelt(mt,i,j)
                q := denom(r)
                totalDegree(q,ls) ^= 0 =>
                  error "structuralConstants: entries of second argument _
  must be (linear) polynomials in the generators"
                p := numer(r)
                totalDegree(p,ls) > 1 =>
                  error "structuralConstants: entries of second argument _
  must be linear polynomials in the generators"
                if (c := coefficient(p, s, 1) ) ^= 0 then qsetelt_!(mat,i,j,c/q)
              gamma := cons(mat, gamma)
              lscopy := rest lscopy
            vector reverse gamma
```

⟨*SCPKG.dotabb*⟩≡

```
"SCPKG" [color="#FF4488",href="bookvol10.4.pdf#nameddest=SCPKG"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"SCPKG" -> "PFECAT"
```

20.47 package SHP SturmHabichtPackage

20.48 SturmHabichtPackage



1.00

Exports:

SturmHabicht	SturmHabichtCoefficients
SturmHabichtMultiple	SturmHabichtSequence
countRealRoots	countRealRootsMultiple
subresultantSequence	

⟨*package SHP SturmHabichtPackage*⟩≡

```
)abbrev package SHP SturmHabichtPackage
++ Author: Lalo Gonzalez-Vega
++ Date Created: 1994?
++ Date Last Updated: 30 January 1996
++ Basic Functions:
++ Related Constructors:
++ Also See:
++ AMS Classifications:
++ Keywords: localization
++ References:
++ Description:
++ This package produces functions for counting
++ etc. real roots of univariate polynomials in x over R, which must
++ be an OrderedIntegralDomain
SturmHabichtPackage(R,x): T == C where
  R: OrderedIntegralDomain
  x: Symbol

  UP ==> UnivariatePolynomial
  L ==> List
  INT ==> Integer
  NNI ==> NonNegativeInteger

  T == with
```

```
--     subresultantSequenceBegin:(UP(x,R),UP(x,R)) -> L UP(x,R)
--       ++ \spad{subresultantSequenceBegin(p1,p2)} computes the  initial terms
--       ++ of the Subresultant sequence Sres(j)(P,deg(P),Q,deg(P)-1)
--       ++ when deg(Q)<deg(P)
--     subresultantSequenceNext:L UP(x,R) -> L UP(x,R)
--     subresultantSequenceInner:(UP(x,R),UP(x,R)) -> L UP(x,R)
      subresultantSequence:(UP(x,R),UP(x,R)) -> L UP(x,R)
        ++ subresultantSequence(p1,p2) computes the (standard)
        ++ subresultant sequence of p1 and p2
--     sign:R -> R
--     delta:NNI -> R

--     polsth1:(UP(x,R),NNI,UP(x,R),NNI,R) -> L UP(x,R)
--     polsth2:(UP(x,R),NNI,UP(x,R),NNI,R) -> L UP(x,R)
--     polsth3:(UP(x,R),NNI,UP(x,R),NNI,R) -> L UP(x,R)
      SturmHabichtSequence:(UP(x,R),UP(x,R)) -> L UP(x,R)
        ++ SturmHabichtSequence(p1,p2) computes the Sturm-Habicht
        ++ sequence of p1 and p2
      SturmHabichtCoefficients:(UP(x,R),UP(x,R)) -> L R
        ++ SturmHabichtCoefficients(p1,p2) computes the principal
        ++ Sturm-Habicht coefficients of p1 and p2

--     variation:L R -> INT
--     permanence:L R -> INT
--     qzeros:L R -> L R
--     epsil:(NNI,R,R) -> INT
--     numbnce:L R -> NNI
--     numbce:L R -> NNI
--     wfunctaux:L R -> INT
--     wfunct:L R -> INT

      SturmHabicht:(UP(x,R),UP(x,R)) -> INT
        ++ SturmHabicht(p1,p2) computes c_{+}-c_{-} where
        ++ c_{+} is the number of real roots of p1 with p2>0 and c_{-}
        ++ is the number of real roots of p1 with p2<0. If p2=1 what
        ++ you get is the number of real roots of p1.
      countRealRoots:(UP(x,R)) -> INT
        ++ countRealRoots(p) says how many real roots p has
      if R has GcdDomain then
         SturmHabichtMultiple:(UP(x,R),UP(x,R)) -> INT
           ++ SturmHabichtMultiple(p1,p2) computes c_{+}-c_{-} where
           ++ c_{+} is the number of real roots of p1 with p2>0 and c_{-}
           ++ is the number of real roots of p1 with p2<0. If p2=1 what
           ++ you get is the number of real roots of p1.
         countRealRootsMultiple:(UP(x,R)) -> INT
           ++ countRealRootsMultiple(p) says how many real roots p has,
```

```
          ++ counted with multiplicity


C == add
   p1,p2: UP(x,R)
   Ex ==> OutputForm
   import OutputForm

   subresultantSequenceBegin(p1,p2):L UP(x,R) ==
     d1:NNI:=degree(p1)
     d2:NNI:=degree(p2)
     n:NNI:=(d1-1)::NNI
     d2 = n =>
       Pr:UP(x,R):=pseudoRemainder(p1,p2)
       append([p1,p2]::L UP(x,R),[Pr]::L UP(x,R))
     d2 = (n-1)::NNI =>
       Lc1:UP(x,R):=leadingCoefficient(p1)*leadingCoefficient(p2)*p2
       Lc2:UP(x,R):=-leadingCoefficient(p1)*pseudoRemainder(p1,p2)
       append([p1,p2]::L UP(x,R),[Lc1,Lc2]::L UP(x,R))
     LSubr:L UP(x,R):=[p1,p2]
     in1:INT:=(d2+1)::INT
     in2:INT:=(n-1)::INT
     for i in in1..in2 repeat
       LSubr:L UP(x,R):=append(LSubr::L UP(x,R),[0]::L UP(x,R))
     c1:R:=(leadingCoefficient(p1)*leadingCoefficient(p2))**((n-d2)::NNI)
     Lc1:UP(x,R):=monomial(c1,0)*p2
     Lc2:UP(x,R):=
       (-leadingCoefficient(p1))**((n-d2)::NNI)*pseudoRemainder(p1,p2)
     append(LSubr::L UP(x,R),[Lc1,Lc2]::L UP(x,R))

   subresultantSequenceNext(LcsI:L UP(x,R)):L UP(x,R) ==
     p2:UP(x,R):=last LcsI
     p1:UP(x,R):=first rest reverse LcsI
     d1:NNI:=degree(p1)
     d2:NNI:=degree(p2)
     in1:NNI:=(d1-1)::NNI
     d2 = in1 =>
       pr1:UP(x,R):=
         (pseudoRemainder(p1,p2) exquo (leadingCoefficient(p1))**2)::UP(x,R)
       append(LcsI:L UP(x,R),[pr1]:L UP(x,R))
     d2 < in1 =>
       c1:R:=leadingCoefficient(p1)
       pr1:UP(x,R):=
        (leadingCoefficient(p2)**((in1-d2)::NNI)*p2 exquo
            c1**((in1-d2)::NNI))::UP(x,R)
       pr2:UP(x,R):=
```

```
            (pseudoRemainder(p1,p2) exquo (-c1)**((in1-d2+2)::NNI))::UP(x,R)
          LSub:L UP(x,R):=[pr1,pr2]
          for k in ((d2+1)::INT)..((in1-1)::INT) repeat
            LSub:L UP(x,R):=append([0]:L UP(x,R),LSub:L UP(x,R))
          append(LcsI:L UP(x,R),LSub:L UP(x,R))

      subresultantSequenceInner(p1,p2):L UP(x,R) ==
        Lin:L UP(x,R):=subresultantSequenceBegin(p1:UP(x,R),p2:UP(x,R))
        indf:NNI:= if not(Lin.last::UP(x,R) = 0) then degree(Lin.last::UP(x,R))
                                                 else 0
        while not(indf = 0) repeat
          Lin:L UP(x,R):=subresultantSequenceNext(Lin:L UP(x,R))
          indf:NNI:= if not(Lin.last::UP(x,R)=0) then degree(Lin.last::UP(x,R))
                                                 else 0
        for j in #(Lin:L UP(x,R))..degree(p1) repeat
          Lin:L UP(x,R):=append(Lin:L UP(x,R),[0]:L UP(x,R))
        Lin


  -- Computation of the subresultant sequence Sres(j)(P,p,Q,q) when:
  --             deg(P) = p   and   deg(Q) = q   and   p > q

      subresultantSequence(p1,p2):L UP(x,R) ==
        p:NNI:=degree(p1)
        q:NNI:=degree(p2)
        List1:L UP(x,R):=subresultantSequenceInner(p1,p2)
        List2:L UP(x,R):=[p1,p2]
        c1:R:=leadingCoefficient(p1)
        for j in 3..#(List1) repeat
          Pr0:UP(x,R):=List1.j
          Pr1:UP(x,R):=(Pr0 exquo c1**((p-q-1)::NNI))::UP(x,R)
          List2:L UP(x,R):=append(List2:L UP(x,R),[Pr1]:L UP(x,R))
        List2


  -- Computation of the sign (+1,0,-1) of an element in an ordered integral
  -- domain

  --     sign(r:R):R ==
  --       r =$R 0 => 0
  --       r >$R 0 => 1
  --       -1


  -- Computation of the delta function:
```

```
      delta(int1:NNI):R ==
        (-1)**((int1*(int1+1) exquo 2)::NNI)


  -- Computation of the Sturm-Habicht sequence of two polynomials P and Q
  -- in R[x] where R is an ordered integral domaine

      polsth1(p1,p:NNI,p2,q:NNI,c1:R):L UP(x,R) ==
        sc1:R:=(sign(c1))::R
        Pr1:UP(x,R):=pseudoRemainder(differentiate(p1)*p2,p1)
        Pr2:UP(x,R):=(Pr1 exquo c1**(q::NNI))::UP(x,R)
        c2:R:=leadingCoefficient(Pr2)
        r:NNI:=degree(Pr2)
        Pr3:UP(x,R):=monomial(sc1**((p-r-1)::NNI),0)*p1
        Pr4:UP(x,R):=monomial(sc1**((p-r-1)::NNI),0)*Pr2
        Listf:L UP(x,R):=[Pr3,Pr4]
        if r < p-1 then
          Pr5:UP(x,R):=monomial(delta((p-r-1)::NNI)*c2**((p-r-1)::NNI),0)*Pr2
          for j in ((r+1)::INT)..((p-2)::INT) repeat
            Listf:L UP(x,R):=append(Listf:L UP(x,R),[0]:L UP(x,R))
          Listf:L UP(x,R):=append(Listf:L UP(x,R),[Pr5]:L UP(x,R))
        if Pr1=0 then List1:L UP(x,R):=Listf
                 else List1:L UP(x,R):=subresultantSequence(p1,Pr2)
        List2:L UP(x,R):=[]
        for j in 0..((r-1)::INT) repeat
          Pr6:UP(x,R):=monomial(delta((p-j-1)::NNI),0)*List1.((p-j+1)::NNI)
          List2:L UP(x,R):=append([Pr6]:L UP(x,R),List2:L UP(x,R))
        append(Listf:L UP(x,R),List2:L UP(x,R))

      polsth2(p1,p:NNI,p2,q:NNI,c1:R):L UP(x,R) ==
        sc1:R:=(sign(c1))::R
        Pr1:UP(x,R):=monomial(sc1,0)*p1
        Pr2:UP(x,R):=differentiate(p1)*p2
        Pr3:UP(x,R):=monomial(sc1,0)*Pr2
        Listf:L UP(x,R):=[Pr1,Pr3]
        List1:L UP(x,R):=subresultantSequence(p1,Pr2)
        List2:L UP(x,R):=[]
        for j in 0..((p-2)::INT) repeat
          Pr4:UP(x,R):=monomial(delta((p-j-1)::NNI),0)*List1.((p-j+1)::NNI)
          Pr5:UP(x,R):=(Pr4 exquo c1)::UP(x,R)
          List2:L UP(x,R):=append([Pr5]:L UP(x,R),List2:L UP(x,R))
        append(Listf:L UP(x,R),List2:L UP(x,R))

      polsth3(p1,p:NNI,p2,q:NNI,c1:R):L UP(x,R) ==
        sc1:R:=(sign(c1))::R
        q1:NNI:=(q-1)::NNI
```

```
        v:NNI:=(p+q1)::NNI
        Pr1:UP(x,R):=monomial(delta(q1::NNI)*sc1**((q+1)::NNI),0)*p1
        Listf:L UP(x,R):=[Pr1]
        List1:L UP(x,R):=subresultantSequence(differentiate(p1)*p2,p1)
        List2:L UP(x,R):=[]
        for j in 0..((p-1)::NNI) repeat
          Pr2:UP(x,R):=monomial(delta((v-j)::NNI),0)*List1.((v-j+1)::NNI)
          Pr3:UP(x,R):=(Pr2 exquo c1)::UP(x,R)
          List2:L UP(x,R):=append([Pr3]:L UP(x,R),List2:L UP(x,R))
        append(Listf:L UP(x,R),List2:L UP(x,R))

      SturmHabichtSequence(p1,p2):L UP(x,R) ==
        p:NNI:=degree(p1)
        q:NNI:=degree(p2)
        c1:R:=leadingCoefficient(p1)
        c1 = 1 or q = 1 => polsth1(p1,p,p2,q,c1)
        q = 0 => polsth2(p1,p,p2,q,c1)
        polsth3(p1,p,p2,q,c1)


  -- Computation of the Sturm-Habicht principal coefficients of two
  -- polynomials P and Q in R[x] where R is an ordered integral domain

      SturmHabichtCoefficients(p1,p2):L R ==
        List1:L UP(x,R):=SturmHabichtSequence(p1,p2)
  --      List2:L R:=[]
        qp:NNI:=#(List1)::NNI
        [coefficient(p,(qp-j)::NNI) for p in List1 for j in 1..qp]
  --      for j in 1..qp repeat
  --        Ply:R:=coefficient(List1.j,(qp-j)::NNI)
  --        List2:L R:=append(List2,[Ply])
  --      List2


  -- Computation of the number of sign variations of a list of non zero
  -- elements in an ordered integral domain

      variation(Lsig:L R):INT ==
        size?(Lsig,1) => 0
        elt1:R:=first Lsig
        elt2:R:=Lsig.2
        sig1:R:=(sign(elt1*elt2))::R
        List1:L R:=rest Lsig
        sig1 = 1 => variation List1
        1+variation List1
```

```
-- Computation of the number of sign permanences of a list of non zero
-- elements in an ordered integral domain

     permanence(Lsig:L R):INT ==
       size?(Lsig,1) => 0
       elt1:R:=first Lsig
       elt2:R:=Lsig.2
       sig1:R:=(sign(elt1*elt2))::R
       List1:L R:=rest Lsig
       sig1 = -1 => permanence List1
       1+permanence List1


-- Computation of the functional W which works over a list of elements
-- in an ordered integral domain, with non zero first element

     qzeros(Lsig:L R):L R ==
       while last Lsig = 0 repeat
         Lsig:L R:=reverse rest reverse Lsig
       Lsig

     epsil(int1:NNI,elt1:R,elt2:R):INT ==
       int1 = 0 => 0
       odd? int1 => 0
       ct1:INT:=if elt1 > 0 then 1 else -1
       ct2:INT:=if elt2 > 0 then 1 else -1
       ct3:NNI:=(int1 exquo 2)::NNI
       ct4:INT:=(ct1*ct2)::INT
       ((-1)**(ct3::NNI))*ct4

     numbnce(Lsig:L R):NNI ==
       null Lsig => 0
       eltp:R:=Lsig.1
       eltp = 0 => 0
       1 + numbnce(rest Lsig)

     numbce(Lsig:L R):NNI ==
       null Lsig => 0
       eltp:R:=Lsig.1
       not(eltp = 0) => 0
       1 + numbce(rest Lsig)

     wfunctaux(Lsig:L R):INT ==
       null Lsig => 0
       List2:L R:=[]
```

```
        List1:L R:=Lsig:L R
        cont1:NNI:=numbnce(List1:L R)
        for j in 1..cont1 repeat
          List2:L R:=append(List2:L R,[first List1]:L R)
          List1:L R:=rest List1
        ind2:INT:=0
        cont2:NNI:=numbce(List1:L R)
        for j in 1..cont2 repeat
          List1:L R:=rest List1
          ind2:INT:=epsil(cont2:NNI,last List2,first List1)
        ind3:INT:=permanence(List2:L R)-variation(List2:L R)
        ind4:INT:=ind2+ind3
        ind4+wfunctaux(List1:L R)

      wfunct(Lsig:L R):INT ==
        List1:L R:=qzeros(Lsig:L R)
        wfunctaux(List1:L R)


  -- Computation of the integer number:
  --    #[{a in Rc(R)/P(a)=0 Q(a)>0}] - #[{a in Rc(R)/P(a)=0 Q(a)<0}]
  -- where:
  --    - R is an ordered integral domain,
  --    - Rc(R) is the real clousure of R,
  --    - P and Q are polynomials in R[x],
  --    - by #[A] we note the cardinal of the set A

  -- In particular:
  --      - SturmHabicht(P,1) is the number of "real" roots of P,
  --      - SturmHabicht(P,Q**2) is the number of "real" roots of P making Q neq 0

       SturmHabicht(p1,p2):INT ==
  --     print("+" :: Ex)
         p2 = 0 => 0
         degree(p1:UP(x,R)) = 0 => 0
         List1:L UP(x,R):=SturmHabichtSequence(p1,p2)
         qp:NNI:=#(List1)::NNI
         wfunct [coefficient(p,(qp-j)::NNI) for p in List1 for j in 1..qp]

       countRealRoots(p1):INT == SturmHabicht(p1,1)

       if R has GcdDomain then
          SturmHabichtMultiple(p1,p2):INT ==
    --      print("+" :: Ex)
          p2 = 0 => 0
          degree(p1:UP(x,R)) = 0 => 0
```

```
        SH:L UP(x,R):=SturmHabichtSequence(p1,p2)
        qp:NNI:=#(SH)::NNI
        ans:= wfunct [coefficient(p,(qp-j)::NNI) for p in SH for j in 1..qp]
        SH:=reverse SH
        while first SH = 0 repeat SH:=rest SH
        degree first SH = 0 => ans
        -- OK: it probably wasn't square free, so this item is probably the
        -- gcd of p1 and p1'
        -- unless p1 and p2 have a factor in common (naughty!)
        differentiate(p1) exquo first SH case UP(x,R) =>
           -- it was the gcd of p1 and p1'
           ans+SturmHabichtMultiple(first SH,p2)
        sqfr:=factorList squareFree p1
        #sqfr = 1 and sqfr.first.xpnt=1 => ans
        reduce("+",[f.xpnt*SturmHabicht(f.fctr,p2) for f in sqfr])

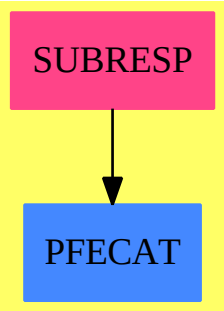
      countRealRootsMultiple(p1):INT == SturmHabichtMultiple(p1,1)
```

⟨*SHP.dotabb*⟩≡

```
"SHP" [color="#FF4488",href="bookvol10.4.pdf#nameddest=SHP"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"SHP" -> "PFECAT"
```

20.49 package SUBRESP SubResultantPackage

20.50 SubResultantPackage



1.00

Exports:
primitivePart subresultantVector

⟨*package SUBRESP SubResultantPackage*⟩≡
```
)abbrev package SUBRESP SubResultantPackage
++ Subresultants
++ Author: Barry Trager, Renaud Rioboo
++ Date Created: 1987
++ Date Last Updated: August 2000
++ Description:
++ This package computes the subresultants of two polynomials which is needed
++ for the 'Lazard Rioboo' enhancement to Tragers integrations formula
++ For efficiency reasons this has been rewritten to call Lionel Ducos
++ package which is currently the best one.
++
SubResultantPackage(R, UP): Exports == Implementation where
  R : IntegralDomain
  UP: UnivariatePolynomialCategory R

  Z   ==> Integer
  N   ==> NonNegativeInteger

  Exports ==> with
    subresultantVector: (UP, UP) -> PrimitiveArray UP
      ++ subresultantVector(p, q) returns \spad{[p0,...,pn]}
      ++ where pi is the i-th subresultant of p and q.
      ++ In particular, \spad{p0 = resultant(p, q)}.
    if R has EuclideanDomain then
      primitivePart     : (UP,  R) -> UP
        ++ primitivePart(p, q) reduces the coefficient of p
        ++ modulo q, takes the primitive part of the result,
```

```
        ++ and ensures that the leading coefficient of that
        ++ result is monic.

    Implementation ==> add

      Lionel ==> PseudoRemainderSequence(R,UP)

      if R has EuclideanDomain then
        primitivePart(p, q) ==
          rec := extendedEuclidean(leadingCoefficient p, q,
                                   1)::Record(coef1:R, coef2:R)
          unitCanonical primitivePart map((rec.coef1 * #1) rem q, p)

      subresultantVector(p1, p2) ==
         F : UP -- auxiliary stuff !
         res : PrimitiveArray(UP) := new(2+max(degree(p1),degree(p2)), 0)
         --
         -- kind of stupid interface to Lionel's  Package !!!!!!!!!!!!
         -- might have been wiser to rewrite the loop ...
         -- But I'm too lazy. [rr]
         --
         l := chainSubResultants(p1,p2)$Lionel
         --
         -- this returns the chain of non null subresultants !
         -- we must  rebuild subresultants from this.
         -- we really hope Lionel Ducos minded what he wrote
         -- since we are fully blind !
         --
         null l =>
           -- Hum it seems that Lionel returns [] when min(|p1|,|p2|) = 0
           zero?(degree(p1)) =>
             res.degree(p2) := p2
             if degree(p2) > 0
             then
               res.((degree(p2)-1)::NonNegativeInteger) := p1
               res.0 := (leadingCoefficient(p1)**(degree p2)) :: UP
             else
               -- both are of degree 0 the resultant is 1 according to Loos
               res.0 := 1
             res
           zero?(degree(p2)) =>
             if degree(p1) > 0
             then
               res.((degree(p1)-1)::NonNegativeInteger) := p2
               res.0 := (leadingCoefficient(p2)**(degree p1)) :: UP
             else
```

```
      -- both are of degree 0 the resultant is 1 according to Loos
      res.0 := 1
    res
  error "SUBRESP: strange Subresultant chain from PRS"
Sn := first(l)
--
-- as of Loos definitions last subresultant should not be defective
--
l := rest(l)
n := degree(Sn)
F := Sn
null l => error "SUBRESP: strange Subresultant chain from PRS"
zero? Sn => error "SUBRESP: strange Subresultant chain from PRS"
while (l ^= []) repeat
  res.(n) := Sn
  F := first(l)
  l := rest(l)
  -- F is potentially defective
  if degree(F) = n
  then
    --
    -- F is defective
    --
    null l => error "SUBRESP: strange Subresultant chain from PRS"
    Sn := first(l)
    l := rest(l)
    n := degree(Sn)
    res.((n-1)::NonNegativeInteger) := F
  else
    --
    -- F is non defective
    --
    degree(F) < n => error "strange result !"
    Sn := F
    n := degree(Sn)
--
-- Lionel forgets about p1 if |p1| > |p2|
-- forgets about p2 if |p2| > |p1|
-- but he reminds p2 if |p1| = |p2|
-- a glance at Loos should correct this !
--
res.n := Sn
--
-- Loos definition
--
if degree(p1) = degree(p2)
```

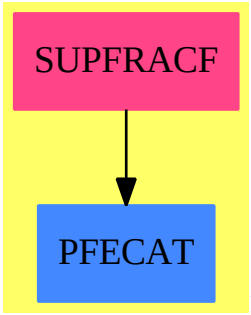
```
        then
          res.((degree p1)+1) := p1
        else
          if degree(p1) > degree(p2)
          then
            res.(degree p1) := p1
          else
            res.(degree p2) := p2
        res
```

⟨*SUBRESP.dotabb*⟩≡

```
"SUBRESP" [color="#FF4488",href="bookvol10.4.pdf#nameddest=SUBRESP"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"SUBRESP" -> "PFECAT"
```

20.51 package SUPFRACF SupFractionFactorizer

20.52 SupFractionFactorizer



1.00

Exports:
factor squareFree

⟨*package SUPFRACF SupFractionFactorizer*⟩≡
```
)abbrev package SUPFRACF SupFractionFactorizer
++ Author: P. Gianni
++ Date Created: October 1993
++ Date Last Updated: March 1995
++ Basic Functions:
++ Related Constructors: MultivariateFactorize
++ Also See:
++ AMS Classifications:
++ Keywords:
++ References:
++ Description:  SupFractionFactorize
++ contains the factor function for univariate
++ polynomials over the quotient field of a ring S such that the package
++ MultivariateFactorize works for S

SupFractionFactorizer(E,OV,R,P) : C == T
 where
  E   :   OrderedAbelianMonoidSup
  OV  :   OrderedSet
  R   :   GcdDomain
  P   :   PolynomialCategory(R,E,OV)
  FP  ==> Fraction P
  SUP ==> SparseUnivariatePolynomial

  C  == with
     factor      : SUP FP   ->  Factored SUP FP
```

```
      ++ factor(p) factors the univariate polynomial p with coefficients
      ++ which are fractions of polynomials over R.
    squareFree  : SUP FP   ->  Factored SUP FP
      ++ squareFree(p) returns the square-free factorization of the univariate polynomial p
      ++ which are fractions of polynomials over R. Each factor has no repeated roots and t
      ++ pairwise relatively prime.

  T  == add
    MFACT  ==> MultivariateFactorize(OV,E,R,P)
    MSQFR  ==> MultivariateSquareFree(E,OV,R,P)
    UPCF2  ==> UnivariatePolynomialCategoryFunctions2

    factor(p:SUP FP) : Factored SUP FP  ==
      p=0 => 0
      R has CharacteristicZero and R has EuclideanDomain =>
        pden : P := lcm [denom c for c in coefficients p]
        pol  : SUP FP := (pden::FP)*p
        ipol: SUP P := map(numer,pol)$UPCF2(FP,SUP FP,P,SUP P)
        ffact: Factored SUP P := 0
        ffact := factor(ipol)$MFACT
        makeFR((1/pden * map(coerce,unit ffact)$UPCF2(P,SUP P,FP,SUP FP)),
        [["prime",map(coerce,u.factor)$UPCF2(P,SUP P,FP,SUP FP),
           u.exponent] for u in factors ffact])
      squareFree p

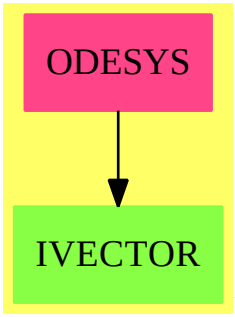
    squareFree(p:SUP FP) : Factored SUP FP  ==
      p=0 => 0
      pden : P := lcm [denom c for c in coefficients p]
      pol  : SUP FP := (pden::FP)*p
      ipol: SUP P := map(numer,pol)$UPCF2(FP,SUP FP,P,SUP P)
      ffact: Factored SUP P := 0
      if R has CharacteristicZero and R has EuclideanDomain then
        ffact := squareFree(ipol)$MSQFR
      else ffact := squareFree(ipol)
      makeFR((1/pden * map(coerce,unit ffact)$UPCF2(P,SUP P,FP,SUP FP)),
        [["sqfr",map(coerce,u.factor)$UPCF2(P,SUP P,FP,SUP FP),
           u.exponent] for u in factors ffact])
```

⟨*SUPFRACF.dotabb*⟩≡

```
"SUPFRACF" [color="#FF4488",href="bookvol10.4.pdf#nameddest=SUPFRACF"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"SUPFRACF" -> "PFECAT"
```

20.53 package ODESYS SystemODESolver

20.54 SystemODESolver



1.00

Exports:
solve solveInField triangulate

⟨*package ODESYS SystemODESolver*⟩≡
```
)abbrev package ODESYS SystemODESolver
++ Author: Manuel Bronstein
++ Date Created: 11 June 1991
++ Date Last Updated: 13 April 1994
++ Description: SystemODESolver provides tools for triangulating
++ and solving some systems of linear ordinary differential equations.
++ Keywords: differential equation, ODE, system
SystemODESolver(F, LO): Exports == Implementation where
  F : Field
  LO: LinearOrdinaryDifferentialOperatorCategory F

  N   ==> NonNegativeInteger
  Z   ==> Integer
  MF  ==> Matrix F
  M   ==> Matrix LO
  V   ==> Vector F
  UF  ==> Union(F, "failed")
  UV  ==> Union(V, "failed")
  REC ==> Record(mat: M, vec: V)
  FSL ==> Record(particular: UF, basis: List F)
  VSL ==> Record(particular: UV, basis: List V)
  SOL ==> Record(particular: F, basis: List F)
  USL ==> Union(SOL, "failed")
  ER  ==> Record(C: MF, g: V, eq: LO, rh: F)

  Exports ==> with
    triangulate: (MF, V) -> Record(A:MF, eqs: List ER)
```

```
      ++ triangulate(M,v) returns
      ++ \spad{A,[[C_1,g_1,L_1,h_1],...,[C_k,g_k,L_k,h_k]]}
      ++ such that under the change of variable \spad{y = A z}, the first
      ++ order linear system \spad{D y = M y + v} is uncoupled as
      ++ \spad{D z_i = C_i z_i + g_i} and each \spad{C_i} is a companion
      ++ matrix corresponding to the scalar equation \spad{L_i z_j = h_i}.
    triangulate: (M, V) -> REC
      ++ triangulate(m, v) returns \spad{[m_0, v_0]} such that \spad{m_0}
      ++ is upper triangular and the system \spad{m_0 x = v_0} is equivalent
      ++ to \spad{m x = v}.
    solve: (MF,V,(LO,F)->USL) -> Union(Record(particular:V, basis:MF),"failed")
      ++ solve(m, v, solve) returns \spad{[[v_1,...,v_m], v_p]} such that
      ++ the solutions in \spad{F} of the system \spad{D x = m x + v} are
      ++ \spad{v_p + c_1 v_1 + ... + c_m v_m} where the \spad{c_i's} are
      ++ constants, and the \spad{v_i's} form a basis for the solutions of
      ++ \spad{D x = m x}.
      ++ Argument \spad{solve} is a function for solving a single linear
      ++ ordinary differential equation in \spad{F}.
    solveInField: (M, V, (LO, F) -> FSL) -> VSL
      ++ solveInField(m, v, solve) returns \spad{[[v_1,...,v_m], v_p]} such
      ++ that the solutions in \spad{F} of the system \spad{m x = v} are
      ++ \spad{v_p + c_1 v_1 + ... + c_m v_m} where the \spad{c_i's} are
      ++ constants, and the \spad{v_i's} form a basis for the solutions of
      ++ \spad{m x = 0}.
      ++ Argument \spad{solve} is a function for solving a single linear
      ++ ordinary differential equation in \spad{F}.

  Implementation ==> add
    import PseudoLinearNormalForm F

    applyLodo   : (M, Z, V, N) -> F
    applyLodo0  : (M, Z, Matrix F, Z, N) -> F
    backsolve   : (M, V, (LO, F) -> FSL) -> VSL
    firstnonzero: (M, Z) -> Z
    FSL2USL     : FSL -> USL
    M2F         : M -> Union(MF, "failed")

    diff := D()$LO

    solve(mm, v, solve) ==
      rec  := triangulate(mm, v)
      sols:List(SOL) := empty()
      for e in rec.eqs repeat
          (u := solve(e.eq, e.rh)) case "failed" => return "failed"
          sols := concat(u::SOL, sols)
      n := nrows(rec.A)    -- dimension of original vectorspace
```

```
        k:N := 0             -- sum of sizes of visited companionblocks
        i:N := 0             -- number of companionblocks
        m:N := 0             -- number of Solutions
        part:V := new(n, 0)
        -- count first the different solutions
        for sol in sols repeat m := m + count(#1 ^= 0, sol.basis)$List(F)
        SolMatrix:MF := new(n, m, 0)
        m := 0
        for sol in reverse_! sols repeat
            i := i+1
            er := rec.eqs.i
            nn := #(er.g)           -- size of active companionblock
            for s in sol.basis repeat
                solVec:V := new(n, 0)
                -- compute corresponding solution base with recursion (24)
                solVec(k+1) := s
                for l in 2..nn repeat solVec(k+l) := diff solVec(k+l-1)
                m := m+1
                setColumn!(SolMatrix, m, solVec)
            -- compute with (24) the corresponding components of the part. sol.
            part(k+1) := sol.particular
            for l in 2..nn repeat part(k+l) := diff part(k+l-1) - (er.g)(l-1)
            k := k+nn
        -- transform these values back to the original system
        [rec.A * part, rec.A * SolMatrix]

      triangulate(m:MF, v:V) ==
        k:N := 0       -- sum of companion-dimensions
        rat := normalForm(m, 1, - diff #1)
        l   := companionBlocks(rat.R, rat.Ainv * v)
        ler:List(ER) := empty()
        for er in l repeat
          n := nrows(er.C)         -- dimension of this companion vectorspace
          op:LO := 0               -- compute homogeneous equation
          for j in 0..n-1 repeat op := op + monomial((er.C)(n, j + 1), j)
          op := monomial(1, n) - op
          sum:V := new(n::N, 0)    -- compute inhomogen Vector (25)
          for j in 1..n-1 repeat sum(j+1) := diff(sum j) + (er.g) j
          h0:F := 0                 -- compute inhomogenity (26)
          for j in 1..n repeat h0 := h0 - (er.C)(n, j) * sum j
          h0 := h0 + diff(sum n) + (er.g) n
          ler := concat([er.C, er.g, op, h0], ler)
          k := k + n
        [rat.A, ler]

    -- like solveInField, but expects a system already triangularized
```

```
      backsolve(m, v, solve) ==
        part:V
        r := maxRowIndex m
        offset := minIndex v - (mr := minRowIndex m)
        while r >= mr and every?(zero?, row(m, r))$Vector(LO) repeat r := r - 1
        r < mr => error "backsolve: system has a 0 matrix"
        (c := firstnonzero(m, r)) ^= maxColIndex m =>
          error "backsolve: undetermined system"
        rec := solve(m(r, c), v(r + offset))
        dim := (r - mr + 1)::N
        if (part? := ((u := rec.particular) case F)) then
          part := new(dim, 0)                          -- particular solution
          part(r + offset) :=  u::F
   -- hom is the basis for the homogeneous solutions, each column is a solution
        hom:Matrix(F) := new(dim, #(rec.basis), 0)
        for i in minColIndex hom .. maxColIndex hom for b in rec.basis repeat
          hom(r, i) := b
        n:N := 1                 -- number of equations already solved
        while r > mr repeat
          r := r - 1
          c := c - 1
          firstnonzero(m, r) ^= c => error "backsolve: undetermined system"
          degree(eq := m(r, c)) > 0 => error "backsolve: pivot of order > 0"
          a := leadingCoefficient(eq)::F
          if part? then
             part(r + offset) := (v(r + offset) - applyLodo(m, r, part, n)) / a
          for i in minColIndex hom .. maxColIndex hom repeat
            hom(r, i) := - applyLodo0(m, r, hom, i, n)
          n := n + 1
        bas:List(V) := [column(hom,i) for i in minColIndex hom..maxColIndex hom]
        part? => [part, bas]
        ["failed", bas]

      solveInField(m, v, solve) ==
        ((n := nrows m) = ncols m) and
           ((u := M2F(diagonalMatrix [diff for i in 1..n] - m)) case MF) =>
               (uu := solve(u::MF, v, FSL2USL solve(#1, #2))) case "failed" =>
                    ["failed", empty()]
               rc := uu::Record(particular:V, basis:MF)
               [rc.particular, [column(rc.basis, i) for i in 1..ncols(rc.basis)]]
        rec := triangulate(m, v)
        backsolve(rec.mat, rec.vec, solve)

      M2F m ==
          mf:MF := new(nrows m, ncols m, 0)
          for i in minRowIndex m .. maxRowIndex m repeat
```

```
            for j in minColIndex m .. maxColIndex m repeat
                (u := retractIfCan(m(i, j))@Union(F, "failed")) case "failed" =>
                     return "failed"
                mf(i, j) := u::F
        mf

    FSL2USL rec ==
        rec.particular case "failed" => "failed"
        [rec.particular::F, rec.basis]

-- returns the index of the first nonzero entry in row r of m
    firstnonzero(m, r) ==
      for c in minColIndex m .. maxColIndex m repeat
        m(r, c) ^= 0 => return c
      error "firstnonzero: zero row"

-- computes +/[m(r, i) v(i) for i ranging over the last n columns of m]
    applyLodo(m, r, v, n) ==
      ans:F := 0
      c := maxColIndex m
      cv := maxIndex v
      for i in 1..n repeat
        ans := ans + m(r, c) (v cv)
        c := c - 1
        cv := cv - 1
      ans

-- computes +/[m(r, i) mm(i, c) for i ranging over the last n columns of m]
    applyLodo0(m, r, mm, c, n) ==
      ans := 0
      rr := maxRowIndex mm
      cc := maxColIndex m
      for i in 1..n repeat
        ans := ans + m(r, cc) mm(rr, c)
        cc := cc - 1
        rr := rr - 1
      ans

    triangulate(m:M, v:V) ==
      x := copy m
      w := copy v
      nrows := maxRowIndex x
      ncols := maxColIndex x
      minr  := i := minRowIndex x
      offset := minIndex w - minr
      for j in minColIndex x .. ncols repeat
```

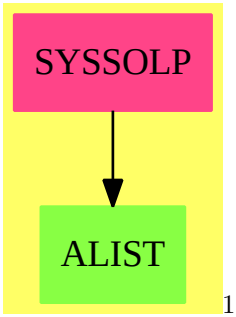
```
      if i > nrows then leave x
      rown := minr - 1
      for k in i .. nrows repeat
        if (x(k, j) ^= 0) and ((rown = minr - 1) or
                          degree x(k,j) < degree x(rown,j)) then rown := k
        rown = minr - 1 => "enuf"
        x := swapRows_!(x, i, rown)
        swap_!(w, i + offset, rown + offset)
      for k in i+1 .. nrows | x(k, j) ^= 0 repeat
        l := rightLcm(x(i,j), x(k,j))
        a := rightQuotient(l, x(i, j))
        b := rightQuotient(l, x(k, j))
        -- l = a x(i,j) = b x(k,j)
        for k1 in j+1 .. ncols repeat
          x(k, k1) :=  a * x(i, k1) - b * x(k, k1)
        x(k, j) := 0
        w(k + offset) := a(w(i + offset)) - b(w(k + offset))
      i := i+1
    [x, w]
```

⟨*ODESYS.dotabb*⟩≡

```
"ODESYS" [color="#FF4488",href="bookvol10.4.pdf#nameddest=ODESYS"]
"IVECTOR" [color="#88FF44",href="bookvol10.3.pdf#nameddest=IVECTOR"]
"ODESYS" -> "IVECTOR"
```

20.55 package SYSSOLP SystemSolvePackage

20.56 SystemSolvePackage



1.00

Exports:
solve triangularSystems

⟨*package SYSSOLP SystemSolvePackage*⟩≡
```
)abbrev package SYSSOLP SystemSolvePackage
++ Author: P. Gianni
++ Date Created: summer 1988
++ Date Last Updated: summer 1990
++ Basic Functions:
++ Related Constructors: Fraction, Polynomial, FloatingRealPackage,
++ FloatingComplexPackage, RadicalSolvePackage
++ Also See: LinearSystemMatrixPackage, GroebnerSolve
++ AMS Classifications:
++ Keywords:
++ References:
++ Description:
++ Symbolic solver for systems of rational functions with coefficients
++ in an integral domain R.
++ The systems are solved in the field of rational functions over R.
++ Solutions are exact of the form variable = value when the value is
++ a member of the coefficient domain R. Otherwise the solutions
++ are implicitly expressed as roots of univariate polynomial equations over R.
++ Care is taken to guarantee that the denominators of the input
++ equations do not vanish on the solution sets.
++ The arguments to solve can either be given as equations or
++ as rational functions interpreted as equal
++ to zero. The user can specify an explicit list of symbols to
++ be solved for, treating all other symbols appearing as parameters
++ or omit the list of symbols in which case the system tries to
++ solve with respect to all symbols appearing in the input.
```

```
NNI      ==> NonNegativeInteger
P        ==> Polynomial
EQ       ==> Equation
L        ==> List
V        ==> Vector
M        ==> Matrix
UP       ==> SparseUnivariatePolynomial
SE       ==> Symbol
IE       ==> IndexedExponents Symbol
SUP      ==> SparseUnivariatePolynomial

SystemSolvePackage(R): Cat == Cap where
    R : IntegralDomain
    F   ==> Fraction Polynomial R
    PP2 ==> PolynomialFunctions2(R,F)
    PPR ==> Polynomial Polynomial R

    Cat == with
       solve:    (L F,    L SE) -> L L EQ F
         ++ solve(lp,lv) finds the solutions of the list lp of
         ++ rational functions with respect to the list of symbols lv.

       solve:    (L EQ F, L SE) -> L L EQ F
         ++ solve(le,lv) finds the solutions of the
         ++ list le of equations of rational functions
         ++ with respect to the list of symbols lv.

       solve:         L F       -> L L EQ F
         ++ solve(lp) finds the solutions of the list lp of rational
         ++ functions with respect to all symbols appearing in lp.

       solve:        L EQ F     -> L L EQ F
         ++ solve(le) finds the solutions of the list le of equations of
         ++ rational functions with respect to all symbols appearing in le.

       solve:    (F, SE) ->  L EQ F
         ++ solve(p,v) solves the equation p=0, where p is a rational function
         ++ with respect to the variable v.

       solve:    (EQ F,SE) -> L EQ F
         ++ solve(eq,v) finds the solutions of the equation
         ++ eq with respect to the variable v.

       solve:         F      -> L EQ F
         ++ solve(p) finds the solution of a rational function p = 0
         ++ with respect to the unique variable appearing in p.
```

```
    solve:         EQ F     -> L EQ F
      ++ solve(eq) finds the solutions of the equation eq
      ++ with respect to the unique variable appearing in eq.

    triangularSystems: (L F,    L SE) -> L L P R
      ++ triangularSystems(lf,lv) solves the system of equations
      ++ defined by lf with respect to the list of symbols lv;
      ++ the system of equations is obtaining
      ++ by equating to zero the list of rational functions lf.
      ++ The output is a list of solutions where
      ++ each solution is expressed as a "reduced" triangular system of
      ++ polynomials.

  Cap == add

    import MPolyCatRationalFunctionFactorizer(IE,SE,R,P F)

                  ---- Local Functions ----
    linSolve: (L F,    L SE) -> Union(L EQ F, "failed")
    makePolys :   L EQ F     ->  L F

    makeR2F(r : R) : F == r :: (P R) :: F

    makeP2F(p:P F):F ==
      lv:=variables p
      lv = [] => retract p
      for v in lv repeat p:=pushdown(p,v)
      retract p
                  ---- Local Functions ----
    makeEq(p:P F,lv:L SE): EQ F ==
      z:=last lv
      np:=numer makeP2F p
      lx:=variables np
      for x in lv repeat if member?(x,lx) then leave x
      up:=univariate(np,x)
      (degree up)=1 =>
        equation(x::P(R)::F,-coefficient(up,0)/leadingCoefficient up)
      equation(np::F,0$F)

    varInF(v: SE): F == v::P(R) :: F

    newInF(n: Integer):F==varInF new()$SE

    testDegree(f :P R , lv :L SE) : Boolean ==
      "or"/[degree(f,vv)>0 for vv in lv]
```

```
                ---- Exported Functions ----

 -- solve a system of rational functions
 triangularSystems(lf: L F,lv:L SE) : L L P R ==
     empty? lv => empty()
     empty? lf => empty()
     #lf = 1 =>
        p:= numer(first lf)
        fp:=(factor p)$GeneralizedMultivariateFactorize(SE,IE,R,R,P R)
        [[ff.factor] for ff in factors fp | testDegree(ff.factor,lv)]
     dmp:=DistributedMultivariatePolynomial(lv,P R)
     OV:=OrderedVariableList(lv)
     DP:=DirectProduct(#lv, NonNegativeInteger)
     push:=PushVariables(R,DP,OV,dmp)
     lq : L dmp
     lvv:L OV:=[variable(vv)::OV for vv in lv]
     lq:=[pushup(df::dmp,lvv)$push for f in lf|(df:=denom f)^=1]
     lp:=[pushup(numer(f)::dmp,lvv)$push for f in lf]
     parRes:=groebSolve(lp,lvv)$GroebnerSolve(lv,P R,R)
     if lq^=[] then
       gb:=GroebnerInternalPackage(P R,DirectProduct(#lv,NNI),OV,dmp)
       parRes:=[pr for pr in parRes|
                  and/[(redPol(fq,pr pretend List(dmp))$gb) ^=0
                    for fq in lq]]
     [[retract pushdown(pf,lvv)$push for pf in pr] for pr in parRes]

-- One polynomial. Implicit variable --
 solve(pol : F) ==
   zero? pol =>
      error "equation is always satisfied"
   lv:=removeDuplicates
       concat(variables numer pol, variables denom pol)
   empty? lv => error "inconsistent equation"
   #lv>1 => error "too many variables"
   solve(pol,first lv)

 -- general solver. Input in equation style. Implicit variables --
 solve(eq : EQ F) ==
   pol:= lhs eq - rhs eq
   zero? pol =>
      error "equation is always satisfied"
   lv:=removeDuplicates
       concat(variables numer pol, variables denom pol)
   empty? lv => error "inconsistent equation"
   #lv>1 => error "too many variables"
   solve(pol,first lv)
```

```
-- general solver. Input in equation style  --
solve(eq:EQ F,var:SE)  == solve(lhs eq - rhs eq,var)

-- general solver. Input in polynomial style  --
solve(pol:F,var:SE) ==
  if R has GcdDomain then
    p:=primitivePart(numer pol,var)
    fp:=(factor p)$GeneralizedMultivariateFactorize(SE,IE,R,R,P R)
    [makeEq(map(makeR2F,ff.factor)$PP2,[var]) for ff in factors fp]
  else empty()

-- Convert a list of Equations in a list of Polynomials
makePolys(l: L EQ F):L F == [lhs e - rhs e for e in l]

-- linear systems solver. Input as list of polynomials  --
linSolve(lp:L F,lv:L SE) ==
    rec:Record(particular:Union(V F,"failed"),basis:L V F)
    lr : L P R:=[numer f for f in lp]
    rec:=linSolve(lr,lv)$LinearSystemPolynomialPackage(R,IE,SE,P R)
    rec.particular case "failed" => "failed"
    rhs := rec.particular :: V F
    zeron:V F:=zero(#lv)
    for p in rec.basis | p ^= zeron repeat
        sym := newInF(1)
        for i in 1..#lv repeat
            rhs.i := rhs.i + sym*p.i
    eqs: L EQ F := []
    for i in 1..#lv repeat
      eqs := append(eqs,[(lv.i)::(P R)::F = rhs.i])
    eqs

-- general solver. Input in polynomial style. Implicit variables --
solve(lr : L F) ==
  lv :="setUnion"/[setUnion(variables numer p, variables denom p)
        for p in lr]
  solve(lr,lv)

-- general solver. Input in equation style. Implicit variables --
solve(le : L EQ F) ==
  lr:=makePolys le
  lv :="setUnion"/[setUnion(variables numer p, variables denom p)
        for p in lr]
  solve(lr,lv)

-- general solver. Input in equation style  --
```

```
        solve(le:L EQ F,lv:L SE)  == solve(makePolys le, lv)

        checkLinear(lr:L F,vl:L SE):Boolean ==
          ld:=[denom pol for pol in lr]
          for f in ld repeat
            if (or/[member?(x,vl) for x in variables f]) then return false
          and/[totalDegree(numer pol,vl) < 2 for pol in lr]

        -- general solver. Input in polynomial style  --
        solve(lr:L F,vl:L SE) ==
            empty? vl => empty()
            checkLinear(lr,vl) =>
                            -- linear system --
                soln := linSolve(lr, vl)
                soln case "failed" => []
                eqns: L EQ F := []
                for i in 1..#vl repeat
                    lhs := (vl.i::(P R))::F
                    rhs :=  rhs soln.i
                    eqns := append(eqns, [lhs = rhs])
                [eqns]

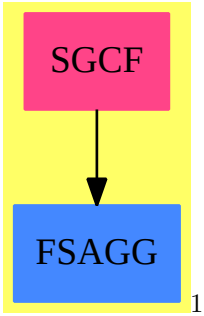
                        -- polynomial system --
            if R has GcdDomain then
              parRes:=triangularSystems(lr,vl)
              [[makeEq(map(makeR2F,f)$PP2,vl) for f in pr]
                                                    for pr in parRes]
            else [[]]
```

⟨*SYSSOLP.dotabb*⟩≡

```
"SYSSOLP" [color="#FF4488",href="bookvol10.4.pdf#nameddest=SYSSOLP"]
"ALIST" [color="#88FF44",href="bookvol10.3.pdf#nameddest=ALIST"]
"SYSSOLP" -> "ALIST"
```

20.57 package SGCF SymmetricGroupCombinatoricFunctions

20.58 SymmetricGroupCombinatoricFunctions



1.00

Exports:

coleman	inverseColeman
listYoungTableaus	makeYoungTableau
nextColeman	nextLatticePermutation
nextPartition	numberOfImproperPartitions
subSet	unrankImproperPartitions0
unrankImproperPartitions1	

⟨*package SGCF SymmetricGroupCombinatoricFunctions*⟩≡

```
)abbrev package SGCF SymmetricGroupCombinatoricFunctions
++ Authors: Johannes Grabmeier, Thorsten Werther
++ Date Created: 03 September 1988
++ Date Last Updated: 07 June 1990
++ Basic Operations: nextPartition, numberOfImproperPartitions,
++   listYoungTableaus, subSet, unrankImproperPartitions0
++ Related Constructors: IntegerCombinatoricFunctions
++ Also See: RepresentationTheoryPackage1, RepresentationTheoryPackage2,
++   IrrRepSymNatPackage
++ AMS Classifications:
++ Keywords: improper partition, partition, subset, Coleman
++ References:
++   G. James/ A. Kerber: The Representation Theory of the Symmetric
++    Group. Encycl. of Math. and its Appl., Vol. 16., Cambridge
++    Univ. Press 1981, ISBN 0-521-30236-6.
++   S.G. Williamson: Combinatorics for Computer Science,
++    Computer Science Press, Rockville, Maryland, USA, ISBN 0-88175-020-4.
++   A. Nijenhuis / H.S. Wilf: Combinatoral Algorithms, Academic Press 1978.
++    ISBN 0-12-519260-6.
++   H. Gollan, J. Grabmeier: Algorithms in Representation Theory and
++    their Realization in the Computer Algebra System Scratchpad,
```

```
++    Bayreuther Mathematische Schriften, Heft 33, 1990, 1-23.
++ Description:
++   SymmetricGroupCombinatoricFunctions contains combinatoric
++   functions concerning symmetric groups and representation
++   theory: list young tableaus, improper partitions, subsets
++   bijection of Coleman.

SymmetricGroupCombinatoricFunctions(): public == private where

  NNI  ==> NonNegativeInteger
  I    ==> Integer
  L    ==> List
  M    ==> Matrix
  V    ==> Vector
  B    ==> Boolean
  ICF  ==> IntegerCombinatoricFunctions Integer

  public ==> with

--    IS THERE A WORKING DOMAIN Tableau ??
--    coerce : M I -> Tableau(I)
--      ++ coerce(ytab) coerces the Young-Tableau ytab to an element of
--      ++ the domain Tableau(I).

    coleman : (L I, L I, L I) -> M I
      ++ coleman(alpha,beta,pi):
      ++ there is a bijection from the set of matrices having nonnegative
      ++ entries and row sums {\em alpha}, column sums {\em beta}
      ++ to the set of {\em Salpha - Sbeta} double cosets of the
      ++ symmetric group {\em Sn}. ({\em Salpha} is the Young subgroup
      ++ corresponding to the improper partition {\em alpha}).
      ++ For a representing element {\em pi} of such a double coset,
      ++ coleman(alpha,beta,pi) generates the Coleman-matrix
      ++ corresponding to {\em alpha, beta, pi}.
      ++ Note: The permutation {\em pi} of {\em {1,2,...,n}} has to be given
      ++ in list form.
      ++ Note: the inverse of this map is {\em inverseColeman}
      ++ (if {\em pi} is the lexicographical smallest permutation
      ++ in the coset). For details see James/Kerber.
    inverseColeman : (L I, L I, M I) -> L I
      ++ inverseColeman(alpha,beta,C):
      ++ there is a bijection from the set of matrices having nonnegative
      ++ entries and row sums {\em alpha}, column sums {\em beta}
      ++ to the set of {\em Salpha - Sbeta} double cosets of the
      ++ symmetric group {\em Sn}. ({\em Salpha} is the Young subgroup
      ++ corresponding to the improper partition {\em alpha}).
```

```
  ++ For such a matrix C, inverseColeman(alpha,beta,C)
  ++ calculates the lexicographical smallest {\em pi} in the
  ++ corresponding double coset.
  ++ Note: the resulting permutation {\em pi} of {\em {1,2,...,n}}
  ++ is given in list form.
  ++ Notes: the inverse of this map is {\em coleman}.
  ++ For details, see James/Kerber.
listYoungTableaus : (L I) -> L M I
  ++ listYoungTableaus(lambda) where {\em lambda} is a proper partition
  ++ generates the list of all standard tableaus of shape {\em lambda}
  ++ by means of lattice permutations. The numbers of the lattice
  ++ permutation are interpreted as column labels. Hence the
  ++ contents of these lattice permutations are the conjugate of
  ++ {\em lambda}.
  ++ Notes: the functions {\em nextLatticePermutation} and
  ++ {\em makeYoungTableau} are used.
  ++ The entries are from {\em 0,...,n-1}.
makeYoungTableau : (L I,L I) -> M I
  ++ makeYoungTableau(lambda,gitter) computes for a given lattice
  ++ permutation {\em gitter} and for an improper partition {\em lambda}
  ++ the corresponding standard tableau of shape {\em lambda}.
  ++ Notes: see {\em listYoungTableaus}.
  ++ The entries are from {\em 0,...,n-1}.
nextColeman : (L I, L I, M I) -> M I
  ++ nextColeman(alpha,beta,C) generates the next Coleman matrix
  ++ of column sums {\em alpha} and row sums {\em beta} according
  ++ to the lexicographical order from bottom-to-top.
  ++ The first Coleman matrix is achieved by {\em C=new(1,1,0)}.
  ++ Also, {\em new(1,1,0)} indicates that C is the last Coleman matrix.
nextLatticePermutation : (L I, L I, B) -> L I
  ++ nextLatticePermutation(lambda,lattP,constructNotFirst) generates
  ++ the lattice permutation according to the proper partition
  ++ {\em lambda} succeeding the lattice permutation {\em lattP} in
  ++ lexicographical order as long as {\em constructNotFirst} is true.
  ++ If {\em constructNotFirst} is false, the first lattice permutation
  ++ is returned.
  ++ The result {\em nil} indicates that {\em lattP} has no successor.
nextPartition : (V I, V I, I) -> V I
  ++ nextPartition(gamma,part,number) generates the partition of
  ++ {\em number} which follows {\em part} according to the right-to-left
  ++ lexicographical order. The partition has the property that
  ++ its components do not exceed the corresponding components of
  ++ {\em gamma}. The first partition is achieved by {\em part=[]}.
  ++ Also, {\em []} indicates that {\em part} is the last partition.
nextPartition : (L I, V I, I) -> V I
  ++ nextPartition(gamma,part,number) generates the partition of
```

```
      ++ {\em number} which follows {\em part} according to the right-to-left
      ++ lexicographical order. The partition has the property that
      ++ its components do not exceed the corresponding components of
      ++ {\em gamma}. the first partition is achieved by {\em part=[]}.
      ++ Also, {\em []} indicates that {\em part} is the last partition.
    numberOfImproperPartitions: (I,I) -> I
      ++ numberOfImproperPartitions(n,m) computes the number of partitions
      ++ of the nonnegative integer n in m nonnegative parts with regarding
      ++ the order (improper partitions).
      ++ Example: {\em numberOfImproperPartitions (3,3)} is 10,
      ++ since {\em [0,0,3], [0,1,2], [0,2,1], [0,3,0], [1,0,2], [1,1,1],
      ++ [1,2,0], [2,0,1], [2,1,0], [3,0,0]} are the possibilities.
      ++ Note: this operation has a recursive implementation.
    subSet : (I,I,I) -> L I
      ++ subSet(n,m,k) calculates the {\em k}-th {\em m}-subset of the set
      ++ {\em 0,1,...,(n-1)} in the lexicographic order considered as
      ++ a decreasing map from {\em 0,...,(m-1)} into {\em 0,...,(n-1)}.
      ++ See S.G. Williamson: Theorem 1.60.
      ++ Error: if not {\em (0 <= m <= n and 0 < = k < (n choose m))}.
    unrankImproperPartitions0 : (I,I,I) -> L I
      ++ unrankImproperPartitions0(n,m,k) computes the {\em k}-th improper
      ++ partition of nonnegative n in m nonnegative parts in reverse
      ++ lexicographical order.
      ++ Example: {\em [0,0,3] < [0,1,2] < [0,2,1] < [0,3,0] <
      ++ [1,0,2] < [1,1,1] < [1,2,0] < [2,0,1] < [2,1,0] < [3,0,0]}.
      ++ Error: if k is negative or too big.
      ++ Note: counting of subtrees is done by
      ++ \spadfunFrom{numberOfImproperPartitions}{SymmetricGroupCombinatoricFunctions}.

    unrankImproperPartitions1: (I,I,I) -> L I
      ++ unrankImproperPartitions1(n,m,k) computes the {\em k}-th improper
      ++ partition of nonnegative n in at most m nonnegative parts
      ++ ordered as follows: first, in reverse
      ++ lexicographically according to their non-zero parts, then
      ++ according to their positions (i.e. lexicographical order
      ++ using {\em subSet}: {\em [3,0,0] < [0,3,0] < [0,0,3] < [2,1,0] <
      ++ [2,0,1] < [0,2,1] < [1,2,0] < [1,0,2] < [0,1,2] < [1,1,1]}).
      ++ Note: counting of subtrees is done by
      ++ {\em numberOfImproperPartitionsInternal}.

  private == add

    import Set I

    -- declaration of local functions
```

```
numberOfImproperPartitionsInternal: (I,I,I) -> I
  -- this is used as subtree counting function in
  -- "unrankImproperPartitions1". For (n,m,cm) it counts
  -- the following set of m-tuples: The  first (from left
  -- to right) m-cm non-zero entries are equal, the remaining
  -- positions sum up to n. Example: (3,3,2) counts
  -- [x,3,0], [x,0,3], [0,x,3], [x,2,1], [x,1,2], x non-zero.


-- definition of local functions


numberOfImproperPartitionsInternal(n,m,cm) ==
  n = 0 => binomial(m,cm)$ICF
  cm = 0 and n > 0 => 0
  s := 0
  for i in 0..n-1 repeat
    s := s + numberOfImproperPartitionsInternal(i,m,cm-1)
  s


-- definition of exported functions

numberOfImproperPartitions(n,m) ==
  if n < 0 or m < 1 then return 0
  if m = 1 or n = 0 then return 1
  s := 0
  for i in 0..n repeat
    s := s + numberOfImproperPartitions(n-i,m-1)
  s


unrankImproperPartitions0(n,m,k) ==
  l : L I  := nil$(L I)
  k < 0 => error"counting of partitions is started at 0"
  k >= numberOfImproperPartitions(n,m) =>
    error"there are not so many partitions"
  for t in 0..(m-2) repeat
    s : I := 0
    for y in 0..n repeat
      sOld := s
      s := s + numberOfImproperPartitions(n-y,m-t-1)
      if s > k then leave
    l := append(l,list(y)$(L I))$(L I)
    k := k - sOld
```

```
      n := n - y
    l := append(l,list(n)$(L I))$(L I)
    l


  unrankImproperPartitions1(n,m,k) ==
    -- we use the counting procedure of the leaves in a tree
    -- having the following structure: First of all non-zero
    -- labels for the sons. If addition along a path gives n,
    -- then we go on creating the subtree for (n choose cm)
    -- where cm is the length of the path. These subsets determine
    -- the positions for the non-zero labels for the partition
    -- to be formeded. The remaining positions are filled by zeros.
    nonZeros   : L I := nil$(L I)
    partition  : V I :=  new(m::NNI,0$I)$(V I)
    k < 0 => nonZeros
    k >= numberOfImproperPartitions(n,m) => nonZeros
    cm : I := m    --cm gives the depth of the tree
    while n ^= 0 repeat
      s : I := 0
      cm := cm - 1
      for y in n..1 by -1 repeat   --determination of the next son
        sOld := s  -- remember old s
        -- this functions counts the number of elements in a subtree
        s := s + numberOfImproperPartitionsInternal(n-y,m,cm)
        if s > k then leave
      -- y is the next son, so put it into the pathlist "nonZero"
      nonZeros := append(nonZeros,list(y)$(L I))$(L I)
      k := k - sOld    --updating
      n := n - y       --updating
    --having found all m-cm non-zero entries we change the structure
    --of the tree and determine the non-zero positions
    nonZeroPos : L I := reverse subSet(m,m-cm,k)
    --building the partition
    for i in 1..m-cm  repeat partition.(1+nonZeroPos.i) := nonZeros.i
    entries partition


  subSet(n,m,k) ==
    k < 0 or n < 0 or m < 0 or m > n =>
      error "improper argument to subSet"
    bin : I := binomial$ICF (n,m)
    k >= bin =>
      error "there are not so many subsets"
    l : L I  := []
    n = 0 => l
```

```
    mm : I := k
    s  : I := m
    for t in 0..(m-1) repeat
       for y in (s-1)..(n+1) repeat
          if binomial$ICF (y,s) > mm then leave
       l := append (l,list(y-1)$(L I))
       mm := mm - binomial$ICF (y-1,s)
       s := s-1
    l


  nextLatticePermutation(lambda, lattP, constructNotFirst) ==

    lprime  : L I  := conjugate(lambda)$PartitionsAndPermutations
    columns : NNI := (first(lambda)$(L I))::NNI
    rows    : NNI := (first(lprime)$(L I))::NNI
    n       : NNI :=(+/lambda)::NNI

    not constructNotFirst =>   -- first lattice permutation
      lattP := nil$(L I)
      for i in columns..1 by -1 repeat
        for l in 1..lprime(i) repeat
          lattP := cons(i,lattP)
      lattP

    help : V I := new(columns,0) -- entry help(i) stores the number
    -- of occurences of number i on our way from right to left
    rightPosition  : NNI := n
    leftEntry : NNI := lattP(rightPosition)::NNI
    ready  : B  := false
    until (ready or (not constructNotFirst)) repeat
      rightEntry : NNI := leftEntry
      leftEntry := lattP(rightPosition-1)::NNI
      help(rightEntry) := help(rightEntry) + 1
      -- search backward decreasing neighbour elements
      if rightEntry > leftEntry then
        if ((lprime(leftEntry)-help(leftEntry)) >_
          (lprime(rightEntry)-help(rightEntry)+1)) then
          -- the elements may be swapped because the number of occurances
          -- of leftEntry would still be greater than those of rightEntry
          ready := true
          j : NNI := leftEntry + 1
          -- search among the numbers leftEntry+1..rightEntry for the
          -- smallest one which can take the place of leftEntry.
          -- negation of condition above:
          while (help(j)=0) or ((lprime(leftEntry)-lprime(j))
```

```
            < (help(leftEntry)-help(j)+2)) repeat j := j + 1
          lattP(rightPosition-1) := j
          help(j) := help(j)-1
          help(leftEntry) := help(leftEntry) + 1
          -- reconstruct the rest of the list in increasing order
          for l in rightPosition..n repeat
            j := 0
            while help(1+j) = 0 repeat j := j + 1
            lattP(l::NNI) := j+1
            help(1+j) := help(1+j) - 1
        -- end of "if rightEntry > leftEntry"
        rightPosition := (rightPosition-1)::NNI
        if rightPosition = 1 then constructNotFirst := false
      -- end of repeat-loop
      not constructNotFirst =>  nil$(L I)
      lattP


    makeYoungTableau(lambda,gitter) ==
      lprime  : L I  := conjugate(lambda)$PartitionsAndPermutations
      columns : NNI := (first(lambda)$(L I))::NNI
      rows    : NNI := (first(lprime)$(L I))::NNI
      ytab    : M I  := new(rows,columns,0)
      help    : V I  := new(columns,1)
      i : I := -1     -- this makes the entries ranging from 0,..,n-1
                      -- i := 0 would make it from 1,..,n.
      j : I := 0
      for l in 1..maxIndex gitter repeat
        j := gitter(l)
        i := i + 1
        ytab(help(j),j) := i
        help(j) := help(j) + 1
      ytab


--    coerce(ytab) ==
--      lli := listOfLists(ytab)$(M I)
--      -- remove the filling zeros in each row. It is assumed that
--      -- that there are no such in row 0.
--      for i in 2..maxIndex lli repeat
--        THIS IS DEFINIVELY WRONG, I NEED A FUNCTION WHICH DELETES THE
--        0s, in my version there are no mapping facilities yet.
--        deleteInPlace(not zero?,lli i)
--      tableau(lli)$Tableau(I)
```

```
listYoungTableaus(lambda) ==
  lattice   : L I
  ytab      : M I
  younglist : L M I := nil$(L M I)
  lattice   := nextLatticePermutation(lambda,lattice,false)
  until null lattice repeat
    ytab      := makeYoungTableau(lambda,lattice)
    younglist := append(younglist,[ytab]$(L M I))$(L M I)
    lattice   := nextLatticePermutation(lambda,lattice,true)
  younglist


nextColeman(alpha,beta,C) ==
  nrow  : NNI := #beta
  ncol  : NNI := #alpha
  vnull : V I  := vector(nil()$(L I)) -- empty vector
  vzero : V I  := new(ncol,0)
  vrest : V I  := new(ncol,0)
  cnull : M I  := new(1,1,0)
  coleman := copy C
  if coleman ^= cnull then
    -- look for the first row of "coleman" that has a succeeding
    -- partition, this can be atmost row nrow-1
    i : NNI := (nrow-1)::NNI
    vrest := row(coleman,i) + row(coleman,nrow)
    --for k in 1..ncol repeat
    --  vrest(k) := coleman(i,k) + coleman(nrow,k)
    succ := nextPartition(vrest,row(coleman, i),beta(i))
    while (succ = vnull) repeat
      if i = 1 then return cnull -- part is last partition
      i := (i - 1)::NNI
      --for k in 1..ncol repeat
      --  vrest(k) := vrest(k) + coleman(i,k)
      vrest := vrest + row(coleman,i)
      succ := nextPartition(vrest, row(coleman, i), beta(i))
    j : I := i
    coleman := setRow_!(coleman, i, succ)
    --for k in 1..ncol repeat
    --  vrest(k) := vrest(k) - coleman(i,k)
    vrest := vrest - row(coleman,i)
  else
    vrest := vector alpha
    -- for k in 1..ncol repeat
    --  vrest(k) := alpha(k)
    coleman := new(nrow,ncol,0)
    j : I := 0
```

```
  for i in (j+1)::NNI..nrow-1 repeat
    succ := nextPartition(vrest,vnull,beta(i))
    coleman := setRow_!(coleman, i, succ)
    vrest := vrest - succ
    --for k in 1..ncol repeat
    --  vrest(k) := vrest(k) - succ(k)
  setRow_!(coleman, nrow, vrest)


nextPartition(gamma:V I, part:V I, number:I) ==
  nextPartition(entries gamma, part, number)


nextPartition(gamma:L I,part:V I,number:I) ==
  n : NNI := #gamma
  vnull : V I := vector(nil()$(L I)) -- empty vector
  if part ^= vnull then
    i : NNI := 2
    sum := part(1)
    while (part(i) = gamma(i)) or (sum = 0) repeat
      sum := sum + part(i)
      i := i + 1
      if i = 1+n then return vnull -- part is last partition
    sum := sum - 1
    part(i) := part(i) + 1
  else
    sum := number
    part := new(n,0)
    i := 1+n
  j : NNI := 1
  while sum > gamma(j) repeat
    part(j) := gamma(j)
    sum := sum - gamma(j)
    j := j + 1
  part(j) := sum
  for k in j+1..i-1 repeat
    part(k) := 0
  part


inverseColeman(alpha,beta,C) ==
  pi   : L I  := nil$(L I)
  nrow : NNI := #beta
  ncol : NNI := #alpha
  help : V I  := new(nrow,0)
  sum  : I   := 1
```

```
    for i in 1..nrow repeat
      help(i) := sum
      sum := sum + beta(i)
    for j in 1..ncol repeat
      for i in 1..nrow repeat
        for k in 2..1+C(i,j) repeat
          pi := append(pi,list(help(i))$(L I))
          help(i) := help(i) + 1
    pi

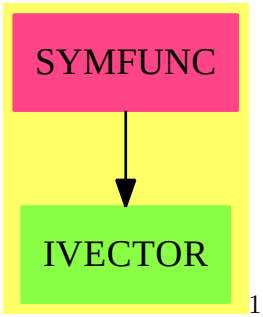

  coleman(alpha,beta,pi) ==
    nrow : NNI := #beta
    ncol : NNI := #alpha
    temp : L L I := nil$(L L I)
    help : L I  := nil$(L I)
    colematrix : M I := new(nrow,ncol,0)
    betasum  : NNI := 0
    alphasum : NNI := 0
    for i in 1..ncol repeat
      help := nil$(L I)
      for j in alpha(i)..1 by-1 repeat
        help := cons(pi(j::NNI+alphasum),help)
      alphasum := (alphasum + alpha(i))::NNI
      temp := append(temp,list(help)$(L L I))
    for i in 1..nrow repeat
      help := nil$(L I)
      for j in beta(i)..1 by-1 repeat
        help := cons(j::NNI+betasum, help)
      betasum := (betasum + beta(i))::NNI
      for j in 1..ncol repeat
        colematrix(i,j) := #intersect(brace(help),brace(temp(j)))
    colematrix
```

⟨*SGCF.dotabb*⟩≡

```
"SGCF" [color="#FF4488",href="bookvol10.4.pdf#nameddest=SGCF"]
"FSAGG" [color="#4488FF",href="bookvol10.2.pdf#nameddest=FSAGG"]
"SGCF" -> "FSAGG"
```

20.59 package SYMFUNC SymmetricFunctions

20.60 SymmetricFunctions



1.00

Exports:
symFunc

⟨*package SYMFUNC SymmetricFunctions*⟩≡
```
)abbrev package SYMFUNC SymmetricFunctions
++ The elementary symmetric functions
++ Author: Manuel Bronstein
++ Date Created: 13 Feb 1989
++ Date Last Updated: 28 Jun 1990
++ Description: Computes all the symmetric functions in n variables.
SymmetricFunctions(R:Ring): Exports == Implementation where
  UP  ==> SparseUnivariatePolynomial R

  Exports ==> with
    symFunc: List R  -> Vector R
      ++ symFunc([r1,...,rn]) returns the vector of the
      ++ elementary symmetric functions in the \spad{ri's}:
      ++ \spad{[r1 + ... + rn, r1 r2 + ... + r(n-1) rn, ..., r1 r2 ... rn]}.
    symFunc: (R, PositiveInteger) -> Vector R
      ++ symFunc(r, n) returns the vector of the elementary
      ++ symmetric functions in \spad{[r,r,...,r]} \spad{n} times.

  Implementation ==> add
    signFix: (UP, NonNegativeInteger) -> Vector R

    symFunc(x, n) == signFix((monomial(1, 1)$UP - x::UP) ** n, 1 + n)

    symFunc l ==
      signFix(*/[monomial(1, 1)$UP - a::UP for a in l], 1 + #l)

    signFix(p, n) ==
```

```
        m := minIndex(v := vectorise(p, n)) + 1
        for i in 0..((#v quo 2) - 1)::NonNegativeInteger repeat
          qsetelt_!(v, 2*i + m, - qelt(v, 2*i + m))
        reverse_! v
```

⟨*SYMFUNC.dotabb*⟩≡

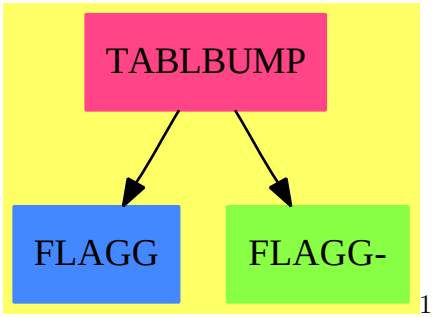
```
  "SYMFUNC" [color="#FF4488",href="bookvol10.4.pdf#nameddest=SYMFUNC"]
  "IVECTOR" [color="#88FF44",href="bookvol10.3.pdf#nameddest=IVECTOR"]
  "SYMFUNC" -> "IVECTOR"
```

Chapter 21

Chapter T

21.1 package TABLBUMP TableauxBumpers

21.2 TableauxBumpers



1.00

Exports:

bat	bat1	bumprow	bumptab	bumptab1
inverse	lex	maxrow	mr	slex
tab	tab1	untab		

⟨*package TABLBUMP TableauxBumpers*⟩≡

```
)abbrev package TABLBUMP TableauxBumpers
++ Author: William H. Burge
++ Date Created: 1987
++ Date Last Updated: 23 Sept 1991
++ Basic Functions:
++ Related Constructors:
++ Also See:
++ AMS Classifications:
++ Keywords: Young tableau
```

```
++ References:
++ Description:
++ TableauBumpers implements the Schenstead-Knuth
++ correspondence between sequences and pairs of Young tableaux.
++ The 2 Young tableaux are represented as a single tableau with
++ pairs as components.
TableauxBumpers(S:OrderedSet):T==C where
     L==>List
     ST==>Stream
     B==>Boolean
     ROW==>Record(fs:B,sd:L S,td:L L S)
     RC==>Record(f1:L S,f2:L L L S,f3:L L S,f4:L L L S)
     PAIR==>L S
     T== with
       bumprow:((S,S)->B,PAIR,L PAIR)->ROW
         ++ bumprow(cf,pr,r) is an auxiliary function which
         ++ bumps a row r with a pair pr
         ++ using comparison function cf, and returns a record
       bumptab:((S,S)->B,PAIR,L L PAIR)->L L PAIR
         ++ bumptab(cf,pr,t) bumps a tableau t with a pair pr
         ++ using comparison function cf, returning a new tableau
       bumptab1:(PAIR,L L PAIR)->L L PAIR
         ++ bumptab1(pr,t) bumps a tableau t with a pair pr
         ++ using comparison function \spadfun{<},
         ++ returning a new tableau
       untab: (L PAIR,L L PAIR)->L PAIR
         ++ untab(lp,llp) is an auxiliary function
         ++ which unbumps a tableau llp,
         ++ using lp to accumulate pairs
       bat1:L L PAIR->L PAIR
         ++ bat1(llp) unbumps a tableau llp.
         ++ Operation bat1 is the inverse of tab1.
       bat:Tableau(L S)->L L S
         ++ bat(ls) unbumps a tableau ls
       tab1:L PAIR->L L PAIR
         ++ tab1(lp) creates a tableau from a list of pairs lp
       tab:L S->Tableau(L S)
         ++ tab(ls) creates a tableau from ls by first creating
         ++ a list of pairs using \spadfunFrom{slex}{TableauBumpers},
         ++ then creating a tableau using \spadfunFrom{tab1}{TableauBumpers}.
       lex:L PAIR->L PAIR
         ++ lex(ls) sorts a list of pairs to lexicographic order
       slex:L S->L PAIR
         ++ slex(ls) sorts the argument sequence ls, then
         ++ zips (see \spadfunFrom{map}{ListFunctions3}) the
         ++ original argument sequence with the sorted result to
```

```
      ++ a list of pairs
    inverse:L S->L S
      ++ inverse(ls) forms the inverse of a sequence ls
    maxrow:(PAIR,L L PAIR,L PAIR,L L PAIR,L L PAIR,L L PAIR)->RC
      ++ maxrow(a,b,c,d,e) is an auxiliary function for mr
    mr:L L PAIR->RC
      ++ mr(t) is an auxiliary function which
      ++ finds the position of the maximum element of a tableau t
      ++ which is in the lowest row, producing a record of results
  C== add
    cf:(S,S)->B
    bumprow(cf,x:(PAIR),lls:(L PAIR))==
      if null lls
      then [false,x,[x]]$ROW
      else (y:(PAIR):=first lls;
            if cf(x.2,y.2)
            then [true,[x.1,y.2],cons([y.1,x.2],rest lls)]$ROW
            else (rw:ROW:=bumprow(cf,x,rest lls);
                    [rw.fs,rw.sd,cons(first lls,rw.td)]$ROW ))

    bumptab(cf,x:(PAIR),llls:(L L PAIR))==
        if null llls
        then [[x]]
        else (rw:ROW:= bumprow(cf,x,first llls);
              if rw.fs
              then cons(rw.td, bumptab(cf,rw.sd,rest llls))
              else cons(rw.td,rest llls))

    bumptab1(x,llls)==bumptab(#1<#2,x,llls)

    rd==> reduce$StreamFunctions2(PAIR,L L PAIR)
    tab1(lls:(L PAIR))== rd([],bumptab1,lls::(ST PAIR))

    srt==>sort$(PAIR)
    lexorder:(PAIR,PAIR)->B
    lexorder(p1,p2)==if p1.1=p2.1 then p1.2<p2.2 else p1.1<p2.1
    lex lp==(sort$(L PAIR))(lexorder(#1,#2),lp)
    slex ls==lex([[i,j] for i in srt(#1<#2,ls) for j in ls])
    inverse ls==[lss.2 for lss in
                lex([[j,i] for i in srt(#1<#2,ls) for j in ls])]

    tab(ls:(PAIR))==(tableau tab1 slex ls )

    maxrow(n,a,b,c,d,llls)==
     if null llls or null(first llls)
     then [n,a,b,c]$RC
```

```
 else (fst:=first first llls;rst:=rest first llls;
       if fst.1>n.1
       then maxrow(fst,d,rst,rest llls,cons(first llls,d),rest llls)
       else maxrow(n,a,b,c,cons(first llls,d),rest llls))

mr llls==maxrow(first first llls,[],rest first llls,rest llls,
                                                          [],llls)

untab(lp, llls)==
  if null llls
  then lp
  else (rc:RC:=mr llls;
        rv:=reverse (bumptab(#2<#1,rc.f1,rc.f2));
        untab(cons(first first rv,lp)
              ,append(rest rv,
                                if null rc.f3
                                then []
                                else cons(rc.f3,rc.f4))))

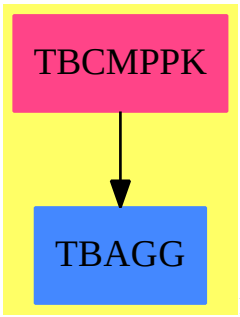
bat1 llls==untab([],[reverse lls for lls in llls])
bat tb==bat1(listOfLists tb)
```

⟨*TABLBUMP.dotabb*⟩≡

```
"TABLBUMP" [color="#FF4488",href="bookvol10.4.pdf#nameddest=TABLBUMP"]
"FLAGG" [color="#4488FF",href="bookvol10.2.pdf#nameddest=FLAGG"]
"FLAGG-" [color="#88FF44",href="bookvol10.3.pdf#nameddest=FLAGG"]
"TABLBUMP" -> "FLAGG-"
"TABLBUMP" -> "FLAGG"
```

21.3 package TBCMPPK TabulatedComputationPackage

21.4 TabulatedComputationPackage



1.00

Exports:

clearTable!	extractIfCan	initTable!	insert!	makingStats?
printInfo!	printStats!	printingInfo?	startStats!	usingTable?

⟨*package TBCMPPK TabulatedComputationPackage*⟩≡

```
)abbrev package TBCMPPK TabulatedComputationPackage
++ Author: Marc Moreno Maza
++ Date Created: 09/09/1998
++ Date Last Updated: 12/16/1998
++ Basic Functions:
++ Related Constructors:
++ Also See:
++ AMS Classifications:
++ Keywords:
++ References:
++ Description:
++ \axiom{TabulatedComputationPackage(Key ,Entry)} provides some modest support
++   for dealing with operations with type \axiom{Key -> Entry}. The result of
++   such operations can be stored and retrieved with this package by using
++   a hash-table. The user does not need to worry about the management of
++   this hash-table. However, onnly one hash-table is built by calling
++   \axiom{TabulatedComputationPackage(Key ,Entry)}.
++ Version: 2.

TabulatedComputationPackage(Key ,Entry): Exports == Implementation where
  Key: SetCategory
  Entry: SetCategory
  N ==> NonNegativeInteger
  H ==> HashTable(Key, Entry, "UEQUAL")
  iprintpack ==> InternalPrintPackage()
```

```
Exports ==  with
   initTable!: () -> Void
     ++ \axiom{initTable!()} initializes the hash-table.
   printInfo!: (String, String) -> Void
     ++ \axiom{printInfo!(x,y)} initializes the mesages to be printed
     ++ when manipulating items from the hash-table. If
     ++ a key is retrieved then \axiom{x} is displayed. If an item is
     ++ stored then \axiom{y} is displayed.
   startStats!: (String) -> Void
     ++ \axiom{startStats!(x)} initializes the statisitics process and
     ++ sets the comments to display when statistics are printed
   printStats!: () -> Void
     ++ \axiom{printStats!()} prints the statistics.
   clearTable!: () -> Void
     ++ \axiom{clearTable!()} clears the hash-table and assumes that
     ++ it will no longer be used.
   usingTable?: () -> Boolean
     ++ \axiom{usingTable?()} returns true iff the hash-table is used
   printingInfo?: () -> Boolean
     ++ \axiom{printingInfo?()} returns true iff messages are printed
     ++ when manipulating items from the hash-table.
   makingStats?: () -> Boolean
     ++ \axiom{makingStats?()} returns true iff the statisitics process
     ++ is running.
   extractIfCan: Key -> Union(Entry,"failed")
     ++ \axiom{extractIfCan(x)} searches the item whose key is \axiom{x}.
   insert!: (Key, Entry) -> Void
     ++ \axiom{insert!(x,y)} stores the item whose key is \axiom{x} and whose
     ++ entry is \axiom{y}.

Implementation == add
   table?: Boolean := false
   t: H := empty()
   info?: Boolean := false
   stats?: Boolean := false
   used: NonNegativeInteger := 0
   ok: String := "o"
   ko: String := "+"
   domainName: String := empty()$String

   initTable!(): Void ==
     table? := true
     t := empty()
     void()
   printInfo!(s1: String, s2: String): Void ==
```

```
    (empty? s1) or (empty? s2) => void()
    not usingTable? =>
      error "in printInfo!()$TBCMPPK: not allowed to use hashtable"
    info? := true
    ok := s1
    ko := s2
    void()
  startStats!(s: String): Void ==
    empty? s => void()
    not table? =>
      error "in startStats!()$TBCMPPK: not allowed to use hashtable"
    stats? := true
    used := 0
    domainName := s
    void()
  printStats!(): Void ==
    not table? =>
      error "in printStats!()$TBCMPPK: not allowed to use hashtable"
    not stats? =>
      error "in printStats!()$TBCMPPK: statistics not started"
    output(" ")$OutputPackage
    title: String := concat("*** ", concat(domainName," Statistics ***"))
    output(title)$OutputPackage
    n: N := #t
    output("   Table     size: ", n::OutputForm)$OutputPackage
    output("   Entries reused: ", used::OutputForm)$OutputPackage
  clearTable!(): Void ==
    not table? =>
      error "in clearTable!()$TBCMPPK: not allowed to use hashtable"
    t := empty()
    table? := false
    info? := false
    stats? := false
    domainName := empty()$String
    void()
  usingTable?() == table?
  printingInfo?() == info?
  makingStats?() == stats?
  extractIfCan(k: Key): Union(Entry,"failed") ==
    not table? => "failed" :: Union(Entry,"failed")
    s: Union(Entry,"failed") := search(k,t)
    s case Entry =>
      if info? then iprint(ok)$iprintpack
      if stats? then used := used + 1
      return s
    "failed" :: Union(Entry,"failed")
```

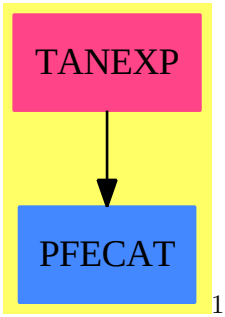
```
      insert!(k: Key, e:Entry): Void ==
        not table? => void()
        t.k := e
        if info? then iprint(ko)$iprintpack
        void()
```

⟨*TBCMPPK.dotabb*⟩≡

```
"TBCMPPK" [color="#FF4488",href="bookvol10.4.pdf#nameddest=TBCMPPK"]
"TBAGG" [color="#4488FF",href="bookvol10.2.pdf#nameddest=TBAGG"]
"TBCMPPK" -> "TBAGG"
```

21.5 package TANEXP TangentExpansions

21.6 TangentExpansions



1.00

Exports:
tanAn tanNa tanSum

⟨*package TANEXP TangentExpansions*⟩≡

```
)abbrev package TANEXP TangentExpansions
++ Expansions of tangents of sums and quotients
++ Author: Manuel Bronstein
++ Date Created: 13 Feb 1989
++ Date Last Updated: 20 Apr 1990
++ Description: Expands tangents of sums and scalar products.
TangentExpansions(R:Field): Exports == Implementation where
  PI ==> PositiveInteger
  Z  ==> Integer
  UP ==> SparseUnivariatePolynomial R
  QF ==> Fraction UP

  Exports ==> with
    tanSum: List R -> R
      ++ tanSum([a1,...,an]) returns \spad{f(a1,...,an)} such that
      ++ if \spad{ai = tan(ui)} then \spad{f(a1,...,an) = tan(u1 + ... + un)}.
    tanAn : (R, PI) -> UP
      ++ tanAn(a, n) returns \spad{P(x)} such that
      ++ if \spad{a = tan(u)} then \spad{P(tan(u/n)) = 0}.
    tanNa : (R,  Z) -> R
      ++ tanNa(a, n) returns \spad{f(a)} such that
      ++ if \spad{a = tan(u)} then \spad{f(a) = tan(n * u)}.

  Implementation ==> add
    import SymmetricFunctions(R)
    import SymmetricFunctions(UP)
```

```
      m1toN : Integer -> Integer
      tanPIa: PI -> QF

      m1toN n     == (odd? n => -1; 1)
      tanAn(a, n) == a * denom(q := tanPIa n) - numer q

      tanNa(a, n) ==
        zero? n => 0
        negative? n => - tanNa(a, -n)
        (numer(t := tanPIa(n::PI)) a) / ((denom t) a)

      tanSum l ==
        m := minIndex(v := symFunc l)
        +/[m1toN(i+1) * v(2*i - 1 + m) for i in 1..(#v quo 2)]
          / +/[m1toN(i) * v(2*i + m) for i in 0..((#v - 1) quo 2)]

  -- tanPIa(n) returns P(a)/Q(a) such that
  -- if a = tan(u) then P(a)/Q(a) = tan(n * u);
      tanPIa n ==
        m := minIndex(v := symFunc(monomial(1, 1)$UP, n))
        +/[m1toN(i+1) * v(2*i - 1 + m) for i in 1..(#v quo 2)]
          / +/[m1toN(i) * v(2*i + m) for i in 0..((#v - 1) quo 2)]
```

⟨*TANEXP.dotabb*⟩≡

```
  "TANEXP" [color="#FF4488",href="bookvol10.4.pdf#nameddest=TANEXP"]
  "PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
  "TANEXP" -> "PFECAT"
```

21.7 package UTSSOL TaylorSolve

UTSSOL is a facility to compute the first few coefficients of a Taylor series given only implicitely by a function f that vanishes when applied to the Taylor series.

It uses the method of undetermined coefficients.

```
Could I either
\begin{itemize}
\item take a function {\tt{}UTSCAT\ F\ ->\ UTSCAT\ F} and still be able to compute
  with undetermined coefficients, or
\item take a function {\tt{}F\ ->\ F}, and do likewise?
\end{itemize}

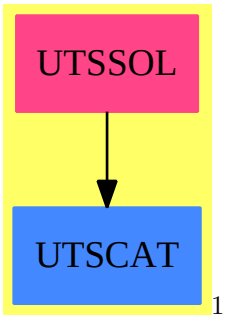
Let's see.

Try to compute the equation without resorting to power series. I.e., %
{\tt{}c:\ SUP\ SUP\ F}, and {\tt{}f:\ SUP\ SUP\ F\ ->\ SUP\ SUP\ F}. Won't this make the
computation of coefficients terribly slow?

I could also try to replace transcendental kernels with variables\dots

Unfortunately, {\tt{}SUP\ F} does not have {\tt{}TRANFUN} -- well, it can't, of
course. However, I'd like to be able to compute %
{\tt{}sin(1+monomial(1,1){\char36}UFPS\ SUP\ EXPR\ INT)}.
```

21.8 TaylorSolve



1.00

Exports:
seriesSolve

⟨*package UTSSOL TaylorSolve*⟩≡

```
)abbrev package UTSSOL TaylorSolve
TaylorSolve(F, UTSF, UTSSUPF): Exports == Implementation where
    F: Field
    SUP  ==> SparseUnivariatePolynomialExpressions
    UTSF: UnivariateTaylorSeriesCategory F
```

```
    UTSSUPF: UnivariateTaylorSeriesCategory SUP F
    NNI  ==> NonNegativeInteger

    Exports == with
        seriesSolve: (UTSSUPF -> UTSSUPF, List F) -> UTSF

    Implementation == add
```
⟨*implementation: UTSSOL TaylorSolve*⟩

⟨*implementation: UTSSOL TaylorSolve*⟩≡
```
        seriesSolve(f, l) ==
            c1 := map(#1::(SUP F), l)$ListFunctions2(F, SUP F)::(Stream SUP F)
            coeffs: Stream SUP F := concat(c1, generate(monomial(1$F,1$NNI)))
--            coeffs: Stream SUP F := concat(c1, monomial(1$F,1$NNI))
```

coeffs is the stream of the already computed coefficients of the solution, plus one which is so far undetermined. We store in st.2 the complete stream and in st.1 the stream starting with the first coefficient that has possibly not yet been computed.

```
The mathematics is not quite worked out. If {\tt{}coeffs} is initialized as
stream with all coefficients set to the \emph{same} transcendental value,
and not enough initial values are given, then the missing ones are
implicitely assumed to be all identical. It may well happen that a solution
is produced, although it is not uniquely determined\dots
```

⟨*implementation: UTSSOL TaylorSolve*⟩+≡
```
            st: List Stream SUP F := [coeffs, coeffs]
```

Consider an arbitrary equation $f\big(x, y(x)\big) = 0$. When setting $x = 0$, we obtain $f\big(0, y(0)\big) = 0$. It is not necessarily the case that this determines $y(0)$ uniquely, so we need one initial value that satisfies this equation.

```
{\tt{}seriesSolve} should check that the given initial values satisfy $f\big(0, y(0),
y'(0),...\big) = 0$.
```

Now consider the derivatives of f, where we write y instead of $y(x)$ for better readability:

$$\frac{d}{dx} f(x, y) = f_1(x, y) + f_2(x, y)y'$$

and

```
\frac{d^2}{dx^2}f(x, y)&=f_{1,1}(x, y)\\
                       &+f_{1,2}(x, y)y^\prime\\
                       &+f_{2,1}(x, y)y^\prime\\
                       &+f_{2,2}(x, y)(y^\prime)^2\\
                       &+f_2(x, y)y^{\prime\prime}.
```

In general, $\frac{d^2}{dx^2} f(x, y)$ depends only linearly on y''.

```
This points to another possibility: Since we know that we only need to solve
linear equations, we could compute two values and then use interpolation.
This might be a bit slower, but more importantly: can we still check that we
have enough initial values? Furthermore, we then really need that $f$ is
analytic, i.e., operators are not necessarily allowed anymore. However, it
seems that composition is allowed.
```

⟨*implementation: UTSSOL TaylorSolve*⟩+≡

```
            next: () -> F :=
                nr := st.1
                res: F

                if ground?(coeff: SUP F := nr.1)$(SUP F)
```

If the next element was already calculated, we can simply return it:

⟨*implementation: UTSSOL TaylorSolve*⟩+≡

```
                then
                    res := ground coeff
                    st.1 := rest nr
                else
```

Otherwise, we have to find the first non-satisfied relation and solve it. It should be linear, or a single non-constant monomial. That is, the solution should be unique.

⟨*implementation: UTSSOL TaylorSolve*⟩+≡

```
                    ns := st.2
                    eqs: Stream SUP F := coefficients f series ns
                    while zero? first eqs repeat eqs := rest eqs
                    eq: SUP F := first eqs
                    if degree eq > 1 then
                        if monomial? eq then res := 0
                        else
                            output(hconcat("The equation is: ", eq::OutputForm))
                                  $OutputPackage
                            error "seriesSolve: equation for coefficient not line
                    else res := (-coefficient(eq, 0$NNI)$(SUP F)
                                 /coefficient(eq, 1$NNI)$(SUP F))

                    nr.1 := res::SUP F
--                    concat!(st.2, monomial(1$F,1$NNI))
                    st.1 := rest nr

                res

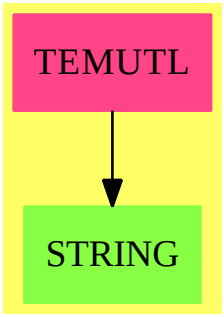
            series generate next
```

⟨*UTSSOL.dotabb*⟩≡

```
"UTSSOL" [color="#FF4488",href="bookvol10.4.pdf#nameddest=UTSSOL"]
"UTSCAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=UTSCAT"]
"UTSSOL" -> "UTSCAT"
```

21.9 package TEMUTL TemplateUtilities

21.10 TemplateUtilities



1.00

Exports:
interpretString stripCommentsAndBlanks

⟨*package TEMUTL TemplateUtilities*⟩≡
```
)abbrev package TEMUTL TemplateUtilities
++ Author: Mike Dewar
++ Date Created:  October 1992
++ Date Last Updated:
++ Basic Operations:
++ Related Domains:
++ Also See:
++ AMS Classifications:
++ Keywords:
++ Examples:
++ References:
++ Description: This package provides functions for template manipulation
TemplateUtilities(): Exports == Implementation where

  Exports == with
    interpretString : String -> Any
      ++ interpretString(s) treats a string as a piece of AXIOM input, by
      ++ parsing and interpreting it.
    stripCommentsAndBlanks : String -> String
      ++ stripCommentsAndBlanks(s) treats s as a piece of AXIOM input, and
      ++ removes comments, and leading and trailing blanks.

  Implementation == add

    import InputForm

    stripC(s:String,u:String):String ==
```

```
      i : Integer := position(u,s,1)
      i = 0 => s
      delete(s,i..)

    stripCommentsAndBlanks(s:String):String ==
      trim(stripC(stripC(s,"++"),"--"),char " ")

    parse(s:String):InputForm ==
      ncParseFromString(s)$Lisp::InputForm

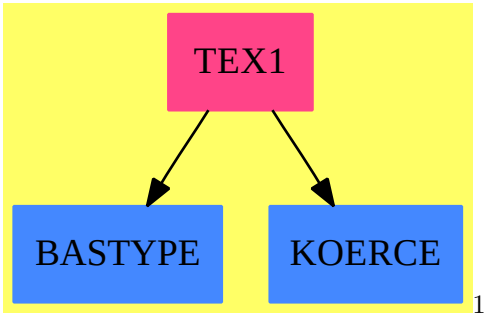
    interpretString(s:String):Any ==
      interpret parse s
```

⟨*TEMUTL.dotabb*⟩≡

```
"TEMUTL" [color="#FF4488",href="bookvol10.4.pdf#nameddest=TEMUTL"]
"STRING" [color="#88FF44",href="bookvol10.3.pdf#nameddest=STRING"]
"TEMUTL" -> "STRING"
```

21.11 package TEX1 TexFormat1

21.12 TexFormat1



1.00

Exports:
coerce

⟨*package TEX1 TexFormat1*⟩≡
```
)abbrev package TEX1 TexFormat1
++ Author: Robert S. Sutor
++ Date Created: 1987 through 1990
++ Change History:
++ Basic Operations: coerce
++ Related Constructors: TexFormat
++ Also See: ScriptFormulaFormat, ScriptFormulaFormat1
++ AMS Classifications:
++ Keywords: TeX, output, format
++ References: \TeX{} is a trademark of the American Mathematical
++   Society.
++ Description:
++   \spadtype{TexFormat1} provides a utility coercion for changing
++   to TeX format anything that has a coercion to the standard output
++   format.

TexFormat1(S : SetCategory): public == private where
  public  ==  with
    coerce: S -> TexFormat()
      ++ coerce(s) provides a direct coercion from a domain S to
      ++ TeX format. This allows the user to skip the step of first
      ++ manually coercing the object to standard output format before
      ++ it is coerced to TeX format.

  private == add
    import TexFormat()
```

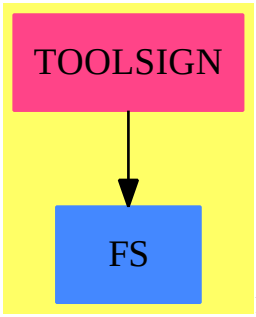
```
      coerce(s : S): TexFormat ==
        coerce(s :: OutputForm)$TexFormat
```

⟨*TEX1.dotabb*⟩≡

```
"TEX1" [color="#FF4488",href="bookvol10.4.pdf#nameddest=TEX1"]
"BASTYPE" [color="#4488FF",href="bookvol10.2.pdf#nameddest=BASTYPE"]
"KOERCE" [color="#4488FF",href="bookvol10.2.pdf#nameddest=KOERCE"]
"TEX1" -> "BASTYPE"
"TEX1" -> "KOERCE"
```

21.13 package TOOLSIGN ToolsForSign

21.14 ToolsForSign



1.00

Exports:
direction nonQsign sign

⟨*package TOOLSIGN ToolsForSign*⟩≡
```
)abbrev package TOOLSIGN ToolsForSign
++ Tools for the sign finding utilities
++ Author: Manuel Bronstein
++ Date Created: 25 August 1989
++ Date Last Updated: 26 November 1991
++ Description: Tools for the sign finding utilities.
ToolsForSign(R:Ring): with
    sign     : R      -> Union(Integer, "failed")
        ++ sign(r) \undocumented
    nonQsign : R      -> Union(Integer, "failed")
        ++ nonQsign(r) \undocumented
    direction: String -> Integer
        ++ direction(s) \undocumented
 == add

    if R is AlgebraicNumber then
      nonQsign r ==
        sign((r pretend AlgebraicNumber)::Expression(
                  Integer))$ElementaryFunctionSign(Integer, Expression Integer)
    else
      nonQsign r == "failed"

    if R has RetractableTo Fraction Integer then
      sign r ==
        (u := retractIfCan(r)@Union(Fraction Integer, "failed"))
          case Fraction(Integer) => sign(u::Fraction Integer)
        nonQsign r
```

```
      else
        if R has RetractableTo Integer then
          sign r ==
            (u := retractIfCan(r)@Union(Integer, "failed"))
              case "failed" => "failed"
            sign(u::Integer)

        else
          sign r ==
            zero? r => 0
--            one? r => 1
            r = 1 => 1
            r = -1 => -1
            "failed"

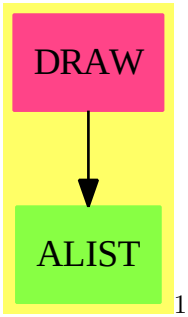
      direction st ==
        st = "right" => 1
        st = "left" => -1
        error "Unknown option"
```

⟨*TOOLSIGN.dotabb*⟩≡

```
"TOOLSIGN" [color="#FF4488",href="bookvol10.4.pdf#nameddest=TOOLSIGN"]
"FS" [color="#4488FF",href="bookvol10.2.pdf#nameddest=FS"]
"TOOLSIGN" -> "FS"
```

21.15 package DRAW TopLevelDrawFunctions

21.16 TopLevelDrawFunctions



1.00

Exports:
draw makeObject

⟨*package DRAW TopLevelDrawFunctions*⟩≡
```
)abbrev package DRAW TopLevelDrawFunctions
++ Author: Clifton J. Williamson
++ Date Created: 23 January 1990
++ Date Last Updated: October 1991 by Jon Steinbach
++ Basic Operations: draw
++ Related Constructors:
++ Also See:
++ AMS Classifications:
++ Keywords:
++ References:
++ Description: TopLevelDrawFunctions provides top level functions for
++ drawing graphics of expressions.
TopLevelDrawFunctions(Ex:Join(ConvertibleTo InputForm,SetCategory)):
 Exports == Implementation where
  B    ==> Boolean
  BIND ==> SegmentBinding Float
  L    ==> List
  SF   ==> DoubleFloat
  DROP ==> DrawOption

  PPC  ==> ParametricPlaneCurve Ex
  PPCF ==> ParametricPlaneCurve(SF -> SF)
  PSC  ==> ParametricSpaceCurve Ex
  PSCF ==> ParametricSpaceCurve(SF -> SF)
  PSF  ==> ParametricSurface Ex
  PSFF ==> ParametricSurface((SF,SF) -> SF)
  SPACE3 ==> ThreeSpace(SF)
```

```
    VIEW2 ==> TwoDimensionalViewport
    VIEW3 ==> ThreeDimensionalViewport

    Exports ==> with

  --% Two Dimensional Function Plots

      draw: (Ex,BIND,L DROP) -> VIEW2
        ++ draw(f(x),x = a..b,l) draws the graph of \spad{y = f(x)} as x
        ++ ranges from \spad{min(a,b)} to \spad{max(a,b)}; \spad{f(x)} is the
        ++ default title, and the options contained in the list l of
        ++ the domain \spad{DrawOption} are applied.
      draw: (Ex,BIND) -> VIEW2
        ++ draw(f(x),x = a..b) draws the graph of \spad{y = f(x)} as x
        ++ ranges from \spad{min(a,b)} to \spad{max(a,b)}; \spad{f(x)} appears
        ++ in the title bar.

  --% Parametric Plane Curves

      draw: (PPC,BIND,L DROP) -> VIEW2
        ++ draw(curve(f(t),g(t)),t = a..b,l) draws the graph of the parametric
        ++ curve \spad{x = f(t), y = g(t)} as t ranges from \spad{min(a,b)} to
        ++ \spad{max(a,b)}; \spad{(f(t),g(t))} is the default title, and the
        ++ options contained in the list l of the domain \spad{DrawOption}
        ++ are applied.
      draw: (PPC,BIND) -> VIEW2
        ++ draw(curve(f(t),g(t)),t = a..b) draws the graph of the parametric
        ++ curve \spad{x = f(t), y = g(t)} as t ranges from \spad{min(a,b)} to
        ++ \spad{max(a,b)}; \spad{(f(t),g(t))} appears in the title bar.

  --% Parametric Space Curves

      draw: (PSC,BIND,L DROP) -> VIEW3
        ++ draw(curve(f(t),g(t),h(t)),t = a..b,l) draws the graph of the
        ++ parametric curve \spad{x = f(t)}, \spad{y = g(t)}, \spad{z = h(t)}
        ++ as t ranges from \spad{min(a,b)} to \spad{max(a,b)}; \spad{h(t)}
        ++ is the default title, and the options contained in the list l of
        ++ the domain \spad{DrawOption} are applied.
      draw: (PSC,BIND) -> VIEW3
        ++ draw(curve(f(t),g(t),h(t)),t = a..b) draws the graph of the parametric
        ++ curve \spad{x = f(t)}, \spad{y = g(t)}, \spad{z = h(t)} as t ranges
        ++ from \spad{min(a,b)} to \spad{max(a,b)}; \spad{h(t)} is the default
        ++ title.
      makeObject: (PSC,BIND,L DROP) -> SPACE3
        ++ makeObject(curve(f(t),g(t),h(t)),t = a..b,l) returns a space of
        ++ the domain \spadtype{ThreeSpace} which contains the graph of the
```

```
      ++ parametric curve \spad{x = f(t)}, \spad{y = g(t)}, \spad{z = h(t)}
      ++ as t ranges from \spad{min(a,b)} to \spad{max(a,b)}; \spad{h(t)}
      ++ is the default title, and the options contained in the list l of
      ++ the domain \spad{DrawOption} are applied.
    makeObject: (PSC,BIND) -> SPACE3
      ++ makeObject(curve(f(t),g(t),h(t)),t = a..b) returns a space of the
      ++ domain \spadtype{ThreeSpace} which contains the graph of the
      ++ parametric curve \spad{x = f(t)}, \spad{y = g(t)}, \spad{z = h(t)}
      ++ as t ranges from \spad{min(a,b)} to \spad{max(a,b)}; \spad{h(t)} is
      ++ the default title.

  --% Three Dimensional Function Plots

    draw: (Ex,BIND,BIND,L DROP) -> VIEW3
      ++ draw(f(x,y),x = a..b,y = c..d,l) draws the graph of \spad{z = f(x,y)}
      ++ as x ranges from \spad{min(a,b)} to \spad{max(a,b)} and y ranges from
      ++ \spad{min(c,d)} to \spad{max(c,d)}; \spad{f(x,y)} is the default
      ++ title, and the options contained in the list l of the domain
      ++ \spad{DrawOption} are applied.
    draw: (Ex,BIND,BIND) -> VIEW3
      ++ draw(f(x,y),x = a..b,y = c..d) draws the graph of \spad{z = f(x,y)}
      ++ as x ranges from \spad{min(a,b)} to \spad{max(a,b)} and y ranges from
      ++ \spad{min(c,d)} to \spad{max(c,d)}; \spad{f(x,y)} appears in the title bar.
    makeObject: (Ex,BIND,BIND,L DROP) -> SPACE3
      ++ makeObject(f(x,y),x = a..b,y = c..d,l) returns a space of the
      ++ domain \spadtype{ThreeSpace} which contains the graph of
      ++ \spad{z = f(x,y)} as x ranges from \spad{min(a,b)} to \spad{max(a,b)}
      ++ and y ranges from \spad{min(c,d)} to \spad{max(c,d)}; \spad{f(x,y)}
      ++ is the default title, and the options contained in the list l of the
      ++ domain \spad{DrawOption} are applied.
    makeObject: (Ex,BIND,BIND) -> SPACE3
      ++ makeObject(f(x,y),x = a..b,y = c..d) returns a space of the domain
      ++ \spadtype{ThreeSpace} which contains the graph of \spad{z = f(x,y)}
      ++ as x ranges from \spad{min(a,b)} to \spad{max(a,b)} and y ranges from
      ++ \spad{min(c,d)} to \spad{max(c,d)}; \spad{f(x,y)} appears as the
      ++ default title.

  --% Parametric Surfaces

    draw: (PSF,BIND,BIND,L DROP) -> VIEW3
      ++ draw(surface(f(u,v),g(u,v),h(u,v)),u = a..b,v = c..d,l) draws the
      ++ graph of the parametric surface \spad{x = f(u,v)}, \spad{y = g(u,v)},
      ++ \spad{z = h(u,v)} as u ranges from \spad{min(a,b)} to \spad{max(a,b)}
      ++ and v ranges from \spad{min(c,d)} to \spad{max(c,d)}; \spad{h(t)}
      ++ is the default title, and the options contained in the list l of
      ++ the domain \spad{DrawOption} are applied.
```

```
    draw: (PSF,BIND,BIND) -> VIEW3
      ++ draw(surface(f(u,v),g(u,v),h(u,v)),u = a..b,v = c..d) draws the
      ++ graph of the parametric surface \spad{x = f(u,v)}, \spad{y = g(u,v)},
      ++ \spad{z = h(u,v)} as u ranges from \spad{min(a,b)} to \spad{max(a,b)}
      ++ and v ranges from \spad{min(c,d)} to \spad{max(c,d)}; \spad{h(t)} is
      ++ the default title.
    makeObject: (PSF,BIND,BIND,L DROP) -> SPACE3
      ++ makeObject(surface(f(u,v),g(u,v),h(u,v)),u = a..b,v = c..d,l) returns
      ++ a space of the domain \spadtype{ThreeSpace} which contains the graph
      ++ of the parametric surface \spad{x = f(u,v)}, \spad{y = g(u,v)},
      ++ \spad{z = h(u,v)} as u ranges from \spad{min(a,b)} to \spad{max(a,b)}
      ++ and v ranges from \spad{min(c,d)} to \spad{max(c,d)}; \spad{h(t)} is
      ++ the default title, and the options contained in the list l of
      ++ the domain \spad{DrawOption} are applied.
    makeObject: (PSF,BIND,BIND) -> SPACE3
      ++ makeObject(surface(f(u,v),g(u,v),h(u,v)),u = a..b,v = c..d) returns
      ++ a space of the domain \spadtype{ThreeSpace} which contains the
      ++ graph of the parametric surface \spad{x = f(u,v)}, \spad{y = g(u,v)},
      ++ \spad{z = h(u,v)} as u ranges from \spad{min(a,b)} to \spad{max(a,b)}
      ++ and v ranges from \spad{min(c,d)} to \spad{max(c,d)}; \spad{h(t)} is
      ++ the default title.

  Implementation ==> add
    import TopLevelDrawFunctionsForCompiledFunctions
    import MakeFloatCompiledFunction(Ex)
    import ParametricPlaneCurve(SF -> SF)
    import ParametricSpaceCurve(SF -> SF)
    import ParametricSurface((SF,SF) -> SF)
    import ThreeSpace(SF)

--------------------------------------------------------------------------
--                      2D - draw's  (given by formulae)
--------------------------------------------------------------------------

--% Two Dimensional Function Plots

    draw(f:Ex,bind:BIND,l:L DROP) ==
      -- create title if necessary
      if not option?(l,"title" :: Symbol) then
        s:String := unparse(convert(f)@InputForm)
        if sayLength(s)$DisplayPackage > 50 then
          l := concat(title "AXIOM2D",l)
        else l := concat(title s,l)
      -- call 'draw'
      draw(makeFloatFunction(f,variable bind),segment bind,l)
```

```
    draw(f:Ex,bind:BIND) == draw(f,bind,nil())

--% Parametric Plane Curves

    draw(ppc:PPC,bind:BIND,l:L DROP) ==
      f := coordinate(ppc,1); g := coordinate(ppc,2)
      -- create title if necessary
      if not option?(l,"title" :: Symbol) then
        s:String := unparse(convert(f)@InputForm)
        if sayLength(s)$DisplayPackage > 50 then
          l := concat(title "AXIOM2D",l)
        else l := concat(title s,l)
      -- create curve with functions as coordinates
      curve : PPCF := curve(makeFloatFunction(f,variable bind),_
                            makeFloatFunction(g,variable bind))$PPCF
      -- call 'draw'
      draw(curve,segment bind,l)

    draw(ppc:PPC,bind:BIND) == draw(ppc,bind,nil())

------------------------------------------------------------------------
--                    3D - Curves  (given by formulas)
------------------------------------------------------------------------

    makeObject(psc:PSC,tBind:BIND,l:L DROP) ==
      -- obtain dependent variable and coordinate functions
      t := variable tBind; tSeg := segment tBind
      f := coordinate(psc,1); g := coordinate(psc,2); h := coordinate(psc,3)
      -- create title if necessary
      if not option?(l,"title" :: Symbol) then
        s:String := unparse(convert(f)@InputForm)
        if sayLength(s)$DisplayPackage > 50 then
          l := concat(title "AXIOM3D",l)
        else l := concat(title s,l)
      -- indicate draw style if necessary
      if not option?(l,"style" :: Symbol) then
        l := concat(style unparse(convert(f)@InputForm),l)
      -- create curve with functions as coordinates
      curve : PSCF := curve(makeFloatFunction(f,t),_
                            makeFloatFunction(g,t),_
                            makeFloatFunction(h,t))
      -- call 'draw'
      makeObject(curve,tSeg,l)

    makeObject(psc:PSC,tBind:BIND) ==
      makeObject(psc,tBind,nil())
```

```
    draw(psc:PSC,tBind:BIND,l:L DROP) ==
      -- obtain dependent variable and coordinate functions
      t := variable tBind; tSeg := segment tBind
      f := coordinate(psc,1); g := coordinate(psc,2); h := coordinate(psc,3)
      -- create title if necessary
      if not option?(l,"title" :: Symbol) then
        s:String := unparse(convert(f)@InputForm)
        if sayLength(s)$DisplayPackage > 50 then
          l := concat(title "AXIOM3D",l)
        else l := concat(title s,l)
      -- indicate draw style if necessary
      if not option?(l,"style" :: Symbol) then
        l := concat(style unparse(convert(f)@InputForm),l)
      -- create curve with functions as coordinates
      curve : PSCF := curve(makeFloatFunction(f,t),_
                            makeFloatFunction(g,t),_
                            makeFloatFunction(h,t))
      -- call 'draw'
      draw(curve,tSeg,l)

    draw(psc:PSC,tBind:BIND) ==
      draw(psc,tBind,nil())

------------------------------------------------------------------------
--                    3D - Surfaces  (given by formulas)
------------------------------------------------------------------------

--% Three Dimensional Function Plots

    makeObject(f:Ex,xBind:BIND,yBind:BIND,l:L DROP) ==
      -- create title if necessary
      if not option?(l,"title" :: Symbol) then
        s:String := unparse(convert(f)@InputForm)
        if sayLength(s)$DisplayPackage > 50 then
          l := concat(title "AXIOM3D",l)
        else l := concat(title s,l)
      -- indicate draw style if necessary
      if not option?(l,"style" :: Symbol) then
        l := concat(style unparse(convert(f)@InputForm),l)
      -- obtain dependent variables and their ranges
      x := variable xBind; xSeg := segment xBind
      y := variable yBind; ySeg := segment yBind
      -- call 'draw'
      makeObject(makeFloatFunction(f,x,y),xSeg,ySeg,l)
```

```
    makeObject(f:Ex,xBind:BIND,yBind:BIND) ==
      makeObject(f,xBind,yBind,nil())

    draw(f:Ex,xBind:BIND,yBind:BIND,l:L DROP) ==
      -- create title if necessary
      if not option?(l,"title" :: Symbol) then
        s:String := unparse(convert(f)@InputForm)
        if sayLength(s)$DisplayPackage > 50 then
          l := concat(title "AXIOM3D",l)
        else l := concat(title s,l)
      -- indicate draw style if necessary
      if not option?(l,"style" :: Symbol) then
        l := concat(style unparse(convert(f)@InputForm),l)
      -- obtain dependent variables and their ranges
      x := variable xBind; xSeg := segment xBind
      y := variable yBind; ySeg := segment yBind
      -- call 'draw'
      draw(makeFloatFunction(f,x,y),xSeg,ySeg,l)

    draw(f:Ex,xBind:BIND,yBind:BIND) ==
      draw(f,xBind,yBind,nil())

  --% parametric surface

    makeObject(s:PSF,uBind:BIND,vBind:BIND,l:L DROP) ==
      f := coordinate(s,1); g := coordinate(s,2); h := coordinate(s,3)
      if not option?(l,"title" :: Symbol) then
        s:String := unparse(convert(f)@InputForm)
        if sayLength(s)$DisplayPackage > 50 then
          l := concat(title "AXIOM3D",l)
        else l := concat(title s,l)
      if not option?(l,"style" :: Symbol) then
        l := concat(style unparse(convert(f)@InputForm),l)
      u := variable uBind; uSeg := segment uBind
      v := variable vBind; vSeg := segment vBind
      surf : PSFF := surface(makeFloatFunction(f,u,v),_
                             makeFloatFunction(g,u,v),_
                             makeFloatFunction(h,u,v))
      makeObject(surf,uSeg,vSeg,l)

    makeObject(s:PSF,uBind:BIND,vBind:BIND) ==
      makeObject(s,uBind,vBind,nil())

    draw(s:PSF,uBind:BIND,vBind:BIND,l:L DROP) ==
      f := coordinate(s,1); g := coordinate(s,2); h := coordinate(s,3)
      -- create title if necessary
```

```
    if not option?(l,"title" :: Symbol) then
      s:String := unparse(convert(f)@InputForm)
      if sayLength(s)$DisplayPackage > 50 then
        l := concat(title "AXIOM3D",l)
      else l := concat(title s,l)
    -- indicate draw style if necessary
    if not option?(l,"style" :: Symbol) then
      l := concat(style unparse(convert(f)@InputForm),l)
    -- obtain dependent variables and their ranges
    u := variable uBind; uSeg := segment uBind
    v := variable vBind; vSeg := segment vBind
    -- create surface with functions as coordinates
    surf : PSFF := surface(makeFloatFunction(f,u,v),_
                           makeFloatFunction(g,u,v),_
                           makeFloatFunction(h,u,v))
    -- call 'draw'
    draw(surf,uSeg,vSeg,l)

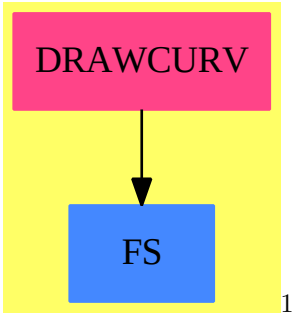
  draw(s:PSF,uBind:BIND,vBind:BIND) ==
    draw(s,uBind,vBind,nil())
```

⟨*DRAW.dotabb*⟩≡

```
"DRAW" [color="#FF4488",href="bookvol10.4.pdf#nameddest=DRAW"]
"ALIST" [color="#88FF44",href="bookvol10.3.pdf#nameddest=ALIST"]
"DRAW" -> "ALIST"
```

21.17 package DRAWCURV TopLevelDrawFunctionsForAlgebraicCurves

21.18 TopLevelDrawFunctionsForAlgebraicCurves



1.00

Exports:
draw

⟨*package DRAWCURV TopLevelDrawFunctionsForAlgebraicCurves*⟩≡

```
)abbrev package DRAWCURV TopLevelDrawFunctionsForAlgebraicCurves
++ Author: Clifton J. Williamson
++ Date Created: 26 June 1990
++ Date Last Updated:  October 1991 by Jon Steinbach
++ Basic Operations: draw
++ Related Constructors:
++ Also See:
++ AMS Classifications:
++ Keywords:
++ References:
++ Description: TopLevelDrawFunctionsForAlgebraicCurves provides top level
++ functions for drawing non-singular algebraic curves.

TopLevelDrawFunctionsForAlgebraicCurves(R,Ex): Exports == Implementation where
  R  : Join(IntegralDomain, OrderedSet, RetractableTo Integer)
  Ex : FunctionSpace(R)

  ANY1  ==> AnyFunctions1
  DROP  ==> DrawOption
  EQ    ==> Equation
  F     ==> Float
  FRAC  ==> Fraction
  I     ==> Integer
  L     ==> List
  P     ==> Polynomial
  RN    ==> Fraction Integer
```

```
SEG   ==> Segment
SY    ==> Symbol
VIEW2 ==> TwoDimensionalViewport

Exports ==> with

  draw: (EQ Ex,SY,SY,L DROP) -> VIEW2
    ++ draw(f(x,y) = g(x,y),x,y,l) draws the graph of a polynomial
    ++ equation.  The list l of draw options must specify a region
    ++ in the plane in which the curve is to sketched.

Implementation ==> add
  import ViewportPackage
  import PlaneAlgebraicCurvePlot
  import ViewDefaultsPackage
  import GraphicsDefaults
  import DrawOptionFunctions0
  import SegmentFunctions2(RN,F)
  import SegmentFunctions2(F,RN)
  import AnyFunctions1(L SEG RN)

  drawToScaleRanges: (SEG F,SEG F) -> L SEG F
  drawToScaleRanges(xVals,yVals) ==
    -- warning: assumes window is square
    xHi := hi xVals; xLo := lo xVals
    yHi := hi yVals; yLo := lo yVals
    xDiff := xHi - xLo; yDiff := yHi - yLo
    pad := abs(yDiff - xDiff)/2
    yDiff > xDiff =>
      [segment(xLo - pad,xHi + pad),yVals]
    [xVals,segment(yLo - pad,yHi + pad)]

  intConvert: R -> I
  intConvert r ==
    (nn := retractIfCan(r)@Union(I,"failed")) case "failed" =>
      error "draw: polynomial must have rational coefficients"
    nn :: I

  polyEquation: EQ Ex -> P I
  polyEquation eq ==
    ff := lhs(eq) - rhs(eq)
    (r := retractIfCan(ff)@Union(FRAC P R,"failed")) case "failed" =>
      error "draw: not a polynomial equation"
    rat := r :: FRAC P R
    retractIfCan(denom rat)@Union(R,"failed") case "failed" =>
      error "draw: non-constant denominator"
```

```
    map(intConvert,numer rat)$PolynomialFunctions2(R,I)

  draw(eq,x,y,l) ==
    -- obtain polynomial equation
    p := polyEquation eq
    -- extract ranges from option list
    floatRange := option(l,"rangeFloat" :: Symbol)
    ratRange := option(l,"rangeRat" :: Symbol)
    (floatRange case "failed") and (ratRange case "failed") =>
      error "draw: you must specify ranges for an implicit plot"
    ranges : L SEG RN := nil()              -- dummy value
    floatRanges : L SEG F := nil()          -- dummy value
    xRange : SEG RN := segment(0,0)         -- dummy value
    yRange : SEG RN := segment(0,0)         -- dummy value
    xRangeFloat : SEG F := segment(0,0)     -- dummy value
    yRangeFloat : SEG F := segment(0,0)     -- dummy value
    if not ratRange case "failed" then
      ranges := retract(ratRange :: Any)$ANY1(L SEG RN)
      not size?(ranges,2) => error "draw: you must specify two ranges"
      xRange := first ranges; yRange := second ranges
      xRangeFloat := map(convert(#1)@Float,xRange)@(SEG F)
      yRangeFloat := map(convert(#1)@Float,yRange)@(SEG F)
      floatRanges := [xRangeFloat,yRangeFloat]
    else
      floatRanges := retract(floatRange :: Any)$ANY1(L SEG F)
      not size?(floatRanges,2) =>
        error "draw: you must specify two ranges"
      xRangeFloat := first floatRanges
      yRangeFloat := second floatRanges
      xRange := map(retract(#1)@RN,xRangeFloat)@(SEG RN)
      yRange := map(retract(#1)@RN,yRangeFloat)@(SEG RN)
      ranges := [xRange,yRange]
    -- create curve plot
    acplot := makeSketch(p,x,y,xRange,yRange)
    -- process scaling information
    if toScale(l,drawToScale()) then
      scaledRanges := drawToScaleRanges(xRangeFloat,yRangeFloat)
      -- add scaled ranges to list of options
      l := concat(ranges scaledRanges,l)
    else
      -- add ranges to list of options
      l := concat(ranges floatRanges,l)
    -- process color information
    ptCol := pointColorPalette(l,pointColorDefault())
    crCol := curveColorPalette(l,lineColorDefault())
    -- draw
```

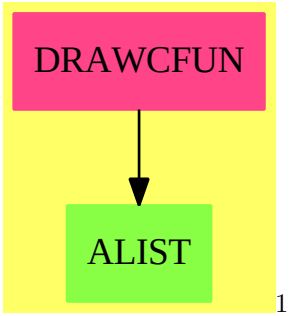
```
        drawCurves(listBranches acplot,ptCol,crCol,pointSizeDefault(),l)
```

⟨*DRAWCURV.dotabb*⟩≡

```
  "DRAWCURV" [color="#FF4488",href="bookvol10.4.pdf#nameddest=DRAWCURV"]
  "FS" [color="#4488FF",href="bookvol10.2.pdf#nameddest=FS"]
  "DRAWCURV" -> "FS"
```

21.19 package DRAWCFUN TopLevelDrawFunctionsForCompiledFunctions

21.20 TopLevelDrawFunctionsForCompiledFunctions



1.00

Exports:
draw makeObject recolor

⟨*package DRAWCFUN TopLevelDrawFunctionsForCompiledFunctions*⟩≡

```
)abbrev package DRAWCFUN TopLevelDrawFunctionsForCompiledFunctions
++ Author: Clifton J. Williamson
++ Date Created: 22 June 1990
++ Date Last Updated: January 1992 by Scott Morrison
++ Basic Operations: draw, recolor
++ Related Constructors:
++ Also See:
++ AMS Classifications:
++ Keywords:
++ References:
++ Description: TopLevelDrawFunctionsForCompiledFunctions provides top level
++ functions for drawing graphics of expressions.
TopLevelDrawFunctionsForCompiledFunctions():
 Exports == Implementation where
  ANY1 ==> AnyFunctions1
  B    ==> Boolean
  F    ==> Float
  L    ==> List
  SEG  ==> Segment Float
  SF   ==> DoubleFloat
  DROP ==> DrawOption
  PLOT ==> Plot
  PPC  ==> ParametricPlaneCurve(SF -> SF)
  PSC  ==> ParametricSpaceCurve(SF -> SF)
  PSF  ==> ParametricSurface((SF,SF) -> SF)
  Pt   ==> Point SF
```

```
    PSFUN ==> (SF, SF) -> Pt
    PCFUN ==> SF -> Pt
    SPACE3 ==> ThreeSpace(SF)
    VIEW2 ==> TwoDimensionalViewport
    VIEW3 ==> ThreeDimensionalViewport

    Exports ==> with

  --% Two Dimensional Function Plots

      draw: (SF -> SF,SEG,L DROP) -> VIEW2
        ++ draw(f,a..b,l) draws the graph of \spad{y = f(x)} as x
        ++ ranges from \spad{min(a,b)} to \spad{max(a,b)}.
        ++ The options contained in the list l of
        ++ the domain \spad{DrawOption} are applied.
      draw: (SF -> SF,SEG) -> VIEW2
        ++ draw(f,a..b) draws the graph of \spad{y = f(x)} as x
        ++ ranges from \spad{min(a,b)} to \spad{max(a,b)}.

  --% Parametric Plane Curves

      draw: (PPC,SEG,L DROP) -> VIEW2
        ++ draw(curve(f,g),a..b,l) draws the graph of the parametric
        ++ curve \spad{x = f(t), y = g(t)} as t ranges from \spad{min(a,b)} to
        ++ \spad{max(a,b)}.
        ++ The options contained in the list l of the domain \spad{DrawOption}
        ++ are applied.
      draw: (PPC,SEG) -> VIEW2
        ++ draw(curve(f,g),a..b) draws the graph of the parametric
        ++ curve \spad{x = f(t), y = g(t)} as t ranges from \spad{min(a,b)} to
        ++ \spad{max(a,b)}.

  --% Parametric Space Curves

      draw: (PSC,SEG,L DROP) -> VIEW3
        ++ draw(curve(f,g,h),a..b,l) draws the graph of the parametric
        ++ curve \spad{x = f(t), y = g(t), z = h(t)} as t ranges from
        ++ \spad{min(a,b)} to \spad{max(a,b)}.
        ++ The options contained in the list l of the domain
        ++ \spad{DrawOption} are applied.
      draw: (PSC,SEG) -> VIEW3
        ++ draw(curve(f,g,h),a..b,l) draws the graph of the parametric
        ++ curve \spad{x = f(t), y = g(t), z = h(t)} as t ranges from
        ++ \spad{min(a,b)} to \spad{max(a,b)}.
      draw: (PCFUN,SEG,L DROP) -> VIEW3
        ++ draw(f,a..b,l) draws the graph of the parametric
```

```
      ++ curve \spad{f} as t ranges from
      ++ \spad{min(a,b)} to \spad{max(a,b)}.
      ++ The options contained in the list l of the domain
      ++ \spad{DrawOption} are applied.
    draw: (PCFUN,SEG) -> VIEW3
      ++ draw(f,a..b,l) draws the graph of the parametric
      ++ curve \spad{f} as t ranges from
      ++ \spad{min(a,b)} to \spad{max(a,b)}.

    makeObject: (PSC,SEG,L DROP) -> SPACE3
      ++ makeObject(curve(f,g,h),a..b,l) returns a space of the
      ++ domain \spadtype{ThreeSpace} which contains the graph of the
      ++ parametric curve \spad{x = f(t), y = g(t), z = h(t)} as t ranges from
      ++ \spad{min(a,b)} to \spad{max(a,b)};
      ++ The options contained in the list l of the domain
      ++ \spad{DrawOption} are applied.
    makeObject: (PSC,SEG) -> SPACE3
      ++ makeObject(sp,curve(f,g,h),a..b) returns the space \spad{sp}
      ++ of the domain \spadtype{ThreeSpace} with the addition of the graph
      ++ of the parametric curve \spad{x = f(t), y = g(t), z = h(t)} as t
      ++ ranges from \spad{min(a,b)} to \spad{max(a,b)}.
    makeObject: (PCFUN,SEG,L DROP) -> SPACE3
      ++ makeObject(curve(f,g,h),a..b,l) returns a space of the
      ++ domain \spadtype{ThreeSpace} which contains the graph of the
      ++ parametric curve \spad{x = f(t), y = g(t), z = h(t)} as t ranges from
      ++ \spad{min(a,b)} to \spad{max(a,b)}.
      ++ The options contained in the list l of the domain
      ++ \spad{DrawOption} are applied.
    makeObject: (PCFUN,SEG) -> SPACE3
      ++ makeObject(sp,curve(f,g,h),a..b) returns the space \spad{sp}
      ++ of the domain \spadtype{ThreeSpace} with the addition of the graph
      ++ of the parametric curve \spad{x = f(t), y = g(t), z = h(t)} as t
      ++ ranges from \spad{min(a,b)} to \spad{max(a,b)}.

  --% Three Dimensional Function Plots

    draw: ((SF,SF) -> SF,SEG,SEG,L DROP) -> VIEW3
      ++ draw(f,a..b,c..d,l) draws the graph of \spad{z = f(x,y)}
      ++ as x ranges from \spad{min(a,b)} to \spad{max(a,b)} and y ranges from
      ++ \spad{min(c,d)} to \spad{max(c,d)}.
      ++ and the options contained in the list l of the domain
      ++ \spad{DrawOption} are applied.
    draw: ((SF,SF) -> SF,SEG,SEG) -> VIEW3
      ++ draw(f,a..b,c..d) draws the graph of \spad{z = f(x,y)}
      ++ as x ranges from \spad{min(a,b)} to \spad{max(a,b)} and y ranges from
      ++ \spad{min(c,d)} to \spad{max(c,d)}.
```

```
    makeObject: ((SF,SF) -> SF,SEG,SEG,L DROP) -> SPACE3
      ++ makeObject(f,a..b,c..d,l) returns a space of the domain
      ++ \spadtype{ThreeSpace} which contains the graph of \spad{z = f(x,y)}
      ++ as x ranges from \spad{min(a,b)} to \spad{max(a,b)} and y ranges from
      ++ \spad{min(c,d)} to \spad{max(c,d)}, and the options contained in the
      ++ list l of the domain \spad{DrawOption} are applied.
    makeObject: ((SF,SF) -> SF,SEG,SEG) -> SPACE3
      ++ makeObject(f,a..b,c..d) returns a space of the domain
      ++ \spadtype{ThreeSpace} which contains the graph of \spad{z = f(x,y)}
      ++ as x ranges from \spad{min(a,b)} to \spad{max(a,b)} and y ranges from
      ++ \spad{min(c,d)} to \spad{max(c,d)}.

  --% Parametric Surfaces

    draw: (PSFUN, SEG, SEG, L DROP) -> VIEW3
      ++ draw(f,a..b,c..d) draws the
      ++ graph of the parametric surface \spad{f(u,v)}
      ++ as u ranges from \spad{min(a,b)} to \spad{max(a,b)}
      ++ and v ranges from \spad{min(c,d)} to \spad{max(c,d)}.
      ++ The options contained in the
      ++ list l of the domain \spad{DrawOption} are applied.
    draw: (PSFUN, SEG, SEG) -> VIEW3
      ++ draw(f,a..b,c..d) draws the
      ++ graph of the parametric surface \spad{f(u,v)}
      ++ as u ranges from \spad{min(a,b)} to \spad{max(a,b)}
      ++ and v ranges from \spad{min(c,d)} to \spad{max(c,d)}
      ++ The options contained in the list
      ++ l of the domain \spad{DrawOption} are applied.
    makeObject: (PSFUN, SEG, SEG, L DROP) -> SPACE3
      ++ makeObject(f,a..b,c..d,l) returns a
      ++ space of the domain \spadtype{ThreeSpace} which contains the
      ++ graph of the parametric surface \spad{f(u,v)}
      ++ as u ranges from \spad{min(a,b)} to
      ++ \spad{max(a,b)} and v ranges from \spad{min(c,d)} to \spad{max(c,d)};
      ++ The options contained in the
      ++ list l of the domain \spad{DrawOption} are applied.
    makeObject: (PSFUN, SEG, SEG) -> SPACE3
      ++ makeObject(f,a..b,c..d,l) returns a
      ++ space of the domain \spadtype{ThreeSpace} which contains the
      ++ graph of the parametric surface \spad{f(u,v)}
      ++ as u ranges from \spad{min(a,b)} to
      ++ \spad{max(a,b)} and v ranges from \spad{min(c,d)} to \spad{max(c,d)}.
    draw: (PSF,SEG,SEG,L DROP) -> VIEW3
      ++ draw(surface(f,g,h),a..b,c..d) draws the
      ++ graph of the parametric surface \spad{x = f(u,v)}, \spad{y = g(u,v)},
      ++ \spad{z = h(u,v)} as u ranges from \spad{min(a,b)} to \spad{max(a,b)}
```

```
    ++ and v ranges from \spad{min(c,d)} to \spad{max(c,d)};
    ++ The options contained in the
    ++ list l of the domain \spad{DrawOption} are applied.
  draw: (PSF,SEG,SEG) -> VIEW3
    ++ draw(surface(f,g,h),a..b,c..d) draws the
    ++ graph of the parametric surface \spad{x = f(u,v)}, \spad{y = g(u,v)},
    ++ \spad{z = h(u,v)} as u ranges from \spad{min(a,b)} to \spad{max(a,b)}
    ++ and v ranges from \spad{min(c,d)} to \spad{max(c,d)};
  makeObject: (PSF,SEG,SEG,L DROP) -> SPACE3
    ++ makeObject(surface(f,g,h),a..b,c..d,l) returns a
    ++ space of the domain \spadtype{ThreeSpace} which contains the
    ++ graph of the parametric surface \spad{x = f(u,v)}, \spad{y = g(u,v)},
    ++ \spad{z = h(u,v)} as u ranges from \spad{min(a,b)} to
    ++ \spad{max(a,b)} and v ranges from \spad{min(c,d)} to \spad{max(c,d)}.
    ++ The options contained in the
    ++ list l of the domain \spad{DrawOption} are applied.
  makeObject: (PSF,SEG,SEG) -> SPACE3
    ++ makeObject(surface(f,g,h),a..b,c..d,l) returns a
    ++ space of the domain \spadtype{ThreeSpace} which contains the
    ++ graph of the parametric surface \spad{x = f(u,v)}, \spad{y = g(u,v)},
    ++ \spad{z = h(u,v)} as u ranges from \spad{min(a,b)} to
    ++ \spad{max(a,b)} and v ranges from \spad{min(c,d)} to \spad{max(c,d)}.
  recolor: ((SF,SF) -> Pt,(SF,SF,SF) -> SF) -> ((SF,SF) -> Pt)
    ++ recolor(), uninteresting to top level user; exported in order to
    ++ compile package.

Implementation ==> add
```

I have had to work my way around the following bug in the compiler: When a local variable is given a mapping as a value, e.g.

```
foo : SF -> SF := makeFloatFunction(f,t),
```

the compiler cannot distinguish that local variable from a local function defined elsewhere in the package. Thus, when 'foo' is passed to a function, e.g.

```
bird := fcn(foo),
```

foo will often be compiled as `|DRAW;foo|` rather than `|foo|`. This, of course, causes a run-time error.

To avoid this problem, local variables are not given mappings as values, but rather (singleton) lists of mappings. The first element of the list can always be extracted and everything goes through as before. There is no major loss in efficiency, as the computation of points will always dominate the computation time.
- cjw, 22 June MCMXC
⟨*package DRAWCFUN TopLevelDrawFunctionsForCompiledFunctions*⟩+≡

```
import PLOT
import TwoDimensionalPlotClipping
import GraphicsDefaults
import ViewportPackage
import ThreeDimensionalViewport
import DrawOptionFunctions0
import MakeFloatCompiledFunction(Ex)
import MeshCreationRoutinesForThreeDimensions
import SegmentFunctions2(SF,Float)
import ViewDefaultsPackage
import AnyFunctions1(Pt -> Pt)
import AnyFunctions1((SF,SF,SF) -> SF)
import DrawOptionFunctions0
import SPACE3

EXTOVARERROR : String := _
  "draw: when specifying function, left hand side must be a variable"
SMALLRANGEERROR : String := _
  "draw: range is in interval with only one point"
DEPVARERROR : String := _
  "draw: independent variable appears on lhs of function definition"

--------------------------------------------------------------------------
--                      2D - draw's
--------------------------------------------------------------------------
```

```
drawToScaleRanges: (Segment SF,Segment SF) -> L SEG
drawToScaleRanges(xVals,yVals) ==
  -- warning: assumes window is square
  xHi := convert(hi xVals)@Float; xLo := convert(lo xVals)@Float
  yHi := convert(hi yVals)@Float; yLo := convert(lo yVals)@Float
  xDiff := xHi - xLo; yDiff := yHi - yLo
  pad := abs(yDiff - xDiff)/2
  yDiff > xDiff =>
    [segment(xLo - pad,xHi + pad),map(convert(#1)@Float,yVals)]
  [map(convert(#1)@Float,xVals),segment(yLo - pad,yHi + pad)]

drawPlot: (PLOT,L DROP) -> VIEW2
drawPlot(plot,l) ==
  branches := listBranches plot
  xRange := xRange plot; yRange := yRange plot
  -- process clipping information
  if (cl := option(l,"clipSegment" :: Symbol)) case "failed" then
    if clipBoolean(l,clipPointsDefault()) then
      clipInfo :=
        parametric? plot => clipParametric plot
        clip plot
      branches := clipInfo.brans
      xRange := clipInfo.xValues; yRange := clipInfo.yValues
    else
      "No explicit user-specified clipping"
  else
    segList := retract(cl :: Any)$ANY1(L SEG)
    empty? segList =>
      error "draw: you may specify at least 1 segment for 2D clipping"
    more?(segList,2) =>
      error "draw: you may specify at most 2 segments for 2D clipping"
    xLo : SF := 0; xHi : SF := 0; yLo : SF := 0; yHi : SF := 0
    if empty? rest segList then
      xLo := lo xRange; xHi := hi xRange
      yRangeF := first segList
      yLo := convert(lo yRangeF)@SF; yHi := convert(hi yRangeF)@SF
    else
      xRangeF := first segList
      xLo := convert(lo xRangeF)@SF; xHi := convert(hi xRangeF)@SF
      yRangeF := second segList
      yLo := convert(lo yRangeF)@SF; yHi := convert(hi yRangeF)@SF
    clipInfo := clipWithRanges(branches,xLo,xHi,yLo,yHi)
    branches := clipInfo.brans
    xRange := clipInfo.xValues; yRange := clipInfo.yValues
  -- process scaling information
  if toScale(l,drawToScale()) then
```

```
      scaledRanges := drawToScaleRanges(xRange,yRange)
      -- add scaled ranges to list of options
      l := concat(ranges scaledRanges,l)
    else
      xRangeFloat : SEG := map(convert(#1)@Float,xRange)
      yRangeFloat : SEG := map(convert(#1)@Float,yRange)
      -- add ranges to list of options
      l := concat(ranges(ll : L SEG := [xRangeFloat,yRangeFloat]),l)
    -- process color information
    ptCol := pointColorPalette(l,pointColorDefault())
    crCol := curveColorPalette(l,lineColorDefault())
    -- draw
    drawCurves(branches,ptCol,crCol,pointSizeDefault(),l)

  normalize: SEG -> Segment SF
  normalize seg ==
    -- normalize [a,b]:
    -- error if a = b, returns [a,b] if a < b, returns [b,a] if b > a
    a := convert(lo seg)@SF; b := convert(hi seg)@SF
    a = b => error SMALLRANGEERROR
    a < b => segment(a,b)
    segment(b,a)
```

The function myTrap1 is a local function for used in creating maps SF -¿ Point SF (two dimensional). The range of this function is SingleFloat. As originally coded it would return $NaNvalue$Lisp which is outside the range. Since this function is only used internallly by the draw package we handle the "failed" case by returning zero. We handle the out-of-range case by returning the maximum or minimum SingleFloat value.

⟨*package DRAWCFUN TopLevelDrawFunctionsForCompiledFunctions*⟩+≡

```
      myTrap1: (SF-> SF, SF) -> SF
      myTrap1(ff:SF-> SF, f:SF):SF ==
        s := trapNumericErrors(ff(f))$Lisp :: Union(SF, "failed")
        s case "failed" => 0
        r:=s::SF
        r >max()$SF => max()$SF
        r < min()$SF => min()$SF
        r

      makePt2: (SF,SF) -> Point SF
      makePt2(x,y) == point(l : List SF := [x,y])

  --% Two Dimensional Function Plots

      draw(f:SF -> SF,seg:SEG,l:L DROP) ==
        -- set adaptive plotting off or on
        oldAdaptive := adaptive?()$PLOT
        setAdaptive(adaptive(l,oldAdaptive))$PLOT
        -- create function SF -> Point SF
        ff : L(SF -> Point SF) := [makePt2(myTrap1(f,#1),#1)]
        -- process change of coordinates
        if (c := option(l,"coordinates" :: Symbol)) case "failed" then
          -- default coordinate transformation
          ff := [makePt2(#1,myTrap1(f,#1))]
        else
          cc : L(Pt -> Pt) := [retract(c :: Any)$ANY1(Pt -> Pt)]
          ff := [(first cc)((first ff)(#1))]
        -- create PLOT
        pl := pointPlot(first ff,normalize seg)
        -- reset adaptive plotting
        setAdaptive(oldAdaptive)$PLOT
        -- draw
        drawPlot(pl,l)

      draw(f:SF -> SF,seg:SEG) == draw(f,seg,nil())

  --% Parametric Plane Curves
```

```
    draw(ppc:PPC,seg:SEG,l:L DROP) ==
      -- set adaptive plotting off or on
      oldAdaptive := adaptive?()$PLOT
      setAdaptive(adaptive(l,oldAdaptive))$PLOT
      -- create function SF -> Point SF
      f := coordinate(ppc,1); g := coordinate(ppc,2)
      fcn : L(SF -> Pt) := [makePt2(myTrap1(f,#1),myTrap1(g,#1))]
      -- process change of coordinates
      if not (c := option(l,"coordinates" :: Symbol)) case "failed" then
        cc : L(Pt -> Pt) := [retract(c :: Any)$ANY1(Pt -> Pt)]
        fcn := [(first cc)((first fcn)(#1))]
      -- create PLOT
      pl := pointPlot(first fcn,normalize seg)
      -- reset adaptive plotting
      setAdaptive(oldAdaptive)$PLOT
      -- draw
      drawPlot(pl,l)

    draw(ppc:PPC,seg:SEG) == draw(ppc,seg,nil())

--------------------------------------------------------------------------
--                      3D - Curves
--------------------------------------------------------------------------

--% functions for creation of maps SF -> Point SF (three dimensional)

    makePt4: (SF,SF,SF,SF) -> Point SF
    makePt4(x,y,z,c) == point(l : List SF := [x,y,z,c])

--% Parametric Space Curves

    id: SF -> SF
    id x == x

    zCoord: (SF,SF,SF) -> SF
    zCoord(x,y,z) == z

    colorPoints: (List List Pt,(SF,SF,SF) -> SF) -> List List Pt
    colorPoints(llp,func) ==
      for lp in llp repeat for p in lp repeat
        p.4 := func(p.1,p.2,p.3)
      llp

    makeObject(psc:PSC,seg:SEG,l:L DROP) ==
      sp := space l
```

```
      -- obtain dependent variable and coordinate functions
      f := coordinate(psc,1); g := coordinate(psc,2); h := coordinate(psc,3)
      -- create function SF -> Point SF with default or user-specified
      -- color function
      fcn : L(SF -> Pt) := [makePt4(myTrap1(f,#1),myTrap1(g,#1),myTrap1(h,#1),_
                           myTrap1(id,#1))]
      pointsColored? : Boolean := false
      if not (c1 := option(l,"colorFunction1" :: Symbol)) case "failed" then
        pointsColored? := true
        fcn := [makePt4(myTrap1(f,#1),myTrap1(g,#1),myTrap1(h,#1),_
                retract(c1 :: Any)$ANY1(SF -> SF)(#1))]
      -- process change of coordinates
      if not (c := option(l,"coordinates" :: Symbol)) case "failed" then
        cc : L(Pt -> Pt) := [retract(c :: Any)$ANY1(Pt -> Pt)]
        fcn := [(first cc)((first fcn)(#1))]
      -- create PLOT
      pl := pointPlot(first fcn,normalize seg)$Plot3D
      -- create ThreeSpace
      s := sp
      -- draw Tube
--      print(pl::OutputForm)
      option?(l,"tubeRadius" :: Symbol) =>
        pts := tubePoints(l,8)
        rad := convert(tubeRadius(l,0.25))@DoubleFloat
        tub := tube(pl,rad,pts)$NumericTubePlot(Plot3D)
        loops := listLoops tub
        -- color points if this has not been done already
        if not pointsColored? then
          if (c3 := option(l,"colorFunction3" :: Symbol)) case "failed"
            then colorPoints(loops,zCoord)  -- default color function
            else colorPoints(loops,retract(c3 :: Any)$ANY1((SF,SF,SF) -> SF))
        mesh(s,loops,false,false)
        s
      -- draw curve
      br := listBranches pl
      for b in br repeat curve(s,b)
      s

    makeObject(psc:PCFUN,seg:SEG,l:L DROP) ==
      sp := space l
      -- create function SF -> Point SF with default or user-specified
      -- color function
      fcn : L(SF -> Pt) := [psc]
      pointsColored? : Boolean := false
      if not (c1 := option(l,"colorFunction1" :: Symbol)) case "failed" then
        pointsColored? := true
```

```
      fcn := [concat(psc(#1), retract(c1 :: Any)$ANY1(SF -> SF)(#1))]
    -- process change of coordinates
    if not (c := option(l,"coordinates" :: Symbol)) case "failed" then
      cc : L(Pt -> Pt) := [retract(c :: Any)$ANY1(Pt -> Pt)]
      fcn := [(first cc)((first fcn)(#1))]
    -- create PLOT
    pl := pointPlot(first fcn,normalize seg)$Plot3D
    -- create ThreeSpace
    s := sp
    -- draw Tube
    option?(l,"tubeRadius" :: Symbol) =>
      pts := tubePoints(l,8)
      rad := convert(tubeRadius(l,0.25))@DoubleFloat
      tub := tube(pl,rad,pts)$NumericTubePlot(Plot3D)
      loops := listLoops tub
      -- color points if this has not been done already
      mesh(s,loops,false,false)
      s
    -- draw curve
    br := listBranches pl
    for b in br repeat curve(s,b)
    s

  makeObject(psc:PSC,seg:SEG) ==
    makeObject(psc,seg,nil())

  makeObject(psc:PCFUN,seg:SEG) ==
    makeObject(psc,seg,nil())

  draw(psc:PSC,seg:SEG,l:L DROP) ==
    sp := makeObject(psc,seg,l)
    makeViewport3D(sp, l)

  draw(psc:PSC,seg:SEG) ==
    draw(psc,seg,nil())

  draw(psc:PCFUN,seg:SEG,l:L DROP) ==
    sp := makeObject(psc,seg,l)
    makeViewport3D(sp, l)

  draw(psc:PCFUN,seg:SEG) ==
    draw(psc,seg,nil())

--------------------------------------------------------------------------
--                      3D - Surfaces
--------------------------------------------------------------------------
```

The function myTrap2 is a local function for used in creating maps SF -¿ Point SF (three dimensional). The range of this function is SingleFloat. As originally coded it would return $NaNvalue$Lisp which is outside the range. Since this function is only used internallly by the draw package we handle the "failed" case by returning zero. We handle the out-of-range case by returning the maximum or minimum SingleFloat value.

⟨*package DRAWCFUN TopLevelDrawFunctionsForCompiledFunctions*⟩+≡

```
    myTrap2: ((SF, SF) -> SF, SF, SF) -> SF
    myTrap2(ff:(SF, SF) -> SF, u:SF, v:SF):SF ==
      s := trapNumericErrors(ff(u, v))$Lisp :: Union(SF, "failed")
      s case "failed" => 0
      r:SF := s::SF
      r >max()$SF => max()$SF
      r < min()$SF => min()$SF
      r

    recolor(ptFunc,colFunc) ==
      pt := ptFunc(#1,#2)
      pt.4 := colFunc(pt.1,pt.2,pt.3)
      pt

    xCoord: (SF,SF) -> SF
    xCoord(x,y) == x

--% Three Dimensional Function Plots

    makeObject(f:(SF,SF) -> SF,xSeg:SEG,ySeg:SEG,l:L DROP) ==
      sp := space l
      -- process color function of two variables
      col2 : L((SF,SF) -> SF) := [xCoord]     -- dummy color function
      pointsColored? : Boolean := false
      if not (c2 := option(l,"colorFunction2" :: Symbol)) case "failed" then
        pointsColored? := true
        col2 := [retract(c2 :: Any)$ANY1((SF,SF) -> SF)]
      fcn : L((SF,SF) -> Pt) :=
        [makePt4(myTrap2(f,#1,#2),#1,#2,(first col2)(#1,#2))]
      -- process change of coordinates
      if (c := option(l,"coordinates" :: Symbol)) case "failed" then
        -- default coordinate transformation
        fcn := [makePt4(#1,#2,myTrap2(f,#1,#2),(first col2)(#1,#2))]
      else
        cc : L(Pt -> Pt) := [retract(c :: Any)$ANY1(Pt -> Pt)]
        fcn := [(first cc)((first fcn)(#1,#2))]
      -- process color function of three variables, if there was no
```

```
      -- color function of two variables
      if not pointsColored? then
        c := option(l,"colorFunction3" :: Symbol)
        fcn :=
          c case "failed" => [recolor((first fcn),zCoord)]
          [recolor((first fcn),retract(c :: Any)$ANY1((SF,SF,SF) -> SF))]
      -- create mesh
      mesh := meshPar2Var(sp,first fcn,normalize xSeg,normalize ySeg,l)
      mesh

    makeObject(f:(SF,SF) -> SF,xSeg:SEG,ySeg:SEG) ==
      makeObject(f,xSeg,ySeg,nil())

    draw(f:(SF,SF) -> SF,xSeg:SEG,ySeg:SEG,l:L DROP) ==
      sp := makeObject(f, xSeg, ySeg, l)
      makeViewport3D(sp, l)

    draw(f:(SF,SF) -> SF,xSeg:SEG,ySeg:SEG) ==
      draw(f,xSeg,ySeg,nil())

  --% parametric surface

    makeObject(s:PSF,uSeg:SEG,vSeg:SEG,l:L DROP) ==
      sp := space l
      -- create functions from expressions
      f : L((SF,SF) -> SF) := [coordinate(s,1)]
      g : L((SF,SF) -> SF) := [coordinate(s,2)]
      h : L((SF,SF) -> SF) := [coordinate(s,3)]
      -- process color function of two variables
      col2 : L((SF,SF) -> SF) := [xCoord]     -- dummy color function
      pointsColored? : Boolean := false
      if not (c2 := option(l,"colorFunction2" :: Symbol)) case "failed" then
        pointsColored? := true
        col2 := [retract(c2 :: Any)$ANY1((SF,SF) -> SF)]
      fcn : L((SF,SF) -> Pt) :=
        [makePt4(myTrap2((first f),#1,#2),myTrap2((first g),#1,#2),myTrap2((first h),#1,#2),
                 myTrap2((first col2),#1,#2))]
      -- process change of coordinates
      if not (c := option(l,"coordinates" :: Symbol)) case "failed" then
        cc : L(Pt -> Pt) := [retract(c :: Any)$ANY1(Pt -> Pt)]
        fcn := [(first cc)((first fcn)(#1,#2))]
      -- process color function of three variables, if there was no
      -- color function of two variables
      if not pointsColored? then
        col3 : L((SF,SF,SF) -> SF) := [zCoord]  -- default color function
        if not (c := option(l,"colorFunction3" :: Symbol)) case "failed" then
```

```
      col3 := [retract(c :: Any)$ANY1((SF,SF,SF) -> SF)]
    fcn := [recolor((first fcn),(first col3))]
  -- create mesh
  mesh := meshPar2Var(sp,first fcn,normalize uSeg,normalize vSeg,l)
  mesh

makeObject(s:PSFUN,uSeg:SEG,vSeg:SEG,l:L DROP) ==
  sp := space l
  -- process color function of two variables
  col2 : L((SF,SF) -> SF) := [xCoord]     -- dummy color function
  pointsColored? : Boolean := false
  if not (c2 := option(l,"colorFunction2" :: Symbol)) case "failed" then
    pointsColored? := true
    col2 := [retract(c2 :: Any)$ANY1((SF,SF) -> SF)]
  fcn : L((SF,SF) -> Pt) :=
    pointsColored? => [concat(s(#1, #2), (first col2)(#1, #2))]
    [s]
  -- process change of coordinates
  if not (c := option(l,"coordinates" :: Symbol)) case "failed" then
    cc : L(Pt -> Pt) := [retract(c :: Any)$ANY1(Pt -> Pt)]
    fcn := [(first cc)((first fcn)(#1,#2))]
  -- create mesh
  mesh := meshPar2Var(sp,first fcn,normalize uSeg,normalize vSeg,l)
  mesh

makeObject(s:PSF,uSeg:SEG,vSeg:SEG) ==
  makeObject(s,uSeg,vSeg,nil())

draw(s:PSF,uSeg:SEG,vSeg:SEG,l:L DROP) ==
  -- draw
  mesh := makeObject(s,uSeg,vSeg,l)
  makeViewport3D(mesh,l)

draw(s:PSF,uSeg:SEG,vSeg:SEG) ==
  draw(s,uSeg,vSeg,nil())

makeObject(s:PSFUN,uSeg:SEG,vSeg:SEG) ==
  makeObject(s,uSeg,vSeg,nil())

draw(s:PSFUN,uSeg:SEG,vSeg:SEG,l:L DROP) ==
  -- draw
  mesh := makeObject(s,uSeg,vSeg,l)
  makeViewport3D(mesh,l)

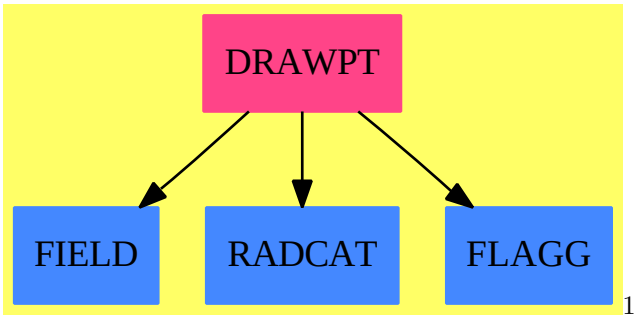
draw(s:PSFUN,uSeg:SEG,vSeg:SEG) ==
  draw(s,uSeg,vSeg,nil())
```

⟨*DRAWCFUN.dotabb*⟩≡

```
"DRAWCFUN" [color="#FF4488",href="bookvol10.4.pdf#nameddest=DRAWCFUN"]
"ALIST" [color="#88FF44",href="bookvol10.3.pdf#nameddest=ALIST"]
"DRAWCFUN" -> "ALIST"
```

21.21 package DRAWPT TopLevelDrawFunctionsForPoints

21.22 TopLevelDrawFunctionsForPoints



1.00

Exports:
draw

⟨*package DRAWPT TopLevelDrawFunctionsForPoints*⟩≡

```
)abbrev package DRAWPT TopLevelDrawFunctionsForPoints
++ Author: Mike Dewar
++ Date Created: 24 May 1995
++ Date Last Updated: 25 November 1996
++ Basic Operations: draw
++ Related Constructors:
++ Also See:
++ AMS Classifications:
++ Keywords:
++ References:
++ Description: TopLevelDrawFunctionsForPoints provides top level functions
++ for drawing curves and surfaces described by sets of points.

TopLevelDrawFunctionsForPoints(): Exports == Implementation where

  DROP  ==> DrawOption
  L     ==> List
  SF    ==> DoubleFloat
  Pt    ==> Point SF
  VIEW2 ==> TwoDimensionalViewport
  VIEW3 ==> ThreeDimensionalViewport

  Exports ==> with
    draw: (L SF,L SF) -> VIEW2
      ++ draw(lx,ly) plots the curve constructed of points (x,y) for x
      ++ in \spad{lx} for y in \spad{ly}.
```

```
    draw: (L SF,L SF,L DROP) -> VIEW2
      ++ draw(lx,ly,l) plots the curve constructed of points (x,y) for x
      ++ in \spad{lx} for y in \spad{ly}.
      ++ The options contained in the list l of
      ++ the domain \spad{DrawOption} are applied.
    draw: (L Pt) -> VIEW2
      ++ draw(lp) plots the curve constructed from the list of points lp.
    draw: (L Pt,L DROP) -> VIEW2
      ++ draw(lp,l) plots the curve constructed from the list of points lp.
      ++ The options contained in the list l of the domain \spad{DrawOption}
      ++ are applied.
    draw: (L SF, L SF, L SF) -> VIEW3
      ++ draw(lx,ly,lz) draws the surface constructed by projecting the values
      ++ in the \axiom{lz} list onto the rectangular grid formed by the
      ++ \axiom{lx x ly}.
    draw: (L SF, L SF, L SF, L DROP) -> VIEW3
      ++ draw(lx,ly,lz,l) draws the surface constructed by
      ++ projecting the values
      ++ in the \axiom{lz} list onto the rectangular grid formed by the
      ++ The options contained in the list l of the domain \spad{DrawOption}
      ++ are applied.

  Implementation ==> add

    draw(lp:L Pt,l:L DROP):VIEW2 ==
      makeViewport2D(makeGraphImage([lp])$GraphImage,l)$VIEW2

    draw(lp:L Pt):VIEW2 == draw(lp,[])

    draw(lx: L SF, ly: L SF, l:L DROP):VIEW2 ==
      draw([point([x,y])$Pt for x in lx for y in ly],l)

    draw(lx: L SF, ly: L SF):VIEW2 == draw(lx,ly,[])

    draw(x:L SF,y:L SF,z:L SF):VIEW3 == draw(x,y,z,[])

    draw(x:L SF,y:L SF,z:L SF,l:L DROP):VIEW3 ==
      m  : Integer := #x
      zero? m => error "No X values"
      n  : Integer := #y
      zero? n => error "No Y values"
      zLen : Integer := #z
      zLen ~= (m*n) =>
        zLen > (m*n) => error "Too many Z-values to fit grid"
        error "Not enough Z-values to fit grid"
      points : L L Pt := []
```

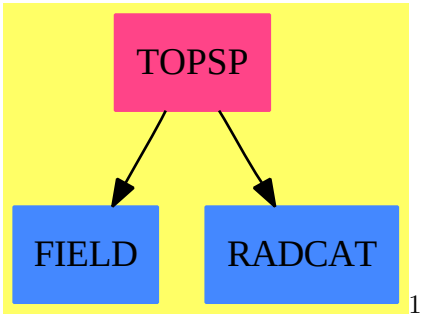
```
        for j in n..1 by -1 repeat
          row : L Pt := []
          for i in m..1 by -1 repeat
            zval := (j-1)*m+i
            row := cons(point([x.i,y.j,z.zval,z.zval]),row)
          points := cons(row,points)
        makeViewport3D(mesh points,l)
```

⟨*DRAWPT.dotabb*⟩≡

```
  "DRAWPT" [color="#FF4488",href="bookvol10.4.pdf#nameddest=DRAWPT"]
  "FIELD"  [color="#4488FF",href="bookvol10.2.pdf#nameddest=FIELD"]
  "RADCAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=RADCAT"]
  "FLAGG" [color="#4488FF",href="bookvol10.2.pdf#nameddest=FLAGG"]
  "DRAWPT" -> "FIELD"
  "DRAWPT" -> "RADCAT"
  "DRAWPT" -> "FLAGG"
```

21.23 package TOPSP TopLevelThreeSpace

21.24 TopLevelThreeSpace



1.00

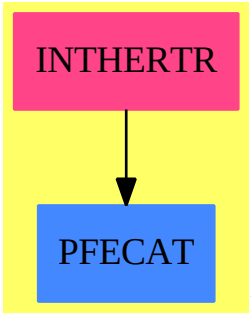
Exports:
createThreeSpace

⟨*package TOPSP TopLevelThreeSpace*⟩≡
```
)abbrev package TOPSP TopLevelThreeSpace
++ Description:
++ This package exports a function for making a \spadtype{ThreeSpace}
TopLevelThreeSpace(): with
    createThreeSpace: () -> ThreeSpace DoubleFloat
      ++ createThreeSpace() creates a \spadtype{ThreeSpace(DoubleFloat)} object
      ++ capable of holding point, curve, mesh components and any combination.
  == add
    createThreeSpace() == create3Space()$ThreeSpace(DoubleFloat)
```

⟨*TOPSP.dotabb*⟩≡
```
"TOPSP" [color="#FF4488",href="bookvol10.4.pdf#nameddest=TOPSP"]
"FIELD"  [color="#4488FF",href="bookvol10.2.pdf#nameddest=FIELD"]
"RADCAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=RADCAT"]
"TOPSP" -> "FIELD"
"TOPSP" -> "RADCAT"
```

21.25 package INTHERTR TranscendentalHermiteIntegration

21.26 TranscendentalHermiteIntegration



1.00

Exports:
HermiteIntegrate

⟨*package INTHERTR TranscendentalHermiteIntegration*⟩≡

```
)abbrev package INTHERTR TranscendentalHermiteIntegration
++ Hermite integration, transcendental case
++ Author: Manuel Bronstein
++ Date Created: 1987
++ Date Last Updated: 12 August 1992
++ Description: Hermite integration, transcendental case.
TranscendentalHermiteIntegration(F, UP): Exports == Implementation where
  F  : Field
  UP : UnivariatePolynomialCategory F

  N   ==> NonNegativeInteger
  RF  ==> Fraction UP
  REC ==> Record(answer:RF, lognum:UP, logden:UP)
  HER ==> Record(answer:RF, logpart:RF, specpart:RF, polypart:UP)

  Exports ==> with
    HermiteIntegrate: (RF, UP -> UP) -> HER
         ++ HermiteIntegrate(f, D) returns \spad{[g, h, s, p]}
         ++ such that \spad{f = Dg + h + s + p},
         ++ h has a squarefree denominator normal w.r.t. D,
         ++ and all the squarefree factors of the denominator of s are
         ++ special w.r.t. D. Furthermore, h and s have no polynomial parts.
         ++ D is the derivation to use on \spadtype{UP}.

  Implementation ==> add
    import MonomialExtensionTools(F, UP)
```

```
    normalHermiteIntegrate: (RF,UP->UP) -> Record(answer:RF,lognum:UP,logden:UP)

    HermiteIntegrate(f, derivation) ==
      rec := decompose(f, derivation)
      hi  := normalHermiteIntegrate(rec.normal, derivation)
      qr  := divide(hi.lognum, hi.logden)
      [hi.answer, qr.remainder / hi.logden, rec.special, qr.quotient + rec.poly]

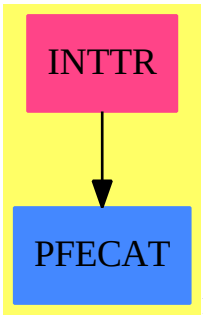
  -- Hermite Reduction on f, every squarefree factor of denom(f) is normal wrt D
  -- this is really a "parallel" Hermite reduction, in the sense that
  -- every multiple factor of the denominator gets reduced at each pass
  -- so if the denominator is P1 P2**2 ... Pn**n, this requires O(n)
  -- reduction steps instead of O(n**2), like Mack's algorithm
  -- (D.Mack, On Rational Integration, Univ. of Utah C.S. Tech.Rep. UCP-38,1975)
  -- returns [g, b, d] s.t. f = g' + b/d and d is squarefree and normal wrt D
    normalHermiteIntegrate(f, derivation) ==
      a := numer f
      q := denom f
      p:UP    := 0
      mult:UP := 1
      qhat := (q exquo (g0 := g := gcd(q, differentiate q)))::UP
      while(degree(qbar := g) > 0) repeat
        qbarhat := (qbar exquo (g := gcd(qbar, differentiate qbar)))::UP
        qtil:= - ((qhat * (derivation qbar)) exquo qbar)::UP
        bc :=
         extendedEuclidean(qtil, qbarhat, a)::Record(coef1:UP, coef2:UP)
        qr := divide(bc.coef1, qbarhat)
        a  := bc.coef2 + qtil * qr.quotient - derivation(qr.remainder)
              * (qhat exquo qbarhat)::UP
        p  := p + mult * qr.remainder
        mult:= mult * qbarhat
      [p / g0, a, qhat]
```

⟨*INTHERTR.dotabb*⟩≡

```
"INTHERTR" [color="#FF4488",href="bookvol10.4.pdf#nameddest=INTHERTR"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"INTHERTR" -> "PFECAT"
```

21.27 package INTTR TranscendentalIntegration

21.28 TranscendentalIntegration



1.00

Exports:

expextendedint	expintegrate
expintfldpoly	explimitedint
monomialIntPoly	monomialIntegrate
primextendedint	primextintfrac
primintegrate	primintfldpoly
primlimintfrac	primlimitedint
tanintegrate	

⟨*package INTTR TranscendentalIntegration*⟩≡

```
)abbrev package INTTR TranscendentalIntegration
++ Risch algorithm, transcendental case
++ Author: Manuel Bronstein
++ Date Created: 1987
++ Date Last Updated: 24 October 1995
++ Description:
++   This package provides functions for the transcendental
++   case of the Risch algorithm.
-- Internally used by the integrator
TranscendentalIntegration(F, UP): Exports == Implementation where
  F  : Field
  UP : UnivariatePolynomialCategory F

  N   ==> NonNegativeInteger
  Z   ==> Integer
  Q   ==> Fraction Z
  GP  ==> LaurentPolynomial(F, UP)
  UP2 ==> SparseUnivariatePolynomial UP
  RF  ==> Fraction UP
  UPR ==> SparseUnivariatePolynomial RF
```

```
IR  ==> IntegrationResult RF
LOG ==> Record(scalar:Q, coeff:UPR, logand:UPR)
LLG ==> List Record(coeff:RF, logand:RF)
NE  ==> Record(integrand:RF, intvar:RF)
NL  ==> Record(mainpart:RF, limitedlogs:LLG)
UPF ==> Record(answer:UP, a0:F)
RFF ==> Record(answer:RF, a0:F)
IRF ==> Record(answer:IR, a0:F)
NLF ==> Record(answer:NL, a0:F)
GPF ==> Record(answer:GP, a0:F)
UPUP==> Record(elem:UP, notelem:UP)
GPGP==> Record(elem:GP, notelem:GP)
RFRF==> Record(elem:RF, notelem:RF)
FF  ==> Record(ratpart:F,  coeff:F)
FFR ==> Record(ratpart:RF, coeff:RF)
UF  ==> Union(FF,  "failed")
UF2 ==> Union(List F, "failed")
REC ==> Record(ir:IR, specpart:RF, polypart:UP)
PSOL==> Record(ans:F, right:F, sol?:Boolean)
FAIL==> error "Sorry - cannot handle that integrand yet"

Exports ==> with
  primintegrate  : (RF, UP -> UP, F -> UF)          -> IRF
    ++ primintegrate(f, ', foo) returns \spad{[g, a]} such that
    ++ \spad{f = g' + a}, and \spad{a = 0} or \spad{a} has no integral in UP.
    ++ Argument foo is an extended integration function on F.
  expintegrate   : (RF, UP -> UP, (Z, F) -> PSOL) -> IRF
    ++ expintegrate(f, ', foo) returns \spad{[g, a]} such that
    ++ \spad{f = g' + a}, and \spad{a = 0} or \spad{a} has no integral in F;
    ++ Argument foo is a Risch differential equation solver on F;
  tanintegrate   : (RF, UP -> UP, (Z, F, F) -> UF2) -> IRF
    ++ tanintegrate(f, ', foo) returns \spad{[g, a]} such that
    ++ \spad{f = g' + a}, and \spad{a = 0} or \spad{a} has no integral in F;
    ++ Argument foo is a Risch differential system solver on F;
  primextendedint:(RF, UP -> UP, F->UF, RF) -> Union(RFF,FFR,"failed")
    ++ primextendedint(f, ', foo, g) returns either \spad{[v, c]} such that
    ++ \spad{f = v' + c g} and \spad{c' = 0}, or \spad{[v, a]} such that
    ++ \spad{f = g' + a}, and \spad{a = 0} or \spad{a} has no integral in UP.
    ++ Returns "failed" if neither case can hold.
    ++ Argument foo is an extended integration function on F.
  expextendedint:(RF,UP->UP,(Z,F)->PSOL, RF) -> Union(RFF,FFR,"failed")
    ++ expextendedint(f, ', foo, g) returns either \spad{[v, c]} such that
    ++ \spad{f = v' + c g} and \spad{c' = 0}, or \spad{[v, a]} such that
    ++ \spad{f = g' + a}, and \spad{a = 0} or \spad{a} has no integral in F.
    ++ Returns "failed" if neither case can hold.
    ++ Argument foo is a Risch differential equation function on F.
```

```
    primlimitedint:(RF, UP -> UP, F->UF, List RF) -> Union(NLF,"failed")
      ++ primlimitedint(f, ', foo, [u1,...,un]) returns
      ++ \spad{[v, [c1,...,cn], a]} such that \spad{ci' = 0},
      ++ \spad{f = v' + a + reduce(+,[ci * ui'/ui])},
      ++ and \spad{a = 0} or \spad{a} has no integral in UP.
      ++ Returns "failed" if no such v, ci, a exist.
      ++ Argument foo is an extended integration function on F.
    explimitedint:(RF, UP->UP,(Z,F)->PSOL,List RF) -> Union(NLF,"failed")
      ++ explimitedint(f, ', foo, [u1,...,un]) returns
      ++ \spad{[v, [c1,...,cn], a]} such that \spad{ci' = 0},
      ++ \spad{f = v' + a + reduce(+,[ci * ui'/ui])},
      ++ and \spad{a = 0} or \spad{a} has no integral in F.
      ++ Returns "failed" if no such v, ci, a exist.
      ++ Argument foo is a Risch differential equation function on F.
    primextintfrac    : (RF, UP -> UP, RF)  -> Union(FFR, "failed")
      ++ primextintfrac(f, ', g) returns \spad{[v, c]} such that
      ++ \spad{f = v' + c g} and \spad{c' = 0}.
      ++ Error: if \spad{degree numer f >= degree denom f} or
      ++ if \spad{degree numer g >= degree denom g} or
      ++ if \spad{denom g} is not squarefree.
    primlimintfrac    : (RF, UP -> UP, List RF) -> Union(NL, "failed")
      ++ primlimintfrac(f, ', [u1,...,un]) returns \spad{[v, [c1,...,cn]]}
      ++ such that \spad{ci' = 0} and \spad{f = v' + +/[ci * ui'/ui]}.
      ++ Error: if \spad{degree numer f >= degree denom f}.
    primintfldpoly    : (UP, F -> UF, F)    -> Union(UP, "failed")
      ++ primintfldpoly(p, ', t') returns q such that \spad{p' = q} or
      ++ "failed" if no such q exists. Argument \spad{t'} is the derivative of
      ++ the primitive generating the extension.
    expintfldpoly     : (GP, (Z, F) -> PSOL) -> Union(GP, "failed")
      ++ expintfldpoly(p, foo) returns q such that \spad{p' = q} or
      ++ "failed" if no such q exists.
      ++ Argument foo is a Risch differential equation function on F.
    monomialIntegrate : (RF, UP -> UP) -> REC
      ++ monomialIntegrate(f, ') returns \spad{[ir, s, p]} such that
      ++ \spad{f = ir' + s + p} and all the squarefree factors of the
      ++ denominator of s are special w.r.t the derivation '.
    monomialIntPoly   : (UP, UP -> UP) -> Record(answer:UP, polypart:UP)
      ++ monomialIntPoly(p, ') returns [q, r] such that
      ++ \spad{p = q' + r} and \spad{degree(r) < degree(t')}.
      ++ Error if \spad{degree(t') < 2}.

  Implementation ==> add
    import SubResultantPackage(UP, UP2)
    import MonomialExtensionTools(F, UP)
    import TranscendentalHermiteIntegration(F, UP)
    import CommuteUnivariatePolynomialCategory(F, UP, UP2)
```

```
    primintegratepoly  : (UP, F -> UF, F) -> Union(UPF, UPUP)
    expintegratepoly   : (GP, (Z, F) -> PSOL) -> Union(GPF, GPGP)
    expextintfrac      : (RF, UP -> UP, RF) -> Union(FFR, "failed")
    explimintfrac      : (RF, UP -> UP, List RF) -> Union(NL, "failed")
    limitedLogs        : (RF, RF -> RF, List RF) -> Union(LLG, "failed")
    logprmderiv        : (RF, UP -> UP)    -> RF
    logexpderiv        : (RF, UP -> UP, F) -> RF
    tanintegratespecial: (RF, RF -> RF, (Z, F, F) -> UF2) -> Union(RFF, RFRF)
    UP2UP2             : UP -> UP2
    UP2UPR             : UP -> UPR
    UP22UPR            : UP2 -> UPR
    notelementary      : REC -> IR
    kappa              : (UP, UP -> UP) -> UP

    dummy:RF := 0

    logprmderiv(f, derivation) == differentiate(f, derivation) / f

    UP2UP2 p ==
      map(#1::UP, p)$UnivariatePolynomialCategoryFunctions2(F, UP, UP, UP2)

    UP2UPR p ==
      map(#1::UP::RF, p)$UnivariatePolynomialCategoryFunctions2(F, UP, RF, UPR)

    UP22UPR p == map(#1::RF, p)$SparseUnivariatePolynomialFunctions2(UP, RF)

-- given p in k[z] and a derivation on k[t] returns the coefficient lifting
-- in k[z] of the restriction of D to k.
    kappa(p, derivation) ==
      ans:UP := 0
      while p ^= 0 repeat
        ans := ans + derivation(leadingCoefficient(p)::UP)*monomial(1,degree p)
        p := reductum p
      ans

-- works in any monomial extension
    monomialIntegrate(f, derivation) ==
      zero? f => [0, 0, 0]
      r := HermiteIntegrate(f, derivation)
      zero?(inum := numer(r.logpart)) => [r.answer::IR, r.specpart, r.polypart]
      iden  := denom(r.logpart)
      x := monomial(1, 1)$UP
      resultvec := subresultantVector(UP2UP2 inum -
                                  (x::UP2) * UP2UP2 derivation iden, UP2UP2 iden)
      respoly := primitivePart leadingCoefficient resultvec 0
```

```
      rec := splitSquarefree(respoly, kappa(#1, derivation))
      logs:List(LOG) := [
              [1, UP2UPR(term.factor),
               UP22UPR swap primitivePart(resultvec(term.exponent),term.factor)]
                    for term in factors(rec.special)]
      dlog :=
--           one? derivation x => r.logpart
          ((derivation x) = 1) => r.logpart
          differentiate(mkAnswer(0, logs, empty()),
                        differentiate(#1, derivation))
      (u := retractIfCan(p := r.logpart - dlog)@Union(UP, "failed")) case UP =>
        [mkAnswer(r.answer, logs, empty), r.specpart, r.polypart + u::UP]
      [mkAnswer(r.answer, logs, [[p, dummy]]), r.specpart, r.polypart]

-- returns [q, r] such that p = q' + r and degree(r) < degree(dt)
-- must have degree(derivation t) >= 2
    monomialIntPoly(p, derivation) ==
      (d := degree(dt := derivation monomial(1,1))::Z) < 2 =>
        error "monomIntPoly: monomial must have degree 2 or more"
      l := leadingCoefficient dt
      ans:UP := 0
      while (n := 1 + degree(p)::Z - d) > 0 repeat
        ans := ans + (term := monomial(leadingCoefficient(p) / (n * l), n::N))
        p   := p - derivation term      -- degree(p) must drop here
      [ans, p]

-- returns either
--   (q in GP, a in F)  st p = q' + a, and a=0 or a has no integral in F
-- or (q in GP, r in GP) st p = q' + r, and r has no integral elem/UP
    expintegratepoly(p, FRDE) ==
      coef0:F := 0
      notelm := answr := 0$GP
      while p ^= 0 repeat
        ans1 := FRDE(n := degree p, a := leadingCoefficient p)
        answr := answr + monomial(ans1.ans, n)
        if ~ans1.sol? then         -- Risch d.e. has no complete solution
               missing := a - ans1.right
               if zero? n then coef0 := missing
                          else notelm := notelm + monomial(missing, n)
        p   := reductum p
      zero? notelm => [answr, coef0]
      [answr, notelm]

-- f is either 0 or of the form p(t)/(1 + t**2)**n
-- returns either
--   (q in RF, a in F)  st f = q' + a, and a=0 or a has no integral in F
```

```
  -- or (q in RF, r in RF) st f = q' + r, and r has no integral elem/UP
      tanintegratespecial(f, derivation, FRDE) ==
        ans:RF := 0
        p := monomial(1, 2)$UP + 1
        while (n := degree(denom f) quo 2) ^= 0 repeat
          r := numer(f) rem p
          a := coefficient(r, 1)
          b := coefficient(r, 0)
          (u := FRDE(n, a, b)) case "failed" => return [ans, f]
          l := u::List(F)
          term:RF := (monomial(first l, 1)$UP + second(l)::UP) / denom f
          ans := ans + term
          f   := f - derivation term    -- the order of the pole at 1+t^2 drops
        zero?(c0 := retract(retract(f)@UP)@F) or
          (u := FRDE(0, c0, 0)) case "failed" => [ans, c0]
        [ans + first(u::List(F))::UP::RF, 0::F]

  -- returns (v in RF, c in RF) s.t. f = v' + cg, and c' = 0, or "failed"
  -- g must have a squarefree denominator (always possible)
  -- g must have no polynomial part and no pole above t = 0
  -- f must have no polynomial part and no pole above t = 0
      expextintfrac(f, derivation, g) ==
        zero? f => [0, 0]
        degree numer f >= degree denom f => error "Not a proper fraction"
        order(denom f,monomial(1,1)) ^= 0 => error "Not integral at t = 0"
        r := HermiteIntegrate(f, derivation)
        zero? g =>
          r.logpart ^= 0 => "failed"
          [r.answer, 0]
        degree numer g >= degree denom g => error "Not a proper fraction"
        order(denom g,monomial(1,1)) ^= 0 => error "Not integral at t = 0"
        differentiate(c := r.logpart / g, derivation) ^= 0 => "failed"
        [r.answer, c]

      limitedLogs(f, logderiv, lu) ==
        zero? f => empty()
        empty? lu => "failed"
        empty? rest lu =>
          logderiv(c0 := f / logderiv(u0 := first lu)) ^= 0 => "failed"
          [[c0, u0]]
        num := numer f
        den := denom f
        l1:List Record(logand2:RF, contrib:UP) :=
  --        [[u, numer v] for u in lu | one? denom(v := den * logderiv u)]
          [[u, numer v] for u in lu | (denom(v := den * logderiv u) = 1)]
        rows := max(degree den,
```

```
                1 + reduce(max, [degree(u.contrib) for u in l1], 0)$List(N))
      m:Matrix(F) := zero(rows, cols := 1 + #l1)
      for i in 0..rows-1 repeat
        for pp in l1 for j in minColIndex m .. maxColIndex m - 1 repeat
          qsetelt_!(m, i + minRowIndex m, j, coefficient(pp.contrib, i))
        qsetelt_!(m,i+minRowIndex m, maxColIndex m, coefficient(num, i))
      m := rowEchelon m
      ans := empty()$LLG
      for i in minRowIndex m .. maxRowIndex m |
       qelt(m, i, maxColIndex m) ^= 0 repeat
        OK := false
        for pp in l1 for j in minColIndex m .. maxColIndex m - 1
         while not OK repeat
          if qelt(m, i, j) ^= 0 then
            OK := true
            c := qelt(m, i, maxColIndex m) / qelt(m, i, j)
            logderiv(c0 := c::UP::RF) ^= 0 => return "failed"
            ans := concat([c0, pp.logand2], ans)
        not OK => return "failed"
      ans

-- returns q in UP s.t. p = q', or "failed"
    primintfldpoly(p, extendedint, t') ==
      (u := primintegratepoly(p, extendedint, t')) case UPUP => "failed"
      u.a0 ^= 0 => "failed"
      u.answer

-- returns q in GP st p = q', or "failed"
    expintfldpoly(p, FRDE) ==
      (u := expintegratepoly(p, FRDE)) case GPGP => "failed"
      u.a0 ^= 0 => "failed"
      u.answer

-- returns (v in RF, c1...cn in RF, a in F) s.t. ci' = 0,
-- and f = v' + a + +/[ci * ui'/ui]
--                                  and a = 0 or a has no integral in UP
    primlimitedint(f, derivation, extendedint, lu) ==
      qr := divide(numer f, denom f)
      (u1 := primlimintfrac(qr.remainder / (denom f), derivation, lu))
        case "failed" => "failed"
      (u2 := primintegratepoly(qr.quotient, extendedint,
               retract derivation monomial(1, 1))) case UPUP => "failed"
      [[u1.mainpart + u2.answer::RF, u1.limitedlogs], u2.a0]

-- returns (v in RF, c1...cn in RF, a in F) s.t. ci' = 0,
-- and f = v' + a + +/[ci * ui'/ui]
```

```
--                                    and a = 0 or a has no integral in F
    explimitedint(f, derivation, FRDE, lu) ==
      qr := separate(f)$GP
      (u1 := explimintfrac(qr.fracPart,derivation, lu)) case "failed" =>
                                                              "failed"
      (u2 := expintegratepoly(qr.polyPart, FRDE)) case GPGP => "failed"
      [[u1.mainpart + convert(u2.answer)@RF, u1.limitedlogs], u2.a0]

-- returns [v, c1...cn] s.t. f = v' + +/[ci * ui'/ui]
-- f must have no polynomial part (degree numer f < degree denom f)
    primlimintfrac(f, derivation, lu) ==
      zero? f => [0, empty()]
      degree numer f >= degree denom f => error "Not a proper fraction"
      r := HermiteIntegrate(f, derivation)
      zero?(r.logpart) => [r.answer, empty()]
      (u := limitedLogs(r.logpart, logprmderiv(#1, derivation), lu))
        case "failed" => "failed"
      [r.answer, u::LLG]

-- returns [v, c1...cn] s.t. f = v' + +/[ci * ui'/ui]
-- f must have no polynomial part (degree numer f < degree denom f)
-- f must be integral above t = 0
    explimintfrac(f, derivation, lu) ==
      zero? f => [0, empty()]
      degree numer f >= degree denom f => error "Not a proper fraction"
      order(denom f, monomial(1,1)) > 0 => error "Not integral at t = 0"
      r  := HermiteIntegrate(f, derivation)
      zero?(r.logpart) => [r.answer, empty()]
      eta' := coefficient(derivation monomial(1, 1), 1)
      (u := limitedLogs(r.logpart, logexpderiv(#1,derivation,eta'), lu))
        case "failed" => "failed"
      [r.answer - eta'::UP *
        +/[((degree numer(v.logand))::Z - (degree denom(v.logand))::Z) *
                                          v.coeff for v in u], u::LLG]

    logexpderiv(f, derivation, eta') ==
      (differentiate(f, derivation) / f) -
            (((degree numer f)::Z - (degree denom f)::Z) * eta')::UP::RF

    notelementary rec ==
      rec.ir + integral(rec.polypart::RF + rec.specpart, monomial(1,1)$UP :: RF)

-- returns
--   (g in IR, a in F)  st f = g'+ a, and a=0 or a has no integral in UP
    primintegrate(f, derivation, extendedint) ==
      rec := monomialIntegrate(f, derivation)
```

```
      not elem?(i1 := rec.ir) => [notelementary rec, 0]
      (u2 := primintegratepoly(rec.polypart, extendedint,
                       retract derivation monomial(1, 1))) case UPUP =>
             [i1 + u2.elem::RF::IR
                 + integral(u2.notelem::RF, monomial(1,1)$UP :: RF), 0]
      [i1 + u2.answer::RF::IR, u2.a0]

  -- returns
  --   (g in IR, a in F)  st f = g' + a, and a = 0 or a has no integral in F
    expintegrate(f, derivation, FRDE) ==
      rec := monomialIntegrate(f, derivation)
      not elem?(i1 := rec.ir) => [notelementary rec, 0]
  -- rec.specpart is either 0 or of the form p(t)/t**n
      special := rec.polypart::GP +
                   (numer(rec.specpart)::GP exquo denom(rec.specpart)::GP)::GP
      (u2 := expintegratepoly(special, FRDE)) case GPGP =>
        [i1 + convert(u2.elem)@RF::IR + integral(convert(u2.notelem)@RF,
                                                 monomial(1,1)$UP :: RF), 0]
      [i1 + convert(u2.answer)@RF::IR, u2.a0]

  -- returns
  --   (g in IR, a in F)  st f = g' + a, and a = 0 or a has no integral in F
    tanintegrate(f, derivation, FRDE) ==
      rec := monomialIntegrate(f, derivation)
      not elem?(i1 := rec.ir) => [notelementary rec, 0]
      r := monomialIntPoly(rec.polypart, derivation)
      t := monomial(1, 1)$UP
      c := coefficient(r.polypart, 1) / leadingCoefficient(derivation t)
      derivation(c::UP) ^= 0 =>
        [i1 + mkAnswer(r.answer::RF, empty(),
                       [[r.polypart::RF + rec.specpart, dummy]$NE]), 0]
      logs:List(LOG) :=
         zero? c => empty()
         [[1, monomial(1,1)$UPR - (c/(2::F))::UP::RF::UPR, (1 + t**2)::RF::UPR]]
      c0 := coefficient(r.polypart, 0)
      (u := tanintegratespecial(rec.specpart, differentiate(#1, derivation),
       FRDE)) case RFRF =>
        [i1 + mkAnswer(r.answer::RF + u.elem, logs, [[u.notelem,dummy]$NE]), c0]
      [i1 + mkAnswer(r.answer::RF + u.answer, logs, empty()), u.a0 + c0]

  -- returns either (v in RF, c in RF) s.t. f = v' + cg, and c' = 0
  --             or (v in RF, a in F)  s.t. f = v' + a
  --                                   and a = 0 or a has no integral in UP
    primextendedint(f, derivation, extendedint, g) ==
      fqr := divide(numer f, denom f)
      gqr := divide(numer g, denom g)
```

```
      (u1 := primextintfrac(fqr.remainder / (denom f), derivation,
                  gqr.remainder / (denom g))) case "failed" => "failed"
      zero?(gqr.remainder) =>
      -- the following FAIL cannot occur if the primitives are all logs
         degree(gqr.quotient) > 0 => FAIL
         (u3 := primintegratepoly(fqr.quotient, extendedint,
               retract derivation monomial(1, 1))) case UPUP => "failed"
         [u1.ratpart + u3.answer::RF, u3.a0]
      (u2 := primintfldpoly(fqr.quotient - retract(u1.coeff)@UP *
        gqr.quotient, extendedint, retract derivation monomial(1, 1)))
          case "failed" => "failed"
      [u2::UP::RF + u1.ratpart, u1.coeff]

-- returns either (v in RF, c in RF) s.t. f = v' + cg, and c' = 0
--             or (v in RF, a in F)  s.t. f = v' + a
--                                   and a = 0 or a has no integral in F
    expextendedint(f, derivation, FRDE, g) ==
      qf := separate(f)$GP
      qg := separate g
      (u1 := expextintfrac(qf.fracPart, derivation, qg.fracPart))
         case "failed" => "failed"
      zero?(qg.fracPart) =>
      --the following FAIL's cannot occur if the primitives are all logs
        retractIfCan(qg.polyPart)@Union(F,"failed") case "failed"=> FAIL
        (u3 := expintegratepoly(qf.polyPart,FRDE)) case GPGP => "failed"
        [u1.ratpart + convert(u3.answer)@RF, u3.a0]
      (u2 := expintfldpoly(qf.polyPart - retract(u1.coeff)@UP :: GP
        * qg.polyPart, FRDE)) case "failed" => "failed"
      [convert(u2::GP)@RF + u1.ratpart, u1.coeff]

-- returns either
--   (q in UP, a in F)  st p = q'+ a, and a=0 or a has no integral in UP
-- or (q in UP, r in UP) st p = q'+ r, and r has no integral elem/UP
    primintegratepoly(p, extendedint, t') ==
      zero? p => [0, 0$F]
      ans:UP := 0
      while (d := degree p) > 0 repeat
        (ans1 := extendedint leadingCoefficient p) case "failed" =>
          return([ans, p])
        p   := reductum p - monomial(d * t' * ans1.ratpart, (d - 1)::N)
        ans := ans + monomial(ans1.ratpart, d)
                           + monomial(ans1.coeff / (d + 1)::F, d + 1)
      (ans1:= extendedint(rp := retract(p)@F)) case "failed" => [ans,rp]
      [monomial(ans1.coeff, 1) + ans1.ratpart::UP + ans, 0$F]

-- returns (v in RF, c in RF) s.t. f = v' + cg, and c' = 0
```

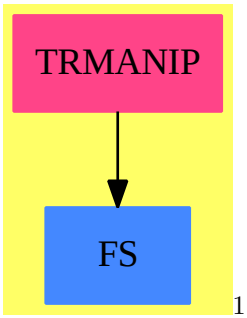
```
  -- g must have a squarefree denominator (always possible)
  -- g must have no polynomial part (degree numer g < degree denom g)
  -- f must have no polynomial part (degree numer f < degree denom f)
      primextintfrac(f, derivation, g) ==
        zero? f => [0, 0]
        degree numer f >= degree denom f => error "Not a proper fraction"
        r := HermiteIntegrate(f, derivation)
        zero? g =>
          r.logpart ^= 0 => "failed"
          [r.answer, 0]
        degree numer g >= degree denom g => error "Not a proper fraction"
        differentiate(c := r.logpart / g, derivation) ^= 0 => "failed"
        [r.answer, c]
```

⟨*INTTR.dotabb*⟩≡

```
"INTTR" [color="#FF4488",href="bookvol10.4.pdf#nameddest=INTTR"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"INTTR" -> "PFECAT"
```

21.29 package TRMANIP TranscendentalManipulations

21.30 TranscendentalManipulations



1.00

Exports:

cos2sec	cosh2sech	cot2tan
cot2trig	coth2tanh	coth2trigh
csc2sin	csch2sinh	expand
expandLog	expandPower	htrigs
removeCosSq	removeCoshSq	removeSinSq
removeSinhSq	sec2cos	sech2cosh
simplify	simplifyExp	simplifyLog
sin2csc	sinh2csch	tan2cot
tan2trig	tanh2coth	tanh2trigh
expandTrigProducts		

⟨*package TRMANIP TranscendentalManipulations*⟩≡

```
)abbrev package TRMANIP TranscendentalManipulations
++ Transformations on transcendental objects
++ Author: Bob Sutor, Manuel Bronstein
++ Date Created: Way back
++ Date Last Updated: 22 January 1996, added simplifyLog MCD.
++ Description:
++   TranscendentalManipulations provides functions to simplify and
++   expand expressions involving transcendental operators.
++ Keywords: transcendental, manipulation.
TranscendentalManipulations(R, F): Exports == Implementation where
  R : Join(OrderedSet, GcdDomain)
  F : Join(FunctionSpace R, TranscendentalFunctionCategory)

  Z       ==> Integer
  K       ==> Kernel F
  P       ==> SparseMultivariatePolynomial(R, K)
  UP      ==> SparseUnivariatePolynomial P
```

```
    POWER   ==> "%power"::Symbol
    POW     ==> Record(val: F,exponent: Z)
    PRODUCT ==> Record(coef : Z, var : K)
    FPR     ==> Fraction Polynomial R

    Exports ==> with
      expand     : F -> F
        ++ expand(f) performs the following expansions on f:\begin{items}
        ++ \item 1. logs of products are expanded into sums of logs,
        ++ \item 2. trigonometric and hyperbolic trigonometric functions
        ++ of sums are expanded into sums of products of trigonometric
        ++ and hyperbolic trigonometric functions.
        ++ \item 3. formal powers of the form \spad{(a/b)**c} are expanded into
        ++ \spad{a**c * b**(-c)}.
        ++ \end{items}
      simplify   : F -> F
        ++ simplify(f) performs the following simplifications on f:\begin{items}
        ++ \item 1. rewrites trigs and hyperbolic trigs in terms
        ++ of \spad{sin} ,\spad{cos}, \spad{sinh}, \spad{cosh}.
        ++ \item 2. rewrites \spad{sin**2} and \spad{sinh**2} in terms
        ++ of \spad{cos} and \spad{cosh},
        ++ \item 3. rewrites \spad{exp(a)*exp(b)} as \spad{exp(a+b)}.
        ++ \item 4. rewrites \spad{(a**(1/n))**m * (a**(1/s))**t} as a single
        ++ power of a single radical of \spad{a}.
        ++ \end{items}
      htrigs     : F -> F
        ++ htrigs(f) converts all the exponentials in f into
        ++ hyperbolic sines and cosines.
      simplifyExp: F -> F
        ++ simplifyExp(f) converts every product \spad{exp(a)*exp(b)}
        ++ appearing in f into \spad{exp(a+b)}.
      simplifyLog : F -> F
        ++ simplifyLog(f) converts every \spad{log(a) - log(b)} appearing in f
        ++ into \spad{log(a/b)}, every \spad{log(a) + log(b)} into \spad{log(a*b)}
        ++ and every \spad{n*log(a)} into \spad{log(a^n)}.
      expandPower: F -> F
        ++ expandPower(f) converts every power \spad{(a/b)**c} appearing
        ++ in f into \spad{a**c * b**(-c)}.
      expandLog  : F -> F
        ++ expandLog(f) converts every \spad{log(a/b)} appearing in f into
        ++ \spad{log(a) - log(b)}, and every \spad{log(a*b)} into
        ++ \spad{log(a) + log(b)}..
      cos2sec    : F -> F
        ++ cos2sec(f) converts every \spad{cos(u)} appearing in f into
        ++ \spad{1/sec(u)}.
      cosh2sech  : F -> F
```

```
    ++ cosh2sech(f) converts every \spad{cosh(u)} appearing in f into
    ++ \spad{1/sech(u)}.
  cot2trig   : F -> F
    ++ cot2trig(f) converts every \spad{cot(u)} appearing in f into
    ++ \spad{cos(u)/sin(u)}.
  coth2trigh : F -> F
    ++ coth2trigh(f) converts every \spad{coth(u)} appearing in f into
    ++ \spad{cosh(u)/sinh(u)}.
  csc2sin    : F -> F
    ++ csc2sin(f) converts every \spad{csc(u)} appearing in f into
    ++ \spad{1/sin(u)}.
  csch2sinh  : F -> F
    ++ csch2sinh(f) converts every \spad{csch(u)} appearing in f into
    ++ \spad{1/sinh(u)}.
  sec2cos    : F -> F
    ++ sec2cos(f) converts every \spad{sec(u)} appearing in f into
    ++ \spad{1/cos(u)}.
  sech2cosh  : F -> F
    ++ sech2cosh(f) converts every \spad{sech(u)} appearing in f into
    ++ \spad{1/cosh(u)}.
  sin2csc    : F -> F
    ++ sin2csc(f) converts every \spad{sin(u)} appearing in f into
    ++ \spad{1/csc(u)}.
  sinh2csch  : F -> F
    ++ sinh2csch(f) converts every \spad{sinh(u)} appearing in f into
    ++ \spad{1/csch(u)}.
  tan2trig   : F -> F
    ++ tan2trig(f) converts every \spad{tan(u)} appearing in f into
    ++ \spad{sin(u)/cos(u)}.
  tanh2trigh : F -> F
    ++ tanh2trigh(f) converts every \spad{tanh(u)} appearing in f into
    ++ \spad{sinh(u)/cosh(u)}.
  tan2cot    : F -> F
    ++ tan2cot(f) converts every \spad{tan(u)} appearing in f into
    ++ \spad{1/cot(u)}.
  tanh2coth  : F -> F
    ++ tanh2coth(f) converts every \spad{tanh(u)} appearing in f into
    ++ \spad{1/coth(u)}.
  cot2tan    : F -> F
    ++ cot2tan(f) converts every \spad{cot(u)} appearing in f into
    ++ \spad{1/tan(u)}.
  coth2tanh  : F -> F
    ++ coth2tanh(f) converts every \spad{coth(u)} appearing in f into
    ++ \spad{1/tanh(u)}.
  removeCosSq: F -> F
    ++ removeCosSq(f) converts every \spad{cos(u)**2} appearing in f into
```

```
      ++ \spad{1 - sin(x)**2}, and also reduces higher
      ++ powers of \spad{cos(u)} with that formula.
    removeSinSq: F -> F
      ++ removeSinSq(f) converts every \spad{sin(u)**2} appearing in f into
      ++ \spad{1 - cos(x)**2}, and also reduces higher powers of
      ++ \spad{sin(u)} with that formula.
    removeCoshSq:F -> F
      ++ removeCoshSq(f) converts every \spad{cosh(u)**2} appearing in f into
      ++ \spad{1 - sinh(x)**2}, and also reduces higher powers of
      ++ \spad{cosh(u)} with that formula.
    removeSinhSq:F -> F
      ++ removeSinhSq(f) converts every \spad{sinh(u)**2} appearing in f into
      ++ \spad{1 - cosh(x)**2}, and also reduces higher powers
      ++ of \spad{sinh(u)} with that formula.
    if R has PatternMatchable(R) and R has ConvertibleTo(Pattern(R)) and F has Co
      expandTrigProducts : F -> F
        ++ expandTrigProducts(e) replaces \axiom{sin(x)*sin(y)} by
        ++ \spad{(cos(x-y)-cos(x+y))/2}, \axiom{cos(x)*cos(y)} by
        ++ \spad{(cos(x-y)+cos(x+y))/2}, and \axiom{sin(x)*cos(y)} by
        ++ \spad{(sin(x-y)+sin(x+y))/2}.  Note that this operation uses
        ++ the pattern matcher and so is relatively expensive.  To avoid
        ++ getting into an infinite loop the transformations are applied
        ++ at most ten times.

  Implementation ==> add
    import FactoredFunctions(P)
    import PolynomialCategoryLifting(IndexedExponents K, K, R, P, F)
    import
      PolynomialCategoryQuotientFunctions(IndexedExponents K,K,R,P,F)

    smpexp    : P -> F
    termexp   : P -> F
    exlog     : P -> F
    smplog    : P -> F
    smpexpand : P -> F
    smp2htrigs: P -> F
    kerexpand : K -> F
    expandpow : K -> F
    logexpand : K -> F
    sup2htrigs: (UP, F) -> F
    supexp    : (UP, F, F, Z) -> F
    ueval     : (F, String, F -> F) -> F
    ueval2    : (F, String, F -> F) -> F
    powersimp : (P, List K) -> F
    t2t       : F -> F
    c2t       : F -> F
```

```
c2s       : F -> F
s2c       : F -> F
s2c2      : F -> F
th2th     : F -> F
ch2th     : F -> F
ch2sh     : F -> F
sh2ch     : F -> F
sh2ch2    : F -> F
simplify0 : F -> F
simplifyLog1 : F -> F
logArgs   : List F -> F

import F
import List F

if R has PatternMatchable R and R has ConvertibleTo Pattern R and F has ConvertibleTo(Pa
  XX : F := coerce new()$Symbol
  YY : F := coerce new()$Symbol
  sinCosRule : RewriteRule(R,R,F) :=
    rule(cos(XX)*sin(YY),(sin(XX+YY)-sin(XX-YY))/2::F)
  sinSinRule : RewriteRule(R,R,F) :=
    rule(sin(XX)*sin(YY),(cos(XX-YY)-cos(XX+YY))/2::F)
  cosCosRule : RewriteRule(R,R,F) :=
    rule(cos(XX)*cos(YY),(cos(XX-YY)+cos(XX+YY))/2::F)
  expandTrigProducts(e:F):F ==
    applyRules([sinCosRule,sinSinRule,cosCosRule],e,10)$ApplyRules(R,R,F)

logArgs(l:List F):F ==
  -- This function will take a list of Expressions (implicitly a sum) and
  -- add them up, combining log terms.  It also replaces n*log(x) by
  -- log(x^n).
  import K
  sum  : F := 0
  arg  : F := 1
  for term in l repeat
    is?(term,"log"::Symbol) =>
      arg := arg * simplifyLog(first(argument(first(kernels(term)))))
    -- Now look for multiples, including negative ones.
    prod : Union(PRODUCT, "failed") := isMult(term)
    (prod case PRODUCT) and is?(prod.var,"log"::Symbol) =>
        arg := arg * simplifyLog ((first argument(prod.var))**(prod.coef))
    sum := sum+term
  sum+log(arg)

simplifyLog(e:F):F ==
  simplifyLog1(numerator e)/simplifyLog1(denominator e)
```

```
    simplifyLog1(e:F):F ==
      freeOf?(e,"log"::Symbol) => e

      -- Check for n*log(u)
      prod : Union(PRODUCT, "failed") := isMult(e)
      (prod case PRODUCT) and is?(prod.var,"log"::Symbol) =>
        log simplifyLog ((first argument(prod.var))**(prod.coef))

      termList : Union(List(F),"failed") := isTimes(e)
      -- I'm using two variables, termList and terms, to work round a
      -- bug in the old compiler.
      not (termList case "failed") =>
        -- We want to simplify each log term in the product and then multiply
        -- them together.  However, if there is a constant or arithmetic
        -- expression (i.e. somwthing which looks like a Polynomial) we would
        -- like to combine it with a log term.
        terms :List F := [simplifyLog(term) for term in termList::List(F)]
        exprs :List F := []
        for i in 1..#terms repeat
          if retractIfCan(terms.i)@Union(FPR,"failed") case FPR then
            exprs := cons(terms.i,exprs)
            terms := delete!(terms,i)
        if not empty? exprs then
          foundLog := false
          i : NonNegativeInteger := 0
          while (not(foundLog) and (i < #terms)) repeat
            i := i+1
            if is?(terms.i,"log"::Symbol) then
              args : List F := argument(retract(terms.i)@K)
              setelt(terms,i, log simplifyLog1(first(args)**(*/exprs)))
              foundLog := true
          -- The next line deals with a situation which shouldn't occur,
          -- since we have checked whether we are freeOf log already.
          if not foundLog then terms := append(exprs,terms)
        */terms

      terms : Union(List(F),"failed") := isPlus(e)
      not (terms case "failed") => logArgs(terms)

      expt : Union(POW, "failed") := isPower(e)
--      (expt case POW) and not one? expt.exponent =>
      (expt case POW) and not (expt.exponent = 1) =>
        simplifyLog(expt.val)**(expt.exponent)

      kers : List K := kernels e
```

```
--      not(one?(#kers)) => e -- Have a constant
      not(((#kers) = 1)) => e -- Have a constant
      kernel(operator first kers,[simplifyLog(u) for u in argument first kers])


    if R has RetractableTo Integer then
      simplify x == rootProduct(simplify0 x)$AlgebraicManipulations(R,F)

    else simplify x == simplify0 x

    expandpow k ==
      a := expandPower first(arg := argument k)
      b := expandPower second arg
--      ne:F := (one? numer a => 1; numer(a)::F ** b)
      ne:F := (((numer a) = 1) => 1; numer(a)::F ** b)
--      de:F := (one? denom a => 1; denom(a)::F ** (-b))
      de:F := (((denom a) = 1) => 1; denom(a)::F ** (-b))
      ne * de

    termexp p ==
      exponent:F := 0
      coef := (leadingCoefficient p)::P
      lpow := select(is?(#1, POWER)$K, lk := variables p)$List(K)
      for k in lk repeat
        d := degree(p, k)
        if is?(k, "exp"::Symbol) then
          exponent := exponent + d * first argument k
        else if not is?(k, POWER) then
          -- Expand arguments to functions as well ... MCD 23/1/97
          --coef := coef * monomial(1, k, d)
          coef := coef * monomial(1, kernel(operator k,[simplifyExp u for u in argument k],
      coef::F * exp exponent * powersimp(p, lpow)

    expandPower f ==
      l := select(is?(#1, POWER)$K, kernels f)$List(K)
      eval(f, l, [expandpow k for k in l])

-- l is a list of pure powers appearing as kernels in p
    powersimp(p, l) ==
      empty? l => 1
      k := first l                          -- k = a**b
      a := first(arg := argument k)
      exponent := degree(p, k) * second arg
      empty?(lk := select(a = first argument #1, rest l)) =>
        (a ** exponent) * powersimp(p, rest l)
      for k0 in lk repeat
```

```
    exponent := exponent + degree(p, k0) * second argument k0
  (a ** exponent) * powersimp(p, setDifference(rest l, lk))

t2t x         == sin(x) / cos(x)
c2t x         == cos(x) / sin(x)
c2s x         == inv sin x
s2c x         == inv cos x
s2c2 x        == 1 - cos(x)**2
th2th x       == sinh(x) / cosh(x)
ch2th x       == cosh(x) / sinh(x)
ch2sh x       == inv sinh x
sh2ch x       == inv cosh x
sh2ch2 x      == cosh(x)**2 - 1
ueval(x, s,f) == eval(x, s::Symbol, f)
ueval2(x,s,f) == eval(x, s::Symbol, 2, f)
cos2sec x     == ueval(x, "cos", inv sec #1)
sin2csc x     == ueval(x, "sin", inv csc #1)
csc2sin x     == ueval(x, "csc", c2s)
sec2cos x     == ueval(x, "sec", s2c)
tan2cot x     == ueval(x, "tan", inv cot #1)
cot2tan x     == ueval(x, "cot", inv tan #1)
tan2trig x    == ueval(x, "tan", t2t)
cot2trig x    == ueval(x, "cot", c2t)
cosh2sech x   == ueval(x, "cosh", inv sech #1)
sinh2csch x   == ueval(x, "sinh", inv csch #1)
csch2sinh x   == ueval(x, "csch", ch2sh)
sech2cosh x   == ueval(x, "sech", sh2ch)
tanh2coth x   == ueval(x, "tanh", inv coth #1)
coth2tanh x   == ueval(x, "coth", inv tanh #1)
tanh2trigh x  == ueval(x, "tanh", th2th)
coth2trigh x  == ueval(x, "coth", ch2th)
removeCosSq x == ueval2(x, "cos", 1 - (sin #1)**2)
removeSinSq x == ueval2(x, "sin", s2c2)
removeCoshSq x== ueval2(x, "cosh", 1 + (sinh #1)**2)
removeSinhSq x== ueval2(x, "sinh", sh2ch2)
expandLog x   == smplog(numer x) / smplog(denom x)
simplifyExp x == (smpexp numer x) / (smpexp denom x)
expand x      == (smpexpand numer x) / (smpexpand denom x)
smpexpand p   == map(kerexpand, #1::F, p)
smplog p      == map(logexpand, #1::F, p)
smp2htrigs p  == map(htrigs(#1::F), #1::F, p)

htrigs f ==
  (m := mainKernel f) case "failed" => f
  op  := operator(k := m::K)
  arg := [htrigs x for x in argument k]$List(F)
```

```
    num := univariate(numer f, k)
    den := univariate(denom f, k)
    is?(op, "exp"::Symbol) =>
      g1 := cosh(a := first arg) + sinh(a)
      g2 := cosh(a) - sinh(a)
      supexp(num,g1,g2,b:= (degree num)::Z quo 2)/supexp(den,g1,g2,b)
    sup2htrigs(num, g1:= op arg) / sup2htrigs(den, g1)

  supexp(p, f1, f2, bse) ==
    ans:F := 0
    while p ^= 0 repeat
      g := htrigs(leadingCoefficient(p)::F)
      if ((d := degree(p)::Z - bse) >= 0) then
           ans := ans + g * f1 ** d
      else ans := ans + g * f2 ** (-d)
      p := reductum p
    ans

  sup2htrigs(p, f) ==
    (map(smp2htrigs, p)$SparseUnivariatePolynomialFunctions2(P, F)) f

  exlog p == +/[r.coef * log(r.logand::F) for r in log squareFree p]

  logexpand k ==
    nullary?(op := operator k) => k::F
    is?(op, "log"::Symbol) =>
       exlog(numer(x := expandLog first argument k)) - exlog denom x
    op [expandLog x for x in argument k]$List(F)

  kerexpand k ==
    nullary?(op := operator k) => k::F
    is?(op, POWER) => expandpow k
    arg := first argument k
    is?(op, "sec"::Symbol) => inv expand cos arg
    is?(op, "csc"::Symbol) => inv expand sin arg
    is?(op, "log"::Symbol) =>
       exlog(numer(x := expand arg)) - exlog denom x
    num := numer arg
    den := denom arg
    (b := (reductum num) / den) ^= 0 =>
      a := (leadingMonomial num) / den
      is?(op, "exp"::Symbol) => exp(expand a) * expand(exp b)
      is?(op, "sin"::Symbol) =>
         sin(expand a) * expand(cos b) + cos(expand a) * expand(sin b)
      is?(op, "cos"::Symbol) =>
         cos(expand a) * expand(cos b) - sin(expand a) * expand(sin b)
```

```
          is?(op, "tan"::Symbol) =>
            ta := tan expand a
            tb := expand tan b
            (ta + tb) / (1 - ta * tb)
          is?(op, "cot"::Symbol) =>
            cta := cot expand a
            ctb := expand cot b
            (cta * ctb - 1) / (ctb + cta)
          op [expand x for x in argument k]$List(F)
        op [expand x for x in argument k]$List(F)

      smpexp p ==
        ans:F := 0
        while p ^= 0 repeat
          ans := ans + termexp leadingMonomial p
          p   := reductum p
        ans

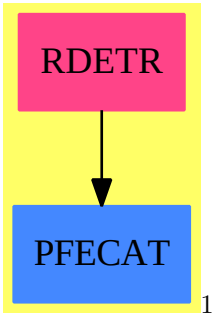
      -- this now works in 3 passes over the expression:
      --   pass1 rewrites trigs and htrigs in terms of sin,cos,sinh,cosh
      --   pass2 rewrites sin**2 and sinh**2 in terms of cos and cosh.
      --   pass3 groups exponentials together
      simplify0 x ==
        simplifyExp eval(eval(x,
            ["tan"::Symbol,"cot"::Symbol,"sec"::Symbol,"csc"::Symbol,
             "tanh"::Symbol,"coth"::Symbol,"sech"::Symbol,"csch"::Symbol],
                [t2t,c2t,s2c,c2s,th2th,ch2th,sh2ch,ch2sh]),
                  ["sin"::Symbol, "sinh"::Symbol], [2, 2], [s2c2, sh2ch2])
```

⟨*TRMANIP.dotabb*⟩≡

```
"TRMANIP" [color="#FF4488",href="bookvol10.4.pdf#nameddest=TRMANIP"]
"FS" [color="#4488FF",href="bookvol10.2.pdf#nameddest=FS"]
"TRMANIP" -> "FS"
```

21.31 package RDETR TranscendentalRischDE

21.32 TranscendentalRischDE



1.00

Exports:
baseRDE monomRDE polyRDE

⟨*package RDETR TranscendentalRischDE*⟩≡
```
)abbrev package RDETR TranscendentalRischDE
++ Risch differential equation, transcendental case.
++ Author: Manuel Bronstein
++ Date Created: Jan 1988
++ Date Last Updated: 2 November 1995
TranscendentalRischDE(F, UP): Exports == Implementation where
  F  : Join(Field, CharacteristicZero, RetractableTo Integer)
  UP : UnivariatePolynomialCategory F

  N   ==> NonNegativeInteger
  Z   ==> Integer
  RF  ==> Fraction UP
  REC ==> Record(a:UP, b:UP, c:UP, t:UP)
  SPE ==> Record(b:UP, c:UP, m:Z, alpha:UP, beta:UP)
  PSOL==> Record(ans:UP, nosol:Boolean)
  ANS ==> Union(ans:PSOL, eq:SPE)
  PSQ ==> Record(ans:RF, nosol:Boolean)

  Exports ==> with
    monomRDE: (RF,RF,UP->UP) -> Union(Record(a:UP,b:RF,c:RF,t:UP), "failed")
      ++ monomRDE(f,g,D) returns \spad{[A, B, C, T]} such that
      ++ \spad{y' + f y = g} has a solution if and only if \spad{y = Q / T},
      ++ where Q satisfies \spad{A Q' + B Q = C} and has no normal pole.
      ++ A and T are polynomials and B and C have no normal poles.
      ++ D is the derivation to use.
    baseRDE : (RF, RF) -> PSQ
      ++ baseRDE(f, g) returns a \spad{[y, b]} such that \spad{y' + fy = g}
```

```
      ++ if \spad{b = true}, y is a partial solution otherwise (no solution
      ++ in that case).
      ++ D is the derivation to use.
    polyRDE : (UP, UP, UP, Z, UP -> UP) -> ANS
      ++ polyRDE(a, B, C, n, D) returns either:
      ++ 1. \spad{[Q, b]} such that \spad{degree(Q) <= n} and
      ++    \spad{a Q'+ B Q = C} if \spad{b = true}, Q is a partial solution
      ++    otherwise.
      ++ 2. \spad{[B1, C1, m, \alpha, \beta]} such that any polynomial solution
      ++    of degree at most n of \spad{A Q' + BQ = C} must be of the form
      ++    \spad{Q = \alpha H + \beta} where \spad{degree(H) <= m} and
      ++    H satisfies \spad{H' + B1 H = C1}.
      ++ D is the derivation to use.

  Implementation ==> add
    import MonomialExtensionTools(F, UP)

    getBound     : (UP, UP, Z) -> Z
    SPDEnocancel1: (UP, UP, Z, UP -> UP) -> PSOL
    SPDEnocancel2: (UP, UP, Z, Z, F, UP -> UP) -> ANS
    SPDE         : (UP, UP, UP, Z, UP -> UP) -> Union(SPE, "failed")

-- cancellation at infinity is possible, A is assumed nonzero
-- needs tagged union because of branch choice problem
-- always returns a PSOL in the base case (never a SPE)
    polyRDE(aa, bb, cc, d, derivation) ==
      n:Z
      (u := SPDE(aa, bb, cc, d, derivation)) case "failed" => [[0, true]]
      zero?(u.c) => [[u.beta, false]]
--      baseCase? := one?(dt := derivation monomial(1, 1))
      baseCase? := ((dt := derivation monomial(1, 1)) = 1)
      n := degree(dt)::Z - 1
      b0? := zero?(u.b)
      (~b0?) and (baseCase? or degree(u.b) > max(0, n)) =>
          answ := SPDEnocancel1(u.b, u.c, u.m, derivation)
          [[u.alpha * answ.ans + u.beta, answ.nosol]]
      (n > 0) and (b0? or degree(u.b) < n) =>
          uansw := SPDEnocancel2(u.b,u.c,u.m,n,leadingCoefficient dt,derivation)
          uansw case ans=> [[u.alpha * uansw.ans.ans + u.beta, uansw.ans.nosol]]
          [[uansw.eq.b, uansw.eq.c, uansw.eq.m,
            u.alpha * uansw.eq.alpha, u.alpha * uansw.eq.beta + u.beta]]
      b0? and baseCase? =>
          degree(u.c) >= u.m => [[0, true]]
          [[u.alpha * integrate(u.c) + u.beta, false]]
      [u::SPE]
```

```
-- cancellation at infinity is possible, A is assumed nonzero
-- if u.b = 0 then u.a = 1 already, but no degree check is done
-- returns "failed" if a p' + b p = c has no soln of degree at most d,
-- otherwise [B, C, m, \alpha, \beta] such that any soln p of degree at
-- most d of  a p' + b p = c  must be of the form p = \alpha h + \beta,
-- where h' + B h = C and h has degree at most m
    SPDE(aa, bb, cc, d, derivation) ==
      zero? cc => [0, 0, 0, 0, 0]
      d < 0 => "failed"
      (u := cc exquo (g := gcd(aa, bb))) case "failed" => "failed"
      aa := (aa exquo g)::UP
      bb := (bb exquo g)::UP
      cc := u::UP
      (ra := retractIfCan(aa)@Union(F, "failed")) case F =>
        a1 := inv(ra::F)
        [a1 * bb, a1 * cc, d, 1, 0]
      bc := extendedEuclidean(bb, aa, cc)::Record(coef1:UP, coef2:UP)
      qr := divide(bc.coef1, aa)
      r  := qr.remainder         -- z = bc.coef2 + b * qr.quotient
      (v  := SPDE(aa, bb + derivation aa,
                  bc.coef2 + bb * qr.quotient - derivation r,
                   d - degree(aa)::Z, derivation)) case "failed" => "failed"
      [v.b, v.c, v.m, aa * v.alpha, aa * v.beta + r]

-- solves q' + b q = c  with deg(q) <= d
-- case (B <> 0) and (D = d/dt or degree(B) > max(0, degree(Dt) - 1))
-- this implies no cancellation at infinity, BQ term dominates
-- returns [Q, flag] such that Q is a solution if flag is false,
-- a partial solution otherwise.
    SPDEnocancel1(bb, cc, d, derivation) ==
      q:UP := 0
      db := (degree bb)::Z
      lb := leadingCoefficient bb
      while cc ^= 0 repeat
        d < 0 or (n := (degree cc)::Z - db) < 0 or n > d => return [q, true]
        r := monomial((leadingCoefficient cc) / lb, n::N)
        cc := cc - bb * r - derivation r
        d := n - 1
        q := q + r
      [q, false]

-- case (t is a nonlinear monomial) and (B = 0 or degree(B) < degree(Dt) - 1)
-- this implies no cancellation at infinity, DQ term dominates or degree(Q) = 0
-- dtm1 = degree(Dt) - 1
    SPDEnocancel2(bb, cc, d, dtm1, lt, derivation) ==
      q:UP := 0
```

```
      while cc ^= 0 repeat
        d < 0 or (n := (degree cc)::Z - dtm1) < 0 or n > d => return [[q, true]]
        if n > 0 then
          r  := monomial((leadingCoefficient cc) / (n * lt), n::N)
          cc := cc - bb * r - derivation r
          d  := n - 1
          q  := q + r
        else        -- n = 0 so solution must have degree 0
          db:N := (zero? bb => 0; degree bb);
          db ^= degree(cc) => return [[q, true]]
          zero? db => return [[bb, cc, 0, 1, q]]
          r  := leadingCoefficient(cc) / leadingCoefficient(bb)
          cc := cc - r * bb - derivation(r::UP)
          d  := - 1
          q := q + r::UP
      [[q, false]]

    monomRDE(f, g, derivation) ==
      gg := gcd(d := normalDenom(f,derivation), e := normalDenom(g,derivation))
      tt := (gcd(e, differentiate e) exquo gcd(gg,differentiate gg))::UP
      (u := ((tt * (aa := d * tt)) exquo e)) case "failed" => "failed"
      [aa, aa * f - (d * derivation tt)::RF, u::UP * e * g, tt]

-- solve y' + f y = g for y in RF
-- assumes that f is weakly normalized (no finite cancellation)
-- base case: F' = 0
    baseRDE(f, g) ==
      (u := monomRDE(f, g, differentiate)) case "failed" => [0, true]
      n := getBound(u.a,bb := retract(u.b)@UP,degree(cc := retract(u.c)@UP)::Z)
      v := polyRDE(u.a, bb, cc, n, differentiate).ans
      [v.ans / u.t, v.nosol]

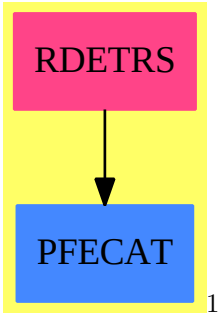
-- return an a bound on the degree of a solution of A P'+ B P = C,A ^= 0
-- cancellation at infinity is possible
-- base case: F' = 0
    getBound(a, b, dc) ==
      da := (degree a)::Z
      zero? b => max(0, dc - da + 1)
      db := (degree b)::Z
      da > (db + 1) => max(0, dc - da + 1)
      da < (db + 1) => dc - db
      (n := retractIfCan(- leadingCoefficient(b) / leadingCoefficient(a)
                      )@Union(Z, "failed")) case Z => max(n::Z, dc - db)
      dc - db
```

⟨*RDETR.dotabb*⟩≡

```
"RDETR" [color="#FF4488",href="bookvol10.4.pdf#nameddest=RDETR"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"RDETR" -> "PFECAT"
```

21.33 package RDETRS TranscendentalRischDESystem

21.34 TranscendentalRischDESystem



1.00

Exports:
baseRDEsys monomRDEsys

⟨*package RDETRS TranscendentalRischDESystem*⟩≡

```
)abbrev package RDETRS TranscendentalRischDESystem
++ Risch differential equation system, transcendental case.
++ Author: Manuel Bronstein
++ Date Created: 17 August 1992
++ Date Last Updated: 3 February 1994
TranscendentalRischDESystem(F, UP): Exports == Implementation where
  F  : Join(Field, CharacteristicZero, RetractableTo Integer)
  UP : UnivariatePolynomialCategory F

  N   ==> NonNegativeInteger
  Z   ==> Integer
  RF  ==> Fraction UP
  V   ==> Vector UP
  U   ==> Union(List UP, "failed")
  REC ==> Record(z1:UP, z2:UP, r1:UP, r2:UP)

  Exports ==> with
    monomRDEsys: (RF, RF, RF, UP -> UP) -> _
                    Union(Record(a:UP, b:RF, h:UP, c1:RF, c2:RF, t:UP),"failed")
      ++ monomRDEsys(f,g1,g2,D) returns \spad{[A, B, H, C1, C2, T]} such that
      ++ \spad{(y1', y2') + ((0, -f), (f, 0)) (y1,y2) = (g1,g2)} has a solution
      ++ if and only if \spad{y1 = Q1 / T, y2 = Q2 / T},
      ++ where \spad{B,C1,C2,Q1,Q2} have no normal poles and satisfy
      ++ A \spad{(Q1', Q2') + ((H, -B), (B, H)) (Q1,Q2) = (C1,C2)}
      ++ D is the derivation to use.
    baseRDEsys: (RF, RF, RF)   -> Union(List RF, "failed")
```

```
      ++ baseRDEsys(f, g1, g2) returns fractions \spad{y_1.y_2} such that
      ++ \spad{(y1', y2') + ((0, -f), (f, 0)) (y1,y2) = (g1,g2)}
      ++ if \spad{y_1,y_2} exist, "failed" otherwise.

  Implementation ==> add
    import MonomialExtensionTools(F, UP)
    import SmithNormalForm(UP, V, V, Matrix UP)

    diophant: (UP, UP, UP, UP, UP) -> Union(REC, "failed")
    getBound: (UP, UP, UP, UP, UP) -> Z
    SPDEsys : (UP, UP, UP, UP, UP, Z, UP -> UP, (F, F, F, UP, UP, Z) -> U) -> U
    DSPDEsys: (F, UP, UP, UP, UP, Z, UP -> UP) -> U
    DSPDEmix: (UP, UP, F, F, N, Z, F) -> U
    DSPDEhdom: (UP, UP, F, F, N, Z) -> U
    DSPDEbdom: (UP, UP, F, F, N, Z) -> U
    DSPDEsys0: (F, UP, UP, UP, UP, F, F, Z, UP -> UP, (UP,UP,F,F,N) -> U) -> U

-- reduces (y1', y2') + ((0, -f), (f, 0)) (y1,y2) = (g1,g2) to
-- A (Q1', Q2') + ((H, -B), (B, H)) (Q1,Q2) = (C1,C2), Q1 = y1 T, Q2 = y2 T
-- where A and H are polynomials, and B,C1,C2,Q1 and Q2 have no normal poles.
-- assumes that f is weakly normalized (no finite cancellation)
    monomRDEsys(f, g1, g2, derivation) ==
      gg := gcd(d := normalDenom(f, derivation),
                e := lcm(normalDenom(g1,derivation),normalDenom(g2,derivation)))
      tt := (gcd(e, differentiate e) exquo gcd(gg,differentiate gg))::UP
      (u := ((tt * (aa := d * tt)) exquo e)) case "failed" => "failed"
      [aa, tt * d * f, - d * derivation tt, u::UP * e * g1, u::UP * e * g2, tt]

-- solve (y1', y2') + ((0, -f), (f, 0)) (y1,y2) = (g1,g2) for y1,y2 in RF
-- assumes that f is weakly normalized (no finite cancellation) and nonzero
-- base case: F' = 0
    baseRDEsys(f, g1, g2) ==
      zero? f => error "baseRDEsys: f must be nonzero"
      zero? g1 and zero? g2 => [0, 0]
      (u := monomRDEsys(f, g1, g2, differentiate)) case "failed" => "failed"
      n := getBound(u.a, bb := retract(u.b), u.h,
                    cc1 := retract(u.c1), cc2 := retract(u.c2))
      (v := SPDEsys(u.a, bb, u.h, cc1, cc2, n, differentiate,
                    DSPDEsys(#1, #2::UP, #3::UP, #4, #5, #6, differentiate)))
                          case "failed" => "failed"
      l := v::List(UP)
      [first(l) / u.t, second(l) / u.t]

-- solve
--   D1 = A Z1 + B R1 - C R2
--   D2 = A Z2 + C R1 + B R2
```

```
  -- i.e. (D1,D2) = ((A, 0, B, -C), (0, A, C, B)) (Z1, Z2, R1, R2)
  -- for R1, R2 with degree(Ri) < degree(A)
  -- assumes (A,B,C) = (1) and A and C are nonzero
      diophant(a, b, c, d1, d2) ==
        (u := diophantineSystem(matrix [[a,0,b,-c], [0,a,c,b]],
                            vector [d1,d2]).particular) case "failed" => "failed"
        v := u::V
        qr1 := divide(v 3, a)
        qr2 := divide(v 4, a)
        [v.1 + b * qr1.quotient - c * qr2.quotient,
         v.2 + c * qr1.quotient + b * qr2.quotient, qr1.remainder, qr2.remainder]

  -- solve
  -- A (Q1', Q2') + ((H, -B), (B, H)) (Q1,Q2) = (C1,C2)
  -- for polynomials Q1 and Q2 with degree <= n
  -- A and B are nonzero
  -- cancellation at infinity is possible
      SPDEsys(a, b, h, c1, c2, n, derivation, degradation) ==
        zero? c1 and zero? c2 => [0, 0]
        n < 0 => "failed"
        g := gcd(a, gcd(b, h))
        ((u1 := c1 exquo g) case "failed") or
          ((u2 := c2 exquo g) case "failed") => "failed"
        a := (a exquo g)::UP
        b := (b exquo g)::UP
        h := (h exquo g)::UP
        c1 := u1::UP
        c2 := u2::UP
        (da := degree a) > 0 =>
          (u := diophant(a, h, b, c1, c2)) case "failed" => "failed"
          rec := u::REC
          v := SPDEsys(a, b, h + derivation a, rec.z1 - derivation(rec.r1),
                       rec.z2 - derivation(rec.r2),n-da::Z,derivation,degradation)
          v case "failed" => "failed"
          l := v::List(UP)
          [a * first(l) + rec.r1, a * second(l) + rec.r2]
        ra := retract(a)@F
        ((rb := retractIfCan(b)@Union(F, "failed")) case "failed") or
          ((rh := retractIfCan(h)@Union(F, "failed")) case "failed") =>
                                  DSPDEsys(ra, b, h, c1, c2, n, derivation)
        degradation(ra, rb::F, rh::F, c1, c2, n)

  -- solve
  -- a (Q1', Q2') + ((H, -B), (B, H)) (Q1,Q2) = (C1,C2)
  -- for polynomials Q1 and Q2 with degree <= n
  -- a and B are nonzero, either B or H has positive degree
```

```
-- cancellation at infinity is not possible
    DSPDEsys(a, b, h, c1, c2, n, derivation) ==
      bb := degree(b)::Z
      hh:Z :=
        zero? h => 0
        degree(h)::Z
      lb := leadingCoefficient b
      lh := leadingCoefficient h
      bb < hh =>
        DSPDEsys0(a,b,h,c1,c2,lb,lh,n,derivation,DSPDEhdom(#1,#2,#3,#4,#5,hh))
      bb > hh =>
        DSPDEsys0(a,b,h,c1,c2,lb,lh,n,derivation,DSPDEbdom(#1,#2,#3,#4,#5,bb))
      det := lb * lb + lh * lh
      DSPDEsys0(a,b,h,c1,c2,lb,lh,n,derivation,DSPDEmix(#1,#2,#3,#4,#5,bb,det))

    DSPDEsys0(a, b, h, c1, c2, lb, lh, n, derivation, getlc) ==
      ans1 := ans2 := 0::UP
      repeat
        zero? c1 and zero? c2 => return [ans1, ans2]
        n < 0 or (u := getlc(c1,c2,lb,lh,n::N)) case "failed" => return "failed"
        lq := u::List(UP)
        q1 := first lq
        q2 := second lq
        c1 := c1 - a * derivation(q1) - h * q1 + b * q2
        c2 := c2 - a * derivation(q2) - b * q1 - h * q2
        n := n - 1
        ans1 := ans1 + q1
        ans2 := ans2 + q2

    DSPDEmix(c1, c2, lb, lh, n, d, det) ==
      rh1:F :=
        zero? c1 => 0
        (d1 := degree(c1)::Z - d) < n => 0
        d1 > n => return "failed"
        leadingCoefficient c1
      rh2:F :=
        zero? c2 => 0
        (d2 := degree(c2)::Z - d) < n => 0
        d2 > n => return "failed"
        leadingCoefficient c2
      q1 := (rh1 * lh + rh2 * lb) / det
      q2 := (rh2 * lh - rh1 * lb) / det
      [monomial(q1, n), monomial(q2, n)]


    DSPDEhdom(c1, c2, lb, lh, n, d) ==
```

```
      q1:UP :=
        zero? c1 => 0
        (d1 := degree(c1)::Z - d) < n => 0
        d1 > n => return "failed"
        monomial(leadingCoefficient(c1) / lh, n)
      q2:UP :=
        zero? c2 => 0
        (d2 := degree(c2)::Z - d) < n => 0
        d2 > n => return "failed"
        monomial(leadingCoefficient(c2) / lh, n)
      [q1, q2]

    DSPDEbdom(c1, c2, lb, lh, n, d) ==
      q1:UP :=
        zero? c2 => 0
        (d2 := degree(c2)::Z - d) < n => 0
        d2 > n => return "failed"
        monomial(leadingCoefficient(c2) / lb, n)
      q2:UP :=
        zero? c1 => 0
        (d1 := degree(c1)::Z - d) < n => 0
        d1 > n => return "failed"
        monomial(- leadingCoefficient(c1) / lb, n)
      [q1, q2]

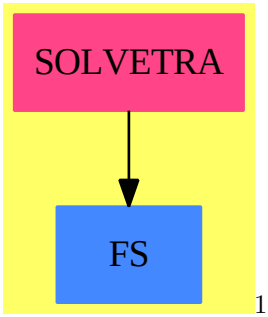
-- return a common bound on the degrees of a solution of
-- A (Q1', Q2') + ((H, -B), (B, H)) (Q1,Q2) = (C1,C2), Q1 = y1 T, Q2 = y2 T
-- cancellation at infinity is possible
-- a and b are nonzero
-- base case: F' = 0
    getBound(a, b, h, c1, c2) ==
      da := (degree a)::Z
      dc :=
        zero? c1 => degree(c2)::Z
        zero? c2 => degree(c1)::Z
        max(degree c1, degree c2)::Z
      hh:Z :=
        zero? h => 0
        degree(h)::Z
      db := max(hh, bb := degree(b)::Z)
      da < db + 1 => dc - db
      da > db + 1 => max(0, dc - da + 1)
      bb >= hh => dc - db
      (n := retractIfCan(leadingCoefficient(h) / leadingCoefficient(a)
                    )@Union(Z, "failed")) case Z => max(n::Z, dc - db)
      dc - db
```

⟨*RDETRS.dotabb*⟩≡

```
"RDETRS" [color="#FF4488",href="bookvol10.4.pdf#nameddest=RDETRS"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"RDETRS" -> "PFECAT"
```

21.35 package SOLVETRA TransSolvePackage

21.36 TransSolvePackage



1.00

Exports:
solve

⟨*package SOLVETRA TransSolvePackage*⟩≡

```
)abbrev package SOLVETRA TransSolvePackage
++ Author: W. Wiwianka, Martin Rubey
++ Date Created: Summer 1991
++ Change History: 9/91
++ Basic Operations: solve
++ Related Constructors: RadicalSolvePackage, FloatingRealPackage
++ Keywords:
++ Description:
++  This package tries to find solutions of equations of type Expression(R).
++  This means expressions involving transcendental, exponential, logarithmic
++  and nthRoot functions.
++  After trying to transform different kernels to one kernel by applying
++  several rules, it calls zerosOf for the SparseUnivariatePolynomial in
++  the remaining kernel.
++  For example the expression  \spad{sin(x)*cos(x)-2}  will be transformed to
++     \spad{-2 tan(x/2)**4 -2 tan(x/2)**3 -4 tan(x/2)**2 +2 tan(x/2) -2}
++  by  using the function  normalize  and then to
++     \spad{-2 tan(x)**2 + tan(x) -2}
++  with help of subsTan. This function tries to express the given function
++  in terms of \spad{tan(x/2)}  to express in terms of \spad{tan(x)} .
++  Other examples are the expressions    \spad{sqrt(x+1)+sqrt(x+7)+1}   or
++   \spad{sqrt(sin(x))+1} .


TransSolvePackage(R) : Exports == Implementation where
   R : Join(OrderedSet, EuclideanDomain, RetractableTo Integer,
            LinearlyExplicitRingOver Integer, CharacteristicZero)
```

```
    I   ==> Integer
    NNI ==> NonNegativeInteger
    RE  ==> Expression R
    EQ  ==> Equation
    S   ==> Symbol
    V   ==> Variable
    L   ==> List
    K   ==> Kernel RE
    SUP ==> SparseUnivariatePolynomial
    C   ==> Complex
    F   ==> Float
    INT ==> Interval
    SMP ==> SparseMultivariatePolynomial


    Exports == with

      solve :  RE           ->  L EQ RE
        ++ solve(expr) finds the solutions of the equation expr = 0
        ++ where expr is a function of type Expression(R)
        ++ with respect to the unique symbol x appearing in eq.
      solve :  EQ RE        ->  L EQ RE
        ++ solve(eq) finds the solutions of the equation  eq
        ++ where  eq  is  an equation of functions of type Expression(R)
        ++ with respect to the unique symbol x appearing in eq.
      solve : ( EQ RE , S ) ->  L EQ RE
        ++ solve(eq,x) finds the solutions of the equation  eq
        ++ where  eq  is  an equation of functions of type Expression(R)
        ++ with respect to the symbol x.
      solve : ( RE , S)     ->  L EQ RE
        ++ solve(expr,x) finds the solutions of the equation expr = 0
        ++ with respect to the symbol x where expr is a function
        ++ of type Expression(R).
      solve :  (L EQ RE, L S) -> L L EQ RE
        ++ solve(leqs, lvar) returns a list of solutions to the list of
        ++ equations leqs with respect to the list of symbols lvar.
--    solve :  (L EQ RE, L Kernel RE) -> L L EQ RE
--      ++ solve(leqs, lker) returns a list of solutions to the list
--      ++ of equations leqs with respect to the list of kernels lker.

    Implementation == add
      import ACF
      import HomogeneousAggregate(R)
      import AlgebraicManipulations(R, RE)
      import TranscendentalManipulations(R, RE)
```

```
import TrigonometricManipulations(R, RE)
import ElementaryFunctionStructurePackage(R, RE)
import SparseUnivariatePolynomial(R)
import LinearSystemMatrixPackage(RE,Vector RE,Vector RE,Matrix RE)
import TransSolvePackageService(R)
import MultivariateFactorize(K, IndexedExponents K, R, SMP(R, K))



   ---- Local Function Declarations ----

solveInner : (RE, S) -> L EQ RE
tryToTrans : ( RE , S)     ->  RE

eliminateKernRoot: (RE , K) -> RE
eliminateRoot: (RE , S) -> RE

combineLog : ( RE , S ) -> RE
testLog : ( RE , S ) -> Boolean
splitExpr : ( RE ) -> L RE
buildnexpr : ( RE , S ) -> L RE
logsumtolog : RE -> RE
logexpp : ( RE , RE ) -> RE

testRootk : ( RE, S) -> Boolean
testkernel : ( RE , S ) -> Boolean
funcinv : ( RE , RE ) -> Union(RE,"failed")
testTrig : ( RE , S ) -> Boolean
testHTrig : ( RE , S ) -> Boolean
tableXkernels : ( RE , S ) -> L RE
subsTan : ( RE , S )  -> RE


-- exported functions


solve(oside: RE) : L EQ RE ==
  zero? oside => error "equation is always satisfied"
  lv := variables oside
  empty? lv => error "inconsistent equation"
  #lv>1 => error "too many variables"
  solve(oside,lv.first)

solve(equ:EQ RE) : L EQ RE ==
  solve(lhs(equ)-rhs(equ))
```

```
solve(equ:EQ RE, x:S) : L EQ RE ==
  oneside:=lhs(equ)-rhs(equ)
  solve(oneside,x)

testZero?(lside:RE,sol:EQ RE):Boolean ==
  if R has QuotientFieldCategory(Integer) then
    retractIfCan(rhs sol)@Union(Integer,"failed") case "failed" => true
  else
    retractIfCan(rhs sol)@Union(Fraction Integer,"failed") case "failed" => true
  zero? eval(lside,sol) => true
  false

solve(lside: RE, x:S) : L EQ RE ==
  [sol for sol in solveInner(lside,x) | testZero?(lside,sol)]

solveInner(lside: RE, x:S) : L EQ RE ==
  lside:=eliminateRoot(lside,x)
  ausgabe1:=tableXkernels(lside,x)

  X:=new()@Symbol
  Y:=new()@Symbol::RE
  (#ausgabe1) = 1 =>
     bigX:= (first ausgabe1)::RE
     eq1:=eval(lside,bigX=(X::RE))
         -- Type  :  Expression R
     f:=univariate(eq1,first kernels (X::RE))
         -- Type  :  Fraction SparseUnivariatePolynomial Expression R
     lfatt:= factors factorPolynomial numer f
     lr:L RE := "append" /[zerosOf(fatt.factor,x) for fatt in lfatt]
         -- Type  :  List Expression R
     r1:=[]::L RE
     for i in 1..#lr repeat
        finv := funcinv(bigX,lr(i))
        if finv case RE then r1:=cons(finv::RE,r1)
     bigX_back:=funcinv(bigX,bigX)::RE
     if not testkernel(bigX_back,x) then
       if bigX = bigX_back then return []::L EQ RE
       return
         "append"/[solve(bigX_back-ri, x) for ri in r1]
     newlist:=[]::L EQ RE

     for i in 1..#r1 repeat
        elR :=  eliminateRoot((numer(bigX_back - r1(i))::RE ),x)
        f:=univariate(elR, kernel(x))
         -- Type  :  Fraction SparseUnivariatePolynomial Expression R
        lfatt:= factors factorPolynomial numer f
```

```
            secondsol:="append" /[zerosOf(ff.factor,x) for ff in lfatt]
            for j in 1..#secondsol repeat
               newlist:=cons((x::RE)=rootSimp( secondsol(j) ),newlist)
         newlist
      newlside:=tryToTrans(lside,x) ::RE
      listofkernels:=tableXkernels(newlside,x)
      (#listofkernels) = 1 => solve(newlside,x)
      lfacts := factors factor(numer lside)
      #lfacts > 1 =>
         sols : L EQ RE := []
         for frec in lfacts repeat
             sols := append(solve(frec.factor :: RE, x), sols)
         sols
      return []::L EQ RE

   -- local functions

    --  This function was suggested by Manuel Bronstein as a simpler
    --  alternative to normalize.
    simplifyingLog(f:RE):RE ==
      (u:=isExpt(f,"exp"::Symbol)) case Record(var:Kernel RE,exponent:Integer) =
        rec := u::Record(var:Kernel RE,exponent:Integer)
        rec.exponent * first argument(rec.var)
      log f


    testkernel(var1:RE,y:S) : Boolean ==
      var1:=eliminateRoot(var1,y)
      listvar1:=tableXkernels(var1,y)
      if (#listvar1 = 1) and ((listvar1(1) = (y::RE))@Boolean ) then
           true
      else if #listvar1 = 0 then true
           else false

    solveRetract(lexpr:L RE, lvar:L S):Union(L L EQ RE, "failed") ==
       nlexpr : L Fraction Polynomial R := []
       for expr in lexpr repeat
          rf:Union(Fraction Polynomial R, "failed") := retractIfCan(expr)$RE
          rf case "failed" => return "failed"
          nlexpr := cons(rf, nlexpr)
       radicalSolve(nlexpr, lvar)$RadicalSolvePackage(R)

    tryToTrans(lside: RE, x:S) : RE ==
      if testTrig(lside,x) or testHTrig(lside,x) then
         convLside:=( simplify(lside) )::RE
         resultLside:=convLside
```

```
      listConvLside:=tableXkernels(convLside,x)
      if (#listConvLside) > 1  then
        NormConvLside:=normalize(convLside,x)
        NormConvLside:=( NormConvLside ) :: RE
        resultLside:=subsTan(NormConvLside , x)

  else if testLog(lside,x) then
         numlside:=numer(lside)::RE
         resultLside:=combineLog(numlside,x)
       else
         NormConvLside:=normalize(lside,x)
         NormConvLside:=( NormConvLside ) :: RE
         resultLside:=NormConvLside
         listConvLside:=tableXkernels(NormConvLside,x)
         if  (#listConvLside) > 1  then
           cnormConvLside:=complexNormalize(lside,x)
           cnormConvLside:=cnormConvLside::RE
           resultLside:=cnormConvLside
           listcnorm:=tableXkernels(cnormConvLside,x)
           if (#listcnorm) > 1 then
             if testLog(cnormConvLside,x) then
               numlside:=numer(cnormConvLside)::RE
               resultLside:=combineLog(numlside,x)
  resultLside


subsTan(exprvar:RE,y:S) : RE ==
  Z:=new()@Symbol
  listofkern:=tableXkernels(exprvar,y)
  varkern:=(first listofkern)::RE
  Y:=(numer first argument first (kernels(varkern)))::RE
  test : Boolean := varkern=tan(((Y::RE)/(2::RE))::RE)
  if not( (#listofkern=1) and test) then
    return exprvar
  fZ:=eval(exprvar,varkern=(Z::RE))
  fN:=(numer fZ)::RE
  f:=univariate(fN, first kernels(Z::RE))
  secondfun:=(-2*(Y::RE)/((Y::RE)**2-1) )::RE
  g:=univariate(secondfun,first kernels(y::RE))
  H:=(new()@Symbol)::RE
  newH:=univariate(H,first kernels(Z::RE))
  result:=decomposeFunc(f,g,newH)
  if not ( result = f ) then
    result1:=result( H::RE )
    resultnew:=eval(result1,H=(( tan((Y::RE))::RE ) ))
  else return exprvar
```

```
eliminateKernRoot(var: RE, varkern: K) : RE ==
  X:=new()@Symbol
  var1:=eval(var, (varkern::RE)=(X::RE) )
  var2:=numer univariate(var1, first kernels(X::RE))
  var3:= monomial(1, ( retract( second argument varkern)@I )::NNI)@SUP RE_
         - monomial(first argument varkern, 0::NNI)@SUP RE
  resultvar:=resultant(var2, var3)

eliminateRoot(var:RE, y:S) : RE ==
  var1:=var
  while testRootk(var1,y) repeat
    varlistk1:=tableXkernels(var1,y)
    for i in varlistk1 repeat
       if is?(i, "nthRoot"::S) then
         var1:=eliminateKernRoot(var1,first kernels(i::RE))
  var1


logsumtolog(var:RE) : RE ==
  (listofexpr:=isPlus(var)) case "failed" => var
  listofexpr:= listofexpr ::L RE
  listforgcd:=[]::L R
  for i in listofexpr repeat
     exprcoeff:=leadingCoefficient(numer(i))
     listforgcd:=cons(exprcoeff, listforgcd)
  gcdcoeff:=gcd(listforgcd)::RE
  newexpr:RE :=0
  for i in listofexpr repeat
     exprlist:=splitExpr(i::RE)
     newexpr:=newexpr + logexpp(exprlist.2, exprlist.1/gcdcoeff)
  kernelofvar:=kernels(newexpr)
  var2:=1::RE
  for i in kernelofvar repeat
     var2:=var2*(first argument i)
  gcdcoeff * log(var2)


testLog(expr:RE,Z:S) : Boolean ==
  testList:=[log]::L S
  kernelofexpr:=tableXkernels(expr,Z)
  if #kernelofexpr = 0 then
    return false
  for i in kernelofexpr repeat
     if not member?(name(first kernels(i)),testList) or _
```

```
          not testkernel( (first argument first kernels(i)) ,Z) then
        return false
  true

splitExpr(expr:RE) : L RE ==
  lcoeff:=leadingCoefficient((numer expr))
  exprwcoeff:=expr
  listexpr:=isTimes(exprwcoeff)
  if listexpr case "failed" then
    [1::RE , expr]
  else
    listexpr:=remove_!(lcoeff::RE , listexpr)
    cons(lcoeff::RE , listexpr)

buildnexpr(expr:RE, Z:S) : L RE ==
  nlist:=splitExpr(expr)
  n2list:=remove_!(nlist.1, nlist)
  anscoeff:RE:=1
  ansmant:RE:=0
  for i in n2list repeat
     if freeOf?(i::RE,Z) then
       anscoeff:=(i::RE)*anscoeff
     else
       ansmant:=(i::RE)
  [anscoeff, ansmant * nlist.1 ]

logexpp(expr1:RE, expr2:RE) : RE ==
  log( (first argument first kernels(expr1))**expr2 )

combineLog(expr:RE,Y:S) : RE ==
  exprtable:Table(RE,RE):=table()
  (isPlus(expr)) case "failed" => expr
  ans:RE:=0
  while expr ^= 0 repeat
    loopexpr:RE:=leadingMonomial(numer(expr))::RE
    if testLog(loopexpr,Y) and (#tableXkernels(loopexpr,Y)=1) then
      exprr:=buildnexpr(loopexpr,Y)
      if search(exprr.1,exprtable) case "failed" then
        exprtable.(exprr.1):=0
      exprtable.(exprr.1):= exprtable.(exprr.1) + exprr.2
    else
      ans:=ans+loopexpr
    expr:=(reductum(numer expr))::RE
  ansexpr:RE:=0
  for i in keys(exprtable) repeat
     ansexpr:=ansexpr + logsumtolog(exprtable.i) * (i::RE)
```

```
  ansexpr:=ansexpr + ans


testRootk(varlistk:RE,y:S) : Boolean ==
  testList:=[nthRoot]::L S
  kernelofeqnvar:=tableXkernels(varlistk,y)
  if #kernelofeqnvar = 0 then
    return false
  for i in kernelofeqnvar repeat
     if member?(name(first kernels(i)),testList) then
       return true
  false

tableXkernels(evar:RE,Z:S) : L RE ==
  kOfvar:=kernels(evar)
  listkOfvar:=[]::L RE
  for i in kOfvar repeat
     if not freeOf?(i::RE,Z) then
         listkOfvar:=cons(i::RE,listkOfvar)
  listkOfvar

testTrig(eqnvar:RE,Z:S) : Boolean ==
  testList:=[sin , cos , tan , cot , sec , csc]::L S
  kernelofeqnvar:=tableXkernels(eqnvar,Z)
  if #kernelofeqnvar = 0 then
    return false
  for i in kernelofeqnvar repeat
     if not member?(name(first kernels(i)),testList) or _
        not testkernel( (first argument first kernels(i)) ,Z) then
       return false
  true


testHTrig(eqnvar:RE,Z:S) : Boolean ==
  testList:=[sinh , cosh , tanh , coth , sech , csch]::L S
  kernelofeqnvar:=tableXkernels(eqnvar,Z)
  if #kernelofeqnvar = 0 then
    return false
  for i in kernelofeqnvar repeat
     if not member?(name(first kernels(i)),testList) or _
        not testkernel( (first argument first kernels(i)) ,Z) then
       return false
  true

-- Auxiliary local function for use in funcinv.
makeInterval(l:R):C INT F ==
```

```
    if R has complex and R has ConvertibleTo(C F) then
      map(interval$INT(F),convert(l)$R)$ComplexFunctions2(F,INT F)
    else
      error "This should never happen"

  funcinv(k:RE,l:RE) : Union(RE,"failed") ==
    is?(k, "sin"::Symbol)   => asin(l)
    is?(k, "cos"::Symbol)   => acos(l)
    is?(k, "tan"::Symbol)   => atan(l)
    is?(k, "cot"::Symbol)   => acot(l)
    is?(k, "sec"::Symbol)   =>
        l = 0 => "failed"
        asec(l)
    is?(k, "csc"::Symbol)   =>
        l = 0 => "failed"
        acsc(l)
    is?(k, "sinh"::Symbol)  => asinh(l)
    is?(k, "cosh"::Symbol)  => acosh(l)
    is?(k, "tanh"::Symbol)  => atanh(l)
    is?(k, "coth"::Symbol)  => acoth(l)
    is?(k, "sech"::Symbol)  => asech(l)
    is?(k, "csch"::Symbol)  => acsch(l)
    is?(k, "atan"::Symbol)  => tan(l)
    is?(k, "acot"::Symbol)  =>
        l = 0 => "failed"
        cot(l)
    is?(k, "asin"::Symbol)  => sin(l)
    is?(k, "acos"::Symbol)  => cos(l)
    is?(k, "asec"::Symbol)  => sec(l)
    is?(k, "acsc"::Symbol)  =>
        l = 0 => "failed"
        csc(l)
    is?(k, "asinh"::Symbol) => sinh(l)
    is?(k, "acosh"::Symbol) => cosh(l)
    is?(k, "atanh"::Symbol) => tanh(l)
    is?(k, "acoth"::Symbol) =>
        l = 0 => "failed"
        coth(l)
    is?(k, "asech"::Symbol) => sech(l)
    is?(k, "acsch"::Symbol) =>
        l = 0 => "failed"
        csch(l)
    is?(k, "exp"::Symbol)   =>
        l = 0 => "failed"
        simplifyingLog l
    is?(k, "log"::Symbol)   =>
```

```
    if R has complex and R has ConvertibleTo(C F) then
      -- We will check to see if the imaginary part lies in [-Pi,Pi)
      ze : Expression C INT F
      ze := map(makeInterval,l)$ExpressionFunctions2(R,C INT F)
      z : Union(C INT F,"failed") := retractIfCan ze
      z case "failed" => exp l
      im := imag z
      fpi : Float := pi()
      (-fpi < inf(im)) and (sup(im) <= fpi) => exp l
      "failed"
    else -- R not Complex or something which doesn't map to Complex Floats
      exp l
  is?(k, "%power"::Symbol)   =>
       (t:=normalize(l)) = 0 => "failed"
       log t
  l

import SystemSolvePackage(RE)

ker2Poly(k:Kernel RE, lvar:L S):Polynomial RE ==
   member?(nm:=name k, lvar) => nm :: Polynomial RE
   k :: RE :: Polynomial RE

smp2Poly(pol:SMP(R,Kernel RE), lvar:L S):Polynomial RE ==
   map(ker2Poly(#1, lvar),
      #1::RE::Polynomial RE, pol)$PolynomialCategoryLifting(
         IndexedExponents Kernel RE, Kernel RE, R, SMP(R, Kernel RE),
                 Polynomial RE)

makeFracPoly(expr:RE, lvar:L S):Fraction Polynomial RE ==
   smp2Poly(numer expr, lvar) / smp2Poly(denom expr, lvar)

makeREpol(pol:Polynomial RE):RE ==
   lvar := variables pol
   lval : List RE := [v::RE for v in lvar]
   ground eval(pol,lvar,lval)

makeRE(frac:Fraction Polynomial RE):RE ==
   makeREpol(numer frac)/makeREpol(denom frac)

solve1Pol(pol:Polynomial RE, var: S, sol:L EQ RE):L L EQ RE ==
   repol := eval(makeREpol pol, sol)
   vsols := solve(repol, var)
   [cons(vsol, sol) for vsol in vsols]

solve1Sys(plist:L Polynomial RE, lvar:L S):L L EQ RE ==
```

```
      rplist := reverse plist
      rlvar := reverse lvar
      sols : L L EQ RE := list(empty())
      for p in rplist for v in rlvar repeat
         sols := "append"/[solve1Pol(p,v,sol) for sol in sols]
      sols
```

The input

```
solve(sinh(z)=cosh(z),z)
```

generates the error (reported as bug # 102):

```
>> Error detected within library code:
   No identity element for reduce of empty list using operation append
```

⟨*package SOLVETRA TransSolvePackage*⟩+≡

```
    solveList(lexpr:L RE, lvar:L S):L L EQ RE ==
       ans1 := solveRetract(lexpr, lvar)
       not(ans1 case "failed") => ans1 :: L L EQ RE
       lfrac:L Fraction Polynomial RE :=
          [makeFracPoly(expr, lvar) for expr in lexpr]
       trianglist := triangularSystems(lfrac, lvar)
--       "append"/[solve1Sys(plist, lvar) for plist in trianglist]
       l: L L L EQ RE := [solve1Sys(plist, lvar) for plist in trianglist]
       reduce(append, l, [])

    solve(leqs:L EQ RE, lvar:L S):L L EQ RE ==
       lexpr:L RE := [lhs(eq)-rhs(eq) for eq in leqs]
       solveList(lexpr, lvar)

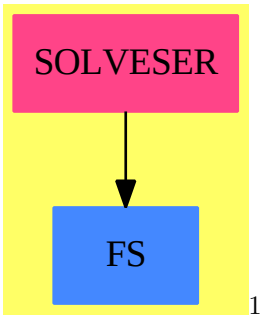
--    solve(leqs:L EQ RE, lker:L Kernel RE):L L EQ RE ==
--       lexpr:L RE := [lhs(eq)-rhs(eq) for eq in leqs]
--       lvar :L S := [new()$S for k in lker]
--       lval :L RE := [kernel v for v in lvar]
--       nlexpr := [eval(expr,lker,lval) for expr in lexpr]
--       ans := solveList(nlexpr, lvar)
--       lker2 :L Kernel RE := [v::Kernel(RE) for v in lvar]
--       lval2 := [k::RE for k in lker]
--       [[map(eval(#1,lker2,lval2), neq) for neq in sol] for sol in ans]
```

⟨*SOLVETRA.dotabb*⟩≡

```
"SOLVETRA" [color="#FF4488",href="bookvol10.4.pdf#nameddest=SOLVETRA"]
"FS" [color="#4488FF",href="bookvol10.2.pdf#nameddest=FS"]
"SOLVETRA" -> "FS"
```

21.37 package SOLVESER TransSolvePackage-Service

21.38 TransSolvePackageService



1.00

Exports:
decomposeFunc unvectorise

⟨*package SOLVESER TransSolvePackageService*⟩≡

```
)abbrev package SOLVESER TransSolvePackageService
++ Author: W. Wiwianka
++ Date Created: Summer 1991
++ Change History: 9/91
++ Basic Operations: decomposeFunc, unvectorise
++ Related Constructors:
++ Keywords:
++ Description: This package finds the function func3 where  func1 and func2
++  are given and  func1 = func3(func2) .  If there is no solution then
++  function func1 will be returned.
++  An example would be  \spad{func1:= 8*X**3+32*X**2-14*X ::EXPR INT} and
++  \spad{func2:=2*X ::EXPR INT} convert them via univariate
++  to FRAC SUP EXPR INT and then the solution is \spad{func3:=X**3+X**2-X}
++  of type FRAC SUP EXPR INT
TransSolvePackageService(R) : Exports == Implementation where
   R   :   Join(IntegralDomain, OrderedSet)

   RE  ==> Expression R
   EQ  ==> Equation
   S   ==> Symbol
   V   ==> Variable
   L   ==> List
   SUP ==> SparseUnivariatePolynomial
   ACF      ==> AlgebraicallyClosedField()
```

```
Exports == with

  decomposeFunc : ( Fraction SUP RE , Fraction SUP RE, Fraction SUP RE )  -> F
    ++ decomposeFunc(func1, func2, newvar)  returns a function  func3 where
    ++ func1 = func3(func2)  and expresses it in the new variable  newvar.
    ++ If there is no solution then func1 will be returned.
  unvectorise : ( Vector RE , Fraction SUP RE , Integer ) -> Fraction SUP RE
    ++ unvectorise(vect, var, n) returns
    ++ \spad{vect(1) + vect(2)*var + ... + vect(n+1)*var**(n)} where
    ++ vect  is the vector of the coefficients of the polynomail , var
    ++ the new variable and n the degree.


Implementation == add
  import ACF
  import TranscendentalManipulations(R, RE)
  import ElementaryFunctionStructurePackage(R, RE)
  import SparseUnivariatePolynomial(R)
  import LinearSystemMatrixPackage(RE,Vector RE,Vector RE,Matrix RE)
  import HomogeneousAggregate(R)

     ---- Local Function Declarations ----

  subsSolve : ( SUP RE, NonNegativeInteger, SUP RE, SUP RE, Integer, Fraction
    --++ subsSolve(f, degf, g1, g2, m, h)


 -- exported functions


  unvectorise(vect:Vector RE, var:Fraction SUP RE,n:Integer) : Fraction SUP RE
    Z:=new()@Symbol
    polyvar: Fraction SUP RE :=0
    for i in 1..((n+1)::Integer) repeat
       vecti:=univariate(vect( i ),first kernels(Z::RE))
       polyvar:=polyvar + ( vecti )*( var )**( (n-i+1)::NonNegativeInteger )
    polyvar


  decomposeFunc(exprf:Fraction SUP RE , exprg:Fraction SUP RE, newH:Fraction S
    X:=new()@Symbol
    f1:=numer(exprf)
    f2:=denom(exprf)
    g1:=numer(exprg)
    g2:=denom(exprg)
    degF:=max(degree(numer(exprf)),degree(denom(exprf)))
```

```
      degG:=max(degree(g1),degree(g2))
      newF1,newF2 : Union(SUP RE, "failed")
      N:= degF exquo degG
      if not ( N case "failed" ) then
        m:=N::Integer
        newF1:=subsSolve(f1,degF,g1,g2,m,newH)
        if f2 = 1 then
          newF2:= 1 :: SUP RE
        else newF2:=subsSolve(f2,degF,g1,g2,m,newH)
        if ( not ( newF1 case "failed" ) ) and ( not ( newF2 case "failed" ) ) then
          newF:=newF1/newF2
        else return exprf
      else return exprf


    -- local functions

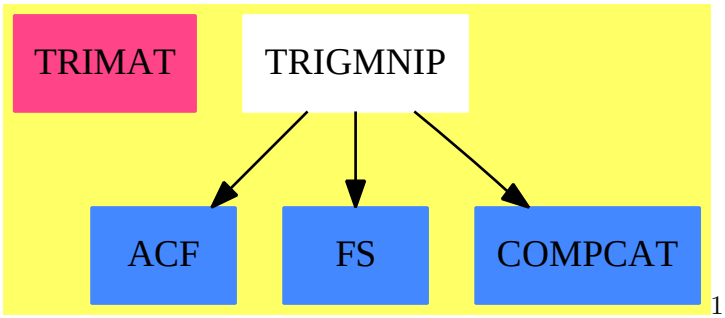

     subsSolve(F:SUP RE, DegF:NonNegativeInteger, G1:SUP RE, G2:SUP RE, M:Integer, HH: Fract
      coeffmat:=new((DegF+1),1,0)@Matrix RE
      for i in 0..M repeat
         coeffmat:=horizConcat(coeffmat, (vectorise( ( ( G1**((M-i)::NonNegativeInteger) )*
      vec:= vectorise(F,DegF+1)
      coeffma:=subMatrix(coeffmat,1,(DegF+1),2,(M+2))
      solvar:=solve(coeffma,vec)
      if not ( solvar.particular  case  "failed" ) then
        solvevarlist:=(solvar.particular)::Vector RE
        resul:= numer(unvectorise(solvevarlist,( HH ),M))
        resul
      else return "failed"
```

⟨*SOLVESER.dotabb*⟩≡

```
"SOLVESER" [color="#FF4488",href="bookvol10.4.pdf#nameddest=SOLVESER"]
"FS" [color="#4488FF",href="bookvol10.2.pdf#nameddest=FS"]
"SOLVESER" -> "FS"
```

21.39 package TRIMAT TriangularMatrixOperations

21.40 TriangularMatrixOperations



1.00

Exports:
LowTriBddDenomInv UpTriBddDenomInv

⟨*package TRIMAT TriangularMatrixOperations*⟩≡

```
)abbrev package TRIMAT TriangularMatrixOperations
++ Fraction free inverses of triangular matrices
++ Author: Victor Miller
++ Date Created:
++ Date Last Updated: 24 Jul 1990
++ Keywords:
++ Examples:
++ References:
++ Description:
++ This package provides functions that compute "fraction-free"
++ inverses of upper and lower triangular matrices over a integral
++ domain.  By "fraction-free inverses" we mean the following:
++ given a matrix B with entries in R and an element d of R such that
++ d * inv(B) also has entries in R, we return d * inv(B).  Thus,
++ it is not necessary to pass to the quotient field in any of our
++ computations.


TriangularMatrixOperations(R,Row,Col,M): Exports == Implementation where
  R   : IntegralDomain
  Row : FiniteLinearAggregate R
  Col : FiniteLinearAggregate R
  M   : MatrixCategory(R,Row,Col)

  Exports ==> with
```

```
    UpTriBddDenomInv: (M,R) -> M
      ++ UpTriBddDenomInv(B,d) returns M, where
      ++ B is a non-singular upper triangular matrix and d is an
      ++ element of R such that \spad{M = d * inv(B)} has entries in R.
    LowTriBddDenomInv:(M,R) -> M
      ++ LowTriBddDenomInv(B,d) returns M, where
      ++ B is a non-singular lower triangular matrix and d is an
      ++ element of R such that \spad{M = d * inv(B)} has entries in R.

  Implementation ==> add

    UpTriBddDenomInv(A,denom) ==
      AI := zero(nrows A, nrows A)$M
      offset := minColIndex AI - minRowIndex AI
      for i in minRowIndex AI .. maxRowIndex AI
        for j in minColIndex AI .. maxColIndex AI repeat
          qsetelt_!(AI,i,j,(denom exquo qelt(A,i,j))::R)
      for i in minRowIndex AI .. maxRowIndex AI repeat
        for j in offset + i + 1 .. maxColIndex AI repeat
          qsetelt_!(AI,i,j, - (((+/[qelt(AI,i,k) * qelt(A,k-offset,j)
                                   for k in i+offset..(j-1)])
                                     exquo qelt(A, j-offset, j))::R))
      AI

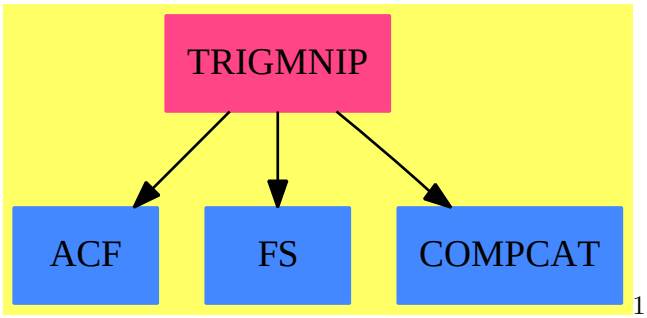
    LowTriBddDenomInv(A, denom) ==
      AI := zero(nrows A, nrows A)$M
      offset := minColIndex AI - minRowIndex AI
      for i in minRowIndex AI .. maxRowIndex AI
        for j in minColIndex AI .. maxColIndex AI repeat
          qsetelt_!(AI,i,j,(denom exquo qelt(A,i,j))::R)
      for i in minColIndex AI .. maxColIndex AI repeat
        for j in i - offset + 1 .. maxRowIndex AI repeat
          qsetelt_!(AI,j,i, - (((+/[qelt(A,j,k+offset) * qelt(AI,k,i)
                                    for k in i-offset..(j-1)])
                                      exquo qelt(A, j, j+offset))::R))
      AI
```

⟨ *TRIMAT.dotabb* ⟩≡

```
"TRIMAT" [color="#FF4488",href="bookvol10.4.pdf#nameddest=TRIMAT"]
"ACF" [color="#4488FF",href="bookvol10.2.pdf#nameddest=ACF"]
"FS" [color="#4488FF",href="bookvol10.2.pdf#nameddest=FS"]
"COMPCAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=COMPCAT"]
"TRIGMNIP" -> "ACF"
"TRIGMNIP" -> "FS"
"TRIGMNIP" -> "COMPCAT"
```

21.41 package TRIGMNIP TrigonometricManipulations

21.42 TrigonometricManipulations



1.00

Exports:

complexElementary	complexForm	complexNormalize	imag	real
real?	trigs			

⟨*package TRIGMNIP TrigonometricManipulations*⟩≡

```
)abbrev package TRIGMNIP TrigonometricManipulations
++ Trigs to/from exps and logs
++ Author: Manuel Bronstein
++ Date Created: 4 April 1988
++ Date Last Updated: 14 February 1994
++ Description:
++   \spadtype{TrigonometricManipulations} provides transformations from
++   trigonometric functions to complex exponentials and logarithms, and back.
++ Keywords: trigonometric, function, manipulation.
TrigonometricManipulations(R, F): Exports == Implementation where
  R : Join(GcdDomain, OrderedSet, RetractableTo Integer,
           LinearlyExplicitRingOver Integer)
  F : Join(AlgebraicallyClosedField, TranscendentalFunctionCategory,
           FunctionSpace R)

  Z   ==> Integer
  SY  ==> Symbol
  K   ==> Kernel F
  FG  ==> Expression Complex R

  Exports ==> with
    complexNormalize: F -> F
      ++ complexNormalize(f) rewrites \spad{f} using the least possible number
      ++ of complex independent kernels.
    complexNormalize: (F, SY) -> F
```

```
      ++ complexNormalize(f, x) rewrites \spad{f} using the least possible
      ++ number of complex independent kernels involving \spad{x}.
    complexElementary: F -> F
      ++ complexElementary(f) rewrites \spad{f} in terms of the 2 fundamental
      ++ complex transcendental elementary functions: \spad{log, exp}.
    complexElementary: (F, SY) -> F
      ++ complexElementary(f, x) rewrites the kernels of \spad{f} involving
      ++ \spad{x} in terms of the 2 fundamental complex
      ++ transcendental elementary functions: \spad{log, exp}.
    trigs  : F -> F
      ++ trigs(f) rewrites all the complex logs and exponentials
      ++ appearing in \spad{f} in terms of trigonometric functions.
    real   : F -> F
      ++ real(f) returns the real part of \spad{f} where \spad{f} is a complex
      ++ function.
    imag   : F -> F
      ++ imag(f) returns the imaginary part of \spad{f} where \spad{f}
      ++ is a complex function.
    real?  : F -> Boolean
      ++ real?(f) returns \spad{true} if \spad{f = real f}.
    complexForm: F -> Complex F
      ++ complexForm(f) returns \spad{[real f, imag f]}.

  Implementation ==> add
    import ElementaryFunctionSign(R, F)
    import InnerTrigonometricManipulations(R,F,FG)
    import ElementaryFunctionStructurePackage(R, F)
    import ElementaryFunctionStructurePackage(Complex R, FG)

    s1  := sqrt(-1::F)
    ipi := pi()$F * s1

    K2KG          : K -> Kernel FG
    kcomplex      : K -> Union(F, "failed")
    locexplogs    : F -> FG
    localexplogs  : (F, F, List SY) -> FG
    complexKernels: F -> Record(ker: List K, val: List F)

    K2KG k           == retract(tan F2FG first argument k)@Kernel(FG)
    real? f          == empty?(complexKernels(f).ker)
    real f           == real complexForm f
    imag f           == imag complexForm f

  -- returns [[k1,...,kn], [v1,...,vn]] such that ki should be replaced by vi
    complexKernels f ==
      lk:List(K) := empty()
```

```
        lv:List(F) := empty()
        for k in tower f repeat
          if (u := kcomplex k) case F then
             lk := concat(k, lk)
             lv := concat(u::F, lv)
        [lk, lv]

  -- returns f if it is certain that k is not a real kernel and k = f,
  -- "failed" otherwise
      kcomplex k ==
        op := operator k
        is?(k, "nthRoot"::SY) =>
          arg := argument k
          even?(retract(n := second arg)@Z) and ((u := sign(first arg)) case Z)
            and (u::Z < 0) => op(s1, n / 2::F) * op(- first arg, n)
          "failed"
        is?(k, "log"::SY) and ((u := sign(a := first argument k)) case Z)
            and (u::Z < 0) => op(- a) + ipi
        "failed"

      complexForm f ==
        empty?((l := complexKernels f).ker) => complex(f, 0)
        explogs2trigs locexplogs eval(f, l.ker, l.val)

      locexplogs f ==
        any?(has?(#1, "rtrig"),
             operators(g := realElementary f))$List(BasicOperator) =>
                localexplogs(f, g, variables g)
        F2FG g

      complexNormalize(f, x) ==
        any?(has?(operator #1, "rtrig"),
         [k for k in tower(g := realElementary(f, x))
                 | member?(x, variables(k::F))]$List(K))$List(K) =>
                     FG2F(rischNormalize(localexplogs(f, g, [x]), x).func)
        rischNormalize(g, x).func

      complexNormalize f ==
        l := variables(g := realElementary f)
        any?(has?(#1, "rtrig"), operators g)$List(BasicOperator) =>
          h := localexplogs(f, g, l)
          for x in l repeat h := rischNormalize(h, x).func
          FG2F h
        for x in l repeat g := rischNormalize(g, x).func
        g
```

```
complexElementary(f, x) ==
  any?(has?(operator #1, "rtrig"),
   [k for k in tower(g := realElementary(f, x))
              | member?(x, variables(k::F))]$List(K))$List(K) =>
                  FG2F localexplogs(f, g, [x])
  g

complexElementary f ==
  any?(has?(#1, "rtrig"),
    operators(g := realElementary f))$List(BasicOperator) =>
      FG2F localexplogs(f, g, variables g)
  g

localexplogs(f, g, lx) ==
  trigs2explogs(F2FG g, [K2KG k for k in tower f
                       | is?(k, "tan"::SY) or is?(k, "cot"::SY)], lx)

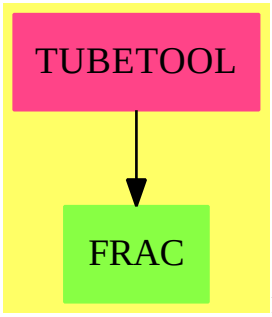
trigs f ==
  real? f => f
  g := explogs2trigs F2FG f
  real g + s1 * imag g
```

⟨*TRIGMNIP.dotabb*⟩≡

```
"TRIGMNIP" [color="#FF4488",href="bookvol10.4.pdf#nameddest=TRIGMNIP"]
"ACF" [color="#4488FF",href="bookvol10.2.pdf#nameddest=ACF"]
"FS" [color="#4488FF",href="bookvol10.2.pdf#nameddest=FS"]
"COMPCAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=COMPCAT"]
"TRIGMNIP" -> "ACF"
"TRIGMNIP" -> "FS"
"TRIGMNIP" -> "COMPCAT"
```

21.43 package TUBETOOL TubePlotTools

21.44 TubePlotTools



1.00

Exports:

cosSinInfo	cross	dot	loopPoints	point
unitVector	?*?	?+?	?-?	

⟨*package TUBETOOL TubePlotTools*⟩≡

```
)abbrev package TUBETOOL TubePlotTools
++ Author: Clifton J. Williamson
++ Date Created: Bastille Day 1989
++ Date Last Updated: 5 June 1990
++ Keywords:
++ Examples:
++ Description:
++   Tools for constructing tubes around 3-dimensional parametric curves.
TubePlotTools(): Exports == Implementation where
  I   ==> Integer
  SF  ==> DoubleFloat
  L   ==> List
  Pt  ==> Point SF

  Exports ==> with
    point: (SF,SF,SF,SF) -> Pt
      ++ point(x1,x2,x3,c) creates and returns a point from the three
      ++ specified coordinates \spad{x1}, \spad{x2}, \spad{x3}, and also
      ++ a fourth coordinate, c, which is generally used to specify the
      ++ color of the point.
    "*" : (SF,Pt) -> Pt
      ++ s * p returns a point whose coordinates are the scalar multiple
      ++ of the point p by the scalar s, preserving the color, or fourth
      ++ coordinate, of p.
    "+" : (Pt,Pt) -> Pt
      ++ p + q computes and returns a point whose coordinates are the sums
```

```
      ++ of the coordinates of the two points \spad{p} and \spad{q}, using
      ++ the color, or fourth coordinate, of the first point \spad{p}
      ++ as the color also of the point \spad{q}.
    "-" : (Pt,Pt) -> Pt
      ++ p - q computes and returns a point whose coordinates are the
      ++ differences of the coordinates of two points \spad{p} and \spad{q},
      ++ using the color, or fourth coordinate, of the first point \spad{p}
      ++ as the color also of the point \spad{q}.
    dot : (Pt,Pt) -> SF
      ++ dot(p,q) computes the dot product of the two points \spad{p}
      ++ and \spad{q} using only the first three coordinates, and returns
      ++ the resulting \spadtype{DoubleFloat}.
    cross : (Pt,Pt) -> Pt
      ++ cross(p,q) computes the cross product of the two points \spad{p}
      ++ and \spad{q} using only the first three coordinates, and keeping
      ++ the color of the first point p.  The result is returned as a point.
    unitVector: Pt -> Pt
      ++ unitVector(p) creates the unit vector of the point p and returns
      ++ the result as a point.  Note: \spad{unitVector(p) = p/|p|}.
    cosSinInfo: I -> L L SF
      ++ cosSinInfo(n) returns the list of lists of values for n, in the
      ++ form:  \spad{[[cos(n - 1) a,sin(n - 1) a],...,[cos 2 a,sin 2 a],[cos a,s
      ++ where \spad{a = 2 pi/n}.  Note: n should be greater than 2.
    loopPoints: (Pt,Pt,Pt,SF,L L SF) -> L Pt
      ++ loopPoints(p,n,b,r,lls) creates and returns a list of points
      ++ which form the loop with radius r, around the center point
      ++ indicated by the point p, with the principal normal vector of
      ++ the space curve at point p given by the point(vector) n, and the
      ++ binormal vector given by the point(vector) b, and a list of lists,
      ++ lls, which is the \spadfun{cosSinInfo} of the number of points
      ++ defining the loop.

  Implementation ==> add
    import PointPackage(SF)

    point(x,y,z,c) == point(l : L SF := [x,y,z,c])

    getColor: Pt -> SF
    getColor pt == (maxIndex pt > 3 => color pt; 0)

    getColor2: (Pt,Pt) -> SF
    getColor2(p0,p1) ==
      maxIndex p0 > 3 => color p0
      maxIndex p1 > 3 => color p1
      0
```

```
a * p ==
  l : L SF := [a * xCoord p,a * yCoord p,a * zCoord p,getColor p]
  point l

p0 + p1 ==
  l : L SF := [xCoord p0 + xCoord p1,yCoord p0 + yCoord p1,_
               zCoord p0 + zCoord p1,getColor2(p0,p1)]
  point l

p0 - p1 ==
  l : L SF := [xCoord p0 - xCoord p1,yCoord p0 - yCoord p1,_
               zCoord p0 - zCoord p1,getColor2(p0,p1)]
  point l

dot(p0,p1) ==
  (xCoord p0 * xCoord p1) + (yCoord p0 * yCoord p1) +_
    (zCoord p0 * zCoord p1)

cross(p0,p1) ==
  x0 := xCoord p0; y0 := yCoord p0; z0 := zCoord p0;
  x1 := xCoord p1; y1 := yCoord p1; z1 := zCoord p1;
  l : L SF := [y0 * z1 - y1 * z0,z0 * x1 - z1 * x0,_
               x0 * y1 - x1 * y0,getColor2(p0,p1)]
  point l

unitVector p == (inv sqrt dot(p,p)) * p

cosSinInfo n ==
  ans : L L SF := nil()
  theta : SF := 2 * pi()/n
  for i in 1..(n-1) repeat              --!! make more efficient
    angle := i * theta
    ans := concat([cos angle,sin angle],ans)
  ans

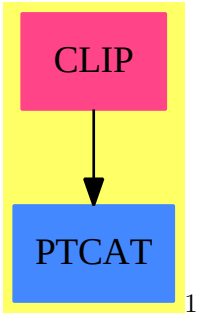
loopPoints(ctr,pNorm,bNorm,rad,cosSin) ==
  ans : L Pt := nil()
  while not null cosSin repeat
    cossin := first cosSin; cos := first cossin; sin := second cossin
    ans := cons(ctr + rad * (cos * pNorm + sin * bNorm),ans)
    cosSin := rest cosSin
  pt := ctr + rad * pNorm
  concat(pt,concat(ans,pt))
```

⟨ *TUBETOOL.dotabb* ⟩≡

```
"TUBETOOL" [color="#FF4488",href="bookvol10.4.pdf#nameddest=TUBETOOL"]
"FRAC" [color="#88FF44",href="bookvol10.3.pdf#nameddest=FRAC"]
"TUBETOOL" -> "FRAC"
```

21.45 package CLIP TwoDimensionalPlotClipping

21.46 TwoDimensionalPlotClipping



1.00

Exports:
clip clipParametric clipWithRanges

⟨*package CLIP TwoDimensionalPlotClipping*⟩≡

```
)abbrev package CLIP TwoDimensionalPlotClipping
++ Automatic clipping for 2-dimensional plots
++ Author: Clifton J. Williamson
++ Date Created: 22 December 1989
++ Date Last Updated: 10 July 1990
++ Keywords: plot, singularity
++ Examples:
++ References:

TwoDimensionalPlotClipping(): Exports == Implementation where
  ++ The purpose of this package is to provide reasonable plots of
  ++ functions with singularities.
  B      ==> Boolean
  L      ==> List
  SEG    ==> Segment
  RN     ==> Fraction Integer
  SF     ==> DoubleFloat
  Pt     ==> Point DoubleFloat
  PLOT   ==> Plot
  CLIPPED ==> Record(brans: L L Pt,xValues: SEG SF,yValues: SEG SF)

  Exports ==> with
    clip: PLOT -> CLIPPED
      ++ clip(p) performs two-dimensional clipping on a plot, p, from
      ++ the domain \spadtype{Plot} for the graph of one variable,
      ++ \spad{y = f(x)}; the default parameters \spad{1/4} for the fraction
```

```
      ++ and \spad{5/1} for the scale are used in the \spadfun{clip} function.
    clip: (PLOT,RN,RN) -> CLIPPED
      ++ clip(p,frac,sc) performs two-dimensional clipping on a plot, p,
      ++ from the domain \spadtype{Plot} for the graph of one variable
      ++ \spad{y = f(x)}; the fraction parameter is specified by \spad{frac}
      ++ and the scale parameter is specified by \spad{sc} for use in the
      ++ \spadfun{clip} function.
    clipParametric: PLOT -> CLIPPED
      ++ clipParametric(p) performs two-dimensional clipping on a plot,
      ++ p, from the domain \spadtype{Plot} for the parametric curve
      ++ \spad{x = f(t)}, \spad{y = g(t)}; the default parameters \spad{1/2}
      ++ for the fraction and \spad{5/1} for the scale are used in the
      ++ \fakeAxiomFun{iClipParametric} subroutine, which is called by this
      ++ function.
    clipParametric: (PLOT,RN,RN) -> CLIPPED
      ++ clipParametric(p,frac,sc) performs two-dimensional clipping on a
      ++ plot, p, from the domain \spadtype{Plot} for the parametric curve
      ++ \spad{x = f(t)}, \spad{y = g(t)}; the fraction parameter is
      ++ specified by \spad{frac} and the scale parameter is specified
      ++ by \spad{sc} for use in the \fakeAxiomFun{iClipParametric} subroutine,
      ++ which is called by this function.
    clipWithRanges: (L L Pt,SF,SF,SF,SF) -> CLIPPED
      ++ clipWithRanges(pointLists,xMin,xMax,yMin,yMax) performs clipping
      ++ on a list of lists of points, \spad{pointLists}.  Clipping is
      ++ done within the specified ranges of \spad{xMin}, \spad{xMax} and
      ++ \spad{yMin}, \spad{yMax}.  This function is used internally by
      ++ the \fakeAxiomFun{iClipParametric} subroutine in this package.
    clip: L Pt -> CLIPPED
      ++ clip(l) performs two-dimensional clipping on a curve l, which is
      ++ a list of points; the default parameters \spad{1/2} for the
      ++ fraction and \spad{5/1} for the scale are used in the
      ++ \fakeAxiomFun{iClipParametric} subroutine, which is called by this
      ++ function.
    clip: L L Pt -> CLIPPED
      ++ clip(ll) performs two-dimensional clipping on a list of lists
      ++ of points, \spad{ll}; the default parameters \spad{1/2} for
      ++ the fraction and \spad{5/1} for the scale are used in the
      ++ \fakeAxiomFun{iClipParametric} subroutine, which is called by this
      ++ function.

  Implementation ==> add
    import PointPackage(DoubleFloat)
    import ListFunctions2(Point DoubleFloat,DoubleFloat)

    point:(SF,SF) -> Pt
    intersectWithHorizLine:(SF,SF,SF,SF,SF) -> Pt
```

```
intersectWithVertLine:(SF,SF,SF,SF,SF) -> Pt
intersectWithBdry:(SF,SF,SF,SF,Pt,Pt) -> Pt
discardAndSplit: (L Pt,Pt -> B,SF,SF,SF,SF) -> L L Pt
norm: Pt -> SF
iClipParametric: (L L Pt,RN,RN) -> CLIPPED
findPt: L L Pt -> Union(Pt,"failed")
Fnan?: SF ->Boolean
Pnan?:Pt ->Boolean

Fnan? x == x~=x
Pnan? p == any?(Fnan?,p)

iClipParametric(pointLists,fraction,scale) ==
  -- error checks and special cases
  (fraction < 0) or (fraction > 1) =>
    error "clipDraw: fraction should be between 0 and 1"
  empty? pointLists => [nil(),segment(0,0),segment(0,0)]
  -- put all points together , sort them according to norm
  sortedList := sort(norm(#1) < norm(#2),select(not Pnan? #1,concat pointLists))
  empty? sortedList => [nil(),segment(0,0),segment(0,0)]
  n := # sortedList
  num := numer fraction
  den := denom fraction
  clipNum := (n * num) quo den
  lastN := n - 1 - clipNum
  firstPt := first sortedList
  xMin : SF := xCoord firstPt
  xMax : SF := xCoord firstPt
  yMin : SF := yCoord firstPt
  yMax : SF := yCoord firstPt
  -- calculate min/max for the first (1-fraction)*N points
  -- this contracts the range
  -- this unnecessarily clips monotonic functions (step-function, x^(high power),etc.)
  for k in 0..lastN  for pt in rest sortedList repeat
    xMin := min(xMin,xCoord pt)
    xMax := max(xMax,xCoord pt)
    yMin := min(yMin,yCoord pt)
    yMax := max(yMax,yCoord pt)
  xDiff := xMax - xMin; yDiff := yMax - yMin
  xDiff = 0 =>
    yDiff = 0 =>
      [pointLists,segment(xMin-1,xMax+1),segment(yMin-1,yMax+1)]
    [pointLists,segment(xMin-1,xMax+1),segment(yMin,yMax)]
  yDiff = 0 =>
    [pointLists,segment(xMin,xMax),segment(yMin-1,yMax+1)]
  numm := numer scale; denn := denom scale
```

```
  -- now expand the range by scale
  xMin := xMin - (numm :: SF) * xDiff / (denn :: SF)
  xMax := xMax + (numm :: SF) * xDiff / (denn :: SF)
  yMin := yMin - (numm :: SF) * yDiff / (denn :: SF)
  yMax := yMax + (numm :: SF) * yDiff / (denn :: SF)
  -- clip with the calculated range
  newclip:=clipWithRanges(pointLists,xMin,xMax,yMin,yMax)
  -- if we split the lists use the new clip
  # (newclip.brans) > # pointLists   => newclip
  -- calculate extents
  xs :L SF:= map (xCoord,sortedList)
  ys :L SF:= map (yCoord,sortedList)
  xMin :SF :=reduce (min,xs)
  yMin :SF :=reduce (min,ys)
  xMax :SF :=reduce (max,xs)
  yMax :SF :=reduce (max,ys)
  xseg:SEG SF :=xMin..xMax
  yseg:SEG SF :=yMin..yMax
  -- return original
  [pointLists,xseg,yseg]@CLIPPED



point(xx,yy) == point(l : L SF := [xx,yy])

intersectWithHorizLine(x1,y1,x2,y2,yy) ==
  x1 = x2 => point(x1,yy)
  point(x1 + (x2 - x1)*(yy - y1)/(y2 - y1),yy)

intersectWithVertLine(x1,y1,x2,y2,xx) ==
  y1 = y2 => point(xx,y1)
  point(xx,y1 + (y2 - y1)*(xx - x1)/(x2 - x1))

intersectWithBdry(xMin,xMax,yMin,yMax,pt1,pt2) ==
  -- pt1 is in rectangle, pt2 is not
  x1 := xCoord pt1; y1 := yCoord pt1
  x2 := xCoord pt2; y2 := yCoord pt2
  if y2 > yMax then
    pt2 := intersectWithHorizLine(x1,y1,x2,y2,yMax)
    x2 := xCoord pt2; y2 := yCoord pt2
  if y2 < yMin then
    pt2 := intersectWithHorizLine(x1,y1,x2,y2,yMin)
    x2 := xCoord pt2; y2 := yCoord pt2
  if x2 > xMax then
    pt2 := intersectWithVertLine(x1,y1,x2,y2,xMax)
```

```
      x2 := xCoord pt2; y2 := yCoord pt2
    if x2 < xMin then
      pt2 := intersectWithVertLine(x1,y1,x2,y2,xMin)
    pt2

  discardAndSplit(pointList,pred,xMin,xMax,yMin,yMax) ==
    ans : L L Pt := nil()
    list : L Pt  := nil()
    lastPt? : B  := false
    lastPt : Pt  := point(0,0)
    while not empty? pointList repeat
      pt := first pointList
      pointList := rest pointList
      pred(pt) =>
        if (empty? list) and lastPt? then
          bdryPt := intersectWithBdry(xMin,xMax,yMin,yMax,pt,lastPt)
          -- print bracket [ coerce bdryPt ,coerce pt ]
          --list := cons(bdryPt,list)
        list := cons(pt,list)
      if not empty? list then
        bdryPt := intersectWithBdry(xMin,xMax,yMin,yMax,first list,pt)
        -- print bracket [ coerce bdryPt,coerce first list]
        --list := cons(bdryPt,list)
        ans := cons( list,ans)
      lastPt := pt
      lastPt? := true
      list := nil()
    empty? list => ans
    reverse_! cons(reverse_! list,ans)

  clip(plot,fraction,scale) ==
--      sayBrightly(["   clip: "::OutputForm]$List(OutputForm))$Lisp
    (fraction < 0) or (fraction > 1/2) =>
      error "clipDraw: fraction should be between 0 and 1/2"
    xVals := xRange plot
    empty?(pointLists := listBranches plot) =>
      [nil(),xVals,segment(0,0)]
    more?(pointLists := listBranches plot,1) =>
      error "clipDraw: plot has more than one branch"
    empty?(pointList := first pointLists) =>
      [nil(),xVals,segment(0,0)]
    sortedList := sort(yCoord(#1) < yCoord(#2),pointList)
    n := # sortedList; num := numer fraction; den := denom fraction
    clipNum := (n * num) quo den
    -- throw out points with large and small y-coordinates
    yMin := yCoord(sortedList.clipNum)
```

```
    yMax := yCoord(sortedList.(n - 1 - clipNum))
    if Fnan? yMin then yMin : SF := 0
    if Fnan? yMax then yMax : SF := 0
    (yDiff := yMax - yMin) = 0 =>
      [pointLists,xRange plot,segment(yMin - 1,yMax + 1)]
    numm := numer scale; denn := denom scale
    xMin := lo xVals; xMax := hi xVals
    yMin := yMin - (numm :: SF) * yDiff / (denn :: SF)
    yMax := yMax + (numm :: SF) * yDiff / (denn :: SF)
    lists := discardAndSplit(pointList,_
       (yCoord(#1) < yMax) and (yCoord(#1) > yMin),xMin,xMax,yMin,yMax)
    yMin := yCoord(sortedList.clipNum)
    yMax := yCoord(sortedList.(n - 1 - clipNum))
    if Fnan? yMin then yMin : SF := 0
    if Fnan? yMax then yMax : SF := 0
    for list in lists repeat
      for pt in list repeat
        if not Fnan?(yCoord pt) then
          yMin := min(yMin,yCoord pt)
          yMax := max(yMax,yCoord pt)
    [lists,xVals,segment(yMin,yMax)]

  clip(plot:PLOT) == clip(plot,1/4,5/1)

  norm(pt) ==
    x := xCoord(pt); y := yCoord(pt)
    if Fnan? x then
      if Fnan? y then
        r:SF := 0
      else
        r:SF := y**2
    else
      if Fnan? y then
        r:SF := x**2
      else
        r:SF := x**2 + y**2
    r

  findPt lists ==
    for list in lists repeat
      not empty? list =>
           for p in list repeat
             not Pnan? p => return p
    "failed"

  clipWithRanges(pointLists,xMin,xMax,yMin,yMax) ==
```

```
    lists : L L Pt := nil()
    for pointList in pointLists repeat
      lists := concat(lists,discardAndSplit(pointList,_
         (xCoord(#1) <= xMax) and (xCoord(#1) >= xMin) and _
         (yCoord(#1) <= yMax) and (yCoord(#1) >= yMin), _
         xMin,xMax,yMin,yMax))
    (pt := findPt lists) case "failed" =>
      [nil(),segment(0,0),segment(0,0)]
    firstPt := pt :: Pt
    xMin : SF := xCoord firstPt; xMax : SF := xCoord firstPt
    yMin : SF := yCoord firstPt; yMax : SF := yCoord firstPt
    for list in lists repeat
      for pt in list repeat
        if not Pnan? pt then
          xMin := min(xMin,xCoord pt)
          xMax := max(xMax,xCoord pt)
          yMin := min(yMin,yCoord pt)
          yMax := max(yMax,yCoord pt)
    [lists,segment(xMin,xMax),segment(yMin,yMax)]

  clipParametric(plot,fraction,scale) ==
    iClipParametric(listBranches plot,fraction,scale)

  clipParametric plot == clipParametric(plot,1/2,5/1)

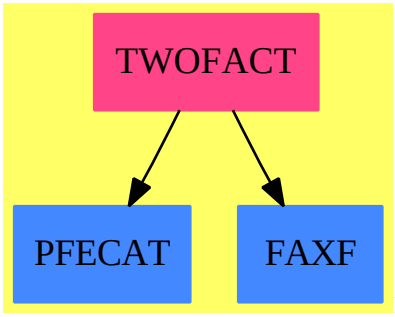
  clip(l: L Pt)   == iClipParametric(list l,1/2,5/1)
  clip(l: L L Pt) == iClipParametric(l,1/2,5/1)
```

⟨*CLIP.dotabb*⟩≡

```
"CLIP" [color="#FF4488",href="bookvol10.4.pdf#nameddest=CLIP"]
"PTCAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PTCAT"]
"CLIP" -> "PTCAT"
```

21.47 package TWOFACT TwoFactorize

21.48 TwoFactorize



1.00

Exports:
generalSqFr generalTwoFactor twoFactor

⟨*package TWOFACT TwoFactorize*⟩≡
```
)abbrev package TWOFACT TwoFactorize
++ Authors : P.Gianni, J.H.Davenport
++ Date Created : May 1990
++ Date Last Updated: March 1992
++ Description:
++ A basic package for the factorization of bivariate polynomials
++ over a finite field.
++ The functions here represent the base step for the multivariate factorizer.

TwoFactorize(F) : C == T
 where
  F           :   FiniteFieldCategory
  SUP         ==>  SparseUnivariatePolynomial
  R           ==>  SUP F
  P           ==>  SUP R
  UPCF2       ==>  UnivariatePolynomialCategoryFunctions2

  C == with
    generalTwoFactor    : (P)  ->  Factored P
      ++ generalTwoFactor(p) returns the factorisation of polynomial p,
      ++ a sparse univariate polynomial (sup) over a
      ++ sup over F.
    generalSqFr    : (P)  ->  Factored P
      ++ generalSqFr(p) returns the square-free factorisation of polynomial p,
      ++ a sparse univariate polynomial (sup) over a
      ++ sup over F.
    twoFactor    : (P,Integer)  ->  Factored P
```

```
      ++ twoFactor(p,n) returns the factorisation of polynomial p,
      ++ a sparse univariate polynomial (sup) over a
      ++ sup over F.
      ++ Also, p is assumed primitive and square-free and n is the
      ++ degree of the inner variable of p (maximum of the degrees
      ++ of the coefficients of p).

  T == add
    PI ==> PositiveInteger
    NNI ==> NonNegativeInteger
    import CommuteUnivariatePolynomialCategory(F,R,P)

                  ----  Local Functions  ----
    computeDegree  :  (P,Integer,Integer) -> PI
    exchangeVars   :           P          -> P
    exchangeVarTerm:        (R, NNI)      -> P
    pthRoot        :     (R, NNI, NNI)    -> R

    -- compute the degree of the extension to reduce the polynomial to a
    -- univariate one
    computeDegree(m : P,mx:Integer,q:Integer): PI ==
      my:=degree m
      n1:Integer:=length(10*mx*my)
      n2:Integer:=length(q)-1
      n:=(n1 quo n2)+1
      n::PI
--      n=1 => 1$PositiveInteger
--      (nextPrime(max(n,min(mx,my)))$IntegerPrimesPackage(Integer))::PI

    exchangeVars(p : P) : P ==
       p = 0 => 0
       exchangeVarTerm(leadingCoefficient p, degree p) +
          exchangeVars(reductum p)

    exchangeVarTerm(c:R, e:NNI) : P ==
       c = 0 => 0
       monomial(monomial(leadingCoefficient c, e)$R, degree c)$P +
          exchangeVarTerm(reductum c, e)

    pthRoot(poly:R,p:NonNegativeInteger,PthRootPow:NonNegativeInteger):R ==
       tmp:=divideExponents(map((#1::F)**PthRootPow,poly),p)
       tmp case "failed" => error "consistency error in TwoFactor"
       tmp

    fUnion ==> Union("nil", "sqfr", "irred", "prime")
    FF     ==> Record(flg:fUnion, fctr:P, xpnt:Integer)
```

```
generalSqFr(m:P): Factored P ==
   m = 0 => 0
   degree m = 0 =>
     l:=squareFree(leadingCoefficient m)
     makeFR(unit(l)::P,[[u.flg,u.fctr::P,u.xpnt] for u in factorList l])
   cont := content m
   m := (m exquo cont)::P
   sqfrm := squareFree m
   pfaclist : List FF := empty()
   unitPart := unit sqfrm
   for u in factorList sqfrm repeat
      u.flg = "nil" =>
         uexp:NNI:=(u.xpnt):NNI
         nfacs:=squareFree(exchangeVars u.fctr)
         for v in factorList nfacs repeat
            pfaclist:=cons([v.flg, exchangeVars v.fctr, v.xpnt*uexp],
                           pfaclist)
         unitPart := unit(nfacs)**uexp * unitPart
      pfaclist := cons(u,pfaclist)
   cont ^= 1 =>
       sqp := squareFree cont
       contlist:=[[w.flg,(w.fctr)::P,w.xpnt] for w in factorList sqp]
       pfaclist:= append(contlist, pfaclist)
       makeFR(unit(sqp)*unitPart,pfaclist)
   makeFR(unitPart,pfaclist)


generalTwoFactor(m:P): Factored P ==
   m = 0 => 0
   degree m = 0 =>
     l:=factor(leadingCoefficient m)$DistinctDegreeFactorize(F,R)
     makeFR(unit(l)::P,[[u.flg,u.fctr::P,u.xpnt] for u in factorList l])
   ll:List FF
   ll:=[]
   unitPart:P
   cont:=content m
   if degree(cont)>0 then
      m1:=m exquo cont
      m1 case "failed" => error "content doesn't divide"
      m:=m1
      contfact:=factor(cont)$DistinctDegreeFactorize(F,R)
      unitPart:=(unit contfact)::P
      ll:=[[w.flg,(w.fctr)::P,w.xpnt] for w in factorList contfact]
   else
      unitPart:=cont::P
```

```
      sqfrm:=squareFree m
      for u in factors sqfrm repeat
          expo:=u.exponent
          if expo < 0 then error "negative exponent in a factorisation"
          expon:NonNegativeInteger:=expo::NonNegativeInteger
          fac:=u.factor
          degree fac = 1 => ll:=[["irred",fac,expon],:ll]
          differentiate fac = 0 =>
             -- the polynomial is  inseparable w.r.t. its main variable
             map(differentiate,fac) = 0 =>
               p:=characteristic$F
               PthRootPow:=(size$F exquo p)::NonNegativeInteger
               m1:=divideExponents(map(pthRoot(#1,p,PthRootPow),fac),p)
               m1 case "failed" => error "consistency error in TwoFactor"
               res:=generalTwoFactor m1
               unitPart:=unitPart*unit(res)**((p*expon)::NNI)
               ll:=[:[[v.flg,v.fctr,expon *p*v.xpnt] for v in factorList res],:ll]
             m2:=generalTwoFactor swap fac
             unitPart:=unitPart*unit(m2)**(expon::NNI)
             ll:=[:[[v.flg,swap v.fctr,expon*v.xpnt] for v in factorList m2],:ll]
          ydeg:="max"/[degree w for w in coefficients fac]
          twoF:=twoFactor(fac,ydeg)
          unitPart:=unitPart*unit(twoF)**expon
          ll:=[:[[v.flg,v.fctr,expon*v.xpnt] for v in factorList twoF],
               :ll]
      makeFR(unitPart,ll)

   -- factorization of a primitive square-free bivariate polynomial --
   twoFactor(m:P,dx:Integer):Factored P ==
      -- choose the degree for the extension
      n:PI:=computeDegree(m,dx,size()$F)
      -- extend the field
      -- find the substitution for x
      look:Boolean:=true
      dm:=degree m
      try:Integer:=min(5,size()$F)
      i:Integer:=0
      lcm := leadingCoefficient m
      umv : R
      while look and i < try repeat
         vval := random()$F
         i:=i+1
         zero? elt(lcm, vval) => "next value"
         umv := map(elt(#1,vval), m)$UPCF2(R, P, F, R)
         degree(gcd(umv,differentiate umv))^=0 => "next val"
         n := 1
```

```
   look := false
extField:=FiniteFieldExtension(F,n)
SUEx:=SUP extField
TP:=SparseUnivariatePolynomial SUEx
mm:TP:=0
m1:=m
while m1^=0 repeat
  mm:=mm+monomial(map(coerce,leadingCoefficient m1)$UPCF2(F,R,
         extField,SUEx),degree m1)
  m1:=reductum m1
lcmm := leadingCoefficient mm
val : extField
umex : SUEx
if not look then
   val := vval :: extField
   umex := map(coerce, umv)$UPCF2(F, R, extField, SUEx)
while look repeat
  val:=random()$extField
  i:=i+1
  elt(lcmm,val)=0 => "next value"
  umex := map(elt(#1,val), mm)$UPCF2(SUEx, TP, extField, SUEx)
  degree(gcd(umex,differentiate umex))^=0 => "next val"
  look:=false
prime:SUEx:=monomial(1,1)-monomial(val,0)
fumex:=factor(umex)$DistinctDegreeFactorize(extField,SUEx)
lfact1:=factors fumex

#lfact1=1 => primeFactor(m,1)
lfact : List TP :=
   [map(coerce,lf.factor)$UPCF2(extField,SUEx,SUEx,TP)
    for lf in lfact1]
lfact:=cons(map(coerce,unit fumex)$UPCF2(extField,SUEx,SUEx,TP),
           lfact)
import GeneralHenselPackage(SUEx,TP)
dx1:PI:=(dx+1)::PI
lfacth:=completeHensel(mm,lfact,prime,dx1)
lfactk: List P :=[]
Normp := NormRetractPackage(F, extField, SUEx, TP, n)

while not empty? lfacth repeat
  ff := first lfacth
  lfacth := rest lfacth
  if (c:=leadingCoefficient leadingCoefficient ff) ^=1 then
    ff:=((inv c)::SUEx)* ff
  not ((ffu:= retractIfCan(ff)$Normp) case "failed") =>
            lfactk := cons(ffu::P, lfactk)
```

```
      normfacs := normFactors(ff)$Normp
      lfacth := [g for g in lfacth | not member?(g, normfacs)]
      ffn := */normfacs
      lfactk:=cons(retractIfCan(ffn)$Normp :: P, lfactk)
    */[primeFactor(ff1,1) for ff1 in lfactk]
```

⟨*TWOFACT.dotabb*⟩≡

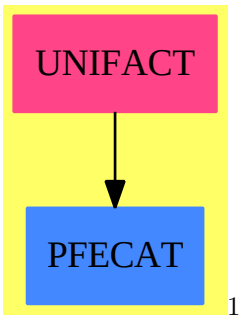
```
"TWOFACT" [color="#FF4488",href="bookvol10.4.pdf#nameddest=TWOFACT"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"FAXF" [color="#4488FF",href="bookvol10.2.pdf#nameddest=FAXF"]
"TWOFACT" -> "PFECAT"
"TWOFACT" -> "FAXF"
```

Chapter 22

Chapter U

22.1 package UNIFACT UnivariateFactorize

22.2 UnivariateFactorize



1.00

Exports:
factor factorSquareFree henselFact

⟨*package UNIFACT UnivariateFactorize*⟩≡
```
)abbrev package UNIFACT UnivariateFactorize
++ Factorisation of univariate polynomials  with integer coefficients
++ Author: Patrizia Gianni
++ Date Created: ???
++ Date Last Updated: December 1993
++ Description:
++ Package for the factorization of univariate polynomials with integer
++ coefficients. The factorization is done by "lifting" (HENSEL) the
++ factorization over a finite field.
UnivariateFactorize(ZP) : public == private where
  Z    ==>  Integer
```

```
PI   ==>  PositiveInteger
NNI  ==>  NonNegativeInteger
SUPZ ==>  SparseUnivariatePolynomial Z

ZP : UnivariatePolynomialCategory Z

FR        ==>  Factored ZP
fUnion    ==>  Union("nil", "sqfr", "irred", "prime")
FFE       ==>  Record(flg:fUnion, fctr:ZP, xpnt:Z)
ParFact   ==>  Record(irr: ZP,pow: Z)
FinalFact ==>  Record(contp: Z,factors:List(ParFact))


public == with
   factor            :           ZP              -> FR
     ++ factor(m) returns the factorization of m
   factorSquareFree  :           ZP              -> FR
     ++ factorSquareFree(m) returns the factorization of m square free
     ++ polynomial
   henselFact        :      (ZP,Boolean)         -> FinalFact
     ++ henselFact(m,flag) returns the factorization of m,
     ++ FinalFact is a Record s.t. FinalFact.contp=content m,
     ++ FinalFact.factors=List of irreducible factors
     ++ of m with exponent , if flag =true the polynomial is
     ++ assumed square free.

private == add
               --- local functions ---

   henselfact  :           ZP      -> List(ZP)
   quadratic   :           ZP      -> List(ZP)
   remp        :        (Z, PI)    -> Z
   negShiftz   :        (Z, PI)    -> Z
   negShiftp   :        (ZP,PI)    -> ZP
   bound       :           ZP      -> PI
   choose      :           ZP      -> FirstStep
   eisenstein  :           ZP      -> Boolean
   isPowerOf2  :           Z       -> Boolean
   subMinusX   :          SUPZ     -> ZP
   sqroot      :           Z       -> Z

               ---   declarations  ---
   CYC       ==> CyclotomicPolynomialPackage()
   DDRecord  ==> Record(factor: ZP,degree: Z)
   DDList    ==> List DDRecord
   FirstStep ==> Record(prime:PI,factors:DDList)
```

```
ContPrim  ==> Record(cont: Z,prim: ZP)

import GeneralHenselPackage(Z,ZP)
import ModularDistinctDegreeFactorizer ZP


factor(m: ZP) ==
  flist := henselFact(m,false)
  ctp:=unitNormal flist.contp
  makeFR((ctp.unit)::ZP,cons(["nil",ctp.canonical::ZP,1$Z]$FFE,
                [["prime",u.irr,u.pow]$FFE for u in flist.factors]))

factorSquareFree(m: ZP) ==
  flist := henselFact(m,true)
  ctp:=unitNormal flist.contp
  makeFR((ctp.unit)::ZP,cons(["nil",ctp.canonical::ZP,1$Z]$FFE,
               [["prime",u.irr,u.pow]$FFE for u in flist.factors]))


-- Integer square root: returns 0 if t is non-positive
sqroot(t: Z): Z  ==
 t <= 0 => 0
 s:Integer:=t::Integer
 s:=approxSqrt(s)$IntegerRoots(Integer)
 t:=s::Z
 t

-- Eisenstein criterion: returns true if polynomial is
-- irreducible. Result of false in inconclusive.
eisenstein(m : ZP): Boolean ==
  -- calculate the content of the terms after the first
  c := content reductum m
  c = 0 => false
  c = 1 => false
  -- factor the content
  -- if there is a prime in the factorization that does not divide
  -- the leading term and appears to multiplicity 1, and the square
  -- of this does not divide the last coef, return true.
  -- Otherwise reurn false.
  lead := leadingCoefficient m
  trail := lead
  m := reductum m
  while m ^= 0 repeat
    trail := leadingCoefficient m
    m:= reductum m
  fc := factor(c) :: Factored(Z)
```

```
  for r in factors fc repeat
    if (r.exponent = 1) and (0 ^= (lead rem r.factor)) and
      (0 ^= (trail rem (r.factor ** 2))) then return true
  false

negShiftz(n: Z,Modulus:PI): Z ==
  if n < 0 then n := n+Modulus
  n > (Modulus quo 2) => n-Modulus
  n

negShiftp(pp: ZP,Modulus:PI): ZP ==
  map(negShiftz(#1,Modulus),pp)

-- Choose the bound for the coefficients of factors
bound(m: ZP):PI ==
  nm,nmq2,lcm,bin0,bin1:NNI
  cbound,j : PI
  k:NNI
  lcm := abs(leadingCoefficient m)::NNI
  nm := (degree m)::NNI
  nmq2:NNI := nm quo 2
  norm: Z := sqroot(+/[coefficient(m,k)**2 for k in 0..nm])
  if nmq2^=1 then nm := (nmq2-1):NNI
  else nm := nmq2
  bin0 := nm
  cbound := (bin0*norm+lcm)::PI
  for i in 2..(nm-1)::NNI repeat
    bin1 := bin0
    bin0 := (bin0*(nm+1-i):NNI) quo i
    j := (bin0*norm+bin1*lcm)::PI
    if cbound<j then cbound := j
  (2*cbound*lcm)::PI -- adjusted by lcm to prepare for exquo in ghensel

remp(t: Z,q:PI): Z == ((t := t rem q)<0 => t+q ;t)

numFactors(ddlist:DDList): Z ==
  ans: Z := 0
  for dd in ddlist repeat
    (d := degree(dd.factor)) = 0 => nil
    ans := ans + ((d pretend Z) exquo dd.degree):: Z
  ans

-- select the prime,try up to 4 primes,
-- choose the one yielding the fewest factors, but stopping if
-- fewer than 9 factors
choose(m: ZP):FirstStep ==
```

```
        qSave:PI := 1
        ddSave:DDList := []
        numberOfFactors: Z := 0
        lcm := leadingCoefficient m
        k: Z := 1
        ddRep := 5
        disc:ZP:=0
        q:PI:=2
        while k<ddRep repeat
          -- q must be a new prime number at each iteration
          q:=nextPrime(q)$IntegerPrimesPackage(Z) pretend PI
          (rr:=lcm rem q) = 0$Z => "next prime"
          disc:=gcd(m,differentiate m,q)
          (degree disc)^=0 => "next prime"
          k := k+1
          newdd := ddFact(m,q)
          ((n := numFactors(newdd)) < 9) =>
            ddSave := newdd
            qSave := q
            k := 5
          (numberOfFactors = 0) or (n < numberOfFactors) =>
            ddSave := newdd
            qSave := q
            numberOfFactors := n
        [qSave,ddSave]$FirstStep

      -- Find the factors of m,primitive, square-free, with lc positive
      -- and mindeg m = 0
      henselfact1(m: ZP):List(ZP) ==
       zero? degree m =>
--           one? m => []
           (m = 1) => []
           [m]
       selected := choose(m)
       (numFactors(selected.factors) = 1$Z) => [m]
       q := selected.prime
       fl := separateFactors(selected.factors,q)
       --choose the bound
       cbound := bound(m)
       completeHensel(m,fl,q,cbound)

      -- check for possible degree reduction
      -- could use polynomial decomposition ?
      henselfact(m: ZP):List ZP ==
       deggcd:=degree m
       mm:= m
```

```
        while not zero? mm repeat (deggcd:=gcd(deggcd, degree mm); mm:=reductum mm)
        deggcd>1 and deggcd<degree m =>
           faclist := henselfact1(divideExponents(m, deggcd)::ZP)
           "append"/[henselfact1 multiplyExponents(mm, deggcd) for mm in faclist]
        henselfact1 m

      quadratic(m: ZP):List(ZP) ==
        d,d2: Z
        d := coefficient(m,1)**2-4*coefficient(m,0)*coefficient(m,2)
        d2 := sqroot(d)
        (d-d2**2)^=0 => [m]
        alpha: Z := coefficient(m,1)+d2
        beta: Z := 2*coefficient(m,2)
        d := gcd(alpha,beta)
        if d ^=1 then
          alpha := alpha quo d
          beta := beta quo d
        m0: ZP := monomial(beta,1)+monomial(alpha,0)
        cons(m0,[(m exquo m0):: ZP])

      isPowerOf2(n : Z): Boolean ==
        n = 1 => true
        qr : Record(quotient: Z, remainder: Z) := divide(n,2)
        qr.remainder = 1 => false
        isPowerOf2 qr.quotient

      subMinusX(supPol : SUPZ): ZP ==
        minusX : SUPZ := monomial(-1,1)$SUPZ
        (elt(supPol,minusX)$SUPZ) : ZP

--    Factorize the polynomial m, test=true if m is known to be
--    square-free, false otherwise.
--    FinalFact.contp=content m, FinalFact.factors=List of irreducible
--    factors with exponent .
      henselFact(m: ZP,test:Boolean):FinalFact ==
       factorlist : List(ParFact) := []
       c : Z

       -- make m primitive
       c := content m
       m := (m exquo c)::ZP

       -- make the lc m positive
       if leadingCoefficient m < 0 then
         c := -c
         m := -m
```

```
-- is x**d factor of m?
if (d := minimumDegree m) >0 then
  m := (monicDivide(m,monomial(1,d))).quotient
  factorlist := [[monomial(1,1),d]$ParFact]

d := degree m

-- is m constant?
d=0 => [c,factorlist]$FinalFact

-- is m linear?
d=1 => [c,cons([m,1]$ParFact,factorlist)]$FinalFact

-- does m satisfy Eisenstein's criterion?
eisenstein m => [c,cons([m,1]$ParFact,factorlist)]$FinalFact

lcPol : ZP := leadingCoefficient(m) :: ZP

-- is m cyclotomic (x**n - 1)?
-lcPol = reductum(m) =>    -- if true, both will = 1
  for fac in
    (cyclotomicDecomposition(degree m)$CYC : List ZP) repeat
      factorlist := cons([fac,1]$ParFact,factorlist)
  [c,factorlist]$FinalFact

-- is m odd cyclotomic (x**(2*n+1) + 1)?
odd?(d) and (lcPol = reductum(m)) =>
  for sfac in cyclotomicDecomposition(degree m)$CYC repeat
     fac:=subMinusX sfac
     if leadingCoefficient fac < 0 then fac := -fac
     factorlist := cons([fac,1]$ParFact,factorlist)
  [c,factorlist]$FinalFact

-- is the poly of the form x**n + 1 with n a power of 2?
-- if so, then irreducible
isPowerOf2(d) and (lcPol = reductum(m)) =>
  factorlist := cons([m,1]$ParFact,factorlist)
  [c,factorlist]$FinalFact

-- is m quadratic?
d=2 =>
 lfq:List(ZP) := quadratic m
 #lfq=1 => [c,cons([lfq.first,1]$ParFact,factorlist)]$FinalFact
 (lf0,lf1) := (lfq.first,second lfq)
 if lf0=lf1 then factorlist := cons([lf0,2]$ParFact,factorlist)
```

```
 else factorlist := append([[v,1]$ParFact for v in lfq],factorlist)
 [c,factorlist]$FinalFact

-- m is square-free
test =>
  fln := henselfact(m)
  [c,append(factorlist,[[pf,1]$ParFact for pf in fln])]$FinalFact

-- find the square-free decomposition of m
irrFact := squareFree(m)
llf := factors irrFact

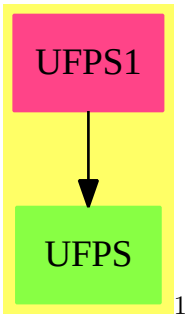
-- factorize the square-free primitive terms
for l1 in llf repeat
  d1 := l1.exponent
  pol := l1.factor
  degree pol=1 => factorlist := cons([pol,d1]$ParFact,factorlist)
  degree pol=2 =>
    fln := quadratic(pol)
    factorlist := append([[pf,d1]$ParFact for pf in fln],factorlist)
  fln := henselfact(pol)
  factorlist := append([[pf,d1]$ParFact for pf in fln],factorlist)
[c,factorlist]$FinalFact
```

⟨*UNIFACT.dotabb*⟩≡

```
"UNIFACT" [color="#FF4488",href="bookvol10.4.pdf#nameddest=UNIFACT"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"UNIFACT" -> "PFECAT"
```

22.3 package UFPS1 UnivariateFormalPowerSeriesFunctions

22.4 UnivariateFormalPowerSeriesFunctions



1.00

Exports:
hadamard

⟨*package UFPS1 UnivariateFormalPowerSeriesFunctions*⟩≡
```
)abbrev package UFPS1 UnivariateFormalPowerSeriesFunctions
UnivariateFormalPowerSeriesFunctions(Coef: Ring): Exports == Implementation
  where

    UFPS ==> UnivariateFormalPowerSeries Coef

    Exports == with

      hadamard: (UFPS, UFPS) -> UFPS

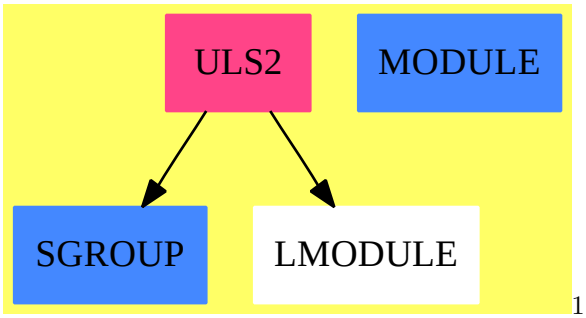
    Implementation == add

      hadamard(f, g) ==
        series map(#1*#2, coefficients f, coefficients g)
                  $StreamFunctions3(Coef, Coef, Coef)
```

⟨*UFPS1.dotabb*⟩≡
```
"UFPS1" [color="#FF4488",href="bookvol10.4.pdf#nameddest=UFPS1"]
"UFPS" [color="#88FF44",href="bookvol10.3.pdf#nameddest=UFPS"]
"UFPS1" -> "UFPS"
```

22.5 package ULS2 UnivariateLaurentSeriesFunctions2

22.6 UnivariateLaurentSeriesFunctions2



1.00

Exports:
map

⟨*package ULS2 UnivariateLaurentSeriesFunctions2*⟩≡

```
)abbrev package ULS2 UnivariateLaurentSeriesFunctions2
++ Author: Clifton J. Williamson
++ Date Created: 5 March 1990
++ Date Last Updated: 5 March 1990
++ Basic Operations:
++ Related Domains:
++ Also See:
++ AMS Classifications:
++ Keywords: Laurent series, map
++ Examples:
++ References:
++ Description: Mapping package for univariate Laurent series
++   This package allows one to apply a function to the coefficients of
++   a univariate Laurent series.
UnivariateLaurentSeriesFunctions2(Coef1,Coef2,var1,var2,cen1,cen2):_
 Exports == Implementation where
  Coef1 : Ring
  Coef2 : Ring
  var1: Symbol
  var2: Symbol
  cen1: Coef1
  cen2: Coef2
  ULS1  ==> UnivariateLaurentSeries(Coef1, var1, cen1)
  ULS2  ==> UnivariateLaurentSeries(Coef2, var2, cen2)
  UTS1  ==> UnivariateTaylorSeries(Coef1, var1, cen1)
  UTS2  ==> UnivariateTaylorSeries(Coef2, var2, cen2)
```

```
    UTSF2 ==> UnivariateTaylorSeriesFunctions2(Coef1, Coef2, UTS1, UTS2)

    Exports ==> with
      map: (Coef1 -> Coef2,ULS1) -> ULS2
        ++ \spad{map(f,g(x))} applies the map f to the coefficients of the Laurent
        ++ series \spad{g(x)}.

    Implementation ==> add

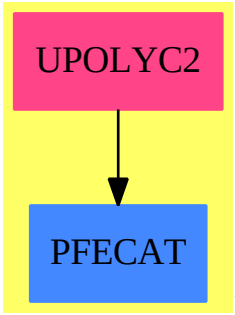
      map(f,ups) == laurent(degree ups, map(f, taylorRep ups)$UTSF2)
```

⟨*ULS2.dotabb*⟩≡

```
"ULS2" [color="#FF4488",href="bookvol10.4.pdf#nameddest=ULS2"]
"MODULE" [color="#4488FF",href="bookvol10.2.pdf#nameddest=MODULE"]
"SGROUP" [color="#4488FF",href="bookvol10.2.pdf#nameddest=SGROUP"]
"ULS2" -> "LMODULE"
"ULS2" -> "SGROUP"
```

22.7 package UPOLYC2 UnivariatePolynomialCategoryFunctions2

22.8 UnivariatePolynomialCategoryFunctions2



1.00

Exports:
map

⟨*package UPOLYC2 UnivariatePolynomialCategoryFunctions2*⟩≡

```
)abbrev package UPOLYC2 UnivariatePolynomialCategoryFunctions2
++ Author:
++ Date Created:
++ Date Last Updated:
++ Basic Functions:
++ Related Constructors:
++ Also See:
++ AMS Classifications:
++ Keywords:
++ References:
++ Description:
++ Mapping from polynomials over R to polynomials over S
++ given a map from R to S assumed to send zero to zero.

UnivariatePolynomialCategoryFunctions2(R,PR,S,PS): Exports == Impl where
  R, S: Ring
  PR  : UnivariatePolynomialCategory R
  PS  : UnivariatePolynomialCategory S

  Exports ==> with
    map: (R -> S, PR) -> PS
     ++ map(f, p) takes a function f from R to S,
     ++ and applies it to each (non-zero) coefficient of a polynomial p
     ++ over R, getting a new polynomial over S.
     ++ Note: since the map is not applied to zero elements, it may map zero
     ++ to zero.
```

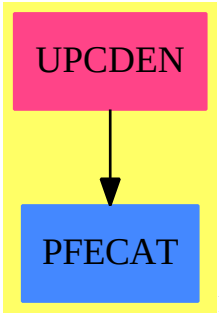
```
    Impl ==> add
      map(f, p) ==
        ans:PS := 0
        while p ^= 0 repeat
          ans := ans + monomial(f leadingCoefficient p, degree p)
          p   := reductum p
        ans
```

⟨*UPOLYC2.dotabb*⟩≡

```
"UPOLYC2" [color="#FF4488",href="bookvol10.4.pdf#nameddest=UPOLYC2"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"UPOLYC2" -> "PFECAT"
```

22.9 package UPCDEN UnivariatePolynomialCommonDenominator

22.10 UnivariatePolynomialCommonDenominator



1.00

Exports:
clearDenominator commonDenominator splitDenominator

⟨*package UPCDEN UnivariatePolynomialCommonDenominator*⟩≡

```
)abbrev package UPCDEN UnivariatePolynomialCommonDenominator
--% UnivariatePolynomialCommonDenominator
++ Author: Manuel Bronstein
++ Date Created: 2 May 1988
++ Date Last Updated: 22 Feb 1990
++ Description: UnivariatePolynomialCommonDenominator provides
++ functions to compute the common denominator of the coefficients of
++ univariate polynomials over the quotient field of a gcd domain.
++ Keywords: gcd, quotient, common, denominator, polynomial.

UnivariatePolynomialCommonDenominator(R, Q, UP): Exports == Impl where
  R : IntegralDomain
  Q : QuotientFieldCategory R
  UP: UnivariatePolynomialCategory Q

  Exports ==> with
    commonDenominator: UP -> R
      ++ commonDenominator(q) returns a common denominator d for
      ++ the coefficients of q.
    clearDenominator : UP -> UP
      ++ clearDenominator(q) returns p such that \spad{q = p/d} where d is
      ++ a common denominator for the coefficients of q.
    splitDenominator : UP -> Record(num: UP, den: R)
      ++ splitDenominator(q) returns \spad{[p, d]} such that \spad{q = p/d} and d
      ++ is a common denominator for the coefficients of q.
```

```
Impl ==> add
  import CommonDenominator(R, Q, List Q)

  commonDenominator p == commonDenominator coefficients p

  clearDenominator p ==
    d := commonDenominator p
    map(numer(d * #1)::Q, p)

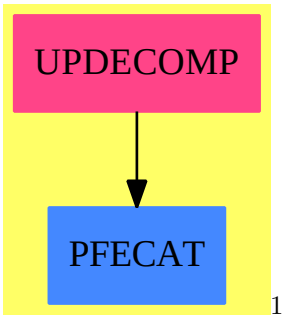
  splitDenominator p ==
    d := commonDenominator p
    [map(numer(d * #1)::Q, p), d]
```

⟨*UPCDEN.dotabb*⟩≡

```
"UPCDEN" [color="#FF4488",href="bookvol10.4.pdf#nameddest=UPCDEN"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"UPCDEN" -> "PFECAT"
```

22.11 package UPDECOMP UnivariatePolynomialDecompositionPackage

22.12 UnivariatePolynomialDecompositionPackage



1.00

Exports:

leftFactorIfCan	monicCompleteDecompose	monicDecomposeIfCan
monicRightFactorIfCan	rightFactorIfCan	

⟨*package UPDECOMP UnivariatePolynomialDecompositionPackage*⟩≡

```
)abbrev package UPDECOMP UnivariatePolynomialDecompositionPackage
++ Author: Frederic Lehobey
++ Date Created: 17 June 1996
++ Date Last Updated: 4 June 1997
++ Basic Operations:
++ Related Domains:
++ Also See:
++ AMS Classifications:
++ Keyword:
++ Exemples:
++ References:
++ [1] Peter Henrici, Automatic Computations with Power Series,
++ Journal of the Association for Computing Machinery, Volume 3, No. 1,
++ January 1956, 10-15
++ [2] Dexter Kozen and Susan Landau, Polynomial Decomposition
++ Algorithms, Journal of Symbolic Computation (1989) 7, 445-456
-- Decomposition would be speeded up (O(n log n) instead of O(n^2)) by
-- implementing the algorithm described in [3] based on [4] and [5].
++ [3] Joachim von zur Gathen, Functional Decomposition Polynomials:
++ the Tame Case, Journal of Symbolic Computation (1990) 9, 281-299
++ [4] R. P. Brent and H. T. Kung, Fast Algorithms for Manipulating
++ Formal Power Series, Journal of the Association for Computing
++ Machinery, Vol. 25, No. 4, October 1978, 581-595
++ [5] R. P. Brent, Multiple-Precision Zero-Finding Methods and the
++ Complexity of Elementary Function Evaluation, Analytic
```

```
++ Computational Complexity, J. F. Traub, Ed., Academic Press,
++ New York 1975, 151-176
++ Description: UnivariatePolynomialDecompositionPackage implements
++ functional decomposition of univariate polynomial with coefficients
++ in an \spad{IntegralDomain} of \spad{CharacteristicZero}.
UnivariatePolynomialDecompositionPackage(R,UP): Exports == Implementation where
  R : Join(IntegralDomain,CharacteristicZero)
  UP : UnivariatePolynomialCategory(R)
  N ==> NonNegativeInteger
  LR ==> Record(left: UP, right: UP)
  QR ==> Record(quotient: UP, remainder: UP)


  Exports ==> with

    monicRightFactorIfCan: (UP,N) -> Union(UP,"failed")
      ++ monicRightFactorIfCan(f,d) returns a candidate to be the
      ++ monic right factor (h in f = g o h) of degree d of a
      ++ functional decomposition of the polynomial f or
      ++ \spad{"failed"} if no such candidate.
    rightFactorIfCan: (UP,N,R) -> Union(UP,"failed")
      ++ rightFactorIfCan(f,d,c) returns a candidate to be the
      ++ right factor (h in f = g o h) of degree d with leading
      ++ coefficient c of a functional decomposition of the
      ++ polynomial f or \spad{"failed"} if no such candidate.
    leftFactorIfCan: (UP,UP) -> Union(UP,"failed")
      ++ leftFactorIfCan(f,h) returns the left factor (g in f = g o h)
      ++ of the functional decomposition of the polynomial f with
      ++ given h or \spad{"failed"} if g does not exist.
    monicDecomposeIfCan: UP -> Union(LR,"failed")
      ++ monicDecomposeIfCan(f) returns a functional decomposition
      ++ of the monic polynomial f of "failed" if it has not found any.
    monicCompleteDecompose: UP -> List UP
      ++ monicCompleteDecompose(f) returns a list of factors of f for
      ++ the functional decomposition ([ f1, ..., fn ] means
      ++ f = f1 o ... o fn).

  Implementation ==> add

    rightFactorIfCan(p,dq,lcq) ==
      dp := degree p
      zero? lcq =>
       error "rightFactorIfCan: leading coefficient may not be zero"
      (zero? dp) or (zero? dq) => "failed"
      nc := dp exquo dq
      nc case "failed" => "failed"
```

```
    n := nc::N
    s := subtractIfCan(dq,1)::N
    lcp := leadingCoefficient p
    q: UP := monomial(lcq,dq)
    k: N
    for k in 1..s repeat
      c: R := 0
      i: N
      for i in 0..subtractIfCan(k,1)::N repeat
       c := c+(k::R-(n::R+1)*(i::R))*
        coefficient(q,subtractIfCan(dq,i)::N)*
         coefficient(p,subtractIfCan(dp+i,k)::N)
      cquo := c exquo ((k*n)::R*lcp)
      cquo case "failed" => return "failed"
      q := q+monomial(cquo::R,subtractIfCan(dq,k)::N)
    q

  monicRightFactorIfCan(p,dq) == rightFactorIfCan(p,dq,1$R)

  import UnivariatePolynomialDivisionPackage(R,UP)

  leftFactorIfCan(f,h) ==
    g: UP := 0
    zero? degree h => "failed"
    for i in 0.. while not zero? f repeat
      qrf := divideIfCan(f,h)
      qrf case "failed" => return "failed"
      qr := qrf :: QR
      r := qr.remainder
      not ground? r => return "failed"
      g := g+monomial(ground(r),i)
      f := qr.quotient
    g

  monicDecomposeIfCan f ==
    df := degree f
    zero? df => "failed"
    for dh in 2..subtractIfCan(df,1)::N | zero?(df rem dh) repeat
      h := monicRightFactorIfCan(f,dh)
      h case UP =>
       g := leftFactorIfCan(f,h::UP)
       g case UP => return [g::UP,h::UP]
    "failed"

  monicCompleteDecompose f ==
    cf := monicDecomposeIfCan f
```

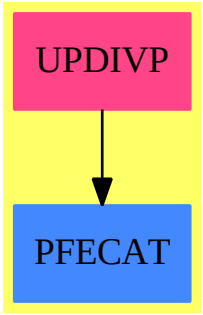
```
        cf case "failed" => [ f ]
        lr := cf :: LR
        append(monicCompleteDecompose lr.left,[lr.right])
```

⟨*UPDECOMP.dotabb*⟩≡

```
"UPDECOMP" [color="#FF4488",href="bookvol10.4.pdf#nameddest=UPDECOMP"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"UPDECOMP" -> "PFECAT"
```

22.13 package UPDIVP UnivariatePolynomialDivisionPackage

22.14 UnivariatePolynomialDivisionPackage



1.00

Exports:
divideIfCan

⟨*package UPDIVP UnivariatePolynomialDivisionPackage*⟩≡

```
)abbrev package UPDIVP UnivariatePolynomialDivisionPackage
++ Author: Frederic Lehobey
++ Date Created: 3 June 1997
++ Date Last Updated: 3 June 1997
++ Basic Operations:
++ Related Domains:
++ Also See:
++ AMS Classifications:
++ Keyword:
++ Exemples:
++ References:
++ Description: UnivariatePolynomialDivisionPackage provides a
++ division for non monic univarite polynomials with coefficients in
++ an \spad{IntegralDomain}.
UnivariatePolynomialDivisionPackage(R,UP): Exports == Implementation where
  R : IntegralDomain
  UP : UnivariatePolynomialCategory(R)
  N ==> NonNegativeInteger
  QR ==> Record(quotient: UP, remainder: UP)

  Exports ==> with

    divideIfCan: (UP,UP) -> Union(QR,"failed")
      ++ divideIfCan(f,g) returns quotient and remainder of the
      ++ division of f by g or "failed" if it has not succeeded.
```

```
    Implementation ==> add

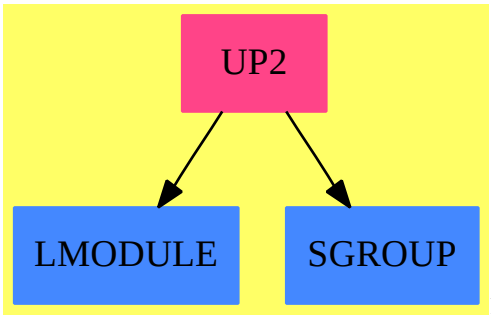
      divideIfCan(p1:UP,p2:UP):Union(QR,"failed") ==
        zero? p2 => error "divideIfCan: division by zero"
  --      one? (lc := leadingCoefficient p2) => monicDivide(p1,p2)
        ((lc := leadingCoefficient p2) = 1) => monicDivide(p1,p2)
        q: UP := 0
        while not ((e := subtractIfCan(degree(p1),degree(p2))) case "failed")
         repeat
          c := leadingCoefficient(p1) exquo lc
          c case "failed" => return "failed"
          ee := e::N
          q := q+monomial(c::R,ee)
          p1 := p1-c*mapExponents(#1+ee,p2)
        [q,p1]
```

⟨*UPDIVP.dotabb*⟩≡

```
"UPDIVP" [color="#FF4488",href="bookvol10.4.pdf#nameddest=UPDIVP"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"UPDIVP" -> "PFECAT"
```

22.15 package UP2 UnivariatePolynomialFunctions2

22.16 UnivariatePolynomialFunctions2



1.00

Exports:
map

⟨*package UP2 UnivariatePolynomialFunctions2*⟩≡

```
)abbrev package UP2 UnivariatePolynomialFunctions2
++ Author:
++ Date Created:
++ Date Last Updated:
++ Basic Functions:
++ Related Constructors:
++ Also See:
++ AMS Classifications:
++ Keywords:
++ References:
++ Description:
++ This package lifts a mapping from coefficient rings R to S to
++ a mapping from \spadtype{UnivariatePolynomial}(x,R) to
++ \spadtype{UnivariatePolynomial}(y,S). Note that the mapping is assumed
++ to send zero to zero, since it will only be applied to the non-zero
++ coefficients of the polynomial.

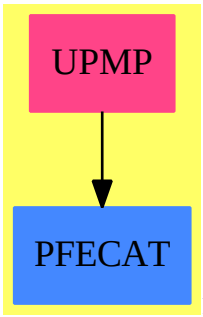
UnivariatePolynomialFunctions2(x:Symbol, R:Ring, y:Symbol, S:Ring): with
  map: (R -> S, UnivariatePolynomial(x,R)) -> UnivariatePolynomial(y,S)
    ++ map(func, poly) creates a new polynomial by applying func to
    ++ every non-zero coefficient of the polynomial poly.
 == add
  map(f, p) == map(f, p)$UnivariatePolynomialCategoryFunctions2(R,
               UnivariatePolynomial(x, R), S, UnivariatePolynomial(y, S))
```

⟨*UP2.dotabb*⟩≡

```
"UP2" [color="#FF4488",href="bookvol10.4.pdf#nameddest=UP2"]
"LMODULE" [color="#4488FF",href="bookvol10.2.pdf#nameddest=LMODULE"]
"SGROUP" [color="#4488FF",href="bookvol10.2.pdf#nameddest=SGROUP"]
"UP2" -> "LMODULE"
"UP2" -> "SGROUP"
```

22.17 package UPMP UnivariatePolynomialMultiplicationPackage

22.18 UnivariatePolynomialMultiplicationPackage



1.00

Exports:
karatsuba karatsubaOnce noKaratsuba

⟨*package UPMP UnivariatePolynomialMultiplicationPackage*⟩≡

```
)abbrev package UPMP UnivariatePolynomialMultiplicationPackage
++ Author: Marc Moreno Maza
++ Date Created: 14.08.2000
++ Description:
++ This package implements Karatsuba's trick for multiplying
++ (large) univariate polynomials. It could be improved with
++ a version doing the work on place and also with a special
++ case for squares. We've done this in Basicmath, but we
++ believe that this out of the scope of AXIOM.

UnivariatePolynomialMultiplicationPackage(R: Ring, U: UnivariatePolynomialCategor
  where
    HL ==> Record(quotient:U,remainder:U)
    C == with
      noKaratsuba: (U, U) -> U
        ++ \spad{noKaratsuba(a,b)} returns \spad{a*b} without
        ++ using Karatsuba's trick at all.
      karatsubaOnce: (U, U) -> U
        ++ \spad{karatsuba(a,b)} returns \spad{a*b} by applying
        ++ Karatsuba's trick once. The other multiplications
        ++ are performed by calling \spad{*} from \spad{U}.
      karatsuba: (U, U, NonNegativeInteger, NonNegativeInteger) -> U;
        ++ \spad{karatsuba(a,b,l,k)} returns \spad{a*b} by applying
        ++ Karatsuba's trick provided that both \spad{a} and \spad{b}
        ++ have at least \spad{l} terms and \spad{k > 0} holds
        ++ and by calling \spad{noKaratsuba} otherwise. The other
```

```
      ++ multiplications are performed by recursive calls with
      ++ the same third argument and \spad{k-1} as fourth argument.

    T == add
      noKaratsuba(a,b) ==
        zero? a => a
        zero? b => b
        zero?(degree(a)) => leadingCoefficient(a) * b
        zero?(degree(b)) => a * leadingCoefficient(b)
        lu: List(U) := reverse monomials(a)
        res: U := 0;
        for u in lu repeat
          res := pomopo!(res, leadingCoefficient(u), degree(u), b)
        res
      karatsubaOnce(a:U,b:U): U ==
        da := minimumDegree(a)
        db := minimumDegree(b)
        if not zero? da then a := shiftRight(a,da)
        if not zero? db then b := shiftRight(b,db)
        d := da + db
        n: NonNegativeInteger := min(degree(a),degree(b)) quo 2
        rec: HL := karatsubaDivide(a, n)
        ha := rec.quotient
        la := rec.remainder
        rec := karatsubaDivide(b, n)
        hb := rec.quotient
        lb := rec.remainder
        w: U := (ha - la) * (lb - hb)
        u: U := la * lb
        v: U := ha * hb
        w := w + (u + v)
        w := shiftLeft(w,n) + u
        zero? d => shiftLeft(v,2*n) + w
        shiftLeft(v,2*n + d) + shiftLeft(w,d)
      karatsuba(a:U,b:U,l:NonNegativeInteger,k:NonNegativeInteger): U ==
        zero? k => noKaratsuba(a,b)
        degree(a) < l => noKaratsuba(a,b)
        degree(b) < l => noKaratsuba(a,b)
        numberOfMonomials(a) < l => noKaratsuba(a,b)
        numberOfMonomials(b) < l => noKaratsuba(a,b)
        da := minimumDegree(a)
        db := minimumDegree(b)
        if not zero? da then a := shiftRight(a,da)
        if not zero? db then b := shiftRight(b,db)
        d := da + db
        n: NonNegativeInteger := min(degree(a),degree(b)) quo 2
```

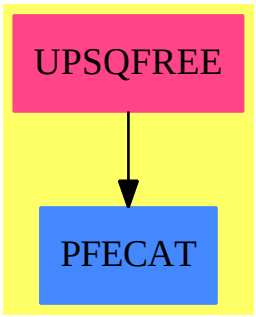
```
k := subtractIfCan(k,1)::NonNegativeInteger
rec: HL := karatsubaDivide(a, n)
ha := rec.quotient
la := rec.remainder
rec := karatsubaDivide(b, n)
hb := rec.quotient
lb := rec.remainder
w: U := karatsuba(ha - la, lb - hb, l, k)
u: U := karatsuba(la, lb, l, k)
v: U := karatsuba(ha, hb, l, k)
w := w + (u + v)
w := shiftLeft(w,n) + u
zero? d => shiftLeft(v,2*n) + w
shiftLeft(v,2*n + d) + shiftLeft(w,d)
```

⟨*UPMP.dotabb*⟩≡

```
"UPMP" [color="#FF4488",href="bookvol10.4.pdf#nameddest=UPMP"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"UPMP" -> "PFECAT"
```

22.19 package UPSQFREE UnivariatePolynomialSquareFree

22.20 UnivariatePolynomialSquareFree



1.00

Exports:
squareFree squareFreePart BumInSepFFE

⟨*package UPSQFREE UnivariatePolynomialSquareFree*⟩≡

```
)abbrev package UPSQFREE UnivariatePolynomialSquareFree
++ Author: Dave Barton, Barry Trager
++ Date Created:
++ Date Last Updated:
++ Basic Functions: squareFree, squareFreePart
++ Related Constructors:
++ Also See:
++ AMS Classifications:
++ Keywords:
++ References:
++ Description:
++ This package provides for square-free decomposition of
++ univariate polynomials over arbitrary rings, i.e.
++ a partial factorization such that each factor is a product
++ of irreducibles with multiplicity one and the factors are
++ pairwise relatively prime. If the ring
++ has characteristic zero, the result is guaranteed to satisfy
++ this condition. If the ring is an infinite ring of
++ finite characteristic, then it may not be possible to decide when
++ polynomials contain factors which are pth powers. In this
++ case, the flag associated with that polynomial is set to "nil"
++ (meaning that that polynomials are not guaranteed to be square-free).

UnivariatePolynomialSquareFree(RC:IntegralDomain,P):C == T
  where
    fUnion ==> Union("nil", "sqfr", "irred", "prime")
```

```
FF     ==> Record(flg:fUnion, fctr:P, xpnt:Integer)
P:Join(UnivariatePolynomialCategory(RC),IntegralDomain) with
  gcd: (%,%) -> %
    ++ gcd(p,q) computes the greatest-common-divisor of p and q.

C == with
  squareFree: P -> Factored(P)
    ++ squareFree(p) computes the square-free factorization of the
    ++ univariate polynomial p. Each factor has no repeated roots, and the
    ++ factors are pairwise relatively prime.
  squareFreePart: P -> P
    ++ squareFreePart(p) returns a polynomial which has the same
    ++ irreducible factors as the univariate polynomial p, but each
    ++ factor has multiplicity one.
  BumInSepFFE: FF -> FF
    ++ BumInSepFFE(f) is a local function, exported only because
    ++ it has multiple conditional definitions.

T == add

  if RC has CharacteristicZero then
    squareFreePart(p:P) == (p exquo gcd(p, differentiate p))::P
  else
    squareFreePart(p:P) ==
      unit(s := squareFree(p)$%) * */[f.factor for f in factors s]

  if RC has FiniteFieldCategory then
    BumInSepFFE(ffe:FF) ==
       ["sqfr", map(charthRoot,ffe.fctr), characteristic$P*ffe.xpnt]
  else if RC has CharacteristicNonZero then
     BumInSepFFE(ffe:FF) ==
        np := multiplyExponents(ffe.fctr,characteristic$P:NonNegativeInteger)
        (nthrp := charthRoot(np)) case "failed" =>
           ["nil", np, ffe.xpnt]
        ["sqfr", nthrp, characteristic$P*ffe.xpnt]

  else
    BumInSepFFE(ffe:FF) ==
      ["nil",
       multiplyExponents(ffe.fctr,characteristic$P:NonNegativeInteger),
        ffe.xpnt]


  if RC has CharacteristicZero then
    squareFree(p:P) ==             --Yun's algorithm - see SYMSAC '76, p.27
       --Note ci primitive is, so GCD's don't need to %do contents.
```

```
      --Change gcd to return cofctrs also?
      ci:=p; di:=differentiate(p); pi:=gcd(ci,di)
      degree(pi)=0 =>
        (u,c,a):=unitNormal(p)
        makeFR(u,[["sqfr",c,1]])
      i:NonNegativeInteger:=0; lffe:List FF:=[]
      lcp := leadingCoefficient p
      while degree(ci)^=0 repeat
         ci:=(ci exquo pi)::P
         di:=(di exquo pi)::P - differentiate(ci)
         pi:=gcd(ci,di)
         i:=i+1
         degree(pi) > 0 =>
            lcp:=(lcp exquo (leadingCoefficient(pi)**i))::RC
            lffe:=[["sqfr",pi,i],:lffe]
      makeFR(lcp::P,lffe)

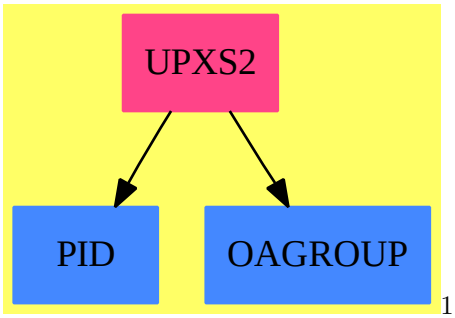
    else
      squareFree(p:P) ==           --Musser's algorithm - see SYMSAC '76, p.27
        --p MUST BE PRIMITIVE, Any characteristic.
        --Note ci primitive, so GCD's don't need to %do contents.
        --Change gcd to return cofctrs also?
      ci := gcd(p,differentiate(p))
      degree(ci)=0 =>
        (u,c,a):=unitNormal(p)
        makeFR(u,[["sqfr",c,1]])
      di := (p exquo ci)::P
      i:NonNegativeInteger:=0; lffe:List FF:=[]
      dunit : P := 1
      while degree(di)^=0 repeat
         diprev := di
         di := gcd(ci,di)
         ci:=(ci exquo di)::P
         i:=i+1
         degree(diprev) = degree(di) =>
            lc := (leadingCoefficient(diprev) exquo leadingCoefficient(di))::RC
            dunit := lc**i * dunit
         pi:=(diprev exquo di)::P
         lffe:=[["sqfr",pi,i],:lffe]
      dunit := dunit * di ** (i+1)
      degree(ci)=0 => makeFR(dunit*ci,lffe)
      redSqfr:=squareFree(divideExponents(ci,characteristic$P)::P)
      lsnil:= [BumInSepFFE(ffe) for ffe in factorList redSqfr]
      lffe:=append(lsnil,lffe)
      makeFR(dunit*(unit redSqfr),lffe)
```

⟨ *UPSQFREE.dotabb* ⟩≡

```
"UPSQFREE" [color="#FF4488",href="bookvol10.4.pdf#nameddest=UPSQFREE"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"UPSQFREE" -> "PFECAT"
```

22.21 package UPXS2 UnivariatePuiseuxSeries-Functions2

22.22 UnivariatePuiseuxSeriesFunctions2



1.00

Exports:
map

⟨*package UPXS2 UnivariatePuiseuxSeriesFunctions2*⟩≡

```
)abbrev package UPXS2 UnivariatePuiseuxSeriesFunctions2
++ Mapping package for univariate Puiseux series
++ Author: Scott C. Morrison
++ Date Created: 5 April 1991
++ Date Last Updated: 5 April 1991
++ Keywords: Puiseux series, map
++ Examples:
++ References:
++ Description:
++   Mapping package for univariate Puiseux series.
++   This package allows one to apply a function to the coefficients of
++   a univariate Puiseux series.
UnivariatePuiseuxSeriesFunctions2(Coef1,Coef2,var1,var2,cen1,cen2):_
 Exports == Implementation where
  Coef1 : Ring
  Coef2 : Ring
  var1: Symbol
  var2: Symbol
  cen1: Coef1
  cen2: Coef2
  UPS1  ==> UnivariatePuiseuxSeries(Coef1, var1, cen1)
  UPS2  ==> UnivariatePuiseuxSeries(Coef2, var2, cen2)
  ULSP2 ==> UnivariateLaurentSeriesFunctions2(Coef1, Coef2, var1, var2, cen1, cen2)

  Exports ==> with
    map: (Coef1 -> Coef2,UPS1) -> UPS2
```

```
        ++ \spad{map(f,g(x))} applies the map f to the coefficients of the
        ++ Puiseux series \spad{g(x)}.

    Implementation ==> add

      map(f,ups) == puiseux(rationalPower ups, map(f, laurentRep ups)$ULSP2)
```

⟨*UPXS2.dotabb*⟩≡

```
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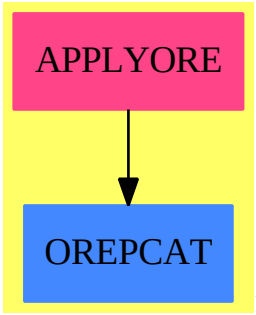
"PID" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PID"]
"OAGROUP" [color="#4488FF",href="bookvol10.2.pdf#nameddest=OAGROUP"]
"UPXS2" -> "PID"
"UPXS2" -> "OAGROUP"
```

22.23 package OREPCTO UnivariateSkewPolynomialCategoryOps

22.24 UnivariateSkewPolynomialCategoryOps



1.00

Exports:

apply	leftDivide	monicLeftDivide
monicRightDivide	rightDivide	times

⟨*package OREPCTO UnivariateSkewPolynomialCategoryOps*⟩≡

```
)abbrev package OREPCTO UnivariateSkewPolynomialCategoryOps
++ Author: Manuel Bronstein
++ Date Created: 1 February 1994
++ Date Last Updated: 1 February 1994
++ Description:
++   \spad{UnivariateSkewPolynomialCategoryOps} provides products and
++    divisions of univariate skew polynomials.
-- Putting those operations here rather than defaults in OREPCAT allows
-- OREPCAT to be defined independently of sigma and delta.
-- MB 2/94
UnivariateSkewPolynomialCategoryOps(R, C): Exports == Implementation where
    R: Ring
    C: UnivariateSkewPolynomialCategory R

    N   ==> NonNegativeInteger
    MOR ==> Automorphism R
    QUOREM ==> Record(quotient: C, remainder: C)

    Exports ==> with
        times: (C, C, MOR, R -> R) -> C
           ++ times(p, q, sigma, delta) returns \spad{p * q}.
           ++ \spad{\sigma} and \spad{\delta} are the maps to use.
        apply: (C, R, R, MOR, R -> R) -> R
          ++ apply(p, c, m, sigma, delta) returns \spad{p(m)} where the action
          ++ is given by \spad{x m = c sigma(m) + delta(m)}.
```

```
        if R has IntegralDomain then
            monicLeftDivide: (C, C, MOR) -> QUOREM
                ++ monicLeftDivide(a, b, sigma) returns the pair \spad{[q,r]}
                ++ such that \spad{a = b*q + r} and the degree of \spad{r} is
                ++ less than the degree of \spad{b}.
                ++ \spad{b} must be monic.
                ++ This process is called ``left division''.
                ++ \spad{\sigma} is the morphism to use.
            monicRightDivide: (C, C, MOR) -> QUOREM
                ++ monicRightDivide(a, b, sigma) returns the pair \spad{[q,r]}
                ++ such that \spad{a = q*b + r} and the degree of \spad{r} is
                ++ less than the degree of \spad{b}.
                ++ \spad{b} must be monic.
                ++ This process is called ``right division''.
                ++ \spad{\sigma} is the morphism to use.
        if R has Field then
            leftDivide: (C, C, MOR) -> QUOREM
                ++ leftDivide(a, b, sigma) returns the pair \spad{[q,r]} such
                ++ that \spad{a = b*q + r} and the degree of \spad{r} is
                ++ less than the degree of \spad{b}.
                ++ This process is called ``left division''.
                ++ \spad{\sigma} is the morphism to use.
            rightDivide: (C, C, MOR) -> QUOREM
                ++ rightDivide(a, b, sigma) returns the pair \spad{[q,r]} such
                ++ that \spad{a = q*b + r} and the degree of \spad{r} is
                ++ less than the degree of \spad{b}.
                ++ This process is called ``right division''.
                ++ \spad{\sigma} is the morphism to use.

    Implementation ==> add
        termPoly:         (R, N, C, MOR, R -> R) -> C
        localLeftDivide : (C, C, MOR, R) -> QUOREM
        localRightDivide: (C, C, MOR, R) -> QUOREM

        times(x, y, sigma, delta) ==
          zero? y => 0
          z:C := 0
          while x ^= 0 repeat
            z := z + termPoly(leadingCoefficient x, degree x, y, sigma, delta)
            x := reductum x
          z

        termPoly(a, n, y, sigma, delta) ==
          zero? y => 0
          (u := subtractIfCan(n, 1)) case "failed" => a * y
          n1 := u::N
```

```
    z:C := 0
    while y ^= 0 repeat
      m := degree y
      b := leadingCoefficient y
      z := z + termPoly(a, n1, monomial(sigma b, m + 1), sigma, delta)
             + termPoly(a, n1, monomial(delta b, m), sigma, delta)
      y := reductum y
    z

  apply(p, c, x, sigma, delta) ==
    w:R  := 0
    xn:R := x
    for i in 0..degree p repeat
      w  := w + coefficient(p, i) * xn
      xn := c * sigma xn + delta xn
    w

  -- localLeftDivide(a, b) returns [q, r] such that a = q b + r
  -- b1 is the inverse of the leadingCoefficient of b
  localLeftDivide(a, b, sigma, b1) ==
      zero? b => error "leftDivide: division by 0"
      zero? a or
       (n := subtractIfCan(degree(a),(m := degree b))) case "failed" =>
              [0,a]
      q  := monomial((sigma**(-m))(b1 * leadingCoefficient a), n::N)
      qr := localLeftDivide(a - b * q, b, sigma, b1)
      [q + qr.quotient, qr.remainder]

  -- localRightDivide(a, b) returns [q, r] such that a = q b + r
  -- b1 is the inverse of the leadingCoefficient of b
  localRightDivide(a, b, sigma, b1) ==
      zero? b => error "rightDivide: division by 0"
      zero? a or
        (n := subtractIfCan(degree(a),(m := degree b))) case "failed" =>
              [0,a]
      q := monomial(leadingCoefficient(a) * (sigma**n) b1, n::N)
      qr := localRightDivide(a - q * b, b, sigma, b1)
      [q + qr.quotient, qr.remainder]

  if R has IntegralDomain then
      monicLeftDivide(a, b, sigma) ==
          unit?(u := leadingCoefficient b) =>
              localLeftDivide(a, b, sigma, recip(u)::R)
          error "monicLeftDivide: divisor is not monic"

      monicRightDivide(a, b, sigma) ==
```

```
                unit?(u := leadingCoefficient b) =>
                    localRightDivide(a, b, sigma, recip(u)::R)
                error "monicRightDivide: divisor is not monic"

        if R has Field then
            leftDivide(a, b, sigma) ==
                localLeftDivide(a, b, sigma, inv leadingCoefficient b)

            rightDivide(a, b, sigma) ==
                localRightDivide(a, b, sigma, inv leadingCoefficient b)
```

⟨*OREPCTO.dotabb*⟩≡

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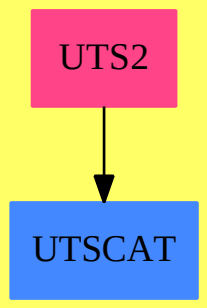
"OREPCTO" [color="#FF4488",href="bookvol10.4.pdf#nameddest=OREPCTO"]
"OREPCAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=OREPCAT"]
"OREPCTO" -> "OREPCAT"
```

22.25 package UTS2 UnivariateTaylorSeriesFunctions2

22.26 UnivariateTaylorSeriesFunctions2



1.00

Exports:
map

⟨*package UTS2 UnivariateTaylorSeriesFunctions2*⟩≡
```
)abbrev package UTS2 UnivariateTaylorSeriesFunctions2
++ Author: Clifton J. Williamson
++ Date Created: 9 February 1990
++ Date Last Updated: 9 February 1990
++ Basic Operations:
++ Related Domains: UnivariateTaylorSeries(Coef1,var,cen)
++ Also See:
++ AMS Classifications:
++ Keywords: Taylor series, map
++ Examples:
++ References:
++ Description: Mapping package for univariate Taylor series.
++   This package allows one to apply a function to the coefficients of
++   a univariate Taylor series.
UnivariateTaylorSeriesFunctions2(Coef1,Coef2,UTS1,UTS2):_
 Exports == Implementation where
  Coef1 : Ring
  Coef2 : Ring
  UTS1  : UnivariateTaylorSeriesCategory Coef1
  UTS2  : UnivariateTaylorSeriesCategory Coef2
  ST2 ==> StreamFunctions2(Coef1,Coef2)

  Exports ==> with
    map: (Coef1 -> Coef2,UTS1) -> UTS2
      ++\spad{map(f,g(x))} applies the map f to the coefficients of
      ++ the Taylor series \spad{g(x)}.
```

```
    Implementation ==> add

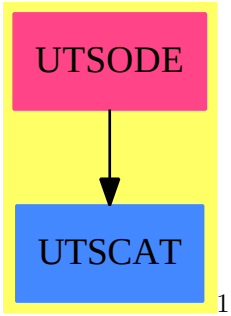
      map(f,uts) == series map(f,coefficients uts)$ST2
```

⟨*UTS2.dotabb*⟩≡

```
"UTS2" [color="#FF4488",href="bookvol10.4.pdf#nameddest=UTS2"]
"UTSCAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=UTSCAT"]
"UTS2" -> "UTSCAT"
```

22.27 package UTSODE UnivariateTaylorSeriesODESolver

22.28 UnivariateTaylorSeriesODESolver



1.00

Exports:

fixedPointExquo	mpsode	ode	ode1	ode2
stFunc1	stFunc2	stFuncN		

⟨*package UTSODE UnivariateTaylorSeriesODESolver*⟩≡

```
)abbrev package UTSODE UnivariateTaylorSeriesODESolver
++ Taylor series solutions of explicit ODE's.
++ Author: Stephen Watt (revised by Clifton J. Williamson)
++ Date Created: February 1988
++ Date Last Updated: 30 September 1993
++ Keywords: differential equation, ODE, Taylor series
++ Examples:
++ References:
UnivariateTaylorSeriesODESolver(Coef,UTS):_
 Exports == Implementation where
  ++ This package provides Taylor series solutions to regular
  ++ linear or non-linear ordinary differential equations of
  ++ arbitrary order.
  Coef  : Algebra Fraction Integer
  UTS   : UnivariateTaylorSeriesCategory Coef
  L   ==> List
  L2  ==> ListFunctions2
  FN  ==> (L UTS) -> UTS
  ST  ==> Stream Coef
  YS  ==> Y$ParadoxicalCombinatorsForStreams(Coef)
  STT ==> StreamTaylorSeriesOperations(Coef)

  Exports ==> with
    stFunc1: (UTS -> UTS) -> (ST -> ST)
      ++ stFunc1(f) is a local function exported due to compiler problem.
```

```
      ++ This function is of no interest to the top-level user.
    stFunc2: ((UTS,UTS) -> UTS) -> ((ST,ST) -> ST)
      ++ stFunc2(f) is a local function exported due to compiler problem.
      ++ This function is of no interest to the top-level user.
    stFuncN: FN -> ((L ST) -> ST)
      ++ stFuncN(f) is a local function xported due to compiler problem.
      ++ This function is of no interest to the top-level user.
    fixedPointExquo: (UTS,UTS) -> UTS
      ++ fixedPointExquo(f,g) computes the exact quotient of \spad{f} and
      ++ \spad{g} using a fixed point computation.
    ode1: ((UTS -> UTS),Coef) -> UTS
      ++ ode1(f,c) is the solution to \spad{y' = f(y)}
      ++ such that \spad{y(a) = c}.
    ode2: ((UTS, UTS) -> UTS,Coef,Coef) -> UTS
      ++ ode2(f,c0,c1) is the solution to \spad{y'' = f(y,y')} such that
      ++ \spad{y(a) = c0} and \spad{y'(a) = c1}.
    ode: (FN,List Coef) -> UTS
      ++ ode(f,cl) is the solution to \spad{y<n>=f(y,y',..,y<n-1>)} such that
      ++ \spad{y<i>(a) = cl.i} for i in 1..n.
    mpsode:(L Coef,L FN) -> L UTS
      ++ mpsode(r,f) solves the system of differential equations
      ++ \spad{dy[i]/dx =f[i] [x,y[1],y[2],...,y[n]]},
      ++ \spad{y[i](a) = r[i]} for i in 1..n.

  Implementation ==> add

    stFunc1 f == coefficients f series(#1)
    stFunc2 f == coefficients f(series(#1),series(#2))
    stFuncN f == coefficients f map(series,#1)$ListFunctions2(ST,UTS)

    import StreamTaylorSeriesOperations(Coef)
    divloopre:(Coef,ST,Coef,ST,ST) -> ST
    divloopre(hx,tx,hy,ty,c) == delay(concat(hx*hy,hy*(tx-(ty*c))))
    divloop: (Coef,ST,Coef,ST) -> ST
    divloop(hx,tx,hy,ty) == YS(divloopre(hx,tx,hy,ty,#1))

    sdiv:(ST,ST) -> ST
    sdiv(x,y) == delay
      empty? x => empty()
      empty? y => error "stream division by zero"
      hx := frst x; tx := rst x
      hy := frst y; ty := rst y
      zero? hy =>
        zero? hx => sdiv(tx,ty)
        error "stream division by zero"
      rhy := recip hy
```

```
        rhy case "failed" => error "stream division:no reciprocal"
        divloop(hx,tx,rhy::Coef,ty)

      fixedPointExquo(f,g) == series sdiv(coefficients f,coefficients g)

  -- first order

      ode1re: (ST -> ST,Coef,ST) -> ST
      ode1re(f,c,y) == lazyIntegrate(c,f y)$STT

      iOde1: ((ST -> ST),Coef) -> ST
      iOde1(f,c) == YS ode1re(f,c,#1)

      ode1(f,c) == series iOde1(stFunc1 f,c)

  -- second order

      ode2re: ((ST,ST)-> ST,Coef,Coef,ST) -> ST
      ode2re(f,c0,c1,y)==
        yi := lazyIntegrate(c1,f(y,deriv(y)$STT))$STT
        lazyIntegrate(c0,yi)$STT

      iOde2: ((ST,ST) -> ST,Coef,Coef) -> ST
      iOde2(f,c0,c1) == YS ode2re(f,c0,c1,#1)

      ode2(f,c0,c1) == series iOde2(stFunc2 f,c0,c1)

  -- nth order

      odeNre: (List ST -> ST,List Coef,List ST) -> List ST
      odeNre(f,cl,yl) ==
        -- yl is [y, y', ..., y<n>]
        -- integrate [y',..,y<n>] to get [y,..,y<n-1>]
        yil := [lazyIntegrate(c,y)$STT for c in cl for y in rest yl]
        -- use y<n> = f(y,..,y<n-1>)
        concat(yil,[f yil])

      iOde: ((L ST) -> ST,List Coef) -> ST
      iOde(f,cl) == first YS(odeNre(f,cl,#1),#cl + 1)

      ode(f,cl) == series iOde(stFuncN f,cl)

      simulre:(L Coef,L ((L ST) -> ST),L ST) -> L ST
      simulre(cst,lsf,c) ==
        [lazyIntegrate(csti,lsfi concat(monom(1,1)$STT,c))_
            for csti in cst for lsfi in lsf]
```

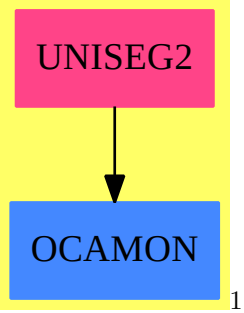
```
      iMpsode:(L Coef,L ((L ST) -> ST)) -> L ST
      iMpsode(cs,lsts) == YS(simulre(cs,lsts,#1),# cs)
      mpsode(cs,lsts) ==
  --       stSol := iMpsode(cs,map(stFuncN,lsts)$L2(FN,(L ST) -> ST))
        stSol := iMpsode(cs,[stFuncN(lst) for lst in lsts])
        map(series,stSol)$L2(ST,UTS)
```

⟨*UTSODE.dotabb*⟩≡

```
"UTSODE" [color="#FF4488",href="bookvol10.4.pdf#nameddest=UTSODE"]
"UTSCAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=UTSCAT"]
"UTSODE" -> "UTSCAT"
```

22.29 package UNISEG2 UniversalSegmentFunctions2

22.30 UniversalSegmentFunctions2



1.00

Exports:
map

⟨*package UNISEG2 UniversalSegmentFunctions2*⟩≡

```
)abbrev package UNISEG2 UniversalSegmentFunctions2
++ Author:
++ Date Created:
++ Date Last Updated: June 4, 1991
++ Basic Operations:
++ Related Domains: Segment, UniversalSegment
++ Also See:
++ AMS Classifications:
++ Keywords: equation
++ Examples:
++ References:
++ Description:
++   This package provides operations for mapping functions onto segments.

UniversalSegmentFunctions2(R:Type, S:Type): with
    map: (R -> S, UniversalSegment R) -> UniversalSegment S
        ++ map(f,seg) returns the new segment obtained by applying
        ++ f to the endpoints of seg.

    if R has OrderedRing then
      map: (R -> S, UniversalSegment R) -> Stream S
        ++ map(f,s) expands the segment s, applying \spad{f} to each value.


  == add
    map(f:R -> S, u:UniversalSegment R):UniversalSegment S ==
```

```
        s := f lo u
        hasHi u => segment(s, f hi u)
        segment s

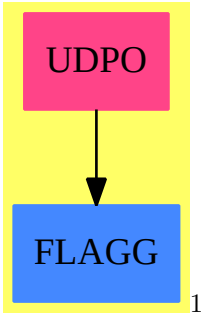
      if R has OrderedRing then
        map(f:R -> S, u:UniversalSegment R): Stream S ==
          map(f, expand u)$StreamFunctions2(R, S)
```

⟨*UNISEG2.dotabb*⟩≡

```
"UNISEG2" [color="#FF4488",href="bookvol10.4.pdf#nameddest=UNISEG2"]
"OCAMON" [color="#4488FF",href="bookvol10.2.pdf#nameddest=OCAMON"]
"UNISEG2" -> "OCAMON"
```

22.31 package UDPO UserDefinedPartialOrdering

22.32 UserDefinedPartialOrdering



1.00

Exports:
setOrder userOrdered? getOrder largest less? more?

⟨*package UDPO UserDefinedPartialOrdering*⟩≡

```
)abbrev package UDPO UserDefinedPartialOrdering
++ Author: Manuel Bronstein
++ Date Created: March 1990
++ Date Last Updated: 9 April 1991
++ Description:
++  Provides functions to force a partial ordering on any set.
UserDefinedPartialOrdering(S:SetCategory): with
  setOrder    : List S -> Void
    ++ setOrder([a1,...,an]) defines a partial ordering on S given by:
    ++    (1)  \spad{a1 < a2 < ... < an}.
    ++    (2)  \spad{b < ai   for i = 1..n} and b not among the ai's.
    ++    (3)  undefined on \spad{(b, c)} if neither is among the ai's.
  setOrder    : (List S, List S) -> Void
    ++ setOrder([b1,...,bm], [a1,...,an]) defines a partial
    ++ ordering on S given by:
    ++    (1)  \spad{b1 < b2 < ... < bm < a1 < a2 < ... < an}.
    ++    (2)  \spad{bj < c < ai}  for c not among the ai's and bj's.
    ++    (3)  undefined on \spad{(c,d)} if neither is among the ai's,bj's.
  getOrder    : () -> Record(low: List S, high: List S)
    ++ getOrder() returns \spad{[[b1,...,bm], [a1,...,an]]} such that the
    ++ partial ordering on S was given by
    ++ \spad{setOrder([b1,...,bm],[a1,...,an])}.
  less?       : (S, S) -> Union(Boolean, "failed")
    ++ less?(a, b) compares \spad{a} and b in the partial ordering induced by
    ++ setOrder.
  less?       : (S, S, (S, S) -> Boolean) -> Boolean
```

```
    ++ less?(a, b, fn) compares \spad{a} and b in the partial ordering induced
    ++ by setOrder, and returns \spad{fn(a, b)} if \spad{a}
    ++ and b are not comparable
    ++ in that ordering.
  largest     : (List S, (S, S) -> Boolean) -> S
    ++ largest(l, fn) returns the largest element of l where the partial
    ++ ordering induced by setOrder is completed into a total one by fn.
  userOrdered?: () -> Boolean
    ++ userOrdered?() tests if the partial ordering induced by
    ++ \spadfunFrom{setOrder}{UserDefinedPartialOrdering} is not empty.
  if S has OrderedSet then
    largest: List S -> S
      ++ largest l returns the largest element of l where the partial
      ++ ordering induced by setOrder is completed into a total one by
      ++ the ordering on S.
    more?  : (S, S) -> Boolean
      ++ more?(a, b) compares \spad{a} and b in the partial ordering induced
      ++ by setOrder, and uses the ordering on S if \spad{a} and b are not
      ++ comparable in the partial ordering.

 == add
  llow :Reference List S := ref nil()
  lhigh:Reference List S := ref nil()

  userOrdered?() == not(empty? deref llow) or not(empty? deref lhigh)
  getOrder()     == [deref llow, deref lhigh]
  setOrder l     == setOrder(nil(), l)

  setOrder(l, h) ==
    setref(llow, removeDuplicates l)
    setref(lhigh, removeDuplicates h)
    void

  less?(a, b, f) ==
    (u := less?(a, b)) case "failed" => f(a, b)
    u::Boolean

  largest(x, f) ==
    empty? x => error "largest: empty list"
    empty? rest x => first x
    a := largest(rest x, f)
    less?(first x, a, f) => a
    first x

  less?(a, b) ==
    for x in deref llow repeat
```

```
        x = a => return(a ^= b)
        x = b => return false
      aa := bb := false$Boolean
      for x in deref lhigh repeat
        if x = a then
          bb => return false
          aa := true
        if x = b then
          aa => return(a ^= b)
          bb := true
      aa => false
      bb => true
      "failed"

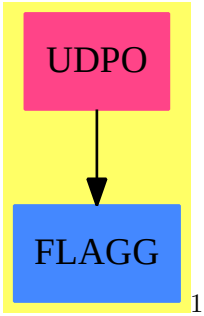
    if S has OrderedSet then
      more?(a, b) == not less?(a, b, #1 <$S #2)
      largest x   == largest(x, #1 <$S #2)
```

⟨*UDPO.dotabb*⟩≡

```
"UDPO" [color="#FF4488",href="bookvol10.4.pdf#nameddest=UDPO"]
"FLAGG" [color="#4488FF",href="bookvol10.2.pdf#nameddest=FLAGG"]
"UDPO" -> "FLAGG"
```

22.33 package UDVO UserDefinedVariableOrdering

22.34 UserDefinedVariableOrdering



1.00

Exports:
resetVariableOrder getVariableOrder setVariableOrder

⟨*package UDVO UserDefinedVariableOrdering*⟩≡
```
)abbrev package UDVO UserDefinedVariableOrdering
++ Author: Manuel Bronstein
++ Date Created: March 1990
++ Date Last Updated: 9 April 1991
++ Description:
++  This packages provides functions to allow the user to select the ordering
++  on the variables and operators for displaying polynomials,
++  fractions and expressions. The ordering affects the display
++  only and not the computations.
UserDefinedVariableOrdering(): with
  setVariableOrder  : List Symbol -> Void
    ++ setVariableOrder([a1,...,an]) defines an ordering on the
    ++ variables given by \spad{a1 > a2 > ... > an > other variables}.
  setVariableOrder  : (List Symbol, List Symbol) -> Void
    ++ setVariableOrder([b1,...,bm], [a1,...,an]) defines an ordering
    ++ on the variables given by
    ++ \spad{b1 > b2 > ... > bm >} other variables \spad{>  a1 > a2 > ... > an}.
  getVariableOrder  : () -> Record(high:List Symbol, low:List Symbol)
    ++ getVariableOrder() returns \spad{[[b1,...,bm], [a1,...,an]]} such that
    ++ the ordering on the variables was given by
    ++ \spad{setVariableOrder([b1,...,bm], [a1,...,an])}.
  resetVariableOrder: () -> Void
    ++ resetVariableOrder() cancels any previous use of
    ++ setVariableOrder and returns to the default system ordering.
 == add
  import UserDefinedPartialOrdering(Symbol)
```

```
    setVariableOrder l       == setOrder reverse l
    setVariableOrder(l1, l2) == setOrder(reverse l2, reverse l1)
    resetVariableOrder()     == setVariableOrder(nil(), nil())

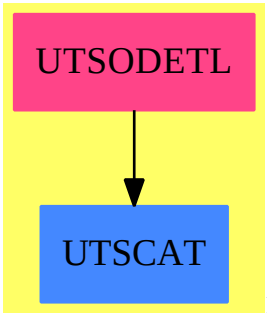
    getVariableOrder() ==
      r := getOrder()
      [reverse(r.high), reverse(r.low)]
```

⟨*UDVO.dotabb*⟩≡

```
  "UDVO" [color="#FF4488",href="bookvol10.4.pdf#nameddest=UDVO"]
  "BASTYPE" [color="#4488FF",href="bookvol10.2.pdf#nameddest=BASTYPE"]
  "KOERCE" [color="#4488FF",href="bookvol10.2.pdf#nameddest=KOERCE"]
  "UDVO" -> "BASTYPE"
  "UDVO" -> "KOERCE"
```

22.35 package UTSODETL UTSodetools

22.36 UTSodetools



1.00

Exports:
LODO2FUN RF2UTS UTS2UP

⟨*package UTSODETL UTSodetools*⟩≡
```
)abbrev package UTSODETL UTSodetools
++ Author: Manuel Bronstein
++ Date Created: 31 January 1994
++ Date Last Updated: 3 February 1994
++ Description:
++  \spad{RUTSodetools} provides tools to interface with the series
++   ODE solver when presented with linear ODEs.
UTSodetools(F, UP, L, UTS): Exports == Implementation where
  F  : Ring
  UP : UnivariatePolynomialCategory F
  L  : LinearOrdinaryDifferentialOperatorCategory UP
  UTS: UnivariateTaylorSeriesCategory F

  Exports ==> with
      UP2UTS:   UP -> UTS
          ++ UP2UTS(p) converts \spad{p} to a Taylor series.
      UTS2UP:   (UTS, NonNegativeInteger) -> UP
          ++ UTS2UP(s, n) converts the first \spad{n} terms of \spad{s}
          ++ to a univariate polynomial.
      LODO2FUN: L -> (List UTS -> UTS)
          ++ LODO2FUN(op) returns the function to pass to the series ODE
          ++ solver in order to solve \spad{op y = 0}.
      if F has IntegralDomain then
          RF2UTS: Fraction UP -> UTS
              ++ RF2UTS(f) converts \spad{f} to a Taylor series.

  Implementation ==> add
```

```
        fun: (Vector UTS, List UTS) -> UTS

        UP2UTS p ==
          q := p(monomial(1, 1) + center(0)::UP)
          +/[monomial(coefficient(q, i), i)$UTS for i in 0..degree q]

        UTS2UP(s, n) ==
          xmc     := monomial(1, 1)$UP - center(0)::UP
          xmcn:UP := 1
          ans:UP  := 0
          for i in 0..n repeat
              ans  := ans + coefficient(s, i) * xmcn
              xmcn := xmc * xmcn
          ans

        LODO2FUN op ==
            a := recip(UP2UTS(- leadingCoefficient op))::UTS
            n := (degree(op) - 1)::NonNegativeInteger
            v := [a * UP2UTS coefficient(op, i) for i in 0..n]$Vector(UTS)
            fun(v, #1)

        fun(v, l) ==
            ans:UTS := 0
            for b in l for i in 1.. repeat ans := ans + v.i * b
            ans

        if F has IntegralDomain then
            RF2UTS f == UP2UTS(numer f) * recip(UP2UTS denom f)::UTS
```

⟨*UTSODETL.dotabb*⟩≡

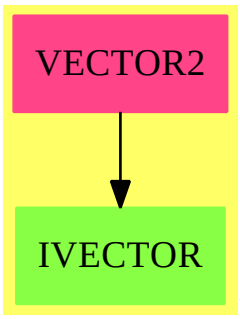
```
"UTSODETL" [color="#FF4488",href="bookvol10.4.pdf#nameddest=UTSODETL"]
"UTSCAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=UTSCAT"]
"UTSODETL" -> "UTSCAT"
```

Chapter 23

Chapter V

23.1 package VECTOR2 VectorFunctions2

23.2 VectorFunctions2



1.00

Exports:
map reduce scan

⟨*package VECTOR2 VectorFunctions2*⟩≡
```
)abbrev package VECTOR2 VectorFunctions2
++ Author:
++ Date Created:
++ Date Last Updated:
++ Basic Functions:
++ Related Constructors:
++ Also See:
++ AMS Classifications:
++ Keywords:
++ References:
++ Description:
```

```
++   This package provides operations which all take as arguments
++ vectors of elements of some type \spad{A} and functions from \spad{A} to
++ another of type B. The operations all iterate over their vector argument
++ and either return a value of type B or a vector over B.

VectorFunctions2(A, B): Exports == Implementation where
  A, B: Type

  VA ==> Vector A
  VB ==> Vector B
  O2 ==> FiniteLinearAggregateFunctions2(A, VA, B, VB)
  UB ==> Union(B,"failed")

  Exports ==> with
    scan   : ((A, B) -> B, VA, B) -> VB
      ++ scan(func,vec,ident) creates a new vector whose elements are
      ++ the result of applying reduce to the binary function func,
      ++ increasing initial subsequences of the vector vec,
      ++ and the element ident.
    reduce : ((A, B) -> B, VA, B) -> B
      ++ reduce(func,vec,ident) combines the elements in vec using the
      ++ binary function func. Argument ident is returned if vec is empty.
    map    : (A -> B, VA) -> VB
      ++ map(f, v) applies the function f to every element of the vector v
      ++ producing a new vector containing the values.
    map : (A -> UB, VA) -> Union(VB,"failed")
      ++ map(f, v) applies the function f to every element of the vector v
      ++ producing a new vector containing the values or \spad{"failed"}.

  Implementation ==> add
    scan(f, v, b)   == scan(f, v, b)$O2
    reduce(f, v, b) == reduce(f, v, b)$O2
    map(f:(A->B), v:VA):VB == map(f, v)$O2

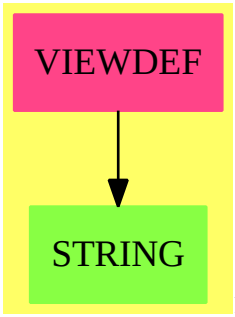
    map(f:(A -> UB), a:VA):Union(VB,"failed") ==
     res : List B  := []
     for u in entries(a) repeat
       r := f u
       r = "failed" => return "failed"
       res := [r::B,:res]
     vector reverse! res
```

⟨*VECTOR2.dotabb*⟩≡

```
"VECTOR2" [color="#FF4488",href="bookvol10.4.pdf#nameddest=VECTOR2"]
"IVECTOR" [color="#88FF44",href="bookvol10.3.pdf#nameddest=IVECTOR"]
"VECTOR2" -> "IVECTOR"
```

23.3 package VIEWDEF ViewDefaultsPackage

23.4 ViewDefaultsPackage



1.00

Exports:

axesColorDefault	lineColorDefault	pointColorDefault
pointSizeDefault	tubePointsDefault	tubeRadiusDefault
unitsColorDefault	var1StepsDefault	var2StepsDefault
unitsColorDefault	viewDefaults	viewPosDefault
viewSizeDefault	viewWriteAvailable	viewWriteDefault

⟨*package VIEWDEF ViewDefaultsPackage*⟩≡

```
)abbrev package VIEWDEF ViewDefaultsPackage
++ Author: Jim Wen
++ Date Created: 15 January 1990
++ Date Last Updated:
++ Basic Operations: pointColorDefault, lineColorDefault, axesColorDefault,
++ unitsColorDefault, pointSizeDefault, viewPosDefault, viewSizeDefault,
++ viewDefaults, viewWriteDefault, viewWriteAvailable, var1StepsDefault,
++ var2StepsDefault, tubePointsDefault, tubeRadiusDefault
++ Related Constructors:
++ Also See:
++ AMS Classifications:
++ Keywords:
++ References:
++ Description: ViewportDefaultsPackage describes default and user definable
++ values for graphics

ViewDefaultsPackage():Exports == Implementation where
  I       ==> Integer
  C       ==> Color
  PAL     ==> Palette
  L       ==> List
  S       ==> String
  E       ==> Expression
```

```
PI      ==> PositiveInteger
NNI     ==> NonNegativeInteger
SF      ==> DoubleFloat
B       ==> Boolean

writeAvailable ==>  (["PIXMAP","BITMAP","POSTSCRIPT","IMAGE"]::L S)
  -- need not worry about case of letters

Exports ==> with
  pointColorDefault   :  ()                          -> PAL
    ++ pointColorDefault() returns the default color of points in a 2D
    ++ viewport.
  pointColorDefault   :  PAL                         -> PAL
    ++ pointColorDefault(p) sets the default color of points in a 2D viewport
    ++ to the palette p.
  lineColorDefault    :  ()                          -> PAL
    ++ lineColorDefault() returns the default color of lines connecting
    ++ points in a 2D viewport.
  lineColorDefault    :  PAL                         -> PAL
    ++ lineColorDefault(p) sets the default color of lines connecting points
    ++ in a 2D viewport to the palette p.
  axesColorDefault    :  ()                          -> PAL
    ++ axesColorDefault() returns the default color of the axes in a
    ++ 2D viewport.
  axesColorDefault    :  PAL                         -> PAL
    ++ axesColorDefault(p) sets the default color of the axes in a 2D
    ++ viewport to the palette p.
  unitsColorDefault   :  ()                          -> PAL
    ++ unitsColorDefault() returns the default color of the unit ticks in
    ++ a 2D viewport.
  unitsColorDefault   :  PAL                         -> PAL
    ++ unitsColorDefault(p) sets the default color of the unit ticks in
    ++ a 2D viewport to the palette p.
  pointSizeDefault    :  ()                          -> PI
    ++ pointSizeDefault() returns the default size of the points in
    ++ a 2D viewport.
  pointSizeDefault    :  PI                          -> PI
    ++ pointSizeDefault(i) sets the default size of the points in a 2D
    ++ viewport to i.
  viewPosDefault      :  () -> L NNI
    ++ viewPosDefault() returns the default X and Y position of a
    ++ viewport window unless overriden explicityly, newly created
    ++ viewports will have this X and Y coordinate.
  viewPosDefault      :  L NNI -> L NNI
    ++ viewPosDefault([x,y]) sets the default X and Y position of a
    ++ viewport window unless overriden explicityly, newly created
```

```
  ++ viewports will have th X and Y coordinates x, y.
viewSizeDefault : () -> L PI
  ++ viewSizeDefault() returns the default viewport width and height.
viewSizeDefault : L PI -> L PI
  ++ viewSizeDefault([w,h]) sets the default viewport width to w and height
  ++ to h.
viewDefaults :  ()                          -> Void
  ++ viewDefaults() resets all the default graphics settings.
viewWriteDefault  :  ()                          -> L S
  ++ viewWriteDefault() returns the list of things to write in a viewport
  ++ data file; a viewalone file is always generated.
viewWriteDefault  :  L S                         -> L S
  ++ viewWriteDefault(l) sets the default list of things to write in a
  ++ viewport data file to the strings in l; a viewalone file is always
  ++ genereated.
viewWriteAvailable : ()                    -> L S
  ++ viewWriteAvailable() returns a list of available methods for writing,
  ++ such as BITMAP, POSTSCRIPT, etc.
var1StepsDefault       : () -> PI
  ++ var1StepsDefault() is the current setting for the number of steps to
  ++ take when creating a 3D mesh in the direction of the first defined
  ++ free variable (a free variable is considered defined when its
  ++ range is specified (e.g. x=0..10)).
var2StepsDefault       : () -> PI
  ++ var2StepsDefault() is the current setting for the number of steps to
  ++ take when creating a 3D mesh in the direction of the first defined
  ++ free variable (a free variable is considered defined when its
  ++ range is specified (e.g. x=0..10)).
var1StepsDefault       : PI -> PI
  ++ var1StepsDefault(i) sets the number of steps to take when creating a
  ++ 3D mesh in the direction of the first defined free variable to i
  ++ (a free variable is considered defined when its range is specified
  ++ (e.g. x=0..10)).
var2StepsDefault       : PI -> PI
  ++ var2StepsDefault(i) sets the number of steps to take when creating a
  ++ 3D mesh in the direction of the first defined free variable to i
  ++ (a free variable is considered defined when its range is specified
  ++ (e.g. x=0..10)).
tubePointsDefault  : PI -> PI
  ++ tubePointsDefault(i) sets the number of points to use when creating
  ++ the circle to be used in creating a 3D tube plot to i.
tubePointsDefault  : () -> PI
  ++ tubePointsDefault() returns the number of points to be used when
  ++ creating the circle to be used in creating a 3D tube plot.
tubeRadiusDefault  : Float -> SF   -- current tube.spad asks for SF
  ++ tubeRadiusDefault(r) sets the default radius for a 3D tube plot to r.
```

```
      tubeRadiusDefault  : () -> SF
        ++ tubeRadiusDefault() returns the radius used for a 3D tube plot.

    Implementation ==> add

      import Color()
      import Palette()
      --import StringManipulations()

      defaultPointColor : Reference(PAL)  := ref bright red()
      defaultLineColor  : Reference(PAL)  := ref pastel green() --bright blue()
      defaultAxesColor  : Reference(PAL)  := ref dim red()
      defaultUnitsColor : Reference(PAL)  := ref dim yellow()
      defaultPointSize  : Reference(PI)   := ref(3::PI)
      defaultXPos       : Reference(NNI)  := ref(0::NNI)
      defaultYPos       : Reference(NNI)  := ref(0::NNI)
      defaultWidth      : Reference(PI)   := ref(400::PI)
      defaultHeight     : Reference(PI)   := ref(400::PI)
      defaultThingsToWrite : Reference(L S) := ref([]::L S)
      defaultVar1Steps  : Reference(PI)   := ref(27::PI)
      defaultVar2Steps  : Reference(PI)   := ref(27::PI)
      defaultTubePoints : Reference(PI)   := ref(6::PI)
      defaultTubeRadius : Reference(SF)   := ref(convert(0.5)@SF)
      defaultClosed     : Reference(B)    := ref(false)

  --%Viewport window dimensions specifications
      viewPosDefault   == [defaultXPos(),defaultYPos()]
      viewPosDefault l ==
        #l < 2 => error "viewPosDefault expects a list with two elements"
        [defaultXPos() := first l,defaultYPos() := last l]

      viewSizeDefault   == [defaultWidth(),defaultHeight()]
      viewSizeDefault l ==
        #l < 2 => error "viewSizeDefault expects a list with two elements"
        [defaultWidth() := first l,defaultHeight() := last l]

      viewDefaults ==
        defaultPointColor : Reference(PAL)  := ref bright red()
        defaultLineColor  : Reference(PAL)  := ref pastel green() --bright blue()
        defaultAxesColor  : Reference(PAL)  := ref dim red()
        defaultUnitsColor : Reference(PAL)  := ref dim yellow()
        defaultPointSize  : Reference(PI)   := ref(3::PI)
        defaultXPos       : Reference(NNI)  := ref(0::NNI)
        defaultYPos       : Reference(NNI)  := ref(0::NNI)
        defaultWidth      : Reference(PI)   := ref(400::PI)
        defaultHeight     : Reference(PI)   := ref(427::PI)
```

```
--%2D graphical output specifications
    pointColorDefault   == defaultPointColor()
    pointColorDefault p == defaultPointColor() := p

    lineColorDefault   == defaultLineColor()
    lineColorDefault p == defaultLineColor() := p

    axesColorDefault   == defaultAxesColor()
    axesColorDefault p == defaultAxesColor() := p

    unitsColorDefault   == defaultUnitsColor()
    unitsColorDefault p == defaultUnitsColor() := p

    pointSizeDefault   == defaultPointSize()
    pointSizeDefault x == defaultPointSize() := x


--%3D specific stuff
    var1StepsDefault   == defaultVar1Steps()
    var1StepsDefault i == defaultVar1Steps() := i

    var2StepsDefault   == defaultVar2Steps()
    var2StepsDefault i == defaultVar2Steps() := i

    tubePointsDefault   == defaultTubePoints()
    tubePointsDefault i == defaultTubePoints() := i

    tubeRadiusDefault   == defaultTubeRadius()
    tubeRadiusDefault f == defaultTubeRadius() := convert(f)@SF

--%File output stuff
    viewWriteAvailable == writeAvailable

    viewWriteDefault == defaultThingsToWrite()

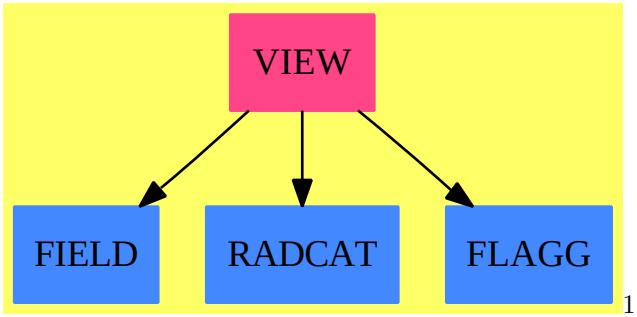
    viewWriteDefault listOfThings ==
      thingsToWrite : L S := []
      for aTypeOfFile in listOfThings repeat
        if (writeTypeInt := position(upperCase aTypeOfFile,viewWriteAvailable()))
          sayBrightly(["  > ",concat(aTypeOfFile,
                       " is not a valid file type for writing a viewport")])$Lisp
        else
          thingsToWrite := append(thingsToWrite,[aTypeOfFile])
      defaultThingsToWrite() := thingsToWrite
```

⟨*VIEWDEF.dotabb*⟩≡

```
"VIEWDEF" [color="#FF4488",href="bookvol10.4.pdf#nameddest=VIEWDEF"]
"STRING" [color="#88FF44",href="bookvol10.3.pdf#nameddest=STRING"]
"VIEWDEF" -> "STRING"
```

23.5 package VIEW ViewportPackage

23.6 ViewportPackage



1.00

Exports:
coerce drawCurves graphCurves

⟨*package VIEW ViewportPackage*⟩≡

```
)abbrev package VIEW ViewportPackage
++ Author: Jim Wen
++ Date Created: 30 April 1989
++ Date Last Updated: 15 June 1990
++ Basic Operations: graphCurves, drawCurves, coerce
++ Related Constructors:
++ Also See:
++ AMS Classifications:
++ Keywords:
++ References:
++ Description: ViewportPackage provides functions for creating GraphImages
++ and TwoDimensionalViewports from lists of lists of points.

ViewportPackage():Exports == Implementation where
  DROP    ==> DrawOption
  GRIMAGE ==> GraphImage
  L       ==> List
  P       ==> Point DoubleFloat
  PAL     ==> Palette
  PI      ==> PositiveInteger
  VIEW2D  ==> TwoDimensionalViewport

  Exports ==> with

    graphCurves : (L L P,PAL,PAL,PI,L DROP) -> GRIMAGE
      ++ graphCurves([[p0],[p1],...,[pn]],ptColor,lineColor,ptSize,[options])
      ++ creates a \spadtype{GraphImage} from the list of lists of points, p0
```

```
      ++ throught pn, using the options specified in the list \spad{options}.
      ++ The graph point color is specified by \spad{ptColor}, the graph line
      ++ color is specified by \spad{lineColor}, and the size of the points is
      ++ specified by \spad{ptSize}.
    graphCurves : L L P -> GRIMAGE
      ++ graphCurves([[p0],[p1],...,[pn]]) creates a \spadtype{GraphImage} from
      ++ the list of lists of points indicated by p0 through pn.
    graphCurves : (L L P,L DROP) -> GRIMAGE
      ++ graphCurves([[p0],[p1],...,[pn]],[options]) creates a
      ++ \spadtype{GraphImage} from the list of lists of points, p0 throught pn,
      ++ using the options specified in the list \spad{options}.
    drawCurves : (L L P,PAL,PAL,PI,L DROP) -> VIEW2D
      ++ drawCurves([[p0],[p1],...,[pn]],ptColor,lineColor,ptSize,[options])
      ++ creates a \spadtype{TwoDimensionalViewport} from the list of lists of
      ++ points, p0 throught pn, using the options specified in the list
      ++ \spad{options}. The point color is specified by \spad{ptColor}, the
      ++ line color is specified by \spad{lineColor}, and the point size is
      ++ specified by \spad{ptSize}.
    drawCurves : (L L P,L DROP) -> VIEW2D
      ++ drawCurves([[p0],[p1],...,[pn]],[options]) creates a
      ++ \spadtype{TwoDimensionalViewport} from the list of lists of points,
      ++ p0 throught pn, using the options specified in the list \spad{options};
    coerce : GRIMAGE -> VIEW2D
      ++ coerce(gi) converts the indicated \spadtype{GraphImage}, gi, into the
      ++ \spadtype{TwoDimensionalViewport} form.

  Implementation ==> add

    import ViewDefaultsPackage
    import DrawOptionFunctions0

  --% Functions that return GraphImages

    graphCurves(listOfListsOfPoints) ==
      graphCurves(listOfListsOfPoints, pointColorDefault(),_
                  lineColorDefault(), pointSizeDefault(),nil())

    graphCurves(listOfListsOfPoints,optionsList) ==
      graphCurves(listOfListsOfPoints, pointColorDefault(),_
                  lineColorDefault(), pointSizeDefault(),optionsList)

    graphCurves(listOfListsOfPoints,ptColor,lineColor,ptSize,optionsList) ==
      len := #listOfListsOfPoints
      listOfPointColors : L PAL := [ptColor for i in 1..len]
      listOfLineColors  : L PAL := [lineColor for i in 1..len]
      listOfPointSizes  : L PI  := [ptSize  for i in 1..len]
```

```
      makeGraphImage(listOfListsOfPoints,listOfPointColors, _
                     listOfLineColors,listOfPointSizes,optionsList)

  --% Functions that return Two Dimensional Viewports

    drawCurves(listOfListsOfPoints,optionsList) ==
      drawCurves(listOfListsOfPoints,pointColorDefault(),_
                 lineColorDefault(),pointSizeDefault(),optionsList)

    drawCurves(ptLists:L L P,ptColor:PAL,lColor:PAL,ptSize:PI,optList:L DROP) ==
      v := viewport2D()
      options(v,optList)
      g :=  graphCurves(ptLists,ptColor,lColor,ptSize,optList)
      putGraph(v,g,1)
      makeViewport2D v

  --% Coercions

    coerce(graf:GRIMAGE):VIEW2D ==
      if (key graf = 0) then makeGraphImage graf
      v := viewport2D()
      title(v,"VIEW2D")
  --    dimensions(v,viewPosDefault().1,viewPosDefault().2,viewSizeDefault().1,vi
      putGraph(v,graf,1::PI)
      makeViewport2D v
```

⟨*VIEW.dotabb*⟩≡

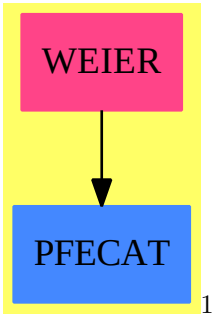
```
"VIEW" [color="#FF4488",href="bookvol10.4.pdf#nameddest=VIEW"]
"FIELD"  [color="#4488FF",href="bookvol10.2.pdf#nameddest=FIELD"]
"RADCAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=RADCAT"]
"FLAGG" [color="#4488FF",href="bookvol10.2.pdf#nameddest=FLAGG"]
"VIEW" -> "FIELD"
"VIEW" -> "RADCAT"
"VIEW" -> "FLAGG"
```

Chapter 24

Chapter W

24.1 package WEIER WeierstrassPreparation

24.2 WeierstrassPreparation



1.00

Exports:

cfirst	clikeUniv	crest	qqq
sts2stst	weierstrass		

⟨*package WEIER WeierstrassPreparation*⟩≡

```
)abbrev package WEIER WeierstrassPreparation
++ Author:William H. Burge
++ Date Created:Sept 1988
++ Date Last Updated:Feb 15 1992
++ Basic Operations:
++ Related Domains:
++ Also See:
++ AMS Classifications:
++ Keywords:
++ Examples:
```

```
++ References:
++ Description: This package implements the Weierstrass preparation
++ theorem f or multivariate power series.
++ weierstrass(v,p) where v is a variable, and p is a
++ TaylorSeries(R) in which the terms
++ of lowest degree s must include c*v**s where c is a constant,s>0,
++ is a list of TaylorSeries coefficients A[i] of the
++ equivalent polynomial
++ A = A[0] + A[1]*v + A[2]*v**2 + ... + A[s-1]*v**(s-1) + v**s
++ such that p=A*B , B being a TaylorSeries of minimum degree 0
WeierstrassPreparation(R): Defn == Impl where
    R : Field
    VarSet==>Symbol
    SMP ==> Polynomial R
    PS  ==> InnerTaylorSeries SMP
    NNI ==> NonNegativeInteger
    ST  ==> Stream
    StS ==> Stream SMP
    STPS==>StreamTaylorSeriesOperations
    STTAYLOR==>StreamTaylorSeriesOperations
    SUP==> SparseUnivariatePolynomial(SMP)
    ST2==>StreamFunctions2
    SMPS==>  TaylorSeries(R)
    L==>List
    null ==> empty?
    likeUniv ==> univariate
    coef ==> coefficient$SUP
    nil ==> empty


    Defn ==>  with

        crest:(NNI->( StS-> StS))
          ++\spad{crest n} is used internally.
        cfirst:(NNI->( StS-> StS))
          ++\spad{cfirst n} is used internally.
        sts2stst:(VarSet,StS)->ST StS
          ++\spad{sts2stst(v,s)} is used internally.
        clikeUniv:VarSet->(SMP->SUP)
          ++\spad{clikeUniv(v)} is used internally.
        weierstrass:(VarSet,SMPS)->L SMPS
          ++\spad{weierstrass(v,ts)} where v is a variable and ts is
          ++ a TaylorSeries, impements the Weierstrass Preparation
          ++ Theorem. The result is a list of TaylorSeries that
          ++ are the coefficients of the equivalent series.
        qqq:(NNI,SMPS,ST SMPS)->((ST SMPS)->ST SMPS)
```

```
      ++\spad{qqq(n,s,st)} is used internally.

Impl ==>  add
    import TaylorSeries(R)
    import StreamTaylorSeriesOperations SMP
    import StreamTaylorSeriesOperations SMPS


    map1==>map$(ST2(SMP,SUP))
    map2==>map$(ST2(StS,SMP))
    map3==>map$(ST2(StS,StS))
    transback:ST SMPS->L SMPS
    transback smps==
        if null smps
        then nil()$(L SMPS)
        else
          if null first (smps:(ST StS))
          then nil()$(L SMPS)
          else
            cons(map2(first,smps:ST StS):SMPS,
               transback(map3(rest,smps:ST StS):(ST SMPS)))$(L SMPS)


    clikeUniv(var)==likeUniv(#1,var)
    mind:(NNI,StS)->NNI
    mind(n, sts)==
       if null sts
       then error "no mindegree"
       else if first sts=0
            then mind(n+1,rest sts)
            else n
    mindegree (sts:StS):NNI== mind(0,sts)


    streamlikeUniv:(SUP,NNI)->StS
    streamlikeUniv(p:SUP,n:NNI): StS ==
      if n=0
      then cons(coef (p,0),nil()$StS)
      else cons(coef (p,n),streamlikeUniv(p,(n-1):NNI))

    transpose:ST StS->ST StS
    transpose(s:ST StS)==delay(
       if null s
       then nil()$(ST StS)
       else cons(map2(first,s),transpose(map3(rest,rst s))))
```

```
        zp==>map$StreamFunctions3(SUP,NNI,StS)

        sts2stst(var, sts)==
           zp(streamlikeUniv(#1,#2),
             map1(clikeUniv var, sts),(integers 0):(ST NNI))

        tp:(VarSet,StS)->ST StS
        tp(v,sts)==transpose sts2stst(v,sts)
        map4==>map$(ST2 (StS,StS))
        maptake:(NNI,ST StS)->ST SMPS
        maptake(n,p)== map4(cfirst n,p) pretend ST SMPS
        mapdrop:(NNI,ST StS)->ST SMPS
        mapdrop(n,p)== map4(crest n,p) pretend ST SMPS
        YSS==>Y$ParadoxicalCombinatorsForStreams(SMPS)
        weier:(VarSet,StS)->ST SMPS
        weier(v,sts)==
             a:=mindegree sts
             if a=0
             then error "has constant term"
             else
               p:=tp(v,sts) pretend (ST SMPS)
               b:StS:=rest(((first p pretend StS)),a::NNI)
               c:=retractIfCan first b
               c case "failed"=>_
   error "the coefficient of the lowest degree of the variable should _
   be a constant"
               e:=recip b
               f:= if e case "failed"
                   then error "no reciprocal"
                   else e::StS
               q:=(YSS qqq(a,f:SMPS,rest p))
               maptake(a,(p*q) pretend ST StS)

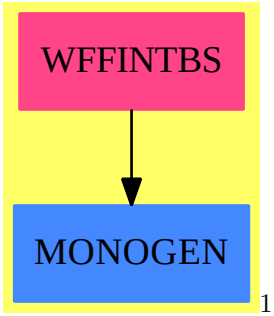
        cfirst n== first(#1,n)$StS
        crest n== rest(#1,n)$StS
        qq:(NNI,SMPS,ST SMPS,ST SMPS)->ST SMPS
        qq(a,e,p,c)==
            cons(e,(-e)*mapdrop(a,(p*c)pretend(ST StS)))
        qqq(a,e,p)==  qq(a,e,p,#1)
        wei:(VarSet,SMPS)->ST SMPS
        wei(v:VarSet,s:SMPS)==weier(v,s:StS)
        weierstrass(v,smps)== transback wei (v,smps)
```

⟨ *WEIER.dotabb* ⟩≡

```
"WEIER" [color="#FF4488",href="bookvol10.4.pdf#nameddest=WEIER"]
"PFECAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=PFECAT"]
"WEIER" -> "PFECAT"
```

24.3 package WFFINTBS WildFunctionFieldIntegralBasis

24.4 WildFunctionFieldIntegralBasis



1.00

Exports:
integralBasis localIntegralBasis

⟨*package WFFINTBS WildFunctionFieldIntegralBasis*⟩≡

```
)abbrev package WFFINTBS WildFunctionFieldIntegralBasis
++ Authors: Victor Miller, Clifton Williamson
++ Date Created: 24 July 1991
++ Date Last Updated: 20 September 1994
++ Basic Operations: integralBasis, localIntegralBasis
++ Related Domains: IntegralBasisTools(R,UP,F),
++   TriangularMatrixOperations(R,Vector R,Vector R,Matrix R)
++ Also See: FunctionFieldIntegralBasis, NumberFieldIntegralBasis
++ AMS Classifications:
++ Keywords: function field, integral basis
++ Examples:
++ References:
++ Description:
++ In this package K is a finite field, R is a ring of univariate
++ polynomials over K, and F is a framed algebra over R.  The package
++ provides a function to compute the integral closure of R in the quotient
++ field of F as well as a function to compute a "local integral basis"
++ at a specific prime.

WildFunctionFieldIntegralBasis(K,R,UP,F): Exports == Implementation where
  K  : FiniteFieldCategory
  --K  : Join(Field,Finite)
  R  : UnivariatePolynomialCategory K
  UP : UnivariatePolynomialCategory R
  F  : FramedAlgebra(R,UP)
```

```
I       ==> Integer
Mat     ==> Matrix R
NNI     ==> NonNegativeInteger
SAE     ==> SimpleAlgebraicExtension
RResult ==> Record(basis: Mat, basisDen: R, basisInv:Mat)
IResult ==> Record(basis: Mat, basisDen: R, basisInv:Mat,discr: R)
MATSTOR ==> StorageEfficientMatrixOperations

Exports ==> with
  integralBasis : () -> RResult
    ++ \spad{integralBasis()} returns a record
    ++ \spad{[basis,basisDen,basisInv]} containing information regarding
    ++ the integral closure of R in the quotient field of F, where
    ++ F is a framed algebra with R-module basis \spad{w1,w2,...,wn}.
    ++ If \spad{basis} is the matrix \spad{(aij, i = 1..n, j = 1..n)}, then
    ++ the \spad{i}th element of the integral basis is
    ++ \spad{vi = (1/basisDen) * sum(aij * wj, j = 1..n)}, i.e. the
    ++ \spad{i}th row of \spad{basis} contains the coordinates of the
    ++ \spad{i}th basis vector.  Similarly, the \spad{i}th row of the
    ++ matrix \spad{basisInv} contains the coordinates of \spad{wi} with
    ++ respect to the basis \spad{v1,...,vn}: if \spad{basisInv} is the
    ++ matrix \spad{(bij, i = 1..n, j = 1..n)}, then
    ++ \spad{wi = sum(bij * vj, j = 1..n)}.
  localIntegralBasis : R -> RResult
    ++ \spad{integralBasis(p)} returns a record
    ++ \spad{[basis,basisDen,basisInv]} containing information regarding
    ++ the local integral closure of R at the prime \spad{p} in the quotient
    ++ field of F, where F is a framed algebra with R-module basis
    ++ \spad{w1,w2,...,wn}.
    ++ If \spad{basis} is the matrix \spad{(aij, i = 1..n, j = 1..n)}, then
    ++ the \spad{i}th element of the local integral basis is
    ++ \spad{vi = (1/basisDen) * sum(aij * wj, j = 1..n)}, i.e. the
    ++ \spad{i}th row of \spad{basis} contains the coordinates of the
    ++ \spad{i}th basis vector.  Similarly, the \spad{i}th row of the
    ++ matrix \spad{basisInv} contains the coordinates of \spad{wi} with
    ++ respect to the basis \spad{v1,...,vn}: if \spad{basisInv} is the
    ++ matrix \spad{(bij, i = 1..n, j = 1..n)}, then
    ++ \spad{wi = sum(bij * vj, j = 1..n)}.

Implementation ==> add
  import IntegralBasisTools(R, UP, F)
  import ModularHermitianRowReduction(R)
  import TriangularMatrixOperations(R, Vector R, Vector R, Matrix R)
  import DistinctDegreeFactorize(K,R)

  listSquaredFactors: R -> List R
```

```
listSquaredFactors px ==
  -- returns a list of the factors of px which occur with
  -- exponent > 1
  ans : List R := empty()
  factored := factor(px)$DistinctDegreeFactorize(K,R)
  for f in factors(factored) repeat
    if f.exponent > 1 then ans := concat(f.factor,ans)
  ans

iLocalIntegralBasis: (Vector F,Vector F,Matrix R,Matrix R,R,R) -> IResult
iLocalIntegralBasis(bas,pows,tfm,matrixOut,disc,prime) ==
  n := rank()$F; standardBasis := basis()$F
  -- 'standardBasis' is the basis for F as a FramedAlgebra;
  -- usually this is [1,y,y**2,...,y**(n-1)]
  p2 := prime * prime; sae := SAE(K,R,prime)
  p := characteristic()$F; q := size()$sae
  lp := leastPower(q,n)
  rb := scalarMatrix(n,1); rbinv := scalarMatrix(n,1)
  -- rb    = basis matrix of current order
  -- rbinv = inverse basis matrix of current order
  -- these are wrt the orginal basis for F
  rbden : R := 1; index : R := 1; oldIndex : R := 1
  -- rbden = denominator for current basis matrix
  -- index = index of original order in current order
  repeat
    -- pows = [(w1 * rbden) ** q,...,(wn * rbden) ** q], where
    -- bas = [w1,...,wn] is 'rbden' times the basis for the order B = 'rb'
    for i in 1..n repeat
      bi : F := 0
      for j in 1..n repeat
        bi := bi + qelt(rb,i,j) * qelt(standardBasis,j)
      qsetelt_!(bas,i,bi)
      qsetelt_!(pows,i,bi ** p)
    coor0 := transpose coordinates(pows,bas)
    denPow := rbden ** ((p - 1) :: NNI)
    (coMat0 := coor0 exquo denPow) case "failed" =>
      error "can't happen"
    -- the jth column of coMat contains the coordinates of (wj/rbden)**q
    -- with respect to the basis [w1/rbden,...,wn/rbden]
    coMat := coMat0 :: Matrix R
    -- the ith column of 'pPows' contains the coordinates of the pth power
    -- of the ith basis element for B/prime.B over 'sae' = R/prime.R
    pPows := map(reduce,coMat)$MatrixCategoryFunctions2(R,Vector R,
                Vector R,Matrix R,sae,Vector sae,Vector sae,Matrix sae)
    -- 'frob' will eventually be the Frobenius matrix for B/prime.B over
    -- 'sae' = R/prime.R; at each stage of the loop the ith column will
```

```
    -- contain the coordinates of p^k-th powers of the ith basis element
    frob := copy pPows; tmpMat : Matrix sae := new(n,n,0)
    for r in 2..leastPower(p,q) repeat
      for i in 1..n repeat for j in 1..n repeat
        qsetelt_!(tmpMat,i,j,qelt(frob,i,j) ** p)
      times_!(frob,pPows,tmpMat)$MATSTOR(sae)
    frobPow := frob ** lp
    -- compute the p-radical
    ns := nullSpace frobPow
    for i in 1..n repeat for j in 1..n repeat qsetelt_!(tfm,i,j,0)
    for vec in ns for i in 1.. repeat
      for j in 1..n repeat
        qsetelt_!(tfm,i,j,lift qelt(vec,j))
    id := squareTop rowEchelon(tfm,prime)
    -- id = basis matrix of the p-radical
    idinv := UpTriBddDenomInv(id, prime)
    -- id * idinv = prime * identity
    -- no need to check for inseparability in this case
    rbinv := idealiser(id * rb, rbinv * idinv, prime * rbden)
    index := diagonalProduct rbinv
    rb := rowEchelon LowTriBddDenomInv(rbinv,rbden * prime)
    if divideIfCan_!(rb,matrixOut,prime,n) = 1
      then rb := matrixOut
      else rbden := rbden * prime
    rbinv := UpTriBddDenomInv(rb,rbden)
    indexChange := index quo oldIndex
    oldIndex := index
    disc := disc quo (indexChange * indexChange)
    (not sizeLess?(1,indexChange)) or ((disc exquo p2) case "failed") =>
      return [rb, rbden, rbinv, disc]

integralBasis() ==
  traceMat := traceMatrix()$F; n := rank()$F
  disc := determinant traceMat        -- discriminant of current order
  zero? disc => error "integralBasis: polynomial must be separable"
  singList := listSquaredFactors disc -- singularities of relative Spec
  runningRb := scalarMatrix(n,1); runningRbinv := scalarMatrix(n,1)
  -- runningRb    = basis matrix of current order
  -- runningRbinv = inverse basis matrix of current order
  -- these are wrt the original basis for F
  runningRbden : R := 1
  -- runningRbden = denominator for current basis matrix
  empty? singList => [runningRb, runningRbden, runningRbinv]
  bas : Vector F := new(n,0); pows : Vector F := new(n,0)
  -- storage for basis elements and their powers
  tfm : Matrix R := new(n,n,0)
```

```
      -- 'tfm' will contain the coordinates of a lifting of the kernel
      -- of a power of Frobenius
      matrixOut : Matrix R := new(n,n,0)
      for prime in singList repeat
        lb := iLocalIntegralBasis(bas,pows,tfm,matrixOut,disc,prime)
        rb := lb.basis; rbinv := lb.basisInv; rbden := lb.basisDen
        disc := lb.discr
        -- update 'running integral basis' if newly computed
        -- local integral basis is non-trivial
        if sizeLess?(1,rbden) then
          mat := vertConcat(rbden * runningRb,runningRbden * rb)
          runningRbden := runningRbden * rbden
          runningRb := squareTop rowEchelon(mat,runningRbden)
          runningRbinv := UpTriBddDenomInv(runningRb,runningRbden)
      [runningRb, runningRbden, runningRbinv]

    localIntegralBasis prime ==
      traceMat := traceMatrix()$F; n := rank()$F
      disc := determinant traceMat        -- discriminant of current order
      zero? disc => error "localIntegralBasis: polynomial must be separable"
      (disc exquo (prime * prime)) case "failed" =>
        [scalarMatrix(n,1), 1, scalarMatrix(n,1)]
      bas : Vector F := new(n,0); pows : Vector F := new(n,0)
      -- storage for basis elements and their powers
      tfm : Matrix R := new(n,n,0)
      -- 'tfm' will contain the coordinates of a lifting of the kernel
      -- of a power of Frobenius
      matrixOut : Matrix R := new(n,n,0)
      lb := iLocalIntegralBasis(bas,pows,tfm,matrixOut,disc,prime)
      [lb.basis, lb.basisDen, lb.basisInv]
```

⟨*WFFINTBS.dotabb*⟩≡

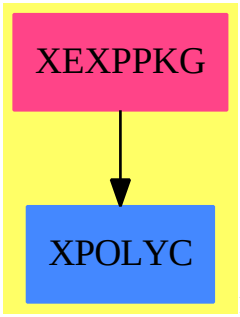
```
"WFFINTBS" [color="#FF4488",href="bookvol10.4.pdf#nameddest=WFFINTBS"]
"MONOGEN" [color="#4488FF",href="bookvol10.2.pdf#nameddest=MONOGEN"]
"WFFINTBS" -> "MONOGEN"
```

Chapter 25

Chapter X

25.1 package XEXPPKG XExponentialPackage

25.2 XExponentialPackage



1.00

Exports:
exp Hausdorff log

⟨*XEXPPKG.dotabb*⟩≡
```
"XEXPPKG" [color="#FF4488",href="bookvol10.4.pdf#nameddest=XEXPPKG"]
```

⟨*package XEXPPKG XExponentialPackage*⟩≡

```
)abbrev package XEXPPKG XExponentialPackage
++ Author: Michel Petitot (petitot@lifl.fr).
++ Date Created: 91
++ Date Last Updated: 7 Juillet 92
++ Fix History: compilation v 2.1 le 13 dec 98
++ Basic Functions:
++ Related Constructors:
++ Also See:
++ AMS Classifications:
++ Keywords:
++ References:
++ Description:
++ This package provides computations of logarithms and exponentials
++ for polynomials in non-commutative
++ variables. \newline Author: Michel Petitot (petitot@lifl.fr).

XExponentialPackage(R, VarSet, XPOLY): Public == Private where
    RN     ==> Fraction Integer
    NNI    ==> NonNegativeInteger
    I      ==> Integer
    R      : Join(Ring, Module RN)
    -- R      : Field
    VarSet : OrderedSet
    XPOLY  : XPolynomialsCat(VarSet, R)

    Public == with
       exp:  (XPOLY, NNI) -> XPOLY
         ++ \axiom{exp(p, n)} returns the exponential of \axiom{p}
         ++ truncated at order \axiom{n}.
       log:  (XPOLY, NNI) -> XPOLY
         ++ \axiom{log(p, n)} returns the logarithm of \axiom{p}
         ++ truncated at order \axiom{n}.
       Hausdorff: (XPOLY, XPOLY, NNI) -> XPOLY
         ++ \axiom{Hausdorff(a,b,n)} returns log(exp(a)*exp(b))
         ++ truncated at order \axiom{n}.

    Private == add

        log (p,n) ==
           p1 : XPOLY := p - 1
           not quasiRegular? p1 =>
             error "constant term <> 1, impossible log"
           s : XPOLY := 0       -- resultat
           k : I := n :: I
           for i in 1 .. n repeat
```

```
            k1 :RN := 1/k
            k2 : R := k1 * 1$R
            s := trunc( trunc(p1,i) * (k2 :: XPOLY - s) , i)
            k := k - 1
         s

      exp (p,n) ==
         not quasiRegular? p =>
           error "constant term <> 0, exp impossible"
         p = 0 => 1
         s : XPOLY := 1$XPOLY       -- resultat
         k : I := n :: I
         for i in 1 .. n repeat
            k1 :RN := 1/k
            k2 : R := k1 * 1$R
            s := trunc( 1 +$XPOLY k2 * trunc(p,i) * s , i)
            k := k - 1
         s

      Hausdorff(p,q,n) ==
         p1: XPOLY := exp(p,n)
         q1: XPOLY := exp(q,n)
         log(p1*q1, n)
```

⟨*XEXPPKG.dotabb*⟩+≡

```
"XEXPPKG" [color="#FF4488",href="bookvol10.4.pdf#nameddest=XEXPPKG"]
"XPOLYC" [color="#4488FF",href="bookvol10.2.pdf#nameddest=XPOLYC"]
"XEXPPKG" -> "XPOLYC"
```

Chapter 26

Chapter Y

Chapter 27

Chapter Z

27.1 package ZDSOLVE ZeroDimensionalSolvePackage

Based on triangular decompositions and the **RealClosure** constructor, the pacakge **ZeroDimensionalSolvePackage** provides operations for computing symbolically the real or complex roots of polynomial systems with finitely many solutions.

⟨*ZeroDimensionalSolvePackage.input*⟩≡

```
)set break resume
)spool ZeroDimensionalSolvePackage.output
)set message test on
)set message auto off
)clear all
--S 1 of 28
R := Integer
--R
--R
--R   (1)  Integer
--R                                                    Type: Domain
--E 1

--S 2 of 28
ls : List Symbol := [x,y,z,t]
--R
--R
--R   (2)  [x,y,z,t]
--R                                                 Type: List Symbol
--E 2
```

```
--S 3 of 28
ls2 : List Symbol := [x,y,z,t,new()$Symbol]
--R
--R
--R   (3)  [x,y,z,t,%A]
--R                                                        Type: List Symbol
--E 3

--S 4 of 28
pack := ZDSOLVE(R,ls,ls2)
--R
--R
--R   (4)  ZeroDimensionalSolvePackage(Integer,[x,y,z,t],[x,y,z,t,%A])
--R                                                              Type: Domain
--E 4

--S 5 of 28
p1 := x**2*y*z + x*y**2*z + x*y*z**2 + x*y*z + x*y + x*z + y*z
--R
--R
--R             2       2     2
--R   (5)  x y z  + (x y  + (x  + x + 1)y + x)z + x y
--R                                                  Type: Polynomial Integer
--E 5

--S 6 of 28
p2 := x**2*y**2*z + x*y**2*z**2 + x**2*y*z + x*y*z + y*z + x + z
--R
--R
--R           2 2     2 2     2
--R   (6)  x y z  + (x y  + (x  + x + 1)y + 1)z + x
--R                                                  Type: Polynomial Integer
--E 6

--S 7 of 28
p3 := x**2*y**2*z**2 + x**2*y**2*z + x*y**2*z + x*y*z + x*z + z + 1
--R
--R
--R         2 2 2      2      2
--R   (7)  x y z  + ((x  + x)y  + x y + x + 1)z + 1
--R                                                  Type: Polynomial Integer
--E 7

--S 8 of 28
lp := [p1, p2, p3]
```

```
--R
--R
--R   (8)
--R         2       2     2
--R   [x y z  + (x y  + (x  + x + 1)y + x)z + x y,
--R       2 2     2 2     2
--R    x y z  + (x y  + (x  + x + 1)y + 1)z + x,
--R     2 2 2      2      2
--R    x y z  + ((x  + x)y  + x y + x + 1)z + 1]
--R                                              Type: List Polynomial Integer
--E 8

--S 9 of 28
triangSolve(lp)$pack
--R
--R
--R   (9)
--R   [
--R     {
--R          20     19      18      17       16      15       14       13       12
--R         z   - 6z   - 41z   + 71z   + 106z   + 92z   + 197z   + 145z   + 257z
--R       +
--R              11       10       9       8       7       6      5       4      3
--R         278z   + 201z   + 278z  + 257z  + 145z  + 197z  + 92z  + 106z  + 71z
--R       +
--R               2
--R         - 41z  - 6z + 1
--R       ,
--R
--R                       19            18             17             16
--R             14745844z   + 50357474z   - 130948857z   - 185261586z
--R           +
--R                           15             14             13             12
--R             - 180077775z   - 338007307z   - 275379623z   - 453190404z
--R           +
--R                           11             10             9              8
--R             - 474597456z   - 366147695z   - 481433567z  - 430613166z
--R           +
--R                           7             6             5             4
--R             - 261878358z  - 326073537z  - 163008796z  - 177213227z
--R           +
--R                           3            2
--R             - 104356755z  + 65241699z  + 9237732z - 1567348
--R        *
--R           y
--R       +
```

```
--R                19           18            17            16            15
--R         1917314z   + 6508991z   - 16973165z   - 24000259z   - 23349192z
--R       +
--R                   14            13            12            11            10
--R         - 43786426z   - 35696474z   - 58724172z   - 61480792z   - 47452440z
--R       +
--R                  9            8            7           6           5
--R         - 62378085z  - 55776527z  - 33940618z  - 42233406z  - 21122875z
--R       +
--R                  4           3          2
--R         - 22958177z  - 13504569z  + 8448317z  + 1195888z - 202934
--R       ,
--R         3       2       3    2              2             2
--R      ((z  - 2z)y  + (- z  - z  - 2z - 1)y - z  - z + 1)x + z  - 1}
--R     ]
--R                                    Type: List RegularChain(Integer,[x,y,z,t])
--E 9

--S 10 of 28
univariateSolve(lp)$pack
--R
--R
--R   (10)
--R   [
--R     [
--R       complexRoots =
--R            12      11      10     9     8      7      6      5     4     3
--R          ?   - 12?   + 24?   + 4?  - 9?  + 27?  - 21?  + 27?  - 9?  + 4?
--R        +
--R             2
--R          24?  - 12? + 1
--R       ,
--R
--R       coordinates =
--R         [
--R                        11         10         9         8         7         6
--R             63x + 62%A   - 721%A   + 1220%A  + 705%A  - 285%A  + 1512%A
--R           +
--R                    5         4       3        2
--R             - 735%A  + 1401%A  - 21%A  + 215%A  + 1577%A - 142
--R           ,
--R
--R                        11         10         9         8         7         6
--R             63y - 75%A   + 890%A   - 1682%A  - 516%A  + 588%A  - 1953%A
--R           +
--R                    5         4        3        2
```

```
--R             1323%A  - 1815%A  + 426%A  - 243%A  - 1801%A + 679
--R           ,
--R          z - %A]
--R       ]
--R     ,
--R
--R                      6    5    4    3    2
--R     [complexRoots= ?  + ?  + ?  + ?  + ?  + ? + 1,
--R                           5       3
--R      coordinates= [x - %A ,y - %A ,z - %A]]
--R     ,
--R                     2
--R    [complexRoots= ?  + 5? + 1,coordinates= [x - 1,y - 1,z - %A]]]
--RType: List Record(complexRoots: SparseUnivariatePolynomial Integer,coordinates: List Poly
--E 10

--S 11 of 28
lr := realSolve(lp)$pack
--R
--R
--R   (11)
--R   [
--R     [%B1,
--R
--R         1184459    19   2335702    18   5460230    17   79900378    16
--R         ------- %B1   - ------- %B1   - ------- %B1   + -------- %B1
--R         1645371          548457          182819          1645371
--R       +
--R         43953929    15   13420192    14   553986    13   193381378    12
--R         -------- %B1   + -------- %B1   + ------ %B1   + --------- %B1
--R          548457           182819           3731           1645371
--R       +
--R         35978916    11   358660781    10   271667666    9   118784873    8
--R         -------- %B1   + --------- %B1   + --------- %B1  + --------- %B1
--R          182819           1645371           1645371           548457
--R       +
--R         337505020    7   1389370    6   688291    5   3378002    4
--R         --------- %B1  + ------- %B1  + ------ %B1  + ------- %B1
--R          1645371          11193          4459          42189
--R       +
--R         140671876    3   32325724    2   8270       9741532
--R         --------- %B1  + -------- %B1  - ---- %B1 - -------
--R          1645371          548457          343       1645371
--R       ,
--R
--R           91729    19   487915    18   4114333    17   1276987    16
```

```
--R          - ------ %B1   + ------ %B1   + ------- %B1   - ------- %B1
--R            705159         705159          705159          235053
--R       +
--R            13243117    15   16292173    14   26536060    13   722714    12
--R          - -------- %B1   - -------- %B1   - -------- %B1   - ------ %B1
--R             705159          705159          705159          18081
--R       +
--R            5382578    11   15449995    10   14279770    9   6603890    8
--R          - ------- %B1   - -------- %B1   - -------- %B1  - ------- %B1
--R             100737          235053          235053          100737
--R       +
--R            409930    7   37340389    6   34893715    5   26686318    4
--R          - ------ %B1  - -------- %B1  - -------- %B1  - -------- %B1
--R             6027          705159          705159          705159
--R       +
--R            801511    3   17206178    2   4406102       377534
--R          - ------ %B1  - -------- %B1  - ------- %B1 + ------
--R             26117         705159          705159       705159
--R       ]
--R     ,
--R
--R     [%B2,
--R
--R          1184459    19   2335702    18   5460230    17   79900378    16
--R          ------- %B2   - ------- %B2   - ------- %B2   + -------- %B2
--R          1645371          548457          182819          1645371
--R       +
--R          43953929    15   13420192    14   553986    13   193381378    12
--R          -------- %B2   + -------- %B2   + ------ %B2   + --------- %B2
--R           548457           182819           3731           1645371
--R       +
--R          35978916    11   358660781    10   271667666    9   118784873    8
--R          -------- %B2   + --------- %B2   + --------- %B2  + --------- %B2
--R           182819           1645371          1645371          548457
--R       +
--R          337505020    7   1389370    6   688291    5   3378002    4
--R          --------- %B2  + ------- %B2  + ------ %B2  + ------- %B2
--R           1645371          11193          4459          42189
--R       +
--R          140671876    3   32325724    2   8270       9741532
--R          --------- %B2  + -------- %B2  - ---- %B2 - -------
--R           1645371          548457          343       1645371
--R       ,
--R
--R             91729    19   487915    18   4114333    17   1276987    16
--R          - ------ %B2   + ------ %B2   + ------- %B2   - ------- %B2
```

```
--R           705159          705159          705159          235053
--R       +
--R           13243117    15   16292173    14   26536060    13   722714    12
--R         - -------- %B2   - -------- %B2   - -------- %B2   - ------ %B2
--R            705159           705159           705159           18081
--R       +
--R           5382578    11   15449995    10   14279770    9   6603890    8
--R         - ------- %B2   - -------- %B2   - -------- %B2  - ------- %B2
--R            100737           235053           235053          100737
--R       +
--R           409930    7   37340389    6   34893715    5   26686318    4
--R         - ------ %B2  - -------- %B2  - -------- %B2  - -------- %B2
--R            6027          705159          705159          705159
--R       +
--R           801511    3   17206178    2   4406102       377534
--R         - ------ %B2  - -------- %B2  - ------- %B2 + ------
--R            26117         705159          705159       705159
--R       ]
--R     ,
--R
--R     [%B3,
--R
--R         1184459    19   2335702    18   5460230    17   79900378    16
--R         ------- %B3   - ------- %B3   - ------- %B3   + -------- %B3
--R         1645371          548457          182819          1645371
--R       +
--R         43953929    15   13420192    14   553986    13   193381378    12
--R         -------- %B3   + -------- %B3   + ------ %B3   + --------- %B3
--R          548457           182819           3731           1645371
--R       +
--R         35978916    11   358660781    10   271667666    9   118784873    8
--R         -------- %B3   + --------- %B3   + --------- %B3  + --------- %B3
--R          182819           1645371           1645371           548457
--R       +
--R         337505020    7   1389370    6   688291    5   3378002    4
--R         --------- %B3  + ------- %B3  + ------ %B3  + ------- %B3
--R          1645371          11193          4459          42189
--R       +
--R         140671876    3   32325724    2   8270       9741532
--R         --------- %B3  + -------- %B3  - ---- %B3 - -------
--R          1645371          548457          343       1645371
--R       ,
--R
--R           91729    19   487915    18   4114333    17   1276987    16
--R         - ------ %B3   + ------ %B3   + ------- %B3   - ------- %B3
--R           705159         705159          705159          235053
```

```
--R       +
--R           13243117    15   16292173    14   26536060    13   722714    12
--R         - -------- %B3   - -------- %B3   - -------- %B3   - ------ %B3
--R            705159           705159           705159           18081
--R       +
--R           5382578    11   15449995    10   14279770    9   6603890    8
--R         - ------- %B3   - -------- %B3   - -------- %B3  - ------- %B3
--R            100737           235053           235053          100737
--R       +
--R           409930    7   37340389    6   34893715    5   26686318    4
--R         - ------ %B3  - -------- %B3  - -------- %B3  - -------- %B3
--R            6027          705159          705159          705159
--R       +
--R           801511    3   17206178    2   4406102       377534
--R         - ------ %B3  - -------- %B3  - ------- %B3 + ------
--R            26117         705159          705159       705159
--R       ]
--R     ,
--R
--R     [%B4,
--R
--R         1184459    19   2335702    18   5460230    17   79900378    16
--R         ------- %B4   - ------- %B4   - ------- %B4   + -------- %B4
--R         1645371          548457          182819          1645371
--R       +
--R         43953929    15   13420192    14   553986    13   193381378    12
--R         -------- %B4   + -------- %B4   + ------ %B4   + ---------- %B4
--R          548457           182819           3731           1645371
--R       +
--R         35978916    11   358660781    10   271667666    9   118784873    8
--R         -------- %B4   + ---------- %B4   + ---------- %B4  + ---------- %B4
--R          182819           1645371           1645371           548457
--R       +
--R         337505020    7   1389370    6   688291    5   3378002    4
--R         ---------- %B4  + ------- %B4  + ------ %B4  + ------- %B4
--R          1645371          11193          4459          42189
--R       +
--R         140671876    3   32325724    2   8270       9741532
--R         --------- %B4  + -------- %B4  - ---- %B4 - -------
--R          1645371          548457          343       1645371
--R       ,
--R
--R            91729    19   487915    18   4114333    17   1276987    16
--R         - ------ %B4   + ------ %B4   + ------- %B4   - ------- %B4
--R           705159         705159          705159          235053
--R       +
```

```
--R           13243117     15    16292173     14    26536060     13    722714     12
--R         - -------- %B4   - -------- %B4   - -------- %B4   - ------ %B4
--R            705159            705159            705159            18081
--R       +
--R           5382578     11    15449995     10    14279770     9    6603890     8
--R         - ------- %B4   - -------- %B4   - -------- %B4  - ------- %B4
--R            100737           235053            235053           100737
--R       +
--R           409930     7    37340389     6    34893715     5    26686318     4
--R         - ------ %B4  - -------- %B4  - -------- %B4  - -------- %B4
--R            6027           705159           705159           705159
--R       +
--R           801511     3    17206178     2    4406102        377534
--R         - ------ %B4  - -------- %B4  - ------- %B4 + ------
--R            26117          705159           705159        705159
--R       ]
--R     ,
--R
--R     [%B5,
--R
--R         1184459     19    2335702     18    5460230     17    79900378     16
--R         ------- %B5   - ------- %B5   - ------- %B5   + -------- %B5
--R         1645371           548457           182819           1645371
--R       +
--R         43953929     15    13420192     14    553986     13    193381378     12
--R         -------- %B5   + -------- %B5   + ------ %B5   + ---------- %B5
--R          548457            182819            3731            1645371
--R       +
--R         35978916     11    358660781     10    271667666     9    118784873     8
--R         -------- %B5   + --------- %B5   + --------- %B5  + --------- %B5
--R          182819            1645371            1645371            548457
--R       +
--R         337505020     7    1389370     6    688291     5    3378002     4
--R         --------- %B5  + ------- %B5  + ------ %B5  + ------- %B5
--R          1645371           11193           4459           42189
--R       +
--R         140671876     3    32325724     2    8270        9741532
--R         --------- %B5  + -------- %B5  - ---- %B5 - -------
--R          1645371           548457           343        1645371
--R       ,
--R
--R           91729     19    487915     18    4114333     17    1276987     16
--R         - ------ %B5   + ------ %B5   + ------- %B5   - ------- %B5
--R           705159          705159           705159           235053
--R       +
--R           13243117     15    16292173     14    26536060     13    722714     12
```

```
--R          - -------- %B5   - -------- %B5   - -------- %B5   - ------ %B5
--R             705159           705159           705159           18081
--R       +
--R             5382578    11   15449995    10   14279770    9   6603890    8
--R          - ------- %B5   - -------- %B5   - -------- %B5  - ------- %B5
--R             100737           235053           235053           100737
--R       +
--R             409930    7   37340389    6   34893715    5   26686318    4
--R          - ------ %B5  - -------- %B5  - -------- %B5  - -------- %B5
--R             6027           705159           705159           705159
--R       +
--R             801511    3   17206178    2   4406102       377534
--R          - ------ %B5  - -------- %B5  - ------- %B5 + ------
--R             26117          705159           705159       705159
--R       ]
--R     ,
--R
--R     [%B6,
--R
--R          1184459    19   2335702    18   5460230    17   79900378    16
--R          ------- %B6   - ------- %B6   - ------- %B6   + -------- %B6
--R          1645371          548457           182819           1645371
--R       +
--R          43953929    15   13420192    14   553986    13   193381378    12
--R          -------- %B6   + -------- %B6   + ------ %B6   + ---------- %B6
--R           548457           182819           3731           1645371
--R       +
--R          35978916    11   358660781    10   271667666    9   118784873    8
--R          -------- %B6   + ---------- %B6   + ---------- %B6  + ---------- %B6
--R           182819           1645371           1645371           548457
--R       +
--R          337505020    7   1389370    6   688291    5   3378002    4
--R          ---------- %B6  + ------- %B6  + ------ %B6  + ------- %B6
--R           1645371          11193          4459          42189
--R       +
--R          140671876    3   32325724    2   8270       9741532
--R          ---------- %B6  + -------- %B6  - ---- %B6 - -------
--R           1645371          548457          343       1645371
--R       ,
--R
--R             91729    19   487915    18   4114333    17   1276987    16
--R          - ------ %B6   + ------ %B6   + ------- %B6   - ------- %B6
--R            705159         705159           705159           235053
--R       +
--R             13243117    15   16292173    14   26536060    13   722714    12
--R          - -------- %B6   - -------- %B6   - -------- %B6   - ------ %B6
```

```
--R            705159            705159            705159            18081
--R       +
--R           5382578    11   15449995    10   14279770    9   6603890    8
--R         - ------- %B6   - -------- %B6   - -------- %B6  - ------- %B6
--R            100737           235053            235053           100737
--R       +
--R           409930    7   37340389    6   34893715    5   26686318    4
--R         - ------ %B6  - -------- %B6  - -------- %B6  - -------- %B6
--R            6027           705159           705159           705159
--R       +
--R           801511    3   17206178    2   4406102       377534
--R         - ------ %B6  - -------- %B6  - ------- %B6 + ------
--R            26117          705159           705159       705159
--R       ]
--R     ,
--R
--R     [%B7,
--R
--R         1184459    19   2335702    18   5460230    17   79900378    16
--R         ------- %B7   - ------- %B7   - ------- %B7   + -------- %B7
--R         1645371           548457           182819           1645371
--R       +
--R         43953929    15   13420192    14   553986    13   193381378    12
--R         -------- %B7   + -------- %B7   + ------ %B7   + ---------- %B7
--R          548457            182819            3731            1645371
--R       +
--R         35978916    11   358660781    10   271667666    9   118784873    8
--R         -------- %B7   + ---------- %B7   + ---------- %B7  + ---------- %B7
--R          182819            1645371           1645371           548457
--R       +
--R         337505020    7   1389370    6   688291    5   3378002    4
--R         ---------- %B7  + ------- %B7  + ------ %B7  + ------- %B7
--R          1645371           11193           4459           42189
--R       +
--R         140671876    3   32325724    2   8270       9741532
--R         ---------- %B7  + -------- %B7  - ---- %B7 - -------
--R          1645371           548457           343       1645371
--R       ,
--R
--R            91729    19   487915    18   4114333    17   1276987    16
--R         - ------ %B7   + ------ %B7   + ------- %B7   - ------- %B7
--R           705159          705159           705159           235053
--R       +
--R           13243117    15   16292173    14   26536060    13   722714    12
--R         - -------- %B7   - -------- %B7   - -------- %B7   - ------ %B7
--R            705159            705159            705159            18081
```

```
--R       +
--R           5382578    11   15449995    10   14279770    9   6603890    8
--R         - ------- %B7   - -------- %B7   - -------- %B7  - ------- %B7
--R            100737          235053           235053          100737
--R       +
--R           409930    7   37340389    6   34893715    5   26686318    4
--R         - ------ %B7  - -------- %B7  - -------- %B7  - -------- %B7
--R            6027          705159          705159          705159
--R       +
--R           801511    3   17206178    2   4406102       377534
--R         - ------ %B7  - -------- %B7  - ------- %B7 + ------
--R            26117         705159          705159       705159
--R       ]
--R     ,
--R
--R     [%B8,
--R
--R         1184459    19   2335702    18   5460230    17   79900378    16
--R         ------- %B8   - ------- %B8   - ------- %B8   + -------- %B8
--R         1645371          548457          182819          1645371
--R       +
--R         43953929    15   13420192    14   553986    13   193381378    12
--R         -------- %B8   + -------- %B8   + ------ %B8   + --------- %B8
--R          548457           182819           3731           1645371
--R       +
--R         35978916    11   358660781    10   271667666    9   118784873    8
--R         -------- %B8   + --------- %B8   + --------- %B8  + --------- %B8
--R          182819           1645371          1645371           548457
--R       +
--R         337505020    7   1389370    6   688291    5   3378002    4
--R         --------- %B8  + ------- %B8  + ------ %B8  + ------- %B8
--R          1645371          11193          4459          42189
--R       +
--R         140671876    3   32325724    2   8270       9741532
--R         --------- %B8  + -------- %B8  - ---- %B8 - -------
--R          1645371          548457          343       1645371
--R       ,
--R
--R            91729    19   487915    18   4114333    17   1276987    16
--R         - ------ %B8   + ------ %B8   + ------- %B8   - ------- %B8
--R           705159         705159          705159          235053
--R       +
--R           13243117    15   16292173    14   26536060    13   722714    12
--R         - -------- %B8   - -------- %B8   - -------- %B8   - ------ %B8
--R            705159           705159          705159          18081
--R       +
```

```
--R           5382578    11   15449995    10   14279770    9   6603890    8
--R         - ------- %B8   - -------- %B8   - -------- %B8  - ------- %B8
--R            100737          235053           235053          100737
--R       +
--R           409930    7   37340389    6   34893715    5   26686318    4
--R         - ------ %B8  - -------- %B8  - -------- %B8  - -------- %B8
--R            6027          705159          705159          705159
--R       +
--R           801511    3   17206178    2   4406102       377534
--R         - ------ %B8  - -------- %B8  - ------- %B8 + ------
--R            26117         705159          705159       705159
--R       ]
--R     ]
--R                                Type: List List RealClosure Fraction Integer
--E 11

--S 12 of 28
# lr
--R
--R
--R   (12)  8
--R                                                        Type: PositiveInteger
--E 12

--S 13 of 28
[ [approximate(r,1/1000000) for r in point] for point in lr]
--R
--R
--R   (13)
--R   [
--R        10048059
--R     [- --------,
--R         2097152
--R
--R        450305731698538794352439791383896641459673197621176821933588120838551 6_
--R         314058924567176091423629695777403099833360761048898228916578137094309_
--R         838597331137202584846939132376157019506760357601165917454986815382098_
--R         789094851523420392811293126141329856546977145464661495487825919941188_
--R         447041722440491921567263542158028061437758844364634410045253024786561_
--R         923163288214175
--R      /
--R        450305728302524548851651180698582663508310069375732046528055470686564 4_
--R         949577509916867201889438090408354817931718593862797624551518983570793_
--R         048774424291488708829840324189200301436123314860200821443733790755311_
--R         243632919864895421704228949571290016119498807957023663865443069392027_
--R         148979688266712323356043491523434068924275280417338574817381189277066_
```

```
--R          143312396681216
--R       ,
--R
--R        21062607688234750738947986804860165962496071486906855387636837150206392
--R         68085864965079005588950564689330944709709993780218732909532589878524
--R         24902071750498366048207515661873872451468533306001120296463516638135
--R         54325598220025030528398108683711061484230702609121129792987689628568
--R         83047905476005638076266490561846205530604781619178201158870378913898
--R         1895
--R      /
--R        21062606094984641924721138048164741753419629532964341024139031423687572
--R         96768527388858559097596521177886218987288195394364024629735706195981
--R         32610365979902512686325867656720234210687703171018424748418142328892
--R         83768123706270847029570621848592886740077193782849920092376059331416
--R         90100066637389634759811822855673103707202647449677622838376299392328
--R         0768
--R       ]
--R     ,
--R
--R        2563013
--R     [- -------,
--R        2097152
--R
--R       -
--R           26113461767919277896986176932377577192382599630635417819227523304402
--R            18989966807292833849076862359320744212592598673381593224350480929
--R            83752303023733723680666816744617300172727135331157124289 7
--R         /
--R           11652254005052225305839819160045891437572266102768589900087901348192
--R            91494092241375398397139401952343332040813992815318882949575545516
--R            96341761930839597754479714023146923426903492193805559398 4
--R       ,
--R
--R        35725945502759172210965887296157882729985170546756032395781981410060342
--R         09173528282659062190230446696394197103892330452627332931637375745006
--R         97898922861109769970872504662353 73
--R      /
--R        10395482693455989368770712448340260558008145511201705922005223665917592
--R         40965948644233914102945295026517998996010481187582253020534650513158
--R         24390172472891738650147029663088 64
--R       ]
--R     ,
--R
--R        1715967
--R     [- -------,
--R        2097152
```

```
--R
--R       -
--R           42130935337843035210848395179770823903772615039695862248289984366O6_
--R            03065607635937456481377349837660312126782256580143620693951995146 5_
--R            18222580524697287410022543952491
--R         /
--R           94418141441853744586496920343492240524365974709662536639306419607 95_
--R            80588258549319984019169991765944326482464113518738358388814786734 0_
--R            19307857605820364195856822304768
--R       ,
--R
--R        76358333471126442225156254244108312253474756690085893388341621725019 04_
--R         99437634673087680904284520891991992530210572097145391898273138907259 1_
--R         4035
--R      /
--R        26241887640860971997842976104780666339342304678958516022785809785037 84_
--R         54920578849901964060226696602689158010354356762503901862988714128491 6_
--R         75648
--R       ]
--R     ,
--R
--R         437701
--R     [- -------,
--R        2097152
--R
--R        16831069086383495883221723326542259135629863131819510314527501614414 97_
--R         47345532815072136486835557964678160350777719907507783521336648453365 4_
--R         91383623741304759
--R      /
--R        16831068680952133890017099827059136389630776687312261111677851880049 07_
--R         42522629868032588781096261414029859736698426488799890837706879999845 4_
--R         23381649008099328
--R       ,
--R
--R        49615501098350101864226810134221087359587148010037606397079680966469 12_
--R         82670847283444311723917219104249213450966312411133
--R      /
--R        49615498727577383155091920782102090298528971186110971262363840408293 76_
--R         59261914313170254867464792718363492160482442215424
--R       ]
--R     ,
--R
--R       222801
--R     [-------,
--R      2097152
--R
```

```
--R       -
--R           899488488040242826510759512197069142713604569254197827557300186521
--R            759921588137716696126349101655220195142994932299137183241705867672
--R            383477
--R         /
--R           116788999866502637217776510069188858270896996022993476969083575245
--R            077779416435209473767866507769405888942764587718542434255625992456
--R            372224
--R       ,
--R
--R       -
--R           238970488813315687832080154437380839561277150920849101984745299188
--R            509546519525467839016613593999693886640036283570552321155037871291
--R            458703265
--R         /
--R           535548727364509632609040328668993190598822544468541143322159383368
--R            192957562833671468654290340746993656285925599117602120446183443145
--R            479421952
--R       ]
--R     ,
--R
--R       765693
--R     [-------,
--R      2097152
--R
--R        855896921981671626787324476117819808872469895861667014021376575432200
--R         303251685786118678330840203328837654339523418704917749518340772512899
--R         000391009630373148561
--R      /
--R        294144244553301079097642841137639349981558021594585691790645253549572
--R         013856818941702330228779890141296236721138154231997238917322156711965
--R         2444639331719460159488
--R       ,
--R
--R       -
--R           205761823058257210124765032486024256111130258154358880884392366276
--R            549382241659362712290777612800192921420574408948085193743688582762
--R            2246433251878894899015
--R         /
--R           267159820332573553809795235350145022057631375989083509709172252064
--R            710198771902667183948906289863714759678360292483949204616471537777
--R            775324180661095366656
--R       ]
--R     ,
--R
--R      5743879
```

```
--R     [-------,
--R      2097152
--R
--R        10762888169689068479555463947735702081714567249426186140236631235747 68_
--R         96085043426397139807254659277266215883344979769861745539788756290007 2_
--R         98476800060834355318980169340872720504761255988923275756383052868895 3_
--R         53542180948277105891754260289006094194962087408300785836666945350176 6_
--R         24841488732463225
--R      /
--R        31317689570803179466484619400235520441903766134585849862285496319161 96_
--R         60161621978176561553253229474652964827643058381089407937456646075782 3_
--R         14688858119555602920851521883888320031865840746939942606326058982861 2_
--R         30923159666912970798648131985157194292723034062293402392348670304206 8_
--R         1530440845099008
--R       ,
--R
--R       -
--R           21132866991857509183641204755654584378701724898654859943898281353 35_
--R            26444466528455752649273493169173140787270143293550347334817207609 8_
--R            72054584900878007756416053431789468836611952973998050294416266855 0_
--R            09812796195049621022194287808935967492585059442776850225178975870 6_
--R            752831632503615
--R         /
--R           16276155849379875802429066243471045808891444661684597180431538394 08_
--R            37252553330980807036369958550221601121108710326360955102602776941 4_
--R            08739114812622116813978168258743807532259146613193997545720052234 9_
--R            83856896428563444801856203827237878735446010610614151801093561720 5_
--R            1706396253618176
--R       ]
--R     ,
--R
--R      19739877
--R     [--------,
--R       2097152
--R
--R       -
--R           29972499368327033037990158048615209492150403875007071777012857667 20_
--R            19253057942247895356602435986014310154780163808277161116037221287 4_
--R            84777803580987284314922548423836585801362934170532170258233335091 8_
--R            00960178993702398593530490046049338987383703085341034708990888081 4_
--R            85398113201846458245880061539477074169948729587596021075021589194 8_
--R            81447685487103153093129546733219013370267109820090228230051075186 0_
--R            71859284570302778073977965258138627622392869961068097280236 75
--R         /
--R           23084332748522785907289100811918110239065041413214326461239367948 73_
--R            93331927060896070213819341764789836062022951917663293763178685145 5_
```

```
--R            014766027206259022525055517418236889688380663660257443176047224
--R            292093196729475160247268834121141893318848728661844434927287285112
--R            897080767552864895056585864033178565910387065006112801516403522741
--R            037360990556054476949527059227070809593049491257519554708879259595
--R            529299201108585608125566354854294715540316759795426563813539 84
--R       ,
--R
--R       -
--R           5128189263548228489096276397868940080600938410663080459407966335845
--R            009264109490520459825316250084723010047035024497436523038925818959
--R            289312931584701353927621435434398674263047293909122850133851990696
--R            490231566094371994333795070782624011727587749989296611277318372294
--R            624207116537910436554574146082884701305543912620419354885410735940
--R            157775896602822364575864611831512943973974715166920465061850603762
--R            875162561958470524125872828391391946429139 55
--R         /
--R           228828193977843933053120879318129047118363109245536899038639082424 3
--R            509463644236249773080647438987739144921607794682653851741189091711
--R            741868145114978337284191822497675868358729486644730856622552687209
--R            203724411800481405702837198310642291275676195774614443815996713502
--R            629391749783590041470860127752372996488627742672487622480063268808
--R            889324891850842494934347337603075939980268208482904859678177751444
--R            657499798278726169630532176732017172372520 6
--R       ]
--R     ]
--R                                                  Type: List List Fraction Integer
--E 13

--S 14 of 28
lpr := positiveSolve(lp)$pack
--R
--R
--R   (14)  []
--R                                    Type: List List RealClosure Fraction Integer
--E 14

--S 15 of 28
f0 := x**3 + y + z + t- 1
--R
--R
--R                  3
--R   (15)  z + y + x  + t - 1
--R                                                        Type: Polynomial Integer
--E 15

--S 16 of 28
```

```
f1 := x + y**3 + z + t -1
--R
--R
--R              3
--R   (16)  z + y  + x + t - 1
--R                                                  Type: Polynomial Integer
--E 16

--S 17 of 28
f2 := x + y + z**3 + t-1
--R
--R
--R          3
--R   (17)  z  + y + x + t - 1
--R                                                  Type: Polynomial Integer
--E 17

--S 18 of 28
f3 := x + y + z + t**3 -1
--R
--R
--R                       3
--R   (18)  z + y + x + t  - 1
--R                                                  Type: Polynomial Integer
--E 18

--S 19 of 28
lf := [f0, f1, f2, f3]
--R
--R
--R   (19)
--R             3             3             3                             3
--R   [z + y + x  + t - 1,z + y  + x + t - 1,z  + y + x + t - 1,z + y + x + t  - 1]
--R                                                Type: List Polynomial Integer
--E 19

--S 20 of 28
lts := triangSolve(lf)$pack
--R
--R
--R   (20)
--R   [
--R       2           3        3
--R     {t  + t + 1, z  - z - t  + t,
--R
--R                3     2     2     3          6     3           3     2
```

```
--R         (3z + 3t  - 3)y  + (3z  + (6t  - 6)z + 3t  - 6t  + 3)y + (3t  - 3)z
--R       +
--R            6     3          9     6     3
--R         (3t  - 6t  + 3)z + t  - 3t  + 5t  - 3t
--R       ,
--R      x + y + z}
--R     ,
--R
--R       16     13     10     7      4      2
--R     {t   - 6t   + 9t   + 4t  + 15t  - 54t  + 27,
--R
--R                     15            14             13            12
--R              4907232t   + 40893984t   - 115013088t   + 22805712t   + 36330336t
--R           +
--R                        10             9             8             7
--R              162959040t   - 159859440t  - 156802608t  + 117168768t
--R           +
--R                       6             5             4             3
--R              126282384t  - 129351600t  + 306646992t  + 475302816t
--R           +
--R                          2
--R              - 1006837776t  - 237269088t + 480716208
--R        *
--R           z
--R       +
--R            54       51        48      46         45        43          42
--R         48t   - 912t   + 8232t   - 72t   - 46848t   + 1152t   + 186324t
--R       +
--R                 40          39        38         37           36         35
--R         - 3780t   - 543144t   - 3168t   - 21384t   + 1175251t   + 41184t
--R       +
--R                34           33          32           31           30
--R         278003t   - 1843242t   - 301815t   - 1440726t   + 1912012t
--R       +
--R                 29           28          27           26            25
--R         1442826t   + 4696262t   - 922481t   - 4816188t   - 10583524t
--R       +
--R                  24            23            22          21            20
--R         - 208751t   + 11472138t   + 16762859t   - 857663t   - 19328175t
--R       +
--R                    19           18            17            16           15
--R         - 18270421t   + 4914903t   + 22483044t   + 12926517t   - 8605511t
--R       +
--R                    14           13           12           11          10
--R         - 17455518t   - 5014597t   + 8108814t   + 8465535t   + 190542t
--R       +
```

```
--R                   9           8          7           6          5          4
--R         - 4305624t  - 2226123t  + 661905t  + 1169775t  + 226260t  - 209952t
--R       +
--R                  3
--R         - 141183t  + 27216t
--R       ,
--R
--R                 3      2      2      3           6     3            3      2
--R         (3z + 3t  - 3)y  + (3z  + (6t  - 6)z + 3t  - 6t  + 3)y + (3t  - 3)z
--R       +
--R             6     3          9     6     3
--R         (3t  - 6t  + 3)z + t  - 3t  + 5t  - 3t
--R       ,
--R                  3
--R      x + y + z + t  - 1}
--R     ,
--R              2                        2                     2
--R    {t,z - 1,y  - 1,x + y}, {t - 1,z,y  - 1,x + y}, {t - 1,z  - 1,z y + 1,x},
--R
--R       16     13     10     7      4      2
--R     {t   - 6t   + 9t   + 4t  + 15t  - 54t  + 27,
--R
--R                     29            28             27           26             25
--R            4907232t   + 40893984t   - 115013088t   - 1730448t   - 168139584t
--R           +
--R                       24             23             22              21
--R            738024480t   - 195372288t   + 315849456t   - 2567279232t
--R           +
--R                      20              19              18              17
--R            937147968t   + 1026357696t   + 4780488240t   - 2893767696t
--R           +
--R                        16              15              14              13
--R            - 5617160352t   - 3427651728t   + 5001100848t   + 8720098416t
--R           +
--R                       12             11              10             9
--R            2331732960t   - 499046544t   - 16243306272t   - 9748123200t
--R           +
--R                      8               7               6              5
--R            3927244320t  + 25257280896t  + 10348032096t  - 17128672128t
--R           +
--R                        4             3              2
--R            - 14755488768t  + 544086720t  + 10848188736t  + 1423614528t
--R           +
--R            - 2884297248
--R        *
--R           z
```

```
--R       +
--R              68         65          62        60           59         57
--R         - 48t   + 1152t   - 13560t   + 360t   + 103656t   - 7560t   - 572820t
--R       +
--R              54           53         52           51           50          49
--R         71316t   + 2414556t   + 2736t   - 402876t   - 7985131t   - 49248t
--R       +
--R               48            47          46           45            44
--R         1431133t   + 20977409t   + 521487t   - 2697635t   - 43763654t
--R       +
--R                  43           42            41            40            39
--R         - 3756573t   - 2093410t   + 71546495t   + 19699032t   + 35025028t
--R       +
--R                   38            37             36            35
--R         - 89623786t   - 77798760t   - 138654191t   + 87596128t   + 235642497t
--R       +
--R                  33            32             31             30
--R         349607642t   - 93299834t   - 551563167t   - 630995176t   + 186818962t
--R       +
--R                  28             27             26             25
--R         995427468t   + 828416204t   - 393919231t   - 1076617485t
--R       +
--R                     24             23              22              21
--R         - 1609479791t   + 595738126t   + 1198787136t   + 4342832069t
--R       +
--R                     20              19              18              17
--R         - 2075938757t   - 4390835799t   - 4822843033t   + 6932747678t
--R       +
--R                   16              15              14              13
--R         6172196808t   + 1141517740t   - 4981677585t   - 9819815280t
--R       +
--R                     12             11              10               9
--R         - 7404299976t   - 157295760t   + 29124027630t   + 14856038208t
--R       +
--R                      8              7              6              5
--R         - 16184101410t  - 26935440354t  - 3574164258t  + 10271338974t
--R       +
--R                    4              3              2
--R         11191425264t  + 6869861262t  - 9780477840t  - 3586674168t + 288429724
--R       ,
--R
--R            3      3      2      6      3           9     6    3
--R         (3z  + (6t  - 6)z  + (6t  - 12t  + 3)z + 2t  - 6t  + t  + 3t)y
--R       +
--R            3      3      6      3      2      9      6      3          12
--R         (3t  - 3)z  + (6t  - 12t  + 6)z  + (4t  - 12t  + 11t  - 3)z + t   - 4
```

```
--R       +
--R           6     3
--R         5t  - 2t
--R       ,
--R                    3
--R      x + y + z + t  - 1}
--R     ,
--R            2
--R    {t - 1,z  - 1,y,x + z},
--R
--R       8    7    6     5     4     3      2
--R     {t  + t  + t  - 2t  - 2t  - 2t  + 19t  + 19t - 8,
--R
--R                      7           6           5            4           3
--R             2395770t  + 3934440t  - 3902067t  - 10084164t  - 1010448t
--R           +
--R                       2
--R             32386932t  + 22413225t - 10432368
--R        *
--R           z
--R       +
--R                   7           6           5           4            3
--R         - 463519t  + 3586833t  + 9494955t  - 8539305t  - 33283098t
--R       +
--R                   2
--R         35479377t  + 46263256t - 17419896
--R       ,
--R
--R               4      3      3       6      3      2          3
--R             3z  + (9t  - 9)z  + (12t  - 24t  + 9)z  + (- 152t  + 219t - 67)z
--R           +
--R                  6      4      3
--R             - 41t  + 57t  + 25t  - 57t + 16
--R        *
--R           y
--R       +
--R            3      4      6      3      3          3              2
--R         (3t  - 3)z  + (9t  - 18t  + 9)z  + (- 181t  + 270t - 89)z
--R       +
--R               6       4      3                     7      6      4       3
--R         (- 92t  + 135t  + 49t  - 135t + 43)z + 27t  - 27t  - 54t  + 396t
--R       +
--R         - 486t + 144
--R       ,
--R                    3
--R      x + y + z + t  - 1}
```

```
--R     ,
--R            3
--R    {t,z - t  + 1,y - 1,x - 1}, {t - 1,z,y,x}, {t,z - 1,y,x}, {t,z,y - 1,x},
--R    {t,z,y,x - 1}]
--R                                    Type: List RegularChain(Integer,[x,y,z,t])
--E 20

--S 21 of 28
univariateSolve(lf)$pack
--R
--R
--R   (21)
--R   [[complexRoots= ?,coordinates= [x - 1,y - 1,z + 1,t - %A]],
--R    [complexRoots= ?,coordinates= [x,y - 1,z,t - %A]],
--R    [complexRoots= ? - 1,coordinates= [x,y,z,t - %A]],
--R    [complexRoots= ?,coordinates= [x - 1,y,z,t - %A]],
--R    [complexRoots= ?,coordinates= [x,y,z - 1,t - %A]],
--R    [complexRoots= ? - 2,coordinates= [x - 1,y + 1,z,t - 1]],
--R    [complexRoots= ?,coordinates= [x + 1,y - 1,z,t - 1]],
--R    [complexRoots= ? - 1,coordinates= [x - 1,y + 1,z - 1,t]],
--R    [complexRoots= ? + 1,coordinates= [x + 1,y - 1,z - 1,t]],
--R
--R                      6     3     2
--R     [complexRoots= ?  - 2?  + 3?  - 3,
--R                            3                 3
--R      coordinates= [2x + %A  + %A - 1,2y + %A  + %A - 1,z - %A,t - %A]]
--R     ,
--R
--R                      5     3     2
--R     [complexRoots= ?  + 3?  - 2?  + 3? - 3,
--R                                    3
--R      coordinates= [x - %A,y - %A,z + %A  + 2%A - 1,t - %A]]
--R     ,
--R
--R                      4    3     2
--R     [complexRoots= ?  - ?  - 2?  + 3,
--R                           3                3                3
--R      coordinates= [x + %A  - %A - 1,y + %A  - %A - 1,z - %A  + 2%A + 1,t - %A
--R     ,
--R    [complexRoots= ? + 1,coordinates= [x - 1,y - 1,z,t - %A]],
--R
--R                      6     3     2
--R     [complexRoots= ?  + 2?  + 3?  - 3,
--R                           3                        3
--R      coordinates= [2x - %A  - %A - 1,y + %A,2z - %A  - %A - 1,t + %A]]
--R     ,
```

```
--R
--R                    6      4      3      2
--R     [complexRoots= ?  + 12?  + 20?  - 45?  - 42? - 953,
--R
--R       coordinates =
--R                         5       4       3        2
--R         [12609x + 23%A  + 49%A  - 46%A  + 362%A  - 5015%A - 8239,
--R                         5       4       3        2
--R          25218y + 23%A  + 49%A  - 46%A  + 362%A  + 7594%A - 8239,
--R                         5       4       3        2
--R          25218z + 23%A  + 49%A  - 46%A  + 362%A  + 7594%A - 8239,
--R                         5       4       3        2
--R          12609t + 23%A  + 49%A  - 46%A  + 362%A  - 5015%A - 8239]
--R       ]
--R     ,
--R
--R                    5      3      2
--R     [complexRoots= ?  + 12?  - 16?  + 48? - 96,
--R                           3
--R      coordinates= [8x + %A  + 8%A - 8,2y - %A,2z - %A,2t - %A]]
--R     ,
--R
--R                    5    4     3     2
--R     [complexRoots= ?  + ?  - 5?  - 3?  + 9? + 3,
--R
--R       coordinates =
--R                 3                    3                    3
--R         [2x - %A  + 2%A - 1, 2y + %A  - 4%A + 1, 2z - %A  + 2%A - 1,
--R                 3
--R          2t - %A  + 2%A - 1]
--R       ]
--R     ,
--R
--R                    4     3     2
--R     [complexRoots= ?  - 3?  + 4?  - 6? + 13,
--R
--R       coordinates =
--R                   3      2                    3      2
--R         [9x - 2%A  + 4%A  - %A + 2, 9y + %A  - 2%A  + 5%A - 1,
--R                 3      2                    3      2
--R          9z + %A  - 2%A  + 5%A - 1, 9t + %A  - 2%A  - 4%A - 1]
--R       ]
--R     ,
--R
--R                    4      2
--R     [complexRoots= ?  - 11?  + 37,
```

```
--R
--R       coordinates =
--R                 2            2                   2            2
--R         [3x - %A  + 7,6y + %A  + 3%A - 7,3z - %A  + 7,6t + %A  - 3%A - 7]
--R       ]
--R     ,
--R    [complexRoots= ? + 1,coordinates= [x - 1,y,z - 1,t + 1]],
--R    [complexRoots= ? + 2,coordinates= [x,y - 1,z - 1,t + 1]],
--R    [complexRoots= ? - 2,coordinates= [x,y - 1,z + 1,t - 1]],
--R    [complexRoots= ?,coordinates= [x,y + 1,z - 1,t - 1]],
--R    [complexRoots= ? - 2,coordinates= [x - 1,y,z + 1,t - 1]],
--R    [complexRoots= ?,coordinates= [x + 1,y,z - 1,t - 1]],
--R
--R                     4     3      2
--R     [complexRoots= ?  + 5?  + 16?  + 30? + 57,
--R
--R       coordinates =
--R                     3       2                          3       2
--R         [151x + 15%A  + 54%A  + 104%A + 93, 151y - 10%A  - 36%A  - 19%A - 62,
--R                    3       2                         3       2
--R          151z - 5%A  - 18%A  - 85%A - 31, 151t - 5%A  - 18%A  - 85%A - 31]
--R       ]
--R     ,
--R
--R                     4    3     2
--R     [complexRoots= ?  - ?  - 2?  + 3,
--R                          3                  3                       3
--R      coordinates= [x - %A  + 2%A + 1,y + %A  - %A - 1,z - %A,t + %A  - %A - 1
--R     ,
--R
--R                     4     3     2
--R     [complexRoots= ?  + 2?  - 8?  + 48,
--R
--R       coordinates =
--R                 3                           3                  3
--R         [8x - %A  + 4%A - 8,2y + %A,8z + %A  - 8%A + 8,8t - %A  + 4%A - 8]
--R       ]
--R     ,
--R
--R                     5    4     3     2
--R     [complexRoots= ?  + ?  - 2?  - 4?  + 5? + 8,
--R                           3            3            3
--R      coordinates= [3x + %A  - 1,3y + %A  - 1,3z + %A  - 1,t - %A]]
--R     ,
--R                    3
--R    [complexRoots= ?  + 3? - 1,coordinates= [x - %A,y - %A,z - %A,t - %A]]]
```

```
--RType: List Record(complexRoots: SparseUnivariatePolynomial Integer,coordinates: List Poly
--E 21

--S 22 of 28
ts := lts.1
--R
--R
--R   (22)
--R     2           3        3
--R   {t  + t + 1, z  - z - t  + t,
--R
--R               3      2      2      3           6     3            3      2
--R       (3z + 3t  - 3)y  + (3z  + (6t  - 6)z + 3t  - 6t  + 3)y + (3t  - 3)z
--R     +
--R          6     3          9     6     3
--R       (3t  - 6t  + 3)z + t  - 3t  + 5t  - 3t
--R     ,
--R    x + y + z}
--R                                        Type: RegularChain(Integer,[x,y,z,t])
--E 22

univariateSolve(ts)$pack
--S 23 of 28
--R
--R
--R   (23)
--R   [
--R                      4     3      2
--R     [complexRoots= ?  + 5?  + 16?  + 30? + 57,
--R
--R       coordinates =
--R                      3       2                           3       2
--R         [151x + 15%A  + 54%A  + 104%A + 93, 151y - 10%A  - 36%A  - 19%A - 62,
--R                     3       2                           3       2
--R          151z - 5%A  - 18%A  - 85%A - 31, 151t - 5%A  - 18%A  - 85%A - 31]
--R       ]
--R     ,
--R
--R                      4    3     2
--R     [complexRoots= ?  - ?  - 2?  + 3,
--R                           3                 3                         3
--R      coordinates= [x - %A  + 2%A + 1,y + %A  - %A - 1,z - %A,t + %A  - %A - 1]]
--R     ,
--R
--R                      4     3     2
--R     [complexRoots= ?  + 2?  - 8?  + 48,
```

```
--R
--R       coordinates =
--R                 3                              3                   3
--R         [8x - %A  + 4%A - 8,2y + %A,8z + %A  - 8%A + 8,8t - %A  + 4%A - 8]
--R       ]
--R     ]
--RType: List Record(complexRoots: SparseUnivariatePolynomial Integer,coordinates
--E 23

--S 24 of 28
realSolve(ts)$pack
--R
--R
--R   (24)  []
--R                                    Type: List List RealClosure Fraction Integer
--E 24

--S 25 of 28
lr2 := realSolve(lf)$pack
--R
--R
--R   (25)
--R   [[0,- 1,1,1], [0,0,1,0], [1,0,0,0], [0,0,0,1], [0,1,0,0], [1,0,%B37,- %B37]
--R    [1,0,%B38,- %B38], [0,1,%B35,- %B35], [0,1,%B36,- %B36], [- 1,0,1,1],
--R
--R     [%B32,
--R
--R          1     15    2     14    1     13    4     12   11     11    4     10
--R         -- %B32   + -- %B32   + -- %B32   - -- %B32   - -- %B32   - -- %B32
--R         27          27          27          27          27          27
--R       +
--R          1     9   14     8    1     7   2     6   1     5   2     4       3
--R         -- %B32  + -- %B32  + -- %B32  + - %B32  + - %B32  + - %B32  + %B32
--R         27         27         27         9         3         9
--R       +
--R         4     2
--R         - %B32  - %B32 - 2
--R         3
--R       ,
--R
--R            1     15    1     14    1     13    2     12   11     11    2
--R         - -- %B32   - -- %B32   - -- %B32   + -- %B32   + -- %B32   + -- %B32
--R           54          27          54          27          54          27
--R       +
--R            1     9    7     8    1     7   1     6   1     5   1     4
--R         - -- %B32  - -- %B32  - -- %B32  - - %B32  - - %B32  - - %B32  - %B32
```

```
--R           54          27          54          9          6          9
--R       +
--R           2     2   1         3
--R         - - %B32  + - %B32 + -
--R           3         2         2
--R       ,
--R
--R            1     15    1     14    1     13    2     12   11     11    2     10
--R         - -- %B32   - -- %B32   - -- %B32   + -- %B32   + -- %B32   + -- %B32
--R           54          27          54          27          54          27
--R       +
--R            1     9    7     8    1     7   1     6   1     5   1     4       3
--R         - -- %B32  - -- %B32  - -- %B32  - - %B32  - - %B32  - - %B32  - %B32
--R           54          27          54          9          6          9
--R       +
--R           2     2   1         3
--R         - - %B32  + - %B32 + -
--R           3         2         2
--R       ]
--R     ,
--R
--R     [%B33,
--R
--R          1     15    2     14    1     13    4     12   11     11    4     10
--R         -- %B33   + -- %B33   + -- %B33   - -- %B33   - -- %B33   - -- %B33
--R         27          27          27          27          27          27
--R       +
--R          1     9   14     8    1     7   2     6   1     5   2     4       3
--R         -- %B33  + -- %B33  + -- %B33  + - %B33  + - %B33  + - %B33  + %B33
--R         27          27          27          9          3          9
--R       +
--R         4     2
--R         - %B33  - %B33 - 2
--R         3
--R       ,
--R
--R            1     15    1     14    1     13    2     12   11     11    2     10
--R         - -- %B33   - -- %B33   - -- %B33   + -- %B33   + -- %B33   + -- %B33
--R           54          27          54          27          54          27
--R       +
--R            1     9    7     8    1     7   1     6   1     5   1     4       3
--R         - -- %B33  - -- %B33  - -- %B33  - - %B33  - - %B33  - - %B33  - %B33
--R           54          27          54          9          6          9
--R       +
--R           2     2   1         3
--R         - - %B33  + - %B33 + -
```

```
--R           3         2        2
--R       ,
--R
--R            1     15    1     14    1     13    2     12   11     11    2
--R         - -- %B33   - -- %B33   - -- %B33   + -- %B33   + -- %B33   + -- %B33
--R           54          27          54          27          54          27
--R       +
--R            1     9    7     8    1     7   1     6   1     5   1     4       3
--R         - -- %B33  - -- %B33  - -- %B33  - - %B33  - - %B33  - - %B33  - %B33
--R           54         27         54         9         6         9
--R       +
--R           2     2   1        3
--R         - - %B33  + - %B33 + -
--R           3         2        2
--R       ]
--R     ,
--R
--R     [%B34,
--R
--R          1     15    2     14    1     13    4     12   11     11    4     10
--R         -- %B34   + -- %B34   + -- %B34   - -- %B34   - -- %B34   - -- %B34
--R         27          27          27          27          27          27
--R       +
--R          1     9   14     8    1     7   2     6   1     5   2     4       3
--R         -- %B34  + -- %B34  + -- %B34  + - %B34  + - %B34  + - %B34  + %B34
--R         27         27         27         9         3         9
--R       +
--R         4     2
--R         - %B34  - %B34 - 2
--R         3
--R       ,
--R
--R            1     15    1     14    1     13    2     12   11     11    2
--R         - -- %B34   - -- %B34   - -- %B34   + -- %B34   + -- %B34   + -- %B34
--R           54          27          54          27          54          27
--R       +
--R            1     9    7     8    1     7   1     6   1     5   1     4       3
--R         - -- %B34  - -- %B34  - -- %B34  - - %B34  - - %B34  - - %B34  - %B34
--R           54         27         54         9         6         9
--R       +
--R           2     2   1        3
--R         - - %B34  + - %B34 + -
--R           3         2        2
--R       ,
--R
--R            1     15    1     14    1     13    2     12   11     11    2
```

```
--R          - -- %B34   - -- %B34   - -- %B34   + -- %B34   + -- %B34   + -- %B34
--R            54          27          54          27          54          27
--R       +
--R             1     9    7     8    1     7   1     6   1     5   1     4       3
--R          - -- %B34  - -- %B34  - -- %B34  - - %B34  - - %B34  - - %B34  - %B34
--R            54         27         54         9         6         9
--R       +
--R            2     2   1        3
--R          - - %B34  + - %B34 + -
--R            3         2        2
--R       ]
--R     ,
--R    [- 1,1,0,1], [- 1,1,1,0],
--R
--R     [%B23,
--R
--R             1     15    1     14    1     13    2     12   11     11    2     10
--R          - -- %B23   - -- %B23   - -- %B23   + -- %B23   + -- %B23   + -- %B23
--R            54          27          54          27          54          27
--R       +
--R             1     9    7     8    1     7   1     6   1     5   1     4       3
--R          - -- %B23  - -- %B23  - -- %B23  - - %B23  - - %B23  - - %B23  - %B23
--R            54         27         54         9         6         9
--R       +
--R            2     2   1        3
--R          - - %B23  + - %B23 + -
--R            3         2        2
--R       ,
--R      %B30,
--R
--R                     1     15    1     14    1     13    2     12   11     11
--R          - %B30 + -- %B23   + -- %B23   + -- %B23   - -- %B23   - -- %B23
--R                    54          27          54          27          54
--R       +
--R             2     10    1     9    7     8    1     7   1     6   1     5
--R          - -- %B23   + -- %B23  + -- %B23  + -- %B23  + - %B23  + - %B23
--R            27          54         27         54         9         6
--R       +
--R         1     4   2     2   1        1
--R         - %B23  + - %B23  - - %B23 - -
--R         9         3         2        2
--R       ]
--R     ,
--R
--R     [%B23,
--R
```

```
--R           1     15    1     14    1     13    2     12   11     11    2
--R        - -- %B23   - -- %B23   - -- %B23   + -- %B23   + -- %B23   + -- %B23
--R          54          27          54          27          54          27
--R      +
--R           1     9    7     8    1     7   1     6   1     5   1     4
--R        - -- %B23  - -- %B23  - -- %B23  - - %B23  - - %B23  - - %B23  - %B23
--R          54         27         54         9         6         9
--R      +
--R          2     2   1        3
--R        - - %B23  + - %B23 + -
--R          3         2        2
--R       ,
--R      %B31,
--R
--R                  1     15    1     14    1     13    2     12   11     11
--R        - %B31 + -- %B23   + -- %B23   + -- %B23   - -- %B23   - -- %B23
--R                 54          27          54          27          54
--R      +
--R           2     10    1     9    7     8    1     7   1     6   1     5
--R        - -- %B23   + -- %B23  + -- %B23  + -- %B23  + - %B23  + - %B23
--R          27          54         27         54         9         6
--R      +
--R        1     4   2     2   1        1
--R        - %B23  + - %B23  - - %B23 - -
--R        9         3         2        2
--R       ]
--R     ,
--R
--R     [%B24,
--R
--R           1     15    1     14    1     13    2     12   11     11    2
--R        - -- %B24   - -- %B24   - -- %B24   + -- %B24   + -- %B24   + -- %B24
--R          54          27          54          27          54          27
--R      +
--R           1     9    7     8    1     7   1     6   1     5   1     4
--R        - -- %B24  - -- %B24  - -- %B24  - - %B24  - - %B24  - - %B24  - %B24
--R          54         27         54         9         6         9
--R      +
--R          2     2   1        3
--R        - - %B24  + - %B24 + -
--R          3         2        2
--R       ,
--R      %B28,
--R
--R                  1     15    1     14    1     13    2     12   11     11
--R        - %B28 + -- %B24   + -- %B24   + -- %B24   - -- %B24   - -- %B24
```

```
--R                  54          27          54          27          54
--R       +
--R            2     10    1     9    7     8    1     7   1     6   1     5
--R         - -- %B24   + -- %B24  + -- %B24  + -- %B24  + - %B24  + - %B24
--R           27          54          27          54          9          6
--R       +
--R         1     4   2     2   1        1
--R         - %B24  + - %B24  - - %B24 - -
--R         9         3         2        2
--R       ]
--R     ,
--R
--R     [%B24,
--R
--R            1     15    1     14    1     13    2     12   11     11    2     10
--R         - -- %B24   - -- %B24   - -- %B24   + -- %B24   + -- %B24   + -- %B24
--R           54          27          54          27          54          27
--R       +
--R            1     9    7     8    1     7   1     6   1     5   1     4       3
--R         - -- %B24  - -- %B24  - -- %B24  - - %B24  - - %B24  - - %B24  - %B24
--R           54         27          54         9          6          9
--R       +
--R           2     2   1        3
--R         - - %B24  + - %B24 + -
--R           3         2        2
--R       ,
--R      %B29,
--R
--R                   1     15    1     14    1     13    2     12   11     11
--R         - %B29 + -- %B24   + -- %B24   + -- %B24   - -- %B24   - -- %B24
--R                  54          27          54          27          54
--R       +
--R            2     10    1     9    7     8    1     7   1     6   1     5
--R         - -- %B24   + -- %B24  + -- %B24  + -- %B24  + - %B24  + - %B24
--R           27          54          27          54          9          6
--R       +
--R         1     4   2     2   1        1
--R         - %B24  + - %B24  - - %B24 - -
--R         9         3         2        2
--R       ]
--R     ,
--R
--R     [%B25,
--R
--R            1     15    1     14    1     13    2     12   11     11    2     10
--R         - -- %B25   - -- %B25   - -- %B25   + -- %B25   + -- %B25   + -- %B25
```

```
--R           54          27          54          27          54          27
--R       +
--R            1     9    7     8    1     7   1     6   1     5   1     4
--R         - -- %B25  - -- %B25  - -- %B25  - - %B25  - - %B25  - - %B25  - %B25
--R           54          27          54          9           6           9
--R       +
--R           2     2   1        3
--R         - - %B25  + - %B25 + -
--R           3         2        2
--R        ,
--R       %B26,
--R
--R                    1     15    1     14    1     13    2     12   11     11
--R         - %B26 + -- %B25   + -- %B25   + -- %B25   - -- %B25   - -- %B25
--R                  54          27          54          27          54
--R       +
--R            2     10    1     9    7     8    1     7   1     6   1     5
--R         - -- %B25   + -- %B25  + -- %B25  + -- %B25  + - %B25  + - %B25
--R           27          54          27          54          9           6
--R       +
--R         1     4   2     2   1        1
--R         - %B25  + - %B25  - - %B25 - -
--R         9         3         2        2
--R       ]
--R      ,
--R
--R      [%B25,
--R
--R            1     15    1     14    1     13    2     12   11     11    2
--R         - -- %B25   - -- %B25   - -- %B25   + -- %B25   + -- %B25   + -- %B25
--R           54          27          54          27          54          27
--R       +
--R            1     9    7     8    1     7   1     6   1     5   1     4
--R         - -- %B25  - -- %B25  - -- %B25  - - %B25  - - %B25  - - %B25  - %B25
--R           54          27          54          9           6           9
--R       +
--R           2     2   1        3
--R         - - %B25  + - %B25 + -
--R           3         2        2
--R        ,
--R       %B27,
--R
--R                    1     15    1     14    1     13    2     12   11     11
--R         - %B27 + -- %B25   + -- %B25   + -- %B25   - -- %B25   - -- %B25
--R                  54          27          54          27          54
--R       +
```

```
--R            2     10    1     9    7     8    1     7   1     6   1     5
--R         - -- %B25   + -- %B25  + -- %B25  + -- %B25  + - %B25  + - %B25
--R           27          54         27         54         9         6
--R       +
--R         1     4   2     2   1        1
--R         - %B25  + - %B25  - - %B25 - -
--R         9         3         2        2
--R       ]
--R     ,
--R    [1,%B21,- %B21,0], [1,%B22,- %B22,0], [1,%B19,0,- %B19], [1,%B20,0,- %B20],
--R            1     3   1   1     3   1   1     3   1
--R    [%B17,- - %B17  + -,- - %B17  + -,- - %B17  + -],
--R            3         3   3         3   3         3
--R            1     3   1   1     3   1   1     3   1
--R    [%B18,- - %B18  + -,- - %B18  + -,- - %B18  + -]]
--R            3         3   3         3   3         3
--R                                Type: List List RealClosure Fraction Integer
--E 25

--S 26 of 28
#lr2
--R
--R
--R   (26)  27
--R                                                      Type: PositiveInteger
--E 26

--S 27 of 28
lpr2 := positiveSolve(lf)$pack
--R
--R
--R                  1     3   1   1     3   1   1     3   1
--R   (27)  [[%B40,- - %B40  + -,- - %B40  + -,- - %B40  + -]]
--R                  3         3   3         3   3         3
--R                                Type: List List RealClosure Fraction Integer
--E 27

--S 28 of 28
[approximate(r,1/10**21)::Float for r in lpr2.1]
--R
--R
--R   (28)
--R   [0.3221853546 2608559291, 0.3221853546 2608559291, 0.3221853546 2608559291,
--R    0.3221853546 2608559291]
--R                                                           Type: List Float
--E 28
```

```
)spool
)lisp (bye)
```

⟨*ZeroDimensionalSolvePackage.help*⟩≡

```
==========================================================================
ZeroDimensionalSolvePackage examples
==========================================================================

The ZeroDimensionalSolvePackage package constructor provides
operations for computing symbolically the complex or real roots of
zero-dimensional algebraic systems.

The package provides no multiplicity information (i.e. some returned
roots may be double or higher) but only distinct roots are returned.

Complex roots are given by means of univariate representations of
irreducible regular chains.  These representations are computed by the
univariateSolve operation (by calling the
InternalRationalUnivariateRepresentationPackage package constructor
which does the job).

Real roots are given by means of tuples of coordinates lying in the
RealClosure of the coefficient ring.  They are computed by the
realSolve and positiveSolve operations.  The former computes all the
solutions of the input system with real coordinates whereas the later
concentrate on the solutions with (strictly) positive coordinates.  In
both cases, the computations are performed by the RealClosure constructor.

Both computations of complex roots and real roots rely on triangular
decompositions.  These decompositions can be computed in two different
ways.  First, by a applying the zeroSetSplit operation from the REGSET
domain constructor.  In that case, no Groebner bases are computed.
This strategy is used by default.  Secondly, by applying the zeroSetSplit
from LEXTRIPK.  To use this later strategy with the operations
univariateSolve, realSolve and positiveSolve one just needs to use an
extra boolean argument.

Note that the way of understanding triangular decompositions is
detailed in the example of the RegularTriangularSet constructor.

The ZeroDimensionalSolvePackage constructor takes three arguments.The
first one R is the coefficient ring; it must belong to the categories
OrderedRing, EuclideanDomain, CharacteristicZero and RealConstant. This
means essentially that R is Integer or Fraction(Integer).  The second
argument ls is the list of variables involved in the systems to solve.
The third one MUST BE concat(ls,s) where s is an additional symbol used
for the univariate representations. The abbreviation for
ZeroDimensionalSolvePackage is ZDSOLVE.
```

We illustrate now how to use the constructor ZDSOLVE by two examples: the Arnborg and Lazard system and the L-3 system (Aubry and Moreno Maza). Note that the use of this package is also demonstrated in the example of the LexTriangularPackage constructor.

Define the coefficient ring.

```
R := Integer
  Integer
                        Type: Domain
```

Define the lists of variables:

```
ls : List Symbol := [x,y,z,t]
  [x,y,z,t]
                        Type: List Symbol
```

and:

```
ls2 : List Symbol := [x,y,z,t,new()$Symbol]
  [x,y,z,t,%A]
                        Type: List Symbol
```

Call the package:

```
pack := ZDSOLVE(R,ls,ls2)
  ZeroDimensionalSolvePackage(Integer,[x,y,z,t],[x,y,z,t,%A])
                        Type: Domain
```

Define a polynomial system (Arnborg-Lazard)

```
p1 := x**2*y*z + x*y**2*z + x*y*z**2 + x*y*z + x*y + x*z + y*z
       2       2     2
  x y z  + (x y  + (x  + x + 1)y + x)z + x y
                        Type: Polynomial Integer

p2 := x**2*y**2*z + x*y**2*z**2 + x**2*y*z + x*y*z + y*z + x + z
     2 2     2 2     2
  x y z  + (x y  + (x  + x + 1)y + 1)z + x
                        Type: Polynomial Integer

p3 := x**2*y**2*z**2 + x**2*y**2*z + x*y**2*z + x*y*z + x*z + z + 1
   2 2 2      2      2
  x y z  + ((x  + x)y  + x y + x + 1)z + 1
                        Type: Polynomial Integer
```

```
lp := [p1, p2, p3]
       2       2     2
 [x y z  + (x y  + (x  + x + 1)y + x)z + x y,
     2 2     2 2     2
  x y z  + (x y  + (x  + x + 1)y + 1)z + x,
   2 2 2       2      2
  x y z  + ((x  + x)y  + x y + x + 1)z + 1]
                      Type: List Polynomial Integer
```

Note that these polynomials do not involve the variable t; we will use it in the second example.

First compute a decomposition into regular chains (i.e. regular triangular sets).

```
triangSolve(lp)$pack
 [
  {
       20     19      18      17       16      15       14       13       12
     z   - 6z   - 41z   + 71z   + 106z   + 92z   + 197z   + 145z   + 257z
   +
         11       10       9       8       7       6      5       4      3
     278z   + 201z   + 278z  + 257z  + 145z  + 197z  + 92z  + 106z  + 71z
   +
          2
     - 41z  - 6z + 1
   ,
                  19            18             17             16
       14745844z   + 50357474z   - 130948857z   - 185261586z
     +
                   15             14             13             12
       - 180077775z   - 338007307z   - 275379623z   - 453190404z
     +
                   11             10             9              8
       - 474597456z   - 366147695z   - 481433567z  - 430613166z
     +
                   7             6             5             4
       - 261878358z  - 326073537z  - 163008796z  - 177213227z
     +
                   3            2
       - 104356755z  + 65241699z  + 9237732z - 1567348
    *
       y
   +
            19           18            17            16            15
     1917314z   + 6508991z   - 16973165z   - 24000259z   - 23349192z
```

```
      +
                  14            13            12            11            10
        - 43786426z   - 35696474z   - 58724172z   - 61480792z   - 47452440z
      +
                  9            8            7            6            5
        - 62378085z  - 55776527z  - 33940618z  - 42233406z  - 21122875z
      +
                  4            3           2
        - 22958177z  - 13504569z  + 8448317z  + 1195888z - 202934
      ,
        3       2       3    2              2              2
       ((z  - 2z)y  + (- z  - z  - 2z - 1)y - z  - z + 1)x + z  - 1}
     ]
                      Type: List RegularChain(Integer,[x,y,z,t])
```

We can see easily from this decomposition (consisting of a single regular chain) that the input system has 20 complex roots.

Then we compute a univariate representation of this regular chain.

```
  univariateSolve(lp)$pack
   [
    [
      complexRoots =
           12      11      10     9     8      7      6      5     4     3
          ?   - 12?   + 24?   + 4?  - 9?  + 27?  - 21?  + 27?  - 9?  + 4?
        +
             2
          24?  - 12? + 1
      ,
       coordinates =
        [
                      11         10          9         8         7          6
          63x + 62%A   - 721%A   + 1220%A  + 705%A  - 285%A  + 1512%A
        +
                  5          4        3         2
          - 735%A  + 1401%A  - 21%A  + 215%A  + 1577%A - 142
        ,
                     11         10          9         8         7          6
          63y - 75%A   + 890%A   - 1682%A  - 516%A  + 588%A  - 1953%A
        +
                 5          4         3         2
          1323%A  - 1815%A  + 426%A  - 243%A  - 1801%A + 679
        ,
         z - %A]
      ]
```

```
   ,
                    6    5    4    3    2
   [complexRoots= ?  + ?  + ?  + ?  + ?  + ? + 1,
                         5       3
    coordinates= [x - %A ,y - %A ,z - %A]]
   ,
                  2
  [complexRoots= ?  + 5? + 1,coordinates= [x - 1,y - 1,z - %A]]]
          Type: List Record(complexRoots: SparseUnivariatePolynomial Integer,
                            coordinates: List Polynomial Integer)
```

We see that the zeros of our regular chain are split into three components. This is due to the use of univariate polynomial factorization.

Each of these components consist of two parts. The first one is an irreducible univariate polynomial p(?) which defines a simple algebraic extension of the field of fractions of R. The second one consists of multivariate polynomials pol1(x,%A), pol2(y,%A) and pol3(z,%A). Each of these polynomials involve two variables: one is an indeterminate x, y or z of the input system lp and the other is %A which represents any root of p(?). Recall that this %A is the last element of the third parameter of ZDSOLVE. Thus any complex root ? of p(?) leads to a solution of the input system lp by replacing %A by this ? in pol1(x,%A), pol2(y,%A) and pol3(z,%A). Note that the polynomials pol1(x,%A), pol2(y,%A) and pol3(z,%A) have degree one w.r.t. x, y or z respectively. This is always the case for all univariate representations. Hence the operation univariateSolve replaces a system of multivariate polynomials by a list of univariate polynomials, what justifies its name. Another example of univariate representations illustrates the LexTriangularPackage package constructor.

We now compute the solutions with real coordinates:

```
  lr := realSolve(lp)$pack
   [
    [%B1,
        1184459    19   2335702    18   5460230    17   79900378    16
        ------- %B1   - ------- %B1   - ------- %B1   + -------- %B1
        1645371          548457          182819          1645371
      +
        43953929    15   13420192    14   553986    13   193381378    12
        -------- %B1   + -------- %B1   + ------ %B1   + --------- %B1
         548457           182819           3731           1645371
      +
        35978916    11   358660781    10   271667666    9   118784873    8
        -------- %B1   + --------- %B1   + --------- %B1  + --------- %B1
```

```
   182819           1645371           1645371           548457
 +
   337505020    7   1389370    6   688291    5   3378002    4
   --------- %B1  + ------- %B1  + ------ %B1  + ------- %B1
   1645371          11193          4459          42189
 +
   140671876    3   32325724    2   8270       9741532
   --------- %B1  + -------- %B1  - ---- %B1 - -------
   1645371          548457          343       1645371
 ,
     91729    19   487915    18   4114333    17   1276987    16
   - ------ %B1   + ------ %B1   + ------- %B1   - ------- %B1
     705159         705159          705159          235053
 +
     13243117    15   16292173    14   26536060    13   722714    12
   - -------- %B1   - -------- %B1   - -------- %B1   - ------ %B1
     705159           705159           705159           18081
 +
     5382578    11   15449995    10   14279770    9   6603890    8
   - ------- %B1   - -------- %B1   - -------- %B1  - ------- %B1
     100737          235053           235053          100737
 +
     409930    7   37340389    6   34893715    5   26686318    4
   - ------ %B1  - -------- %B1  - -------- %B1  - -------- %B1
     6027          705159          705159          705159
 +
     801511    3   17206178    2   4406102       377534
   - ------ %B1  - -------- %B1  - ------- %B1 + ------
     26117         705159          705159       705159
 ]
,
 [%B2,
   1184459    19   2335702    18   5460230    17   79900378    16
   ------- %B2   - ------- %B2   - ------- %B2   + -------- %B2
   1645371          548457          182819          1645371
 +
   43953929    15   13420192    14   553986    13   193381378    12
   -------- %B2   + -------- %B2   + ------ %B2   + ---------- %B2
   548457           182819           3731           1645371
 +
   35978916    11   358660781    10   271667666    9   118784873    8
   -------- %B2   + ---------- %B2   + ---------- %B2  + ---------- %B2
   182819           1645371           1645371           548457
 +
   337505020    7   1389370    6   688291    5   3378002    4
   --------- %B2  + ------- %B2  + ------ %B2  + ------- %B2
```

```
   1645371          11193          4459          42189
 +
   140671876    3   32325724    2   8270       9741532
   --------- %B2  + -------- %B2  - ---- %B2 - -------
    1645371          548457          343       1645371
 ,
     91729    19   487915    18   4114333    17   1276987    16
   - ------ %B2   + ------ %B2   + ------- %B2   - ------- %B2
     705159         705159          705159          235053
 +
     13243117    15   16292173    14   26536060    13   722714    12
   - -------- %B2   - -------- %B2   - -------- %B2   - ------ %B2
      705159           705159          705159           18081
 +
     5382578    11   15449995    10   14279770    9   6603890    8
   - ------- %B2   - -------- %B2   - -------- %B2  - ------- %B2
      100737          235053          235053          100737
 +
     409930    7   37340389    6   34893715    5   26686318    4
   - ------ %B2  - -------- %B2  - -------- %B2  - -------- %B2
      6027          705159          705159          705159
 +
     801511    3   17206178    2   4406102       377534
   - ------ %B2  - -------- %B2  - ------- %B2 + ------
      26117         705159          705159       705159
 ]
,
 [%B3,
   1184459    19   2335702    18   5460230    17   79900378    16
   ------- %B3   - ------- %B3   - ------- %B3   + -------- %B3
   1645371          548457          182819          1645371
 +
   43953929    15   13420192    14   553986    13   193381378    12
   -------- %B3   + -------- %B3   + ------ %B3   + --------- %B3
    548457           182819          3731           1645371
 +
   35978916    11   358660781    10   271667666    9   118784873    8
   -------- %B3   + --------- %B3   + --------- %B3  + --------- %B3
    182819           1645371          1645371          548457
 +
   337505020    7   1389370    6   688291    5   3378002    4
   --------- %B3  + ------- %B3  + ------ %B3  + ------- %B3
    1645371          11193          4459          42189
 +
   140671876    3   32325724    2   8270       9741532
   --------- %B3  + -------- %B3  - ---- %B3 - -------
```

```
    1645371           548457           343        1645371

  ,
      91729     19   487915     18   4114333     17   1276987     16
   - ------ %B3    + ------ %B3    + ------- %B3    - ------- %B3
      705159          705159           705159           235053

  +
      13243117     15   16292173     14   26536060     13   722714     12
   - -------- %B3    - -------- %B3    - -------- %B3    - ------ %B3
      705159            705159            705159            18081

  +
      5382578     11   15449995     10   14279770     9   6603890     8
   - ------- %B3    - -------- %B3    - -------- %B3  - ------- %B3
      100737           235053            235053           100737

  +
      409930     7   37340389     6   34893715     5   26686318     4
   - ------ %B3   - -------- %B3   - -------- %B3   - -------- %B3
      6027           705159           705159           705159

  +
      801511     3   17206178     2   4406102        377534
   - ------ %B3   - -------- %B3   - ------- %B3 + ------
      26117          705159           705159        705159

  ]

 ,
  [%B4,
    1184459     19   2335702     18   5460230     17   79900378     16
    ------- %B4    - ------- %B4    - ------- %B4    + -------- %B4
    1645371           548457           182819           1645371

  +
    43953929     15   13420192     14   553986     13   193381378     12
    -------- %B4    + -------- %B4    + ------ %B4    + --------- %B4
     548457            182819            3731            1645371

  +
    35978916     11   358660781     10   271667666     9   118784873     8
    -------- %B4    + --------- %B4    + --------- %B4  + --------- %B4
     182819            1645371           1645371            548457

  +
    337505020     7   1389370     6   688291     5   3378002     4
    --------- %B4  + ------- %B4  + ------ %B4  + ------- %B4
     1645371           11193           4459           42189

  +
    140671876     3   32325724     2   8270        9741532
    --------- %B4  + -------- %B4  - ---- %B4 - -------
     1645371           548457           343        1645371

  ,
      91729     19   487915     18   4114333     17   1276987     16
   - ------ %B4    + ------ %B4    + ------- %B4    - ------- %B4
```

```
      705159          705159          705159          235053
  +
      13243117    15   16292173    14   26536060    13   722714    12
    - -------- %B4   - -------- %B4   - -------- %B4   - ------ %B4
       705159           705159           705159           18081
  +
      5382578    11   15449995    10   14279770    9   6603890    8
    - ------- %B4   - -------- %B4   - -------- %B4  - ------- %B4
       100737           235053           235053          100737
  +
      409930    7   37340389    6   34893715    5   26686318    4
    - ------ %B4  - -------- %B4  - -------- %B4  - -------- %B4
       6027          705159          705159          705159
  +
      801511    3   17206178    2   4406102       377534
    - ------ %B4  - -------- %B4  - ------- %B4 + ------
       26117         705159          705159       705159
  ]
,
  [%B5,
     1184459    19   2335702    18   5460230    17   79900378    16
     ------- %B5   - ------- %B5   - ------- %B5   + -------- %B5
     1645371          548457          182819          1645371
  +
     43953929    15   13420192    14   553986    13   193381378    12
     -------- %B5   + -------- %B5   + ------ %B5   + --------- %B5
      548457           182819           3731           1645371
  +
     35978916    11   358660781    10   271667666    9   118784873    8
     -------- %B5   + --------- %B5   + --------- %B5  + --------- %B5
      182819           1645371           1645371           548457
  +
     337505020    7   1389370    6   688291    5   3378002    4
     --------- %B5  + ------- %B5  + ------ %B5  + ------- %B5
      1645371          11193          4459          42189
  +
     140671876    3   32325724    2   8270       9741532
     --------- %B5  + -------- %B5  - ---- %B5 - -------
      1645371          548457          343       1645371
  ,
       91729    19   487915    18   4114333    17   1276987    16
     - ------ %B5   + ------ %B5   + ------- %B5   - ------- %B5
       705159         705159          705159          235053
  +
       13243117    15   16292173    14   26536060    13   722714    12
     - -------- %B5   - -------- %B5   - -------- %B5   - ------ %B5
```

```
      705159           705159           705159           18081
  +
     5382578    11   15449995    10   14279770    9   6603890    8
   - ------- %B5   - -------- %B5   - -------- %B5  - ------- %B5
      100737           235053           235053          100737
  +
     409930    7   37340389    6   34893715    5   26686318    4
   - ------ %B5  - -------- %B5  - -------- %B5  - -------- %B5
      6027          705159          705159          705159
  +
     801511    3   17206178    2   4406102       377534
   - ------ %B5  - -------- %B5  - ------- %B5 + ------
      26117         705159          705159       705159
  ]
,
  [%B6,
    1184459    19   2335702    18   5460230    17   79900378    16
    ------- %B6   - ------- %B6   - ------- %B6   + -------- %B6
    1645371          548457          182819          1645371
  +
    43953929    15   13420192    14   553986    13   193381378    12
    -------- %B6   + -------- %B6   + ------ %B6   + --------- %B6
     548457           182819           3731           1645371
  +
    35978916    11   358660781    10   271667666    9   118784873    8
    -------- %B6   + --------- %B6   + --------- %B6  + --------- %B6
     182819           1645371           1645371           548457
  +
    337505020    7   1389370    6   688291    5   3378002    4
    --------- %B6  + ------- %B6  + ------ %B6  + ------- %B6
     1645371          11193          4459          42189
  +
    140671876    3   32325724    2   8270       9741532
    --------- %B6  + -------- %B6  - ---- %B6 - -------
     1645371          548457          343       1645371
  ,
     91729    19   487915    18   4114333    17   1276987    16
    - ------ %B6   + ------ %B6   + ------- %B6   - ------- %B6
     705159         705159          705159          235053
  +
     13243117    15   16292173    14   26536060    13   722714    12
    - -------- %B6   - -------- %B6   - -------- %B6   - ------ %B6
      705159           705159           705159           18081
  +
     5382578    11   15449995    10   14279770    9   6603890    8
    - ------- %B6   - -------- %B6   - -------- %B6  - ------- %B6
```

```
      100737           235053           235053           100737
  +
     409930    7   37340389    6   34893715    5   26686318    4
   - ------ %B6  - -------- %B6  - -------- %B6  - -------- %B6
      6027           705159           705159           705159
  +
     801511    3   17206178    2   4406102       377534
   - ------ %B6  - -------- %B6  - ------- %B6 + ------
      26117          705159           705159       705159
  ]
 ,
  [%B7,
     1184459    19   2335702    18   5460230    17   79900378    16
     ------- %B7   - ------- %B7   - ------- %B7   + -------- %B7
     1645371          548457           182819           1645371
  +
     43953929    15   13420192    14   553986    13   193381378    12
     -------- %B7   + -------- %B7   + ------ %B7   + --------- %B7
      548457           182819           3731           1645371
  +
     35978916    11   358660781    10   271667666    9   118784873    8
     -------- %B7   + --------- %B7   + --------- %B7  + --------- %B7
      182819           1645371           1645371           548457
  +
     337505020    7   1389370    6   688291    5   3378002    4
     --------- %B7  + ------- %B7  + ------ %B7  + ------- %B7
      1645371          11193          4459          42189
  +
     140671876    3   32325724    2   8270       9741532
     --------- %B7  + -------- %B7  - ---- %B7 - -------
      1645371          548457          343       1645371
  ,
       91729    19   487915    18   4114333    17   1276987    16
     - ------ %B7   + ------ %B7   + ------- %B7   - ------- %B7
       705159         705159          705159           235053
  +
     13243117    15   16292173    14   26536060    13   722714    12
   - -------- %B7   - -------- %B7   - -------- %B7   - ------ %B7
      705159           705159           705159           18081
  +
     5382578    11   15449995    10   14279770    9   6603890    8
   - ------- %B7   - -------- %B7   - -------- %B7  - ------- %B7
      100737           235053           235053           100737
  +
     409930    7   37340389    6   34893715    5   26686318    4
   - ------ %B7  - -------- %B7  - -------- %B7  - -------- %B7
```

```
     6027          705159          705159          705159
 +
    801511    3   17206178    2   4406102       377534
  - ------ %B7  - -------- %B7  - ------- %B7 + ------
     26117          705159          705159       705159
 ]
,
 [%B8,
   1184459    19   2335702    18   5460230    17   79900378    16
   ------- %B8   - ------- %B8   - ------- %B8   + -------- %B8
   1645371          548457          182819          1645371
 +
   43953929    15   13420192    14   553986    13   193381378    12
   -------- %B8   + -------- %B8   + ------ %B8   + --------- %B8
    548457           182819           3731           1645371
 +
   35978916    11   358660781    10   271667666    9   118784873    8
   -------- %B8   + ---------- %B8   + ---------- %B8  + ---------- %B8
    182819           1645371           1645371           548457
 +
   337505020    7   1389370    6   688291    5   3378002    4
   --------- %B8  + ------- %B8  + ------ %B8  + ------- %B8
    1645371          11193          4459          42189
 +
   140671876    3   32325724    2   8270       9741532
   --------- %B8  + -------- %B8  - ---- %B8 - -------
    1645371          548457          343       1645371
 ,
     91729    19   487915    18   4114333    17   1276987    16
   - ------ %B8   + ------ %B8   + ------- %B8   - ------- %B8
     705159         705159          705159          235053
 +
    13243117    15   16292173    14   26536060    13   722714    12
   - -------- %B8   - -------- %B8   - -------- %B8   - ------ %B8
     705159           705159           705159           18081
 +
    5382578    11   15449995    10   14279770    9   6603890    8
   - ------- %B8   - -------- %B8   - -------- %B8  - ------- %B8
     100737           235053           235053          100737
 +
    409930    7   37340389    6   34893715    5   26686318    4
   - ------ %B8  - -------- %B8  - -------- %B8  - -------- %B8
     6027           705159          705159          705159
 +
    801511    3   17206178    2   4406102       377534
   - ------ %B8  - -------- %B8  - ------- %B8 + ------
```

```
            26117          705159           705159       705159
       ]
     ]
                    Type: List List RealClosure Fraction Integer

  The number of real solutions for the input system is:

    # lr
      8
                    Type: PositiveInteger

  Each of these real solutions is given by a list of elements in
  RealClosure(R).  In these 8 lists, the first element is a value of z,
  the second of y and the last of x.  This is logical since by setting
  the list of variables of the package to [x,y,z,t] we mean that the
  elimination ordering on the variables is t < z < y < x.  Note that
  each system treated by the ZDSOLVE package constructor needs only to
  be zero-dimensional w.r.t. the variables involved in the system it-self
  and not necessarily w.r.t. all the variables used to define the package.

  We can approximate these real numbers as follows.  This computation
  takes between 30 sec. and 5 min, depending on your machine.

    [ [approximate(r,1/1000000) for r in point] for point in lr]
     [
         10048059
       [- --------,
          2097152

         450305731698538794352439791383896641459673197621176821933588120838551 6_
          314058924567176091423629695777403099833360761048898228916578137094309_
          838597331137202584846939132376157019506760357601165917454986815382098_
          789094851523420392811293126141329856546977145464661495487825919941188_
          447041722440491921567263542158028061437758844364634410045253024786561_
          923163288214175
        /
         450305728302524548851651180698582663508310069375732046528055470686564 4_
          949577509916867201889438090408354817931718593862797624551518983570793_
          048774424291488708829840324189200301436123314860200821443733790755311_
          243632919864895421704228949571290016119498807957023663865443069392027_
          148979688266712323356043491523434068924275280417338574817381189277066_
          143312396681216
        ,

         210626076882347507389479868048601659624960714869068553876368371502063 9_
          680858649650790055889505646893309447097099937802187329095325898785247_
```

```
    24902071750498366048207515661873872451468533306001120296463516638135 1_
    54325598220025030528398108683711061484230702609121129792987689628568 1_
    83047905476005638076266490561846205530604781619178201158870378913898 8_
    1895
  /
    21062606094984641924721138048164741753419629532964341024139031423687 57_
    96768527388858559097596521177886218987288195394364024629735706195981 2_
    32610365979902512686325867656720234210687703171018424748418142328892 1_
    83768123706270847029570621848592886740077193782849920092376059331416 8_
    90100066637389634759811822855673103707202647449677622838376299392328 0_
    0768
  ]
,

    2563013
[- -------,
    2097152

  -
      26113461767919277896986176932377577192382599630635417819227523304 40_
       18989966807292833849076862359320744212592598673381593224350480929 4_
       83752303023733723680666816744617300172727135331157124289 7
    /
      11652254005052225305839819160045891437572266102768589900087901348 19_
       91494092241375398397139401952343332040813992815318882949575545516 3_
       96341761930839597754479714023146923426903492193805559398 4
  ,

    35725945502759172210965887296157882729985170546756032395781981410060 34_
    09173528282659062190230446696394197103892330452627332931637375745006 1_
    97898922861109769970872504662353 73
  /
    10395482693455989368770712448340260558008145511201705922005223665917 59_
    40965948644233914102945295026517998996010481187582253020534650513158 1_
    24390172472891738650147029663088 64
  ]
,

    1715967
[- -------,
    2097152

  -
      42130935337843035210848395179770823903772615039695862248289984366 06_
       03065607635937456481377349837660312126782256580143620693951995146 5_
       18222580524697287410022543952491
```

```
      /
        9441814144185374458649692034349224052436597470966253663930641960795_
         8058825854931998401916999176594432648246411351873835838881478673 40_
         1930785760582036419585682230 4768
    ,

     7635833347112644222515625424410831225347475669008589338834162172501904_
      9943763467308768090428452089199199253021057209714539189827313890725 91_
      4035
   /
     2624188764086097199784297610478066633934230467895851602278580978503784_
      5492057884990196406022669660268915801035435676250390186298871412849 16_
      75648
    ]
  ,

      437701
  [- -------,
     2097152

     1683106908638349588322172332654225913562986313181951031452750161441497_
      4734553281507213648683555796467816035077771990750778352133664845336 54_
      91383623741304759
   /
     1683106868095213389001709982705913638963077668731226111167785188004907_
      4252262986803258878109626141402985973669842648879989083770687999984 54_
      23381649008099328
    ,

     4961550109835010186422681013422108735958714801003760639707968096646912_
      8267084728344431172391721910424921345096631241113 3
   /
     4961549872757738315509192078210209029852897118611097126236384040829376_
      5926191431317025486746479271836349216048244221542 4
    ]
  ,

    222801
  [-------,
   2097152

    -
        8994884880402428265107595121970691427136045692541978275573001865213_
         7599215881377166961263491016552201951429949322991371832417058676 72_
         383477
      /
```

```
        11678899986650263721777651006918885827089699602299347696908357524 57_
         07777941643520947376786650776940588894276458771854243425562599245 6_
         372224
      ,
      -
        23897048881331568783208015443738083956127715092084910198474529918 85_
         50954651952546783901661359399969388664003628357055232115503787129 1_
         458703265
       /
        53554872736450963260904032866899319059882254446854114332215938336 81_
         19295756283367146865429034074699365628592559911760212044618344314 5_
         479421952
      ]
    ,

      765693
    [-------,
     2097152

       85589692198167162678732447611781980887246989586166701402137657543220 02_
        30325168578611867833084020332883765433952341870491774951834077251289 9_
        000391009630373148561
     /
       29414424455330107909764284113763934998155802159458569179064525354957 23_
        01385681894170233022877989014129623672113815423199723891732215671196 5_
        244463933171946015948 8
      ,
      -
        20576182305825721012476503248602425611113025815435888088439236627 67_
         54938224165936271229077761280019292142057440894808519374368858276 2_
         2246433251878894899015
       /
        26715982033257355380979523535014502205763137598908350970917225206 42_
         71019877190266718394890628986371475967836029248394920461647153777 7_
         775324180661095366656
      ]
    ,

     5743879
    [-------,
     2097152

       10762888169689068479555463947735702081714567249426186140236631235747 68_
        96085043426397139807254659277266215883344979769861745539788756290007 2_
```

```
      984768000608343553189801693408727205047612559889232757563830528688953_
      535421809482771058917542602890060941949620874083007858366669453501766_
      24841488732463225
    /
     3131768957080317946648461940023552044190376613458584986228549631916196_
      601616219781765615532532294746529648276430583810894079374566460757823_
      146888581195556029208515218838883200318658407469399426063260589828612_
      309231596669129707986481319851571942927230340622934023923486703042068_
      1530440845099008
    ,
    -
        211328669918575091836412047556545843787017248986548599438982813533 5_
         264444665284557526492734931691731407872701432935503473348172076098_
         720545849008780077564160534317894688366119529739980502944162668550_
         098127961950496210221942878089359674925850594427768502251789758706_
         752831632503615
       /
        162761558493798758024290662434710458088914446616845971804315383940 8_
         372525533309808070363699585502216011211087103263609551026027769414_
         087391148126221168139781682587438075322591466131939975457200522349_
         838568964285634448018562038272378787354460106106141518010935617205_
         1706396253618176
    ]
  ,
   19739877
  [--------,
    2097152
    -
        299724993683270330379901580486152094921504038750070717770128576672 0_
         192530579422478953566024359860143101547801638082771611160372212874_
         847778035809872843149225484238365858013629341705321702582333350918_
         009601789937023985935304900460493398738370308534103470899088808148_
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         814476854871031530931295467332190133702671098200902282300510751860_
         718592845703027780739779652581386276223928699610680972802367 5
       /
        230843327485227859072891008119181102390650414132143264612393679487 3_
         933319270608960702138193417647898360620229519176632937631786851455_
         014766027206259022252505551741823688896883806636602574431760472240_
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         897080767552864895056585864033178565910387065006112801516403522741_
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         529299201108585608125566354854294715540316759795426563813539 84
```

```
       ,

       -
           5128189263548228489096276397868940080600938410663080459407966335845_
            0092641094905204598253162500847230100470350244974365230389258189 59_
            2893129315847013539276214354343986742630472939091228501338519906 96_
            4902315660943719943337950707826240117275877499892966112773183722 94_
            6242071165379104365545741460828847013055439126204193548854107359 40_
            1577758966028223645758646118315129439739747151669204650618506037 62_
            8751625619584705241258728283913919464291395 5
         /
           2288281939778439330531208793181290471183631092455368990386390824243_
            5094636442362497730806474389877391449216077946826538517411890917 11_
            7418681451149783372841918224976758683587294866447308566225526872 09_
            2037244118004814057028371983106422912756761957746144438159967135 02_
            6293917497835900414708601277523729964886277426724876224800632688 08_
            8893248918508424949343473376030759399802682084829048596781777514 44_
            6574997982787261696305321767320171723725209 6
       ]
     ]
                        Type: List List Fraction Integer
```

We can also concentrate on the solutions with real (strictly) positive coordinates:

```
  lpr := positiveSolve(lp)$pack
    []
                        Type: List List RealClosure Fraction Integer
```

Thus we have checked that the input system has no solution with strictly positive coordinates.

Let us define another polynomial system (L-3).

```
  f0 := x**3 + y + z + t- 1
              3
     z + y + x  + t - 1
                      Type: Polynomial Integer

  f1 := x + y**3 + z + t -1
          3
     z + y  + x + t - 1
                      Type: Polynomial Integer

  f2 := x + y + z**3 + t-1
      3
```

```
      z  + y + x + t - 1
                        Type: Polynomial Integer

   f3 := x + y + z + t**3 -1
                    3
      z + y + x + t  - 1
                        Type: Polynomial Integer

   lf := [f0, f1, f2, f3]
             3              3              3                                3
   [z + y + x  + t - 1,z + y  + x + t - 1,z  + y + x + t - 1,z + y + x + t  - 1]
                        Type: List Polynomial Integer
```

First compute a decomposition into regular chains (i.e. regular triangular sets).

```
   lts := triangSolve(lf)$pack
   [
       2           3        3
     {t  + t + 1, z  - z - t  + t,
                  3      2      2      3           6     3            3      2
         (3z + 3t  - 3)y  + (3z  + (6t  - 6)z + 3t  - 6t  + 3)y + (3t  - 3)z
       +
            6     3          9     6     3
         (3t  - 6t  + 3)z + t  - 3t  + 5t  - 3t
       ,
        x + y + z}
     ,
       16     13     10     7      4      2
     {t   - 6t   + 9t   + 4t  + 15t  - 54t  + 27,
                     15             14             13            12            11
            4907232t   + 40893984t   - 115013088t   + 22805712t   + 36330336t
          +
                     10             9             8             7
            162959040t   - 159859440t  - 156802608t  + 117168768t
          +
                     6             5             4             3
            126282384t  - 129351600t  + 306646992t  + 475302816t
          +
                        2
            - 1006837776t  - 237269088t + 480716208
       *
          z
       +
            54       51        48      46         45        43          42
         48t   - 912t   + 8232t   - 72t   - 46848t   + 1152t   + 186324t
```

```
  +
           40          39         38          37           36          35
    - 3780t   - 543144t   - 3168t   - 21384t   + 1175251t   + 41184t
  +
           34           33          32           31           30
    278003t   - 1843242t   - 301815t   - 1440726t   + 1912012t
  +
           29           28          27           26           25
    1442826t   + 4696262t   - 922481t   - 4816188t   - 10583524t
  +
            24           23           22          21           20
    - 208751t   + 11472138t   + 16762859t   - 857663t   - 19328175t
  +
              19           18           17           16          15
    - 18270421t   + 4914903t   + 22483044t   + 12926517t   - 8605511t
  +
              14          13          12          11          10
    - 17455518t   - 5014597t   + 8108814t   + 8465535t   + 190542t
  +
             9          8          7          6          5          4
    - 4305624t  - 2226123t  + 661905t  + 1169775t  + 226260t  - 209952t
  +
           3
    - 141183t  + 27216t
  ,
           3      2      2      3          6     3           3      2
    (3z + 3t  - 3)y  + (3z  + (6t  - 6)z + 3t  - 6t  + 3)y + (3t  - 3)z
  +
       6     3          9     6     3
    (3t  - 6t  + 3)z + t  - 3t  + 5t  - 3t
  ,
              3
   x + y + z + t  - 1}
 ,
          2                       2                     2
{t,z - 1,y  - 1,x + y}, {t - 1,z,y  - 1,x + y}, {t - 1,z  - 1,z y + 1,x},
   16     13     10     7      4      2
 {t   - 6t   + 9t   + 4t  + 15t  - 54t  + 27,
                  29           28           27          26           25
          4907232t   + 40893984t   - 115013088t   - 1730448t   - 168139584t
        +
                    24           23           22           21
          738024480t   - 195372288t   + 315849456t   - 2567279232t
        +
                    20           19           18           17
          937147968t   + 1026357696t   + 4780488240t   - 2893767696t
```

```
    +
                  16              15              14              13
      - 5617160352t   - 3427651728t   + 5001100848t   + 8720098416t
    +
                12             11               10              9
      2331732960t   - 499046544t   - 16243306272t   - 9748123200t
    +
                8              7               6               5
      3927244320t  + 25257280896t  + 10348032096t  - 17128672128t
    +
                  4             3               2
      - 14755488768t  + 544086720t  + 10848188736t  + 1423614528t
    +
      - 2884297248
 *
    z
+
      68        65         62        60          59        57          56
 - 48t   + 1152t   - 13560t   + 360t   + 103656t   - 7560t   - 572820t
+
       54           53        52          51           50         49
  71316t   + 2414556t   + 2736t   - 402876t   - 7985131t   - 49248t
+
          48            47          46           45            44
  1431133t   + 20977409t   + 521487t   - 2697635t   - 43763654t
+
            43           42            41            40            39
  - 3756573t   - 2093410t   + 71546495t   + 19699032t   + 35025028t
+
             38            37             36            35             34
  - 89623786t   - 77798760t   - 138654191t   + 87596128t   + 235642497t
+
            33            32             31             30             29
  349607642t   - 93299834t   - 551563167t   - 630995176t   + 186818962t
+
            28             27             26             25
  995427468t   + 828416204t   - 393919231t   - 1076617485t
+
               24             23              22              21
  - 1609479791t   + 595738126t   + 1198787136t   + 4342832069t
+
               20              19              18              17
  - 2075938757t   - 4390835799t   - 4822843033t   + 6932747678t
+
             16              15              14              13
  6172196808t   + 1141517740t   - 4981677585t   - 9819815280t
```

```
    +
                   12             11                10               9
      - 7404299976t   - 157295760t   + 29124027630t   + 14856038208t
    +
                    8              7             6              5
      - 16184101410t  - 26935440354t  - 3574164258t  + 10271338974t
    +
                  4             3             2
      11191425264t  + 6869861262t  - 9780477840t  - 3586674168t + 2884297248
    ,
         3      3      2      6      3           9     6    3
      (3z  + (6t  - 6)z  + (6t  - 12t  + 3)z + 2t  - 6t  + t  + 3t)y
    +
         3      3      6      3      2      9      6      3          12     9
      (3t  - 3)z  + (6t  - 12t  + 6)z  + (4t  - 12t  + 11t  - 3)z + t   - 4t
    +
        6     3
      5t  - 2t
    ,
                3
     x + y + z + t  - 1}
   ,
          2
  {t - 1,z  - 1,y,x + z},
     8    7    6     5     4     3      2
   {t  + t  + t  - 2t  - 2t  - 2t  + 19t  + 19t - 8,
                   7           6           5           4           3
           2395770t  + 3934440t  - 3902067t  - 10084164t  - 1010448t
         +
                     2
           32386932t  + 22413225t - 10432368
      *
         z
    +
              7           6           5           4            3
      - 463519t  + 3586833t  + 9494955t  - 8539305t  - 33283098t
    +
              2
      35479377t  + 46263256t - 17419896
    ,
           4      3      3       6      3      2          3
         3z  + (9t  - 9)z  + (12t  - 24t  + 9)z  + (- 152t  + 219t - 67)z
        +
              6      4      3
         - 41t  + 57t  + 25t  - 57t + 16
      *
```

```
          y
       +
            3      4      6      3      3          3              2
         (3t  - 3)z  + (9t  - 18t  + 9)z  + (- 181t  + 270t - 89)z
       +
              6       4      3                     7      6      4       3
         (- 92t  + 135t  + 49t  - 135t + 43)z + 27t  - 27t  - 54t  + 396t
       +
         - 486t + 144
       ,
                   3
        x + y + z + t  - 1}
      ,
             3
     {t,z - t  + 1,y - 1,x - 1}, {t - 1,z,y,x}, {t,z - 1,y,x}, {t,z,y - 1,x},
     {t,z,y,x - 1}]
                      Type: List RegularChain(Integer,[x,y,z,t])
```

Then we compute a univariate representation.

```
  univariateSolve(lf)$pack
  [[complexRoots= ?,coordinates= [x - 1,y - 1,z + 1,t - %A]],
   [complexRoots= ?,coordinates= [x,y - 1,z,t - %A]],
   [complexRoots= ? - 1,coordinates= [x,y,z,t - %A]],
   [complexRoots= ?,coordinates= [x - 1,y,z,t - %A]],
   [complexRoots= ?,coordinates= [x,y,z - 1,t - %A]],
   [complexRoots= ? - 2,coordinates= [x - 1,y + 1,z,t - 1]],
   [complexRoots= ?,coordinates= [x + 1,y - 1,z,t - 1]],
   [complexRoots= ? - 1,coordinates= [x - 1,y + 1,z - 1,t]],
   [complexRoots= ? + 1,coordinates= [x + 1,y - 1,z - 1,t]],

                    6     3     2
    [complexRoots= ?  - 2?  + 3?  - 3,
                         3                  3
     coordinates= [2x + %A  + %A - 1,2y + %A  + %A - 1,z - %A,t - %A]]
    ,

                    5     3     2
    [complexRoots= ?  + 3?  - 2?  + 3? - 3,
                                      3
     coordinates= [x - %A,y - %A,z + %A  + 2%A - 1,t - %A]]
    ,

                    4    3     2
    [complexRoots= ?  - ?  - 2?  + 3,
                         3                3                3
```

```
 coordinates= [x + %A  - %A - 1,y + %A  - %A - 1,z - %A  + 2%A + 1,t - %A]]
,
[complexRoots= ? + 1,coordinates= [x - 1,y - 1,z,t - %A]],

                6     3     2
[complexRoots= ?  + 2?  + 3?  - 3,
                     3                        3
 coordinates= [2x - %A  - %A - 1,y + %A,2z - %A  - %A - 1,t + %A]]
,

                6      4      3      2
[complexRoots= ?  + 12?  + 20?  - 45?  - 42? - 953,

  coordinates =
                   5       4       3        2
    [12609x + 23%A  + 49%A  - 46%A  + 362%A  - 5015%A - 8239,
                   5       4       3        2
     25218y + 23%A  + 49%A  - 46%A  + 362%A  + 7594%A - 8239,
                   5       4       3        2
     25218z + 23%A  + 49%A  - 46%A  + 362%A  + 7594%A - 8239,
                   5       4       3        2
     12609t + 23%A  + 49%A  - 46%A  + 362%A  - 5015%A - 8239]
  ]
,

                5      3      2
[complexRoots= ?  + 12?  - 16?  + 48? - 96,
                     3
 coordinates= [8x + %A  + 8%A - 8,2y - %A,2z - %A,2t - %A]]
,

                5    4     3     2
[complexRoots= ?  + ?  - 5?  - 3?  + 9? + 3,

  coordinates =
              3                    3                    3
    [2x - %A  + 2%A - 1, 2y + %A  - 4%A + 1, 2z - %A  + 2%A - 1,
              3
     2t - %A  + 2%A - 1]
  ]
,

                4     3     2
[complexRoots= ?  - 3?  + 4?  - 6? + 13,

  coordinates =
```

```
              3      2                   3      2
    [9x - 2%A  + 4%A  - %A + 2, 9y + %A  - 2%A  + 5%A - 1,
             3      2                   3      2
     9z + %A  - 2%A  + 5%A - 1, 9t + %A  - 2%A  - 4%A - 1]
  ]
,

                4      2
[complexRoots= ?  - 11?  + 37,

  coordinates =
            2            2                  2            2
    [3x - %A  + 7,6y + %A  + 3%A - 7,3z - %A  + 7,6t + %A  - 3%A - 7]
  ]
,
[complexRoots= ? + 1,coordinates= [x - 1,y,z - 1,t + 1]],
[complexRoots= ? + 2,coordinates= [x,y - 1,z - 1,t + 1]],
[complexRoots= ? - 2,coordinates= [x,y - 1,z + 1,t - 1]],
[complexRoots= ?,coordinates= [x,y + 1,z - 1,t - 1]],
[complexRoots= ? - 2,coordinates= [x - 1,y,z + 1,t - 1]],
[complexRoots= ?,coordinates= [x + 1,y,z - 1,t - 1]],

                4     3      2
[complexRoots= ?  + 5?  + 16?  + 30? + 57,

  coordinates =
                3       2                          3       2
    [151x + 15%A  + 54%A  + 104%A + 93, 151y - 10%A  - 36%A  - 19%A - 62,
              3       2                          3       2
     151z - 5%A  - 18%A  - 85%A - 31, 151t - 5%A  - 18%A  - 85%A - 31]
  ]
,

                4    3     2
[complexRoots= ?  - ?  - 2?  + 3,
                     3                 3                       3
 coordinates= [x - %A  + 2%A + 1,y + %A  - %A - 1,z - %A,t + %A  - %A - 1]]
,

                4     3     2
[complexRoots= ?  + 2?  - 8?  + 48,

  coordinates =
            3                          3                   3
    [8x - %A  + 4%A - 8,2y + %A,8z + %A  - 8%A + 8,8t - %A  + 4%A - 8]
  ]
```

```
   ,
                    5    4     3     2
    [complexRoots= ?  + ?  - 2?  - 4?  + 5? + 8,
                          3           3           3
     coordinates= [3x + %A  - 1,3y + %A  - 1,3z + %A  - 1,t - %A]]
    ,
                   3
    [complexRoots= ?  + 3? - 1,coordinates= [x - %A,y - %A,z - %A,t - %A]]]
          Type: List Record(complexRoots: SparseUnivariatePolynomial Integer,
                            coordinates: List Polynomial Integer)
```

Note that this computation is made from the input system lf.

However it is possible to reuse a pre-computed regular chain as follows:

```
ts := lts.1
   2          3        3
 {t  + t + 1, z  - z - t  + t,

              3      2      2      3           6     3            3      2
      (3z + 3t  - 3)y  + (3z  + (6t  - 6)z + 3t  - 6t  + 3)y + (3t  - 3)z
   +
        6     3          9     6     3
      (3t  - 6t  + 3)z + t  - 3t  + 5t  - 3t
   ,
  x + y + z}
                     Type: RegularChain(Integer,[x,y,z,t])

univariateSolve(ts)$pack
[
                   4     3      2
  [complexRoots= ?  + 5?  + 16?  + 30? + 57,
    coordinates =
                   3       2                           3       2
      [151x + 15%A  + 54%A  + 104%A + 93, 151y - 10%A  - 36%A  - 19%A - 62,
                   3       2                           3       2
       151z - 5%A  - 18%A  - 85%A - 31, 151t - 5%A  - 18%A  - 85%A - 31]
    ]
  ,
                   4    3     2
  [complexRoots= ?  - ?  - 2?  + 3,
                         3                 3                       3
   coordinates= [x - %A  + 2%A + 1,y + %A  - %A - 1,z - %A,t + %A  - %A - 1]]
  ,
                   4     3     2
```

```
   [complexRoots= ?  + 2?  - 8?  + 48,
     coordinates =
                3                            3                  3
       [8x - %A  + 4%A - 8,2y + %A,8z + %A  - 8%A + 8,8t - %A  + 4%A - 8]
     ]
    ]
         Type: List Record(complexRoots: SparseUnivariatePolynomial Integer,
                           coordinates: List Polynomial Integer)

  realSolve(ts)$pack
    []
                        Type: List List RealClosure Fraction Integer
```

We compute now the full set of points with real coordinates:

```
  lr2 := realSolve(lf)$pack
  [[0,- 1,1,1], [0,0,1,0], [1,0,0,0], [0,0,0,1], [0,1,0,0], [1,0,%B37,- %B37],
   [1,0,%B38,- %B38], [0,1,%B35,- %B35], [0,1,%B36,- %B36], [- 1,0,1,1],
    [%B32,
        1     15    2     14    1     13    4     12   11     11    4     10
       -- %B32   + -- %B32   + -- %B32   - -- %B32   - -- %B32   - -- %B32
       27          27          27          27          27          27
     +
        1     9   14     8    1     7   2     6   1     5   2     4       3
       -- %B32  + -- %B32  + -- %B32  + - %B32  + - %B32  + - %B32  + %B32
       27         27         27         9         3         9
     +
       4     2
       - %B32  - %B32 - 2
       3
     ,
         1     15    1     14    1     13    2     12   11     11    2     10
       - -- %B32   - -- %B32   - -- %B32   + -- %B32   + -- %B32   + -- %B32
         54          27          54          27          54          27
     +
         1     9    7     8    1     7   1     6   1     5   1     4       3
       - -- %B32  - -- %B32  - -- %B32  - - %B32  - - %B32  - - %B32  - %B32
         54         27         54         9         6         9
     +
         2     2   1        3
       - - %B32  + - %B32 + -
         3         2        2
     ,
         1     15    1     14    1     13    2     12   11     11    2     10
       - -- %B32   - -- %B32   - -- %B32   + -- %B32   + -- %B32   + -- %B32
         54          27          54          27          54          27
```

```
  +
       1     9    7     8    1     7   1     6   1     5   1     4       3
    - -- %B32  - -- %B32  - -- %B32  - - %B32  - - %B32  - - %B32  - %B32
       54          27          54          9          6          9
  +
       2     2   1        3
    - - %B32  + - %B32 + -
       3          2        2
  ]
,
[%B33,
      1     15    2     14    1     13    4     12   11     11    4     10
    -- %B33   + -- %B33   + -- %B33   - -- %B33   - -- %B33   - -- %B33
    27          27          27          27          27          27
  +
      1     9   14     8    1     7   2     6   1     5   2     4       3
    -- %B33  + -- %B33  + -- %B33  + - %B33  + - %B33  + - %B33  + %B33
    27          27          27          9          3          9
  +
    4     2
    - %B33  - %B33 - 2
    3
  ,
       1     15    1     14    1     13    2     12   11     11    2     10
    - -- %B33   - -- %B33   - -- %B33   + -- %B33   + -- %B33   + -- %B33
       54          27          54          27          54          27
  +
       1     9    7     8    1     7   1     6   1     5   1     4       3
    - -- %B33  - -- %B33  - -- %B33  - - %B33  - - %B33  - - %B33  - %B33
       54          27          54          9          6          9
  +
       2     2   1        3
    - - %B33  + - %B33 + -
       3          2        2
  ,
       1     15    1     14    1     13    2     12   11     11    2     10
    - -- %B33   - -- %B33   - -- %B33   + -- %B33   + -- %B33   + -- %B33
       54          27          54          27          54          27
  +
       1     9    7     8    1     7   1     6   1     5   1     4       3
    - -- %B33  - -- %B33  - -- %B33  - - %B33  - - %B33  - - %B33  - %B33
       54          27          54          9          6          9
  +
       2     2   1        3
    - - %B33  + - %B33 + -
       3          2        2
```

```
  ]
 ,
 [%B34,
     1     15    2     14    1     13    4     12   11     11    4     10
   -- %B34   + -- %B34   + -- %B34   - -- %B34   - -- %B34   - -- %B34
   27          27          27          27          27          27
  +
    1     9   14     8    1     7   2     6   1     5   2     4       3
   -- %B34  + -- %B34  + -- %B34  + - %B34  + - %B34  + - %B34  + %B34
   27          27          27          9         3         9
  +
   4     2
   - %B34  - %B34 - 2
   3
  ,
      1     15    1     14    1     13    2     12   11     11    2     10
   - -- %B34   - -- %B34   - -- %B34   + -- %B34   + -- %B34   + -- %B34
     54          27          54          27          54          27
  +
      1     9    7     8    1     7   1     6   1     5   1     4       3
   - -- %B34  - -- %B34  - -- %B34  - - %B34  - - %B34  - - %B34  - %B34
     54          27          54          9         6         9
  +
     2     2   1        3
   - - %B34  + - %B34 + -
     3         2        2
  ,
      1     15    1     14    1     13    2     12   11     11    2     10
   - -- %B34   - -- %B34   - -- %B34   + -- %B34   + -- %B34   + -- %B34
     54          27          54          27          54          27
  +
      1     9    7     8    1     7   1     6   1     5   1     4       3
   - -- %B34  - -- %B34  - -- %B34  - - %B34  - - %B34  - - %B34  - %B34
     54          27          54          9         6         9
  +
     2     2   1        3
   - - %B34  + - %B34 + -
     3         2        2
  ]
 ,
[- 1,1,0,1], [- 1,1,1,0],
 [%B23,
      1     15    1     14    1     13    2     12   11     11    2     10
   - -- %B23   - -- %B23   - -- %B23   + -- %B23   + -- %B23   + -- %B23
     54          27          54          27          54          27
  +
```

```
      1     9    7     8    1     7   1     6   1     5   1     4       3
  - -- %B23  - -- %B23  - -- %B23  - - %B23  - - %B23  - - %B23  - %B23
     54          27          54          9          6          9
 +
     2     2   1         3
   - - %B23  + - %B23 + -
     3          2         2
 ,
%B30,
              1     15    1     14    1     13    2     12   11     11
   - %B30 + -- %B23   + -- %B23   + -- %B23   - -- %B23   - -- %B23
             54          27          54          27          54
 +
      2     10    1     9    7     8    1     7   1     6   1     5
   - -- %B23   + -- %B23  + -- %B23  + -- %B23  + - %B23  + - %B23
     27          54          27          54          9          6
 +
   1     4   2     2   1         1
   - %B23  + - %B23  - - %B23 - -
   9          3          2         2
 ]
,
[%B23,
      1     15    1     14    1     13    2     12   11     11    2     10
   - -- %B23   - -- %B23   - -- %B23   + -- %B23   + -- %B23   + -- %B23
     54          27          54          27          54          27
 +
      1     9    7     8    1     7   1     6   1     5   1     4       3
   - -- %B23  - -- %B23  - -- %B23  - - %B23  - - %B23  - - %B23  - %B23
     54          27          54          9          6          9
 +
     2     2   1         3
   - - %B23  + - %B23 + -
     3          2         2
 ,
%B31,
              1     15    1     14    1     13    2     12   11     11
   - %B31 + -- %B23   + -- %B23   + -- %B23   - -- %B23   - -- %B23
             54          27          54          27          54
 +
      2     10    1     9    7     8    1     7   1     6   1     5
   - -- %B23   + -- %B23  + -- %B23  + -- %B23  + - %B23  + - %B23
     27          54          27          54          9          6
 +
   1     4   2     2   1         1
   - %B23  + - %B23  - - %B23 - -
```

```
    9          3          2         2
  ]
,
[%B24,
       1     15    1     14    1     13    2     12   11     11    2     10
   - -- %B24   - -- %B24   - -- %B24   + -- %B24   + -- %B24   + -- %B24
     54           27           54           27           54           27
  +
       1     9    7     8    1     7   1     6   1     5   1     4       3
   - -- %B24  - -- %B24  - -- %B24  - - %B24  - - %B24  - - %B24  - %B24
     54          27          54          9          6          9
  +
     2     2   1         3
   - - %B24  + - %B24 + -
     3          2         2
  ,
 %B28,
              1     15    1     14    1     13    2     12   11     11
   - %B28 + -- %B24   + -- %B24   + -- %B24   - -- %B24   - -- %B24
             54           27           54           27           54
  +
      2     10    1     9    7     8    1     7   1     6   1     5
   - -- %B24   + -- %B24  + -- %B24  + -- %B24  + - %B24  + - %B24
     27           54          27          54          9          6
  +
     1     4   2     2   1         1
   - %B24  + - %B24  - - %B24 - -
     9          3          2         2
  ]
,
[%B24,
       1     15    1     14    1     13    2     12   11     11    2     10
   - -- %B24   - -- %B24   - -- %B24   + -- %B24   + -- %B24   + -- %B24
     54           27           54           27           54           27
  +
       1     9    7     8    1     7   1     6   1     5   1     4       3
   - -- %B24  - -- %B24  - -- %B24  - - %B24  - - %B24  - - %B24  - %B24
     54          27          54          9          6          9
  +
     2     2   1         3
   - - %B24  + - %B24 + -
     3          2         2
  ,
 %B29,
              1     15    1     14    1     13    2     12   11     11
   - %B29 + -- %B24   + -- %B24   + -- %B24   - -- %B24   - -- %B24
```

```
            54          27          54          27          54
  +
       2     10    1     9    7     8    1     7   1     6   1     5
    - -- %B24   + -- %B24  + -- %B24  + -- %B24  + - %B24  + - %B24
       27          54          27          54          9         6
  +
    1     4   2     2   1        1
    - %B24  + - %B24  - - %B24 - -
    9         3         2        2
  ]
,
[%B25,
       1     15    1     14    1     13    2     12   11     11    2     10
    - -- %B25   - -- %B25   - -- %B25   + -- %B25   + -- %B25   + -- %B25
       54          27          54          27          54          27
  +
       1     9    7     8    1     7   1     6   1     5   1     4       3
    - -- %B25  - -- %B25  - -- %B25  - - %B25  - - %B25  - - %B25  - %B25
       54         27          54         9         6         9
  +
       2     2   1        3
    - - %B25  + - %B25 + -
       3         2        2
  ,
 %B26,
              1     15    1     14    1     13    2     12   11     11
    - %B26 + -- %B25   + -- %B25   + -- %B25   - -- %B25   - -- %B25
             54          27          54          27          54
  +
       2     10    1     9    7     8    1     7   1     6   1     5
    - -- %B25   + -- %B25  + -- %B25  + -- %B25  + - %B25  + - %B25
       27          54          27          54          9         6
  +
    1     4   2     2   1        1
    - %B25  + - %B25  - - %B25 - -
    9         3         2        2
  ]
,
[%B25,
       1     15    1     14    1     13    2     12   11     11    2     10
    - -- %B25   - -- %B25   - -- %B25   + -- %B25   + -- %B25   + -- %B25
       54          27          54          27          54          27
  +
       1     9    7     8    1     7   1     6   1     5   1     4       3
    - -- %B25  - -- %B25  - -- %B25  - - %B25  - - %B25  - - %B25  - %B25
       54         27          54         9         6         9
```

```
      +
          2     2   1        3
        - - %B25  + - %B25 + -
          3         2        2
      ,
     %B27,
                  1     15    1     14    1     13    2     12   11     11
       - %B27 + -- %B25   + -- %B25   + -- %B25   - -- %B25   - -- %B25
                 54          27          54          27          54
      +
           2     10    1     9    7     8    1     7   1     6   1     5
       - -- %B25   + -- %B25  + -- %B25  + -- %B25  + - %B25  + - %B25
         27          54         27         54         9         6
      +
        1     4   2     2   1        1
        - %B25  + - %B25  - - %B25 - -
        9         3         2        2
      ]
    ,
    [1,%B21,- %B21,0], [1,%B22,- %B22,0], [1,%B19,0,- %B19], [1,%B20,0,- %B20],
            1     3   1   1     3   1   1     3   1
    [%B17,- - %B17  + -,- - %B17  + -,- - %B17  + -],
            3         3   3         3   3         3
            1     3   1   1     3   1   1     3   1
    [%B18,- - %B18  + -,- - %B18  + -,- - %B18  + -]]
            3         3   3         3   3         3
                        Type: List List RealClosure Fraction Integer
```

The number of real solutions for the input system is:

```
  #lr2
    27
                                Type: PositiveInteger
```

Another example of computation of real solutions illustrates the LexTriangularPackage package constructor.

We concentrate now on the solutions with real (strictly) positive coordinates:

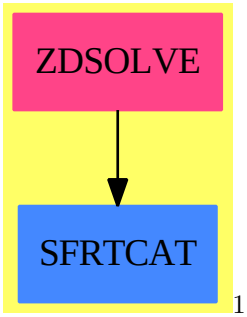
```
  lpr2 := positiveSolve(lf)$pack
              1     3   1   1     3   1   1     3   1
     [[%B40,- - %B40  + -,- - %B40  + -,- - %B40  + -]]
              3         3   3         3   3         3
                              Type: List List RealClosure Fraction Integer
```

```
Finally, we approximate the coordinates of this point with 20 exact digits:

  [approximate(r,1/10**21)::Float for r in lpr2.1]
   [0.3221853546 2608559291, 0.3221853546 2608559291, 0.3221853546 2608559291,
    0.3221853546 2608559291]
                              Type: List Float

See Also:
o )show ZeroDimensionalSolvePackage
```

27.2 ZeroDimensionalSolvePackage



1.00

Exports:
convert positiveSolve realSolve squareFree triangSolve univariateSolve

⟨*package ZDSOLVE ZeroDimensionalSolvePackage*⟩≡
```
)abbrev package ZDSOLVE ZeroDimensionalSolvePackage
++ Author: Marc Moreno Maza
++ Date Created: 23/01/1999
++ Date Last Updated: 08/02/1999
++ Basic Functions:
++ Related Constructors:
++ Also See:
++ AMS Classifications:
++ Keywords:
++ References:
++ Description:
++   A package for computing symbolically the complex and real roots of
++   zero-dimensional algebraic systems over the integer or rational
++   numbers. Complex roots are given by means of univariate representations
++   of irreducible regular chains. Real roots are given by means of tuples
++   of coordinates lying in the \spadtype{RealClosure} of the coefficient ring.
++   This constructor takes three arguments. The first one \spad{R} is the
++   coefficient ring. The second one \spad{ls} is the list of variables involved
++   in the systems to solve. The third one must be \spad{concat(ls,s)} where
++   \spad{s} is an additional symbol used for the univariate representations.
++   WARNING: The third argument is not checked.
++   All operations are based on triangular decompositions.
++   The default is to compute these decompositions directly from the input
++   system by using the \spadtype{RegularChain} domain constructor.
++   The lexTriangular algorithm can also be used for computing these decompositions
++   (see the \spadtype{LexTriangularPackage} package constructor).
++   For that purpose, the operations \axiomOpFrom{univariateSolve}{ZeroDimensionalSolvePack
++   \axiomOpFrom{realSolve}{ZeroDimensionalSolvePackage} and
++   \axiomOpFrom{positiveSolve}{ZeroDimensionalSolvePackage} admit an optional
++   argument. \newline Author: Marc Moreno Maza.
```

```
++ Version: 1.

ZeroDimensionalSolvePackage(R,ls,ls2): Exports == Implementation where
  R : Join(OrderedRing,EuclideanDomain,CharacteristicZero,RealConstant)
  ls: List Symbol
  ls2: List Symbol
  V ==> OrderedVariableList(ls)
  N ==> NonNegativeInteger
  Z ==> Integer
  B ==> Boolean
  P ==> Polynomial R
  LP ==> List P
  LS ==> List Symbol
  Q ==> NewSparseMultivariatePolynomial(R,V)
  U ==> SparseUnivariatePolynomial(R)
  TS ==> RegularChain(R,ls)
  RUR ==> Record(complexRoots: U, coordinates: LP)
  K ==> Fraction R
  RC ==> RealClosure(K)
  PRC ==> Polynomial RC
  REALSOL ==> List RC
  URC ==> SparseUnivariatePolynomial RC
  V2 ==> OrderedVariableList(ls2)
  Q2 ==> NewSparseMultivariatePolynomial(R,V2)
  E2 ==> IndexedExponents V2
  ST ==> SquareFreeRegularTriangularSet(R,E2,V2,Q2)
  Q2WT ==> Record(val: Q2, tower: ST)
  LQ2WT ==> Record(val: List(Q2), tower: ST)
  WIP ==> Record(reals: List(RC), vars: List(Symbol), pols: List(Q2))
  polsetpack ==> PolynomialSetUtilitiesPackage(R,E2,V2,Q2)
  normpack ==> NormalizationPackage(R,E2,V2,Q2,ST)
  rurpack ==> InternalRationalUnivariateRepresentationPackage(R,E2,V2,Q2,ST)
  quasicomppack ==> SquareFreeQuasiComponentPackage(R,E2,V2,Q2,ST)
  lextripack ==> LexTriangularPackage(R,ls)

  Exports ==  with
     triangSolve: (LP,B,B) -> List RegularChain(R,ls)
       ++ \spad{triangSolve(lp,info?,lextri?)} decomposes the variety
       ++ associated with \axiom{lp} into regular chains.
       ++ Thus a point belongs to this variety iff it is a regular
       ++ zero of a regular set in in the output.
       ++ Note that \axiom{lp} needs to generate a zero-dimensional ideal.
       ++ If \axiom{lp} is not zero-dimensional then the result is only
       ++ a decomposition of its zero-set in the sense of the closure
       ++ (w.r.t. Zarisky topology).
```

```
    ++ Moreover, if \spad{info?} is \spad{true} then some information is
    ++ displayed during the computations.
    ++ See \axiomOpFrom{zeroSetSplit}{RegularTriangularSetCategory}(lp,true,info?).
    ++ If \spad{lextri?} is \spad{true} then the lexTriangular algorithm is called
    ++ from the \spadtype{LexTriangularPackage} constructor
    ++ (see \axiomOpFrom{zeroSetSplit}{LexTriangularPackage}(lp,false)).
    ++ Otherwise, the triangular decomposition is computed directly from the input
    ++ system by using the \axiomOpFrom{zeroSetSplit}{RegularChain} from \spadtype{Regula
  triangSolve: (LP,B) -> List RegularChain(R,ls)
    ++ \spad{triangSolve(lp,info?)} returns the same as \spad{triangSolve(lp,false)}
  triangSolve: LP -> List RegularChain(R,ls)
    ++ \spad{triangSolve(lp)} returns the same as \spad{triangSolve(lp,false,false)}
  univariateSolve: RegularChain(R,ls) -> List Record(complexRoots: U, coordinates: LP)
    ++ \spad{univariateSolve(ts)} returns a univariate representation
    ++ of \spad{ts}.
    ++ See \axiomOpFrom{rur}{RationalUnivariateRepresentationPackage}(lp,true).
  univariateSolve: (LP,B,B,B) -> List RUR
    ++ \spad{univariateSolve(lp,info?,check?,lextri?)} returns a univariate
    ++ representation of the variety associated with \spad{lp}.
    ++ Moreover, if \spad{info?} is \spad{true} then some information is
    ++ displayed during the decomposition into regular chains.
    ++ If \spad{check?} is \spad{true} then the result is checked.
    ++ See \axiomOpFrom{rur}{RationalUnivariateRepresentationPackage}(lp,true).
    ++ If \spad{lextri?} is \spad{true} then the lexTriangular algorithm is called
    ++ from the \spadtype{LexTriangularPackage} constructor
    ++ (see \axiomOpFrom{zeroSetSplit}{LexTriangularPackage}(lp,false)).
    ++ Otherwise, the triangular decomposition is computed directly from the input
    ++ system by using the \axiomOpFrom{zeroSetSplit}{RegularChain} from \spadtype{Regula
  univariateSolve: (LP,B,B) -> List RUR
    ++ \spad{univariateSolve(lp,info?,check?)} returns the same as
    ++ \spad{univariateSolve(lp,info?,check?,false)}.
  univariateSolve: (LP,B) -> List RUR
    ++ \spad{univariateSolve(lp,info?)} returns the same as
    ++ \spad{univariateSolve(lp,info?,false,false)}.
  univariateSolve: LP -> List RUR
    ++ \spad{univariateSolve(lp)} returns the same as
    ++ \spad{univariateSolve(lp,false,false,false)}.
  realSolve: RegularChain(R,ls) -> List REALSOL
    ++ \spad{realSolve(ts)} returns the set of the points in the regular
    ++ zero set of \spad{ts} whose coordinates are all real.
    ++ WARNING: For each set of coordinates given by \spad{realSolve(ts)}
    ++ the ordering of the indeterminates is reversed w.r.t. \spad{ls}.
  realSolve: (LP,B,B,B) -> List REALSOL
    ++ \spad{realSolve(ts,info?,check?,lextri?)} returns the set of the points
    ++ in the variety associated with \spad{lp} whose coordinates are all real.
    ++ Moreover, if \spad{info?} is \spad{true} then some information is
```

```
  ++ displayed during decomposition into regular chains.
  ++ If \spad{check?} is \spad{true} then the result is checked.
  ++ If \spad{lextri?} is \spad{true} then the lexTriangular algorithm is ca
  ++ from the \spadtype{LexTriangularPackage} constructor
  ++ (see \axiomOpFrom{zeroSetSplit}{LexTriangularPackage}(lp,false)).
  ++ Otherwise, the triangular decomposition is computed directly from the i
  ++ system by using the \axiomOpFrom{zeroSetSplit}{RegularChain} from \spad
  ++ WARNING: For each set of coordinates given by \spad{realSolve(ts,info?,
  ++ the ordering of the indeterminates is reversed w.r.t. \spad{ls}.
realSolve: (LP,B,B) -> List REALSOL
  ++ \spad{realSolve(ts,info?,check?)} returns the same as \spad{realSolve(t
realSolve: (LP,B) -> List REALSOL
  ++ \spad{realSolve(ts,info?)} returns the same as \spad{realSolve(ts,info?
realSolve: LP -> List REALSOL
  ++ \spad{realSolve(lp)} returns the same as \spad{realSolve(ts,false,false
positiveSolve: RegularChain(R,ls)  -> List REALSOL
  ++ \spad{positiveSolve(ts)} returns the points of the regular
  ++ set of \spad{ts} with (real) strictly positive coordinates.
positiveSolve: (LP,B,B) -> List REALSOL
  ++ \spad{positiveSolve(lp,info?,lextri?)} returns the set of the points
  ++ in the variety associated with \spad{lp} whose coordinates are (real) s
  ++ Moreover, if \spad{info?} is \spad{true} then some information is
  ++ displayed during decomposition into regular chains.
  ++ If \spad{lextri?} is \spad{true} then the lexTriangular algorithm is ca
  ++ from the \spadtype{LexTriangularPackage} constructor
  ++ (see \axiomOpFrom{zeroSetSplit}{LexTriangularPackage}(lp,false)).
  ++ Otherwise, the triangular decomposition is computed directly from the i
  ++ system by using the \axiomOpFrom{zeroSetSplit}{RegularChain} from \spad
  ++ WARNING: For each set of coordinates given by \spad{positiveSolve(lp,in
  ++ the ordering of the indeterminates is reversed w.r.t. \spad{ls}.
positiveSolve: (LP,B) -> List REALSOL
  ++ \spad{positiveSolve(lp)} returns the same as \spad{positiveSolve(lp,inf
positiveSolve: LP -> List REALSOL
  ++ \spad{positiveSolve(lp)} returns the same as \spad{positiveSolve(lp,fal
squareFree: (TS) -> List ST
  ++ \spad{squareFree(ts)} returns the square-free factorization of \spad{ts
  ++ Moreover, each factor is a Lazard triangular set and the decomposition
  ++ is a Kalkbrener split of \spad{ts}, which is enough here for
  ++ the matter of solving zero-dimensional algebraic systems.
  ++ WARNING: \spad{ts} is not checked to be zero-dimensional.
convert: Q -> Q2
  ++ \spad{convert(q)} converts \spad{q}.
convert: P -> PRC
  ++ \spad{convert(p)} converts \spad{p}.
convert: Q2 -> PRC
  ++ \spad{convert(q)} converts \spad{q}.
```

```
      convert: U -> URC
        ++ \spad{convert(u)} converts \spad{u}.
      convert: ST -> List Q2
        ++ \spad{convert(st)} returns the members of \spad{st}.

   Implementation == add
      news: Symbol := last(ls2)$(List Symbol)
      newv: V2 := (variable(news)$V2)::V2
      newq: Q2 :=  newv :: Q2

      convert(q:Q):Q2 ==
        ground? q => (ground(q))::Q2
        q2: Q2 := 0
        while not ground?(q) repeat
          v: V := mvar(q)
          d: N := mdeg(q)
          v2: V2 := (variable(convert(v)@Symbol)$V2)::V2
          iq2: Q2 := convert(init(q))@Q2
          lq2: Q2 := (v2 :: Q2)
          lq2 := lq2 ** d
          q2 := iq2 * lq2 + q2
          q := tail(q)
        q2 + (ground(q))::Q2

      squareFree(ts:TS):List(ST) ==
        irred?: Boolean := false
        st: ST := [[newq]$(List Q2)]
        lq: List(Q2) := [convert(p)@Q2 for p in parts(ts)]
        lq := sort(infRittWu?,lq)
        toSee: List LQ2WT := []
        if irred?
          then
            lf := irreducibleFactors([first lq])$polsetpack
            lq := rest lq
            for f in lf repeat
              toSee := cons([cons(f,lq),st]$LQ2WT, toSee)
          else
            toSee := [[lq,st]$LQ2WT]
        toSave: List ST := []
        while not empty? toSee repeat
          lqwt := first toSee; toSee := rest toSee
          lq := lqwt.val; st := lqwt.tower
          empty? lq =>
            toSave := cons(st,toSave)
          q := first lq; lq := rest lq
          lsfqwt: List Q2WT := squareFreePart(q,st)$ST
```

```
    for sfqwt in lsfqwt repeat
      q := sfqwt.val; st := sfqwt.tower
      if not ground? init(q)
        then
          q := normalizedAssociate(q,st)$normpack
      newts := internalAugment(q,st)$ST
      newlq := [remainder(q,newts).polnum for q in lq]
      toSee := cons([newlq,newts]$LQ2WT,toSee)
  toSave


triangSolve(lp: LP, info?: B, lextri?: B): List TS ==
  lq: List(Q) := [convert(p)$Q for p in lp]
  lextri? => zeroSetSplit(lq,false)$lextripack
  zeroSetSplit(lq,true,info?)$TS

triangSolve(lp: LP, info?: B): List TS == triangSolve(lp,info?,false)

triangSolve(lp: LP): List TS == triangSolve(lp,false)

convert(u: U): URC ==
  zero? u => 0
  ground? u => ((ground(u) :: K)::RC)::URC
  uu: URC := 0
  while not ground? u repeat
    uu := monomial((leadingCoefficient(u) :: K):: RC,degree(u)) + uu
    u := reductum u
  uu + ((ground(u) :: K)::RC)::URC

coerceFromRtoRC(r:R): RC ==
  (r::K)::RC

convert(p:P): PRC ==
  map(coerceFromRtoRC,p)$PolynomialFunctions2(R,RC)

convert(q2:Q2): PRC ==
  p: P := coerce(q2)$Q2
  convert(p)@PRC

convert(sts:ST): List Q2 ==
  lq2: List(Q2) := parts(sts)$ST
  lq2 := sort(infRittWu?,lq2)
  rest(lq2)

realSolve(ts: TS): List REALSOL ==
  lsts: List ST := squareFree(ts)
```

```
    lr: REALSOL := []
    lv: List Symbol := []
    toSee := [[lr,lv,convert(sts)@(List Q2)]$WIP for sts in lsts]
    toSave: List REALSOL := []
    while not empty? toSee repeat
      wip := first toSee; toSee := rest toSee
      lr := wip.reals; lv := wip.vars; lq2 := wip.pols
      (empty? lq2) and (not empty? lr) =>
         toSave := cons(reverse(lr),toSave)
      q2 := first lq2; lq2 := rest lq2
      qrc := convert(q2)@PRC
      if not empty? lr
        then
          for r in reverse(lr) for v in reverse(lv) repeat
            qrc := eval(qrc,v,r)
      lv := cons((mainVariable(qrc) :: Symbol),lv)
      urc: URC := univariate(qrc)@URC
      urcRoots := allRootsOf(urc)$RC
      for urcRoot in urcRoots repeat
        toSee := cons([cons(urcRoot,lr),lv,lq2]$WIP, toSee)
    toSave

  realSolve(lp: List(P), info?:Boolean, check?:Boolean, lextri?: Boolean): List REALSOL
    lts: List TS
    lq: List(Q) := [convert(p)$Q for p in lp]
    if lextri?
      then
        lts := zeroSetSplit(lq,false)$lextripack
      else
        lts := zeroSetSplit(lq,true,info?)$TS
    lsts:  List ST := []
    for ts in lts repeat
      lsts := concat(squareFree(ts), lsts)
    lsts := removeSuperfluousQuasiComponents(lsts)$quasicomppack
    lr: REALSOL := []
    lv: List Symbol := []
    toSee := [[lr,lv,convert(sts)@(List Q2)]$WIP for sts in lsts]
    toSave: List REALSOL := []
    while not empty? toSee repeat
      wip := first toSee; toSee := rest toSee
      lr := wip.reals; lv := wip.vars; lq2 := wip.pols
      (empty? lq2) and (not empty? lr) =>
         toSave := cons(reverse(lr),toSave)
      q2 := first lq2; lq2 := rest lq2
      qrc := convert(q2)@PRC
      if not empty? lr
```

```
      then
        for r in reverse(lr) for v in reverse(lv) repeat
          qrc := eval(qrc,v,r)
    lv := cons((mainVariable(qrc) :: Symbol),lv)
    urc: URC := univariate(qrc)@URC
    urcRoots := allRootsOf(urc)$RC
    for urcRoot in urcRoots repeat
      toSee := cons([cons(urcRoot,lr),lv,lq2]$WIP, toSee)
  if check?
    then
      for p in lp repeat
        for realsol in toSave repeat
          prc: PRC := convert(p)@PRC
          for rr in realsol for symb in reverse(ls) repeat
            prc := eval(prc,symb,rr)
          not zero? prc =>
            error "realSolve$ZDSOLVE: bad result"
  toSave

realSolve(lp: List(P), info?:Boolean, check?:Boolean): List REALSOL  ==
  realSolve(lp,info?,check?,false)

realSolve(lp: List(P), info?:Boolean): List REALSOL  ==
  realSolve(lp,info?,false,false)

realSolve(lp: List(P)): List REALSOL  ==
  realSolve(lp,false,false,false)

positiveSolve(ts: TS): List REALSOL ==
  lsts: List ST := squareFree(ts)
  lr: REALSOL := []
  lv: List Symbol := []
  toSee := [[lr,lv,convert(sts)@(List Q2)]$WIP for sts in lsts]
  toSave: List REALSOL := []
  while not empty? toSee repeat
    wip := first toSee; toSee := rest toSee
    lr := wip.reals; lv := wip.vars; lq2 := wip.pols
    (empty? lq2) and (not empty? lr) =>
       toSave := cons(reverse(lr),toSave)
    q2 := first lq2; lq2 := rest lq2
    qrc := convert(q2)@PRC
    if not empty? lr
      then
        for r in reverse(lr) for v in reverse(lv) repeat
          qrc := eval(qrc,v,r)
    lv := cons((mainVariable(qrc) :: Symbol),lv)
```

```
      urc: URC := univariate(qrc)@URC
      urcRoots := allRootsOf(urc)$RC
      for urcRoot in urcRoots repeat
        if positive? urcRoot
          then
            toSee := cons([cons(urcRoot,lr),lv,lq2]$WIP, toSee)
    toSave

  positiveSolve(lp: List(P), info?:Boolean, lextri?: Boolean): List REALSOL  ==
    lts: List TS
    lq: List(Q) := [convert(p)$Q for p in lp]
    if lextri?
      then
        lts := zeroSetSplit(lq,false)$lextripack
      else
        lts := zeroSetSplit(lq,true,info?)$TS
    lsts:  List ST := []
    for ts in lts repeat
      lsts := concat(squareFree(ts), lsts)
    lsts := removeSuperfluousQuasiComponents(lsts)$quasicomppack
    lr: REALSOL := []
    lv: List Symbol := []
    toSee := [[lr,lv,convert(sts)@(List Q2)]$WIP for sts in lsts]
    toSave: List REALSOL := []
    while not empty? toSee repeat
      wip := first toSee; toSee := rest toSee
      lr := wip.reals; lv := wip.vars; lq2 := wip.pols
      (empty? lq2) and (not empty? lr) =>
         toSave := cons(reverse(lr),toSave)
      q2 := first lq2; lq2 := rest lq2
      qrc := convert(q2)@PRC
      if not empty? lr
        then
          for r in reverse(lr) for v in reverse(lv) repeat
            qrc := eval(qrc,v,r)
      lv := cons((mainVariable(qrc) :: Symbol),lv)
      urc: URC := univariate(qrc)@URC
      urcRoots := allRootsOf(urc)$RC
      for urcRoot in urcRoots repeat
        if positive? urcRoot
          then
            toSee := cons([cons(urcRoot,lr),lv,lq2]$WIP, toSee)
    toSave

  positiveSolve(lp: List(P), info?:Boolean): List REALSOL  ==
    positiveSolve(lp, info?, false)
```

```
positiveSolve(lp: List(P)): List REALSOL  ==
  positiveSolve(lp, false, false)

univariateSolve(ts: TS): List RUR ==
  toSee: List ST := squareFree(ts)
  toSave: List RUR := []
  for st in toSee repeat
    lus: List ST := rur(st,true)$rurpack
    for us in lus repeat
      g: U  := univariate(select(us,newv)::Q2)$Q2
      lc: LP := [convert(q2)@P for q2 in parts(collectUpper(us,newv)$ST)$ST]
      toSave := cons([g,lc]$RUR, toSave)
  toSave

univariateSolve(lp: List(P), info?:Boolean, check?:Boolean, lextri?: Boolean
  lts: List TS
  lq: List(Q) := [convert(p)$Q for p in lp]
  if lextri?
    then
      lts := zeroSetSplit(lq,false)$lextripack
    else
      lts := zeroSetSplit(lq,true,info?)$TS
  toSee:  List ST := []
  for ts in lts repeat
    toSee := concat(squareFree(ts), toSee)
  toSee := removeSuperfluousQuasiComponents(toSee)$quasicomppack
  toSave: List RUR := []
  if check?
    then
      lq2: List(Q2) := [convert(p)$Q2 for p in lp]
  for st in toSee repeat
    lus: List ST := rur(st,true)$rurpack
    for us in lus repeat
       if check?
         then
           rems: List(Q2) := [removeZero(q2,us)$ST for q2 in lq2]
           not every?(zero?,rems) =>
             output(st::OutputForm)$OutputPackage
             output("Has a bad RUR component:")$OutputPackage
             output(us::OutputForm)$OutputPackage
             error "univariateSolve$ZDSOLVE: bad RUR"
       g: U  := univariate(select(us,newv)::Q2)$Q2
       lc: LP := [convert(q2)@P for q2 in parts(collectUpper(us,newv)$ST)$ST
       toSave := cons([g,lc]$RUR, toSave)
  toSave
```

```
    univariateSolve(lp: List(P), info?:Boolean, check?:Boolean): List RUR ==
      univariateSolve(lp,info?,check?,false)

    univariateSolve(lp: List(P), info?:Boolean): List RUR ==
      univariateSolve(lp,info?,false,false)

    univariateSolve(lp: List(P)): List RUR ==
      univariateSolve(lp,false,false,false)
```

⟨*ZDSOLVE.dotabb*⟩≡

```
"ZDSOLVE" [color="#FF4488",href="bookvol10.4.pdf#nameddest=ZDSOLVE"]
"SFRTCAT" [color="#4488FF",href="bookvol10.2.pdf#nameddest=SFRTCAT"]
"ZDSOLVE" -> "SFRTCAT"
```

Chapter 28

Chunk collections

⟨*algebra*⟩≡
⟨*package AF AlgebraicFunction*⟩
⟨*package INTHERAL AlgebraicHermiteIntegration*⟩
⟨*package INTALG AlgebraicIntegrate*⟩
⟨*package INTAF AlgebraicIntegration*⟩
⟨*package ALGMANIP AlgebraicManipulations*⟩
⟨*package ALGMFACT AlgebraicMultFact*⟩
⟨*package ALGPKG AlgebraPackage*⟩
⟨*package ALGFACT AlgFactor*⟩
⟨*package INTPACK AnnaNumericalIntegrationPackage*⟩
⟨*package OPTPACK AnnaNumericalOptimizationPackage*⟩
⟨*package ODEPACK AnnaOrdinaryDifferentialEquationPackage*⟩
⟨*package PDEPACK AnnaPartialDifferentialEquationPackage*⟩
⟨*package ANY1 AnyFunctions1*⟩
⟨*package APPRULE ApplyRules*⟩
⟨*package APPLYORE ApplyUnivariateSkewPolynomial*⟩
⟨*package ASSOCEQ AssociatedEquations*⟩
⟨*package PMPRED AttachPredicates*⟩
⟨*package AXSERV AxiomServer*⟩

⟨*package BALFACT BalancedFactorisation*⟩
⟨*package BOP1 BasicOperatorFunctions1*⟩
⟨*package BEZOUT BezoutMatrix*⟩
⟨*package BOUNDZRO BoundIntegerRoots*⟩
⟨*package BRILL BrillhartTests*⟩

⟨*package CARTEN2 CartesianTensorFunctions2*⟩
⟨*package CHVAR ChangeOfVariable*⟩
⟨*package CPIMA CharacteristicPolynomialInMonogenicalAlgebra*⟩
⟨*package CHARPOL CharacteristicPolynomialPackage*⟩

⟨*package IBACHIN ChineseRemainderToolsForIntegralBases*⟩
⟨*package CVMP CoerceVectorMatrixPackage*⟩
⟨*package COMBF CombinatorialFunction*⟩
⟨*package CDEN CommonDenominator*⟩
⟨*package COMMONOP CommonOperators*⟩
⟨*package COMMUPC CommuteUnivariatePolynomialCategory*⟩
⟨*package COMPFACT ComplexFactorization*⟩
⟨*package COMPLEX2 ComplexFunctions2*⟩
⟨*package CINTSLPE ComplexIntegerSolveLinearPolynomialEquation*⟩
⟨*package COMPLPAT ComplexPattern*⟩
⟨*package CPMATCH ComplexPatternMatch*⟩
⟨*package CRFP ComplexRootFindingPackage*⟩
⟨*package CMPLXRT ComplexRootPackage*⟩
⟨*package CTRIGMNP ComplexTrigonometricManipulations*⟩
⟨*package ODECONST ConstantLODE*⟩
⟨*package COORDSYS CoordinateSystems*⟩
⟨*package CRAPACK CRApackage*⟩
⟨*package CYCLES CycleIndicators*⟩
⟨*package CSTTOOLS CyclicStreamTools*⟩
⟨*package CYCLOTOM CyclotomicPolynomialPackage*⟩

⟨*package DFINTTLS DefiniteIntegrationTools*⟩
⟨*package DEGRED DegreeReductionPackage*⟩
⟨*package DIOSP DiophantineSolutionPackage*⟩
⟨*package DIRPROD2 DirectProductFunctions2*⟩
⟨*package DLP DiscreteLogarithmPackage*⟩
⟨*package DISPLAY DisplayPackage*⟩
⟨*package DDFACT DistinctDegreeFactorize*⟩
⟨*package DFSFUN DoubleFloatSpecialFunctions*⟩
⟨*package DBLRESP DoubleResultantPackage*⟩
⟨*package DRAWCX DrawComplex*⟩
⟨*package DRAWHACK DrawNumericHack*⟩
⟨*package DROPT0 DrawOptionFunctions0*⟩
⟨*package DROPT1 DrawOptionFunctions1*⟩
⟨*package D01AGNT d01AgentsPackage*⟩
⟨*package D01WGTS d01WeightsPackage*⟩
⟨*package D02AGNT d02AgentsPackage*⟩
⟨*package D03AGNT d03AgentsPackage*⟩

⟨*package EP EigenPackage*⟩
⟨*package EF ElementaryFunction*⟩
⟨*package DEFINTEF ElementaryFunctionDefiniteIntegration*⟩
⟨*package LODEEF ElementaryFunctionLODESolver*⟩
⟨*package ODEEF ElementaryFunctionODESolver*⟩
⟨*package SIGNEF ElementaryFunctionSign*⟩
⟨*package EFSTRUC ElementaryFunctionStructurePackage*⟩

⟨*package EFULS ElementaryFunctionsUnivariateLaurentSeries*⟩
⟨*package EFUPXS ElementaryFunctionsUnivariatePuiseuxSeries*⟩
⟨*package INTEF ElementaryIntegration*⟩
⟨*package RDEEF ElementaryRischDE*⟩
⟨*package RDEEFS ElementaryRischDESystem*⟩
⟨*package ELFUTS EllipticFunctionsUnivariateTaylorSeries*⟩
⟨*package EQ2 EquationFunctions2*⟩
⟨*package ERROR ErrorFunctions*⟩
⟨*package GBEUCLID EuclideanGroebnerBasisPackage*⟩
⟨*package EVALCYC EvaluateCycleIndicators*⟩
⟨*package ESCONT ExpertSystemContinuityPackage*⟩
⟨*package ESCONT1 ExpertSystemContinuityPackage1*⟩
⟨*package ESTOOLS ExpertSystemToolsPackage*⟩
⟨*package ESTOOLS1 ExpertSystemToolsPackage1*⟩
⟨*package ESTOOLS2 ExpertSystemToolsPackage2*⟩
⟨*package EXPR2 ExpressionFunctions2*⟩
⟨*package EXPRSOL ExpressionSolve*⟩
⟨*package ES1 ExpressionSpaceFunctions1*⟩
⟨*package ES2 ExpressionSpaceFunctions2*⟩
⟨*package EXPRODE ExpressionSpaceODESolver*⟩
⟨*package OMEXPR ExpressionToOpenMath*⟩
⟨*package EXPR2UPS ExpressionToUnivariatePowerSeries*⟩
⟨*package EXPRTUBE ExpressionTubePlot*⟩
⟨*package E04AGNT e04AgentsPackage*⟩

⟨*package FACTFUNC FactoredFunctions*⟩
⟨*package FR2 FactoredFunctions2*⟩
⟨*package FRUTIL FactoredFunctionUtilities*⟩
⟨*package FACUTIL FactoringUtilities*⟩
⟨*package FGLMICPK FGLMIfCanPackage*⟩
⟨*package FORDER FindOrderFinite*⟩
⟨*package FAMR2 FiniteAbelianMonoidRingFunctions2*⟩
⟨*package FDIV2 FiniteDivisorFunctions2*⟩
⟨*package FFF FiniteFieldFunctions*⟩
⟨*package FFHOM FiniteFieldHomomorphisms*⟩
⟨*package FFPOLY FiniteFieldPolynomialPackage*⟩
⟨*package FFPOLY2 FiniteFieldPolynomialPackage2*⟩
⟨*package FFSLPE FiniteFieldSolveLinearPolynomialEquation*⟩
⟨*package FLAGG2 FiniteLinearAggregateFunctions2*⟩
⟨*package FLASORT FiniteLinearAggregateSort*⟩
⟨*package FSAGG2 FiniteSetAggregateFunctions2*⟩
⟨*package FLOATCP FloatingComplexPackage*⟩
⟨*package FLOATRP FloatingRealPackage*⟩
⟨*package FCPAK1 FortranCodePackage1*⟩
⟨*package FOP FortranOutputStackPackage*⟩
⟨*package FORT FortranPackage*⟩

⟨*package FRIDEAL2 FractionalIdealFunctions2*⟩
⟨*package FFFG FractionFreeFastGaussian*⟩
⟨*package FFFGF FractionFreeFastGaussianFractions*⟩
⟨*package FRAC2 FractionFunctions2*⟩
⟨*package FRNAAF2 FramedNonAssociativeAlgebraFunctions2*⟩
⟨*package FSPECF FunctionalSpecialFunction*⟩
⟨*package FFCAT2 FunctionFieldCategoryFunctions2*⟩
⟨*package FFINTBAS FunctionFieldIntegralBasis*⟩
⟨*package PMASSFS FunctionSpaceAssertions*⟩
⟨*package PMPREDFS FunctionSpaceAttachPredicates*⟩
⟨*package FSCINT FunctionSpaceComplexIntegration*⟩
⟨*package FS2 FunctionSpaceFunctions2*⟩
⟨*package FSINT FunctionSpaceIntegration*⟩
⟨*package FSPRMELT FunctionSpacePrimitiveElement*⟩
⟨*package FSRED FunctionSpaceReduce*⟩
⟨*package SUMFS FunctionSpaceSum*⟩
⟨*package FS2EXPXP FunctionSpaceToExponentialExpansion*⟩
⟨*package FS2UPS FunctionSpaceToUnivariatePowerSeries*⟩
⟨*package FSUPFACT FunctionSpaceUnivariatePolynomialFactor*⟩

⟨*package GALFACTU GaloisGroupFactorizationUtilities*⟩
⟨*package GALFACT GaloisGroupFactorizer*⟩
⟨*package GALPOLYU GaloisGroupPolynomialUtilities*⟩
⟨*package GALUTIL GaloisGroupUtilities*⟩
⟨*package GAUSSFAC GaussianFactorizationPackage*⟩
⟨*package GHENSEL GeneralHenselPackage*⟩
⟨*package GENMFACT GeneralizedMultivariateFactorize*⟩
⟨*package GENPGCD GeneralPolynomialGcdPackage*⟩
⟨*package GENUPS GenerateUnivariatePowerSeries*⟩
⟨*package GENEEZ GenExEuclid*⟩
⟨*package GENUFACT GenUFactorize*⟩
⟨*package INTG0 GenusZeroIntegration*⟩
⟨*package GOSPER GosperSummationMethod*⟩
⟨*package GRDEF GraphicsDefaults*⟩
⟨*package GRAY GrayCode*⟩
⟨*package GBF GroebnerFactorizationPackage*⟩
⟨*package GBINTERN GroebnerInternalPackage*⟩
⟨*package GB GroebnerPackage*⟩
⟨*package GROEBSOL GroebnerSolve*⟩
⟨*package GUESS Guess*⟩
⟨*package GUESSAN GuessAlgebraicNumber*⟩
⟨*package GUESSF GuessFinite*⟩
⟨*package GUESSF1 GuessFiniteFunctions*⟩
⟨*package GUESSINT GuessInteger*⟩
⟨*package GOPT0 GuessOptionFunctions0*⟩
⟨*package GUESSP GuessPolynomial*⟩

⟨*package GUESSUP GuessUnivariatePolynomial*⟩

⟨*package HB HallBasis*⟩
⟨*package HEUGCD HeuGcd*⟩

⟨*package IDECOMP IdealDecompositionPackage*⟩
⟨*package INCRMAPS IncrementingMaps*⟩
⟨*package INFPROD0 InfiniteProductCharacteristicZero*⟩
⟨*package INPRODFF InfiniteProductFiniteField*⟩
⟨*package INPRODPF InfiniteProductPrimeField*⟩
⟨*package ITFUN2 InfiniteTupleFunctions2*⟩
⟨*package ITFUN3 InfiniteTupleFunctions3*⟩
⟨*package INFINITY Infinity*⟩
⟨*package IALGFACT InnerAlgFactor*⟩
⟨*package ICDEN InnerCommonDenominator*⟩
⟨*package IMATLIN InnerMatrixLinearAlgebraFunctions*⟩
⟨*package IMATQF InnerMatrixQuotientFieldFunctions*⟩
⟨*package INMODGCD InnerModularGcd*⟩
⟨*package INNMFACT InnerMultFact*⟩
⟨*package INBFF InnerNormalBasisFieldFunctions*⟩
⟨*package INEP InnerNumericEigenPackage*⟩
⟨*package INFSP InnerNumericFloatSolvePackage*⟩
⟨*package INPSIGN InnerPolySign*⟩
⟨*package ISUMP InnerPolySum*⟩
⟨*package ITRIGMNP InnerTrigonometricManipulations*⟩
⟨*package INFORM1 InputFormFunctions1*⟩
⟨*package INTBIT IntegerBits*⟩
⟨*package COMBINAT IntegerCombinatoricFunctions*⟩
⟨*package INTFACT IntegerFactorizationPackage*⟩
⟨*package ZLINDEP IntegerLinearDependence*⟩
⟨*package INTHEORY IntegerNumberTheoryFunctions*⟩
⟨*package PRIMES IntegerPrimesPackage*⟩
⟨*package INTRET IntegerRetractions*⟩
⟨*package IROOT IntegerRoots*⟩
⟨*package INTSLPE IntegerSolveLinearPolynomialEquation*⟩
⟨*package IBATOOL IntegralBasisTools*⟩
⟨*package IBPTOOLS IntegralBasisPolynomialTools*⟩
⟨*package IR2 IntegrationResultFunctions2*⟩
⟨*package IRRF2F IntegrationResultRFToFunction*⟩
⟨*package IR2F IntegrationResultToFunction*⟩
⟨*package INTTOOLS IntegrationTools*⟩
⟨*package IPRNTPK InternalPrintPackage*⟩
⟨*package IRURPK InternalRationalUnivariateRepresentationPackage*⟩
⟨*package IRREDFFX IrredPolyOverFiniteField*⟩
⟨*package IRSN IrrRepSymNatPackage*⟩
⟨*package INVLAPLA InverseLaplaceTransform*⟩

⟨*package KERNEL2 KernelFunctions2*⟩
⟨*package KOVACIC Kovacic*⟩

⟨*package LAPLACE LaplaceTransform*⟩
⟨*package LAZM3PK LazardSetSolvingPackage*⟩
⟨*package LEADCDET LeadingCoefDetermination*⟩
⟨*package LEXTRIPK LexTriangularPackage*⟩
⟨*package LINDEP LinearDependence*⟩
⟨*package LODOF LinearOrdinaryDifferentialOperatorFactorizer*⟩
⟨*package LODOOPS LinearOrdinaryDifferentialOperatorsOps*⟩
⟨*package LPEFRAC LinearPolynomialEquationByFractions*⟩
⟨*package LSMP LinearSystemMatrixPackage*⟩
⟨*package LSMP1 LinearSystemMatrixPackage1*⟩
⟨*package LSPP LinearSystemPolynomialPackage*⟩
⟨*package LGROBP LinGroebnerPackage*⟩
⟨*package LF LiouvillianFunction*⟩
⟨*package LIST2 ListFunctions2*⟩
⟨*package LIST3 ListFunctions3*⟩
⟨*package LIST2MAP ListToMap*⟩

⟨*package MKBCFUNC MakeBinaryCompiledFunction*⟩
⟨*package MKFLCFN MakeFloatCompiledFunction*⟩
⟨*package MKFUNC MakeFunction*⟩
⟨*package MKRECORD MakeRecord*⟩
⟨*package MKUCFUNC MakeUnaryCompiledFunction*⟩
⟨*package MAPHACK1 MappingPackageInternalHacks1*⟩
⟨*package MAPHACK2 MappingPackageInternalHacks2*⟩
⟨*package MAPHACK3 MappingPackageInternalHacks3*⟩
⟨*package MAPPKG1 MappingPackage1*⟩
⟨*package MAPPKG2 MappingPackage2*⟩
⟨*package MAPPKG3 MappingPackage3*⟩
⟨*package MAPPKG4 MappingPackage4*⟩
⟨*package MMLFORM MathMLForm*⟩
⟨*package MATCAT2 MatrixCategoryFunctions2*⟩
⟨*package MCDEN MatrixCommonDenominator*⟩
⟨*package MATLIN MatrixLinearAlgebraFunctions*⟩
⟨*package MTHING MergeThing*⟩
⟨*package MESH MeshCreationRoutinesForThreeDimensions*⟩
⟨*package MDDFACT ModularDistinctDegreeFactorizer*⟩
⟨*package MHROWRED ModularHermitianRowReduction*⟩
⟨*package MRF2 MonoidRingFunctions2*⟩
⟨*package MONOTOOL MonomialExtensionTools*⟩
⟨*package MSYSCMD MoreSystemCommands*⟩
⟨*package MPCPF MPolyCatPolyFactorizer*⟩
⟨*package MPRFF MPolyCatRationalFunctionFactorizer*⟩

⟨*package MPC2 MPolyCatFunctions2*⟩
⟨*package MPC3 MPolyCatFunctions3*⟩
⟨*package MRATFAC MRationalFactorize*⟩
⟨*package MFINFACT MultFiniteFactorize*⟩
⟨*package MMAP MultipleMap*⟩
⟨*package MCALCFN MultiVariableCalculusFunctions*⟩
⟨*package MULTFACT MultivariateFactorize*⟩
⟨*package MLIFT MultivariateLifting*⟩
⟨*package MULTSQFR MultivariateSquareFree*⟩

⟨*package NAGF02 NagEigenPackage*⟩
⟨*package NAGE02 NagFittingPackage*⟩
⟨*package NAGF04 NagLinearEquationSolvingPackage*⟩
⟨*package NAGSP NAGLinkSupportPackage*⟩
⟨*package NAGD01 NagIntegrationPackage*⟩
⟨*package NAGE01 NagInterpolationPackage*⟩
⟨*package NAGF07 NagLapack*⟩
⟨*package NAGF01 NagMatrixOperationsPackage*⟩
⟨*package NAGE04 NagOptimisationPackage*⟩
⟨*package NAGD02 NagOrdinaryDifferentialEquationsPackage*⟩
⟨*package NAGD03 NagPartialDifferentialEquationsPackage*⟩
⟨*package NAGC02 NagPolynomialRootsPackage*⟩
⟨*package NAGC05 NagRootFindingPackage*⟩
⟨*package NAGC06 NagSeriesSummationPackage*⟩
⟨*package NAGS NagSpecialFunctionsPackage*⟩
⟨*package NSUP2 NewSparseUnivariatePolynomialFunctions2*⟩
⟨*package NEWTON NewtonInterpolation*⟩
⟨*package NCODIV NonCommutativeOperatorDivision*⟩
⟨*package NONE1 NoneFunctions1*⟩
⟨*package NODE1 NonLinearFirstOrderODESolver*⟩
⟨*package NLINSOL NonLinearSolvePackage*⟩
⟨*package NORMPK NormalizationPackage*⟩
⟨*package NORMMA NormInMonogenicAlgebra*⟩
⟨*package NORMRETR NormRetractPackage*⟩
⟨*package NPCOEF NPCoef*⟩
⟨*package NFINTBAS NumberFieldIntegralBasis*⟩
⟨*package NUMFMT NumberFormats*⟩
⟨*package NTPOLFN NumberTheoreticPolynomialFunctions*⟩
⟨*package NUMERIC Numeric*⟩
⟨*package NUMODE NumericalOrdinaryDifferentialEquations*⟩
⟨*package NUMQUAD NumericalQuadrature*⟩
⟨*package NCEP NumericComplexEigenPackage*⟩
⟨*package NCNTFRAC NumericContinuedFraction*⟩
⟨*package NREP NumericRealEigenPackage*⟩
⟨*package NUMTUBE NumericTubePlot*⟩

⟨*package OCTCT2 OctonionCategoryFunctions2*⟩
⟨*package ODEINT ODEIntegration*⟩
⟨*package ODETOOLS ODETools*⟩
⟨*package ARRAY12 OneDimensionalArrayFunctions2*⟩
⟨*package ONECOMP2 OnePointCompletionFunctions2*⟩
⟨*package OMPKG OpenMathPackage*⟩
⟨*package OMSERVER OpenMathServerPackage*⟩
⟨*package OPQUERY OperationsQuery*⟩
⟨*package ORDCOMP2 OrderedCompletionFunctions2*⟩
⟨*package ORDFUNS OrderingFunctions*⟩
⟨*package ORTHPOL OrthogonalPolynomialFunctions*⟩
⟨*package OUT OutputPackage*⟩

⟨*package PADEPAC PadeApproximantPackage*⟩
⟨*package PADE PadeApproximants*⟩
⟨*package PWFFINTB PAdicWildFunctionFieldIntegralBasis*⟩
⟨*package YSTREAM ParadoxicalCombinatorsForStreams*⟩
⟨*package PLEQN ParametricLinearEquations*⟩
⟨*package PARPC2 ParametricPlaneCurveFunctions2*⟩
⟨*package PARSC2 ParametricSpaceCurveFunctions2*⟩
⟨*package PARSU2 ParametricSurfaceFunctions2*⟩
⟨*package PFRPAC PartialFractionPackage*⟩
⟨*package PARTPERM PartitionsAndPermutations*⟩
⟨*package PATTERN1 PatternFunctions1*⟩
⟨*package PATTERN2 PatternFunctions2*⟩
⟨*package PATMATCH PatternMatch*⟩
⟨*package PMASS PatternMatchAssertions*⟩
⟨*package PMFS PatternMatchFunctionSpace*⟩
⟨*package PMINS PatternMatchIntegerNumberSystem*⟩
⟨*package INTPM PatternMatchIntegration*⟩
⟨*package PMKERNEL PatternMatchKernel*⟩
⟨*package PMLSAGG PatternMatchListAggregate*⟩
⟨*package PMPLCAT PatternMatchPolynomialCategory*⟩
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Chapter 29

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