

Weak Cardinality Theorems for First-Order Logic

Till Tantau

Fakultät für Elektrotechnik und Informatik
Technische Universität Berlin

Fundamentals of Computation Theory 2003



Outline

- 1 History
 - Enumerability in Recursion and Automata Theory
 - Known Weak Cardinality Theorem
 - Why Do Cardinality Theorems Hold Only for Certain Models?
- 2 Unification by First-Order Logic
 - Elementary Definitions
 - Enumerability for First-Order Logic
 - Weak Cardinality Theorems for First-Order Logic
- 3 Applications
 - A Separability Result for First-Order Logic



Outline

- 1 History
 - Enumerability in Recursion and Automata Theory
 - Known Weak Cardinality Theorem
 - Why Do Cardinality Theorems Hold Only for Certain Models?
- 2 Unification by First-Order Logic
 - Elementary Definitions
 - Enumerability for First-Order Logic
 - Weak Cardinality Theorems for First-Order Logic
- 3 Applications
 - A Separability Result for First-Order Logic



Motivation of Enumerability

Problem

Many functions are not computable or not efficiently computable.

Example

- #SAT:
How many satisfying assignments does a formula have?

Motivation of Enumerability

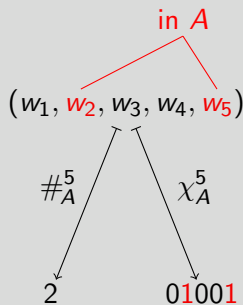
Problem

Many functions are not computable or not efficiently computable.

Example

For difficult languages A :

- Cardinality function $\#_A^n$:
How many input words are in A ?
- Characteristic function χ_A^n :
Which input words are in A ?



Motivation of Enumerability

Problem

Many functions are not computable or not efficiently computable.

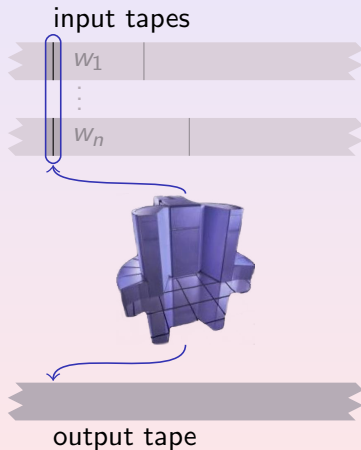
Solutions

Difficult functions can be

- computed using probabilistic algorithms,
- computed efficiently on average,
- approximated, or
- **enumerated.**



Enumerators Output Sets of Possible Function Values



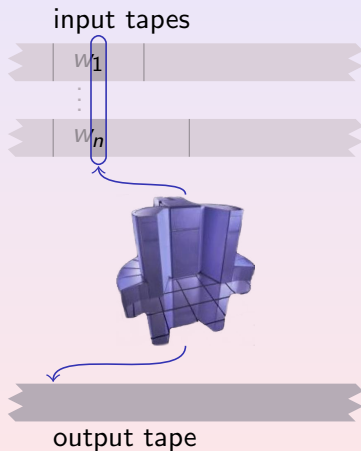
Definition (1987, 1989, 1994, 2001)

An ***m*-enumerator** for a function f

- ① reads n input words w_1, \dots, w_n ,
- ② does a computation,
- ③ outputs at most m values,
- ④ one of which is $f(w_1, \dots, w_n)$.



Enumerators Output Sets of Possible Function Values



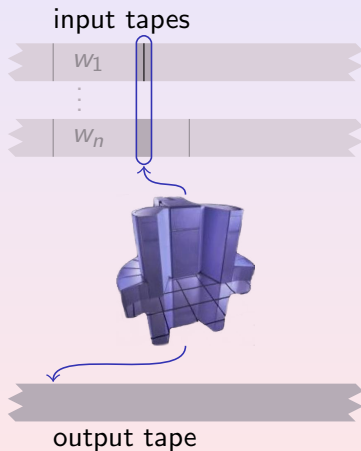
Definition (1987, 1989, 1994, 2001)

An ***m*-enumerator** for a function f

- ① reads n input words w_1, \dots, w_n ,
- ② does a computation,
- ③ outputs at most m values,
- ④ one of which is $f(w_1, \dots, w_n)$.



Enumerators Output Sets of Possible Function Values



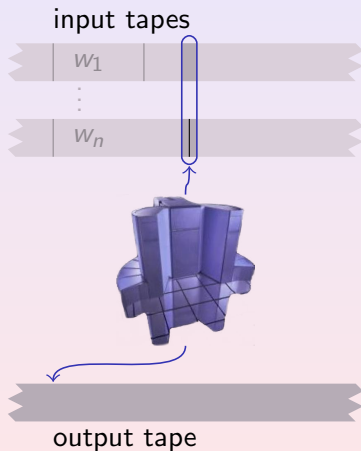
Definition (1987, 1989, 1994, 2001)

An ***m*-enumerator** for a function f

- ① reads n input words w_1, \dots, w_n ,
- ② does a computation,
- ③ outputs at most m values,
- ④ one of which is $f(w_1, \dots, w_n)$.



Enumerators Output Sets of Possible Function Values



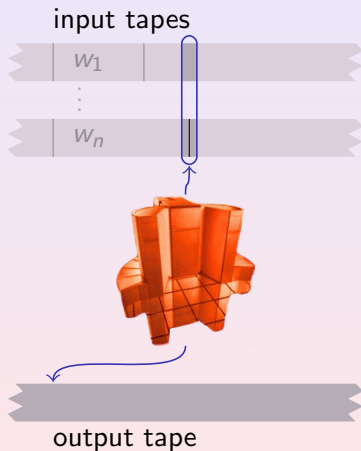
Definition (1987, 1989, 1994, 2001)

An ***m*-enumerator** for a function f

- ① reads n input words w_1, \dots, w_n ,
- ② does a computation,
- ③ outputs at most m values,
- ④ one of which is $f(w_1, \dots, w_n)$.



Enumerators Output Sets of Possible Function Values



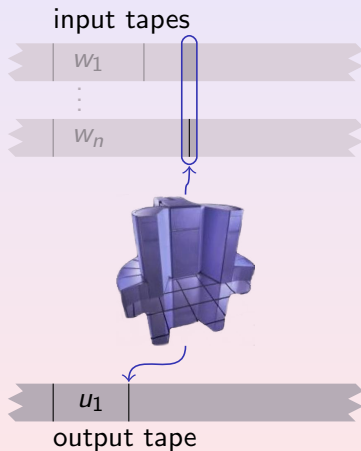
Definition (1987, 1989, 1994, 2001)

An ***m*-enumerator** for a function f

- ① reads n input words w_1, \dots, w_n ,
- ② **does a computation**,
- ③ outputs at most m values,
- ④ one of which is $f(w_1, \dots, w_n)$.



Enumerators Output Sets of Possible Function Values



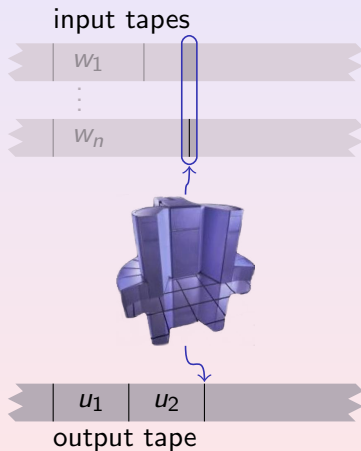
Definition (1987, 1989, 1994, 2001)

An ***m*-enumerator** for a function f

- 1 reads n input words w_1, \dots, w_n ,
- 2 does a computation,
- 3 **outputs at most m values,**
- 4 one of which is $f(w_1, \dots, w_n)$.



Enumerators Output Sets of Possible Function Values



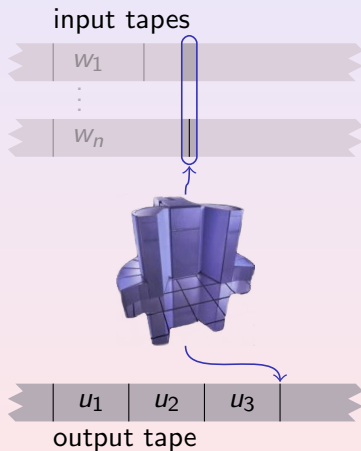
Definition (1987, 1989, 1994, 2001)

An ***m*-enumerator** for a function f

- ① reads n input words w_1, \dots, w_n ,
- ② does a computation,
- ③ **outputs at most m values,**
- ④ one of which is $f(w_1, \dots, w_n)$.



Enumerators Output Sets of Possible Function Values



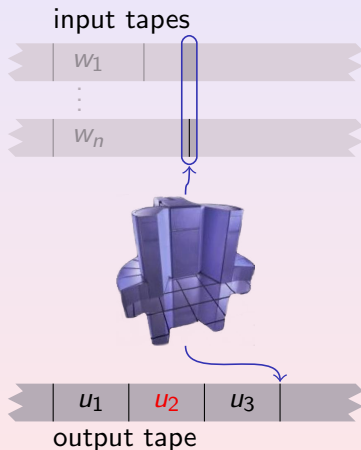
Definition (1987, 1989, 1994, 2001)

An ***m*-enumerator** for a function f

- ① reads n input words w_1, \dots, w_n ,
- ② does a computation,
- ③ **outputs at most m values,**
- ④ one of which is $f(w_1, \dots, w_n)$.



Enumerators Output Sets of Possible Function Values



Definition (1987, 1989, 1994, 2001)

An ***m*-enumerator** for a function f

- ① reads n input words w_1, \dots, w_n ,
- ② does a computation,
- ③ outputs at most m values,
- ④ one of which is $f(w_1, \dots, w_n)$.



How Well Can the Cardinality Function Be Enumerated?

Observation

For fixed n , the cardinality function $\#_A^n$

- can be 1-enumerated by Turing machines only for recursive A , but
- can be $(n + 1)$ -enumerated for every language A .

Question

What about 2-, 3-, 4-, ..., n -enumerability?



How Well Can the Cardinality Function Be Enumerated?

Observation

For fixed n , the cardinality function $\#_A^n$

- can be 1-enumerated by Turing machines only for recursive A , but
- can be $(n + 1)$ -enumerated for every language A .

Question

What about 2-, 3-, 4-, ..., n -enumerability?



How Well Can the Cardinality Function Be Enumerated by Turing Machines?

Cardinality Theorem (Kummer, 1992)

If $\#_A^n$ is n -enumerable by a Turing machine, then A is recursive.

Weak Cardinality Theorems ()

- 1 If χ_A^n is n -enumerable by a Turing machine, then A is recursive.
- 2 If $\#_A^2$ is 2-enumerable by a Turing machine, then A is recursive.
- 3 If $\#_A^n$ is n -enumerable by a Turing machine that never enumerates both 1 and n , then A is recursive.



How Well Can the Cardinality Function Be Enumerated by Turing Machines?

Cardinality Theorem (Kummer, 1992)

If $\#_A^n$ is n -enumerable by a Turing machine, then A is recursive.

Weak Cardinality Theorems (1987, 1989, 1992)

- 1 *If χ_A^n is n -enumerable by a Turing machine, then A is recursive.*
- 2 *If $\#_A^2$ is 2-enumerable by a Turing machine, then A is recursive.*
- 3 *If $\#_A^n$ is n -enumerable by a Turing machine that never enumerates both 0 and n , then A is recursive.*



How Well Can the Cardinality Function Be Enumerated by Turing Machines?

Cardinality Theorem (Kummer, 1992)

If $\#_A^n$ is n -enumerable by a Turing machine, then A is recursive.

Weak Cardinality Theorems (1987, 1989, 1992)

- 1 *If χ_A^n is n -enumerable by a Turing machine, then A is recursive.*
- 2 *If $\#_A^2$ is 2-enumerable by a Turing machine, then A is recursive.*
- 3 *If $\#_A^n$ is n -enumerable by a Turing machine that never enumerates both 0 and n , then A is recursive.*



How Well Can the Cardinality Function Be Enumerated by Turing Machines?

Cardinality Theorem (Kummer, 1992)

If $\#_A^n$ is n -enumerable by a Turing machine, then A is recursive.

Weak Cardinality Theorems (1987, 1989, 1992)

- 1 *If χ_A^n is n -enumerable by a Turing machine, then A is recursive.*
- 2 *If $\#_A^2$ is 2-enumerable by a Turing machine, then A is recursive.*
- 3 *If $\#_A^n$ is n -enumerable by a Turing machine that never enumerates both 0 and n , then A is recursive.*

How Well Can the Cardinality Function Be Enumerated by Finite Automata?

Conjecture

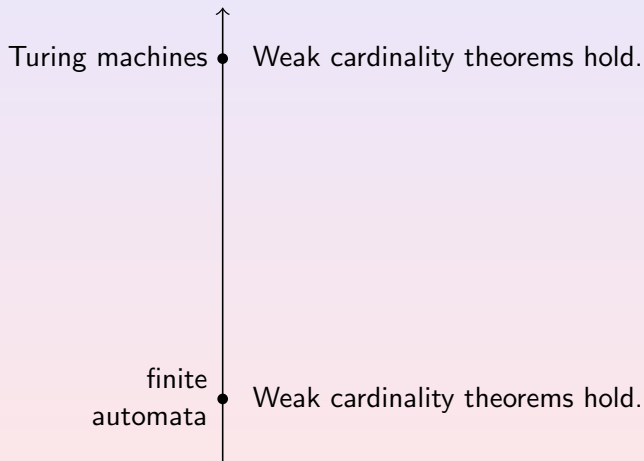
If $\#_A^n$ is n -enumerable by a **finite automaton**, then A is **regular**.

Weak Cardinality Theorems (2001, 2002)

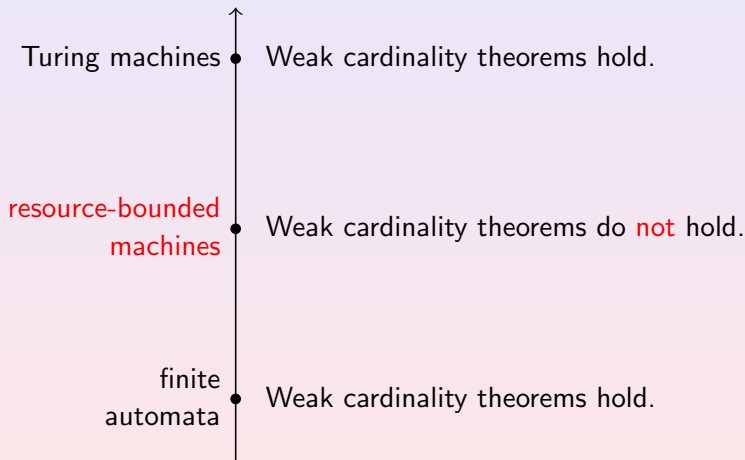
- 1 If χ_A^n is n -enumerable by a **finite automaton**, then A is **regular**.
- 2 If $\#_A^2$ is 2-enumerable by a **finite automaton**, then A is **regular**.
- 3 If $\#_A^n$ is n -enumerable by a **finite automaton** that never enumerates both 0 and n , then A is **regular**.



Cardinality Theorems Do Not Hold for All Models



Cardinality Theorems Do Not Hold for All Models



Why?

First Explanation

The weak cardinality theorems hold both for recursion and automata theory **by coincidence**.

Second Explanation

The weak cardinality theorems hold both for recursion and automata theory, **because they are instantiations of single, unifying theorems**.



Why?

First Explanation

The weak cardinality theorems hold both for recursion and automata theory **by coincidence**.

Second Explanation

The weak cardinality theorems hold both for recursion and automata theory, **because they are instantiations of single, unifying theorems**.

The second explanation is correct.

The theorems can (almost) be unified using first-order logic.



Outline

- 1 History
 - Enumerability in Recursion and Automata Theory
 - Known Weak Cardinality Theorem
 - Why Do Cardinality Theorems Hold Only for Certain Models?
- 2 Unification by First-Order Logic
 - Elementary Definitions
 - Enumerability for First-Order Logic
 - Weak Cardinality Theorems for First-Order Logic
- 3 Applications
 - A Separability Result for First-Order Logic



What Are Elementary Definitions?

Definition

A relation R is **elementarily definable in a logical structure \mathcal{S}** if

- 1 there exists a first-order formula ϕ ,
- 2 that is true exactly for the elements of R .

Example

The set of even numbers is elementarily definable in $(\mathbb{N}, +)$ via the formula $\phi(x) \equiv \exists z. z + z = x$.

Example

The set of powers of 2 is not elementarily definable in $(\mathbb{N}, +)$.



Characterisation of Classes by Elementary Definitions

Theorem (Büchi, 1960)

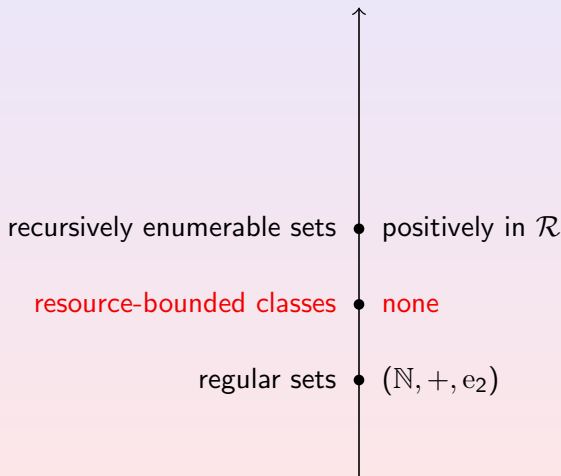
*There exists a logical structure $(\mathbb{N}, +, e_2)$ such that a set $A \subseteq \mathbb{N}$ is **regular** iff it is **elementarily definable** in $(\mathbb{N}, +, e_2)$.*

Theorem

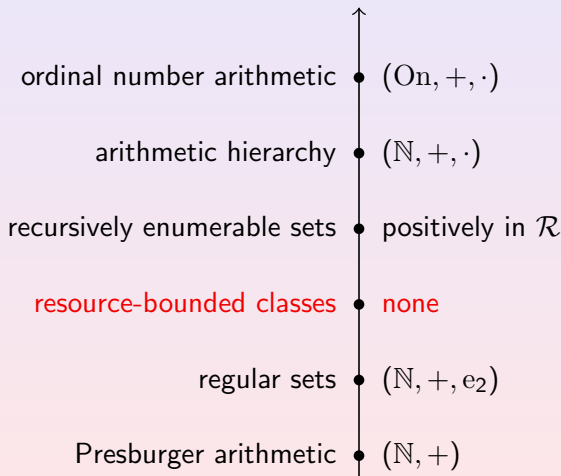
*There exists a logical structure \mathcal{R} such that a set $A \subseteq \mathbb{N}$ is **recursively enumerable** iff it is **positively elementarily definable** in \mathcal{R} .*



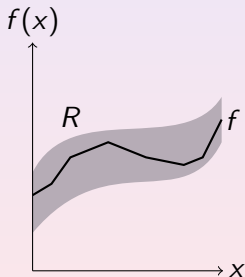
Characterisation of Classes by Elementary Definitions



Characterisation of Classes by Elementary Definitions



Elementary Enumerability is a Generalisation of Elementary Definability



Definition

A function f is

elementarily m -enumerable in a structure \mathcal{S} if

- ① its graph is contained in an **elementarily definable** relation R ,
- ② which is **m -bounded**, i.e., for each x there are at most m different y with $(x, y) \in R$.



The Original Notions of Enumerability are Instantiations

Theorem

*A function is m -enumerable by a **finite automaton** iff it is elementarily m -enumerable in $(\mathbb{N}, +, e_2)$.*

Theorem

*A function is m -enumerable by a **Turing machine** iff it is positively elementarily m -enumerable in \mathcal{R} .*



The First Weak Cardinality Theorem

Theorem

Let S be a logical structure with universe U and let $A \subseteq U$. If

- ① S is well-orderable and
- ② χ_A^n is elementarily n -enumerable in S ,

then A is elementarily definable in S .



The First Weak Cardinality Theorem

Theorem

Let S be a logical structure with universe U and let $A \subseteq U$. If

- 1 S is well-orderable and
- 2 χ_A^n is elementarily n -enumerable in S ,

then A is elementarily definable in S .

Corollary

If χ_A^n is n -enumerable by a finite automaton, then A is regular.



The First Weak Cardinality Theorem

Theorem

Let S be a logical structure with universe U and let $A \subseteq U$. If

- ① S is well-orderable and
- ② χ_A^n is elementarily n -enumerable in S ,

then A is elementarily definable in S .

Corollary (with more effort)

If χ_A^n is n -enumerable by a Turing machine, then A is recursive.



The Second Weak Cardinality Theorem

Theorem

Let S be a logical structure with universe U and let $A \subseteq U$. If

- 1 S is well-orderable,
- 2 every finite relation on U is elementarily definable in S , and
- 3 $\#_A^2$ is elementarily 2-enumerable in S ,

then A is elementarily definable in S .



The Third Weak Cardinality Theorem

Theorem

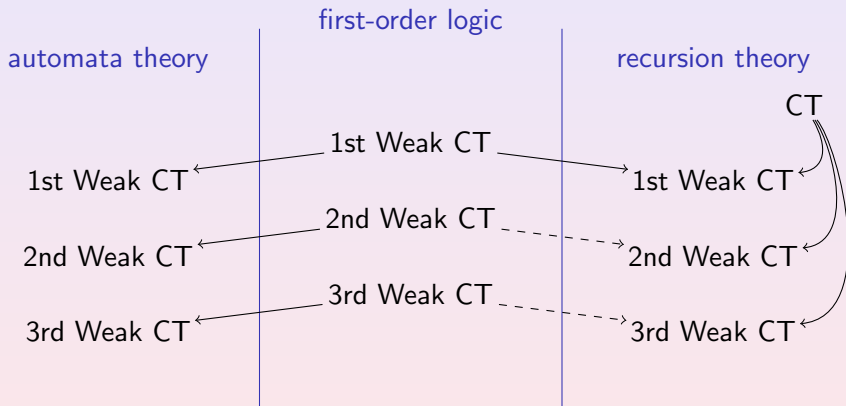
Let S be a logical structure with universe U and let $A \subseteq U$. If

- ① S is well-orderable,
- ② every finite relation on U is elementarily definable in S , and
- ③ $\#_A^n$ is elementarily n -enumerable in S via a relation that *never* 'enumerates' both 0 and n ,

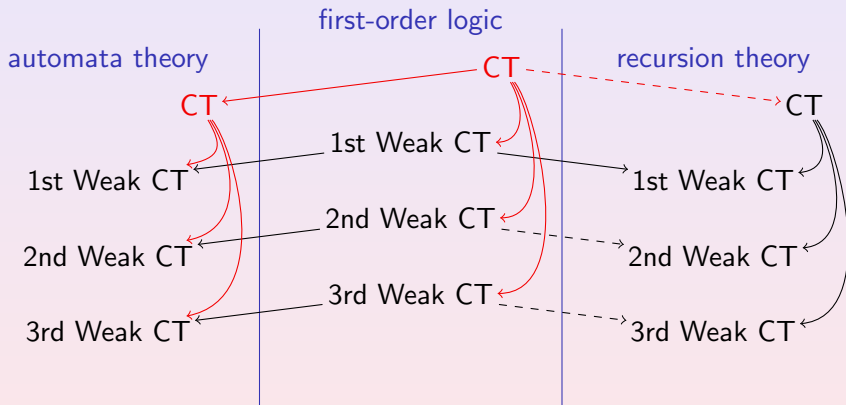
then A is *elementarily definable* in S .



Relationships Between Cardinality Theorems (CT)



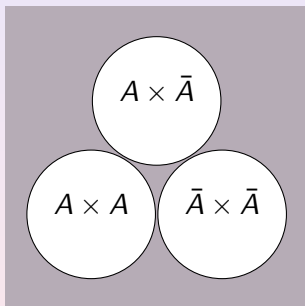
Relationships Between Cardinality Theorems (CT)



Outline

- 1 History
 - Enumerability in Recursion and Automata Theory
 - Known Weak Cardinality Theorem
 - Why Do Cardinality Theorems Hold Only for Certain Models?
- 2 Unification by First-Order Logic
 - Elementary Definitions
 - Enumerability for First-Order Logic
 - Weak Cardinality Theorems for First-Order Logic
- 3 Applications
 - A Separability Result for First-Order Logic





Theorem

Let S be a well-orderable logical structure in which all finite relations are elementarily definable.

If there exist elementarily definable supersets of $A \times A$, $A \times \bar{A}$, and $\bar{A} \times \bar{A}$ whose intersection is empty, then A is elementarily definable in S .

Note

The theorem is no longer true if we add $\bar{A} \times A$ to the list.



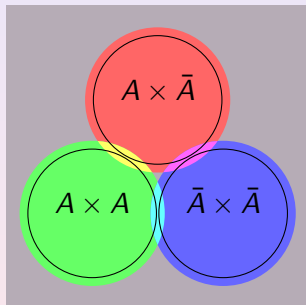
Theorem

Let S be a well-orderable logical structure in which all finite relations are elementarily definable.

If there exist elementarily definable supersets of $A \times A$, $A \times \bar{A}$, and $\bar{A} \times \bar{A}$ whose intersection is empty, then A is elementarily definable in S .

Note

The theorem is no longer true if we add $\bar{A} \times A$ to the list.



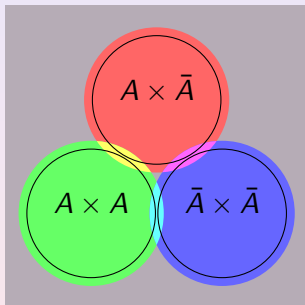
Theorem

Let S be a well-orderable logical structure in which all finite relations are elementarily definable.

If there exist elementarily definable supersets of $A \times A$, $A \times \bar{A}$, and $\bar{A} \times \bar{A}$ whose intersection is empty, then A is elementarily definable in S .

Note

The theorem is no longer true if we add $\bar{A} \times A$ to the list.



Summary

Summary

- The weak cardinality theorems for first-order logic **unify** the weak cardinality theorems of automata and recursion theory.
- The logical approach yields weak cardinality theorems for **other computational models**.
- Cardinality theorems are **separability theorems** in disguise.

Open Problems

- Does a cardinality theorem for first-order logic hold?
- What about non-well-orderable structures like $(\mathbb{R}, +, \cdot)$?

